ANALYSIS & PDE

Volume 10 No. 7 2017

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ANALYSIS AND PDE
Vol. 10, No. 7, 2017
dx.doi.org/10.2140/apde.2017.10.1539

A VECTOR FIELD METHOD FOR RELATIVISTIC TRANSPORT EQUATIONS WITH APPLICATIONS

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We adapt the vector field method of Klainerman to the study of relativistic transport equations. First, we prove robust decay estimates for velocity averages of solutions to the relativistic massive and massless transport equations, without any compact support requirements (in x or v) for the distribution functions. In the second part of this article, we apply our method to the study of the massive and massless Vlasov–Nordström systems. In the massive case, we prove global existence and (almost) optimal decay estimates for solutions in dimensions $n \geq 4$ under some smallness assumptions. In the massless case, the system decouples and we prove optimal decay estimates for the solutions in dimensions $n \geq 4$ for arbitrarily large data, and in dimension 3 under some smallness assumptions, exploiting a certain form of the null condition satisfied by the equations. The 3-dimensional massive case requires an extension of our method and will be treated in future work.

MSC2010: 35B40, 35Q83, 83C30.
Keywords: relativistic kinetic equations, wave equation, vector-field method, asymptotic behaviour, nonlinear stability, Vlasov–Nordström system.

1. Introduction

The vector field method of Klainerman [1985b] provides powerful tools which are at the core of many fundamental results in the study of nonlinear wave equations, such as the famous proof of the stability of the Minkowski space [Christodoulou and Klainerman 1993]. In essence, the method takes advantage of the symmetries of a linear evolution equation to derive in a robust way boundedness and decay estimates of solutions. The robustness is crucial, as the final aim is typically to prove the nonlinear stability of some stationary solution, so that the method should be stable when perturbed by the nonlinearities of the equations.

In this paper, we are interested in the massive and massless relativistic transport equations[1]

$$\boldsymbol{T}_m(f) \equiv \left(\sqrt{m^2+|v|^2}\partial_t + v^i\partial_{x^i}\right)(f) = 0, \tag{1}$$

where $m \geq 0$ is the mass[2] of the particles and f is a function of (t,x,v) defined on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$ if $m > 0$ and $\mathbb{R}_t \times \mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$ otherwise, with $n \geq 1$ being the dimension of the physical space.

Decay estimates via the method of characteristics for relativistic transport equations. For transport equations, the standard method to prove decay estimates is the method of characteristics. The origin of these decay estimates goes back in the nonrelativistic case to the work of Bardos and Degond [1985] on the Vlasov–Poisson system. Recall that if f is a regular solution to say $\boldsymbol{T}_1(f) = 0$ then, for all $(t,x,v) \in \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$,

$$f(t,x,v) = f\left(0, x - \frac{vt}{\sqrt{1+|v|^2}}, v\right),$$

and assuming that f has initially compact support in v, one can easily infer the velocity average estimate, for all $t > 0$ and all $x \in \mathbb{R}^n$,

$$\int_{v\in\mathbb{R}^n} |f|(t,x,v)\,dv \lesssim \frac{C(V)}{t^n}\|f(t=0)\|_{L^1_x L^\infty_v}, \tag{2}$$

[1] We will be using in the whole article the Einstein summation convention. For instance, $v^i\partial_{x^i}$ in (1) stands for $\sum_{i=1}^n v^i\partial_{x^i}$.

[2] In the remainder of this article, we will often normalize the mass to be either 1 or 0 and thus consider mostly $\boldsymbol{T}_1$ and $\boldsymbol{T}_0$. We will however sometimes keep the mass $m > 0$ so that the reader can see how some of the estimates would degenerate as $m \to 0$.

where V is an upper bound on the size of the support in v of f at the initial time and $C(V) \to +\infty$ as $V \to +\infty$.

These estimates, while being relatively easy to derive, suffer from several significant drawbacks when applied to a nonlinear system:

(1) They require a strong control of the characteristics of the system.

(2) The constant $C(V)$ in (2) depends on the size of the v-support of the solutions. Similar, more refined estimates, which do not require a compact support assumption, can nonetheless be derived (see [Schaeffer 2004]), but they are based on an even finer analysis of the characteristics.[3] This explains (partially) why most of the previous works assumed compact support in v. One therefore typically needs to bound an extra quantity, the size of the v-support at time t. In particular, this approach enlarges the number of variables of the system that need to be controlled.

Concerning the first problem, we note that there are many evolution problems for which the characteristics in a neighbourhood of a stationary solution will eventually diverge from the original ones, introducing extra difficulties in the analysis. A famous example of that is the stability of Minkowski space, where there is a logarithmic divergence; see [Christodoulou and Klainerman 1993; Lindblad and Rodnianski 2010]. Moreover, to prove decay estimates such as (2), one needs to control the Jacobian associated with the differential of the characteristic flow[4] and in order to obtain improved decay estimates for derivatives, one also needs estimates on the derivatives of the Jacobian. See, for instance, [Hwang et al. 2011], where such a program is carried out for the Vlasov–Poisson system. In other words, one needs strong control on the characteristics to be able to prove sharp decay estimates via this method in a nonlinear setting. Finally, note that there are many interesting models where the correct assumption, from the point of view of physics (see, for instance, the end of the Introduction in [Villani 2010]), is to allow arbitrarily large velocities.

Decay estimates for the wave equation. In the context of the wave equation

$$\Box\phi \equiv \left[-\partial_t^2 + \sum_{i=1}^{n} \partial_{x^i}^2\right]\phi = 0,$$

several methods exist to prove decay estimates of solutions. For instance, one standard way is to use the Fourier representation of the solution together with estimates for oscillatory integrals. In his fundamental paper, Klainerman [1985b] introduced what is now referred to as *the vector field method*.[5] Instead of relying on an explicit integral representation of the solutions, it uses:

(1) A coercive conservation law. In the case of the wave equation, this is simply the conservation of the energy.

[3] Note also that in [Schaeffer 2004], there is a loss of decay for the velocity averages of f compared to the linear case, which is directly related to the polynomial decay in v of the initial data and independent of the smallness assumptions.

[4] In the context of the Vlasov equation on a curved Lorentzian manifold, this means that one needs estimates on the differential of the exponential map, or at least on its restriction to certain submanifolds.

[5] Let us also mention that, complementary to the method of Klainerman, which uses vector fields as commutators, one can also use vector fields as *multipliers*, in the style of the work of Morawetz [1962; 1968].

(2) A set of vector fields which commute with the equations. In the case of the wave equation, these are the Killing and conformal Killing fields of the Minkowski space.

(3) Weighted Sobolev $L^2 - L^\infty$ inequalities. The standard derivatives ∂_t, ∂_{x^i} are rewritten in terms of the commutator vector fields before applying the usual Sobolev inequalities. The weights in these decompositions together with those arising from the conservation laws are then translated into decay rates.

This leads to the decay estimate

$$|\partial\phi(t,x)| \lesssim \frac{\mathcal{E}^{\frac{1}{2}}(\phi)}{\big(1+\big|t-|x|\big|\big)^{\frac{1}{2}}\big(1+\big|t+|x|\big|\big)^{\frac{n-1}{2}}} \tag{3}$$

for solutions of the wave equation $\Box\phi = 0$, where $E(\phi)$ is an energy norm obtained by integrating ϕ and derivatives of ϕ (with weights) at time $t = 0$.

These types of estimates, being based on conservation laws and commutators, are quite robust, and as a consequence, are applicable in strongly nonlinear settings, such as the Einstein equations or the Euler equations; see, for instance, [Christodoulou 2007] for such an application.

A vector field method for transport equations. In our opinion, the decay estimate (2), being based on an explicit representation of the solutions, given by the method of characteristics, should be compared to the decay estimates for the wave equation obtained via the Fourier or other integral representations. In this paper, we derive an analogue of the vector field method for the massive and massless relativistic transport equations (1). The coercive conservation law is given by the conservation of the L^1-norm of the solution, while the vector fields commuting with the operators are essentially obtained by taking the *complete lifts* of the Killing and conformal Killing fields, a classical operation in differential geometry which takes a vector field on a manifold M to a vector field on the tangent bundle TM. The weighted Sobolev inequalities are slightly more technical. One of the main ingredients is that averages in v possess good commutation properties with the Killing vector fields and their complete lifts. Our decay estimates can then be stated as:

Theorem 1 (decay estimates for velocity averages of massless distribution functions). *For any regular distribution function f, solution to $\boldsymbol{T}_0(f) = 0$, and any $(t,x) \in \mathbb{R}^+_t \times \mathbb{R}^n_x$, we have*

$$\int_{v\in\mathbb{R}^n_v\setminus\{0\}} |f|(t,x,v)\frac{dv}{|v|} \lesssim \frac{1}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}} \sum_{\substack{|\alpha|\le n\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}} \big\| |v|^{-1}\widehat{Z}^\alpha(f)(t=0)\big\|_{L^1(\mathbb{R}^n_x\times(\mathbb{R}^n_v\setminus\{0\}))}, \tag{4}$$

where the α are multi-indices of length $|\alpha|$ and the $\widehat{Z}^\alpha$ are differential operators of order $|\alpha|$ obtained as a composition of $|\alpha|$ vector fields of the algebra $\widehat{\mathbb{K}}$.

The detailed list of the vector fields and their complete lifts used here is given in Section 2G1. For the massive transport equation, we prove:

Theorem 2 (decay estimates for velocity averages of massive distribution functions). *For any regular distribution function f, solution to $\boldsymbol{T}_1(f)=0$, any $x\in\mathbb{R}^n$ and any $t\geq\sqrt{1+|x|^2}$, we have*

$$\int_{v\in\mathbb{R}^n_v}|f|(t,x,v)\frac{dv}{\sqrt{1+|v|^2}}\lesssim\frac{1}{(1+t)^n}\sum_{\substack{|\alpha|\leq n\\ \widehat{Z}^\alpha\in\widehat{\mathbb{P}}^{|\alpha|}}}\left\|\widehat{Z}^\alpha(f)_{|H_1\times\mathbb{R}^n_v}\,v_\alpha\nu_1^\alpha\right\|_{L^1(H_1\times\mathbb{R}^n_v)},\tag{5}$$

where H_1 denotes the unit hyperboloid $H_1\equiv\{(t,x)\in\mathbb{R}_t\times\mathbb{R}^n_x\mid 1=t^2-x^2\}$, $\widehat{Z}^\alpha(f)_{|H_1\times\mathbb{R}^n_v}$ is the restriction to $H_1\times\mathbb{R}^n_v$ of $\widehat{Z}^\alpha(f)$, $v_\alpha\nu_1^\alpha$ is the contraction of the 4*-velocity $(\sqrt{1+|v|^2},v^i)$ with the unit normal ν_1 to H_1 and where the $\widehat{Z}^\alpha$ are differential operators obtained as a composition of $|\alpha|$ vector fields of the algebra $\widehat{\mathbb{P}}$.*

Remark 1.1. No compact support assumptions in x or in v are required for the statements of Theorems 1 or 2, but the norm on the right-hand side of (5) has two extra powers of v compared to the left-hand side of (5). Of course, for the norms on the right-hand sides of (4) or (5) to be finite, some amount of decay in x and v for the data is needed. Note that from the point of view of nonlinear applications, it is sufficient to propagate bounds for the norms appearing on the right-hand sides of (4) or (5), without any need to control pointwise the decay in x or v of the solutions, to get the desired decay estimates for the velocity averages.

Remark 1.2. In (4), the decay is worse near the light cone $t=|x|$. This of course is an analogue of the decay estimate (3) as traditionally obtained for the wave equation by the vector field method.

Remark 1.3. Estimate (5) is the analogue of the decay estimate for Klein–Gordon fields ϕ, solutions to $\Box\phi=\phi$, for which, for all $t\geq\sqrt{1+|x|^2}$,

$$|\partial\phi(t,x)|\lesssim\frac{\mathcal{E}^{\frac{1}{2}}[\phi]}{(1+t)^{\frac{n}{2}}},$$

where $E[\phi]$ is an energy norm obtained by integrating ϕ and derivatives of ϕ (with weights) on an initial hyperboloid, as obtained by Klainerman [1993].

Remark 1.4. As in the case of the Klein–Gordon equation, one can easily prove that for regular solutions f to $\boldsymbol{T}_1(f)=0$ with data given at $t=0$ and decaying sufficiently fast as $|x|\to+\infty$ (in particular, solutions arising from data with compact support in x) the norm on the right-hand side of (5) is finite, so that the decay estimate applies.[6] Thus, the use of hyperboloids is merely a technical issue. The restriction "$t\geq\sqrt{1+|x|^2}$" simply means in the future of the unit hyperboloid. We provide a classical construction in Appendix A, which explains how Theorem 2 can be applied to solutions arising from initial data with compact x-support given at $t=0$ to obtain a $1/t^n$ decay of velocity averages in the whole future of the $t=0$ hypersurface.

Remark 1.5. The reader might wonder whether the same types of techniques could be applied for the classical transport operator $\boldsymbol{T}_{\mathrm{cl}}=\partial_t+v^i\partial_{x^i}$. This question was addressed in [Smulevici 2016], where

[6] See also [Georgiev 1992], where decay estimates for the Klein–Gordon operator were obtained starting from noncompactly supported data at $t=0$ using (mostly) vector-field-type methods.

decay estimates for velocity averages of solutions to the classical transport operator were obtained. As an application, that paper considered the study of small data solutions of the Vlasov–Poisson system and provided an alternative proof (with some additional information on the asymptotic behaviour of the solutions, concerning in particular the decay in $|x|$ and uniform bounds on some global norms) of the optimal time decay for derivatives of velocity averages obtained first in [Hwang et al. 2011]. One of the nice features of the vector field method is that improved decay estimates for derivatives follow typically easily from the main estimates, and [Smulevici 2016] was no exception. In the relativistic case, our vector field method also provides improved decay for derivatives. See Propositions 3.2 and 3.4 in Sections 3B and 3C, respectively.

Applications to the massive and massless Vlasov–Nordström systems. In the second part of this paper, we will apply our vector field method to the massive and massless Vlasov–Nordström systems

$$\square\phi = m^2\int_v f\frac{dv}{\sqrt{m^2+|v|^2}}, \tag{6}$$

$$\boldsymbol{T}_m(f) - (\boldsymbol{T}_m(\phi)v^i + m^2\nabla^i\phi)\frac{\partial f}{\partial v^i} = (n+1)f\,\boldsymbol{T}_m(\phi), \tag{7}$$

where $m=0$ in the massless case and $m>0$ in the massive case, $\square \equiv -\partial_t^2 + \sum_{i=1}^n \partial_{x^i}^2$ is the standard wave operator of Minkowski space, ϕ is a scalar function of (t,x) and f is, as before, a function of (t,x,v^i) with $x\in\mathbb{R}^n$, $v\in\mathbb{R}^n$ if $m>0$, $v\in\mathbb{R}^n\setminus\{0\}$ if $m=0$. A good introduction to this system can be found in [Calogero 2003]. See also the classical works [Calogero 2006; Pallard 2006].

Roughly speaking, the Vlasov–Nordström system can be derived, in the context of scalar gravitation metric theory, by considering only a special class of metrics (that of metrics conformal to the Minkowski metric) and by neglecting some of the nonlinear terms in the equations for the gravitational field (see [Calogero 2003, Section 2] for a detailed discussion on the derivation). Since most of the simplifications concern difficulties which we already know how to handle, in the style of [Christodoulou and Klainerman 1993] or [Lindblad and Rodnianski 2010], and since the method that we are using here is of the same type as the one used to study the Einstein vacuum equations, we believe it is a good model problem before addressing the full Einstein–Vlasov system via vector field methods.

Before presenting our main results for the massive and massless Vlasov–Nordström systems, let us explain the main differences between the $m=0$ and $m>0$ cases. First, as easily seen from (6)–(7), when $m=0$, the system degenerates to a partially decoupled system[7]

$$\square\phi = 0, \tag{8}$$

$$\boldsymbol{T}_0(f) - (\boldsymbol{T}_0(\phi)v^i)\frac{\partial f}{\partial v^i} = (n+1)f\,\boldsymbol{T}_0(\phi). \tag{9}$$

Because of the decoupling, the first equation is simply the wave equation on Minkowski space and the second can be viewed as a linear transport equation, where the transport operator is the massless

[7]In fact, using $e^{-(n+1)\phi}f$ as an unknown, we can obtain an even simpler form of the equations where the right-hand side of (9) is put to 0. See (51).

relativistic transport operator plus some perturbations. In particular, all solutions are necessarily global as long as the initial data is sufficiently regular that the linear equations can be solved. Thus, our objective is solely to derive sharp asymptotics for the solutions of the transport equation. Moreover, since we have in mind future applications to more nonlinear problems, the only estimates that we will use on ϕ will be those compatible with what can be derived via a standard application of the vector field method.

Apart from the decoupling just explained, let us mention also two important pieces of structure present in the above equations. First, another great simplification comes from the existence of an *extra scaling symmetry* present only in the massless case: the vector field $v^i \partial_{v^i}$ commutes with the massless transport operator $\boldsymbol{T_0}$ and it is precisely this combination of derivatives in v which appears in equation (9). This fact will make all the error terms obtained after commutations much better than if a random set of derivatives in v was present in (9). Another property of (8)–(9) is the existence of a *null structure*, similar to the null structure of Klainerman for wave equations. More precisely, we show that $\boldsymbol{T_0}(\phi)$ has roughly the structure

$$\boldsymbol{T_0}(\phi) \simeq |v|\bar{\partial}\phi + \frac{1}{t}\,\partial\phi \cdot z(t,x,v),$$

where $\bar{\partial}\phi$ denotes derivatives tangential to the outgoing cone, $\partial\phi$ denotes arbitrary derivatives of ϕ and $z(t,x,v)$ are weights which are bounded along the characteristics of the linear massless transport operator. Since $\bar{\partial}\phi$ has better decay properties than a random derivative $\partial\phi$, we see that products of the form $\boldsymbol{T_0}(\phi)g$, where g is a solution to $\boldsymbol{T_0}(g) = 0$, have better decay properties than expected.[8] Similar to the study of 3-dimensional wave equations with nonlinearities satisfying the null condition, the extra decay obtained means that in dimension 3 (or greater), all the error terms in the (approximate) conservation laws are now integrable.

We now state our main results for the massless Vlasov–Nordström system.

Theorem 3 (asymptotics in the massless case for dimension $n \geq 4$). *Let $n \geq 4$ and $N \geq \frac{3}{2}n + 1$. Let ϕ be a solution of (8) satisfying $\phi(t=0) = \phi_0$ and $\partial_t\phi(t=0) = \phi_1$ for some sufficiently regular functions (ϕ_0, ϕ_1). Then, if $\mathcal{E}_N[\phi_0,\phi_1] < +\infty$, where $\mathcal{E}_N[\phi_0,\phi_1]$ is an energy norm containing up to N derivatives of $(\partial\phi_0, \phi_1)$ and if $E_N[f_0] < +\infty$, where $E_N[f_0]$ is a norm containing up to N derivatives of f_0, then the unique classical solution f to (9) satisfying $f(t=0) = f_0$ also satisfies*:

(1) *Global bounds. For all $t \geq 0$,*

$$E_N[f](t) \leq e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_N[f_0],$$

where $C > 0$ is a constant depending only on N, n.

[8] It is interesting to compare this form of the null condition to the one uncovered in [Dafermos 2006] for the massless Einstein–Vlasov system in spherical symmetry. In fact, two null conditions were used there. The obvious one consists essentially in understanding why null components of the energy momentum tensor of f decay better than expected. A more secret null condition is used in the analysis of the differential equation satisfied by the part of the velocity vector tangent to the outgoing cone. Our null condition is closely related to this one, even though we exploit it in a different manner since we are not using directly the characteristic system of ordinary differential equations associated with the transport equations.

(2) *Pointwise estimates for velocity averages. For all* $(t,x) \in [0,+\infty) \times \mathbb{R}^n_x$ *and all multi-indices* α *satisfying* $|\alpha| \le N-n$,

$$\int_{v \in \mathbb{R}^n_v \setminus \{0\}} |\widehat{Z}^\alpha f|(t,x,v)|v|\,dv \lesssim \frac{e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_N[f_0]}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}.$$

In dimension $n \ge 3$, similar results can be obtained provided an extra smallness assumption on the initial data for the wave function as well as stronger decay for the initial data of f hold.

Theorem 4 (asymptotics in the massless case for dimension $n = 3$). *Let* $n \ge 3$, $N \ge 7$ *and* $q \ge 1$. *Let* ϕ *be a solution of* (8) *satisfying* $\phi(t=0) = \phi_0$ *and* $\partial_t\phi(t=0) = \phi_1$ *for some sufficiently regular functions* (ϕ_0, ϕ_1). *Then, if* $\mathcal{E}_N[\phi_0,\phi_1] \le \varepsilon$, *where* $\mathcal{E}_N[\phi_0,\phi_1]$ *is an energy norm containing up to* N *derivatives of* $(\partial\phi_0, \phi_1)$ *and if* $E_{N,q}[f_0] < +\infty$, *where* $E_{N,q}[f_0]$ *is a norm*[9] *containing up to* N *derivatives of* f_0, *then the unique classical solution* f *to* (9) *satisfying* $f(t=0) = f_0$ *also satisfies*:

(1) *Global bounds with loss. For all* $t \ge 0$,

$$E_{N,q}[f](t) \le (1+t)^{C\varepsilon^{1/2}} E_{N,q}[f_0],$$

where $C > 0$ *depends only on* N, n.

(2) *Improved global bounds for lower orders. For any* $M \le N - \frac{1}{2}(n+2)$ *and any* $t \ge 0$,

$$E_{M,q-1}[f](t) \le e^{C\varepsilon^{1/2}} E_{N,q}[f_0].$$

(3) *Pointwise estimates for velocity averages. For all* $(t,x) \in [0,+\infty) \times \mathbb{R}^n_x$ *and all multi-indices* $|\alpha| \le N - \frac{1}{2}(3n+2)$,

$$\int_{v \in \mathbb{R}^n_v \setminus \{0\}} |\widehat{Z}^\alpha f|(t,x,v)|v|\,dv \lesssim \frac{E_{N,q}[f_0]}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}.$$

Perhaps counterintuitively, the massive case turned out to be harder to treat. While it is true that in the massive case the pointwise decay of velocity averages is not weaker along the null cone, there are two important extra difficulties, namely:

- The equations are now fully coupled. In particular, one cannot close an energy estimate for (6) unless we have some decay for the right-hand side. On the other hand, our decay estimates, being based on commutators, necessarily lose some derivatives. In turn, this would imply commuting (9) more, but we would then fail to close the estimates at the top order. We resolve this issue by another decay estimate for inhomogeneous transport equations with rough source terms satisfying certain product structures. This other type of decay estimate only provides L^2_x time decay of the velocity averages, which is precisely what is required to close the energy estimate for (6). The proof of this L^2_x decay estimate itself can be reduced to our L^∞ estimates, so that it can also be obtained using purely vector-field-type methods.

[9] The index q refers to powers of certain weights. See (63) for a precise definition of the norms.

- The vector field $v^i \partial_{v^i}$ does not commute with the massive transport operator. This implies that commuting with (some of) the standard vector fields will lose a power of t of decay compared to the massless case.

Because of the last issue, the results that we will present here are restricted to dimension $n \geq 4$. One way to treat the 3-dimensional case would be to improve upon the commutation formulae to eliminate the most dangerous terms. For instance, one could try to use *modified vector fields* in the spirit of [Smulevici 2016]. We plan to address the 3-dimensional case in future work.

A slightly more technical consequence of this last issue is that it introduces t weights in the estimates, which are not constant on the leaves of the hyperboloidal foliation that we wish to use. Together with the fact that the energy estimates are weaker on hyperboloids, this implies that the error terms arising in the top-order approximate conservation laws can be shown to be space-time integrable only in dimension $n \geq 5$. To address the dimension $n = 4$, instead of estimating directly $\widehat{Z}^\alpha(f)$, where f is the unknown distribution function and $\widehat{Z}^\alpha$ is a combination of α vector fields, we estimate instead a renormalized quantity of the form $\widehat{Z}^\alpha(f) + g^\alpha$, where the g^α is a (small) nonlinear term constructed from the solution. The extra terms appearing in the equation when the transport operator hits g^α will then cancel some of the worst terms in the equations.

Our main result in the massive case can then be stated as follows.

Theorem 5. *Let $n \geq 4$ and $m > 0$. Let $N \in \mathbb{N}$ be sufficiently large depending only on n. For any $\rho \geq 1$, denote by H_ρ the hyperboloid*

$$H_\rho = \{(t, x) \in \mathbb{R}_t \times \mathbb{R}^n_x \mid \rho^2 = t^2 - x^2\}.$$

For any sufficiently regular function ψ defined on $\mathbb{R}_t \times \mathbb{R}^n_x$, denote by $\psi_{|H_\rho}$ its restriction to H_ρ. Similarly, for any sufficiently regular function g defined on $\bigcup_{1\leq\rho\leq+\infty} H_\rho \times \mathbb{R}^n_v$, denote by $g_{|H_\rho\times\mathbb{R}^n_v}$ its restriction to $H_\rho \times \mathbb{R}^n_v$. Then, there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, if $E_{N+n}[f_0] + \mathcal{E}_N[\phi_0, \phi_1] \leq \varepsilon$, where $E_{N+n}[f_0]$ and $\mathcal{E}_N[\phi_0, \phi_1]$ are norms depending on respectively $N + n$ derivatives of f_0 and N derivatives of $(\partial\phi_0, \phi_1)$, then there exists a unique classical solution (f, ϕ) to (6)–(7) satisfying the initial conditions

$$\phi_{|H_1} = \phi_0, \quad \partial_t \phi_{|H_1} = \phi_1, \quad f_{|H_1\times\mathbb{R}^n_v} = f_0$$

such that (f, ϕ) exists globally[10] *and satisfies the following estimates*:

(1) *Global bounds. For all $\rho \geq 1$,*

$$\mathcal{E}_N[\phi](\rho) \lesssim \varepsilon \quad \textit{and} \quad E_N[f][\rho] \lesssim \varepsilon \rho^{C\varepsilon^{1/4}},$$

where $C = 1$ when $n = 4$ and $C = 0$ when $n > 4$.

(2) *Pointwise decay for $\partial Z^\alpha \phi$. For all multi-indices $|\alpha|$ such that $|\alpha| \leq N - \frac{1}{2}(n+2)$ and all (t, x) with $t \geq \sqrt{1 + |x|^2}$, we have*

$$|\partial Z^\alpha \phi| \lesssim \frac{\varepsilon}{(1+t)^{\frac{n-1}{2}}(1 + (t - |x|))^{\frac{1}{2}}}.$$

[10] Here, globally means at every point lying in the future of the initial hyperboloid H_1. In 3 dimensions, this, of course, would already follow from [Calogero 2006] for regular initial data of compact support given on a constant-time slice.

(3) *Pointwise decay for $\rho(|\partial Z^\alpha f|)$. For all multi-indices α and β such that $|\alpha| \le N-n$ and $|\beta| \le \lfloor \frac{N}{2} \rfloor - n$ and all (t,x) with $t \ge \sqrt{1+|x|^2}$, we have*

$$\int_v |\widehat{Z}^\alpha f| \frac{dv}{v^0} \lesssim \frac{\varepsilon}{(1+t)^{n-C\varepsilon^{1/4}}} \quad \text{and} \quad \int_v v^0 |\widehat{Z}^\beta f|\, dv \lesssim \frac{\varepsilon}{(1+t)^{n-C\varepsilon^{1/4}}},$$

where $C = 1$ when $n = 4$ and $C = 0$ when $n > 4$.

(4) *Finally, the following L^2-estimates on f hold. For all multi-indices α with $\lfloor \frac{N}{2} \rfloor - n + 1 \le |\alpha| \le N$, and all (t,x) with $t \ge \sqrt{1+|x|^2}$, we have*

$$\int_{H_\rho} \frac{t}{\rho} \left(\int_v |\widehat{Z}^\alpha f| \frac{dv}{v^0} \right)^2 d\mu_{H_\rho} \lesssim \varepsilon^2 \rho^{C\varepsilon^{1/4}-n},$$

where $C = 2$ when $n = 4$ and $C = 0$ when $n > 4$.

Remark 1.6. As for the linear decay estimates of Theorem 2, it is not essential to start on an initial hyperboloid for the conclusions of Theorem 5 to hold. In particular, an easy argument based on finite speed of propagation, similar to that given in Appendix A, shows that our method and results apply to the case of sufficiently small initial data with compact x-support given at $t = 0$.

Remark 1.7. In [Friedrich 2004], solutions of the massive Vlasov–Nordström system in dimension 3 arising from small, regular, compactly supported (in x and v) data given at $t = 0$ were studied and the asymptotics of velocity averages of the Vlasov field and up to two derivatives of the wave function were obtained. However, no estimates were obtained for derivatives of the Vlasov field or for higher derivatives of the wave function. Thus, [Friedrich 2004] is the analogue of [Bardos and Degond 1985] for the Vlasov–Nordström system, while we obtained here (in dimension 4 and greater) results more in the spirit of [Hwang et al. 2011; Smulevici 2016].

Remark 1.8. A posteriori, it is straightforward to propagate higher moments of the solutions in any of the situations of Theorems 3, 4 and 5, provided that these moments are finite initially. Moreover, we recall that improved decay for derivatives of f and ϕ follows from the statements of Theorems 3, 4 and 5. See, for instance, Propositions 3.2 and 3.4 below.

Aside: the Einstein–Vlasov system. As explained above, the Vlasov–Nordström system is a model problem for the more physically relevant Einstein–Vlasov system. We refer to the recent book[11] [Ringström 2013] for a thorough introduction to this system. The small data theory around the Minkowski space is still incomplete for the Einstein–Vlasov system. The spherically symmetric cases in dimension $(3+1)$ have been treated in [Rein and Rendall 1992] for the massive case and in [Dafermos 2006] for the massless case with compactly supported initial data. A proof of stability for the massless case without spherical symmetry but with compact support in both x and v was recently given by M. Taylor [2017]. As in [Dafermos 2006], the compact support assumptions and the fact that the particles are massless are important as they allow one to reduce the proof to that of the vacuum case outside from a strip going to null infinity. Interestingly, Taylor's argument is quite geometric, relying for instance on the double null

[11] Apart from a general introduction to the Einstein–Vlasov system, the main purpose of [Ringström 2013] is to present a proof of stability of exponentially expanding space-times for the Einstein–Vlasov system.

foliation, in the spirit of [Klainerman and Nicolò 2003], as well as several structures associated with the tangent bundle of the tangent bundle of the base manifold.

We hope to address the stability of the Minkowski space for the Einstein–Vlasov system in the massive and massless case (without the compact support assumptions) using the method developed in this paper in future works.

Structure of the article. Section 2 contains preliminary materials, such as basic properties of the transport operators, the definition and properties of the foliation by hyperboloids used for the analysis of the massive distribution function, the commutation vector fields and elementary properties of these vector fields. In Section 3, we introduce the vector field method for relativistic Vlasov fields and prove Theorems 1 and 2. In Section 4, we apply our method first to the massless case in dimension $n \geq 4$ (Section 4B3) and $n = 3$ (Section 4B4) and then to the massive case in dimension $n \geq 4$ (Section 4C). In Appendix A, we provide a classical construction which explains how our decay estimates in the massive case can be applied to data of compact support in x given at $t = 0$. Some integral estimates useful in the course of the paper are proven in Appendix B. Finally, Appendix C contains a general geometrical framework for the analysis of the Vlasov equation on a Lorentzian manifold.

2. Preliminaries

2A. ***Basic notations.*** Throughout this paper we work on the $(n+1)$-dimensional Minkowski space $(\mathbb{R}^{n+1}, \eta)$, where the standard Minkowski metric η is globally defined in Cartesian coordinates (t, x^i) by $\eta = \text{diag}\{-1, 1, \dots, 1\}$. We denote space-time indices by Greek letters $\alpha, \beta, \dots \in \{0, \dots, n\}$ and spatial indices by Latin letters $i, j, \dots \in \{1, \dots, n\}$. We will sometimes use $\partial_{x^\alpha}, \partial_t, \partial_{x^i}, \partial_{v^i}, \dots$ to denote the partial derivatives $\partial/\partial x^\alpha$, $\partial/\partial t, \dots$.

Since we will be interested in either massive particles with $m = 1$ or massless particles $m = 0$, the velocity vector $(v^\beta)_{\beta=0,\dots,n}$ will be parametrized by $(v^i)_{i=1,\dots,n}$ and $v^0 = |v|$ in the massless case, $v^0 = \sqrt{1+|v|^2}$ in the massive case.

The indices 0 and $m > 0$ will be used to denote objects corresponding to the massless and massive cases respectively, such as the massless transport operator $\boldsymbol{T}_0$ and the massive one $\boldsymbol{T}_m$, and should not be confused with spatial or space-time indices for tensor components (we use bold letters on the transport operators to avoid this confusion).

The notation $A \lesssim B$ will be used to denote an inequality of the form $A \leq CB$ for some constant $C > 0$ independent of the solutions (typically C will depend on the number of dimensions, the maximal order of commutations N, the value of the mass m).

2B. ***The relativistic transport operators.*** For any $m > 0$ and any $v \in \mathbb{R}^n$, let us define the *massive relativistic transport operator* $\boldsymbol{T}_m$ by

$$\boldsymbol{T}_m = v^0 \partial_t + v^i \partial_{x^i}, \quad \text{with } v^0 = \sqrt{m^2 + |v|^2}. \tag{10}$$

Similarly, we define for any $v \in \mathbb{R}^3 \setminus \{0\}$, the *massless transport operator* $\boldsymbol{T}_0$ by

$$\boldsymbol{T}_0 = v^0 \partial_t + v^i \partial_{x^i}, \quad \text{with } v^0 = |v|. \tag{11}$$

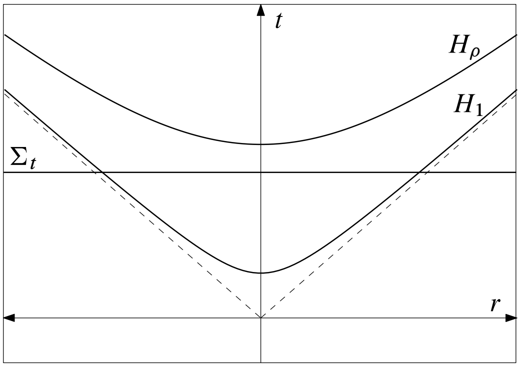

Figure 1. The H_ρ foliations in the (t,r) plane, $\rho > 1$.

For the sake of comparison, let us recall that the *classical* transport operator is given by

$$\boldsymbol{T}_{\mathrm{cl}} = \partial_t + v^i \partial_{x^i}.$$

In the remainder of this work, we will normalize the mass to be either 1 or 0, so that the massive transport operators we will study are

$$\boldsymbol{T}_1 = \sqrt{1+|v|^2}\partial_t + v^i \partial_{x^i} \quad \text{and} \quad \boldsymbol{T}_0 = |v|\partial_t + v^i \partial_{x^i}.$$

2C. ***The foliations.*** We will consider two distinct foliations of (some subsets of) the Minkowski space.

Let us fix global Cartesian coordinates (t, x^i), $1 \le i \le n$, on $\mathbb{R}^{n+1}$ and denote by Σ_t the hypersurface of constant t. The hypersurfaces Σ_t, $t \in \mathbb{R}$, then give a complete foliation of $\mathbb{R}^{n+1}$. The second foliation is defined as follows. For any $\rho > 0$, define H_ρ by

$$H_\rho = \{(t,x) \mid t \ge |x| \text{ and } t^2 - |x|^2 = \rho^2\}.$$

See Figure 1. For any $\rho > 0$, H_ρ is thus only one sheet of a two-sheeted hyperboloid.[12]

Note that

$$\bigcup_{\rho \ge 1} H_\rho = \{(t,x) \in \mathbb{R}^{n+1} \mid t \ge (1+|x|^2)^{\frac{1}{2}}\}.$$

The above subset of $\mathbb{R}^{n+1}$ will be referred to as *the future of the unit hyperboloid*; see Figure 2. On this set, we will use as an alternative to the Cartesian coordinates (t,x) the following two other sets of coordinates:

- Spherical coordinates. We first consider spherical coordinates (r,ω) on $\mathbb{R}^n_x$, where ω denotes spherical coordinates on the $(n-1)$-dimensional spheres and $r = |x|$. Then (ρ, r, ω) defines a coordinate system on the future of the unit hyperboloid. These new coordinates are defined globally on the future of the unit hyperboloid apart from the usual degeneration of spherical coordinates and at $r = 0$.

[12] The hyperboloidal foliation was originally introduced in [Klainerman 1985a] in the context of the nonlinear Klein–Gordon equation. For more recent applications, see [Wang 2015a; LeFloch and Ma 2016], which concern the stability of the Minkowski space for the Einstein–Klein–Gordon system.

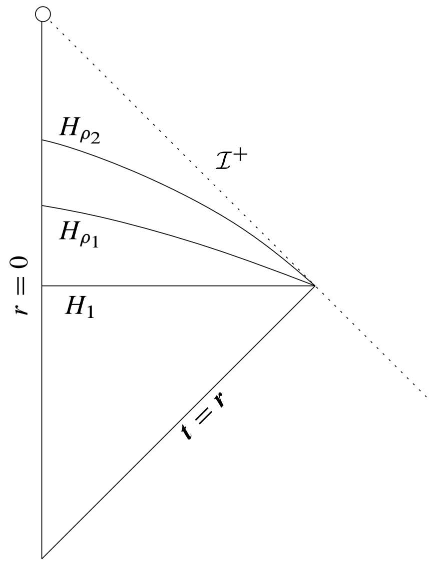

Figure 2. The H_ρ foliations in a Penrose diagram of Minkowski space, $\rho_2 > \rho_1 > 1$.

- Pseudo-Cartesian coordinates. These are the coordinates $(y^0, y^j) \equiv (\rho, x^j)$. These new coordinates are also defined globally on the future of the unit hyperboloid.

For any function defined on (some part of) the future of the unit hyperboloid, we will move freely between these three sets of coordinates.

2D. ***Geometry of the hyperboloids.*** The Minkowski metric η is given in (ρ, r, ω) coordinates by

$$\eta = -\frac{\rho^2}{t^2}(d\rho^2 - dr^2) - \frac{2\rho r}{t^2}\, d\rho\, dr + r^2 \sigma_{\mathbb{S}^{n-1}},$$

where $\sigma_{\mathbb{S}^{n-1}}$ is the standard round metric on the $(n-1)$-dimensional unit sphere, so that, for instance,

$$\sigma_{\mathbb{S}^2} = \sin\theta^2\, d\theta^2 + d\phi^2$$

in standard (θ, ϕ) spherical coordinates for the 2-sphere. The 4-dimensional volume form is thus given by

$$\frac{\rho}{t} r^{n-1}\, d\rho\, dr\, d\sigma_{\mathbb{S}^{n-1}},$$

where $d\sigma_{\mathbb{S}^{n-1}}$ is the standard volume form of the $(n-1)$-dimensional unit sphere.

The Minkowski metric induces on each of the H_ρ a Riemannian metric given by

$$ds^2_{H_\rho} = \frac{\rho^2}{t^2}\, dr^2 + r^2 \sigma_{\mathbb{S}^{n-1}}.$$

A normal differential form to H_ρ is given by $t\, dt - r\, dr$, while $t\partial_t + r\,\partial_r$ is a normal vector field. Since

$$\eta(t\partial_t + r\partial_r,\, t\partial_t + r\partial_r) = -\rho^2,$$

the future unit normal vector field to H_ρ is given by the vector field

$$\nu_\rho \equiv \frac{1}{\rho}(t\,\partial_t + r\,\partial_r). \tag{12}$$

Finally, the induced volume form on H_ρ, denoted by $d\mu_{H_\rho}$, is given by

$$d\mu_{H_\rho} = \frac{\rho}{t} r^{n-1}\,dr\,d\sigma_{\mathbb{S}^{n-1}}.$$

2E. ***Regular distribution functions.*** For the massive transport operator, we will consider distribution functions f as functions of (t, x, v) or (ρ, r, ω, v) defined on

$$\bigcup_{1\le\rho<P} H_\rho \times \mathbb{R}^n_v, \quad P \in [1, +\infty];$$

i.e., we are looking at the future of the unit hyperboloid, or a subset of it, times $\mathbb{R}^n_v$.

For the massless transport operator, we need to exclude $|v| = 0$ and we will only use the Σ_t foliation so that we will consider distribution functions f as functions of (t, x, v) defined on $[0, T) \times \mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$, $T \in [0, +\infty]$.

In the remainder of this article, we will denote by *regular* distribution function any such function f that is sufficiently regular that all the norms appearing on the right-hand sides of the estimates are finite. For simplicity, the reader might assume that f is smooth and decays fast enough in x and v at infinity and in the massless case, that f is integrable near $v = 0$ and similarly for the distribution functions obtained after commutations.

In physics, distribution functions represent the number of particles and are therefore required to be nonnegative. This will play no role in the present article, so we simply assume that distribution functions are real-valued.

2F. ***The linear equations.*** In the first part of this paper, we will study, for any $\boldsymbol{T} = \boldsymbol{T}_0, \boldsymbol{T}_1$, the homogeneous transport equation

$$\boldsymbol{T} f = 0, \tag{13}$$

as well as the inhomogeneous transport equation

$$\boldsymbol{T} f = v^0 h, \tag{14}$$

where $v^0 = \sqrt{1 + |v|^2}$ in the massive case and $v^0 = |v|$ in the massless case and where the source term h is a regular distribution function, as explained in Section 2E.

In the massless case, we will study the solution f to (13) or (14) with the initial data condition $f(t = 0, \cdot) = f_0$, where f_0 is a function defined on $\mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$.

In the massive case, we will study the solution f to (13) or (14) in the future of the unit hyperboloid with the initial data condition $f_{|H_1 \times \mathbb{R}^n_v} = f_0$, where f_0 is a function defined on $H_1 \times \mathbb{R}^n_v$.

Equations (13) and (14) are transport equations and can therefore be solved explicitly (at least for C^1 initial data) via the method of characteristics. If f solves (13), then

$$f(t, x, v) = f\left(0, x - \frac{v}{v^0} t, v\right),$$

where $v^0 = \sqrt{1+|v|^2}$ for the massive case and $v^0 = |v|$ for the massless case. In the inhomogeneous case, we obtain via the Duhamel formula that if f solves (14) with 0 initial data, then

$$f(t,x,v) = \int_0^t h\left(s, x-(t-s)\frac{v}{v^0}, v\right) ds.$$

2G. *The commutation vector fields.*

2G1. *Complete lifts of isometries and conformal isometries.* Let us recall that the set of generators of isometries of the Minkowski space, that is to say, the set of Killing fields, denoted by $\mathbb{P}$, consists of the translations, the rotations and the hyperbolic rotations; i.e.,

$$\mathbb{P} = \{\partial_t, \partial_{x^1}, \dots, \partial_{x^n}\} \cup \{\Omega_{ij} = x^i\partial_{x^j} - x^j\partial_{x^i} \mid 1 \le i,j \le n\} \cup \{\Omega_{0i} = t\partial_{x^i} + x^i\partial_t \mid 1 \le i \le n\}.$$

Mostly in the case of the massless transport operator, it will be useful, as in the study of the wave equation, to add the scaling vector field $S = t\partial_t + x^i\partial_i$ to our set of commutator vector fields. Let us thus define the set

$$\mathbb{K} = \mathbb{P} \cup \{S\}.$$

The vector fields in $\mathbb{K}$ and $\mathbb{P}$ lie in the tangent bundle of the Minkowski space. To any vector field on a manifold, we can associate a *complete lift*, which is a vector field lying on the tangent bundle, to the tangent bundle of the manifold. In Appendix C, we recall the general construction on a Lorentzian manifold. For the sake of simplicity, let us here give a working definition of the complete lifts only in coordinates.

Definition 2.1. Let W be a vector field of the form $W = W^\alpha \partial_{x^\alpha}$. Then let

$$\widehat{W} = W^\alpha \partial_{x^\alpha} + v^\beta \frac{\partial W^i}{\partial x^\beta} \partial_{v^i}, \tag{15}$$

where $(v^\beta)_{\beta=0,\dots,n} = (v^0, v^1, \dots, v^n)$ with $v^0 = |v|$ in the massless case and $v^0 = \sqrt{1+|v|^2}$ in the massive case, be called the *complete lift*[13] of W.

We will denote by

$$\widehat{\mathbb{K}} \equiv \{\widehat{Z} \mid Z \in \mathbb{K}\} \quad \text{and} \quad \widehat{\mathbb{P}} \equiv \{\widehat{Z} \mid Z \in \mathbb{P}\}$$

the sets of the complete lifts of $\mathbb{K}$ and $\mathbb{P}$.

Finally, let us also define $\widehat{\mathbb{P}}_0$ and $\widehat{\mathbb{K}}_0$ as the sets composed respectively of $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{K}}$ and a scaling vector field[14] in (t,x) only:

$$\widehat{\mathbb{P}}_0 \equiv \widehat{\mathbb{P}} \cup \{t\partial_t + x^i\partial_{x^i}\} = \widehat{\mathbb{P}} \cup \{S\}, \tag{16}$$

$$\widehat{\mathbb{K}}_0 \equiv \widehat{\mathbb{K}} \cup \{t\partial_t + x^i\partial_{x^i}\} = \widehat{\mathbb{K}} \cup \{S\}. \tag{17}$$

[13]This is in fact a small abuse of notation, as, with the above definition, $\widehat{W}$ actually corresponds to the restriction of the complete lift of W to the submanifold corresponding to $v^0 = \sqrt{1+|v|^2}$ in the massive case and $v^0 = |v|$ in the massless case. See again Appendix C for a more precise definition of $\widehat{W}$.

[14]Here, by a small abuse of notation, we denote with the same letter S, the vector field $t\,\partial_t + x^i\,\partial_{x^i}$ irrespectively of whether we consider it as a vector field on $\mathbb{R}^{n+1}$ or a vector field on (some subsets of) $\mathbb{R}^{n+1} \times \mathbb{R}^n_v$.

Lemma 2.2. *In Cartesian coordinates, the complete lifts of the elements of $\mathbb{P}$ and $\mathbb{K}$ are given by the following formulae*:

$$\hat{\partial}_t = \partial_t, \quad \hat{\partial}_{x^i} = \partial_{x^i},$$
$$\widehat{\Omega}_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i} + v^i \partial_{v^j} - v^j \partial_{v^i}, \quad \widehat{\Omega}_{0i} = t \partial_{x^i} + x^i \partial_t + v^0 \partial_{v^i},$$
$$\widehat{S} = t \partial_t + x^i \partial_{x^i} + v^i \partial_{v^i}.$$

2G2. *Commutation properties of the complete lifts.* As for the wave equation, the symmetries of the Minkowski space are reflected in the transport operators (10) and (11) through the existence of commutation vector fields. More precisely,

Lemma 2.3.
- *Commutation rules for the massive transport operator*:

$$[\boldsymbol{T}_1, \widehat{Z}] = 0 \qquad \forall \widehat{Z} \in \widehat{\mathbb{P}}, \tag{18}$$
$$[\boldsymbol{T}_1, S] = \boldsymbol{T}_1, \tag{19}$$

  where $S = t \partial_t + x^i \partial_{x^i}$ is the usual scaling vector field.
- *Commutation rules for the massless transport operator*:

$$[\boldsymbol{T}_0, \widehat{Z}] = 0 \qquad \forall \widehat{Z} \in \widehat{\mathbb{K}}, \tag{20}$$
$$[\boldsymbol{T}_0, S] = \boldsymbol{T}_0. \tag{21}$$

Proof. The identities can be verified directly using the explicit expressions for the elements in $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{K}}$, but also follow from the general formula given in Appendix C (see Lemma C.7). $\square$

Remark 2.4. Note that from the expression of $\widehat{S}$ and the two commutation rules for $\boldsymbol{T}_0$ and $\widehat{S}$ and for $\boldsymbol{T}_0$ and S, it follows that

$$[\boldsymbol{T}_0, v^i \partial_{v^i}] = -\boldsymbol{T}_0.$$

Thus, we have in a certain sense two scaling symmetries, one in x and one in v.

Remark 2.5. It is interesting to note that while the Klein–Gordon operator $\square - m^2$ ($m > 0$) does not commute with the scaling vector field, the massive transport equation does commute in the form of equation (19). What does not commute is the second scaling vector field $v^i \partial_{v^i}$.

We also have the following commutation relation within $\widehat{\mathbb{P}}_0$ and $\widehat{\mathbb{K}}_0$.

Lemma 2.6. *For any $Z, Z' \in \widehat{\mathbb{P}}_0$, there exist constant coefficients $C_{ZZ'W}$ such that*

$$[Z, Z'] = \sum_{W \in \widehat{\mathbb{P}}} C_{ZZ'W} W.$$

Similarly, for any $Z, Z' \in \widehat{\mathbb{K}}_0$, there exist constant coefficients $D_{ZZ'W}$ such that

$$[Z, Z'] = \sum_{W \in \widehat{\mathbb{K}}} D_{ZZ'W} W.$$

2H. *Weights preserved by the flow.* Recall that in a general Lorentzian manifold with metric g, if γ is a geodesic with tangent vector $\dot{\gamma}$ and K denotes a Killing field, then $g(\dot{\gamma}, K)$ is preserved along γ. In this section, we explain how to transpose this fact to the transport operators on Minkowski space.

We define the sets of weights

$$\Bbbk_m \equiv \{v^\alpha x^\beta - x^\alpha v^\beta, v^\alpha\}, \tag{22}$$

$$\Bbbk_0 \equiv \{x^\alpha v_\alpha, v^\alpha x^\beta - x^\alpha v^\beta, v^\alpha\}. \tag{23}$$

The following lemma can be easily checked.

Lemma 2.7. (1) *For all $\mathfrak{z} \in \Bbbk_0$, we have $[\boldsymbol{T}_0, \mathfrak{z}] = 0$.*

(2) *For all $\mathfrak{z} \in \Bbbk_m$, we have $[\boldsymbol{T}_m, \mathfrak{z}] = 0$.*

The weights in $\Bbbk_m$ and $\Bbbk_0$ also have good commutation properties with the vector fields in $\widehat{\mathbb{P}}_0$ and $\widehat{\mathbb{K}}_0$.

Lemma 2.8. *For any $\mathfrak{z} \in \Bbbk_m$ and any $\widehat{Z} \in \widehat{\mathbb{P}}_0$,*

$$[\widehat{Z}, \mathfrak{z}] = \sum_{\mathfrak{z}' \in \Bbbk_m} c_{\mathfrak{z}'} \mathfrak{z}',$$

where the $c_{\mathfrak{z}'}$ are constant coefficients.

Similarly for any $\mathfrak{z} \in \Bbbk_0$ and any $\widehat{Z} \in \widehat{\mathbb{K}}_0$,

$$[\widehat{Z}, \mathfrak{z}] = \sum_{\mathfrak{z}' \in \Bbbk_m} d_{\mathfrak{z}'} \mathfrak{z}'$$

for some constant coefficients $d_{\mathfrak{z}'}$.

Proof. This follows from straightforward computations. □

2I. *Multi-index notations.* Recall that a multi-index α of length $|\alpha|$ is an element of $\mathbb{N}^r$ for some $r \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^r \alpha_i = |\alpha|$.

Let Z^i, $i = 1, \dots, 2n + 2 + \frac{1}{2}n(n-1)$, be an ordering of $\mathbb{K}$. For any multi-index α, we will denote by Z^α the differential operator of order $|\alpha|$ given by the composition $Z^{\alpha_1} Z^{\alpha_2} \cdots$.

In view of the above discussion, the complete lift operation defines a bijection between $\mathbb{K}$ and $\widehat{\mathbb{K}}$. Thus, to any ordering of $\mathbb{K}$, we can associate an ordering of $\widehat{\mathbb{K}}$. One extends this ordering to $\widehat{\mathbb{K}}_0$ by setting[15] $\widehat{Z}^{2n+3+\frac{1}{2}n(n-1)} = S$. We will again write $\widehat{Z}^\alpha$ to denote the differential operator of order $|\alpha|$ obtained by the composition $\widehat{Z}^{\alpha_1} \widehat{Z}^{\alpha_2} \cdots$.

Similarly, we consider an ordering of $\mathbb{P}$, which gives us an ordering of $\widehat{\mathbb{P}}$ which can be extended to give an ordering of $\widehat{\mathbb{P}}_0$, and we write Z^α and $\widehat{Z}^\alpha$ for a composition of $|\alpha|$ vector fields in $\mathbb{P}$, $\widehat{\mathbb{P}}$ or $\widehat{\mathbb{P}}_0$.

The notation $\mathbb{K}^N$ will be used to denote the set of all the differential operators of the form Z^α, with $|\alpha| = N$. Similarly, we will use the notations $\mathbb{P}^{|\alpha|}$, $\widehat{\mathbb{K}}_0^{|\alpha|}$ and $\widehat{\mathbb{P}}_0^{|\alpha|}$.

We will also write $\partial_{t,x}^\alpha$ to denote a differential operator of order $|\alpha|$ obtained as a composition of $|\alpha|$ translations among the ∂_t, ∂_{x^i} vector fields.

As for the sets of vector fields, we will also consider orderings of the sets of weights $\Bbbk_m$ and $\Bbbk_0$ and we will write $\mathfrak{z}^\alpha \in \Bbbk_m^{|\alpha|}$ or $\mathfrak{z}^\alpha \in \Bbbk_0^{|\alpha|}$ to denote a product of $|\alpha|$ weights in $\Bbbk_m$ or $\Bbbk_0$.

[15]Note that this is a small abuse of notation, since S is not obtained via the complete lift construction.

2J. ***Vector field identities.*** The following classical vector field identities will be used later in the paper.

Lemma 2.9. *The following identities hold*:

$$(t^2-r^2)\partial_t = tS - x^i\Omega_{0i},\quad (t^2-r^2)\partial_i = -x^j\Omega_{ij} + t\Omega_{0i} - x^iS,\quad (t^2-r^2)\partial_r = t\frac{x^i}{r}\Omega_{0i} - rS.$$

Furthermore,

$$\begin{aligned}\partial_s &\equiv \tfrac{1}{2}(\partial_t+\partial_r) = \frac{S+\omega^i\Omega_{0i}}{2(t+r)},\\ \bar{\partial}_i &\equiv \partial_i - \omega_i\partial_r = \frac{\omega^j\Omega_{ij}}{r} = \frac{-\omega_i\omega^j\Omega_{0j}+\Omega_{0i}}{t}.\end{aligned} \tag{24}$$

2K. ***The particle vector field and the stress energy tensor of Vlasov fields.*** Recall that in the Vlasov–Poisson or Einstein–Vlasov systems, the transport equation for f is coupled to an elliptic equation or a set of evolution equations, via integrals in v of f, often referred to as velocity averages in the classical case. In the relativistic cases, the volume forms[16] in these integrals are defined as

$$d\mu_m \equiv \frac{dv^1\wedge\cdots\wedge dv^n}{v^0} = \frac{dv}{\sqrt{m^2+|v|^2}}, \tag{25}$$

where as usual $m=0$ in the massless case.

Remark 2.10. In the massless case, the volume form $dv/|v|$ is singular near $v=0$. In the remainder of this article, we will however study mostly energy densities, which introduce an additional factor of $|v|^2$ in the relevant integrals and thus remove this singular behaviour near $v=0$.

We now define the particle vector field in the case of massive particles as

$$N_m^\mu \equiv \int_{\mathbb{R}^n} f v^\mu\, d\mu_m,$$

and in the case of massless particles as

$$N_0^\mu \equiv \int_{\mathbb{R}^n\setminus\{0\}} f v^\mu\, d\mu_0,$$

as well as the energy momentum tensors

$$T_m^{\mu\nu} \equiv \int_{\mathbb{R}^n} f v^\mu v^\nu\, d\mu_m \quad\text{and}\quad T_0^{\mu\nu} \equiv \int_{\mathbb{R}^n\setminus\{0\}} f v^\mu v^\nu\, d\mu_0,$$

where $d\mu_m$ and $d\mu_0$ are the volume forms defined in (25). More generally, we can define the higher moments

$$M_m^{\alpha_1\cdots\alpha_p} \equiv \int_{\mathbb{R}^n} f v^{\alpha_1}\cdots v^{\alpha_p}\, d\mu_m,$$

and similarly for the massless system.

[16]For the interested reader, they can be interpreted geometrically as the natural volume forms associated with an induced metric on the manifold on which the averages are computed, together with a choice of normal in the massless case.

The interest in any of the above quantities is that if f is a solution to the associated massless or massive transport equations, then these quantities are divergence free. Indeed, we have

$$\partial_\mu T_0^{\mu\nu} = \int_{\mathbb{R}^n\setminus\{0\}} \boldsymbol{T}_0(f) v^\nu \, d\mu_0, \tag{26}$$

$$\partial_\mu T_m^{\mu\nu} = \int_{\mathbb{R}^n} \boldsymbol{T}_m(f) v^\nu \, d\mu_m. \tag{27}$$

We will be interested in particular in the energy densities

$$\rho_0(f) \equiv T_0(\partial_t, \partial_t) = \int_{\mathbb{R}^n\setminus\{0\}} f|v|\, dv \tag{28}$$

for the massless case, while for the massive case we define

$$\rho_m(f) \equiv T_m(\partial_t, \partial_t) = \int_{\mathbb{R}^n} f v^0 \, dv. \tag{29}$$

In the following, we will denote by $\rho(f)$ either of the quantities $\rho_m(f)$ or $\rho_0(f)$ depending on whether we are looking at the massive or the massless relativistic operator.

In the massive case, we will also make use of the energy density

$$\chi_m(f) \equiv T_m(\partial_t, \nu_\rho), \tag{30}$$

where ν_ρ is the future unit normal to H_ρ introduced in Section 2D. We compute

$$\chi_{\mathrm{m}}(f) = \int_{v\in\mathbb{R}^n} f v_0 \left(\frac{t}{\rho} v_0 + \frac{r}{\rho} v^r\right) d\mu_m = \int_{v\in\mathbb{R}^n} f \left(\frac{t}{\rho} v^0 - \frac{x^i}{\rho} v_i\right) dv.$$

The following lemma will be used later.

Lemma 2.11 (coercivity of the energy density normal to the hyperboloids). *Assuming that $t \geq r$, we have*

$$\chi_m(f) \geq \frac{t}{2\rho} \int_{v\in\mathbb{R}^n} f \left[\left(1 - \frac{r}{t}\right)((v^0)^2 + v_r^2) + r^2 \sigma_{AB} v^A v^B + m^2\right] \frac{dv}{v^0}. \tag{31}$$

Proof. Using that

$$(v^0)^2 = v_r^2 + r^2 \sigma_{AB} v^A v^B + m^2,$$

where σ_{AB} denotes the components of the metric $\sigma_{\mathbb{S}^n}$ and v^A, v^B are the angular velocities, we have

$$(v^0)^2 = \tfrac{1}{2}(v^0)^2 + \tfrac{1}{2}(v_r^2 + r^2 \sigma_{AB} v^A v^B + m^2),$$

and thus

$$v^0 \left(\frac{t}{\rho} v^0 - \frac{x^i}{\rho} v_i\right) = \frac{t}{2\rho} \left((v^0)^2 + v_r^2 + r^2 \sigma_{AB} v^A v^B + m^2 - 2\frac{x^i}{t} v_i v^0\right).$$

The lemma now follows from

$$(v^0)^2 + v_r^2 - 2\frac{x^i}{t} v_i v^0 \geq \left(1 - \frac{r}{t}\right)((v^0)^2 + v_r^2),$$

assuming $t \geq r$. □

Remark 2.12.

- Since $(v^0)^2 \geq v_r^2$, we will use (31) in the form

$$\chi_m(f) \geq \frac{t}{2\rho} \int_{v\in\mathbb{R}^n} f\left[\left(1-\frac{r}{t}\right)(v^0)^2 + r^2\sigma_{AB}v^Av^B + m^2\right] d\mu_m.$$

- We also remark that

$$\chi_m(|f|) \geq \tfrac{1}{2}m^2 \int_v |f| \frac{dv}{v^0} = \tfrac{1}{2}m^2\rho_m\left(\frac{|f|}{(v^0)^2}\right),$$

since $t/(2\rho) \geq \frac{1}{2}$, and, furthermore,

$$\chi_m(|f|) \geq \frac{t-r}{2\rho}\rho_m(|f|) = \frac{\rho}{2(t+r)}\rho_m(|f|).$$

- Finally, independently of Lemma 2.11, since by the Cauchy–Schwarz inequality for Lorentzian metrics, as the vectors ν_ρ and v are both timelike future directed,

$$\left|\frac{tv^0 - x^iv_i}{\rho}\right| = |\langle v, \nu_\rho\rangle| \geq |v||\nu_\rho| = m, \quad \text{where } |v| = |g(v,v)|^{\frac{1}{2}},$$

we get immediately

$$\int_v |f|\,dv \leq \int_v \frac{1}{m}\left|\frac{tv^0 - x^iv_i}{\rho}\right||f|\,dv = \frac{1}{m}\chi_m(|f|).$$

2L. *Commutation vector fields and energy densities.* Vector fields and the operator of averaging in v essentially commute in the following sense.

Lemma 2.13. *Let f be a regular distribution function for the massless case. Then*:

- *For any translation ∂_{x^α}, we have*

$$\partial_{x^\alpha}[\rho_0(f)] = \rho_0(\partial_{x^\alpha}(f)) = \rho_0(\hat{\partial}_{x^\alpha}(f)).$$

- *For any rotation Ω_{ij}, $1 \leq i, j, \leq n$, we have*

$$\Omega_{ij}[\rho_0(f)] = \rho_0(\widehat{\Omega}_{ij}(f)),$$

where $\widehat{\Omega}_{ij}$ is the complete lift of the vector field Ω_{ij}.

- *For any Lorentz boost Ω_{0i}, $1 \leq i \leq n$, we have*

$$\Omega_{0i}[\rho_0(f)] = \rho_0(\widehat{\Omega}_{0i}(f)) + 2\rho_0\left(\frac{v^i}{|v|}f\right).$$

- *For the scaling vector field S, we have*

$$S[\rho_0(f)] = \rho_0(\widehat{S}(f)) + (n+1)\rho_0(f).$$

- *Finally, all the above equalities hold (almost everywhere) with f replaced by $|f|$.*

Proof. Let us consider, for instance, a Lorentz boost $\Omega_{0i} = t\partial_{x^i} + x^i\partial_t$. Then

$$\Omega_{0i}[\rho_0(f)] = \int_v (t\partial_{x^i} + x^i\partial_t)(f)|v|\,dv. \tag{32}$$

On the other hand,

$$\begin{aligned}\int_v (t\partial_{x^i} + x^i\partial_t)(f)|v|\,dv &= \int_v (t\partial_{x^i} + x^i\partial_t + |v|\partial_{v^i})(f)|v|\,dv - \int_v |v|^2\partial_{v^i}(f)\,dv \\ &= \int_v \widehat{\Omega}_{0i}(f)|v|\,dv + 2\int_v \frac{v^i}{|v|}(f)|v|\,dv \\ &= \rho_0(\widehat{\Omega}_{0i}(f)) + 2\rho_0\left(\frac{v^i}{|v|}f\right),\end{aligned}$$

using an integration by parts in v^i. The other cases can all be treated similarly, the translations being trivial since $\hat{\partial}_{x^\alpha} = \partial_{x^\alpha}$. That f can be replaced by $|f|$ follows from the standard property of differentiation of the absolute value.[17] □

In the massive case, we have the following lemma, whose proof is left to the reader since it is very similar to the above.

Lemma 2.14. *Let f be a regular distribution function for the massive case. Then*:

- *For any translation ∂_{x^α}, we have*
$$\partial_{x^\alpha}[\rho_m(f)] = \rho_m(\partial_{x^\alpha}(f)) = \rho_m(\hat{\partial}_{x^\alpha}(f)).$$
- *For any rotation Ω_{ij}, $1 \le i, j, \le n$, we have*
$$\Omega_{ij}[\rho_m(f)] = \rho_m(\widehat{\Omega}_{ij}(f)),$$
where $\widehat{\Omega}_{ij}$ is the complete lift of the vector field Ω_{ij}.
- *For any Lorentz boost Ω_{0i}, $1 \le i \le n$, we have*
$$\Omega_{0i}[\rho_m(f)] = \rho_0(\widehat{\Omega}_{0i}(f)) + 2\rho_m\left(\frac{v^i}{v^0}f\right).$$
- *Finally, all the above equalities holds with f replaced by $|f|$.*

Remark 2.15. Although we do not have for all commutation vector fields $Z\rho = \rho\widehat{Z}$, we do have that $|Z\rho(|f|)| \lesssim \rho(|\widehat{Z}(f)|) + \rho(|f|)$ and this is all we shall need from the above. Note also that if we were looking at other moments, then similar formulae would hold with different coefficients. For instance, we have $\Omega_{0i}\int_v f\,d\mu_m = \int_v \widehat{\Omega}_{0i} f\,d\mu_m$ for sufficiently regular f.

Remark 2.16. In the massless case, we included the scaling vector field, but recall that $\boldsymbol{T_0}$ actually commutes with S (in the sense that $[\boldsymbol{T_0}, S] = \boldsymbol{T_0}$) so we will not really need to replace S by $\widehat{S}$. Note also that S enjoys good commutation properties with $\boldsymbol{T_m}$ and that $S\rho_m = \rho_m S$.

[17] Recall that $f \in W^{1,1}$ implies that $|f| \in W^{1,1}$ with $\partial|f| = (f/|f|)\,\partial f$ almost everywhere. See, for instance, [Lieb and Loss 1997, Chapter 6.17].

2M. ***(Approximate) conservation laws for Vlasov fields.*** The following lemma is easily verified.[18]

Lemma 2.17 (massless case). *Let h be a regular distribution function for the massless case in the sense of Section 2E. Let f be a regular solution to $\boldsymbol{T}_0(f) = v^0 h$, with $v^0 = |v|$, defined on $[0,T]\times\mathbb{R}^n_x\times(\mathbb{R}^n_v\setminus\{0\})$ for some $T>0$. Then, for all $t\in[0,T]$,*

$$\int_{\Sigma_t}\rho_0(f)(t,x)\,dx\left(\equiv\int_{x\in\mathbb{R}^n}\int_{v\in\mathbb{R}^n\setminus\{0\}}|v|f(t,x,v)\,dx\,dv\right)$$
$$=\int_{\Sigma_0}\rho_0(f)(0,x)\,dx+\int_0^t\int_{\Sigma_s}\rho_0(h)(s,x)\,dx\,ds, \tag{33}$$

and

$$\int_{\Sigma_t}\rho_0(|f|)(t,x)\,dx\le\int_{\Sigma_0}\rho_0(|f|)(0,x)\,dx+\int_0^t\int_{\Sigma_s}\rho_0(|h|)(s,x)\,dx\,ds. \tag{34}$$

Proof. The proof of (33) follows from an easy integration by parts (or an application of Stokes' theorem) and (26). A standard regularization argument of the absolute value allows to derive (34) in a similar manner. □

A similar identity holds for the massive case, but we shall need the following variant where we replace the Σ_t foliation by the H_ρ one.

Lemma 2.18 (massive case). *Let h be a regular distribution function for the massive case in the sense of Section 2E. Let f be a regular solution to $\boldsymbol{T}_m(f) = v^0 h$, with $v^0 = \sqrt{m^2+|v|^2}$, $m>0$, defined on $\bigcup_{\rho\in[1,P]} H_\rho\times\mathbb{R}^n_v$ for some $P>1$. Then, for all $\rho\in[1,P]$,*

$$\int_{H_\rho}\chi_m(f)(\rho,r,\omega)\,d\mu_{H_\rho}=\int_{H_1}\chi_m(f)(1,r,\omega)\,d\mu_{H_1}+\int_1^\rho\int_{H_\rho}\rho_m(h)(s,r,\omega)\,d\mu_{H_s}\,ds, \tag{35}$$

$$\int_{H_\rho}\chi_m(|f|)(\rho,r,\omega)\,d\mu_{H_\rho}\le\int_{H_1}\chi_m(|f|)(1,r,\omega)\,d\mu_{H_1}+\int_1^\rho\int_{H_\rho}\rho_m(|h|)(s,r,\omega)\,d\mu_{H_s}\,ds, \tag{36}$$

Proof. Again, the proof of (35) just follows from (27) and an integration by parts, while that of (36) follows similarly after a standard regularization argument. □

3. The vector field method for Vlasov fields

3A. ***The norms.*** We define, in the following, norms of distribution functions obtained from the standard conservation laws for the transport equations and the commutation vector fields introduced in the previous section.

Definition 3.1.
- Let f be a regular distribution function for the massless case in the sense of Section 2E defined on $[0,T]\times\mathbb{R}^n_x\times(\mathbb{R}^n_v\setminus\{0\})$. For $k\in\mathbb{N}$, we define, for all $t\in[0,T]$,

$$\|f\|_{\mathbb{K},k}(t)\equiv\sum_{|\alpha|\le k}\sum_{\widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}\int_{\Sigma_t}\rho_0(|\widehat{Z}^\alpha f|)(t,x)\,dx. \tag{37}$$

[18] Recall that if f is a regular solution to $\boldsymbol{T}(f)=v^0h$, then $|f|$ is a solution, in the sense of distributions, of $\boldsymbol{T}(|f|)=(f/|f|)v^0h$.

- Similarly, let f be a regular distribution function for the massive case in the sense of Section 2E defined on $\bigcup_{1\le\rho\le P} H_\rho\times\mathbb{R}^n_v$. For $k\in\mathbb{N}$, we define, for all $\rho\in[1,P]$,

$$\|f\|_{\mathbb{P},k}(\rho)\equiv\sum_{|\alpha|\le k}\sum_{\widehat{Z}^\alpha\in\widehat{\mathbb{P}}^{|\alpha|}}\int_{H_\rho}\chi_m(|\widehat{Z}^\alpha f|)\,d\mu_{H_\rho}. \tag{38}$$

3B. ***Klainerman–Sobolev inequalities and decay estimates: massless case.*** We are now ready to prove to following variant of the Klainerman–Sobolev inequalities.[19]

Theorem 6 (Klainerman–Sobolev inequalities for velocity averages of massless distribution functions). *Let f be a regular distribution function for the massless case defined on $[0,T]\times\mathbb{R}^n_x\times(\mathbb{R}^n_v\setminus\{0\})$ for some $T>0$. Then, for all $(t,x)\in[0,T]\times\mathbb{R}^n_x$,*

$$\rho_0(|f|)(t,x)\lesssim\frac{1}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}\|f\|_{\mathbb{K},n}(t). \tag{39}$$

Proof. Let $(t,x)\in[0,T]\times\mathbb{R}^n_x$ and assume first that $|x|\notin\big[\frac12 t,\frac32 t\big]$ and $t+|x|\ge1$. Let ψ be defined as

$$\psi: y\to\rho_0\big(|f|(t,x+(t+|x|)y)\big),$$

where $y=(y_1,y_2,\dots,y_n)$. Note that

$$\partial_{y_i}\psi(y)=\partial_{y^i}\big[\rho_0\big(|f|(t,x+(t+|x|)y\big)\big]=(t+|x|)\,\partial_{x^i}(\rho_0[|f|])(t,x+(t+|x|)y).$$

Assume now that $|y|\le\frac14$. Using the fact that we are away from the light cone and the condition on $|y|$, it follows that

$$\frac{1}{C}\le\frac{\big|t+|x|\big|}{\big|t-|x+(t+|x|)y|\big|}\le C$$

for some $C>0$. It then follows from the vector field identities of Lemma 2.9 that

$$\big|\partial_{y^i}\rho_0\big(|f|(t,x+(t+|x|)y)\big)\big|\lesssim\sum_{Z\in\mathbb{K}}\big|Z(\rho_0[|f|])\big|(t,x+(t+|x|)y).$$

From Lemma 2.13, we then obtain that

$$\begin{aligned}\big|\partial_{y^i}\rho_0\big[|f|(t,x+(t+|x|)y)\big]\big|&\lesssim\sum_{\substack{|\alpha|\le1\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}}\big|\rho_0[\widehat{Z}(|f|)]\big|(t,x+(t+|x|)y)+\rho_0(|f|)(t,x+(t+|x|)y)\\&\lesssim\sum_{\substack{|\alpha|\le1\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}}\big|\rho_0[\widehat{Z}^\alpha(|f|)]\big|(t,x+(t+|x|)y)\end{aligned}$$

[19] Note that (in more than one spatial dimension) we cannot apply directly the standard Klainerman–Sobolev inequalities, in fact not even the usual Sobolev inequalities, to quantities such as $\rho(|f|)$ because of the lack of regularity of the absolute value. The aim of this section is therefore to explain how to circumvent this technical issue.

For a very clear introduction to Klainerman–Sobolev inequalities in the classical case of the wave equation, the interested reader may consult [Wang 2015b]. Some of the arguments below have been adapted from those notes.

$$\lesssim \sum_{\substack{|\alpha|\leq 1\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}} \rho_0\big[\big|\widehat{Z}^\alpha(|f|)\big|\big](t,x+(t+|x|)y)$$
$$\lesssim \sum_{\substack{|\alpha|\leq 1\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}} \rho_0\big[|\widehat{Z}^\alpha(f)|\big](t,x+(t+|x|)y),$$

where we have used in the last line that for any vector field $\widehat{Z}$, we have $|\widehat{Z}(|f|)| = |\widehat{Z}(f)|$ (almost everywhere and provided f is sufficiently regular), which essentially follows from the fact that $\partial|f| = (f/|f|)\,\partial f$ almost everywhere if $f\in W^{1,1}$. Let now $\delta = 1/(16n)$, so that if $|y_i|\leq \delta^{\frac{1}{2}}$ for all $1\leq i\leq n$, we then have $|y|\leq \frac{1}{4}$. Applying now a 1-dimensional Sobolev inequality in the variable y_1, we have

$$|\psi(0)| = \rho_0[|f|](t,x) \lesssim \int_{|y_1|\leq\delta^{1/2}} \big(\big|\partial_{y_1}\psi(y_1,0,\dots,0)\big| + \big|\psi(y_1,0,\dots,0)\big|\big)\,dy_1$$
$$\lesssim \int_{|y_1|\leq\delta^{1/2}} \Bigg(\sum_{\substack{|\alpha|\leq 1\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}} \rho_0\big[|\widehat{Z}^\alpha(f)|\big]\big(t,x+(t+|x|)(y_1,0,\dots,0)\big)\Bigg)\,dy_1.$$

We can now apply a 1-dimensional Sobolev inequality in the variable y_2 and repeat the previous argument, with $|Z^\alpha(f)|$ replacing $|f|$, to obtain

$$|\psi(0)| \lesssim \int_{|y_1|\leq\delta^{1/2}}\int_{|y_2|\leq\delta^{1/2}} \Bigg(\sum_{\substack{|\alpha|\leq 2\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}} \rho_0\big[|\widehat{Z}^\alpha(f)|\big]\big(t,x+(t+|x|)(y_1,y_2,\dots,0)\big)\Bigg)\,dy_1\,dy_2.$$

Repeating the argument up to exhaustion of all variables, we obtain that

$$\rho_0[|f|](t,x)$$
$$\lesssim \int_{|y_1|\leq\delta^{1/2}}\int_{|y_2|\leq\delta^{1/2}}\cdots\int_{|y_n|\leq\delta^{1/2}} \Bigg(\sum_{\substack{|\alpha|\leq n\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^{|\alpha|}}} \rho_0\big[|\widehat{Z}^\alpha(f)|\big]\big(t,x+(t+|x|)(y_1,y_2,\dots,y_n)\big)\Bigg)\,dy_1\,dy_2\cdots dy_n.$$

Applying the change of variable $z=(t+|x|)y$ gives us a $(t+|x|)^n$ factor which completes the proof of the inequality in this particular case. The case where $(t+|x|)\leq 1$ follows from simpler considerations and is therefore left to the reader.

Let us thus turn to the case where $x\in\big[\frac{1}{2}t,\frac{3}{2}t\big]$ and $(t+|x|)\geq 1$. Note that it then follows that $t>\frac{2}{5}$ and $|x|>\frac{1}{3}$. Let us introduce spherical coordinates $(r,\omega)\in[0,+\infty)\times\mathbb{S}^{n-1}$ such that $x=r\omega$ and denote by q the optical function $q\equiv r-t$. Let $v(t,q,\omega)\equiv\rho_0(f)(t,(t+q)\omega)$.

Note that $\partial_q v=\partial_r\rho_0$ and $q\,\partial_q v=(r-t)\partial_r$ and that there exist constants C_{ij} such that

$$\partial_\omega v = \partial_\omega\big(\rho_0(f)(t,(q+t)\omega)\big) = \sum_{i<j} C_{ij}\Omega_{ij}\rho_0(f),$$

where the Ω_{ij} are the rotation vector fields.

Let $q_0 = |x| - t$. We need to prove that

$$t^{n-1}(1+|q_0|)\,|v(t,q_0,\omega)| \lesssim \|f\|_{\mathbb{K},n}(t).$$

Using a 1-dimensional Sobolev inequality, we have for any $\omega \in \mathbb{S}^{n-1}$,

$$|v(t,q_0,\omega)| \lesssim \int_{|q|<\frac{1}{4}t} \sum_{|\alpha|\le 1} \left|(\partial_q^\alpha v)(t,q+q_0,\eta)\right| dq.$$

Now

$$(\partial_q v)(t,q+q_0,\omega) = (\partial_r \rho_0(f))(t,q+q_0,\omega) = \rho_0(\partial_r(|f|)),$$

and thus

$$|\partial_q v(t,q+q_0,\omega)| \lesssim \rho_0(|\partial_r f|)(t,q+q_0,\omega),$$

where we have used again the properties of the derivatives of the absolute value. Let now $(\omega_1,\omega_2,\dots,\omega_{n-1})$ be a local coordinate patch in a neighbourhood of the point $\omega \in \mathbb{S}^{n-1}$. Using again a 1-dimensional inequality, we have

$$\begin{aligned}\left|\rho_0(|\partial_r^\alpha f|)(t,q+q_0,\omega)\right| \lesssim \int_{\omega_1} &\left|\partial_{\omega_1}\rho_0(|\partial_r^\alpha f|)\big(t,q+q_0,\omega+(\omega_1,0,\dots,0)\big)\right| d\omega_1 \\ &+ \int_{\omega_1}\left|\rho_0(|\partial_r^\alpha f|)\big(t,q+q_0,\omega+(\omega_1,0,\dots,0)\big)\right| d\omega_1.\end{aligned}$$

Since ∂_{ω_1} can be rewritten in terms of the rotation vector fields, it follows from Lemma 2.13 that

$$\left|\partial_{\omega_1}\rho_0(|\partial_r^\alpha f|)\right| \lesssim \sum_{|\beta|\le 1} \rho_0\big(|\widehat{Z}^\beta \partial_r^\alpha f|\big).$$

Repeating until exhaustion of the number of variables on $\mathbb{S}^{n-1}$ and using that $\partial_r = (x^k/|x|)\partial_{x^k}$ and the commutation properties between $\widehat{Z}^\alpha$ and ∂_r, we obtain that

$$\left|\rho_0(|f|)(t,q+q_0,\omega)\right| \lesssim \int_{|q|\le\frac{1}{4}t}\int_{\eta\in\mathbb{S}^{n-1}} \sum_{|\alpha|\le n} \rho_0\big(|\widehat{Z}^\alpha(f)|\big)(t,q+q_0,\eta)\, dq\, d\sigma_{\mathbb{S}^{n-1}}.$$

Now since in the domain of integration $r = t+q+q_0 = q+|x| \sim t$, we have

$$\begin{aligned} t^{n-1}\left|\rho_0(|f|)(t,q+q_0,\omega)\right| &\lesssim \sum_{|\alpha|\le n}\int_{|q|\le\frac{1}{4}t}\int_{\eta\in\mathbb{S}^{n-1}} \rho_0\big(|\widehat{Z}^\alpha(f)|\big)(t,q+q_0,\eta) r^{n-1}\, dq\, d\sigma_{\mathbb{S}^{n-1}} \\ &\lesssim \sum_{|\alpha|\le n}\int_{\frac{1}{4}t\le r\le\frac{7}{4}t}\int_{\eta\in\mathbb{S}^{n-1}} \rho_0\big(|\widehat{Z}^\alpha(f)|\big)(t,r,\eta) r^{n-1}\, dr\, d\sigma_{\mathbb{S}^{n-1}} \\ &\lesssim \sum_{|\alpha|\le n}\int_{\frac{1}{4}t\le|y|\le\frac{7}{4}t} \rho_0\big(|\widehat{Z}^\alpha(f)|\big)(t,y)\, dy, \end{aligned}$$

which concludes the proof when $|q_0|\le 1$.

Assume now that $|q_0|>1$. Let $\chi \in C_0^\infty\big(-\frac{1}{2},\frac{1}{2}\big)$ be a smooth cut-off function such that $\chi(0)=1$ and define $V_{q_0}(t,q,\omega) \equiv \chi((q-q_0)/q_0)v(t,q,\omega)$. To get the extra factor of $|q_0|$, we apply the method used above replacing the function v by the function $(s,\eta) \to V_{q_0}(t, q_0 + q_0 s, \eta)$ and applying first a 1-dimensional Sobolev inequality in s on $|s| < \frac{1}{2}$. The extra powers of q_0 appearing are then absorbed since $|q_0 + q_0 s| \sim |q_0|$ in the region of integration and since $(r-t)\partial_r$ can be expressed as a linear combination of commutation vector fields from Lemma 2.9 (with coefficients homogeneous of degree 0). The rest of the proof is similar to the one just given when $|q_0| \le 1$ and therefore omitted. □

Since the norm on the right-hand side is conserved for solutions of the homogeneous massless transport equations, we obtain in particular:

Theorem 7 (decay estimates for velocity averages of massless distribution functions). *Let f be a regular distribution function for the massless case, a solution to $\boldsymbol{T}_0(f) = 0$ on $\mathbb{R}_t \times \mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$. Then, for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$,*

$$\rho_0(|f|)(t,x) \lesssim \frac{1}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}} \|f\|_{\mathbb{K},n}(0). \tag{40}$$

Finally, as for the wave equation, we have improved decay for derivatives of the solutions. More precisely, let $\partial = \partial_t$, let ∂_{x^i} be any translation, and let $\bar{\partial}$ be a derivative tangential to the cone $t = |x|$, such as $\partial_t + \partial_r$ or the projection on the angular derivatives of ∂_{x^i}, $\bar{\partial}_{x^i} = \partial_{x^i} - (x^i/r)\partial_r$. Then, we have the following proposition.

Proposition 3.2 (improved decay for derivatives of velocity averages of massless distribution functions). *Let f be a regular distribution function for the massless-case solution to $\boldsymbol{T}_0(f) = 0$ on $\mathbb{R}_t \times \mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$. Then, for all multi-indices l, k and for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$,*

$$|\partial^l \bar{\partial}^k \rho_0(f)(t,x)| \lesssim \frac{1}{\big(1+\big|t-|x|\big|\big)^{1+|l|}\big(1+\big|t+|x|\big|\big)^{n-1+|k|}} \|f\|_{\mathbb{K},n+k+l}(0). \tag{41}$$

Proof. This proof is similar to that of the improved decay estimates for the wave equation, and therefore omitted. □

Remark 3.3. Note that the improved decay estimates (41) only apply to velocity averages of f, because of the lack of regularity of velocity averages of $|f|$.

Finally, let us mention that we can obtain decay for other moments of the solutions, provided the corresponding moments for f and the $\widehat{Z}^\alpha(f)$ are finite initially. For instance, in Theorem 1 on page 1542, the decay estimate was written for the density of particles, while in Theorem 7, we considered the energy density. One can move freely from one to the other by considering $f|v|^q$ instead of f (provided the initial data can support it of course).

3C. *Klainerman–Sobolev inequalities and decay estimates: massive case.*

In the massive case $m > 0$, we will prove:

Theorem 8 (Klainerman–Sobolev inequalities for velocity averages of massive distribution functions). *Let f be a regular distribution function for the massive case defined on $\bigcup_{1 \le \rho < P} H_\rho \times \mathbb{R}^n_v$ for some*

$P \in [1, +\infty]$. *Then, for all* $(t,x) \in \bigcup_{1 \le \rho < P} H_\rho$,

$$\int_{v \in \mathbb{R}^n} |f|(t,x,v) \frac{dv}{v^0} \lesssim \frac{1}{(1+t)^n} \|f\|_{\mathbb{P},n}(\rho(t,x)), \tag{42}$$

where $\rho(t,x) = (t^2 - |x|^2)^{\frac{1}{2}}$ *and the norm* $\|f\|_{\mathbb{P},n}$ *is defined as in Section 3A.*

Proof. Recall from Remark 2.12 that

$$\chi_m(|f|)(t,x) \ge m^2 \frac{t}{2\rho} \int_v f \, d\mu_m = m^2 \frac{t}{2\rho} \int_{v \in \mathbb{R}^n} |f|(t,x,v) \frac{dv}{v^0}, \tag{43}$$

and thus,

$$\int_{H_\rho} \chi_m(|f|)(t,x) d\mu_{H_\rho} \ge \int_{H_\rho} m^2 \frac{t}{2\rho} \int_{v \in \mathbb{R}^n} |f|(t,x,v) \frac{dv}{v^0} \, d\mu_{H_\rho}. \tag{44}$$

Let (t,x) be fixed in $\bigcup_{1 \le \rho \le P} H_\rho$ and define the function ψ in the (y^α)-system of coordinates (see the end of Section 2C) as

$$\psi(y^0, y^j) \equiv \int_{v \in \mathbb{R}^n} |f|(y^0, x^j + t y^j) \, d\mu_m.$$

Similarly to the proof of the massless case, we apply first a 1-dimensional Sobolev inequality in the variable y^1

$$\int_{v \in \mathbb{R}^n} |f|(y^0, x^j) \, d\mu_m = |\psi(y^0, 0)| \lesssim \int_{|y^1| \le 1/(8n)^{\frac{1}{2}}} \big[|\partial_{y^1} \psi|(y^0, y^1, 0, \dots, 0) + |\psi|(y^0, y^1, 0, \dots, 0) \big] \, dy^1.$$

Now

$$\partial_{y^1} \psi = \frac{t}{t(y^0, x^1 + t y^1, x^2, \dots, x^n)} \Omega_{01},$$

where the t in the numerator is that of the point (t,x), while $t(y^0, x^j + t y^j) \equiv \big((y^0)^2 + (x^j + t y^j)^2 \big)^{\frac{1}{2}}$ is the time of the point defined in the y^α-coordinates by $(y^0, x^j + t y^j)$. Now if $|x| \le \frac{1}{2} t$, then it follows from the condition $|y^1| \le 1/(8n^{\frac{1}{2}}) \le \frac{1}{8}$ that $(y^0)^2 \ge \frac{3}{4} t^2$ and thus that

$$\left| \frac{t}{t(y^0, x^1 + t y^1, x^2, \dots, x^n)} \right| \le C$$

for some uniform $C > 0$. On the other hand if $|x| \ge \frac{1}{2} t$, then it follows from the condition $|y^1| \le 1/(8n^{\frac{1}{2}}) \le \frac{1}{8}$ that $|x^j + t y^j| \ge \frac{3}{8} t$, where $y^j = (y^1, 0, \dots, 0)$. Thus, we have, for $|y^1| \le 1/(8n^{\frac{1}{2}})$,

$$|\partial_{y^1} \psi|(y^0, y^1, 0, \dots, 0) \lesssim \left| \int_v \Omega_{01} |f|(y^0, x^1 + t y^1, x^2, \dots, x^n, v) \, d\mu_m \right|.$$

The remainder of the proof is then similar to the massless case. We have

$$\left| \int_v \Omega_{01}(|f|)(y^0, x^1 + t y^1, x^2, \dots, x^n, v) \, d\mu_m \right| \lesssim \sum_{|\alpha| \le 1} \int_v |\hat{Z}^\alpha f|(y^0, x^1 + t y^1, x^2, \dots, x^n, v) \, d\mu_m.$$

Inserting in the Sobolev inequality and repeating up to exhaustion of all the variables $\big($the fact that, for all j, $|y^j| \leq 1/(8n^{\frac{1}{2}})$ guarantees that $|y| = \big(\sum_{j=1}^n |y^j|^2\big)^{\frac{1}{2}} \leq \frac{1}{8}$ so that we still have $t/(t(y^0, x^j + ty^j)) \sim 1\big)$, we obtain

$$\int_v |f|(y^0, x^1, x^2, \cdots, x^n, v)\, d\mu_m \lesssim \sum_{|\alpha| \leq n} \int_{|y| \leq \frac{1}{8}} \int_v |\hat{Z}^\alpha f|(y^0, x^j + ty^j, v)\, d\mu_m\, dy.$$

Recall that the volume form on each of the H_ρ is given in spherical coordinates by $(\rho/t) r^{n-1}\, dr\, d\sigma$, or in y^α-coordinates by $(y^0/t)\, dy$. Thus, we have

$$\begin{aligned} \int_v |f|(y^0, x^1, x^2, \ldots, x^n, v)\, d\mu_m &\lesssim \sum_{|\alpha| \leq n} \int_{|y| \leq \frac{1}{8}} \int_v |\hat{Z}^\alpha f|(y^0, x^j + ty^j, v)\, d\mu_m \frac{t(y^0, x^j + ty^j)}{y^0}\, d\mu_{H_\rho} \\ &\lesssim \frac{t(y^0, x^j)}{y^0} \sum_{|\alpha| \leq n} \int_{|y| \leq \frac{1}{8}} \int_v |\hat{Z}^\alpha f|(y^0, x^j + ty^j, v)\, d\mu_m\, d\mu_{H_\rho}, \end{aligned}$$

where we have used again that $t(y^0, x^j + ty^j) \sim t(y^0, x^j)$ in the region of integration. Applying the change of coordinates $z^j = ty^j$ and noticing that the quantities on the right-hand side are controlled by the estimate (44) applied to $\hat{Z}^\alpha(f)$ completes the proof. ☐

Since the norm on the right-hand side of (42) is conserved if f is a solution to the massive transport equation, we obtain, as a corollary, the following pointwise decay estimate.

Theorem 9 (pointwise decay estimates for velocity averages of massive distribution functions). *Let f be a regular distribution function for the massive case satisfying the massive transport equation $\boldsymbol{T_m}(f) = 0$ on $\bigcup_{1 \leq \rho < +\infty} H_\rho \times \mathbb{R}^n_v$. Then, for all $(t, x) \in \bigcup_{1 \leq \rho < +\infty} H_\rho$,*

$$\int_{v \in \mathbb{R}^n} |f|(t, x, v) \frac{dv}{v^0} \lesssim \frac{1}{(1+t)^n} \|f\|_{\mathbb{P}, n}.$$

Finally, let us mention the following improved decay for derivatives.

Proposition 3.4 (improved decay estimates for derivatives of velocity averages of massive distribution functions). *Let f be a regular distribution function for the massive case satisfying the massive transport equation $\boldsymbol{T_m}(f) = 0$ on $\bigcup_{1 \leq \rho < +\infty} H_\rho \times \mathbb{R}^n_v$. Then, for all $i \in \mathbb{N}$, for all multi-indices l and for all $(t, x) \in \bigcup_{1 \leq \rho < +\infty} H_\rho$,*

$$\left| \nu_\rho^i \partial_y^l \int_{v \in \mathbb{R}^n} f(t, x, v) \frac{dv}{v^0} \right| \lesssim \frac{1}{(1+t)^{n+|l|} \rho^i} \|f\|_{\mathbb{P}, n+i+l},$$

where $\nu_\rho = x^\alpha \partial_{x^\alpha}/\rho$ is the future unit normal to H_ρ and ∂_y^l is a combination of $|l|$ vector fields among the ∂_{y^k}, $1 \leq k \leq n$, which are tangent to the H_ρ.

Proof. We have $\nu_\rho = S/\rho$ with S the scaling vector field. On the other hand, recall that S essentially commutes with the massive transport operator, so that in particular $\boldsymbol{T_m}(S(f)) = 0$ if $\boldsymbol{T_m}(f) = 0$. Thus, $\int_v S(f)/v^0\, dv = S\big(\int_v f/v^0\, dv\big)$ satisfies the same decay estimates as $\int_v f/v^0\, dv$, which shows the improved decay for $\nu_\rho(\int_v f/v^0\, dv)$. The higher-order derivatives follow similarly. Indeed, using that

$S(\rho)=\rho$, we have for instance $S^2(f)=\rho^2 v_\rho^2(f)+S(f)$. Applying the decay estimates for the velocity averages of $S^2(f)$ and $S(f)$ gives the correct improved decay for velocity averages of $v_\rho^2(f)$. Higher normal derivatives can be treated similarly. Finally, the improved decay for tangential derivatives of velocity averages is an easy consequence of the fact that $\partial_{y^k}=(1/t)\Omega_{0k}$. □

4. Applications to the Vlasov–Nordström system

4A. ***Generalities on the Vlasov–Nordström system.*** In 1913, Nordström introduced a gravitation theory based on the replacement of the Poisson equation by a scalar wave equation. The Vlasov–Nordström system describes the coupling of this gravitational theory with collisionless matter.[20]

It can be roughly obtained from the Einstein–Vlasov equations within the class of metrics conformal to the Minkowski metric by neglecting some of the nonlinear self-interactions of the conformal factor. In dimension $n=3$, global existence for sufficiently regular massive distribution functions, with compact support in (x,v), has been proven in [Calogero 2006].

Following [Calogero 2003], it is possible to make a derivation of this system for arbitrary mass, as well as arbitrary dimension. Consider the metric

$$g=e^{2\phi}\eta$$

conformal to the Minkowski metric η, where ϕ is a function on $\mathbb{R}^{n+1}$. For this system, the mass shell is defined by the equation

$$e^{2\phi}\eta_{\alpha\beta}v^\alpha v^\beta=-m^2, \quad \text{which provides } v^0=\sqrt{m^2e^{-2\phi}+\eta_{ij}v^iv^j}.$$

We can introduce the coordinates

$$\hat{v}^i=e^\phi v^i,$$

which consistently also provides

$$\hat{v}^0=\sqrt{m^2+\eta_{ij}v^iv^j}=e^\phi v^0.$$

Considering distributions of particles which are conserved along the geodesic flow of g, we can define the associated transport operator as

$$\boldsymbol{T}_g\equiv v^\alpha\left(\frac{\partial}{\partial x^\alpha}-v^\beta\Gamma^i_{\alpha\beta}\frac{\partial}{\partial v^i}\right),$$

where $\Gamma^i_{\alpha\beta}$ are the Christoffel symbols of the metric g, which are given by

$$\Gamma^i_{\alpha\beta}=\delta^i_\alpha\frac{\partial\phi}{\partial x^\beta}+\delta^i_\beta\frac{\partial\phi}{\partial x^\alpha}-\eta_{\alpha\beta}\frac{\partial\phi}{\partial x^i},$$

so that

$$\boldsymbol{T}_g=v^\alpha\frac{\partial}{\partial x^\alpha}-(2v^\alpha\nabla_\alpha\phi v^i+m^2e^{-2\phi}\nabla^i\phi)\frac{\partial}{\partial v^i}.$$

In the $(t,x,\hat{v})$-system of coordinates, we compute

$$\boldsymbol{T}_g=e^{-\phi}\left(\hat{v}^\alpha\frac{\partial}{\partial x^\alpha}-(\hat{v}^\alpha\nabla_\alpha\phi\hat{v}^i+m^2\nabla^i\phi)\frac{\partial}{\partial\hat{v}^i}\right).$$

[20] See [Calogero 2003] for an introduction to the system.

To couple the Vlasov field and the scalar function ϕ, we follow [Calogero 2003] and require that[21]

$$\Box\phi = m^2 e^{(n+1)\phi} \int_v f \frac{dv}{v^0}. \tag{45}$$

Depending on the value of the mass m, we are thus faced with the following two systems:

- The massless Vlasov–Nordström system.

$$\Box\phi = 0, \tag{46}$$

$$\hat{v}^\alpha \frac{\partial f}{\partial x^\alpha} - \hat{v}^\alpha \nabla_\alpha \phi \hat{v}^i \frac{\partial f}{\partial \hat{v}^i} = 0. \tag{47}$$

  In this case, the equations decouple. We can of course solve the first equation and then think of the second equation as a linear transport equation for f.

- The massive Vlasov–Nordström system. In this case, we can perform yet another change of unknowns by considering

$$\tilde{f}(t, x, \hat{v}) \equiv e^{(n+1)\phi} f(t, x, \hat{v}),$$

  which has the advantage of removing the ϕ-dependence in the right-hand side of equation (45).

  We then obtain the usual expression of the (massive) Vlasov–Nordström system

$$\Box\phi = m^2 \int_{\hat{v}} \tilde{f} \frac{d\hat{v}}{\hat{v}^0}, \tag{48}$$

$$\hat{v}^\alpha \frac{\partial \tilde{f}}{\partial x^\alpha} - (\hat{v}^\alpha \nabla_\alpha \phi \hat{v}^i + m^2 \nabla^i \phi) \frac{\partial \tilde{f}}{\partial \hat{v}^i} = (n+1) \tilde{f} \hat{v}^\alpha \frac{\partial \phi}{\partial x^\alpha}. \tag{49}$$

  From now on, we will drop the ˆ and ˜ on all the variables to ease the notations.

4B. ***The massless Vlasov–Nordström system.*** We consider in this section, the system (46)–(47). We will denote by $\boldsymbol{T}_\phi$ the transport operator defined by

$$\boldsymbol{T}_\phi \equiv v^\alpha \frac{\partial}{\partial x^\alpha} - v^\alpha \nabla_\alpha \phi \cdot v^i \frac{\partial}{\partial v^i};$$

i.e.,

$$\boldsymbol{T}_\phi = \boldsymbol{T}_0 - \boldsymbol{T}_0(\phi) \cdot v^i \partial_{v^i}.$$

The massless Vlasov–Nordström system can then be rewritten as

$$\Box\phi = 0, \tag{50}$$

$$\boldsymbol{T}_\phi(f) = 0, \tag{51}$$

which we complement by the initial conditions

$$\phi(t=0) = \phi_0, \quad \partial_t \phi(t=0) = \phi_1, \tag{52}$$

$$f(t=0) = f_0, \tag{53}$$

[21] It should not be surprising that the right-hand side of this equation vanishes for massless distribution functions, as, up to an overall factor, it corresponds to the trace of the energy-momentum tensor, the latter being of course proportional to m for Vlasov fields.

where (ϕ_0, ϕ_1) are sufficiently regular functions defined on $\mathbb{R}^n_x$ and f_0 is a sufficiently regular function defined on $\mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$.

By sufficiently regular, we mean that all the computations below make sense. We will eventually require that $\mathcal{E}_N[\phi_0, \phi_1] < +\infty$, where $\mathcal{E}_N$ is the energy norm defined by (57) and similarly, we will also require below that $\| f_0\|_{\mathbb{K},N} < +\infty$ (with some additional weights in the case of dimension 3). Provided N is large enough (depending only on n), these two regularity requirements are then enough so that all the computations below are justified. In the remaining, we will therefore omit any further mention of regularity issues.

4B1. *Commutation formula for $\boldsymbol{T}_\phi$.* Recall the algebra of commutation fields $\widehat{\mathbb{K}}_0 = \widehat{\mathbb{K}} \cup \{S\}$, where S is the usual scaling vector field, defined in (17). Similar to Lemma 2.3, we have:

Lemma 4.1. *For any $\widehat{Z} \in \widehat{\mathbb{K}}_0$,*

$$[\boldsymbol{T}_\phi, \widehat{Z}] = c_Z \boldsymbol{T}_0 + \big[-c_Z \boldsymbol{T}_0(\phi) + \boldsymbol{T}_0(Z(\phi))\big] v^i \partial_{v^i} = c_Z \boldsymbol{T}_\phi + \boldsymbol{T}_0(Z(\phi)) v^i \partial_{v^i},$$

where $c_Z = 0$ if $\widehat{Z} \in \widehat{\mathbb{K}}$ and $c_Z = 1$ if $\widehat{Z} = S$, and where Z is the nonlifted field corresponding to $\widehat{Z}$ if $\widehat{Z} \in \widehat{\mathbb{K}}$ and $Z = S$ if $\widehat{Z} = S$.

Proof. Note first that for all $\widehat{Z} \in \widehat{\mathbb{K}}_0$, we have $[\widehat{Z}, v^i \partial_{v^i}] = 0$. We then compute

$$\begin{aligned}
[\boldsymbol{T}_\phi, \widehat{Z}] &= [\boldsymbol{T}_0, \widehat{Z}] - [\boldsymbol{T}_0(\phi) v^i \partial_{v^i}, \widehat{Z}] \\
&= [\boldsymbol{T}_0, \widehat{Z}] + \widehat{Z}(\boldsymbol{T}_0(\phi)) v^i \partial_{v^i} + \boldsymbol{T}_0(\phi)[\widehat{Z}, v^i \partial_{v^i}] \\
&= [\boldsymbol{T}_0, \widehat{Z}] + \big([\widehat{Z}, \boldsymbol{T}_0]\phi + \boldsymbol{T}_0(\widehat{Z}\phi)\big) v^i \partial_{v^i}.
\end{aligned}$$

To conclude the proof, replace all the instances of $[\boldsymbol{T}_0, \widehat{Z}]$ by $c_Z \boldsymbol{T}_0$ according to Lemma 2.3. □

Iterating the above, one obtains:

Lemma 4.2. *Let f be a solution to (51). For any multi-index α, we have the commutator estimate*

$$\big|[\boldsymbol{T}_\phi, \widehat{Z}^\alpha] f\big| \le C \sum_{\substack{|\beta| \le |\alpha|, |\gamma| \le |\alpha| \\ |\beta| + |\gamma| \le |\alpha| + 1}} |\boldsymbol{T}_0(Z^\gamma \phi)| \cdot |\widehat{Z}^\beta f|, \tag{54}$$

where $Z^\gamma \in \mathbb{K}_0^{|\gamma|}$ and $\widehat{Z}^\beta \in \widehat{\mathbb{K}}_0^{|\beta|}$ and $C > 0$ is some constant depending only on $|\alpha|$.

4B2. *Approximate conservation law.* Similar to Lemma 2.17, we have:

Lemma 4.3. *Let h be a regular distribution function for the massless case in the sense of Section 2E. Let g be a regular solution to $\boldsymbol{T}_\phi(g) = v^0 h$, with $v^0 = |v|$, defined on $[0, T] \times \mathbb{R}^n_x \times (\mathbb{R}^n_v \setminus \{0\})$ for some $T > 0$. Then, for all $t \in [0, T]$,*

$$\int_{\Sigma_t} \rho_0(|g|)(t, x)\, dx \le \int_{\Sigma_0} \rho_0(|g|)(0, x)\, dx + \int_0^t \int_{\Sigma_s} \rho_0(|h|)(s, x)\, dx\, ds + (n+1) \int_0^t \int_{\Sigma_s} \int_{v \in \mathbb{R}^n_v \setminus \{0\}} |\boldsymbol{T}_0(\phi) f|\, dv\, dx\, ds. \tag{55}$$

Proof. As for the proof of Lemma 2.17, this follows, after regularization of the absolute value, from integration by parts or an application of Stokes' theorem. The term $\boldsymbol{T}_0(\phi)v^i\,\partial_{v^i}|f|$, which appears in the computation, gives rise, after integration by parts in v, to the last term in (55) since $\partial_{v^i}(v^i\boldsymbol{T}_0(\phi)) = (n+1)\boldsymbol{T}_0(\phi)$. □

4B3. *Massless case in dimension $n \geq 4$.* In this section, we prove Theorem 3, found on page 1545. The $n = 3$ case requires slightly more refined techniques (which of course would work also for $n \geq 4$), but the estimates in the $n \geq 4$ case are slightly stronger and simpler; we therefore provide an independent proof.

If ϕ is a solution to the wave equation, let us consider the energy at time $t = 0$,

$$\mathcal{E}_N[\phi](t=0) \equiv \sum_{\substack{|\alpha|\leq N\\ |\alpha|\in\mathbb{K}^{|\alpha|}}} \left\| Z^\alpha(\partial\phi)(t=0)\right\|^2_{L^2(\mathbb{R}^n_x)}. \tag{56}$$

Now if $\phi(t=0) = \phi_0$ and $\partial_t\phi(t=0) = \phi_1$, for pairs of sufficiently regular functions (ϕ_0, ϕ_1) defined on $\mathbb{R}^n_x$, then the above quantity can be computed purely in terms of ϕ_0, ϕ_1, so we define[22]

$$\mathcal{E}_N[\phi_0,\phi_1] \equiv \mathcal{E}_N[\phi](t=0). \tag{57}$$

Similarly, if f is a solution to (51) arising from initial data f_0 at $t = 0$, then we define

$$E_N[f](t=0) \equiv \|f\|_{\mathbb{K},N}(t=0)\Bigg(= \sum_{\substack{|\alpha|\leq N\\ \widehat{Z}^\alpha\in\widehat{\mathbb{K}}^\alpha}} \left\|\rho_0\big(\widehat{Z}^\alpha(f)(t=0)\big)\right\|_{L^1(\mathbb{R}^n_x)}\Bigg), \tag{58}$$

and we remark that this quantity can be computed purely in terms of f_0, so we will set

$$E_N[f_0] \equiv E_N[f](t=0).$$

We will prove:

Theorem 10. *Let $n \geq 4$ and let $N \geq \frac{3}{2}n + 1$. Let (ϕ_0, ϕ_1, f_0) be an initial data set for the massless Vlasov–Nordström system such that $\mathcal{E}_N[\phi_0,\phi_1] + E_N[f_0] < +\infty$. Then, the unique solution (f, ϕ) to (50)–(51) satisfying the initial conditions (52)–(53) also satisfies the following estimates*:

(1) *Global bounds. For all $t \geq 0$,*

$$E_N[f](t) \leq e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]}E_N[f_0],$$

where $C > 0$ is a constant depending only on N, n.

(2) *Pointwise estimates for velocity averages. For all $(t,x) \in [0,+\infty)\times\mathbb{R}^n_x$ and all multi-indices α satisfying $|\alpha| \leq N - n$,*

$$\rho_0(|\widehat{Z}^\alpha f|)(t,x) \lesssim \frac{e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]}E_N[f_0]}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}.$$

[22] The alternative to the approach we use here is to assume that (ϕ_0, ϕ_1) are regular initial data with decay fast enough in x, for instance by assuming compact support, so that the resulting $\mathcal{E}_N[\phi(t=0)]$ is finite. What we want to emphasize here is that the quantity $\mathcal{E}_N[\phi(t=0)]$ can in fact be computed purely in terms of the initial data (using the equation to rewrite second and higher time derivatives of ϕ in terms of spatial derivatives), and that this is all that is needed in terms of decay in x.

Proof. Let N, n, ϕ_0, ϕ_1, f_0 be as in the statement of the theorem. From the conservation of energy and the commutation properties of the Z^α with the wave operator, we have, for all t,

$$\mathcal{E}_N[\phi](t) = \mathcal{E}_N[\phi_0, \phi_1] \equiv \mathcal{E}_N.$$

Applying the standard decay estimates obtained via the vector field method to ϕ, we have for all multi-indices α satisfying $|\alpha| \le N - \frac{1}{2}(n+2)$ and for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$,

$$|\partial Z^\alpha \phi(t,x)|^2 \lesssim \frac{\mathcal{E}_N[\phi](t)}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}. \tag{59}$$

It follows from a standard existence theory for regular data that for all t, we have $E_N[f(t)] < +\infty$.

Applying the Klainerman–Sobolev inequality (39), we obtain, for all multi-indices α satisfying $|\alpha| \le N - n$ and for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$,

$$\big|\rho_0(\widehat{Z}^\alpha(f))(t,x)\big| \lesssim \frac{E_N[f](t)}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}.$$

From Lemma 4.3 and the commutator estimate (54), we have for all $t \ge 0$ and all multi-indices α,

$$\int_{\Sigma_t} \rho_0(|\widehat{Z}^\alpha f|)(t,x)\,dx \le \int_{\Sigma_0} \rho_0(|\widehat{Z}^\alpha f|)(0,x)\,dx + \int_0^t \int_{\Sigma_s} \rho_0(|h^\alpha|)(s,x)\,dx\,ds, \tag{60}$$

where[23]

$$|h^\alpha| \lesssim \frac{1}{v^0} \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|\\ |\beta|+|\gamma|\le|\alpha|+1}} |\boldsymbol{T}_0(Z^\gamma\phi)|\cdot|\widehat{Z}^\beta f| \lesssim \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|\\ |\beta|+|\gamma|\le|\alpha|+1}} |\partial(Z^\gamma\phi)|\cdot|\widehat{Z}^\beta f|,$$

so that

$$\rho_0(|h^\alpha|) \lesssim \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|\\ |\beta|+|\gamma|\le|\alpha|+1}} |\partial(Z^\gamma\phi)|\rho_0(|\widehat{Z}^\beta f|),$$

since ϕ is independent of v. Integrating over x, we obtain, for all $s \in [0,t]$,

$$\int_{\Sigma_s} \rho_0(|h^\alpha|)(s,x)\,dx \lesssim \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|\\ |\beta|+|\gamma|\le|\alpha|+1}} \int_{\Sigma_s} |\partial(Z^\gamma\phi)|\rho_0(|\widehat{Z}^\beta f|)(s,x)\,dx.$$

We now estimate each term in the above sum depending on the values of $|\gamma|$ and $|\beta|$.

If $|\beta| \le N-n$, we then apply the pointwise estimates on $\rho_0(\widehat{Z}^\beta(f))$ to obtain

$$\int_{\Sigma_s} |\partial(Z^\gamma\phi)|\rho_0(|\widehat{Z}^\beta f|)(s,x)\,dx \lesssim \int_{\Sigma_s} |\partial(Z^\gamma\phi)| \frac{E_N[f](s)}{\big(1+\big|s-|x|\big|\big)\big(1+\big|s+|x|\big|\big)^{n-1}}\,dx.$$

[23] Note that the last term in the right-hand side of Lemma 4.3 is similar to the error terms arising from the commutator estimate of Lemma 4.2 and is therefore accounted for in the h^α error term in equation (60).

Applying the Cauchy–Schwarz inequality and using that[24]

$$\left\| \frac{1}{\big(1+\big|s-|x|\big|\big)\big(1+\big|s+|x|\big|\big)^{n-1}} \right\|_{L^2_x} \lesssim \frac{1}{(1+s)^{\frac{n-1}{2}}}, \tag{61}$$

we obtain

$$\int_{\Sigma_s} |\partial(Z^\gamma \phi)|\rho_0(|\widehat{Z}^\beta f|)(s,x)\,dx \lesssim \mathcal{E}_N^{\frac{1}{2}}[\phi](s)\frac{E_N[f](s)}{(1+s)^{\frac{n-1}{2}}}. \tag{62}$$

If now $|\beta| > N-n$, then $|\gamma| \le |\alpha|+1-|\beta| \le N-\frac{1}{2}(n+2)$ and thus, we also have (62), using this time the pointwise estimates on $\partial(Z^\gamma\phi)$ given by (59). Applying Grönwall's inequality finishes the proof of the theorem. $\square$

4B4. *Massless case in dimension $n=3$.* We now turn to the case of dimension 3, where the slower pointwise decay of solutions to the wave equations leads to a slightly harder analysis. First, let us strengthen our norms for the Vlasov field.

For this, recall the algebra of weights $\Bbbk_0$ introduced in Section 2H and define a rescaled version κ_0 by

$$\kappa_0 \equiv (v^0)^{-1}\Bbbk_0 = \left\{ \frac{\mathfrak{z}}{v^0} \;\middle|\; \mathfrak{z} \in \Bbbk_0 \right\},$$

where we recall that $v^0 = |v|$ in the massless case. If α is a multi-index, we will write $[\mathfrak{z}/v^0]^\alpha \in \kappa_0^{|\alpha|}$ to denote a product $|\alpha|$ elements of κ_0 and $[|\mathfrak{z}|/v^0]^\alpha$ in the case we take the product of the absolute values of these elements.

Let us now define, for any regular distribution function f, the weighted norm

$$\begin{aligned} E_{N,q}[f] &\equiv \sum_{\substack{|\alpha|\le N\\ |\beta|\le q}} \sum_{\widehat{Z}^\alpha \in \widehat{\mathbb{K}}^{|\alpha|}} \int_{\Sigma_t} \rho_0\left(|\widehat{Z}^\alpha f| \left[\frac{|\mathfrak{z}|}{v^0}\right]^\beta\right)(x)\,dx \\ &\left(= \sum_{\substack{|\alpha|\le N\\ |\beta|\le q}} \sum_{\widehat{Z}^\alpha \in \widehat{\mathbb{K}}^{|\alpha|}} \int_{\Sigma_t} \int_{v\in\mathbb{R}^n\setminus\{0\}} \left(|\widehat{Z}^\alpha f|(x,v) \left[\frac{|\mathfrak{z}|}{v^0}\right]^\beta\right) v^0\,dv\,dx\right), \end{aligned} \tag{63}$$

where the weights $\mathfrak{z}/v^0$ lie in κ_0.

Theorem 11 (asymptotic behaviour in dimension $n=3$). *Consider the dimension $n=3$. Let $N\ge 7$ and $q\ge 1$. Let (ϕ_0,ϕ_1,f_0) be an initial data set for the massless Vlasov–Nordström system such that $\mathcal{E}_N[\phi_0,\phi_1]+E_N[f_0]_{N,q} < +\infty$. Then, the unique solution (f,ϕ) to (50)–(51) satisfying the initial conditions (52)–(53) also satisfies the following estimates*:

(1) *Global bounds with growth for the top-order norms. For all $t\in\mathbb{R}_t$,*

$$E_{N,q}[f](t) \le (1+t)^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_{N,q}[f_0], \tag{64}$$

where $C>0$ is a constant depending only on N, n and q.

[24] For the convenience of the reader, we have added in Appendix B certain integral estimates which include (61).

(2) *Small data improvement for the low-order norms. There exists an* ε_0 *(depending only on* n, N, q*) such that if* $\mathcal{E}_N[\phi_0,\phi_1] \le \varepsilon_0$, *then for all* $t \in \mathbb{R}_t$,

$$E_{N-\frac{n+4}{2},q-1}[f](t) \le e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]} E_{N,q}[f_0]. \tag{65}$$

(3) *Under the above smallness assumption, we also have the optimal pointwise estimates for velocity averages. For all* $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$ *and all multi-indices* α *satisfying* $|\alpha| \le N - \frac{1}{2}(3n+4)$ *and all* $|\beta| \le q-1$,

$$\rho_0\left(\left|\widehat{Z}^\alpha(f)\left[\frac{\mathfrak{z}}{v^0}\right]^\beta\right|\right)(t,x) \lesssim \frac{e^{C\mathcal{E}_N^{1/2}[\phi_0,\phi_1]}E_{N,q}[f_0]}{\big(1+\big|t-|x|\big|\big)\big(1+\big|t+|x|\big|\big)^{n-1}}.$$

Proof. First, let us note that for all $\mathfrak{z} \in \Bbbk_0$, we have

$$v^i\partial_{v^i}\left(\frac{\mathfrak{z}}{v^0}f\right) = \frac{\mathfrak{z}}{v^0}\,v^i\partial_{v^i}f,$$

from which it follows that for all regular distribution functions g, we have $[\boldsymbol{T}_\phi, \mathfrak{z}/v^0]g = 0$. Thus, we can upgrade Lemma 4.2 to

$$\left|\left[\boldsymbol{T}_\phi, \left[\frac{\mathfrak{z}}{v^0}\right]^\sigma \widehat{Z}^\alpha\right]f\right| \le C \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|\\|\beta|+|\gamma|\le|\alpha|+1}} |\boldsymbol{T}_0(Z^\gamma\phi)|\cdot\left[\frac{|\mathfrak{z}|}{v^0}\right]^\sigma|\widehat{Z}^\beta f|, \tag{66}$$

where $Z^\gamma \in \mathbb{K}_0^{|\gamma|}$, $\widehat{Z}^\beta \in \widehat{\mathbb{K}}_0^{|\beta|}$, $[\mathfrak{z}/v^0]^\sigma \in \kappa_0^{|\sigma|}$ and $C>0$ is some constant depending only on $|\alpha|$. Applying arguments similar to those used in the $n \ge 4$ case yields

$$\begin{aligned} E_{N,q}[f](t) &\le E_{N,q}[f_0] + C\int_0^t \sum_{\substack{|\beta|\le|\alpha|,|\gamma|\le|\alpha|\\|\beta|+|\gamma|\le|\alpha|+1}}\sum_{|\sigma|\le q}\int_{\Sigma_s}|\partial(Z^\gamma\phi)|\rho_0\left(\left[\frac{|\mathfrak{z}|}{v^0}\right]^\sigma|\widehat{Z}^\beta f|\right)(s,x)\,dx\,ds \\ &\le E_{N,q}[f_0] + C\int_0^t \mathcal{E}_N^{\frac{1}{2}}\frac{E_{N,q}[f](s)}{(1+s)}\,ds. \end{aligned} \tag{67}$$

Applying Grönwall's inequality, we obtain (64).

Now assume that $\mathcal{E}_N \le \varepsilon_0$ with ε_0 small enough that

$$E_{N,q}[f](t) \le (1+t)^\delta E_{N,q}[f_0],$$

with $\delta = C\mathcal{E}_N^{\frac{1}{2}} < \frac{1}{2}$.

The key to the improved estimates is the following decomposition of the transport operator $\boldsymbol{T}_0$:

$$\begin{aligned} \boldsymbol{T}_0 = v^0\partial_t + v^i\partial_{x^i} &= v^0\left(\partial_t + \frac{x^i}{|x|}\partial_{x^i}\right) - v^0\frac{x^i}{|x|}\partial_{x^i} + v^i\partial_{x^i} \\ &= v^0\left(\partial_t + \frac{x^i}{|x|}\partial_{x^i}\right) + \frac{v^0x^i}{t|x|}(|x|-t)\partial_{x^i} + \frac{v^it - x^iv^0}{t}\partial_{x^i}\Bigg| \end{aligned}$$

$$= v^0 \underbrace{\left(\partial_t + \frac{x^i}{|x|}\partial_{x^i}\right)}_{\text{outgoing derivatives}} - \frac{v^0}{t} \underbrace{\frac{x^i}{|x|}}_{\text{bounded}} \underbrace{\left(\frac{-x^j\Omega_{ij} + t\Omega_{0i} - x_i S}{t+r}\right)}_{\leq C(|\Omega_{ij}|+|\Omega_{0i}|+|S|)} + v^0 \frac{\mathfrak{z}}{v^0 t}\partial_{x^i},$$

where the weight $\mathfrak{z}$ in the last term is $v^i t - x^i v^0 \in \Bbbk_0$. Recall[25] now the following improved decay for outgoing derivatives of solutions to the wave equations: for all multi-indices α such that $|\alpha| \leq N - \frac{1}{2}(n+2) - 1$,

$$\left|\left(\partial_t + \frac{x^i}{r}\partial_{x^i}\right)Z^\alpha(\phi)\right| \lesssim \frac{\mathcal{E}_N}{(1+t)^{\frac{3}{2}}}.$$

To estimate the second term, we need to obtain decay for $Z\phi$ as solution to the wave equation. This is done by integrating the decay of $\partial Z\phi$ coming from the Klainerman–Sobolev inequality along ingoing null rays. We do not perform the proof of this fact here, but the reader can refer to the proof of Lemma 4.14, where a similar result is proven (for a Cauchy problem with initial data on a hyperboloid). One then obtains:

$$\left|\left(\frac{-x^j\Omega_{ij} + t\Omega_{0i} - x_i S}{t+r}\right)Z^\alpha\phi\right| \lesssim \frac{\mathcal{E}_N}{(1+t)^{\frac{1}{2}}}.$$

As a consequence, it follows that for all multi-indices $|\alpha| \leq N - \frac{1}{2}(n+2) - 1$,

$$|\boldsymbol{T}_0(Z^\alpha\phi)| \lesssim \mathcal{E}_N v^0\left(\frac{1}{(1+t)^{\frac{3}{2}}} + \sum_{\mathfrak{z}\in\Bbbk_0}\frac{|\mathfrak{z}|}{v^0}\frac{1}{t(1+t)}\right).$$

Repeating the previous ingredients then gives (65). The pointwise estimates then follow from the Klainerman–Sobolev inequality (42). □

4C. ***The massive Vlasov–Nordström system.*** We now turn to the massive case, that is to say the system

$$\Box\phi = m^2\int_v f\frac{dv}{v^0} = m^2\rho_1\left(\frac{f}{(v^0)^2}\right)$$

$$\boldsymbol{T}_m(f) - \left(\boldsymbol{T}_m(\phi)v^i + m^2\nabla^i\phi\right)\frac{\partial f}{\partial v^i} = (n+1)\boldsymbol{T}_m(\phi)f.$$

As in the massless case, we introduce the notation $\boldsymbol{T}_\phi \equiv \boldsymbol{T}_m - (\boldsymbol{T}_m(\phi)v^i + m^2\nabla^i\phi)(\partial/\partial v^i)$ for the transport operator that appears on the left-hand side of the last equation. With this notation, we will seek solutions of the massive Vlasov–Nordström system

$$\Box\phi = m^2\int_v f\frac{dv}{v^0}, \tag{68}$$

$$\boldsymbol{T}_\phi(f) = (n+1)f\,\boldsymbol{T}_m(\phi) \tag{69}$$

completed by the initial conditions

$$\phi_{|H_1} = \phi_0, \quad \partial_t\phi_{|H_1} = \phi_1, \tag{70}$$

$$f_{|H_1\times\mathbb{R}^n_v} = f_0. \tag{71}$$

[25] This can be obtained from the usual Klainerman–Sobolev inequality and the formula for ∂_s in (24) by integration along the constant $t = |x|$ null lines. See, for instance, [Wang 2015b] for details.

As for the massless case, the lower the dimension, the harder it is to close the estimates. We consider here only the dimensions $n \geq 4$. As already explained, to treat the case $n = 3$, we need a refinement of our method, for instance, the use of modified vector fields in the spirit of [Smulevici 2016], and we postpone this to future work. The proof that we shall give below will be enough to close the estimates for $n = 4$ with some ε-growth in the norms, and without any growth if $n > 4$.

In the following, we will set the mass m equal to 1.

4D. *The norms.* In the context of the massive Vlasov–Nordström system, we define the following energies, similar to the energies defined in (56) and (58):

- For the field ϕ, satisfying a wave equation,

$$\mathcal{E}_N[\phi](\rho) \equiv \sum_{\substack{|\alpha|\leq N \\ Z^\alpha \in \mathbb{P}^{|\alpha|}}} \int_{H_\rho} T[Z^\alpha \phi](\partial_t, \nu_\rho)\, d\mu_{H_\rho}, \tag{72}$$

  where, for any scalar function ψ we denote by $T[\psi] = d\psi \otimes d\psi - \frac{1}{2}\eta(\nabla\psi, \nabla\psi)\eta$ its energy-momentum tensor.

- For the field f, satisfying a transport equation,

$$E_N[f](\rho) \equiv \sum_{\substack{|\alpha|\leq \lfloor N/2 \rfloor \\ Z^\alpha \in \widehat{\mathbb{P}}^\alpha}} \left\| \chi_1\big((v^0)^2 |\widehat{Z}^\alpha(f)|\big) \right\|_{L^1(H_\rho)} + \sum_{\substack{\lfloor N/2 \rfloor + 1 \leq |\alpha| \leq N \\ Z^\alpha \in \widehat{\mathbb{P}}^\alpha}} \left\| \chi_1(|\widehat{Z}^\alpha(f)|) \right\|_{L^1(H_\rho)},$$

  where for any regular distribution function g, the energy density $\chi_1(g)$ is defined as in Section 2K.

Remark 4.4. The weight on the lower-order derivatives contained in the norm of f ensures that pointwise estimates can be performed on terms of the form

$$\int_v \left| v^0 \widehat{Z}^\alpha f(t,x,v) \right| dv \lesssim \frac{E_N[f](\rho)}{t^n},$$

according to Theorem 8, given on page 1564, provided that $|\alpha| \leq \left\lfloor \frac{N}{2} \right\rfloor - n$. It should furthermore be noticed that the "unweighted" standard estimates coming from Theorem 8 are still true for $|\alpha| \leq N - n$:

$$\int_v \left| \widehat{Z}^\alpha f(t,x,v) \right| \frac{dv}{v^0} \lesssim \frac{E_N[f](\rho)}{t^n}.$$

They will nonetheless not be used in the following.

4D1. *The main result.* Our main result for the massive Vlasov–Nordström system is contained in the following theorem.

Theorem 12. *Let $n \geq 4$ and $N \geq 3n + 4$. Let (f_0, ϕ_0, ϕ_1) be an initial data set for the system (68)–(71). Then, there exists an $\varepsilon_0 > 0$ such that, for all $0 \leq \varepsilon < \varepsilon_0$, if*

- $\mathcal{E}_N[\phi_0, \phi_1] \leq \varepsilon$ *(initial regularity of ϕ),*
- $E_{N+n}[f_0] \leq \varepsilon$ *(initial regularity of f),*

then, the unique classical solution (f, ϕ) *of* (68)–(71) *exists in the whole of the future unit hyperboloid and satisfies the following estimates*:

(1) *Energy bounds for* ϕ. *For all* $\rho \geq 1$,

$$\mathcal{E}_N[\phi](\rho) \leq 2\varepsilon.$$

(2) *Global bounds for* f *at order less than* N. *For all* $\rho \geq 1$,

$$E_N[f](\rho) \leq \rho^{C\varepsilon^{1/4}} 2\varepsilon,$$

where $C = 1$ *when* $n = 4$, *and* $C = 0$ *when* $n > 4$.

(3) *Pointwise decay for* $\partial Z^\alpha \phi$. *For all multi-indices* $|\alpha|$ *such that* $|\alpha| \leq N - \frac{n+2}{2}$ *and all* (t, x) *with* $t \geq \sqrt{1 + |x|^2}$, *we have*

$$|\partial Z^\alpha \phi| \lesssim \frac{\varepsilon}{(1+t)^{\frac{n-1}{2}}(1 + (t - |x|))^{\frac{1}{2}}}.$$

(4) *Pointwise decay for* $\rho(|\partial Z^\alpha f|)$. *For all multi-indices* α *and* β *such that* $|\alpha| \leq N - n$ *and* $|\beta| \leq \lfloor \frac{N}{2} \rfloor - n$ *and all* (t, x) *with* $t \geq \sqrt{1 + |x|^2}$, *we have*

$$\int_v |\widehat{Z}^\alpha f| \frac{dv}{v^0} \lesssim \frac{\varepsilon}{(1+t)^{n - C\varepsilon^{1/4}}},$$

$$\int_v v^0 |\widehat{Z}^\beta f|\, dv \lesssim \frac{\varepsilon}{(1+t)^{n - C\varepsilon^{1/4}}},$$

where $C = 1$ *when* $n = 4$ *and* $C = 0$ *when* $n > 4$.

(5) *Finally, the following* L^2 *estimates on* f *hold. For all multi-indices* α *with* $\lfloor \frac{N}{2} \rfloor - n + 1 \leq |\alpha| \leq N$, *and all* (t, x) *with* $t \geq \sqrt{1 + |x|^2}$, *we have*

$$\int_{H_\rho} \frac{t}{\rho} \left(\int_v |\widehat{Z}^\alpha f| \frac{dv}{v^0} \right)^2 d\mu_{H_\rho} \lesssim \varepsilon^2 \rho^{C\varepsilon^{1/4} - n},$$

where $C = 2$ *when* $n = 4$ *and* $C = 0$ *when* $n > 4$.

4E. *Proof of Theorem 12.*

4E1. *Structure of the proof and the bootstrap assumptions.* From now on, we consider a solution (f, ϕ) to (68)–(71) arising from initial data satisfying the requirements of Theorem 12. Let P be the largest (hyperboloidal) time so that the following bootstrap assumptions hold on $[1, P]$: assume that there exist an ε small enough and $\delta \in \left[0, \frac{1}{2}\right)$ such that, for all (ρ, r, ω) in $[1, P] \times \mathbb{R}^3$, we have

- energy bounds for ϕ,

$$\mathcal{E}_N[\phi](\rho) \leq 2\varepsilon; \tag{73}$$

- global bounds for f,

$$E_N[f](\rho) \leq \rho^\delta 2\varepsilon. \tag{74}$$

It follows from a continuity argument[26] that $P > 1$ and the remainder of the proof will be devoted to the improvement of each of the above inequalities, establishing the validity of Theorem 12. The proof is organized as follows:

- We first prove the necessary commutation formulae with the transport operator $\boldsymbol{T}_\phi$ in Section 4E2. The fundamental commutator is given in Lemma 4.11.

- A second step consists in rewriting the well-known standard Klainerman–Sobolev estimates for scalar fields using the hyperboloidal foliation (Proposition 4.12), in Section 4E3. These decay estimates for derivatives of scalar fields also provide estimates on the fields themselves after integration along null lines (Lemma 4.14).

- In Section 4E5, the bootstrap assumption (73) is improved, assuming weighted L^2_x decay estimates for the higher-order derivatives of the solution to the transport equation (see Lemma 4.18). The proof is based on energy estimates for which we need the source terms to have sufficient decay. When only low derivatives are involved, our Klainerman–Sobolev inequalities for f are sufficient to close the energy estimates for ϕ, so that the L^2_x decay estimates are only required to handle the high derivatives case (see Lemma 4.18).

- In Section 4E6, the bootstrap assumption (74) is improved. The proof relies on the conservation law for the massive transport equation (Lemma 4.20). Unfortunately, some of the source terms arising from the commutation relations are a priori not space-time integrable. To handle this lack of decay, we use *renormalized* variables by incorporating part of the source term in the original variables; see equation (98). Here we use pointwise estimates for $\partial Z^\alpha \phi$ but also the pointwise estimates on $Z^\alpha \phi$ provided by Lemma 4.14. The improvement of the bootstrap assumption is obtained after returning back to the original variables, provided that the initial data are small enough (Proposition 4.24).

- One finally proves in Section 4E7 the L^2-estimates for the transport equation, which are required in Section 4E5 to improve the bootstrap assumption on the solution of the wave equation. To this end, the equations for the renormalized variables introduced in equation (98) in Section 4E6 are rewritten as a system (Lemma 4.28) of inhomogeneous transport equations. Using the fact that we have control on the initial data for $N + n$ derivatives, it is possible to prove strong pointwise estimates for the homogeneous part of the solution to this system carrying the initial data (Lemma 4.29). The inhomogeneous part of the solution to this system (with no initial data) can be decomposed into an L^2 integrable function and a pointwise decaying function; see equations (114) and (115). This decomposition can then be exploited to prove the decay of weighted L^2-norms of higher-order derivatives of f (see Proposition 4.31). The later decay estimate is then used to improve the bootstrap assumption for the wave equation (see Lemma 4.18).

Note that:

- An estimate for the size of δ in (74) is obtained in Section 4E7 (Lemma 4.30).

- Finally, the maximal regularity is required in Lemma 4.22, when pointwise estimates have to be performed on f.

[26] Note that the methods of this paper show in particular that the system is well-posed in the spaces corresponding to the norms $\mathcal{E}_N^{1/2}[\phi]$ and $E_N[f]$ for N sufficiently large. See also [Calogero and Rein 2004] for another local existence statement.

In the sequel, we will heavily use the following pointwise estimates, which hold under the bootstrap assumptions (73) and (74):

- As a consequence of Proposition 4.12, if $|\gamma| \le N - \lfloor \frac{n}{2} \rfloor - 1$, then

$$|\partial Z^\gamma \phi| \lesssim \frac{\sqrt{\varepsilon}}{(t-|x|)^{\frac{1}{2}}(1+t)^{\frac{n-1}{2}}} = \frac{\sqrt{\varepsilon}}{\rho(1+t)^{\frac{n}{2}-1}}.$$

- As a consequence of Lemma 4.14, if $|\gamma| \le N - \lfloor \frac{n}{2} \rfloor - 1$, then

$$|Z^\gamma \phi| \lesssim \frac{\sqrt{\varepsilon}(t-|x|)^{\frac{1}{2}}}{(1+t)^{\frac{n-1}{2}}} = \frac{\sqrt{\varepsilon}\rho}{(1+t)^{\frac{n}{2}}}.$$

- As a consequence of Theorem 8, given on page 1564, if $|\beta| \le N - n$, then

$$\int_v |\widehat{Z}^\beta f| \frac{dv}{v^0} \lesssim \frac{\varepsilon \rho^\delta}{(1+t)^n}.$$

- Finally, as a consequence of Theorem 8, if $|\beta| \le \lfloor \frac{N}{2} \rfloor - n$, then

$$\int_v |v^0 \widehat{Z}^\beta f|\, dv \lesssim \frac{\varepsilon \rho^\delta}{(1+t)^n}.$$

4E2. *Commutators in the massive case.* Let us start with the following commutation relations.

Lemma 4.5.

$$[\partial_t, \partial_{v^i}] = 0, \tag{75}$$

$$[\partial_{x^i}, \partial_{v^i}] = 0, \tag{76}$$

$$[t\partial_{x^j} + x^j \partial_t + v^0 \partial_{v^j}, \partial_{v^i}] = -\frac{v^i}{v^0} \partial_{v^j}, \tag{77}$$

$$[x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}, \partial_{v^k}] = -\delta^i_k \partial_{v^j} + \delta^j_k \partial_{v^i}, \tag{78}$$

$$[t\partial_{x^j} + x^j \partial_t + v^0 \partial_{v^j}, v^i \partial_{v^i}] = \frac{1}{v^0} \partial_{v^j}, \tag{79}$$

$$[x^i \partial_j - x^j \partial_i + v^i \partial_{v^j} - v^j \partial_{v^i}, v^k \partial_{v^k}] = 0. \tag{80}$$

We now evaluate the commutators $[\boldsymbol{T}_\phi, \widehat{Z}]$ for $\widehat{Z} \in \widehat{\mathbb{P}}$. We have

$$\begin{aligned}
[\boldsymbol{T}_\phi, \widehat{Z}] f &= \underbrace{[\boldsymbol{T}_1, \widehat{Z}] f}_{=0 \text{ if } \widehat{Z} \in \widehat{\mathbb{P}}} - [\boldsymbol{T}_1(\phi) v^i \partial_{v^i}, \widehat{Z}] f - [\nabla^i \phi \cdot \partial_{v^i}, \widehat{Z}] f \\
&= \widehat{Z}[\boldsymbol{T}_1(\phi)] v^i \partial_{v^i} f + \boldsymbol{T}_1(\phi)[\widehat{Z}, v^i \partial_{v^i}] f - [\nabla^i \phi \cdot \partial_{v^i}, \widehat{Z}] f \\
&= \big([\widehat{Z}, \boldsymbol{T}_1]\phi + \boldsymbol{T}_1(\widehat{Z}\phi)\big) v^i \partial_{v^i} f - [\nabla^i \phi \partial_{v^i}, \widehat{Z}] f + \begin{cases} \boldsymbol{T}_1(\phi) \frac{1}{v^0} \partial_{v^j} f & \text{if } \widehat{Z} = t\partial_{x^j} + x^j \partial_t + v^0 \partial_{v^j}, \\ 0 & \text{otherwise} \end{cases} \\
&= \boldsymbol{T}_1(Z\phi) v^i \partial_{v^i} f - [\nabla^i \phi \cdot \partial_{v^i}, \widehat{Z}] f + \begin{cases} \boldsymbol{T}_1(\phi) \frac{1}{v^0} \partial_{v^j} f & \text{if } \widehat{Z} = t\partial_{x^j} + x^j \partial_t + v^0 \partial_{v^j}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{81}$$

We have used that $\widehat{Z}(\phi) = Z(\phi)$ since ϕ is independent of v. To estimate the second term on the right-hand side of the last equation, we need:

Lemma 4.6. *For any $Z \in \mathbb{P}$:*

- *If Z is a translation, then*
$$[\nabla^i \phi \cdot \partial_{v^i}, \widehat{Z}] = -\nabla^i (Z\phi)\partial_{v^i}.$$
- *If $Z = \Omega_{jk}$ is a rotation such that $\widehat{Z} = \Omega_{jk} + v^j \partial_{v^k} - v^k \partial_{v^j}$, then*
$$[\nabla^i \phi \cdot \partial_{v^i}, \widehat{Z}] = -\nabla^i (Z\phi)\partial_{v^i}.$$
- *If $Z = \Omega_{0j}$ is a Lorentz boost such that $\widehat{Z} = \Omega_{0j} + v^0 \partial_{v^j}$, then*
$$[\nabla^i \phi \cdot \partial_{v^i}, \widehat{Z}] = -\nabla^i (Z\phi)\partial_{v^i} + \nabla_i \phi \frac{v^i}{v^0} \partial_{v^j} + \partial_t(\phi)\, \partial_{v^j} f.$$

We now summarize these computations:

Lemma 4.7. *Let $\widehat{Z} \in \widehat{\mathbb{P}}$. Then*
$$[\boldsymbol{T}_\phi, \widehat{Z}] f = \boldsymbol{T}_1(Z\phi) v^i\, \partial_{v^i} f + \sum_{\substack{|\alpha| \le 1 \\ 1 \le j \le n \\ 0 \le \beta \le n}} p^{j\beta}\left(\frac{v}{v^0}\right) \partial_{x^\beta} Z^\alpha(\phi) \cdot \partial_{v^j} f,$$
where the $p^{i\beta}(v/v^0)$ are polynomial of degree at most 1 in the variables v^k/v^0, $1 \le k \le n$.

The terms containing derivatives of v in the above formulae are problematic, since the ∂_v are not part of the algebra $\widehat{\mathbb{P}}$. We use the following decomposition for all $1 \le i \le n$:
$$\partial_{v^i} = \frac{1}{v^0}(t\partial_{x^i} + x^i \partial_t + v^0 \partial_{v^i}) - \frac{1}{v^0}(t\partial_{x^i} + x^i \partial_t) = \frac{1}{v^0}\widehat{\Omega}_{0i} - \frac{1}{v^0}(t\partial_{x^i} + x^i \partial_t). \tag{82}$$

Remark 4.8. Note that
$$\frac{1}{v^0}|t\, \partial_{x^i} f + x^i\, \partial_t f| \le \frac{t}{v^0}\big(|\partial_t f| + |\partial_{x^i} f|\big) \tag{83}$$
for (t, x) in the future of the unit hyperboloid. Now ∂_t and ∂_{x^i} belong to $\widehat{\mathbb{P}}$, but the price to pay is the extra t-factor. It is precisely this extra t-growth which forbids us to close the estimate in dimension 3. A similar obstacle was identified for the Vlasov–Poisson system in dimension 3 and solved by means of *modified vector fields* in [Smulevici 2016]. We hope to treat the 3-dimensional massive Vlasov–Nordström system in future work.

This leads to the commutation formula:

Lemma 4.9. *Let $\widehat{Z} \in \widehat{\mathbb{P}}$. Then*
$$[\boldsymbol{T}_\phi, \widehat{Z}] f = \boldsymbol{T}_1(Z\phi) \sum_{|\alpha|=1} q_\alpha\left(\frac{v}{v^0}, t, x\right) \widehat{Z}^\alpha(f) + \sum_{\substack{|\alpha| \le 1, |\beta| = 1 \\ 0 \le \gamma \le n}} \frac{p^\gamma_{\alpha\beta}(v/v^0, t, x)}{v^0} \partial_{x^\gamma} Z^\alpha(\phi) \cdot \widehat{Z}^\beta(f),$$
where the $q_\alpha(v/v^0, t, x)$ and $p^\gamma_{\alpha\beta}(v/v^0, t, x)$ are polynomial of degree at most 1 in the variables
$$\frac{v^k}{v^0}, \quad \frac{v^k}{v^0} t, \quad \frac{v^k}{v^0} x^i, \quad 1 \le i, k \le n.$$

Iterating the above formula, we obtain:

Lemma 4.10. *Let α be a multi-index and $\widehat{Z}^\alpha \in \widehat{\mathbb{P}}^{|\alpha|}$. Then*

$$[\boldsymbol{T}_\phi, \widehat{Z}^\alpha] f = \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\gamma|\le|\alpha|\\ 1\le|\beta|\le|\alpha|}} \boldsymbol{T}_1(Z^\gamma\phi) q_{\beta\gamma}\left(\frac{v}{v^0}, t, x\right)\widehat{Z}^\beta(f) + \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 0\le\sigma\le n\\ 1\le|\beta|\le|\alpha|\\ 0\le|\gamma|\le|\alpha|}} \frac{1}{v^0} p^\sigma_{\gamma\beta}\left(\frac{v}{v^0}, t, x\right)\partial_{x^\sigma} Z^\gamma(\phi)\cdot\widehat{Z}^\beta f,$$

where

- *the $q_{\beta\gamma}(v/v^0, t, x)$ are linear combinations of the terms*
$$q\left(\frac{v^k}{v^0}\right), \quad q'\left(\frac{v^k}{v^0}\right)t, \quad q''\left(\frac{v^k}{v^0}\right)x^i, \quad 1\le i,k\le n,$$
where q, q', q'' are polynomials of degree at most $|\alpha|$,
- *the $p^\gamma_{\gamma\beta}(v/v^0, t, x)$ are linear combinations with constant coefficient of the terms*
$$p\left(\frac{v^k}{v^0}\right), \quad p'\left(\frac{v^k}{v^0}\right)t, \quad p''\left(\frac{v^k}{v^0}\right)x^i, \quad 1\le i,k\le n,$$
where p, p', p'' are polynomials of degree at most $|\alpha|$.

Proof. This follows by an induction argument on the length of the multi-index α and we therefore only provide some details here. Assume the lemma is true for $|\alpha|$. Recall that, for any $\widehat{Z} \in \widehat{\mathbb{P}}_0$,

$$[\boldsymbol{T}_\phi, \widehat{Z}\widehat{Z}^\alpha](f) = [\boldsymbol{T}_\phi, \widehat{Z}]\widehat{Z}^\alpha(f) + \widehat{Z}[\boldsymbol{T}_\phi, \widehat{Z}^\alpha](f) = I_1 + I_2,$$

with

$$I_1 = [\boldsymbol{T}_\phi, \widehat{Z}]\widehat{Z}^\alpha(f), \quad I_2 = \widehat{Z}[\boldsymbol{T}_\phi, \widehat{Z}^\alpha](f).$$

Using Lemma 4.9, we have for I_1,

$$I_1 = \boldsymbol{T}_1(Z\phi)\sum_{|\gamma|=1} q_\gamma\left(\frac{v}{v^0}, t, x\right)\widehat{Z}^\gamma(\widehat{Z}^\alpha f) + \sum_{\substack{|\gamma|\le1, |\beta|=1\\ 0\le\sigma\le n}} \frac{1}{v^0} p^\sigma_{\gamma\beta}\left(\frac{v}{v^0}, t, x\right)\partial_{x^\sigma} Z^\gamma(\phi)\cdot\widehat{Z}^\beta(\widehat{Z}^\alpha f), \quad (84)$$

with q_γ and $p^\sigma_{\gamma\beta}$ as in the statement of Lemma 4.9. Since all the terms in (84) clearly have the desired form, we turn to I_2. Applying the induction hypothesis, we have

$$\begin{aligned} I_2 &= \widehat{Z}\Bigg[\sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\gamma|\le|\alpha|\\ 1\le|\beta|\le|\alpha|}} \boldsymbol{T}_1(Z^\gamma\phi) q_{\beta\gamma}\left(\frac{v}{v^0}, t, x\right)\widehat{Z}^\beta(f) + \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 0\le\sigma\le n\\ 1\le|\beta|\le|\alpha|}} \frac{1}{v^0} p^\sigma_{\gamma\beta}\left(\frac{v}{v^0}, t, x\right)\partial_{x^\sigma} Z^\gamma(\phi)\cdot\widehat{Z}^\beta f\Bigg] \\ &= \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\gamma|\le|\alpha|\\ 1\le|\beta|\le|\alpha|}} \widehat{Z}\left[\boldsymbol{T}_1(Z^\gamma\phi) q_{\beta\gamma}\left(\frac{v}{v^0}, t, x\right)\widehat{Z}^\beta(f)\right] + \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 0\le\sigma\le n\\ 1\le|\beta|\le|\alpha|}} \widehat{Z}\left[\frac{1}{v^0} p^\sigma_{\gamma\beta}\left(\frac{v}{v^0}, t, x\right)\partial_{x^\sigma} Z^\gamma(\phi)\cdot\widehat{Z}^\beta f\right] \\ &= J_1 + J_2, \end{aligned}$$

where

$$J_1 = \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\gamma|\le|\alpha|\\ 1\le|\beta|\le|\alpha|}} \widehat{Z}\left[\boldsymbol{T}_1(Z^\gamma\phi) q_{\beta\gamma}\left(\frac{v}{v^0}, t, x\right)\widehat{Z}^\beta(f)\right]$$

and

$$J_2 = \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 0\le\sigma\le n\\ 1\le|\beta|\le|\alpha|}} \widehat{Z}\left[\frac{1}{v^0} p^{\sigma}_{\gamma\beta}\left(\frac{v}{v^0}, t, x\right)\partial_{x^\sigma} Z^\gamma(\phi)\cdot \widehat{Z}^\beta f\right].$$

To see that J_1 has the correct form, we distribute $\widehat{Z}$, which gives rise to three types of terms. The terms arising when $\widehat{Z}$ hits $\boldsymbol{T}_1(Z^\gamma\phi)$ or $\widehat{Z}^\beta(f)$ are easily seen to have the right form. It remains to look at the case when $\widehat{Z}$ hits $q_{\beta\gamma}$. If $\widehat{Z}$ is a translation, $\widehat{Z} = \partial_{x^\beta}$ and one easily sees that $\widehat{Z}(q_{\beta\gamma})$ has the correct form. If $\widehat{Z}$ is the lift of a rotation or a Lorentz boost, then we write schematically $\widehat{Z} = {}^xZ + {}^vZ$, where ${}^xZ$ is a homogeneous differential operator of order 1 in (t,x) and ${}^vZ$ is a homogeneous differential operator of order 1 in v. It is then easy to check that ${}^xZ$ applied to a polynomial in the variables v^i/v^0 of degree $\le|\alpha|$, possibly multiplied by the variables t, x^i will produce a polynomial, of the same degree $\le|\alpha|$, in the variables v^i/v^0, possibly multiplied by the variables t, x^i. Similarly, ${}^vZ$ applied to a polynomial in the variable v^i/v^0 of degree $\le|\alpha|$ will produce a polynomial in the variables v^i/v^0 of degree $\le|\alpha|+1$. As a consequence, $\widehat{Z}^\alpha(q_{\beta\gamma})$ is a linear combination of polynomials of degree $|\alpha|+1$, possibly multiplied by t, x^i. The term J_2 can be treated similarly. □

The full expression for $\boldsymbol{T}_\phi(\widehat{Z}^\alpha f)$ can now be computed using the transport equation (69) satisfied by f.

Lemma 4.11. *Let Z^α be in $\mathbb{P}^{|\alpha|}$. Then the following equation holds*:

$$\begin{aligned}\boldsymbol{T}_\phi(\widehat{Z}^\alpha(f)) = &\sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ |\gamma|\ge1,|\beta|\ge1}} \boldsymbol{T}_1(Z^\gamma\phi) q_{\beta\gamma}\left(\frac{v^i}{v^0}, t, x\right)\widehat{Z}^\beta(f)\\ &+ \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 0\le\sigma\le n\\ 1\le|\beta|\le|\alpha|}} \frac{1}{v^0} p^\sigma_{\gamma\beta}\left(\frac{v}{v^0}, t, x\right)\partial_\sigma Z^\gamma(\phi)\widehat{Z}^\beta f + \sum_{|\gamma|+|\beta|=|\alpha|} r_{\gamma\beta}\boldsymbol{T}_1(Z^\gamma\phi)\widehat{Z}^\beta(f), \end{aligned}\tag{85}$$

where the $q_{\beta\gamma}$ and $p^\sigma_{\gamma\beta}$ are as in Lemma 4.10 and the $r_{\gamma\beta}$ are constants.

Proof. We have

$$\boldsymbol{T}_\phi(\widehat{Z}^\alpha f) = [\boldsymbol{T}_\phi, \widehat{Z}^\alpha] f + \widehat{Z}^\alpha \boldsymbol{T}_\phi f = [\boldsymbol{T}_\phi, \widehat{Z}^\alpha] f + \widehat{Z}^\alpha((n+1)\boldsymbol{T}_1(\phi) f). \tag{86}$$

The lemma thus follows from Lemma 4.10 and the fact that

$$\widehat{Z}^\alpha(\boldsymbol{T}_1(\phi) f) = \sum_{|\gamma|+|\beta|=|\alpha|} r_{\gamma\beta}\boldsymbol{T}_1(Z^\gamma\phi)\widehat{Z}^\beta(f), \tag{87}$$

where the $r_{\gamma\beta}$ are constants. □

4E3. *The H_ρ foliation and the wave equation.* The aim of this section is to provide a Klainerman–Sobolev-type inequality, applicable to solutions of the inhomogeneous wave equation if the inhomogeneities decay sufficiently fast, using only energies on the H_ρ foliation. This question was addressed by Klainerman

[1993] for the Klein–Gordon operator and we show here how a similar proof can also be applied to the wave operator. We thus consider a function ψ and its energy-momentum tensor

$$T[\psi] = d\psi \otimes d\psi - \tfrac{1}{2}(\eta(\nabla\psi, \nabla\psi))\eta. \tag{88}$$

If we want to perform energy estimates on H_ρ, we need to multiply $T[\psi]$ by ∂_t and the normal to H_ρ, ν_ρ, and integrate on H_ρ. Let us thus compute the quantity $T(\partial_t, \nu_\rho)$.

We find

$$T[\psi](\partial_t, \nu_\rho) = \frac{t}{2\rho}(\psi_t^2 + \psi_r^2 + |\nabla\!\!\!/\psi|^2) + \frac{r}{\rho}\psi_t\psi_r. \tag{89}$$

Recall also that the volume form on H_ρ is given by $(\rho/t)r^{n-1}\,dr\,d\sigma$. Since we are looking only at the region $\rho \geq 1$, we have that $t > r$ and $T(\partial_t, \nu_\rho)$ is clearly positive definite, with some degeneration as $r \to t$. More precisely, fix (t, x) in the future of the unit hyperboloid. Assume first that $r = |x| \leq \frac{1}{2}t$; then $(\rho/t)T[\psi](\partial_t, \nu_\rho) \geq |\partial\psi|^2$.

Let (Y^0, Y^i) be the coordinates of (t, x) in the (y^α)-system of coordinates adapted to the H_ρ foliation as introduced in Section 2C. Let $\Phi(y) = \partial\psi(Y^0, Y^j + ty^j)$. Then, a classical Sobolev inequality yields

$$\begin{aligned} |\partial\psi(Y^0, Y^j)|^2 = |\Phi(0)|^2 &\lesssim \sum_{k \leq \frac{n+2}{2}} \int_{|y|\leq\delta} |\partial_y^k \Phi(y)|^2\,dy \\ &\lesssim \sum_{k \leq \frac{n+2}{2}} \int_{|y|\leq\delta} \big|Z^k(\partial\psi)(Y^0, Y^j + ty^j)\big|^2\,dy, \end{aligned} \tag{90}$$

using that $\partial_{y^i} = (1/t)\Omega_{0i}$ and the fact that $\partial_{y^i}\Phi = t\,\partial_{y^i}\psi$ together with estimates on $t/(t(Y^0, Y^j + ty^j))$ similar to those of Section 3C. Applying the change of coordinates $y^j \to ty^j$ yields

$$|\partial\psi(Y^0, Y^j)|^2 \lesssim \frac{1}{t^n} \sum_{k \leq \frac{n+2}{2}} \int_{|y|\leq t\delta} \big|Z^k(\partial\psi)(Y^0, Y^j + y^j)\big|^2\,dy. \tag{91}$$

Finally, $|Y^j + y^j| = |x^j + y^j| \lesssim \big(\frac{1}{2} + \delta\big)t$ so that, if $\delta > 0$, we are still away from the light cone. Thus, the right-hand side of the previous equations can be controlled by the energies of $Z^k(\partial\psi)$ on H_ρ. On the other hand, if $\frac{1}{2}t \leq r < t$, we first remark that

$$\frac{\rho}{t}T[\psi](\partial_t, \nu_\rho) \geq \left(1 - \frac{r}{t}\right)|\partial\psi|^2. \tag{92}$$

Thus, we may repeat the previous arguments, losing the factor $(1 - r/t)$ in the process, as follows:

$$\begin{aligned} |\partial\psi(Y^0, Y^j)|^2 &\lesssim \frac{1}{t^n} \sum_{k \leq \frac{n+2}{2}} \int_{|y|\leq t\delta} \big|Z^k(\partial\psi)(Y^0, Y^j + y^j)\big|^2\,dy \\ &\lesssim \frac{1}{t^n} \sum_{k \leq \frac{n+2}{2}} \int_{|y|\leq t\delta} \left(1 - \frac{r}{t}\right)^{-1}\left(1 - \frac{r}{t}\right)\big|Z^k(\partial\psi)(Y^0, Y^j + y^j)\big|^2\,dy. \end{aligned} \tag{93}$$

Since
$$\frac{1}{1-r/t}=\frac{t(t+r)}{\rho^2}\lesssim\frac{t^2}{\rho^2}$$
and since again, we can replace $t(Y^0,Y^j+ty^j)$ by t in all the above computations, we have shown that
$$|\partial\psi(Y^0,Y^j)|^2\lesssim\frac{1}{t^{n-2}\rho^2}.$$
Since $\rho^2=(t+r)(t-r)$ and since $t\geq r$ in the future of the unit hyperboloid, this is exactly the decay estimate predicted by the usual Klainerman–Sobolev inequality using the Σ_t foliation. We summarize this in the following proposition.

Proposition 4.12 (Klainerman–Sobolev inequality for the wave equation using the hyperboloidal foliation). *For any sufficiently regular function ψ of (t,x) defined on the future of the unit hyperboloid, let $\mathcal{E}_{\frac{1}{2}(n+2)}[\psi](\rho)$ denote the energy*
$$\mathcal{E}_{\frac{n+2}{2}}[\psi](\rho)=\sum_{|\alpha|\leq\frac{n+2}{2}}\int_{H_\rho}T[Z^\alpha[\psi]](\partial_t,\nu_\rho)\,d\mu_\rho.$$
Then, for all (t,x) in the future of the unit hyperboloid,
$$|\partial\psi|(t,x)\lesssim\frac{1}{t^{\frac{n-1}{2}}(t-|x|)^{\frac{1}{2}}}\mathcal{E}_N^{\frac{1}{2}}[\psi](\rho(t,x)). \tag{94}$$

It is interesting to note that the above proof does not make use of the scaling vector field.

Remark 4.13. We will use in the following the inequality
$$\frac{\rho}{2t}T[\psi](\partial_t,\nu_\rho)=|\nabla\!\!\!/\psi|^2+\left(\psi_t^2+\psi_r^2+\frac{2r}{t}\psi_t\psi_r\right)=|\nabla\!\!\!/\psi|^2+\left(1-\frac{r}{t}\right)(\psi_t^2+\psi_r^2)+\frac{r}{t}(\psi_t+\psi_r)^2;$$
that is to say,
$$|\partial\psi|^2\lesssim\frac{\rho}{t-r}T[\psi](\partial_t,\nu_\rho)=\frac{t+r}{\rho}T[\psi](\partial_t,\nu_\rho).$$

The inequality (94) provides decay for $\partial\psi$ but not for ψ. By integration along null lines, one can obtain the following decay for ψ.

Lemma 4.14. *Let ψ be such that $\mathcal{E}_{\frac{1}{2}(n+2)}[\psi](\rho)$ is uniformly bounded on $[1,P]$ for some $P>1$. Assume moreover that $\psi_{|\rho=1}$ vanishes at ∞.*

Then ψ satisfies, for all (t,x) in the future of H_1,
$$|\psi(t,x)|\lesssim\sup_{[1,P]}\left[\mathcal{E}_N^{\frac{1}{2}}[\Psi]\right]\frac{(1+u)^{\frac{1}{2}}}{t^{\frac{n-1}{2}}},$$
where $u=t-|x|$.

Proof. The Klainerman–Sobolev estimates provide
$$|\partial\psi(t,x)|\lesssim\frac{\sup_{[1,P]}\left[\mathcal{E}_N^{\frac{1}{2}}\right]}{\rho t^{\frac{n}{2}-1}}.$$

Let $(t,x)=(t,r,\omega\in\mathbb{S}^n)$ be a point in the future of H_1, and consider the point on the hyperboloid lying at the intersection of the past light cone from (t,x), the hyperboloid H_1, and the direction ω. Writing $u=t-|x|$ and $v=t+|x|$, we have

$$(t_1,x_1)=\Big(t_1=\sqrt{1+r_1^2},r_1=\frac{1}{2}\Big(v-\frac{1}{v}\Big),\omega\Big)\in H_1.$$

Note that

$$\frac{1}{\langle r_1\rangle}=\frac{2}{v+1/v}\lesssim\frac{1}{\langle v\rangle}\quad\text{since } v\geq t\geq 1.$$

Integrating along the direction ω along the past light cone from (t,x) from (t_1,r_1,ω) to (t,r,ω), one obtains

$$\psi(t,x)=\psi(t_1,x_1)+\int_{t_1-r_1}^{t-r}(\partial_u\psi)\,du,$$

so that

$$\begin{aligned}|\psi(t,x)|&\lesssim\frac{\mathcal{E}_N^{\frac{1}{2}}(\rho=1)}{\langle r_1\rangle^{\frac{n-1}{2}}}+\int_{t_1-r_1}^{t-r}\frac{\sup_{[1,P]}[\mathcal{E}_N^{\frac{1}{2}}]}{u^{\frac{1}{2}}v^{\frac{n-1}{2}}}\,du\\&\lesssim\frac{\mathcal{E}_N^{\frac{1}{2}}(\rho=1)}{\langle v\rangle^{\frac{n-1}{2}}}+\frac{\sup_{[1,P]}[\mathcal{E}_N^{\frac{1}{2}}]u^{\frac{1}{2}}}{v^{\frac{n-1}{2}}}\lesssim\sup_{[1,P]}[\mathcal{E}_N^{\frac{1}{2}}]\frac{(1+u)^{\frac{1}{2}}}{v^{\frac{n-1}{2}}},\end{aligned}$$

which concludes the proof since $1/v\leq 2/t$. Here we have used that

$$|\psi(t_1,x_1)|\lesssim\frac{\mathcal{E}_N^{\frac{1}{2}}(\rho=1)}{\langle r_1\rangle^{\frac{n-1}{2}}},$$

which follows from usual weighted Sobolev inequalities on $\mathbb{R}^n$ applied to $\partial\psi$ and the assumption that ψ restricted to $\rho=1$ vanishes at infinity. □

4E4. *Commutation of the wave equation.* The commutation of the wave equation with our set of vector fields is straightforward and leads to:

Lemma 4.15. *For any multi-index* α,

$$\square Z^\alpha(\phi)=\int_{v\in\mathbb{R}^n_v}\widehat{Z}^\alpha(f)\frac{dv}{v^0}. \tag{95}$$

Proof. First, recall that the vector fields in the algebra $\mathbb{P}$ commute exactly with $\square$. The lemma then follows from Lemma 2.14 and Remark 2.15 in the case Z^α contains some combinations of Lorentz boosts. □

Remark 4.16. The following inequality will be used later on:

$$\left|\int_{v\in\mathbb{R}^n_v}\widehat{Z}^\alpha(f)\frac{dv}{v^0}\right|\leq\chi_1\big(|\widehat{Z}^\alpha(f)|\big),$$

which is a direct consequence of Remark 2.12.

4E5. *Energy estimates for the wave equation on hyperboloids.* Consider ψ defined in hyperboloidal time for all $\rho \in [1, P]$ and assume that ψ solves $\Box\psi = h$. Recalling the expression for $T[\psi](\partial_t, \nu_\rho)$ given by (89), we have:

Lemma 4.17. *Let $\rho \in [1, P]$. Then,*

$$\int_{H_\rho} T[\psi](\partial_t, \nu_\rho)\, d\mu_{H_\rho} = \int_{H_1} T[\psi](\partial_t, \nu_\rho)\, d\mu_{H_1} + \int_1^\rho \int_{H_{\rho'}} (\partial_t \psi)(\rho')\, h(\rho')\, d\mu_{H_{\rho'}}\, d\rho'.$$

Proof. The proof of this fact is only sketched, since classical. The reader can refer to [Klainerman 1993]. Remember that the divergence of the stress-energy tensor $T[\psi]$ is given by

$$\partial^\alpha T_{\alpha\beta}[\psi] = h\, \partial_\beta \psi$$

when ψ satisfies the equation $\Box\psi = h$. The lemma then follows by integration between the two hyperboloids H_1 and H_ρ and an application of Stokes' theorem. $\square$

To close the energy estimates for $Z^\alpha(\phi)$, we need the right-hand side of (95) to decay. Since for $|\alpha| \le N - n$, the required decay follows from our Klainerman–Sobolev inequality (42) as well as the bootstrap assumption (74), we have the following lemma:

Lemma 4.18. *Assume that $\delta < 1$. Assume moreover that for all multi-indices α such that $N - n + 1 \le |\alpha| \le N$, the following L^2_x decay estimate holds*:

$$\int_{H_\rho} \frac{t}{\rho} \left(\int_v |\widehat{Z}^\alpha f| \frac{dv}{v^0} \right)^2 d\mu_{H_\rho} \lesssim \varepsilon^2 \rho^{\delta - n}. \tag{96}$$

Then, the following inequality holds for all $\rho \in [1, P]$:

$$\mathcal{E}_N[\phi](\rho) \le \varepsilon(1 + C\varepsilon^{\frac{1}{2}}),$$

where C is a constant depending solely on the dimension n and the regularity N. In particular, for ε small enough, for all $\rho \in [1, P]$,

$$\mathcal{E}_N[\phi](\rho) \le \tfrac{3}{2}\varepsilon.$$

Remark 4.19.
- The weighted L^2-estimates (96) will in fact be proven in Section 4E7 for the wider range of multi-indices α with $\lfloor \frac{N}{2} \rfloor - n + 1 \le |\alpha| \le N$.
- Note that the L^2-estimates are needed only for $|\alpha| > N - n$: for lower order, the pointwise decay estimates for the velocity averages are sufficient to conclude.

Proof. The proof of this lemma relies on Lemma 4.17. Applying first Lemmata 4.17 and 4.15 to $Z^\alpha(\phi)$ for all multi-indices of length $|\alpha| \le N$, one obtains immediately, for $\rho \le P$,

$$\begin{aligned}\mathcal{E}_N[\phi](\rho) - \mathcal{E}_N[\phi](1) &\lesssim \sum_{|\alpha| \le N} \sum_{Z^\alpha \in \mathbb{P}^{|\alpha|}} \int_1^\rho \int_{H_{\rho'}} |\partial_t Z^\alpha \phi| \left(\int_v \frac{|\widehat{Z}^\alpha f|}{v^0}\, dv \right) d\mu_{H_{\rho'}}\, d\rho' \\ &\lesssim \sum_{|\alpha| \le N} \sum_{Z^\alpha \in \mathbb{P}^{|\alpha|}} \int_1^\rho \left(\int_{H_{\rho'}} \left| \left(\frac{\rho'}{t}\right)^{\frac{1}{2}} \partial_t Z^\alpha \phi \right| \cdot \left(\frac{t}{\rho'}\right)^{\frac{1}{2}} \int_v \frac{|\widehat{Z}^\alpha f|}{v^0}\, dv \right) d\mu_{H_{\rho'}}\, d\rho' \right|\end{aligned}$$

$$\lesssim \int_1^\rho \mathcal{E}_N^{\frac{1}{2}}[\phi](\rho')\bigg(\sum_{|\alpha|\leq N}\sum_{Z^\alpha\in\mathbb{P}^{|\alpha|}}\bigg(\int_{H_{\rho'}}\Big(\frac{t}{\rho'}\Big)\Big(\int_v \frac{|\widehat{Z}^\alpha f|}{v^0}\,dv\Big)^2 d\mu_{H_{\rho'}}\bigg)^{\frac{1}{2}}\bigg)\,d\rho'.$$

We now apply, for the low derivatives of f, Theorem 8 on page 1564 (in conjunction with Remark 2.12): for $|\alpha|\leq N-n$,

$$\int_{H_{\rho'}}\Big(\frac{t}{\rho'}\Big)\Big(\int_v \frac{|\widehat{Z}^\alpha f|}{v^0}\,dv\Big)^2 d\mu_{H_{\rho'}} \lesssim \int_0^\infty \frac{t}{\rho'}\cdot\frac{\varepsilon^2\rho'^{2\delta}}{t^{2n}}r^{n-1}\frac{\rho'}{t}\,dr \lesssim \varepsilon^2\rho'^{2\delta-n}\int_0^\infty \frac{y^{n-1}}{\langle y\rangle^{2n}}\,dy.$$

Since the last integral is convergent, we obtain

$$\bigg[\int_{H_{\rho'}}\Big(\frac{t}{\rho'}\Big)\Big(\int_v \frac{|\widehat{Z}^\alpha f|}{v^0}\,dv\Big)^2 d\mu_{H_{\rho'}}\bigg]^{\frac{1}{2}} \lesssim \varepsilon\rho'^{\delta-\frac{n}{2}},$$

which, assuming $\delta<1$, is integrable in ρ. For the higher derivatives of f, i.e., for $|\alpha|>N-n$, one uses the assumption of the lemma to obtain

$$\sum_{N-n<|\alpha|\leq N}\sum_{Z^\alpha\in\mathbb{P}^{|\alpha|}}\bigg(\int_{H_\rho}\frac{t}{\rho}\Big(\int_v |\widehat{Z}^\alpha f|\frac{dv}{v^0}\Big)^2 d\mu_{H_\rho}\bigg)^{\frac{1}{2}} \lesssim \varepsilon\rho^{\frac{\delta}{2}-\frac{n}{2}}.$$

We obtain finally

$$\mathcal{E}_N[\phi](\rho)\leq\varepsilon+C\varepsilon\bigg(\int_1^\rho (\rho'^{\frac{\delta-n}{2}})\mathcal{E}_N^{\frac{1}{2}}[\phi](\rho')\,d\rho'\bigg),$$

where C is a constant depending only on the regularity and the dimension. We remark that

$$\frac{\delta-n}{2}\leq\frac{\delta}{2}-\frac{3}{2}<-1$$

for $n>3$ and $\delta<1$. The result then follows using the bootstrap assumptions (73) and integrating in ρ. □

4E6. *L^1-estimates for the transport equation.* In the remainder of the article, we will use the notation

$$\mathbb{E}[g](\rho)=\int_{H_\rho}\chi_1(g)\,d\mu_\rho$$

for any regular distribution function g.

Lemma 4.20. *Let h be a regular distribution function for the massive case in the sense of Section 2E. Let g be a regular solution to $\boldsymbol{T}_\phi g=v^0h$, with $v^0=(1+|v|^2)^{\frac{1}{2}}$, defined on $\bigcup_{\rho\in[1,P]}H_\rho\times\mathbb{R}^n_v$, for some $P>1$. Then, for all $\rho\in[1,P]$,*

$$\int_{H_\rho}\chi_1(|g|)\,d\mu_{H_\rho}-\int_{H_1}\chi_1(|g|)\,d\mu_{H_1}\lesssim\int_1^\rho\int_{H_{\rho'}}\int_v\Big(v^0|h|+\frac{1}{v^0}|\partial_{x^0}\phi g|+|\boldsymbol{T}_1(\phi)g|\Big)\,dv\,d\mu_{H'_\rho}\,d\rho'. \quad (97)$$

Proof. One proves first that

$$\int_{H_\rho}\chi_1(g)\,d\mu_{H_\rho}=\int_{H_1}\chi_1(g)\,d\mu_{H_1}+\int_1^\rho\int_{H_{\rho'}}\int_v\Big(v^0h+\Big(\frac{1}{v^0}\partial_{x^0}\phi-(n+1)\boldsymbol{T}_1(\phi)\Big)g\Big)\,dv\,d\mu_{H'_\rho}\,d\rho',$$

since by integration by parts

$$\int_v (\boldsymbol{T}_1(\phi)v^i + \nabla^i \phi)\,\partial_{v^i} g\,dv = -\int_v g\left((n+1)\boldsymbol{T}_1(\phi) - \frac{\partial_{x^0}\phi}{v^0}\right)dv.$$

This establishes the lemma in the case $g \geq 0$. As in the proof of Lemma 2.18, the conclusion in the general case follows after regularization of the absolute value. □

For any multi-index α, let us now introduce the auxiliary function g^α:

$$g^\alpha = \widehat{Z}^\alpha(f) - \sum_{\substack{|\gamma|+|\beta|\leq|\alpha|+1\\ 1\leq|\beta|\\ 1\leq|\gamma|\leq N-\frac{n+2}{2}}} q_{\beta\gamma} Z^\gamma(\phi)\widehat{Z}^\beta(f), \tag{98}$$

where the $q_{\beta\gamma}$ are as in the statement of Lemma 4.11. One can view g^α as a renormalization of $\widehat{Z}^\alpha(f)$. The extra terms in the definition of g^α will allow us to absorb certain source terms in the equation satisfied by $\boldsymbol{T}_\phi[\widehat{Z}^\alpha(f)]$ (see Lemma 4.11) which cannot be estimated adequately because they carry too much v^0-weight, leading either to a t-loss (see Remark 4.16) or to a v^0-loss.

To perform L^1-estimates on $\widehat{Z}^\alpha(f)$, we therefore proceed as follows:

- We derive the equations satisfied by the g^α and then use them to obtain L^1-estimates for the g^α.
- We then prove the same L^1-estimates for $(v^0)^2 g^\alpha$ with $|\alpha| \leq \left\lfloor \frac{N}{2} \right\rfloor$ to take into account that the lower derivatives of f are weighted by $(v^0)^2$ in the $E_N[f]$ norm (see the definition of the norm in Section 4D).
- Finally, the L^1 estimates on g^α are then transformed into L^1-estimates on $\widehat{Z}^\alpha(f)$ using pointwise estimates on $Z^\gamma(\phi)$ for γ sufficiently small.

We start by deriving the equations for the g^α.

Lemma 4.21. *For any multi-index α, g^α satisfies the equation*

$$\begin{aligned}
\boldsymbol{T}_\phi g^\alpha = &\sum_{\substack{|\gamma|+|\beta|\leq|\alpha|+1\\ |\gamma|\geq 1,\,|\beta|\geq 1\\ |\gamma|>N-\frac{n+2}{2}}} q_{\beta\gamma}\boldsymbol{T}_1(Z^\gamma\phi)\widehat{Z}^\beta(f) + \sum_{\substack{|\gamma|+|\beta|\leq|\alpha|+1\\ 0\leq\sigma\leq n,\,1\leq|\beta|\leq|\alpha|}} \frac{1}{v^0} p^\sigma_{\gamma\beta}\,\partial_{x^\sigma} Z^\gamma(\phi)\widehat{Z}^\beta f\\
&+ \sum_{|\gamma|+|\beta|=|\alpha|} r_{\gamma\beta}\boldsymbol{T}_1(Z^\gamma\phi)\widehat{Z}^\beta f - \sum_{\substack{|\gamma|+|\beta|\leq|\alpha|+1\\ 1\leq|\beta|\\ 1\leq|\gamma|\leq N-\frac{n+2}{2}}} \boldsymbol{T}_\phi(q_{\beta\gamma})Z^\gamma(\phi)\widehat{Z}^\beta f\\
&- \sum_{\substack{|\gamma|+|\beta|\leq|\alpha|+1\\ 1\leq|\beta|\\ 1\leq|\gamma|\leq N-\frac{n+2}{2}}} q_{\beta\gamma}Z^\gamma(\phi)\Bigg(\sum_{\substack{|\kappa|+|\sigma|\leq|\beta|+1\\ 1\leq|\kappa|,\,1\leq|\sigma|\leq|\beta|}} \boldsymbol{T}_1(Z^\kappa\phi)q_{\sigma\kappa}\widehat{Z}^\sigma(f)\\
&\qquad\qquad + \sum_{\substack{|\kappa|+|\sigma|\leq|\beta|+1\\ 0\leq\omega\leq n,\,1\leq|\sigma|\leq|\beta|}} \frac{1}{v^0}p^\omega_{\kappa\sigma}\,\partial_{x^\omega}Z^\kappa(\phi)\widehat{Z}^\sigma f\\
&\qquad\qquad + \sum_{|\kappa|+|\sigma|=|\beta|} r_{\kappa\sigma}\boldsymbol{T}_1(Z^\kappa\phi)\widehat{Z}^\sigma f\Bigg).
\end{aligned}$$

Proof. This formula is a direct consequence of the product rule and a double application of Lemma 4.11. □

Based on Lemma 4.21, we now proceed to the estimates on g:

Lemma 4.22. *Assume that $\delta = \varepsilon^{\frac{1}{4}}$, and let α be a multi-index such that $|\alpha| \geq \lfloor \frac{N}{2} \rfloor + 1$. Then, g^α satisfies, for all $\rho \in [1, P]$,*

$$\mathbb{E}[|g^\alpha|](\rho) \leq (1 + C\varepsilon^{\frac{1}{4}}) e^{C\varepsilon^{1/2}} \varepsilon \rho^{\varepsilon^{1/4}},$$

where C is a constant depending only on the regularity and the dimension. If, furthermore $n > 4$, then δ can be vanishing.

Proof. Applying Lemmata 4.20 and 4.21, we obtain L^1 estimates for g^α provided we can control the source terms. The worse terms that have to be estimated are integrals in ρ of quantities of the forms

$$\int_{H_\rho} \int_v v^0 q_{\beta\gamma} |\partial Z^\gamma \phi| |\widehat{Z}^\beta f| \, dv \, d\mu_{H_\rho} \qquad \text{for } |\gamma| > N - \frac{n+2}{2}, \ |\beta| \leq N+1-|\gamma|, \tag{99}$$

$$\int_{H_\rho} \int_v |\boldsymbol{T}_\phi(q_{\beta\gamma})| |Z^\gamma \phi| |\widehat{Z}^\beta f| \, dv \, d\mu_{H_\rho} \qquad \text{for } |\gamma| \leq N - \frac{n+2}{2}, \ |\beta| \leq N+1-|\gamma|, \tag{100}$$

$$\int_{H_\rho} \int_v v^0 |q_{\beta\gamma} q_{\kappa\sigma}| |Z^\gamma \phi| |\partial Z^\kappa \phi| |\widehat{Z}^\sigma f| \, dv \, d\mu_{H_\rho} \quad \text{for } |\gamma|, |\kappa| \leq N - \frac{n+2}{2}, \ |\sigma| \leq N+1-|\kappa|, \tag{101}$$

$$\int_{H_\rho} \int_v v^0 |q_{\beta\gamma} q_{\kappa\sigma}| |Z^\gamma \phi| |\partial Z^\kappa \phi| |\widehat{Z}^\sigma f| \, dv \, d\mu_{H_\rho} \quad \text{for } |\gamma|, |\kappa| > N - \frac{n+2}{2}, \ |\sigma| \leq N+1-|\kappa|. \tag{102}$$

The other error terms are easier to handle, so, as an illustration, we will only give details below for the extra error terms

$$\int_{H_\rho} \int_v \frac{1}{v^0} |\partial_t \phi| |g^\alpha| \, dv \, d\mu_{H_\rho}, \tag{103}$$

$$\int_{H_\rho} \int_v |\boldsymbol{T}_1 \phi| |g^\alpha| \, dv \, d\mu_{H_\rho}. \tag{104}$$

We deal first with equation (99). To this end, recall that

$$|q_{\beta\gamma}| \lesssim t.$$

Furthermore, since

$$|\beta| \leq \frac{n+2}{2} \leq \left\lfloor \frac{N}{2} \right\rfloor - n \quad \text{since } N \geq 3n+4,$$

the Klainerman–Sobolev estimates of Theorem 8 on page 1564 can be applied, because the bootstrap assumption (74) is satisfied. Note that the theorem is applied estimating by the low-order part of the energy, which allows for the absorption of the additional v^0, as pointed out in Remark 4.4:

$$\begin{aligned}
\int_{H_\rho} \int_v v^0 q_{\beta\gamma} |\partial Z^\gamma \phi| |\widehat{Z}^\beta f| \, dv \, d\mu_{H_\rho} &\lesssim \int_{H_\rho} \int_v v^0 t |\partial Z^\gamma \phi| |\widehat{Z}^\beta f| \, dv \, d\mu_{H_\rho} \\
&\lesssim \int_{H_\rho} \frac{\varepsilon \rho^\delta}{t^{n-1}} \left(\frac{t}{\rho}\right)^{\frac{1}{2}} \left(\frac{\rho}{t}\right)^{\frac{1}{2}} |\partial Z^\gamma \phi| \, d\mu_{H_\rho} \\
&\lesssim \varepsilon^{\frac{3}{2}} \rho^{\delta - \frac{1}{2}} \left(\int_0^\infty t^{3-2n} r^{n-1} \, dr \right)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{3}{2}} \rho^{\delta + 1 - \frac{n}{2}}.
\end{aligned}$$

We now deal with (100). To this end, we notice that since $|\partial\phi|$ decays faster than $1/t$, we have

$$|\boldsymbol{T}_\phi(q_{\beta\gamma})| \lesssim v^0.$$

Using Lemma 4.14 and the bootstrap assumption (73), we obtain

$$\begin{aligned}\int_{H_\rho}\int_v |\boldsymbol{T}_\phi(q_{\beta\gamma})||Z^\gamma\phi||\widehat{Z}^\beta f|\,dv\,d\mu_{H_\rho} &\lesssim \int_{H_\rho}\int_v |Z^\gamma\phi||v^0\widehat{Z}^\beta f|\,dv\,d\mu_{H_\rho}\\ &\lesssim \int_{H_\rho}\frac{\varepsilon^{\frac12}\rho}{t^{\frac n2}}\cdot\frac{t}{\rho}\chi_1(|\widehat{Z}^\beta f|)\,d\mu_{H_\rho} \lesssim \varepsilon^{\frac32}\rho^{\delta+1-\frac n2}.\end{aligned}$$

We have in the course of the estimate used Remark 2.12.

Consider now the term (101) and recall that

$$|q_{\beta\gamma}q_{\kappa\sigma}| \lesssim t^2.$$

Assume first that $|\kappa| \le N-\frac12(n+2)$. Using Lemma 4.14 and the bootstrap assumption (73), we obtain

$$\begin{aligned}\int_{H_\rho}\int_v v^0|q_{\beta\gamma}q_{\kappa\sigma}||Z^\gamma\phi||\partial Z^\kappa\phi||\widehat{Z}^\sigma f|\,dv\,d\mu_{H_\rho} &\lesssim \int_{H_\rho} t^2\cdot\frac{\varepsilon^{\frac12}\rho}{t^{\frac n2}}\cdot\frac{\varepsilon^{\frac12}}{\rho t^{\frac n2-1}}\cdot\frac{t}{\rho}\chi_1(|\widehat{Z}^\sigma f|)\,d\mu_{H_\rho}\\ &\lesssim \frac{\varepsilon}{\rho}\int_{H_\rho} t^{4-n}\chi_1(|\widehat{Z}^\sigma f|)\,d\mu_{H_\rho}\\ &\lesssim \varepsilon^2\rho^{\delta+3-n}.\end{aligned}$$

We have in the course of the estimate used Remark 2.12. Now, if $|\kappa| > N-\frac12(n+2)$, then $|\sigma| \le \frac12(n+2) \le \left\lfloor\frac N2\right\rfloor - n$ since $N \ge 3n+4$.

Thus,

$$\begin{aligned}\int_{H_\rho}\int_v v^0|q_{\beta\gamma}q_{\kappa\sigma}||Z^\gamma\phi||\partial Z^\kappa\phi||\widehat{Z}^\sigma f|\,dv\,d\mu_{H_\rho} &\lesssim \int_{H_\rho} t^2\cdot\frac{\varepsilon^{\frac12}\rho}{t^{\frac n2}}\frac{\varepsilon\rho^\delta}{t^n}\left(\frac{t}{\rho}\right)^{\frac12}\left(\frac{\rho}{t}\right)^{\frac12}|\partial Z^\kappa(\phi)|\,d\mu_\rho\\ &\lesssim \varepsilon^{\frac32}\rho^{\delta+\frac12}\left(\int_{H_\rho}\frac{\rho}{t}|\partial Z^\kappa(\phi)|^2\,d\mu_\rho\right)^{\frac12}\left(\int_{H_\rho}\frac{t^5}{t^{3n}}\,d\mu_\rho\right)^{\frac12}\\ &\lesssim \varepsilon^2\rho^{\delta+3-n}.\end{aligned}$$

The term (102) can be estimated similarly since

$$|\sigma| \le \frac{n+2}{2} \le \left\lfloor\frac N2\right\rfloor - n \quad \text{since } N \ge 3n+4.$$

Finally, for the error terms (103) and (104), we apply Proposition 4.12 and Remark 2.12:

$$\begin{aligned}\int_{H_\rho}\int_v \frac{1}{v^0}|\partial_t\phi||g^\alpha|\,dv\,d\mu_{H_\rho} &\lesssim \frac{\varepsilon^{\frac12}}{\rho^{\frac n2}}\mathbb{E}[|g^\alpha|](\rho),\\ \int_{H_\rho}\int_v |\boldsymbol{T}_1(\phi)g^\alpha|\,dv\,d\mu_{H_\rho} &\lesssim \int_{H_\rho}\frac{\varepsilon^{\frac12}}{\rho t^{\frac n2-1}}\frac{t}{\rho}\chi_1(|g^\alpha|)(\rho)\,d\rho \lesssim \frac{\varepsilon^{\frac12}}{\rho^{\frac n2}}\mathbb{E}[|g^\alpha|](\rho).\end{aligned}$$

After integration in ρ, we then obtain that g^α satisfies the integral inequality

$$\mathbb{E}[|g^\alpha|](\rho) - \mathbb{E}[|g^\alpha|](1) \lesssim \int_1^\rho \varepsilon^{\frac{3}{2}}\delta^{-1}\rho'^{\delta}\,d\rho' + \int_1^\rho \frac{\varepsilon^{\frac{1}{2}}}{\rho'^{\frac{n}{2}}}\mathbb{E}[|g^\alpha|](\rho')\,d\rho'$$

$$\leq \varepsilon + C\left(\frac{\varepsilon^{\frac{3}{2}}}{\delta}\rho^{\delta} + \int_1^\rho \frac{\varepsilon^{\frac{1}{2}}}{\rho'^{\frac{n}{2}}}\mathbb{E}[|g^\alpha|](\rho')\,d\rho'\right),$$

and thus,

$$\mathbb{E}[|g^\alpha|](\rho) \leq \varepsilon\rho^{\delta}\left(1 + C\frac{\varepsilon^{\frac{1}{2}}}{\delta}\right) + C\int_1^\rho \frac{\varepsilon^{\frac{1}{2}}}{\rho'^{\frac{n}{2}}}\mathbb{E}[|g^\alpha|](\rho')\,d\rho'$$

for some constant C depending only on the dimension and the regularity. Grönwall's lemma provides

$$\mathbb{E}[|g^\alpha|](\rho) \leq \varepsilon\rho^{\delta}\left(1 + C\frac{\varepsilon^{\frac{1}{2}}}{\delta}\right)\cdot\exp\left(C\varepsilon^{\frac{1}{2}}\int_1^\rho \frac{d\rho'}{\rho'^{\frac{n}{2}}}\right);$$

that is, if $\delta = \varepsilon^{\frac{1}{4}}$, there exists a constant $\widetilde{C}$ such that

$$\mathbb{E}[|g^\alpha|](\rho) \leq (1 + \widetilde{C}\varepsilon^{\frac{1}{4}})e^{\widetilde{C}\varepsilon^{1/2}}\varepsilon\rho^{\varepsilon^{1/4}}. \qquad \square$$

We then consider the lower-order derivatives of f, particularly since these low derivatives of f are weighted in v^0 in the energy:

Lemma 4.23. *Assume $\delta = \varepsilon^{\frac{1}{4}}$, and let α be a multi-index such that $|\alpha| \leq \lfloor \frac{N}{2} \rfloor$. Then, $(v^0)^2 g^\alpha$ satisfies, for all $\rho \in [1, P]$,*

$$\mathbb{E}\big[|(v^0)^2 g^\alpha|\big](\rho) \leq (1 + C\varepsilon^{\frac{1}{4}})e^{C\varepsilon^{1/2}}\varepsilon\rho^{\delta},$$

where C is a constant depending only on the regularity and the dimension. If, furthermore $n > 4$, then δ can be vanishing.

Proof. Let us compute first $\boldsymbol{T}_\phi((v^0)^2)$:

$$|\boldsymbol{T}_\phi((v^0)^2)| = |2v^0\boldsymbol{T}_\phi(v^0)| = \big|2v^0(\nabla^i\phi + \boldsymbol{T}_1(\phi)v^i)\,\partial_{v^i}v^0\big| = \big|2\big((v^0)^2\boldsymbol{T}_1(\phi) - v^0\partial_{x^0}\phi\big)\big| \lesssim |(v^0)^3\,\partial\phi|.$$

Using Proposition 4.12, one consequently obtains

$$|\boldsymbol{T}_\phi((v^0)^2)| \lesssim (v^0)^3\frac{\varepsilon^{\frac{1}{2}}}{\rho t^{\frac{n}{2}-1}},$$

with $(v^0)^2 g^\alpha$ satisfying the equation

$$\boldsymbol{T}_\phi((v^0)^2 g^\alpha) = (v^0)^2\boldsymbol{T}_\phi(g^\alpha) + \boldsymbol{T}_\phi((v^0)^2)g^\alpha.$$

Furthermore, since $|\alpha| \leq \frac{1}{2}N$, the source terms of the equation satisfied by g^α (see Lemma 4.21) are only of the forms ∂Z^γ or $Z^\gamma\phi$ with

$$|\gamma| \leq |\alpha| \leq \left\lfloor \frac{N}{2} \right\rfloor \leq N - \frac{n+2}{2} \quad \text{since } N \geq n+2.$$

As a consequence, for low derivatives acting on f, all the terms containing ϕ can be estimated pointwise.

The terms to be considered separately in this context are the same as in the proof of Lemma 4.22. The only difference is the term of (99), which is absent, since only low derivatives of ϕ appear in the expression of g^α. The estimates which one obtains are listed below. The arguments to perform the estimates are the same as above:

$$\int_{H_\rho}\int_v |p^i_{\gamma\beta}||\partial_i Z^\gamma\phi|(v^0)^2|\widehat{Z}^\beta f|\,dv\,d\mu_{H_\rho} \lesssim \int_{H_\rho} t\cdot\frac{\varepsilon^{\frac12}}{\rho t^{\frac{n}{2}-1}}\chi_1((v^0)^2|\widehat{Z}^\beta f|)\,d\mu_{H_\rho} \lesssim \varepsilon^{\frac32}\rho^{\delta+1-\frac{n}{2}},$$

$$\int_{H_\rho}\int_v |\boldsymbol{T}_\phi(q_{\beta\gamma})||Z^\gamma\phi|(v^0)^2|\widehat{Z}^\beta f|\,dv\,d\mu_{H_\rho} \lesssim \int_{H_\rho} \frac{\varepsilon^{\frac12}\rho}{t^{\frac{n}{2}}}\cdot\frac{t}{\rho}\chi_1((v^0)^2|\widehat{Z}^\beta f|)\,d\mu_{H_\rho} \lesssim \varepsilon^{\frac32}\rho^{\delta+1-\frac{n}{2}},$$

$$\begin{aligned}\int_{H_\rho}\int_v v^0|q_{\beta\gamma}q_{\kappa\sigma}||Z^\gamma\phi||\partial Z^\kappa\phi|(v^0)^2|\widehat{Z}^\sigma f|\,dv\,d\mu_{H_\rho}&\\ \lesssim \int_{H_\rho} t^2\cdot\frac{\varepsilon^{\frac12}\rho}{t^{\frac{n}{2}}}\cdot\frac{\varepsilon^{\frac12}}{\rho t^{\frac{n}{2}-1}}\cdot\frac{t}{\rho}\chi_1((v^0)^2|\widehat{Z}^\beta f|)\,d\mu_{H_\rho} &\lesssim \varepsilon^2\rho^{\delta+3-n},\end{aligned}$$

$$\int_{H_\rho}\int_v \frac{1}{v^0}|\partial_t\phi|(v^0)^2|g^\alpha|\,dv\,d\mu_{H_\rho} \lesssim \frac{\varepsilon^{\frac12}}{\rho^{\frac{n}{2}}}\mathbb{E}\big[(v^0)^2|g^\alpha|\big](\rho),$$

$$\int_{H_\rho}\int_v \boldsymbol{T}_\phi((v^0)^2)|g^\alpha|\,dv\,d\mu_{H_\rho} \lesssim \int_{H_\rho}\frac{\varepsilon^{\frac12}}{\rho t^{\frac{n}{2}-1}}\cdot\frac{t}{\rho}\chi_1((v^0)^2|g^\alpha|)\,d\mu_{H_\rho} \lesssim \frac{\varepsilon^{\frac12}}{\rho^{\frac{n}{2}}}\mathbb{E}\big[(v^0)^2|g^\alpha|\big](\rho).$$

Altogether $(v^0)^2 g^\alpha$ satisfies the integral inequality

$$\mathbb{E}\big[|(v^0)^2 g^\alpha|\big](\rho) - \mathbb{E}\big[|(v^0)^2 g^\alpha|\big](1) \lesssim \int_1^\rho \varepsilon^{\frac32}\rho'^{\delta+1-\frac{n}{2}}\,d\rho' + \int_1^\rho \frac{\varepsilon^{\frac12}}{\rho'^{\frac{n}{2}}}\mathbb{E}\big[(v^0)^2|g^\alpha|\big](\rho')\,d\rho'.$$

The conclusion is obtained in a similar fashion as in the end of the proof of Lemma 4.22. □

Proposition 4.24. *Assume $\delta = \varepsilon^{\frac14}$. For all $\rho \in [1, P]$, we have*

- *if $n > 4$,*

$$E_N[f](\rho) \le \frac{(1 + C\varepsilon^{\frac14})e^{C\varepsilon^{1/2}}\varepsilon\rho^{\widetilde{C}\varepsilon^{1/4}}}{1 - C\varepsilon^{\frac12}},$$

- *and if $n = 4$,*

$$E_N[f](\rho) \le \frac{(1 + C\varepsilon^{\frac14})e^{C\varepsilon^{1/2}}\varepsilon\rho^{\widetilde{C}\varepsilon^{1/4}}}{1 - C\varepsilon^{\frac12}},$$

where C is a constant depending on the dimension and the regularity, and $\widetilde{C} = 0$ when $n > 4$ and $\widetilde{C} = 1$ when $n = 4$. In particular, for ε small enough, in dimension $n > 4$, for all $\rho \in [1, P]$,

$$E_N[f](\rho) \le \tfrac32\varepsilon,$$

and, in dimension 4,

$$E_N[f](\rho) \le \tfrac32\varepsilon\rho^{\varepsilon^{1/4}}.$$

Proof. Recall the definition of g^α:

$$g^\alpha = Z^\alpha f - \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\beta|\\ 1\le|\gamma|\le N-\frac{n+2}{2}}} q_{\beta\gamma}(v^i/v^0,t,x)Z^\gamma\phi\widehat{Z}^\beta f.$$

By the second triangular inequality, we immediately have

$$\mathbb{E}\big[|(v^0)^p g^\alpha|\big] \ge \left|\mathbb{E}\big[|(v^0)^p\widehat{Z}^\alpha f|\big] - \mathbb{E}\left[\left|\sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\beta|\\ 1\le|\gamma|\le N-\frac{n+2}{2}}} q_{\beta\gamma}(v^i/v^0,t,x)Z^\gamma\phi(v^0)^p\widehat{Z}^\beta f\right|\right]\right|,$$

so that, using Lemma 4.14,

$$\mathbb{E}\big[|(v^0)^p\widehat{Z}^\alpha f|\big] - \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\beta|\\ 1\le|\gamma|\le N-\frac{n+2}{2}}} \mathbb{E}\left[\left|t\cdot\frac{C\varepsilon\rho}{t^{\frac{n}{2}}}(v^0)^p\widehat{Z}^\beta f\right|\right] \le \mathbb{E}\big[|(v^0)^p g^\alpha|\big],$$

$$\mathbb{E}\big[|(v^0)^p\widehat{Z}^\alpha f|\big] - C\varepsilon^{\frac{1}{2}}\rho^{2-\frac{n}{2}} \sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ 1\le|\beta|\\ 1\le|\gamma|\le N-\frac{n+2}{2}}} \mathbb{E}\big[|(v^0)^p\widehat{Z}^\beta f|\big] \le \mathbb{E}\big[|(v^0)^p g^\alpha|\big]$$

for some constants C.

We now split the sum above between $|\beta|=|\alpha|$ and $|\beta|<|\alpha|$ and sum over the multi-indices $|\alpha|\le N$, taking $p=2$ for $|\alpha|\le\lfloor\frac{N}{2}\rfloor$ and $p=0$ otherwise to build the energy $E_N[f]$. One gets, using the bootstrap assumptions (74), as well as Lemmata 4.22 and 4.23, for all ρ in $[1, P]$,

$$(1-C\varepsilon^{\frac{1}{2}})E_N[f](\rho) \le (1+\widetilde{C}\varepsilon^{\frac{1}{4}})e^{\widetilde{C}\varepsilon^{\frac{1}{2}}}\varepsilon\rho^\delta + C\varepsilon^{\frac{3}{2}}\rho^{\delta+2-\frac{n}{2}},$$

where C is a constant (possibly different from the one above) depending only on the dimension and the regularity. Note finally that the ρ-loss is present only in dimension 4.

As a consequence, for all ρ in $[1, P]$, if $n>4$,

$$E_N[f](\rho) \le \frac{\le (1+\widetilde{C}\varepsilon^{\frac{1}{4}})e^{\widetilde{C}\varepsilon^{\frac{1}{2}}}\varepsilon\rho^\delta}{1-C\varepsilon^{\frac{1}{2}}},$$

and, if $n=4$,

$$E_N[f](\rho) \le \frac{\le (1+\widetilde{C}\varepsilon^{\frac{1}{4}})e^{\widetilde{C}\varepsilon^{\frac{1}{2}}}\varepsilon\rho^\delta}{1-C\varepsilon^{\frac{1}{2}}}.$$

□

4E7. *L^2-estimates for the transport equation.* Consider here the vector X defined by

$$X = (\widehat{Z}^{\alpha_1}f,\dots,\widehat{Z}^{\alpha_q}f) \quad \text{with } |\alpha_1| \ge \left\lfloor\frac{N}{2}\right\rfloor - n + 1 \text{ and } |\alpha_q| = N, \tag{105}$$

where the multi-index α goes over all the multi-indices of length larger than $\lfloor \frac{N}{2} \rfloor - n + 1$. Using the same notation, we introduce the vector G^h:

$$G^h = (g^{\alpha_1}, \dots, g^{\alpha_q}) \quad \text{with } |\alpha_1| \geq \left\lfloor \frac{N}{2} \right\rfloor - n + 1 \text{ and } |\alpha_q| = N,$$

where g^α has been defined by equation (98). Consider finally the vector H defined by

$$H = (v^0 \widehat{Z}^{\alpha_1} f, \dots, v^0 \widehat{Z}^{\alpha_q} f) \quad \text{with } |\alpha_1| = 0 \text{ and } |\alpha_q| \leq \left\lfloor \frac{N}{2} \right\rfloor - n.$$

In a similar fashion as above, let us now consider the vector G^l defined by

$$G^l = (v^0 g^{\alpha_1}, \dots, v^0 g^{\alpha_q}) \quad \text{with } |\alpha_1| = 0 \text{ and } |\alpha_q| \leq \left\lfloor \frac{N}{2} \right\rfloor - n.$$

In the following, we will denote by $|A|_\infty$ the supremum over all the components of a vector or matrix A. Recall that, if A and B are two matrices, then,

$$|AB|_\infty \lesssim |A|_\infty |B|_\infty.$$

Throughout this section, an inequality of the form

$$|A| \lesssim \frac{1}{1 - \varepsilon^{\frac{1}{2}}}$$

appears often for some quantity A. Since we have assumed that

$$\varepsilon^{\frac{1}{2}} \leq \tfrac{1}{2},$$

$|A|$ can be bounded by 2, and we can ignore the dependency on the upper bound when $1 - \varepsilon^{\frac{1}{2}}$ appears in the denominator. We can then write

$$|A| \lesssim 1.$$

In the same spirit, $e^{C\sqrt{\varepsilon}}$ is treated as a constant.

The relation between the vectors X, H, G^h, G^l is now stated in the following lemma:

Lemma 4.25. *Assume that ε is sufficiently small. Then, the following relations between G^h, G^l, H, X hold*:

(1) *There exists a square matrix A^l such that*
- $G^l = H - A^l H$,
- *A satisfies*

$$|A^l|_\infty \lesssim \frac{\varepsilon^{\frac{1}{2}} \rho}{t^{\frac{n}{2}-1}},$$

- *if ε is small enough, then $1 - A^l$ is invertible, and*

$$|(1 - A^l)^{-1}|_\infty \lesssim 1.$$

(2) *There exist a square matrix A' and a rectangular matrix A'' such that*
- $G^h = X - A'X - A''H$,

- *(A') and (A'') satisfy*

$$|A'_{ij}|_\infty \lesssim \frac{\varepsilon^{\frac{1}{2}}\rho}{t^{\frac{n}{2}-1}} \quad \textit{and} \quad |A''_{ij}|_\infty \lesssim \frac{\varepsilon^{\frac{1}{2}}\rho}{v^0 t^{\frac{n}{2}-1}},$$

- *if ε is small enough, then $1-A'$ is invertible, and*

$$|(1-A')^{-1}|_\infty \lesssim 1.$$

Proof. The proof of the algebraic relations between G^l, G^h, X, and H is a direct consequence of the definition of g^α, as stated in equation (98). The components of A are of the form $q_{\beta\gamma}Z^\gamma\phi$ with $|\gamma| \leq N - \lfloor \frac{n}{2} \rfloor - 1$ (see Lemma 4.11 for the definition of $q_{\beta\gamma}$). Since $q_{\beta\gamma} \lesssim t$, the decay estimates for $Z^\gamma\phi$ (see Lemma 4.14), as well as the bootstrap assumptions (73), provide the estimates on the components of A^l, A', and A''. Standard algebraic manipulations ensure the invertibility of the square matrices $1-A^l$ and $1-A'$, and

$$|A^l|_\infty \lesssim \frac{1}{1-\varepsilon^{\frac{1}{2}}}, \quad |A'|_\infty \lesssim \frac{1}{1-\varepsilon^{\frac{1}{2}}}. \qquad \square$$

Lemma 4.26. *The commutator relation of Lemma 4.21 can be rewritten as*

$$\boldsymbol{T}_\phi G^h + \bar{\boldsymbol{A}} X = \bar{\boldsymbol{B}} H,$$

where:

- *$\bar{\boldsymbol{A}} = (\bar{A}_{ij})$ is square matrix, depending on (t,x,v), whose components can be bounded by*

$$|\bar{A}_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}}v^0}{\rho t^{\frac{n}{2}-1}} \quad \textit{or} \quad |\bar{A}_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{v^0 \rho t^{\frac{n}{2}-2}} \quad \textit{or} \quad |\bar{A}_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}} v^0 \rho}{t^{\frac{n}{2}}} \quad \textit{or} \quad |\bar{A}_{ij}| \lesssim \frac{\varepsilon v^0}{t^{n-3}};$$

  in particular, for any regular distribution function g,

$$\int_{H_\rho}\int_v |\bar{A}_{ij} g|\, dv\, d\mu_{H_\rho} \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\rho^{\frac{n}{2}-1}} \int_{H_\rho} \chi_1(|g|)\, d\mu_{H_\rho}.$$

- *$\bar{\boldsymbol{B}}$ is rectangular matrix, depending on (t,x,v), whose components can be bounded by*

$$|\bar{B}_{ij}| \lesssim t|f_{ij}| \quad \textit{with } \|f_{ij}\|_{L^2(H_\rho)} \lesssim \varepsilon^{\frac{1}{2}}.$$

Proof. The proof of this lemma consists essentially in rearranging the terms of the commutator formula stated in Lemma 4.21 and in the relation

$$\boldsymbol{T}_\phi(v^0) = v^0(\boldsymbol{T}_1(\phi) - \partial_t\phi),$$

which can be bounded by means of Lemma 4.14 by

$$|\boldsymbol{T}_\phi(v^0)| \lesssim \frac{(v^0)^2\varepsilon^{\frac{1}{2}}}{t^{\frac{n}{2}-1}\rho}.$$

Recall furthermore that, if $\widehat{Z}^\beta f$ can be estimated pointwise, then $\partial Z^\gamma\phi$ should be estimated in energy, as explained in the proof of Lemma 4.22. The estimates which then follow from splitting are obtained much as in the proof of Lemma 4.22. To understand the components of the matrices $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$, and the related estimates, we finally provide the following examples:

- The matrix $\boldsymbol{A}$ contains the terms (here in absolute value)

$$
\begin{aligned}
|r_{\beta\gamma}\boldsymbol{T}_1(Z^\gamma\phi)| &\lesssim \frac{\varepsilon^{\frac{1}{2}}v^0}{\rho t^{\frac{n}{2}-2}} && \text{for } |\gamma|\le N-\frac{n+2}{2},\\
\left|\frac{1}{v^0}p^{\delta}_{\gamma\beta}\partial_\delta(Z^\gamma\phi)\right| &\lesssim \frac{\varepsilon^{\frac{1}{2}}}{v^0\rho t^{\frac{n}{2}-2}} && \text{for } |\gamma|\le N-\frac{n+2}{2},\\
|\boldsymbol{T}_\phi(q_{\beta\gamma}Z^\gamma\phi)| &\lesssim \frac{\varepsilon^{\frac{1}{2}}v^0\rho}{t^{\frac{n}{2}}} && \text{for } |\gamma|,|\kappa|\le N-\frac{n+2}{2},\\
|q_{\beta\gamma}q_{\kappa\sigma}Z^\gamma\phi\boldsymbol{T}_1(Z^\kappa\phi)| &\lesssim \frac{\varepsilon v^0}{t^{n-3}} && \text{for } |\gamma|,|\kappa|\le N-\frac{n+2}{2}.
\end{aligned}
$$

- The matrix $\boldsymbol{B}$ contains the terms (here in absolute value)

$$
\frac{1}{v^0}q_{\beta\gamma}\boldsymbol{T}_1(Z^\gamma\phi)\quad \text{with } |\gamma|>N-\frac{n+2}{2},
$$

  where $|q_{\beta\gamma}|\lesssim t$, and $\|(v^0)^{-1}\boldsymbol{T}_1(Z^\gamma\phi)\|^2_{L^2(H_\rho)}\lesssim\varepsilon$. $\square$

Consider now the case when the operator $\boldsymbol{T}_\phi$ acts on the vector G^l:

Lemma 4.27. *There exists a square matrix $\hat{\boldsymbol{A}}$ such that*

$$
\boldsymbol{T}_\phi G^l = \hat{\boldsymbol{A}}G^l.
$$

The components $(\hat{A}_{ij})$ of the matrix $\hat{\boldsymbol{A}}$ satisfy

$$
|\hat{A}_{ij}|\lesssim\frac{\varepsilon^{\frac{1}{2}}v^0}{\rho t^{\frac{n}{2}-1}}\quad or\quad |\hat{A}_{ij}|\lesssim\frac{\varepsilon^{\frac{1}{2}}}{v^0\rho t^{\frac{n}{2}-2}}\quad or\quad |\hat{A}_{ij}|\lesssim\frac{\varepsilon^{\frac{1}{2}}v^0\rho}{t^{\frac{n}{2}}}\quad or\quad |\hat{A}_{ij}|\lesssim\frac{\varepsilon^{\frac{1}{2}}v^0}{t^{n-3}};
$$

in particular, for any regular distribution function g,

$$
\int_{H_\rho}\int_v |\hat{A}_{ij}g|\,dv\,d\mu_{H_\rho}\lesssim\frac{\varepsilon^{\frac{1}{2}}}{\rho^{\frac{n}{2}-1}}\int_{H_\rho}\chi_1(|g|)\,d\mu_{H_\rho}.
$$

Proof. This equation essentially relies on the commutator formula of Lemma 4.21. Note that for this formula, since α is very low, the first term of the right-hand side of the commutator formula

$$
\sum_{\substack{|\gamma|+|\beta|\le|\alpha|+1\\ |\gamma|\ge1,\,|\beta|\ge1\\ |\gamma|\ge N-\lfloor n/2\rfloor}} q_{\beta\gamma}\boldsymbol{T}_1(Z^\gamma\phi)\widehat{Z}^\beta(f)
$$

does not appear in the formula. Following the arguments of Lemma 4.23, in the situation when the number of derivatives is low $\big(\text{smaller than } \lfloor\frac{N}{2}\rfloor-n\big)$, the derivatives of the wave equation can all be estimated pointwise. Proposition 4.12 and Lemma 4.14 provide a relation of the form

$$\boldsymbol{T}_\phi G^l = \hat{\boldsymbol{A}}' H,$$

where $\hat{\boldsymbol{A}}'$ satisfies the same properties as $\bar{\boldsymbol{A}}$ in Lemma 4.26. Finally, we use the relation between G^l and H stated in Lemma 4.25. □

Lemma 4.28. *There exists a square matrix $\boldsymbol{A}$ and a rectangular matrix $\boldsymbol{B}$ satisfying*

- *$\boldsymbol{A} = (A_{ij})$ is square matrix, depending on (t, x, v) whose components can be bounded by*

$$|A_{ij}| \lesssim \varepsilon^{\frac{1}{2}} \cdot \frac{v^0}{\rho t^{\frac{n}{2}-1}} \quad \text{or} \quad |A_{ij}| \lesssim \varepsilon^{\frac{1}{2}} \cdot \frac{1}{v^0 \rho t^{\frac{n}{2}-2}} \quad \text{or} \quad |A_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}} v^0 \rho}{t^{\frac{n}{2}}} \quad \text{or} \quad |A_{ij}| \lesssim \varepsilon \cdot \frac{v^0}{t^{n-3}};$$

  in particular, for any regular distribution function g,

$$\int_{H_\rho} \int_v |A_{ij} g| \, dv \, d\mu_{H_\rho} \lesssim \frac{1}{\rho^{\frac{n}{2}-1}} \varepsilon^{\frac{1}{2}} \int_{H_\rho} \chi_1(|g|) \, d\mu_{H_\rho},$$

- *$\boldsymbol{B}$ is a rectangular matrix, depending on (t, x, v) whose components can be bounded by*

$$|B_{ij}| \lesssim t |f_{ij}| \quad \text{with } \|f_{ij}\|_{L^2(H_\rho)} \lesssim \varepsilon^{\frac{1}{2}}$$

such that the vector G^h satisfies the equation

$$\boldsymbol{T}_\phi G^h + \boldsymbol{A} G^h = \boldsymbol{B} G^l.$$

Proof. The proof relies on the combination of Lemmata 4.25, 4.26, and 4.27. Assuming ε small enough that the matrices $1 - A$ and $1 - A'$ are invertible, and substituting the expressions of X and H into the functions G^h and G^l in the equation satisfied by G^h stated in Lemma 4.26, one obtains the equation

$$\boldsymbol{T}_\phi G^h + \boldsymbol{A} \cdot (1 - A')^{-1} \cdot G^h = \boldsymbol{B} \cdot (1 - A)^{-1} \cdot G^l + \boldsymbol{A} \cdot (1 - A')^{-1} \cdot A'' \cdot G^l.$$

Since the components of the matrices $(1 - A')^{-1}$ and $(1 - A)^{-1}$ are both bounded by $1/(1 - \varepsilon)$, the components of the matrices $\boldsymbol{A} \cdot (1 - A')^{-1}$ and $\boldsymbol{B} \cdot (1 - A)^{-1}$ satisfy the same properties as the $\boldsymbol{A}$ and $\boldsymbol{B}$, up to constant $1/(1 - \varepsilon)$. We finally consider $\boldsymbol{C} = \boldsymbol{A} \cdot (1 - A')^{-1} \cdot A''$, whose components can be bounded by means of Lemma 4.14, as follows (we remind the reader that A'' contains a $(v^0)^{-1}$):

$$|C_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{1 - \varepsilon^{\frac{1}{2}}} \cdot \frac{\varepsilon^{\frac{1}{2}}}{\rho t^{\frac{n}{2}-1}} \cdot \frac{\varepsilon^{\frac{1}{2}} \rho}{t^{\frac{n}{2}}} \quad \text{or} \quad |C_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{1 - \varepsilon^{\frac{1}{2}}} \cdot \frac{\varepsilon^{\frac{1}{2}}}{\rho t^{\frac{n}{2}-2}} \cdot \frac{\varepsilon^{\frac{1}{2}} \rho}{t^{\frac{n}{2}}} \quad \text{or} \quad |C_{ij}| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{1 - \varepsilon^{\frac{1}{2}}} \cdot \frac{\varepsilon^{\frac{1}{2}}}{t^{n-3}} \cdot \frac{\varepsilon^{\frac{1}{2}} \rho}{t^{\frac{n}{2}}}.$$

One now easily notices that they can all be bounded by terms of the form

$$|C_{ij}| \lesssim t |f_{ij}| \quad \text{with } \|f_{ij}\|_{L^2(H_\rho)} \lesssim \frac{\varepsilon^{\frac{1}{2}}}{1 - \varepsilon^{\frac{1}{2}}}.$$

□

Decomposition of the solution. We consider now the set of equations of the form

$$\boldsymbol{T}_\phi G^h + \boldsymbol{A} G^h = \boldsymbol{B} G^l, \quad \text{with initial data } G^h(\rho{=}1), \tag{106}$$

where all matrices are as above. To obtain estimates on the solution G^h, we split it into two parts

$$G^h = G + G_{\mathsf{hom}}, \tag{107}$$

where G_{hom} and G are respective solutions to the Cauchy problems with initial data on the hyperboloid H_1:

$$\boldsymbol{T}_\phi G_{\mathsf{hom}} + \boldsymbol{A} G_{\mathsf{hom}} = 0 \qquad \text{with } G_{\mathsf{hom}} = G^h, \tag{108}$$

$$\boldsymbol{T}_\phi G + \boldsymbol{A} G = \boldsymbol{B} G^l \quad \text{with } G = 0. \tag{109}$$

We proceed by evaluating both components individually.

Homogeneous part. Commuting equation (108) n-times with the vector field $\widehat{Z}$ yields

$$\boldsymbol{T}_\phi(\widehat{Z}^\alpha G_{\mathsf{hom}}) = -\widehat{Z}^\alpha(\boldsymbol{A} G_{\mathsf{hom}}) + [\boldsymbol{T}_\phi, \widehat{Z}^\alpha] G_{\mathsf{hom}} \tag{110}$$

for $|\alpha| \le n$. Applying estimate (97) directly to this equation would yield problematic terms in the estimate for the same reason as discussed after (98). As before, we introduce an auxiliary function g^α_{hom} analogous to that defined in (97). Applying estimate (97) to the transport equation of g^α_{hom} yields

$$\mathbb{E}[|g^\alpha_{\mathsf{hom}}|](\rho) \le \varepsilon(1 + C\varepsilon) e^{C\sqrt{\varepsilon}} \tag{111}$$

for $|\alpha| \le n$. In turn, for G_{hom} we obtain, for $n > 4$,

$$\sum_{|\alpha| \le n} \mathbb{E}[|\widehat{Z}^\alpha G_{\mathsf{hom}}|] \le \frac{\varepsilon(1 + C\varepsilon^{\frac{1}{2}}) e^{C\varepsilon^{\frac{1}{2}}}}{1 - \varepsilon^{\frac{1}{2}}},$$

and, if $n = 4$,

$$\sum_{|\alpha| \le n} \mathbb{E}[|\widehat{Z}^\alpha G_{\mathsf{hom}}|] \le \frac{\varepsilon(1 + C\varepsilon^{\frac{1}{2}}) e^{C\varepsilon^{\frac{1}{2}}} \rho^\delta}{1 - \varepsilon^{\frac{1}{2}}},$$

where the constant C does depend only on the dimension and the regularity N.

This yields

$$\|G_{\mathsf{hom}}\|_{\mathbb{P},n}(\rho) \le \begin{cases} \dfrac{\varepsilon(1 + C\varepsilon) e^{C\varepsilon}}{1 - \varepsilon^{\frac{1}{2}}} & n > 4, \\[2ex] \dfrac{\varepsilon(1 + C\varepsilon) e^{C\varepsilon} \rho^\delta}{1 - \varepsilon^{\frac{1}{2}}} & n = 4. \end{cases} \tag{112}$$

In combination with the Klainerman–Sobolev estimate (42) this implies the following lemma.

Lemma 4.29. *The following estimate holds*:

$$\int_{v\in\mathbb{R}^n} |G_{\mathsf{hom}}| \frac{dv}{v^0} \leq \begin{cases} \dfrac{\varepsilon(1+C\varepsilon^{\frac{1}{2}})e^{C\varepsilon^{\frac{1}{2}}}}{(1+t)^n} & n>4, \\ \dfrac{\varepsilon(1+C\varepsilon^{\frac{1}{2}})e^{C\varepsilon^{\frac{1}{2}}}\rho^{\delta}}{(1+t)^n} & n=4, \end{cases} \tag{113}$$

where the constant C does depend only on the dimension and the regularity N.

Inhomogeneous part. Before giving the full proof of the actual L^2-decay estimates for the inhomogeneous part of G^h, let us explain the main ideas on a simple model problem. Assume that T is a transport operator such as the relativistic transport operator or even just the classical one and that f is a function of (t,x,v) satisfying

$$\boldsymbol{T}(f) = hg,$$

where $h = h(t,x)$ is uniformly bounded in L^2_x and such that g is itself a solution to the free transport equation $\boldsymbol{T}(g) = 0$ with g regular enough that $L^1_{x,v}$-bounds hold for g and decay estimates similar to our Klainerman–Sobolev inequality can be applied for the velocity averages of g. The aim is to prove L^2_x-decay estimates on $\int_v |f|\,dv$, the difficulty being that h has very little regularity so that we cannot commute the equation. Instead, note that, by uniqueness, $f = gH$, where H is the solution to the inhomogeneous transport equation $\boldsymbol{T}(H) = h$ with zero data. Indeed,

$$\boldsymbol{T}(gH) = \boldsymbol{T}(g)H + g\boldsymbol{T}(H) = gh,$$

since $\boldsymbol{T}(g) = 0$. Now,

$$\left\| \int_v gH\,dv \right\|_{L^2_x} \lesssim \left\| \left(\int_v |g|\,dv \right)^{\frac{1}{2}} \left(\int_v |g|H^2\,dv \right)^{\frac{1}{2}} \right\|_{L^2_x} \lesssim \left\| \left(\int_v |g|\,dv \right)^{\frac{1}{2}} \right\|_{L^\infty_x} \left\| \int_v |g|H^2\,dv \right\|^2_{L^1_x}.$$

Since we have assumed g to solve the free transport equations and to be as regular as needed, we know that we have some decay for $\left\| \left(\int_v |g|\,dv \right)^{\frac{1}{2}} \right\|_{L^\infty_x}$. Thus, it remains only to prove boundedness for $\left\| \int_v |g|H^2\,dv \right\|_{L^1_x}$. This can be obtained using again the transport equation for gH and the associated approximate conservation laws. Indeed, we have

$$\boldsymbol{T}(gH^2) = 2ghH,$$

and thus, we need to estimate an integral of the form $\int_{t,x,v} |ghH|\,dt\,dx\,dv$. This is done as follows. First,

$$\begin{aligned} \int_{t,x,v} |ghH|\,dt\,dx\,dv &= \int_t \int_{x,v} |g|^{\frac{1}{2}}|h||g|^{\frac{1}{2}}H\,dx\,dv\,dt \\ &\lesssim \int_t \left(\int_{x,v} |g||h|^2\,dx\,dv \right)^{\frac{1}{2}} \left(\int_{x,v} |g|H^2\,dx\,dv \right)^{\frac{1}{2}} dt. \end{aligned}$$

It follows that, if one can obtain enough decay for $\left(\int_{x,v}|g|(x,v)|h|^2(x)\,dx\,dv\right)^{\frac{1}{2}}$, then the estimate can close via a Grönwall-type inequality. For the decay estimate, simply note again that

$$\left|\int_{x,v}|g|(t,x,v)|h|^2(t,x)\,dx\,dv\right| \lesssim \left\|\int_v g\,dv\right\|_{L^\infty_x}\|h(t,x)\|^2_{L^2_x}.$$

This concludes the discussion of the estimates for the model problem. To estimate the inhomogeneous part of G^h, we will essentially follow this strategy except that

- we need to work with systems;
- the operator $\boldsymbol{T}$ needs to be replaced by $\boldsymbol{T}_\phi$ (or rather $\boldsymbol{T}_\phi + A$);
- the matrix $\boldsymbol{B}$ replacing h is not uniformly bounded in L^2_x (there is a t-loss);
- the vector replacing g does not quite satisfy a homogeneous transport equation;
- and finally, in all steps, we need to keep track of the exact decay rates in ρ to make sure the time integrals converge.

To perform the estimates on G, we first notice the following useful decomposition. Let $\boldsymbol{K}$ be the matrix solution to the equation

$$\boldsymbol{T}_\phi\boldsymbol{K} + \boldsymbol{A}\boldsymbol{K} + \boldsymbol{K}\hat{\boldsymbol{A}} = \boldsymbol{B}, \quad \text{with } \boldsymbol{K}|_{H_1} = 0. \tag{114}$$

Then, an immediate calculation proves that the vector $\boldsymbol{K}G^l$ satisfies the equation

$$\boldsymbol{T}_\phi(\boldsymbol{K}G^l) + \boldsymbol{A}(\boldsymbol{K}G^l) = \boldsymbol{B}G^l, \quad \text{with } (\boldsymbol{K}G^l)|_{H_1} = 0.$$

By uniqueness of solutions to the Cauchy problem and (109), we obtain

$$G = \boldsymbol{K}G^l. \tag{115}$$

Before performing the estimates on G, let us remind that

- using the bootstrap assumption (74) and Theorem 8 on page 1564, the elements of the vector G^l can be estimated pointwise, and, by Lemma 4.27,
$$\int_v |G^l|_\infty\,dv \lesssim \varepsilon t^{\delta-n},$$
- the components of the source terms in the equation satisfied by $\boldsymbol{K}$ can be estimated in $L^2(H_\rho)$ (see Lemma 4.28).

Following the strategy described in the Introduction, we introduce the scalar function

$$|\boldsymbol{K}\boldsymbol{K}\boldsymbol{G}|_\infty = \sum_{\alpha,\beta,\gamma,\kappa,\mu} |K^\beta_\alpha K^\kappa_\gamma G^l_\mu|,$$

where K^β_α, K^κ_γ are the components of the matrix $\boldsymbol{K}$ and G^l_μ are the components of the vector G^l, and where the sum is taken over all possible combinations of two elements of $\boldsymbol{K}$ and G^l. One furthermore easily checks that each element of this sum satisfies the equation

$$T_\phi(K^\beta_\alpha K^\kappa_\gamma G^l_\mu) = (A^\sigma_\alpha K^\beta_\sigma + K^\sigma_\beta \hat{A}^\alpha_\sigma)K^\kappa_\gamma G^l_\mu + K^\beta_\alpha (A^\sigma_\gamma K^\kappa_\sigma + K^\sigma_\gamma \hat{A}^\kappa_\sigma)G^l_\mu + K^\beta_\alpha K^\kappa_\gamma \hat{A}^\sigma_\mu G^l_\sigma \tag{116}$$

$$- (B^\alpha_\beta K^\kappa_\gamma + K^\alpha_\beta B^\kappa_\gamma)G^l_\mu. \tag{117}$$

As a consequence, we have:

Lemma 4.30. *Assume $\delta = \varepsilon^{\frac{1}{4}}$. The function $|KKG|_\infty$ satisfies*

- *for $n > 4$, for all ρ in $[1, P]$,*
$$\mathbb{E}[|KKG|_\infty](\rho) \lesssim \varepsilon^{\frac{3}{2}};$$
- *for $n = 4$, for all ρ in $[1, P]$,*
$$\mathbb{E}[|KKG|_\infty](\rho) \lesssim \varepsilon\rho^{\varepsilon^{\frac{1}{4}}}.$$

Proof. The right-hand side of (116) can easily be estimated, using the properties of the matrices A and $\hat{A}$ stated in Lemmata 4.28 and 4.27, by

$$\int_{H_\rho} \chi_1\big(\big|(A^\sigma_\alpha K^\beta_\sigma + K^\sigma_\beta \hat{A}^\alpha_\sigma)K^\kappa_\gamma G^l_\mu + K^\beta_\alpha (A^\sigma_\gamma K^\kappa_\sigma + K^\sigma_\gamma \hat{A}^\kappa_\sigma)G^l_\mu + K^\beta_\alpha K^\kappa_\gamma \hat{A}^\sigma_\mu G^l_\sigma\big|\big)\, d\mu_{H_\rho} \lesssim \frac{\sqrt{\varepsilon}}{\rho^{\frac{n}{2}-1}}\mathbb{E}[|KKG|_\infty](\rho).$$

Furthermore, by the Cauchy–Schwarz inequality, as well as the property of the matrix B stated in Lemma 4.28, one gets, when estimating (117):

$$\begin{aligned}\int_v \big(\big|(B^\alpha_\beta K^\kappa_\gamma + K^\alpha_\beta B^\kappa_\gamma)G^l_\mu\big|\big)\, dv &\lesssim |B|_\infty \chi_1(|G^l|)^{\frac{1}{2}}\chi_1(|KKG|_\infty)^{\frac{1}{2}}\\ &\lesssim |B|_\infty \frac{\varepsilon^{\frac{1}{2}}}{t^{\frac{n}{2}-\frac{\delta}{2}}}\chi_1(|KKG|_\infty)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}\int_{H_\rho}\int_v \big(\big|(B^\alpha_\beta K^\kappa_\gamma + K^\alpha_\beta B^\kappa_\gamma)G^l_\mu\big|\big)\, dv\, d\mu_{H_\rho} &\lesssim \varepsilon^{\frac{1}{2}}\rho^{\frac{\delta+2-n}{2}}\|t^{-1}B\|_{L^2(H_\rho)}\big(\mathbb{E}[|KKG|_\infty](\rho)\big)^{\frac{1}{2}}\\ &\lesssim \varepsilon\rho^{\frac{\delta+2-n}{2}}\big(\mathbb{E}[|KKG|_\infty](\rho)\big)^{\frac{1}{2}}.\end{aligned}$$

As a consequence $|KKG|_\infty$ satisfies, for all $\rho \in [1, P]$, the integral inequality

$$\begin{aligned}\mathbb{E}[|KKG|_\infty](\rho) &\lesssim \int_1^\rho \frac{\sqrt{\varepsilon}}{\rho'^{\frac{n}{2}-1}}\mathbb{E}[|KKG|_\infty](\rho') + \varepsilon\rho'^{\frac{\delta}{2}+1-\frac{n}{2}}\big(\mathbb{E}[|KKG|_\infty](\rho')\big)^{\frac{1}{2}}\, d\rho'\\ &\lesssim \int_1^\rho \frac{\sqrt{\varepsilon}}{\rho'^{\frac{n}{2}-1}}\mathbb{E}[|KKG|_\infty](\rho') + \varepsilon^{\frac{3}{4}}\rho^{\frac{\delta}{2}+\frac{1}{2}-\frac{n}{4}}\cdot\varepsilon^{\frac{1}{4}}\rho^{\frac{1}{2}-\frac{n}{4}}\big(\mathbb{E}[|KKG|_\infty](\rho')\big)^{\frac{1}{2}}\, d\rho'\\ &\lesssim \frac{\varepsilon^{\frac{3}{2}}}{\frac{n}{2}-2-\delta}(1-\rho^{\delta+2-\frac{n}{2}}) + \int_1^\rho \frac{\sqrt{\varepsilon}}{\rho'^{\frac{n}{2}-1}}\mathbb{E}[|KKG|_\infty](\rho')\, d\rho'.\end{aligned}$$

Assume now $n > 4$. Then, Grönwall's inequality implies immediately, for all ρ in $[1, P]$,

$$\mathbb{E}[|KKG|_\infty](\rho) \lesssim e^{C\varepsilon^{1/2}}\varepsilon^{\frac{3}{2}} \lesssim \varepsilon^{\frac{3}{2}}$$

for some constant C, depending only on n and N.

In the case $n = 4$, the integral inequality becomes

$$\mathbb{E}[|\boldsymbol{KKG}|_\infty](\rho) \lesssim \frac{\varepsilon^{\frac{3}{2}}}{\delta}\rho^\delta + \int_1^\rho \frac{\sqrt{\varepsilon}}{\rho'}\mathbb{E}[|\boldsymbol{KKG}|_\infty](\rho')\,d\rho'. \tag{118}$$

To perform the estimates in this case, we make the bootstrap assumption: let $P' < P$ be the maximal P such that, for all ρ in $[1, P']$,

$$\mathbb{E}[|\boldsymbol{KKG}|_\infty](\rho) \le C\varepsilon\rho^{\varepsilon^{1/4}}, \tag{119}$$

where C depends only on the dimension n and the regularity N. Inserting the bootstrap assumption (119) in (118), one gets, for $\delta = \varepsilon^{\frac{1}{4}}$,

$$\mathbb{E}[|\boldsymbol{KKG}|_\infty](\rho) \lesssim \varepsilon^{1+\frac{1}{4}}\rho^{\varepsilon^{1/4}}$$

for all $\rho < P'$. The bootstrap assumption (119) can then be improved for ε small enough. □

We can finally state the L^2-estimates for f:

Proposition 4.31. *Assume that ε is sufficiently small. Under the bootstrap assumptions (73) and (74), the following estimate holds for all multi-indices α such that $\lfloor \frac{N}{2} \rfloor - n + 1 \le |\alpha| \le N$:*

$$\int_{H_\rho} \frac{t}{\rho}\left(\int_v |\widehat{Z}^\alpha f|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho} \lesssim \rho^{2\varepsilon^{1/4}-n}\varepsilon^2.$$

Proof. We first notice that, by Lemma 4.25, for α such that $\frac{N}{2} + 1 \le |\alpha| \le N$,

$$\begin{aligned}
&\left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |\widehat{Z}^\alpha f|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}} \\
&\qquad \lesssim \frac{1}{1-\varepsilon^{\frac{1}{2}}}\left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |g^\alpha|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}} \\
&\qquad \lesssim \frac{1}{1-\varepsilon^{\frac{1}{2}}}\left\{\left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |G^\alpha_{\mathsf{hom}}|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}} + \left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |G^\alpha|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}}\right\},
\end{aligned}$$

where G^α_{hom} and G^α are the components of G_{hom} and G respectively. The first term of this sum is estimated by means of Lemma 4.29:

$$\left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |G^\alpha_{\mathsf{hom}}|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}} \lesssim \varepsilon\rho^{\frac{\delta-n}{2}}\left(\int_0^\infty \frac{y^{n-1}}{\langle y\rangle^{2n}}\,dy\right)^{\frac{1}{2}}. \tag{120}$$

The second term of the sum is estimated as follows: Let us denote by G^α the components of the vector G; we have

$$\begin{aligned}
\left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |G^\alpha|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}} &\lesssim \left(\int_{H_\rho} \frac{t}{\rho}\left(\int_v |\boldsymbol{K}^k_\alpha G^l_k|\,\frac{dv}{v^0}\right)^2 d\mu_{H_\rho}\right)^{\frac{1}{2}} \\
&\lesssim \sum_k \int_{H_\rho} \frac{t}{\rho}\left(\int_v |G^l_k|\,\frac{dv}{v^0}\right)\left(\int_v |(\boldsymbol{K}^k_\alpha)^2 G^l_k|\,\frac{dv}{v^0}\right) d\mu_{H_\rho},
\end{aligned} \tag{121}$$

where the last sum over k is taken of all the components of $\boldsymbol{K}$ and G^l and is consequently finite. In combination with the pointwise decay for G^l and the estimate in Lemma 4.30, this implies

$$\int_{H_\rho} \frac{t}{\rho}\left(\int_v |G^\alpha| \frac{dv}{v^0}\right)^2 d\mu_{H_\rho} \lesssim \rho^{2\varepsilon^{1/4}-n}\varepsilon^2. \tag{122}$$

In combination with the bound on the homogeneous part above, this yields the claim. ☐

Appendix A: Distribution functions for massive particles with compact support in x

Theorem 2, on page 1543, and Theorem 5, on page 1547, require that the initial data be given on the initial hyperboloid H_1 instead of a more traditional $t = \text{const}$ hypersurface. In this appendix, we explain how we can go from the $t = 0$ hypersurface to H_1, provided the initial data on $t = 0$ has sufficient decay in x. For simplicity, consider the homogeneous massive transport equation with initial data f_0 given at $t = 0$. Assume that the support of f_0 is contained in the ball of radius R. Without loss of generality, we may translate the problem in time, so that we now consider the problem with data at time $t = \sqrt{R^2+1}$:

$$\boldsymbol{T}_m(f) = 0, \tag{123}$$

$$f(t = \sqrt{R^2+1}) = f_0. \tag{124}$$

Now, by the finite speed of propagation, the solution to this problem vanishes outside of the cone

$$\mathcal{C}(R) \equiv \left\{(t,r,\omega) \,\middle|\, t-r = \sqrt{R^2+1}-R,\ \omega\in\mathbb{S}^{n-1},\ t\geq\sqrt{R^2+1}\right\}$$
$$\cup\left\{(t,r,\omega) \,\middle|\, t+r = \sqrt{R^2+1}+R,\ \omega\in\mathbb{S}^{n-1},\ t\leq\sqrt{R^2+1}\right\}$$

depicted in Figure 3.

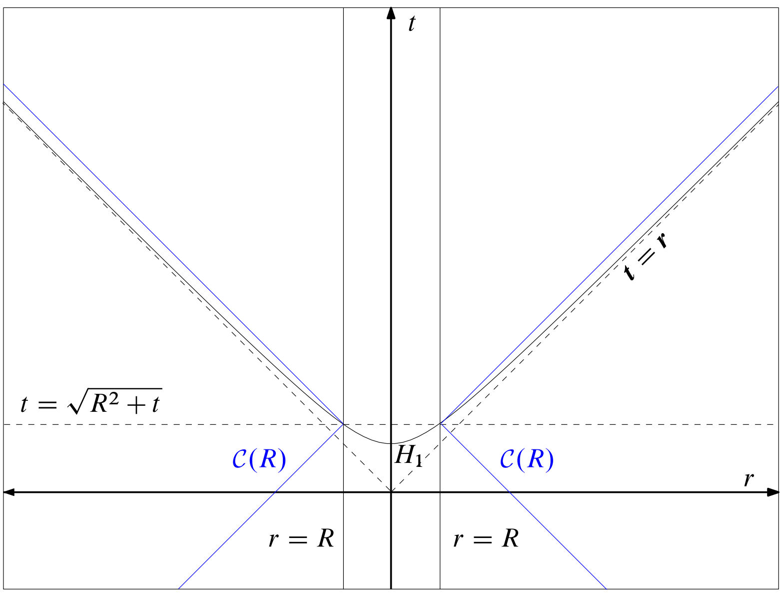

Figure 3. The trace of a distribution function with compact support on H_1.

Thus, the trace of f on H_1 is compactly supported and as a consequence, the norm appearing on the right-hand side of Theorem 2 is finite. Recall also that Theorem 2 gives pointwise estimates for $t \geq \sqrt{1+|x|^2}$. On the other hand, the region $t < \sqrt{1+|x|^2}$ lies in the exterior of $\mathcal{C}(R)$ and hence $f(t,x) = 0$ for $t < \sqrt{1+|x|^2}$. Thus, for compactly supported initial data given on some $t = \text{const}$ hypersurfaces, we can apply Theorem 2 and obtain a $1/t^n$ decay uniformly in x.

Finally, let us mention that the above arguments can be easily adapted to the nonlinear massive Vlasov–Nordström system for small initial data. Thus, once again, the use of hyperboloids in Theorem 5 is merely technical.

Appendix B: Integral estimate

Lemma B.1. *Let n be a positive integer. Consider α, β such that*

$$\alpha + \beta > n.$$

There exists a constant $C_{\alpha,\beta,n}$ such that the following estimate is true: for all $t > 0$, if $\beta \neq 1$, then

$$\int_0^\infty \frac{r^{n-1}\,dr}{(1+t+r)^\alpha(1+|t-r|)^\beta} \leq \frac{C_{\alpha,\beta,n}}{t^{\alpha+\beta-n}}(1+t^{\beta-1}).$$

If $\beta = 1$, then

$$\int_0^\infty \frac{r^{n-1}\,dr}{(1+t+r)^\alpha(1+|t-r|)} \leq \frac{C_{\alpha,n}}{t^{\alpha+1-n}}(1+\log(t+1)).$$

Proof. Let

$$A = \int_0^\infty \frac{r^{n-1}\,dr}{(1+t+r)^\alpha(1+|t-r|)^\beta}.$$

First, let us make the change of variable

$$r = ty.$$

This gives

$$A = \frac{1}{t^{\alpha+\beta-n}} \int_0^\infty \frac{y^{n-1}\,dy}{(1/t+1+y)^\alpha(1/t+|1-y|)^\beta}.$$

The first part of the denominator is bounded below by $(1+y)^\alpha$, so that

$$A \leq \frac{1}{t^{\alpha+\beta-n}} \int_0^\infty \frac{y^{n-1}\,dy}{(1+y)^\alpha(1/t+|1-y|)^\beta}.$$

We then cut the integral in two at the value $r = 2$. Let us thus introduce the constants $K_{\alpha,\beta,n}$ by

$$K_{\alpha,\beta,n} = \int_2^\infty \frac{y^{n-1}\,dy}{(1+y)^\alpha(y-1)^\beta}.$$

A can then be bounded by

$$A \leq \frac{1}{t^{\alpha+\beta-n}}\left(K_{\alpha,\beta,n} + 2^n \int_0^1 \frac{dy}{(1/t+1-y)^\beta}\right).$$

The remaining integral can be computed: for $\beta \neq 1$,

$$\int_0^1 \frac{dy}{(1/t+1-y)^\beta} = \frac{1}{\beta-1}\left(t^{\beta-1} - \frac{1}{(1/t+1)^{\beta-1}}\right) = \frac{t^{\beta-1}}{1-\beta}\left(1 - \frac{1}{1+t^{\beta-1}}\right) \leq C_\beta t^{\beta-1}.$$

If $\beta = 1$, we get

$$\int_0^1 \frac{dy}{(1/t+1-y)} = \log(1+t).$$

We finally get the announced result: if $\beta \neq 1$, then

$$\int_0^\infty \frac{r^{n-1}\,dr}{(1+t+r)^\alpha(1+|t-r|)^\beta} \leq \frac{C_{\alpha,\beta,n}}{t^{\alpha+\beta-n}}(1+t^{\beta-1}).$$

If $\beta = 1$, then

$$\int_0^\infty \frac{r^{n-1}\,dr}{(1+t+r)^\alpha(1+|t-r|)^1} \leq \frac{C_{\alpha,\beta,n}}{t^{\alpha+1-n}}(1+\log(t+1)). \qquad \square$$

Appendix C: Geometry of Vlasov fields

In this section we present the necessary elements to understand the underlying geometry of Vlasov fields on an arbitrary curved manifold. In particular, we will present, with some amount of detail,

- the geometry of the tangent bundle;
- the notion of complete lift, which is an essential tool to understand the commutators with the Vlasov field;
- how the ambient geometry of the tangent bundle can be reduced to the mass shell.

Most of the calculations will be left to the reader.

The reader who wishes to know more about the geometry of the tangent bundle can refer to the book by Crampin and Pirani [1986]. This section has also been greatly inspired by the work of Sarbach et al. [2014a; 2014b].

Throughout this section, let M be an $(n+1)$-dimensional smooth, oriented manifold, endowed with a Lorentzian metric g, of signature $(-,+,+,+)$. The Levi-Civita connection is denoted by ∇. The tangent bundle of M is denoted by TM. We furthermore assume that M is time oriented: there exists a uniformly timelike vector field T chosen, by convention, to be future pointing.

Geometry of the tangent bundle. This section is a reminder of some elementary geometric facts.

Definition C.1. The tangent bundle of M is the disjoint union of the tangent plane to M:

$$TM = \bigsqcup_{x\in M} T_x M.$$

TM is a vector bundle of dimension $2n+2$ over M, with fibre $\mathbb{R}^{n+1}$, and projection given by

$$\pi : V = (x,v) \in TM \longmapsto x \in M.$$

Consider a chart (U, x^α). One defines on the open set TU the system of coordinates

$$(x^\alpha, v^\alpha = dx^\alpha).$$

This system of coordinates provides a local trivialization of TM by

$$V \in TM \longrightarrow (\pi(V), v^\alpha(V)) \in U \times \mathbb{R}^{n+1}.$$

In these coordinates, the metric reads

$$g = g_{\alpha\beta} v^\alpha \otimes v^\beta.$$

We now consider the tangent space to the tangent space of M, denoted by TTM. If $V = (x, v)$ is a point of TM, the tangent space to TM at the point (x, v), denoted by $T_{(x,v)}TM$, is generated by the vectors

$$\left\{ \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial v^\alpha} \right\}.$$

Let (x, v) be in TM. If $t \mapsto \sigma(t)$ is a curve on M, with $\sigma(0) = x, \dot\sigma(0) = v$, the *natural lift* of σ is the curve of TM defined by

$$\sigma^h = t \mapsto (\sigma(t), \dot\sigma(t)).$$

As a consequence, any curve in M can be obtained by projection on M of a curve in TM. We consequently define:

Definition C.2 (vertical space). The push forward $\pi_\star$ of the mapping π defines, for $(x, v) \in TM$, a surjective mapping $T_{(x,v)}TM$ into $T_x M$. The kernel of $\pi_\star : T_{(x,v)}TM \to M$ is the horizontal space $V_{(x,v)}M$ at (x, v). This is a subspace of dimension $n + 1$ of $T_{(x,v)}TM$. In a system of coordinates (x^α, v^α), it is generated by

$$\left\{ \frac{\partial}{\partial v^\alpha} \right\}.$$

If $t \mapsto \sigma(t)$ is a curve on M, with $\sigma(0) = x$ and $\dot\sigma(0) = v$, the *horizontal lift* of σ is the curve of TM defined by

$$\sigma^h = t \mapsto (\sigma(t), V(t)),$$

where $V(t)$ is the vector field along the curve σ obtained by the parallel transport of v along the curve σ. In the coordinates (x^α, v^α), V obeys the differential equation

$$\nabla^\alpha_{\dot\sigma} V = \dot V^\alpha + \Gamma^\alpha_{\beta\gamma} \dot\sigma^\beta V^\gamma = 0 \quad \text{with } V(0) = v,$$

where the $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the connection. The tangent vector to the curve σ is given, in the coordinates (x^α, v^α), at $t = 0$, by

$$\dot\sigma(0)^\alpha \frac{\partial}{\partial x^\alpha} + \dot V^\alpha(0) \frac{\partial}{\partial v^\alpha} = \dot\sigma(0)^\alpha \left(\frac{\partial}{\partial x^\alpha} - \Gamma^\beta_{\alpha\gamma} v^\gamma \frac{\partial}{\partial v^\beta} \right).$$

The vector

$$\dot\sigma(0)^\alpha \left(\frac{\partial}{\partial x^\alpha} - \Gamma^\beta_{\alpha\gamma} v^\gamma \frac{\partial}{\partial v^\beta} \right)$$

is the *horizontal lift* of the vector $\dot\sigma(0)$. This definition depends only on the vector v in $T_x M$.

Definition C.3. The horizontal space $H_{(x,v)}M$ at (x, v) is the subspace of $T_{(x,v)}TM$ generated by the horizontal vectors

$$e_\alpha = \frac{\partial}{\partial x^\alpha} - \Gamma^\beta_{\alpha\gamma} v^\gamma \frac{\partial}{\partial v^\beta}.$$

It is independent of the chosen system of coordinates, and has trivial intersection with the vertical subspace of $T_{(x,v)}TM$.

Finally, the tangent space is endowed with a metric:

Definition C.4. The Sasaki metric on the tangent bundle TM is the metric of signature $(2n, 2)$ defined, in coordinates, by

$$g_s(e_\alpha, e_\beta) = g_s\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right) = g_{\alpha\beta},$$

and

$$g_s\left(e_\alpha, \frac{\partial}{\partial v^\beta}\right) = 0.$$

Geodesic spray, and its commutators. We now turn our attention to the lift of geodesics to the tangent bundle.

Definition C.5. Let γ be a geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = v$. The vector of $H_{(x,v)}M$ obtained by performing the horizontal lift of v, denoted by T, is given by

$$T = v^\alpha e_\alpha = v^\alpha \left(\frac{\partial}{\partial x^\alpha} - \Gamma^\beta_{\alpha\gamma} v^\gamma \frac{\partial}{\partial v^\beta}\right).$$

This defines globally a vector field on TM, called the geodesic spray.

Contrary to the geodesic flow, this vector field is defined globally on the manifold. Furthermore, since its integral curves are the natural lift of geodesics, it naturally models the behaviour of freely falling particles in the context of general relativity.

As we have seen earlier, one key aspect of the this work relies on the commutators with the transport operator T (see Section 2G2). The right tool to understand this is the notion of *complete lift* (see Section 2G1). It can be introduced as follows. Consider a vector field X on M. Assume that (locally) this vector field arises from a flow ϕ^t:

$$\frac{d\phi^t}{dt} = X(\phi^t).$$

The mapping ϕ^t can naturally be lifted into a mapping of TM by the formula

$$\phi^t_\star = (\phi^t, d\phi^t).$$

This immediately defines a vector field $\widehat{X}$ on TM by the formula

$$\frac{d\phi^t_\star}{dt} = \widehat{X}(\phi^t_\star).$$

It is also possible to have a definition relying on Lie transport along curves.

Definition C.6. Let X be a vector field on M. The complete lift of the vector field X on M into a vector field $\widehat{X}$ on TM is done as follows: Let $p \in M$ and consider $X(p)$. Let γ be an integral curve of X with initial data

$$\begin{cases} \gamma(0) = p, \\ \dfrac{d\gamma}{ds}(s) = X(\gamma(s)). \end{cases}$$

Let $v \in T_p M$, and consider the vector field Y defined on γ by Lie transporting the vector v along γ. It obeys the equation

$$\mathcal{L}_X Y = [X, Y] = 0 \quad \text{with } Y(p) = v.$$

This defines a curve $\Gamma = (\gamma, Y(\gamma))$ on TM. The mapping

$$\widehat{X} : TM \to TTM,$$
$$(x, v) \mapsto \frac{d\Gamma}{ds}(0),$$

defines a vector field on TM, defined as being the complete lift of X on TM.

An expression of the complete lift of the vector $W = W^\alpha \partial_{x^\alpha}$ in adapted coordinates is given in [Crampin and Pirani 1986, page 330] (and on page 288 of that work for affine transformations) by

$$\widehat{W} = W^\alpha \frac{\partial}{\partial x^\alpha} + v^\beta \frac{\partial W^\alpha}{\partial x^\beta} \frac{\partial}{\partial v^\alpha}. \tag{125}$$

This expression can also be written as

$$\widehat{W} = W^\alpha e_\alpha + v^\beta \nabla_\beta W^\alpha \frac{\partial}{\partial v^\alpha}. \tag{126}$$

One of the main interests of the complete lift is its relation with the commutators of the geodesic spray. It is possible to give a precise characterization of the commutators with the geodesic spray which arise from vector fields on the base manifold; see [Crampin and Pirani 1986, Chapter 13, Section 6].

Theorem 13. *A complete lift $\widehat{X}$ of a vector field is a symmetry of the geodesic spray Θ, i.e., commutes with the geodesic spray*

$$[\widehat{X}, T] = 0,$$

if, and only if, the vector field X is an infinitesimal affine transformation of the corresponding affine connection, and satisfies the equation, for all vector fields V, W,

$$\mathcal{L}_X \nabla_V W = \nabla_{[X,V]} W + \nabla_V \mathcal{L}_X W.$$

In the presence of a metric, the commutator of a complete lift $\widehat{X}$, of a vector field X, can be written explicitly; see [Sarbach and Zannias 2014b, Formula (74)].

Lemma C.7. *Let X be a vector field on M. The complete lift $\widehat{X}$ of X commutes with the geodesic spray Θ if, and only if,*

$$[T, \widehat{X}] = v^\alpha v^\beta [\nabla_\alpha \nabla_\beta X^\mu - R^\mu{}_{\beta\alpha\nu} X^\nu] \frac{\partial}{\partial v^\mu} = 0.$$

Remark C.8. The equation

$$\nabla_\alpha \nabla_\beta X^\mu - R^\mu{}_{\beta\alpha\nu} X^\nu = 0$$

is the equation for Jacobi fields; see [Crampin and Pirani 1986, page 340].

Geometry of the mass shell. If one considers a set of freely falling particles of given mass m, the 4-velocity of such a particle satisfies

$$g(v,v) = -m^2.$$

It is consequently natural to consider the subset of the tangent bundle TM defined by

$$P_m = \{(x,v) \in TM \mid g_x(v,v) = -m^2,\ v \text{ is future oriented}\},$$

called the *mass shell*. This set is the phase space of the considered set of particles. When the mass m is positive, P_m is a smooth submanifold of TM. When the mass m is vanishing, P_m is no longer a smooth submanifold because of the singularity at the tip of the vertex. If we ignore this fact, P_m is a fibre bundle over M. The projection over M is obtained by the restriction of the canonical projection of the bundle TM over M. The fibre at a point x is the subset of the tangent plane $T_x M$ given by

$$\{v \in T_x M \mid g_x(v,v) = -m^2,\ v \text{ is future oriented}\}.$$

Consider now a local chart (U, x^α) on M. We have seen that this local system of coordinates gives rise to a local chart on TM given by $(TU, x^\alpha, v^\alpha = dx^\alpha)$. This system of coordinates gives rise to a system of coordinates $(\bar{x}^\alpha x^\alpha, \bar{v}^i = v^i)$ on the mass shell by eliminating v^0 in the equation

$$g_{\alpha\beta} v^\alpha v^\beta = -m^2. \tag{127}$$

After one has chosen this system of coordinates, it is necessary to derive the relations between the partial derivatives in the variables (x^α, v^α), and the partial derivatives in the variables $(\bar{x}^\alpha = x^\alpha, \bar{v}^i = v^i)$. This is done by a simple application of the chain rule. Since v^0 does depend on the metric, it is first necessary to derive the following relations first: differentiating (127) gives

$$\frac{\partial v^0}{\partial x^\alpha} = -\frac{1}{2v_0} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} v^\beta v^\gamma, \tag{128}$$

$$\frac{\partial v^0}{\partial \bar{v}^i} = -\frac{v_i}{v_0}, \tag{129}$$

where we have used the notation

$$v_\alpha = g_{\alpha\beta} v^\beta.$$

Consider now a smooth function f on the tangent bundle TM. Its restriction to the mass shell is denoted by $\bar{f}$. An immediate application of the chain rule brings the following relations:

$$\frac{\partial \bar{f}}{\partial \bar{x}^\alpha} = \frac{\partial f}{\partial x^\alpha} + \frac{\partial v^0}{\partial x^\alpha} \frac{\partial f}{\partial v^0} = \frac{\partial f}{\partial x^\alpha} - \frac{1}{2v_0} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} v^\beta v^\gamma \frac{\partial f}{\partial v^0}, \tag{130}$$

$$\frac{\partial \bar{f}}{\partial \bar{v}^i} = \frac{\partial f}{\partial v^i} + \frac{\partial v^0}{\partial \bar{v}^i} \frac{\partial f}{\partial v^0} = \frac{\partial f}{\partial v^i} - \frac{v_i}{v_0} \frac{\partial f}{\partial v^0}. \tag{131}$$

The relations (130), (131) can now be used to determine which vectors are tangent to the mass shell. We notice first that the vector fields e_α, when applied to a function f, satisfy

$$\begin{aligned}
e_\alpha(f) &= \frac{\partial f}{\partial x^\alpha} - v^\beta \Gamma^\gamma_{\beta\alpha} \frac{\partial f}{\partial v^\gamma} \\
&= \frac{\partial \bar{f}}{\partial \bar{x}^\alpha} - v^\beta \Gamma^i_{\beta\alpha} \frac{\partial \bar{f}}{\partial \bar{v}^i} - \left(-\frac{1}{2v_0} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} v^\beta v^\gamma + v^\beta \Gamma^0_{\beta\alpha} + \frac{v_i}{v_0} v^\beta \Gamma^i_{\beta\alpha} \right) \frac{\partial f}{\partial v^0} \\
&= \frac{\partial \bar{f}}{\partial \bar{x}^\alpha} - v^\beta \Gamma^i_{\beta\alpha} \frac{\partial \bar{f}}{\partial \bar{v}^i} + \frac{1}{2v_0} \left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} v^\beta v^\gamma - 2 v^\beta v_\gamma \Gamma^\gamma_{\beta\alpha} \right) \frac{\partial f}{\partial v^0}.
\end{aligned}$$

A quick calculation shows, using the expression of the Christoffel symbols, that

$$v^\beta v_\gamma \Gamma^\gamma_{\beta\alpha} = \frac{1}{2} v^\beta v^\gamma \left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} \right) = \frac{1}{2} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} v^\beta v^\gamma.$$

The expression of e_α is consequently

$$e_\alpha(\bar{f}) = \frac{\partial \bar{f}}{\partial \bar{x}^\alpha} - v^\beta \Gamma^i_{\beta\alpha} \frac{\partial \bar{f}}{\partial \bar{v}^i} = e_\alpha(f).$$

This proves in particular that e_α is tangent to the mass shell, as well as the Liouville vector field

$$T(f) = v^\alpha e_\alpha(f) = v^\alpha e_\alpha(\bar{f}) = v^\alpha \frac{\partial \bar{f}}{\partial \bar{x}^\alpha} - v^\alpha v^\beta \Gamma^i_{\beta\alpha} \frac{\partial \bar{f}}{\partial \bar{v}^i}.$$

In dimension n, the mass shell is of dimension $2n+1$. We have as a consequence completely characterized the generators of the tangent plane to the mass shell, which is generated by the vectors

$$\left\{ e_\alpha, \frac{\partial}{\partial \bar{v}^i} \right\}.$$

From this, it is easy to deduce that the vector

$$v^\alpha \frac{\partial}{\partial v^\alpha}$$

is normal, for the Sasaki metric, to the mass shell (and also tangent in the massless case). The unit normal is consequently given by, for $m > 0$,

$$N = \frac{-1}{m^2} v^\alpha \frac{\partial}{\partial v^\alpha} = \frac{-1}{v_0} \frac{\partial}{\partial v^0} + \frac{1}{m^2} v^i \frac{\partial}{\partial \bar{v}^i}. \tag{132}$$

We will now discuss the conditions ensuring that a complete lift is tangent to the mass shell. The same procedure based on equations (130), (131) can be applied to the complete lift of a vector field X:

$$\begin{aligned}
\widehat{X} &= X^\alpha e_\alpha + v^\beta \nabla_\beta X^i \frac{\partial}{\partial \bar{v}^i} + \frac{1}{v_0} v^\beta v^\gamma \nabla_\beta X_\gamma \frac{\partial}{\partial v^0} \\
&= X^\alpha e_\alpha + v^\beta \nabla_\beta X^i \frac{\partial}{\partial \bar{v}^i} + \frac{1}{v_0} \pi^{(X)}_{\beta\gamma} v^\beta v^\gamma \frac{\partial}{\partial v^0}.
\end{aligned} \tag{133}$$

We immediately get the following lemma:

Lemma C.9. • *If $m > 0$, then X is Killing if, and only if, $\widehat{X}$ is tangent to P_m.*

- *If $m = 0$, then X is conformal Killing if, and only if, $\widehat{X}$ is tangent to P_0.*

Proof. The proof of this fact consists in noticing that

$$g_S(N, \widehat{X}) = v^\mu v^\nu \nabla_{(\mu} X_{\nu)}.$$

Then, if X is Killing in the massive case, or conformal Killing in the massless case,

$$g_S(N, \widehat{X}) = 0,$$

and then $\widehat{X}$ is tangent to $P_m(x)$.

Assume now that

$$g_S(N, \widehat{X}) = v^\mu v^\nu \nabla_{(\mu} X_{\nu)} = 0.$$

Consider now the symmetric 2-form $\nabla_{(\mu} X_{\nu)}$ on the vertical space, which is endowed with the metric g_{ij}. Then, in the massless case, the symmetric 2-form $\nabla_{(\mu} X_{\nu)}$ vanishes on the light cone of g_{ij} and is, as a consequence, proportional to it:

$$\nabla_{(\mu} X_{\nu)} = \phi g_{\mu\nu};$$

i.e., X is conformal Killing. The conclusion in the massive case follows in the same way. □

Acknowledgements

We would like to thank Martin Taylor for several interesting discussions on his work on the massless Einstein–Vlasov system. We would also like to thank Olivier Sarbach for his geometric introduction to Vlasov fields. J. Joudioux and J. Smulevici are partially funded by ANR-12-BS01-012-01 (AARG). J. Smulevici is also partially funded by ANR SIMI-1-003-01. D. Fajman gratefully acknowledges the travel support by ANR-12-BS01-012-01 (AARG). The comments of the referee are gratefully acknowledged.

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Received 28 Apr 2016. Revised 13 Apr 2017. Accepted 9 May 2017.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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ANALYSIS & PDE

Volume 10 No. 7 2017