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BERGMAN KERNEL AND HYPERCONVEXITY INDEX

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Dedicated to Professor John Erik Fornaess on the occasion of his 70th birthday

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with the hyperconvexity index $\alpha(\Omega) > 0$. Let ϱ be the relative extremal function of a fixed closed ball in Ω, and set $\mu := |\varrho|(1 + |\log|\varrho||)^{-1}$ and $\nu := |\varrho|(1 + |\log|\varrho||)^n$. We obtain the following estimates for the Bergman kernel. (1) For every $0 < \alpha < \alpha(\Omega)$ and $2 \le p < 2 + 2\alpha(\Omega)/(2n - \alpha(\Omega))$, there exists a constant $C > 0$ such that $\int_\Omega |K_\Omega(\cdot, w)/\sqrt{K_\Omega(w)}|^p \le C|\mu(w)|^{-(p-2)n/\alpha}$ for all $w \in \Omega$. (2) For every $0 < r < 1$, there exists a constant $C > 0$ such that $|K_\Omega(z, w)|^2/(K_\Omega(z)K_\Omega(w)) \le C(\min\{\nu(z)/\mu(w), \nu(w)/\mu(z)\})^r$ for all $z, w \in \Omega$. Various applications of these estimates are given.

1. Introduction

A domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a negative continuous plurisubharmonic (psh) function ρ on Ω such that $\{\rho < c\} \Subset \Omega$ for any $c < 0$. The class of hyperconvex domains is very wide; e.g., every bounded pseudoconvex domain with Lipschitz boundary is hyperconvex [Demailly 1987]. Although hyperconvex domains already admit a rich function theory (see, e.g., [Ohsawa 1993; Błocki and Pflug 1998; Herbort 1999; Poletsky and Stessin 2008]), it is not enough to get quantitative results unless one imposes certain growth conditions on the bounded exhaustion function ρ (compare [Berndtsson and Charpentier 2000; Błocki 2005; Diederich and Ohsawa 1995]).

A meaningful condition is $-\rho \le C\delta^\alpha$ for some constants $\alpha, C > 0$, where δ denotes the boundary distance. Let $\alpha(\Omega)$ be the supremum of all α. We call it the *hyperconvexity index* of Ω. From the fundamental work of Diederich and Fornaess [1977], we know that if Ω is a bounded pseudoconvex domain with C^2-boundary then there exists a continuous negative psh function ρ on Ω such that $C^{-1}\delta^\eta \le -\rho \le C\delta^\eta$ for some constants $\eta, C > 0$. The supremum $\eta(\Omega)$ of all η is called the *Diederich–Fornaess index* of Ω (see, e.g., [Adachi and Brinkschulte 2015; Fu and Shaw 2016; Harrington 2008]). Clearly, $\alpha(\Omega) \ge \eta(\Omega)$. Recently, Harrington [2008] showed that if Ω is a bounded pseudoconvex domain with Lipschitz boundary then $\eta(\Omega) > 0$.

On the other hand, there are plenty of domains with very irregular boundaries such that $\alpha(\Omega) > 0$, while it is difficult to verify $\eta(\Omega) > 0$. For instance, Koebe's distortion theorem implies $\alpha(\Omega) \ge \frac{1}{2}$ if $\Omega \subsetneq \mathbb{C}$ is a simply connected domain [Carleson and Gamelin 1993, Chapter 1, Theorem 4.4]. Recently, Carleson and Totik [2004] and Totik [2006] obtained various Wiener-type criteria for planar domains with positive

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hyperconvexity indices. In particular, if $\partial\Omega$ is uniformly perfect in the sense of Pommerenke [1979], then $\alpha(\Omega) > 0$ [Carleson and Totik 2004, Theorem 1.7]. Moreover, for domains like $\Omega = \mathbb{C} \setminus E$, where E is a compact set in $\mathbb{R}$ (e.g., Cantor-type sets), the connection between the metric properties of E and the precise value of $\alpha(\Omega)$ (especially the optimal case $\alpha(\Omega) = \frac{1}{2}$) was studied in detail in [Carleson and Totik 2004; Totik 2006]. In the Appendix of this paper, we will provide more examples of higher-dimensional domains with positive hyperconvexity indices. The Teichmüller space of a compact Riemann surface with genus ≥ 2 which is boundedly embedded in $\mathbb{C}^{3g-3}$ probably has a positive hyperconvexity index.

For a domain $\Omega \subset \mathbb{C}^n$, let ϱ be the *relative extremal function* of a (fixed) closed ball $\bar{B} \subset \Omega$; i.e.,

$$\varrho(z) := \varrho_{\bar{B}}(z) := \sup\{u(z) : u \in \mathrm{PSH}^-(\Omega),\ u|_{\bar{B}} \leq -1\},$$

where $\mathrm{PSH}^-(\Omega)$ denotes the set of negative psh functions on Ω. It is known that ϱ is continuous on $\overline{\Omega}$ if Ω is a bounded hyperconvex domain [Błocki 2002, Proposition 3.1.3(vii)]. Furthermore, it is easy to show that if $\alpha(\Omega) > 0$ then for every $0 < \alpha < \alpha(\Omega)$ there exists a constant $C > 0$ such that $-\varrho \leq C\delta^\alpha$.

The goal of this paper is to present some off-diagonal estimates of the Bergman kernel on domains with positive hyperconvexity indices, in terms of ϱ. Usually, off-diagonal behavior of the Bergman kernel is more sensitive to the geometry of a domain than on-diagonal behavior (compare to [Barrett 1992]).

Let $K_\Omega(z, w)$ be the Bergman kernel of Ω. It is well-known that $K_\Omega(\cdot, w) \in L^2(\Omega)$ for all $w \in \Omega$. Thus, it is natural to ask the following:

Problem. For which Ω and $p > 2$ does one have $K_\Omega(\cdot, w) \in L^p(\Omega)$ for all $w \in \Omega$?

For the sake of convenience, we set

$$\beta(\Omega) = \sup\{\beta \geq 2 : K_\Omega(\cdot, w) \in L^\beta(\Omega) \text{ for all } w \in \Omega\}.$$

We call it the *integrability index* of the Bergman kernel. From the well-known works of Kerzman, Catlin and Bell, we know that $\beta(\Omega) = \infty$ if Ω is a bounded pseudoconvex domain of finite D'Angelo type. On the other hand, it is not difficult to see from the work of Barrett [1992] that there exist unbounded Diederich–Fornaess worm domains with $\beta(\Omega)$ arbitrarily close to 2 (see, e.g., [Krantz and Peloso 2008, Lemma 7.5]). Thus, it is meaningful to show the following:

Theorem 1.1. *If $\Omega \subset \mathbb{C}^n$ is pseudoconvex, then $\beta(\Omega) \geq 2 + 2\alpha(\Omega)/(2n - \alpha(\Omega))$. Furthermore, if Ω is a bounded domain with $\alpha(\Omega) > 0$, then for every $0 < \alpha < \alpha(\Omega)$ and $2 \leq p < 2 + 2\alpha(\Omega)/(2n - \alpha(\Omega))$, there exists a constant $C > 0$ such that*

$$\int_\Omega \left|K_\Omega(\cdot, w)/\sqrt{K_\Omega(w)}\right|^p \leq C|\mu(w)|^{-(p-2)n/\alpha}, \quad w \in \Omega, \tag{1-1}$$

where $K_\Omega(w) = K_\Omega(w, w)$ and $\mu := |\varrho|(1 + |\log|\varrho||)^{-1}$.

The lower bound for $\beta(\Omega)$ can be improved substantially when $n = 1$:

Theorem 1.2. *If Ω is a domain in $\mathbb{C}$, then $\beta(\Omega) \geq 2 + \alpha(\Omega)/(1 - \alpha(\Omega))$.*

In particular, we obtain the known fact that if $\Omega \subsetneq \mathbb{C}$ is a simply connected domain then $\beta(\Omega) \geq 3$. A famous conjecture of Brennan [1978] suggests that the bound may be improved to $\beta(\Omega) \geq 4$; an equivalent statement is that, if $f : \Omega \to \mathbb{D}$ is a conformal mapping where $\mathbb{D}$ is the unit disc, then $f' \in L^p(\Omega)$ for all $p < 4$. There has been extensive research on this conjecture (see [Bertilsson 1998; Carleson and Jones 1992; Carleson and Makarov 1994; Pommerenke 1992], etc.).

Nevertheless, Theorem 1.2 is best understood in view of the following:

Proposition 1.3. *Let $E \subset \mathbb{C}$ be a compact set satisfying* $\mathrm{Cap}(E) > 0$ *and* $\dim_H(E) < 1$, *where* Cap *and* $\dim_H$ *denote the logarithmic capacity and the Hausdorff dimension, respectively. Set $\Omega := \mathbb{C} \setminus E$. Then* $\beta(\Omega) \le 2 + \dim_H(E)/(1 - \dim_H(E))$.

Example. There exists a Cantor-type set E with $\dim_H(E) = 0$ and $\mathrm{Cap}(E) > 0$ [Carleson 1967, §4, Theorem 5]. Thus, $\beta(\mathbb{C} \setminus E) = 2$ in view of Proposition 1.3.

Example. Andrievskii [2005] constructed a compact set $E \subset \mathbb{R}$ with $\dim_H(E) = \frac{1}{2}$ and $\alpha(\mathbb{C} \setminus E) = \frac{1}{2}$. It follows from Theorem 1.2 and Proposition 1.3 that $\beta(\mathbb{C} \setminus E) = 3$.

Problem. Is there a bounded domain $\Omega \subset \mathbb{C}$ with $\beta(\Omega) = 2$?

Theorems 1.1 and 1.2 shed some light on the study of the Bergman space

$$A^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f|^p < \infty \right\}$$

for domains with positive hyperconvexity indices. For instance, we can show that $A^p(\Omega) \cap A^2(\Omega)$ lies dense in $A^2(\Omega)$ for suitable $p > 2$ and the reproducing property of $K_\Omega(z, w)$ holds in $A^p(\Omega)$ for suitable $p < 2$ (see Section 4). A related problem is to study whether the Bergman projection can be extended to a bounded projection $L^p(\Omega) \to A^p(\Omega)$ for all p in some nonempty open interval around 2. For flat Hartogs triangles, a complete answer was recently given by Edholm and McNeal [2016]. For more information on this matter, we refer the reader to the review article of Lanzani [2015] and the references therein.

Set

$$K_{\Omega,p}(z) := \sup\{|f(z)| : f \in A^p(\Omega),\ \|f\|_{L^p(\Omega)} \le 1\}.$$

Using $f := (K_\Omega(\cdot\,, z)/\sqrt{K_\Omega(z)})/\|K_\Omega(\cdot\,, z)/\sqrt{K_\Omega(z)}\|_{L^p(\Omega)}$ as a candidate, we conclude from estimate (1-1):

Corollary 1.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $p < 2 + 2\alpha(\Omega)/(2n - \alpha(\Omega))$,*

$$K_{\Omega,p}(z) \ge C_{\alpha,p}\sqrt{K_\Omega(z)}|\mu(z)|^{(p-2)n/(p\alpha)}.$$

Remark. If Ω is a bounded pseudoconvex domain with C^2-boundary, then $K_\Omega(z) \ge C\delta(z)^{-2}$ in view of the Ohsawa–Takegoshi extension theorem [1987]. On the other hand, Hopf's lemma implies $|\varrho| \ge C\delta$. Thus,

$$K_{\Omega,p}(z) \ge C_{\alpha,p}\delta(z)^{-(1-(p-2)n/(p\alpha))}|\log\delta(z)|^{-(p-2)n/(p\alpha)}$$

as $z \to \partial\Omega$. Notice also that $(p-2)n/(p\alpha) < \frac{1}{2}$ if and only if $p < 2 + 2\alpha(\Omega)/(2n - \alpha(\Omega))$.

We would like to mention an interesting connection between the problem on page 1430 and the regularity problem of biholomorphic maps. The starting point is the following:

Theorem 1.5 [Lempert 1986, Theorem 6.2]. *Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with C^2-boundary such that its Bergman projection P_{Ω_1} maps $C_0^\infty(\Omega_1)$ into $L^p(\Omega_1)$ for some $p > 2$. Let $\Omega_2 \subset \mathbb{C}^n$ be a bounded domain with real-analytic boundary. Then any biholomorphic map $F : \Omega_1 \to \Omega_2$ extends to a Hölder-continuous map $\overline{\Omega}_1 \to \overline{\Omega}_2$.*

Notice that if Ω is a domain with $\int_\Omega |K_\Omega(\cdot, w)|^p$ locally uniformly bounded in w for some $p \geq 1$, then for any $\phi \in C_0^\infty(\Omega)$,

$$|P_\Omega(\phi)(z)|^p \leq \int_{\zeta \in \operatorname{supp}\phi} |K_\Omega(\zeta, z)|^p \|\phi\|^p_{L^q(\Omega)}, \quad 1/p + 1/q = 1,$$

so that

$$\int_{z\in\Omega} |P_\Omega(\phi)(z)|^p \leq \|\phi\|^p_{L^q(\Omega)} \int_{\zeta \in \operatorname{supp}\phi} \int_{z\in\Omega} |K_\Omega(z, \zeta)|^p < \infty, \tag{1-2}$$

i.e., P_Ω maps $C_0^\infty(\Omega)$ into $L^p(\Omega)$. Thus, we have:

Corollary 1.6. *Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with C^2-boundary such that the integral $\int_\Omega |K_\Omega(\cdot, w)|^p$ is locally uniformly bounded in w for some $p > 2$. Let $\Omega_2 \subset \mathbb{C}^n$ be a bounded domain with real-analytic boundary. Then any biholomorphic map $F : \Omega_1 \to \Omega_2$ extends to a Hölder-continuous map $\overline{\Omega}_1 \to \overline{\Omega}_2$.*

In particular, it follows from Corollary 1.6 and Theorem 1.1 that any biholomorphic map between a bounded *pseudoconvex* domain with C^2-boundary and a bounded domain with real-analytic boundary extends to a Hölder-continuous map between their closures, which was first proved in [Diederich and Fornaess 1979]. On the other hand, Barrett [1984] constructed a *nonpseudoconvex* bounded smooth domain $\Omega \subset \mathbb{C}^2$ such that P_Ω fails to map $C_0^\infty(\Omega)$ into $L^p(\Omega)$ for any $p > 2$ so that $\int_\Omega |K_\Omega(\cdot, w)|^p$ can not be locally uniformly bounded in w. However, it is still expected that if Ω is a bounded domain with *real-analytic* boundary then there exists $p > 2$ such that $\int_\Omega |K_\Omega(\cdot, w)|^p$ is locally uniformly bounded in w.

With the help of an elegant technique due to Błocki [2005] (see also [Herbort 2000] for prior related techniques) on estimating the pluricomplex Green function, we may prove the following:

Theorem 1.7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. For every $0 < r < 1$, there exists a constant $C > 0$ such that*

$$\mathcal{B}_\Omega(z, w) := \frac{|K_\Omega(z, w)|^2}{K_\Omega(z) K_\Omega(w)} \leq C\left(\min\left\{\frac{\nu(z)}{\mu(w)}, \frac{\nu(w)}{\mu(z)}\right\}\right)^r, \quad z, w \in \Omega, \tag{1-3}$$

where $\mu := |\varrho|/(1 + |\log|\varrho||)$ and $\nu := |\varrho|(1 + |\log|\varrho||)^n$.

We call $\mathcal{B}_\Omega(z, w)$ the normalized Bergman kernel of Ω. There is a long list of papers about pointwise estimates of the *weighted* normalized Bergman kernel $\mathcal{B}_{\Omega,\varphi}(z, w) := |K_{\Omega,\varphi}(z, w)|^2/(K_{\Omega,\varphi}(z) K_{\Omega,\varphi}(w))$ when Ω is $\mathbb{C}^n$ or a compact algebraic manifold, after a seminal paper of Christ [1991] (see [Delin 1998; Lindholm 2001; Ma and Marinescu 2007; Christ 2013; Zelditch 2016], etc.). Quantitative measurements of positivity of $i\partial\bar\partial\varphi$ play a crucial role in these works.

The basic difference between $\mathcal{B}_\Omega(z, w)$ and $\mathcal{B}_{\Omega,\varphi}(z, w)$ is that the former is always a *biholomorphic invariant*. Skwarczyński [1980] showed that

$$d_S(z, w) := \left(1 - \sqrt{\mathcal{B}_\Omega(z, w)}\right)^{1/2}$$

gives an invariant distance on a bounded domain Ω. The relationship between d_S and the Bergman distance d_B is

$$d_B(z, w) \geq \sqrt{2} d_S(z, w) \tag{1-4}$$

(see, e.g., [Jarnicki and Pflug 1993, Corollary 6.4.7]). By Theorem 1.7 and (1-4), we may prove the following:

Corollary 1.8. *If Ω is a bounded domain with $\alpha(\Omega) > 0$, then for fixed $z_0 \in \Omega$, there exists a constant $C > 0$ such that*

$$d_B(z_0, z) \geq C \frac{|\log \delta(z)|}{\log|\log \delta(z)|}, \tag{1-5}$$

provided z sufficiently close to $\partial\Omega$.

Błocki [2005] first proved (1-5) for any bounded domain which admits a continuous negative psh function ρ with $C_1\delta^\alpha \leq -\rho \leq C_2\delta^\alpha$ for some constants $C_1, C_2, \alpha > 0$ (e.g., Ω is a pseudoconvex domain with Lipschitz boundary [Harrington 2008]). Diederich and Ohsawa [1995] proved earlier that the weaker inequality

$$d_B(z_0, z) \geq C \log|\log \delta(z)|$$

holds for more general bounded domains admitting a continuous negative psh function ρ with $C_1\delta^{1/\alpha} \leq -\rho \leq C_2\delta^\alpha$ for some constants $C_1, C_2, \alpha > 0$.

In order to study isometric embedding of Kähler manifolds, Calabi [1953] introduced the notion "diastasis". Marcel Berger [1996] wrote, "It seems to me that the notion of diastasis should make a comeback [...]. For example, it would be interesting to compare the diastasis with the various types of Kobayashi metrics (when they exist)."

Notice that the diastasis $D_B(z, w)$ with respect to the Bergman metric is $-\log \mathcal{B}_\Omega(z, w)$.

Corollary 1.9. *If Ω is a bounded domain with $\alpha(\Omega) > 0$, then for fixed $z_0 \in \Omega$, there exists a constant $C > 0$ such that*

$$D_B(z_0, z) \geq C d_K(z_0, z), \tag{1-6}$$

where d_K denotes the Kobayashi distance.

Problem. Does one have $d_B(z_0, z) \geq C d_K(z_0, z)$ for bounded domains with $\alpha(\Omega) > 0$?

A domain $\Omega \subset \mathbb{C}^n$ is called *weighted circular* if there exists an n-tuple $(a_1, \ldots, a_n)$ of positive numbers such that $z \in \Omega$ implies $(e^{ia_1\theta} z_1, \ldots, e^{ia_n\theta} z_n) \in \Omega$ for any $\theta \in \mathbb{R}$. As a final consequence of Theorem 1.7, we obtain:

Corollary 1.10. *Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega_1) > 0$. Let $\Omega_2 \subset \mathbb{C}^n$ be a bounded weighted circular domain which contains the origin. Let $0 < \alpha < \alpha(\Omega_1)$ be given. Then for any biholomorphic map $F : \Omega_1 \to \Omega_2$, there is a constant $C > 0$ such that*

$$\delta_2(F(z)) \leq C\delta_1(z)^{\alpha/(2n)}, \quad z \in \Omega_1. \tag{1-7}$$

Here δ_1 and δ_2 denote the boundary distances of Ω_1 and Ω_2, respectively.

Remark. Inequalities like (1-7) are crucial in the study of the regularity problem of biholomorphic maps (see, e.g., [Diederich and Fornaess 1979; Lempert 1986]).

2. L^2 boundary decay estimates of the Bergman kernel

Proposition 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain. Let ρ be a negative continuous psh function on Ω. Set*

$$\Omega_t = \{z \in \Omega : -\rho(z) > t\}, \quad t > 0.$$

Let $a > 0$ be given. For every $0 < r < 1$, there exist constants $\varepsilon_r, C_r > 0$ such that

$$\int_{-\rho \le \varepsilon} |K_\Omega(\cdot, w)|^2 \le C_r K_{\Omega_a}(w)(\varepsilon/a)^r \tag{2-1}$$

for all $w \in \Omega_a$ and $\varepsilon \le \varepsilon_r a$.

The proof of the proposition is essentially the same as for Proposition 6.1 in [Chen 2016]. For the sake of completeness, we include a proof here. The key ingredient is the following weighted estimate of the L^2-minimal solution of the $\bar\partial$-equation due to Berndtsson.

Theorem 2.2 [Chen 2016, Corollary 2.3]. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\varphi \in$ PSH(Ω). Let ψ be a continuous psh function on Ω which satisfies $ri\partial\bar\partial\psi \ge i\partial\psi \wedge \bar\partial\psi$ as currents for some $0 < r < 1$. Suppose v is a $\bar\partial$-closed $(0,1)$-form on Ω such that $\int_\Omega |v|^2 e^{-\varphi} < \infty$. Then the $L^2(\Omega, \varphi)$-minimal solution of $\bar\partial u = v$ satisfies*

$$\int_\Omega |u|^2 e^{-\psi-\varphi} \le \frac{1}{1-r} \int_\Omega |v|^2_{i\partial\bar\partial\psi} e^{-\psi-\varphi}. \tag{2-2}$$

Here $|v|^2_{i\partial\bar\partial\psi}$ should be understood as the infimum of nonnegative locally bounded functions H satisfying $i\bar v \wedge v \le Hi\partial\bar\partial\psi$ as currents.

Proof of Proposition 2.1. Assume first that Ω is bounded. Let $\kappa : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\kappa|_{(-\infty,1]} = 1$, $\kappa|_{[3/2,\infty)} = 0$ and $|\kappa'| \le 2$. We then have

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot, w)|^2 \le \int_\Omega \kappa(-\rho/\varepsilon)|K_\Omega(\cdot, w)|^2.$$

By the well-known property of the Bergman projection, we obtain

$$\int_\Omega \kappa(-\rho/\varepsilon) K_\Omega(\cdot, w) \cdot \overline{K_\Omega(\cdot, \zeta)} = \kappa(-\rho(\zeta)/\varepsilon) K_\Omega(\zeta, w) - u(\zeta), \quad \zeta \in \Omega,$$

where u is the $L^2(\Omega)$-minimal solution of the equation

$$\bar\partial u = \bar\partial(\kappa(-\rho/\varepsilon)K_\Omega(\cdot, w)) =: v.$$

Since $\kappa(-\rho(w)/\varepsilon) = 0$ provided $\frac{3}{2}\varepsilon \le a$ (i.e., $\varepsilon \le 2a/3$),

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot, w)|^2 \le -u(w). \tag{2-3}$$

Set

$$\psi = -r\log(-\rho), \quad 0 < r < 1.$$

Clearly, ψ is psh and satisfies $ri\partial\bar\partial\psi \ge i\partial\psi \wedge \bar\partial\psi$ so that

$$i\bar v \wedge v \le C_0 r^{-1}|\kappa'(-\rho/\varepsilon)|^2 |K_\Omega(\cdot, w)|^2 i\partial\bar\partial\psi$$

for some numerical constant $C_0 > 0$. Thus, by Theorem 2.2,

$$\begin{aligned}\int_\Omega |u|^2 e^{-\psi} &\le C_r \int_{\varepsilon\le -\rho\le (3/2)\varepsilon} |K_\Omega(\cdot, w)|^2 e^{-\psi} \\ &\le C_r \varepsilon^r \int_{-\rho\le(3/2)\varepsilon} |K_\Omega(\cdot, w)|^2.\end{aligned}$$

Since $e^{-\psi} \ge a^r$ on Ω_a and u is holomorphic there, it follows that

$$\begin{aligned}|u(w)|^2 &\le K_{\Omega_a}(w) \int_{\Omega_a} |u|^2 \\ &\le K_{\Omega_a}(w) a^{-r} \int_\Omega |u|^2 e^{-\psi} \\ &\le C_r K_{\Omega_a}(w)(\varepsilon/a)^r \int_{-\rho\le(3/2)\varepsilon} |K_\Omega(\cdot, w)|^2.\end{aligned}$$

Thus, by (2-3),

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot, w)|^2 \le C_r K_{\Omega_a}(w)^{1/2} (\varepsilon/a)^{r/2} \left(\int_{-\rho\le(3/2)\varepsilon} |K_\Omega(\cdot, w)|^2\right)^{1/2}.$$

Notice that

$$\int_{-\rho\le(3/2)\varepsilon} |K_\Omega(\cdot, w)|^2 \le \int_\Omega |K_\Omega(\cdot, w)|^2 = K_\Omega(w) \le K_{\Omega_a}(w)$$

provided $\frac{3}{2}\varepsilon \le a$. Thus,

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot, w)|^2 \le C_r K_{\Omega_a}(w)(\varepsilon/a)^{r/2}.$$

Replacing ε by $\frac{3}{2}\varepsilon$ in the argument above, we obtain

$$\int_{-\rho\le(3/2)\varepsilon} |K_\Omega(\cdot, w)|^2 \le C_r K_{\Omega_a}(w)(3/2)^{r/2}(\varepsilon/a)^{r/2}$$

provided $(\frac{3}{2})^2\varepsilon \le a$. Thus, we may improve the upper bound by

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot, w)|^2 \le C_r K_{\Omega_a}(w)(\varepsilon/a)^{r/2+r/4}.$$

By induction, we conclude that, for every $k \in \mathbb{Z}^+$,

$$\int_{-\rho\le\varepsilon} |K_\Omega(\cdot, w)|^2 \le C_{r,k} K_{\Omega_a}(w)(\varepsilon/a)^{r/2+r/4+\cdots+r/2^k}$$

provided $(\frac{3}{2})^k\varepsilon \le a$. Since $r/2+r/4+\cdots+r/2^k \to 1$ as $k\to\infty$ and $r\to 1$, we get the desired estimate under the assumption that Ω is bounded.

In general, Ω may be exhausted by an increasing sequence $\{\Omega_j\}$ of bounded pseudoconvex domains. From the argument above, we know that

$$\int_{\Omega_j\cap\{-\rho\le\varepsilon\}} |K_{\Omega_j}(\cdot\,,w)|^2 \le C_r K_{\Omega_j\cap\Omega_a}(w)(\varepsilon/a)^r$$

holds for all $j\gg 1$. Since $\Omega_j\uparrow\Omega$, it is well-known that $K_{\Omega_j}(\cdot\,,w)\to K_\Omega(\cdot\,,w)$ locally uniformly in Ω and $K_{\Omega_j\cap\Omega_a}(w)\to K_{\Omega_a}(w)$. It follows from Fatou's lemma that

$$\begin{aligned}\int_{-\rho\le\varepsilon}|K_\Omega(\cdot\,,w)|^2 &= \liminf_{j\to\infty}\int_{\Omega_j\cap\{-\rho\le\varepsilon\}}|K_{\Omega_j}(\cdot\,,w)|^2\\ &\le C_r K_{\Omega_a}(w)(\varepsilon/a)^r.\end{aligned}$$

□

Remark. One of the referees kindly suggested an alternative proof as follows. Berndtsson and Charpentier [2000] showed that, if $\int_\Omega |f|^2|\rho|^{-r}<\infty$ for some $0<r<1$, then

$$\int_\Omega |P_\Omega(f)|^2|\rho|^{-r} \le C_r\int_\Omega |f|^2|\rho|^{-r}<\infty$$

where $P_\Omega(f)(z) := \int_\Omega K_\Omega(z,\cdot)f(\cdot)$ is the Bergman projection. If one applies $f=\chi_{\Omega_a}K_{\Omega_a}(\cdot\,,w)$ where χ_{Ω_a} denotes the characteristic function on Ω_a, then $K_\Omega(z,w)=P_\Omega(f)(z)$ and

$$\int_\Omega |K_\Omega(\cdot\,,w)|^2|\rho|^{-r}\le C_r\int_{\Omega_a}|K_{\Omega_a}(\cdot\,,w)|^2|\rho|^{-r},$$

from which the estimate (2-1) immediately follows.

Let ϱ be the relative extremal function of a (fixed) closed ball $\bar B\subset\Omega$. We have:

Proposition 2.3. *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain with $\alpha(\Omega)>0$. For every $0<r<1$, there exist constants $\varepsilon_r, C_r>0$ such that*

$$\int_{-\varrho\le\varepsilon}|K_\Omega(\cdot\,,w)|^2/K_\Omega(w)\le C_r(\varepsilon/\mu(w))^r \tag{2-4}$$

for all $\varepsilon\le\varepsilon_r\mu(w)$, where $\mu=|\varrho|(1+|\log|\varrho||)^{-1}$.

In order to prove this proposition, we need an elementary estimate of the pluricomplex Green function. Recall that the *pluricomplex Green function* $g_\Omega(z,w)$ of a domain $\Omega\subset\mathbb{C}^n$ is defined as

$$g_\Omega(z,w)=\sup\{u(z): u\in \mathrm{PSH}^-(\Omega),\ u(z)\le\log|z-w|+O(1)\text{ near }w\}.$$

We first show the following quasi-Hölder-continuity of ϱ.

Lemma 2.4. *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain with $\alpha(\Omega)>0$. For every $r>1$ and $0<\alpha<\alpha(\Omega)$, there exists a constant $C>0$ such that*

$$\varrho(z_2)\ge r\varrho(z_1)-C|z_1-z_2|^\alpha,\quad z_1,z_2\in\Omega. \tag{2-5}$$

Proof. Choose $\rho \in C(\Omega) \cap \mathrm{PSH}^-(\Omega)$ with $-\rho \le C_\alpha \delta^\alpha$. Clearly

$$\varrho(z) \ge \frac{\rho(z)}{\inf_{\bar B} |\rho|} \ge -C_\alpha \delta^\alpha.$$

To get (2-5), we employ a well-known technique of Walsh [1968] as follows. Set $\varepsilon := |z_1 - z_2|$, $\Omega' := \Omega - (z_1 - z_2)$ and

$$u(z) = \begin{cases} \varrho(z) & \text{if } z \in \Omega \setminus \Omega', \\ \max\{\varrho(z), r\varrho(z + z_1 - z_2) - C\varepsilon^\alpha\} & \text{if } z \in \Omega \cap \Omega'. \end{cases}$$

We claim that $u \in \mathrm{PSH}^-(\Omega)$ provided $C \gg 1$. Indeed, if $z \in \Omega \cap \partial\Omega'$, then $\delta(z) \le \varepsilon$ so that

$$\varrho(z) \ge -C_\alpha \delta(z)^\alpha \ge -C_\alpha \varepsilon^\alpha \ge r\varrho(z + z_1 - z_2) - C_\alpha \varepsilon^\alpha.$$

Moreover, if $\varepsilon \le \varepsilon_r \ll 1$, then $\varrho(z + z_1 - z_2) \le -1/r$ for $z \in \bar B$ since ϱ is continuous on $\overline{\Omega}$. Thus, $u|_{\bar B} \le -1$. Since $z_2 = z_1 - (z_1 - z_2) \in \Omega \cap \Omega'$, it follows that

$$\varrho(z_2) \ge u(z_2) \ge r\varrho(z_1) - C_\alpha \varepsilon^\alpha.$$

If $\varepsilon = |z_1 - z_2| > \varepsilon_r$, then (2-5) trivially holds. □

Remark. It is not known whether ϱ is Hölder-continuous on $\overline{\Omega}$. The answer is positive if $n = 1$ [Carleson and Gamelin 1993, p. 138].

Proposition 2.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\alpha(\Omega) > 0$. There exists a constant $C \gg 1$ such that*

$$\{g_\Omega(\cdot\,, w) < -1\} \subset \{\varrho < -C^{-1}\mu(w)\}, \quad w \in \Omega. \tag{2-6}$$

Proof. Fix $0 < \alpha < \alpha(\Omega)$. We have $-\varrho \le C_\alpha \delta^\alpha$ for some constant $C_\alpha > 0$. Clearly, it suffices to consider the case when $|\varrho(w)| \le \frac{1}{2}$. Applying Lemma 2.4 with $r = \frac{3}{2}$, we see that if $\varrho(z) = \varrho(w)/2$ then

$$C_1 |z - w|^\alpha \ge \tfrac{3}{2}\varrho(z) - \varrho(w) = -\tfrac{1}{4}\varrho(w)$$

so that

$$\log \frac{|z - w|}{R} \ge \frac{1}{\alpha} \log|\varrho(w)|/(4C_1) - \log R \ge C_2 \log|\varrho(w)|$$

for some constant $C_2 \gg 1$. It follows that

$$\psi(z) := \begin{cases} \log|z - w|/R & \text{if } \varrho(z) \le \varrho(w)/2, \\ \max\{\log|z - w|/R, 2C_2(\varrho(w)^{-1} \log|\varrho(w)|)\varrho(z)\} & \text{otherwise} \end{cases}$$

is a well-defined negative psh function on Ω with a logarithmic pole at w, and if $\varrho(z) \ge \varrho(w)/2$, then

$$g_\Omega(z, w) \ge \psi(z) \ge 2C_2(\varrho(w)^{-1} \log|\varrho(w)|)\varrho(z). \tag{2-7}$$

Thus,

$$\{g_\Omega(\cdot\,, w) < -1\} \cap \{\varrho \ge \varrho(w)/2\} \subset \{\varrho < -C^{-1}\mu(w)\}$$

provided $C \gg 1$. Since $\{\varrho < \varrho(w)/2\} \subset \{\varrho < -C^{-1}\mu(w)\}$ if $C \gg 1$, we conclude the proof. □

Proof of Proposition 2.3. Set $A_w := \{g_\Omega(\cdot, w) < -1\}$. It is known from [Herbort 1999] or [Chen 1999] that

$$K_{A_w}(w) \le C_n K_\Omega(w). \tag{2-8}$$

By Proposition 2.5,

$$A_w \subset \Omega_{a(w)} := \{\varrho < -a(w)\} \tag{2-9}$$

where $a(w) := C^{-1}\mu(w)$ with $C \gg 1$. If we choose $\rho = \varrho$ in Proposition 2.1, it follows that, for every $\varepsilon \le \varepsilon_r a(w)$,

$$\begin{aligned} \int_{-\varrho \le \varepsilon} |K_\Omega(\cdot, w)|^2 &\le C_r K_{\Omega_{a(w)}}(w)(\varepsilon/a(w))^r \\ &\le C_{n,r} K_\Omega(w)(\varepsilon/a(w))^r \end{aligned} \tag{2-10}$$

in view of (2-8) and (2-9). □

3. L^p-integrability of the Bergman kernel

Proof of Theorem 1.1. Without loss of generality, we may assume $\alpha(\Omega) > 0$. For every $0 < \alpha < \alpha(\Omega)$, we may choose $\rho \in \mathrm{PSH}^-(\Omega)$ such that

$$-\rho \le C_\alpha \delta^\alpha$$

for some constant $C_\alpha > 0$. Let S be a compact set in Ω, and let $w \in S$. By virtue of Proposition 2.1, we conclude that, for every $0 < r < 1$,

$$\int_{-\rho \le \varepsilon} |K_\Omega(\cdot, w)|^2 \le C\varepsilon^r$$

where $C = C(n, r, \alpha, S) > 0$. Since $\{\delta \le \varepsilon\} \subset \{-\rho \le C_\alpha \varepsilon^\alpha\}$, it follows that

$$\int_{\delta \le \varepsilon} |K_\Omega(\cdot, w)|^2 \le C\varepsilon^{r\alpha}.$$

Since $|\delta(\zeta) - \delta(z)| \le |\zeta - z|$, we have $B(z, \delta(z)) \subset \{\delta \le 2\delta(z)\}$. By the mean value inequality, we get

$$|K_\Omega(z, w)|^2 \le C_n \delta(z)^{-2n} \int_{\delta \le 2\delta(z)} |K_\Omega(\cdot, w)|^2 \le C\delta(z)^{r\alpha - 2n}. \tag{3-1}$$

Thus, for every $\tau > 0$,

$$\begin{aligned} \int_\Omega |K_\Omega(\cdot, w)|^{2+\tau} &= \int_{\delta > 1/2} |K_\Omega(\cdot, w)|^{2+\tau} + \sum_{k=1}^\infty \int_{2^{-k-1} < \delta \le 2^{-k}} |K_\Omega(\cdot, w)|^{2+\tau} \\ &\le C 2^{n\tau} \int_\Omega |K_\Omega(\cdot, w)|^2 + C \sum_{k=1}^\infty 2^{(k+1)\tau(n - r\alpha/2)} \int_{\delta \le 2^{-k}} |K_\Omega(\cdot, w)|^2 \\ &\le C + C 2^{\tau(n - r\alpha/2)} \sum_{k=1}^\infty 2^{-k(r\alpha + \tau(r\alpha/2 - n))} \\ &< \infty \end{aligned}$$

provided $\tau < 2r\alpha/(2n - r\alpha)$. Since r and α can be arbitrarily close to 1 and $\alpha(\Omega)$, respectively, we conclude the proof of the first statement.

Since $\{\delta \le \varepsilon\} \subset \{-\varrho \le C_\alpha \varepsilon^\alpha\}$, it follows from Proposition 2.3 that

$$\int_{\delta\le\varepsilon} |K_\Omega(\cdot\,, w)|^2/K_\Omega(w) \le C_{\alpha,r}(\varepsilon^\alpha/\mu(w))^r \tag{3-2}$$

provided $\varepsilon^\alpha/\mu(w) \le \varepsilon_r \ll 1$. For every $z \in \Omega$,

$$|K_\Omega(z, w)|^2/K_\Omega(w) \le K_\Omega(z) \le C_n \delta(z)^{-2n}, \tag{3-3}$$

and if $(2\delta(z))^\alpha \le \varepsilon_r \mu(w)$,

$$\begin{aligned} |K_\Omega(z, w)|^2 &\le C_n \delta(z)^{-2n} \int_{\delta\le 2\delta(z)} |K_\Omega(\cdot\,, w)|^2 \\ &\le C_{\alpha,r} K_\Omega(w)\mu(w)^{-r}\delta(z)^{\alpha r - 2n}. \end{aligned} \tag{3-4}$$

For every $\tau < 2r\alpha/(2n - r\alpha)$, we conclude from (3-3) that

$$\begin{aligned} \int_{2\delta\ge(\varepsilon_r\mu(w))^{1/\alpha}} |K_\Omega(\cdot\,, w)|^{2+\tau} &\le C_n K_\Omega(w)^{\tau/2} \int_{2\delta\ge(\varepsilon_r\mu(w))^{1/\alpha}} |K_\Omega(\cdot\,, w)|^2 \delta^{-n\tau} \\ &\le C_{\alpha,r} \frac{K_\Omega(w)^{\tau/2}}{\mu(w)^{n\tau/\alpha}} \int_\Omega |K_\Omega(\cdot\,, w)|^2 \\ &\le C_{\alpha,r} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{n\tau/\alpha}}. \end{aligned} \tag{3-5}$$

Now choose $k_w \in \mathbb{Z}^+$ such that $(\varepsilon_r \mu(w))^{1/\alpha} \in (2^{-k_w-1}, 2^{-k_w}]$ (it suffices to consider the case when $\mu(w)$ is sufficiently small). We then have

$$\begin{aligned} \int_{2\delta<(\varepsilon_r\mu(w))^{1/\alpha}} |K_\Omega(\cdot\,, w)|^{2+\tau} &\le \sum_{k=k_w}^\infty \int_{2^{-k-1}<\delta\le 2^{-k}} |K_\Omega(\cdot\,, w)|^{2+\tau} \\ &\le C_{\alpha,r,\tau} \frac{K_\Omega(w)^{\tau/2}}{\mu(w)^{\tau r/2}} \sum_{k=k_w}^\infty 2^{k\tau(n-r\alpha/2)} \int_{\delta\le 2^{-k}} |K_\Omega(\cdot\,, w)|^2 \quad \text{(by (3-4))} \\ &\le C_{\alpha,r,\tau} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{r(1+\tau/2)}} \sum_{k=k_w}^\infty 2^{-k(r\alpha+\tau(r\alpha/2-n))} \quad \text{(by (3-2))} \\ &\le C_{\alpha,r,\tau} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{r(1+\tau/2)}} \mu(w)^{(r\alpha+\tau(r\alpha/2-n))/\alpha} \\ &\le C_{\alpha,r,\tau} \frac{K_\Omega(w)^{1+\tau/2}}{\mu(w)^{\tau n/\alpha}}. \end{aligned} \tag{3-6}$$

By (3-5) and (3-6), (1-1) immediately follows. □

Proof of Theorem 1.2. It suffices to use the following lemma instead of (3-1) in the proof of the first statement in Theorem 1.1. □

Lemma 3.1. *Let Ω be a domain in $\mathbb{C}$. For every compact set $S \subset \Omega$ and $\alpha < \alpha(\Omega)$, there exists a constant $C > 0$ such that*

$$|K_\Omega(z,w)| \le C\delta(z)^{\alpha-1}, \quad z \in \Omega,\ w \in S.$$

Proof. Let $g_\Omega(z,w)$ be the (negative) Green function on Ω. Let $\Delta(c,r)$ be the disc with center c and radius r. Fix $w \in S$ and $z \in \Omega$ for a moment. Clearly, it suffices to consider the case when $\delta(z) \le \delta(w)/4$. Since $g_\Omega(\xi,\zeta)$ is harmonic in $\xi \in \Delta(z,\delta(z))$ and $\zeta \in \Delta(w,\delta(w)/2)$, respectively, we conclude from Poisson's formula that

$$g_\Omega(\xi,\zeta) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} g_\Omega(z+\tfrac{1}{2}\delta(z)e^{i\theta}, w+\tfrac{1}{2}\delta(w)e^{i\vartheta}) \times \frac{\frac{1}{4}\delta(z)^2-|\xi-z|^2}{\left|\frac{1}{2}\delta(z)e^{i\theta}-(\xi-z)\right|^2}\frac{\frac{1}{4}\delta(w)^2-|\zeta-w|^2}{\left|\frac{1}{2}\delta(w)e^{i\vartheta}-(\zeta-w)\right|^2}\,d\theta\,d\vartheta$$

where $\xi \in \Delta(z,\delta(z)/4)$ and $\zeta \in \Delta(w,\delta(w)/4)$. By the extremal property of g_Ω, it is easy to verify that $-g_\Omega \le C\delta(z)^\alpha$ on $\partial\Delta(z,\delta(z)/2)\times\partial\Delta(w,\delta(w)/2)$. Thus,

$$\left|\frac{\partial^2 g_\Omega(\xi,\zeta)}{\partial\xi\,\partial\bar{\zeta}}\right| \le C\delta(z)^{\alpha-1}.$$

Using the formula $K_\Omega(\xi,\zeta) = \dfrac{2}{\pi}\dfrac{\partial^2 g_\Omega(\xi,\zeta)}{\partial\xi\,\partial\bar{\zeta}}$ from [Schiffer 1946], the assertion immediately follows. □

In order to prove Proposition 1.3, we need the following:

Theorem 3.2 [Carleson 1967, §6, Theorem 1]. *Let $\Omega = \mathbb{C}\setminus E$ where $E \subset \mathbb{C}$ is a compact set. Then*

(1) $A^2(\Omega) \ne \{0\}$ *if and only if* $\mathrm{Cap}(E) > 0$, *and*

(2) $A^p(\Omega) = \{0\}$ *if* $\Lambda_{2-q}(E) < \infty$, $2 < p < \infty$ *and* $1/p+1/q=1$. *Here $\Lambda_s(E)$ denotes the s-dimensional Hausdorff measure of E.*

Remark. Let $\Omega \subset \mathbb{C}$ be a domain and E a closed polar set in Ω. It is well-known that E is removable for negative harmonic functions so that $g_{\Omega\setminus E}(z,w) = g_\Omega(z,w)$ for $z,w \in \Omega\setminus E$. Thus, $K_{\Omega\setminus E}(z,w) = K_\Omega(z,w)$ in view of Schiffer's formula. By the reproducing property of the Bergman kernel, we immediately get the known fact that $A^2(\Omega\setminus E) = A^2(\Omega)$.

Proof of Proposition 1.3. Suppose on the contrary $\beta(\Omega) > 2+\dim_H(E)/(1-\dim_H(E))$. Fix

$$\beta(\Omega) > p > 2 + \frac{\dim_H(E)}{1-\dim_H(E)},$$

and let q be the conjugate exponent of p, i.e., $1/p+1/q=1$. We then have $K_\Omega(\cdot,w) \in A^p(\Omega)$ for fixed w. Since

$$\dim_H(E) = \sup\{s : \Lambda_s(E) = \infty\}$$

and $2-q > \dim_H(E)$, it follows that $\Lambda_{2-q}(E) < \infty$ so that $K_\Omega(\cdot,w) = 0$ in view of Theorem 3.2(2). On the other hand, $\mathrm{Cap}(E) > 0$, so $K_\Omega(\cdot,w) \ne 0$ in view of Theorem 3.2(1), which is absurd. □

Theorem 1.2 implies $\beta(\Omega) \to \infty$ as $\alpha(\Omega) \to 1$ for planar domains (notice that $\alpha(\Omega) = 1$ when $\Omega \subset \mathbb{C}$ is convex or $\partial\Omega$ is C^1). It is also known that $\beta(\Omega) = \infty$ if Ω is a bounded smooth convex domain in $\mathbb{C}^n$ [Boas and Straube 1991]. Thus, it is reasonable to make the following:

Conjecture 3.3. *If $\Omega \subset \mathbb{C}^n$ is convex, then $\beta(\Omega) = \infty$.*

4. Applications of L^p-integrability of the Bergman kernel

We first study density of $A^p(\Omega) \cap A^2(\Omega)$ in $A^2(\Omega)$.

Proposition 4.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. For every $1 \le p < 2 + 2\alpha(\Omega)/(2n - \alpha(\Omega))$, $A^p(\Omega) \cap A^2(\Omega)$ lies dense in $A^2(\Omega)$.*

Proof. Choose a sequence of functions $\chi_j \in C_0^\infty(\Omega)$ such that $0 \le \chi_j \le 1$ and the sequence of sets $\{\chi_j = 1\}$ exhausts Ω. Given $f \in A^2(\Omega)$, we set $f_j = P_\Omega(\chi_j f)$. Clearly, $f_j \in A^p(\Omega) \cap A^2(\Omega)$ in view of Theorem 1.1 and (1-2). Moreover,

$$\|f_j - f\|_{L^2(\Omega)} = \|P_\Omega((\chi_j - 1)f)\|_{L^2(\Omega)} \le \|(\chi_j - 1)f\|_{L^2(\Omega)} \to 0. \qquad \square$$

Similarly, we may prove the following:

Proposition 4.2. *Let Ω be a domain in $\mathbb{C}$. For every $1 \le p < 2 + \alpha(\Omega)/(1 - \alpha(\Omega))$, $A^p(\Omega) \cap A^2(\Omega)$ lies dense in $A^2(\Omega)$.*

Next we study the reproducing property of the Bergman kernel in $A^p(\Omega)$.

Proposition 4.3. *Let Ω be a bounded domain in $\mathbb{C}$ with $\alpha(\Omega) > 0$. If $p > 2 - \alpha(\Omega)$, then $f = P_\Omega(f)$ for all $f \in A^p(\Omega)$.*

Proof. Suppose $f \in A^p(\Omega)$ with $p > 2 - \alpha(\Omega)$. Let q be the conjugate exponent of p. Since $q < 2 + \alpha(\Omega)/(1 - \alpha(\Omega))$, the integral $\int_\Omega f(\cdot)K_\Omega(z, \cdot)$ is well-defined in view of Theorem 1.2. Clearly, it suffices to consider the case $p < 2$. By Theorem 1 of [Hedberg 1972], we may find a sequence $f_j \in \mathcal{O}(\overline{\Omega}) \subset A^2(\Omega) \subset A^p(\Omega)$ such that $\|f_j - f\|_{L^p(\Omega)} \to 0$. It follows that, for every $z \in \Omega$,

$$f(z) = \lim_{j\to\infty} f_j(z) = \lim_{j\to\infty} \int_\Omega f_j(\cdot)K_\Omega(z, \cdot) = \int_\Omega f(\cdot)K_\Omega(z, \cdot)$$

since $K_\Omega(z, \cdot) \in L^q(\Omega)$. $\square$

For a bounded domain $\Omega \subset \mathbb{C}^n$, the *Berezin transform* T_Ω of Ω is defined as

$$T_\Omega(f)(z) = \int_\Omega f(\cdot)\frac{|K_\Omega(\cdot, z)|^2}{K_\Omega(z)}, \quad z \in \Omega, \ f \in L^\infty(\Omega).$$

Clearly, one has $f = T_\Omega(f)$ for all $f \in A^\infty(\Omega)$.

Corollary 4.4. *Let Ω be a bounded domain in $\mathbb{C}$ with $\alpha(\Omega) > 0$. If $p > 2/\alpha(\Omega) - 1$, then $f = T_\Omega(f)$ for all $f \in A^p(\Omega)$.*

Proof. Set $p' = 2p/(p+1)$. It follows from Hölder's inequality that

$$\begin{aligned}\int_\Omega |f K_\Omega(\cdot\,,z)|^{p'} &\le \left(\int_\Omega |f|^{p'/(2-p')}\right)^{2-p'} \left(\int_\Omega |K_\Omega(\cdot\,,z)|^{p'/(p'-1)}\right)^{p'-1}\\ &= \left(\int_\Omega |f|^{p}\right)^{2-p'} \left(\int_\Omega |K_\Omega(\cdot\,,z)|^{p'/(p'-1)}\right)^{p'-1}\\ &< \infty\end{aligned}$$

since $p' > 2-\alpha(\Omega)$ and $p'/(p'-1) < 2+\alpha(\Omega)/(1-\alpha(\Omega))$. Thus, $h := f K_\Omega(\cdot\,,z)/K_\Omega(z) \in A^{p'}(\Omega)$ for fixed $z \in \Omega$ so that

$$f(z) = h(z) = \int_\Omega h(\cdot)K_\Omega(z,\cdot) = \int_\Omega f(\cdot)\frac{|K_\Omega(\cdot\,,z)|^2}{K_\Omega(z)}. \qquad \square$$

For higher-dimensional cases, we can only prove the following:

Proposition 4.5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose there exists a negative psh exhaustion function ρ on Ω such that, for suitable constants $C, \alpha > 0$,*

$$|\rho(z) - \rho(w)| \le C|z-w|^\alpha, \quad z, w \in \Omega.$$

For every $p > 4n/(2n+\alpha)$, one has $f = P_\Omega(f)$ for all $f \in A^p(\Omega)$.

Proof. Set $\Omega_t = \{-\rho > t\}$, $t \ge 0$, and $\rho_t := \rho + t$. For every $z \in \Omega_t$, we choose $z^* \in \partial\Omega_t$ such that $|z - z^*| = \delta_t(z) := d(z, \partial\Omega_t)$. We then have

$$|\rho_t(z)| = |\rho_t(z) - \rho_t(z^*)| \le C|z-z^*|^\alpha = C\delta_t(z)^\alpha$$

where C is a constant independent of t. By a similar argument as the proof of Theorem 1.1, we may show that, for fixed $w \in \Omega$,

$$\int_{\Omega_t} |K_{\Omega_t}(\cdot\,,w)|^q \le C = C(q,w) < \infty$$

holds uniformly in $t \ll 1$ for every $q < 2+2\alpha/(2n-\alpha)$. Let $2 > p > 4n/(2n+\alpha)$ and $f \in A^p(\Omega)$. Fix $z \in \Omega$ for a moment. For every $t \ll 1$, we have $z \in \Omega_t$ and

$$f(z) = \int_{\Omega_t} f(\cdot)K_{\Omega_t}(z,\cdot). \tag{4-1}$$

Notice that

$$\begin{aligned}&\left|\int_\Omega f(\cdot)K_\Omega(z,\cdot) - \int_{\Omega_t} f(\cdot)K_{\Omega_t}(z,\cdot)\right|\\ &\qquad \le \int_{\Omega_t} |f||K_\Omega(z,\cdot) - K_{\Omega_t}(z,\cdot)| + \int_{\Omega\setminus\Omega_t} |f||K_\Omega(z,\cdot)|\\ &\qquad \le \|f\|_{L^p(\Omega)}\|K_\Omega(z,\cdot) - K_{\Omega_t}(z,\cdot)\|_{L^q(\Omega_t)} + \|f\|_{L^p(\Omega\setminus\Omega_t)}\|K_\Omega(z,\cdot)\|_{L^q(\Omega)}\end{aligned} \tag{4-2}$$

where $1/p+1/q=1$ (which implies $q<2+2\alpha/(2n-\alpha)$). Take $0<\gamma\ll 1$ so that $(q-\gamma)/(1-\gamma/2)<2+2\alpha/(2n-\alpha)$. We then have

$$\begin{aligned}\int_{\Omega_t}|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^q\\ &=\int_{\Omega_t}|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^\gamma|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^{q-\gamma}\\ &\le\Big(\int_{\Omega_t}|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^2\Big)^{\gamma/2}\Big(\int_{\Omega_t}|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^{(q-\gamma)/(1-\gamma/2)}\Big)^{1-\gamma/2}\end{aligned}$$

in view of Hölder's inequality. Since

$$\begin{aligned}\int_{\Omega_t}|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^2&=\int_{\Omega_t}|K_\Omega(z,\cdot)|^2+\int_{\Omega_t}|K_{\Omega_t}(z,\cdot)|^2-2\,\mathrm{Re}\int_{\Omega_t}K_\Omega(z,\cdot)K_{\Omega_t}(\cdot,z)\\ &\le K_{\Omega_t}(z)-K_\Omega(z)\\ &\to 0\quad (t\to 0)\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_t}|K_\Omega(z,\cdot)-K_{\Omega_t}(z,\cdot)|^{(q-\gamma)/(1-\gamma/2)}\\ &\le 2^{(q-\gamma)/(1-\gamma/2)}\Big(\int_\Omega|K_\Omega(z,\cdot)|^{(q-\gamma)/(1-\gamma/2)}+\int_{\Omega_t}|K_{\Omega_t}(z,\cdot)|^{(q-\gamma)/(1-\gamma/2)}\Big)\\ &\le C,\end{aligned}$$

it follows from (4-1) and (4-2) that $f=P_\Omega(f)$. □

Similarly, we have:

Corollary 4.6. *If $p>2n/\alpha$, then $f=T_\Omega(f)$ for all $f\in A^p(\Omega)$.*

5. Estimate of the pluricomplex Green function

The goal of this section is to show the following:

Proposition 5.1. *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain with $\alpha(\Omega)>0$. There exists a constant $C\gg 1$ such that*

$$\{g_\Omega(\cdot,w)<-1\}\subset\{\varrho>-C\nu(w)\},\qquad w\in\Omega,\tag{5-1}$$

where $\nu=|\varrho|(1+|\log|\varrho||)^n$.

We will follow the argument of Błocki [2005] with necessary modifications. The key observation is the following:

Lemma 5.2 [Błocki 2005]. *Let $\Omega\subset\mathbb{C}^n$ be a bounded hyperconvex domain. Suppose ζ and w are two points in Ω such that the closed balls $\bar B(\zeta,\varepsilon)$, $\bar B(w,\varepsilon)\subset\mathbb{C}^n$ and $\bar B(\zeta,\varepsilon)\cap\bar B(w,\varepsilon)=\varnothing$. Then there exists $\tilde\zeta\in\bar B(\zeta,\varepsilon)$ such that*

$$|g_\Omega(\tilde\zeta,w)|^n\le n!\,(\log R/\varepsilon)^{n-1}|g_\Omega(w,\zeta)|\tag{5-2}$$

where $R:=\mathrm{diam}(\Omega)$.

For the sake of completeness, we include a proof here, which relies heavily on the following fundamental results.

Theorem 5.3 [Demailly 1987]. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$.*

(1) *For every $w \in \Omega$, one has $(dd^c g_\Omega(\cdot, w))^n = (2\pi)^n \delta_w$, where δ_w denotes the Dirac measure at w.*

(2) *For every $\zeta \in \Omega$ and $\eta > 0$, one has $\int_\Omega (dd^c \max\{g_\Omega(\cdot, \zeta), -\eta\})^n = (2\pi)^n$.*

Theorem 5.4 ([Błocki 1993]; see also [Błocki 2002]). *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that $u, v \in \mathrm{PSH}^- \cap L^\infty(\Omega)$ are nonpositive psh functions such that $u = 0$ on $\partial\Omega$. Then*

$$\int_\Omega |u|^n (dd^c v)^n \le n!\, \|v\|_\infty^{n-1} \int_\Omega |v| (dd^c u)^n. \tag{5-3}$$

Proof of Lemma 5.2. Let $\eta = \log R/\varepsilon$. Since $g_\Omega(z, \zeta) \ge \log|z - \zeta|/R$, it follows that

$$\{g_\Omega(\cdot, \zeta) = -\eta\} \subset \bar{B}(\zeta, \varepsilon).$$

First applying Theorem 5.4 with $u = \max\{g_\Omega(\cdot, w), -t\}$ and $v = \max\{g_\Omega(\cdot, \zeta), -\eta\}$ and then letting $t \to +\infty$, we obtain

$$\int_\Omega |g_\Omega(\cdot, w)|^n (dd^c \max\{g_\Omega(\cdot, \zeta), -\eta\})^n \le n!\,(2\pi)^n \eta^{n-1} |g_\Omega(w, \zeta)|$$

in view of Theorem 5.3(1). Since $\bar{B}(\zeta, \varepsilon) \cap \bar{B}(w, \varepsilon) = \varnothing$, it follows that $g_\Omega(\cdot, w)$ is continuous on $\bar{B}(\zeta, \varepsilon)$ so that there exists $\tilde{\zeta} \in \bar{B}(\zeta, \varepsilon)$ such that

$$|g_\Omega(\tilde{\zeta}, w)| = \min_{\bar{B}(\zeta,\varepsilon)} |g_\Omega(\cdot, w)|.$$

Since the measure $(dd^c \max\{g_\Omega(\cdot, \zeta), -\eta\})^n$ is supported on $\{g_\Omega(\cdot, \zeta) = -\eta\}$ with total mass $(2\pi)^n$, we immediately get (5-2). □

Proof of Proposition 5.1. Clearly, it suffices to consider the case when w is sufficiently close to $\partial\Omega$. Fix $\zeta \in \Omega$ with $\varrho(\zeta) \le 2\varrho(w)$ for a moment. Set $\varepsilon := |\varrho(w)|^{2/\alpha}$. Since $\varepsilon \le C_\alpha^{2/\alpha} \delta(w)^2$, we see that $\bar{B}(w, \varepsilon) \subset \Omega$ provided $\delta(w) \le \varepsilon_\alpha \ll 1$. For every $z \in \Omega$ with $\delta(z) \le \varepsilon$, we have

$$|\varrho(z)| \le C_\alpha \delta(z)^\alpha \le C_\alpha \varepsilon^\alpha = C_\alpha |\varrho(w)|^2 \quad (\le |\varrho(w)|/2) \tag{5-4}$$

provided $\delta(w) \le \varepsilon_\alpha \ll 1$. It follows from (2-7) and (5-4) that for every $\tau > 0$ there exists $\varepsilon_\tau \ll \varepsilon_\alpha$ such that

$$\sup_{\delta \le \varepsilon} |g_\Omega(\cdot, w)| \le \tau \tag{5-5}$$

provided $\delta(w) \le \varepsilon_\tau$. Since

$$C_\alpha \delta(\zeta)^\alpha \ge -\varrho(\zeta) \ge -2\varrho(w) = 2\varepsilon^{\alpha/2}$$

and Lemma 2.4 yields

$$C_1 |\zeta - w|^\alpha \ge \tfrac{3}{2}\varrho(w) - \varrho(\zeta) \ge -\tfrac{1}{2}\varrho(w) = \tfrac{1}{2}\varepsilon^{\alpha/2},$$

it follows that if $\delta(w) \le \varepsilon_\tau \ll 1$ then $\bar{B}(\zeta, \varepsilon) \subset \Omega$ and

$$\bar{B}(\zeta, \varepsilon) \cap \bar{B}(w, \varepsilon) = \varnothing. \tag{5-6}$$

By Lemma 5.2, there exists $\tilde{\zeta} \in \bar{B}(\zeta, \varepsilon)$ such that (5-2) holds.

Now set

$$\Psi(z) := \sup\{u(z) : u \in \mathrm{PSH}^-(\Omega),\ u|_{\bar{B}(w,\varepsilon)} \le -1\}.$$

We claim that

$$g_\Omega(z, w) \ge \log R/\varepsilon \Psi(z), \quad z \in \Omega \setminus B(w, \varepsilon), \qquad g_\Omega(z, w) \le \log \delta(w)/\varepsilon \Psi(z), \quad z \in \Omega. \tag{5-7}$$

To see this, first notice that

$$\log \frac{|z-w|}{R} \le g_\Omega(z, w) \le \log \frac{|z-w|}{\delta(w)}, \quad z \in \Omega. \tag{5-8}$$

Since

$$u(z) = \begin{cases} \log|z-w|/R & \text{if } z \in B(w, \varepsilon), \\ \max\{\log|z-w|/R, \log R/\varepsilon \Psi(z)\} & \text{if } z \in \Omega \setminus B(w, \varepsilon) \end{cases}$$

is a negative psh function on Ω with a logarithmic pole at w, it follows that

$$g_\Omega(z, w) \ge \log R/\varepsilon \Psi(z), \quad z \in \Omega \setminus B(w, \varepsilon).$$

Since (5-8) implies $g_\Omega(\cdot\,, w)|_{\bar{B}(w,\varepsilon)} \le \log \varepsilon/\delta(w)$, we have

$$\Psi(z) \ge \frac{g_\Omega(z, w)}{\log \delta(w)/\varepsilon}, \quad z \in \Omega.$$

By (5-5) and (5-7), we obtain

$$\sup_{\delta \le \varepsilon} |\Psi| \le \frac{\tau}{\log \delta(w)/\varepsilon}. \tag{5-9}$$

Set $\widetilde{\Omega} = \Omega - (\tilde{\zeta} - \zeta)$ and

$$v(z) = \begin{cases} \Psi(z) & \text{if } z \in \Omega \setminus \widetilde{\Omega}, \\ \max\{\Psi(z), \Psi(z + \tilde{\zeta} - \zeta) - \tau/(\log \delta(w)/\varepsilon)\} & \text{if } z \in \Omega \cap \widetilde{\Omega}. \end{cases}$$

Since $\Omega \cap \partial\widetilde{\Omega} \subset \{\delta \le \varepsilon\}$, it follows from (5-9) that $v \in \mathrm{PSH}^-(\Omega)$. Since

$$\Psi(z) \le \frac{\log|z-w|/\delta(w)}{\log R/\varepsilon}, \quad z \in \Omega \setminus B(w, \varepsilon),$$

in view of (5-8) and (5-7), and $z + \tilde{\zeta} - \zeta \in \bar{B}(w, 2\varepsilon)$ if $z \in \bar{B}(w, \varepsilon)$, it follows from the maximal principle that

$$v|_{\bar{B}(w,\varepsilon)} \le -\frac{\log \delta(w)/(2\varepsilon)}{\log R/\varepsilon}.$$

Thus,

$$\Psi(\tilde{\zeta}) - \frac{\tau}{\log \delta(w)/\varepsilon} \le v(\zeta) \le \frac{\log \delta(w)/(2\varepsilon)}{\log R/\varepsilon} \Psi(\zeta).$$

Combining with (5-6) and (5-7), we obtain

$$g_\Omega(\zeta, w) \geq \frac{(\log R/\varepsilon)^2}{\log \delta(w)/\varepsilon \cdot \log \delta(w)/(2\varepsilon)}(g_\Omega(\tilde{\zeta}, w) - \tau) \geq C_3(g_\Omega(\tilde{\zeta}, w) - \tau)$$

since $\delta(w) \geq |\varrho(w)/C_\alpha|^{1/\alpha} = \sqrt{\varepsilon}/C_\alpha^{1/\alpha}$. If we choose $\tau = 1/(2C_3)$, then

$$\begin{aligned} g_\Omega(\zeta, w) &\geq -C_3(n!)^{1/n}(\log R/\varepsilon)^{1-1/n}|g_\Omega(w, \zeta)|^{1/n} - \tfrac{1}{2} \quad \text{(by (5-2))} \\ &\geq -C_4|\log|\varrho(w)||^{1-1/n}\frac{|\varrho(w)\log|\varrho(\zeta)||^{1/n}}{|\varrho(\zeta)|^{1/n}} - \tfrac{1}{2} \quad \text{(by (2-7))} \\ &\geq -C_5\frac{|\varrho(w)|^{1/n}|\log|\varrho(w)||}{|\varrho(\zeta)|^{1/n}} - \tfrac{1}{2} \end{aligned}$$

since $\varrho(\zeta) \leq 2\varrho(w)$. Thus,

$$\{g_\Omega(\cdot, w) < -1\} \cap \{\varrho \leq 2\varrho(w)\} \subset \{\varrho > -C\nu(w)\}$$

provided $C \gg 1$. Since $\{\varrho > 2\varrho(w)\} \subset \{\varrho > -C\nu(w)\}$ if $C \gg 1$, we conclude the proof. □

6. Pointwise estimate of the normalized Bergman kernel and applications

Proof of Theorem 1.7. By Proposition 2.3, we know that for every $0 < r < 1$ there exist constants $\varepsilon_r, C_r > 0$ such that

$$\int_{-\varrho \leq \varepsilon} |K_\Omega(\cdot, w)|^2/K_\Omega(w) \leq C_r(\varepsilon/\mu(w))^r$$

for all $\varepsilon \leq \varepsilon_r \mu(w)$. Fix $z \in \Omega$ with $b(z) := C\nu(z) \leq \varepsilon_r \mu(w)$ for a moment, where C is the constant in (5-1). Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function satisfying $\chi|_{(0,\infty)} = 0$ and $\chi|_{(-\infty, -\log 2)} = 1$. We proceed with the proof in a similar way as [Chen 1999]. Notice that $g_\Omega(\cdot, z)$ is a continuous negative psh function on $\Omega \setminus \{z\}$ which satisfies

$$-i\partial\bar{\partial}\log(-g_\Omega(\cdot, z)) \geq i\partial\log(-g_\Omega(\cdot, z)) \wedge \bar{\partial}\log(-g_\Omega(\cdot, z))$$

as currents. By virtue of the Donnelly–Fefferman estimate [1983] (see also [Berndtsson and Charpentier 2000]), there exists a solution of the equation

$$\bar{\partial}u = K_\Omega(\cdot, w)\bar{\partial}\chi(-\log(-g_\Omega(\cdot, z)))$$

such that

$$\begin{aligned} \int_\Omega |u|^2 e^{-2ng_\Omega(\cdot, z)} &\leq C_0 \int_\Omega |K_\Omega(\cdot, w)|^2 |\bar{\partial}\chi(-\log(-g_\Omega(\cdot, z)))|^2_{-i\partial\bar{\partial}\log(-g_\Omega(\cdot, z))} e^{-2ng_\Omega(\cdot, z)} \\ &\leq C_n \int_{\varrho > -b(z)} |K_\Omega(\cdot, w)|^2 \quad \text{(by (5-1))} \\ &\leq C_{n,r} K_\Omega(w)(\nu(z)/\mu(w))^r. \end{aligned}$$

Set

$$f := K_\Omega(\cdot, w)\chi(-\log(-g_\Omega(\cdot, z))) - u.$$

Clearly, we have $f \in \mathcal{O}(\Omega)$. Since $g_\Omega(\zeta, z) = \log|\zeta - z| + O(1)$ as $\zeta \to z$ and u is holomorphic in a neighborhood of z, it follows that $u(z) = 0$, i.e., $f(z) = K_\Omega(z, w)$. Moreover,

$$\begin{aligned}\int_\Omega |f|^2 &\le 2\int_{\varrho > -b(z)} |K_\Omega(\cdot, w)|^2 + 2\int_\Omega |u|^2 \\ &\le C_{n,r} K_\Omega(w)(\nu(z)/\mu(w))^r\end{aligned}$$

since $g_\Omega(\cdot, z) < 0$. Thus, we get

$$K_\Omega(z) \ge \frac{|f(z)|^2}{\|f\|^2_{L^2(\Omega)}} \ge C_{n,r}^{-1} \frac{|K_\Omega(z, w)|^2}{K_\Omega(w)} (\mu(w)/\nu(z))^r,$$

and

$$\mathcal{B}_\Omega(z, w) \le C_{n,r}(\nu(z)/\mu(w))^r.$$

If $b(z) > \varepsilon_r \mu(w)$, then the inequality above trivially holds since $|K_\Omega(z, w)|^2/(K_\Omega(z)K_\Omega(w)) \le 1$. By symmetry of $\mathcal{B}_\Omega$, the assertion immediately follows. □

Remark. It would be interesting to get pointwise estimates for $|S_\Omega(z, w)|^2/(S_\Omega(z)S_\Omega(w))$, where S_Ω is the Szegö kernel (compare to [Chen and Fu 2011]).

Proof of Corollary 1.8. Let $z \in \Omega$ be an arbitrarily fixed point which is sufficiently close to $\partial\Omega$. By the Hopf–Rinow theorem, there exists a Bergman geodesic γ jointing z_0 to z, for ds_B^2 is complete on Ω. We may choose a finite number of points $\{z_k\}_{k=1}^m \subset \gamma$ with the order

$$z_0 \to z_1 \to z_2 \to \cdots \to z_m \to z,$$

where

$$|\varrho(z_{k+1})|(1 + |\log|\varrho(z_{k+1})||)^{n+2} = |\varrho(z_k)|$$

and

$$|\varrho(z)|(1 + |\log|\varrho(z)||)^{n+2} \ge |\varrho(z_m)|.$$

Since

$$\begin{aligned}\frac{\nu(z_{k+1})}{\mu(z_k)} &= \frac{|\varrho(z_{k+1})|}{|\varrho(z_k)|}(1 + |\log|\varrho(z_{k+1})||)^n(1 + |\log|\varrho(z_k)||) \\ &\le \frac{|\varrho(z_{k+1})|}{|\varrho(z_k)|}(1 + |\log|\varrho(z_{k+1})||)^{n+1} \\ &= (1 + |\log|\varrho(z_{k+1})||)^{-1},\end{aligned}$$

it follows from Theorem 1.7 that there exists $k_0 \in \mathbb{Z}^+$ such that $\mathcal{B}_\Omega(z_k, z_{k+1}) \le \frac{1}{4}$ for all $k \ge k_0$. By (1-4),

$$d_B(z_k, z_{k+1}) \ge 1.$$

Notice that

$$\begin{aligned}|\varrho(z_{k_0})| &= |\varrho(z_{k_0+1})||\log|\varrho(z_{k_0+1})||^{n+2} \\ &\le |\varrho(z_{k_0+2})||\log|\varrho(z_{k_0+2})||^{2(n+2)} \\ &\le \cdots \le |\varrho(z_m)||\log|\varrho(z_m)||^{(m-k_0)(n+2)}.\end{aligned}$$

Thus,

$$m-k_0 \geq \text{const.} \frac{|\log|\varrho(z_m)||}{\log|\log|\varrho(z_m)||} \geq \text{const.} \frac{|\log|\varrho(z)||}{\log|\log|\varrho(z)||}$$

so that

$$\begin{aligned} d_B(z, z_0) &\geq \sum_{k=k_0}^{m-1} d_B(z_k, z_{k+1}) \geq m-k_0-1 \\ &\geq \text{const.} \frac{|\log|\varrho(z)||}{|\log|\log|\varrho(z)|||} \\ &\geq \text{const.} \frac{|\log \delta(z)|}{\log|\log \delta(z)|} \end{aligned}$$

since $|\varrho(z)| \leq C_\alpha \delta^\alpha$ for any $\alpha < \alpha(\Omega)$. □

Proof of Corollary 1.9. For every $0 < \alpha < \alpha(\Omega)$, we have $-\varrho \leq C_\alpha \delta^\alpha$. Theorem 1.7 then yields

$$D_B(z_0, z) \geq \alpha |\log \delta(z)|$$

as $z \to \partial\Omega$. Thus, it suffices to show

$$d_K(z, z_0) \leq C|\log \delta(z)| \tag{6-1}$$

as $z \to \partial\Omega$. To see this, let F_K be the Kobayashi–Royden metric. Since F_K is decreasing under holomorphic mappings, we conclude that $F_K(z; X)$ is dominated by the KR metric of the ball $B(z, \delta(z))$. Thus, $F_K(z; X) \leq C|X|/\delta(z)$, from which (6-1) immediately follows (compare to the proof of Proposition 7.3 in [Chen 2016]). □

In order to prove Corollary 1.10, we need the following elementary fact.

Lemma 6.1. *If $\Omega \subset \mathbb{C}^n$ is a bounded weighted circular domain which contains the origin, then $K_\Omega(z, 0) = K_\Omega(0)$ for any $z \in \Omega$.*

Proof. For fixed $\theta \in \mathbb{R}$, we set $F_\theta(z) := (e^{ia_1\theta} z_1, \ldots, e^{ia_n\theta} z_n)$. By the transform formula of the Bergman kernel,

$$K_\Omega(F_\theta(z), 0) = K_\Omega(z, 0), \quad z \in \Omega.$$

It follows that, for any n-tuple $(m_1, \ldots, m_n)$ of nonnegative integers,

$$e^{i(a_1 m_1 + \cdots + a_n m_n)\theta} \frac{\partial^{m_1+\cdots+m_n} K_\Omega(z, 0)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}\bigg|_{z=0} = \frac{\partial^{m_1+\cdots+m_n} K_\Omega(z, 0)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}\bigg|_{z=0} \quad \text{for all } \theta \in \mathbb{R}$$

so that $\frac{\partial^{m_1+\cdots+m_n} K_\Omega(z,0)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}\big|_{z=0} = 0$ if not all m_j are zero. Taylor's expansion of $K_\Omega(z, 0)$ at $z = 0$ and the identity theorem of holomorphic functions yield $K_\Omega(z, 0) = K_\Omega(0)$ for any $z \in \Omega$. □

Proof of Corollary 1.10. By Lemma 6.1,

$$\mathcal{B}_{\Omega_2}(F(z), 0) = K_{\Omega_2}(0) K_{\Omega_2}(F(z))^{-1} \geq C^{-1} \delta_2(F(z))^{2n}.$$

On the other hand, Theorem 1.7 implies

$$\mathcal{B}_{\Omega_1}(z, F^{-1}(0)) \leq C_\alpha \delta_1(z)^\alpha.$$

Since $\mathcal{B}_{\Omega_2}(F(z), 0) = \mathcal{B}_{\Omega_1}(z, F^{-1}(0))$, we conclude the proof. □

Appendix: Examples of domains with positive hyperconvexity indices

We start with the following almost trivial fact.

Proposition A.1. *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{C}^n$ such that there exists a biholomorphic map $F : \Omega_1 \to \Omega_2$ which extends to a Hölder-continuous map $\overline{\Omega}_1 \to \overline{\Omega}_2$. If $\alpha(\Omega_2) > 0$, then $\alpha(\Omega_1) > 0$.*

Proof. Let δ_1 and δ_2 denote the boundary distances of Ω_1 and Ω_2, respectively. Choose $\rho_2 \in \mathrm{PSH}^- \cap C(\Omega_2)$ such that $-\rho_2 \le C\delta_2^\alpha$ for some $C, \alpha > 0$. Set $\rho_1 := \rho_2 \circ F$. Clearly, $\rho_1 \in \mathrm{PSH}^- \cap C(\Omega_1)$. For fixed $z \in \Omega_1$, we choose $z^* \in \partial\Omega_1$ so that $|z - z^*| = \delta_1(z)$. Since $F(z^*) \in \partial\Omega_2$, it follows that

$$\begin{aligned} -\rho_1(z) &\le C\delta_2(F(z))^\alpha = C(\delta_2(F(z)) - \delta_2(F(z^*)))^\alpha \\ &\le C|F(z) - F(z^*)|^\alpha \le C|z - z^*|^{\gamma\alpha} \\ &\le C\delta_1(z)^{\gamma\alpha}, \end{aligned}$$

where γ is the order of Hölder continuity of F on $\overline{\Omega}_1$. □

Example. Let $D \subset \mathbb{C}$ be a bounded Jordan domain which admits a uniformly Hölder-continuous conformal map f onto the unit disc Δ (e.g., a quasidisc with a fractal boundary). Set $F(z_1, \dots, z_n) := (f(z_1), \dots, f(z_n))$. Clearly, F is a biholomorphic map between D^n and Δ^n which extends to a Hölder-continuous map between their closures. Let

$$\Omega_2 := \{z \in \mathbb{C}^n : |z_1|^{a_1} + \cdots + |z_n|^{a_n} < 1\},$$

where $a_j > 0$. Clearly, we have $\alpha(\Omega_2) > 0$. By Proposition A.1, we conclude that the domain $\Omega_1 := F^{-1}(\Omega_2)$ satisfies $\alpha(\Omega_1) > 0$. Notice that some parts of $\partial\Omega_1$ might be highly irregular.

A domain $\Omega \subset \mathbb{C}^n$ is called $\mathbb{C}$-convex if $\Omega \cap L$ is a simply connected domain in L for every affine complex line L. Clearly, every convex domain is $\mathbb{C}$-convex.

Proposition A.2. *If $\Omega \subset \mathbb{C}^n$ is a bounded $\mathbb{C}$-convex domain, then $\alpha(\Omega) \ge \frac{1}{2}$.*

Proof. Let $w \in \Omega$ be an arbitrarily fixed point. Let w^* be a point on $\partial\Omega$ satisfying $\delta(w) = |w - w^*|$. Let L be the complex line determined by w and w^*. Since every $\mathbb{C}$-convex domain is linearly convex [Hörmander 1994, Theorem 4.6.8], it follows that there exists an affine complex hyperplane $H \subset \mathbb{C}^n \setminus \Omega$ with $w^* \in H$. Since $|w - w^*| = \delta(w)$, H has to be *orthogonal* to L. Let π_L denote the natural projection $\mathbb{C}^n \to L$. Notice that $\pi_L(\Omega)$ is a bounded simply connected domain in L in view of [Hörmander 1994, Proposition 4.6.7]. By Proposition 7.3 in [Chen 2016], there exists a negative continuous function ρ_L on $\pi_L(\Omega)$ with

$$(\delta_L/\delta_L(z_L^0))^2 \le -\rho_L \le (\delta_L/\delta_L(z_L^0))^{1/2},$$

where δ_L denotes the boundary distance of $\pi_L(\Omega)$ and $z_L^0 \in \pi_L(\Omega)$ satisfies $\delta_L(z_L^0) = \sup_{\pi_L(\Omega)} \delta_L$. Fix a point $z^0 \in \Omega$. We have

$$\delta_L(z_L^0) \ge \delta_L(\pi_L(z^0)) \ge \delta(z^0).$$

Set

$$\varrho_{z_0}(z) = \sup\{u(z) : u \in \mathrm{PSH}^-(\Omega),\ u(z^0) \le -1\}.$$

Clearly, $\varrho_{z_0} \in \mathrm{PSH}^-(\Omega)$. Since $\Omega \subset \pi_L^{-1}(\pi_L(\Omega))$, it follows that $\pi_L^*(\rho_L) \in \mathrm{PSH}^-(\Omega)$. Since $\pi_L^*(\delta_L)(w) = \delta(w)$ and

$$\pi_L^*(\rho_L)(z^0) = \rho_L(\pi_L(z^0)) \le -(\delta_L(\pi_L(z^0))/\delta_L(z_L^0))^2,$$

then

$$\begin{aligned}\varrho_{z_0}(w) &\ge (\delta_L(z_L^0)/\delta_L(\pi_L(z^0)))^2 \pi_L^*(\rho_L)(w)\\ &\ge -(\delta_L(z_L^0)^{3/2}/\delta_L(\pi_L(z^0))^2)\delta(w)^{1/2}\\ &\ge -(R^{3/2}/\delta(z^0)^2)\delta(w)^{1/2},\end{aligned}$$

where $R = \mathrm{diam}(\Omega)$. Thus, $\alpha(\Omega) \ge \frac{1}{2}$. □

Remark. After the first version of this paper was finished, the author was kindly informed by Nikolai Nikolov that Proposition A.2 follows also from Proposition 3(ii) of [Nikolov and Trybuła 2015].

Complex dynamics also provides interesting examples of domains with $\alpha(\Omega) > 0$. Let $q(z) = \sum_{j=0}^d a_j z^j$ be a complex polynomial of degree $d \ge 2$. Let q^n denote the n-iterates of q. The attracting basin at ∞ of q is defined by

$$F_\infty := \{z \in \overline{\mathbb{C}} : q^n(z) \to \infty \text{ as } n \to \infty\},$$

which is a domain in $\overline{\mathbb{C}}$ with $q(F_\infty) = F_\infty$. The Julia set of q is defined by $J := \partial F_\infty$. It is known that J is always uniformly perfect. Thus, $\alpha(F_\infty) > 0$.

We say that q is *hyperbolic* if there exist constants $C > 0$ and $\gamma > 1$ such that

$$\inf_J |(q^n)'| \ge C\gamma^n \quad \text{for all } n \ge 1.$$

Consider a holomorphic family $\{q_\lambda\}$ of hyperbolic polynomials of constant degree $d \ge 2$ over the unit disc Δ. Let F_∞^λ denote the attracting basin at ∞ of q_λ, and let $J_\lambda := \partial F_\infty^\lambda$. Let Ω_r denote the total space of F_∞^λ over the disc $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$, where $0 < r \le 1$, that is

$$\Omega_r = \{(\lambda, w) : \lambda \in \Delta_r,\ w \in F_\infty^\lambda\}.$$

Proposition A.3. *For every* $0 < r < 1$, Ω_r *is a bounded domain in* $\mathbb{C}^2$ *with* $\alpha(\Omega_r) > 0$.

Proof. We first show that Ω_r is a domain. Mañé, Sad and Sullivan [Mañé et al. 1983] showed that there exists a family of maps $\{f_\lambda\}_{\lambda\in\Delta}$ such that

(1) $f_\lambda : J_0 \to J_\lambda$ is a homeomorphism for each $\lambda \in \Delta$,
(2) $f_0 = \mathrm{id}|_{J_0}$,
(3) $f(\lambda, z) := f_\lambda(z)$ is holomorphic on Δ for each $z \in J_0$ and
(4) $q_\lambda = f_\lambda \circ q_0 \circ f_\lambda^{-1}$ on J_λ, for each $\lambda \in \Delta$.

In other words, properties (1)–(3) say that $\{f_\lambda\}_{\lambda\in\Delta}$ gives a *holomorphic motion* of J_0. By a result of Slodkowski [1991], $\{f_\lambda\}_{\lambda\in\Delta}$ may be extended to a holomorphic motion $\{\tilde f_\lambda\}_{\lambda\in\Delta}$ of $\overline{\mathbb{C}}$ such that

(a) $\tilde f_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasiconformal map of dilatation $\le (1+|\lambda|)/(1-|\lambda|)$, for each $\lambda \in \Delta$,
(b) $\tilde f_\lambda : F_\infty^0 \to F_\infty^\lambda$ is a homeomorphism for each $\lambda \in \Delta$ and
(c) $\tilde f(\lambda, z) := \tilde f_\lambda(z)$ is jointly Hölder-continuous in (λ, z).

It follows immediately that Ω_r is a domain in $\mathbb{C}^n$ for each $r \le 1$. Let δ_λ and δ denote the boundary distances of F^λ_∞ and Ω_1, respectively. We claim that for every $0 < r < 1$ there exists $\gamma > 0$ such that

$$\delta_\lambda(w) \le C\delta(\lambda, w)^\gamma, \quad \lambda \in \Delta_r,\ w \in F^\lambda_\infty. \tag{A-1}$$

To see this, choose $(\lambda', w_{\lambda'})$ where $w_{\lambda'} \in J_{\lambda'}$, such that

$$\delta(\lambda, w) = \sqrt{|\lambda - \lambda'|^2 + |w - w_{\lambda'}|^2}.$$

Write $w_{\lambda'} = \tilde{f}(\lambda', z_0)$ where $z_0 \in J_0$. Since $\tilde{f}(\lambda, z_0) \in J_\lambda$, it follows that

$$\begin{aligned}\delta_\lambda(w) &\le |w - \tilde{f}(\lambda, z_0)| \le |w - w_{\lambda'}| + |\tilde{f}(\lambda', z_0) - \tilde{f}(\lambda, z_0)| \\ &\le |w - w_{\lambda'}| + C|\lambda - \lambda'|^\gamma \\ &\le \delta(\lambda, w) + C\delta(\lambda, w)^\gamma \\ &\le C'\delta(\lambda, w)^\gamma,\end{aligned}$$

where γ is the order of Hölder continuity of $\tilde{f}$ on Ω_r.

Recall that the Green function $g_\lambda(w) := g_{F^\lambda_\infty}(w, \infty)$ at ∞ of F^λ_∞ satisfies

$$g_\lambda(w) = \lim_{n\to\infty} d^{-n} \log|q^n_\lambda(w)|, \quad w \in F^\lambda_\infty, \tag{A-2}$$

where the convergence is uniform on compact subsets of F^λ_∞ [Ransford 1995, Corollary 6.5.4]. Actually the proof of that result shows that the convergence is also uniform on compact subsets of Ω_1. Since $\log|q^n_\lambda(w)|$ is psh in (λ, w), so is $g(\lambda, w) := g_\lambda(w)$. By (A-1) it suffices to verify that for every $0 < r < 1$ there are positive constants C and α such that $-g_\lambda(w) \le C\delta_\lambda(w)^\alpha$ for each $\lambda \in \Delta_r$ and $w \in F^\lambda_\infty$. This can be verified similarly to the proof of Theorem 3.2 in [Carleson and Gamelin 1993]. □

Conjecture A.4. *Let $D \subset \mathbb{C}$ be a domain with $\alpha(D) > 0$. Let $\{f_\lambda\}_{\lambda\in\Delta}$ be a holomorphic motion of D. Let*

$$\Omega_r := \{(\lambda, w) : \lambda \in \Delta_r,\ w \in f_\lambda(D)\}.$$

One has $\alpha(\Omega_r) > 0$ for each $r < 1$.

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