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ANALYTIC HYPOELLIPTICITY FOR SUMS OF SQUARES AND THE TREVES CONJECTURE, II

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We are concerned with the problem of real analytic regularity of the solutions of sums of squares with real analytic coefficients. The Treves conjecture defines a stratification and states that an operator of this type is analytic hypoelliptic if and only if all the strata in the stratification are symplectic manifolds.

Albano, Bove, and Mughetti (2016) produced an example where the operator has a single symplectic stratum, according to the conjecture, but is not analytic hypoelliptic.

If the characteristic manifold has codimension 2 and if it consists of a single symplectic stratum, defined again according to the conjecture, it has been shown that the operator is analytic hypoelliptic.

We show here that the above assertion is true only if the stratum is single, by producing an example with two symplectic strata which is not analytic hypoelliptic.

1. Introduction

The purpose of the paper is to discuss the real analytic regularity of the distribution solutions to sums of squares equations

$$P(x, D)u = \sum_{j=1}^{N} X_j(x, D)^2 u = f, \tag{1-1}$$

where $X_j(x, D)$ denote vector fields with real analytic coefficients defined in an open set $\Omega \subset \mathbb{R}^n$, u is a distribution in Ω and $f \in C^\omega(\Omega)$, the space of all real analytic functions in Ω.

We suppose that the vector fields verify Hörmander's condition

(H) The Lie algebra generated by the vector fields and their commutators has dimension n, equal to the dimension of the ambient space.

In 1996 F. Treves [1999], see also [Bove and Treves 2004] for a formulation closer to the following, as well as [Treves 2006] for variants, stated a conjecture for the sums of squares of vector fields to be analytic hypoelliptic. In this paper we give neither the motivations nor the details about its statement; for both the motivations and a short introduction to the conjecture, as well as a brief review of the existing literature, we refer to [Albano et al. 2016].

Let us first give a very sketchy idea of how the conjecture was formulated. The main concept it uses is a stratification of the characteristic variety. This is a partition of the set $\{(x, \xi) \mid X_j(x, \xi) = 0,\ j = 1, \dots, N\}$ into real analytic manifolds as follows.

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Let P be as in (1-1). Then the characteristic variety of P is

$$\operatorname{Char}(P) = \{(x, \xi) \mid X_j(x, \xi) = 0,\ j = 1, \dots, N\},$$

where $X_j(x, \xi)$ denotes the symbol of the j-th vector field. This is a real analytic variety and, as such, it can be stratified, locally, in real analytic manifolds Σ_i for i in a finite family of indices $\mathscr{I}$. This means that

$$\operatorname{Char}(P) = \bigcup_{i \in \mathscr{I}} \Sigma_i,$$

and the Σ_i have the property that for $i \neq i'$, we have $\Sigma_i \neq \Sigma_{i'}$ and either $\Sigma_i \cap \overline{\Sigma}_{i'} = \varnothing$ or, if $\Sigma_i \cap \overline{\Sigma}_{i'} \neq \varnothing$, then $\Sigma_i \subset \partial \Sigma_{i'}$ (the boundary of $\Sigma_{i'}$). We refer to [Treves ≥ 2017] for more details.

Next we examine the rank of the restriction of the symplectic form, that is, of the form $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$, to the strata Σ_i, meaning that at any point $\rho \in \Sigma_i$ in a certain fixed neighborhood of $\rho_0 \in \operatorname{Char}(P)$, we restrict σ to the tangent space to Σ_i at ρ, denoted by $T_\rho \Sigma_i$. We want σ to have constant rank on each stratum Σ_i.

If this is not the case, we may consider the analytic variety where there is a change of rank, since the symplectic form restricted to Σ_i has a matrix whose entries are the Poisson brackets of the defining functions of Σ_i. Hence the rank is not maximal on a closed analytic subvariety where the determinant of a maximal minor vanishes. We may start over the procedure described above and further stratify this subvariety. The procedure ends after a finite number of steps yielding a stratification of $\operatorname{Char}(P)$ with real analytic manifolds where the restriction of the symplectic form has constant rank.

In the final step one considers the multiple Poisson brackets of the symbols of the vector fields. Let $I = (i_1, i_2, \dots, i_r)$, where $i_j \in \{1, \dots, N\}$. Write $|I| = r$ and define

$$X_I(x, \xi) = \{X_{i_1}(x, \xi), \{X_{i_2}(x, \xi), \{\cdots \{X_{i_{r-1}}(x, \xi), X_{i_r}(x, \xi)\} \cdots\}\}\}.$$

Here r is called the length of the multiple Poisson bracket $X_I(x, \xi)$. We recall that the Poisson bracket is defined as

$$\{X_i(x, \xi), X_j(x, \xi)\} = \sum_{\ell=1}^n \left(\frac{\partial X_i}{\partial \xi_\ell} \frac{\partial X_j}{\partial x_\ell} - \frac{\partial X_i}{\partial x_\ell} \frac{\partial X_j}{\partial \xi_\ell} \right).$$

We recall that L. Hörmander[1967] solved the problem of the C^∞ hypoellipticity of sums of squares formulating his well known condition using the algebra built with the Poisson brackets of the vector fields.

It is clear that since all the strata defined above are submanifolds of the characteristic variety, the symbols of the vector fields vanish on each stratum.

Next we examine all the Poisson brackets of two (symbols of) vector fields on a stratum in a neighborhood U of a fixed point ρ_0. Denote again by Σ_i the stratum. A few things may happen: there is at least a nonzero Poisson bracket at ρ_0 and hence, possibly shrinking U, on all of it, in which case we stop. Otherwise all brackets may vanish identically on $\Sigma_i \cap U$. Finally there may be Poisson brackets that vanish on a subvariety of $\Sigma_i \cap U$.

At this point we repeat the stratification construction above using the equations defining this subvariety. Then we pick a new stratum, say Σ_j, and we have that either there is a nonzero Poisson bracket on $\Sigma_j \cap U$ or every Poisson bracket identically vanishes in $\Sigma_j \cap U$.

This procedure may be iterated by considering Poisson brackets of length 3 etc.

In the end, after a finite number of steps, we wind up with a stratification such that every stratum Σ has the following properties:

(a) Σ is a real analytic submanifold of $\operatorname{Char}(P)$.

(b) The symplectic form restricted to Σ has constant rank.

(c) Denoting again by U a neighborhood of the point $\rho_0 \in \operatorname{Char}(P)$, there is an index say $m = m(\Sigma)$ such that at least one bracket of length m is nonzero on $U \cap \Sigma$ and all the brackets of length less than m identically vanish on $\Sigma \cap U$.

Conjecture 1.1 [Treves 1999]. *Consider the operator P in (1-1) and define the stratification as sketched above. Then P is analytic hypoelliptic if and only if every stratum in the stratification is symplectic.*

Remark 1.2. The above conjecture is subtle and depends essentially on two ingredients: first the way the stratification is defined and, second, the fact that the strata must be symplectic manifolds. The condition that the strata, whatever that might mean, should be symplectic seems quite reasonable, since there are examples of operators with a nonsymplectic characteristic manifold which are known not to be analytic hypoelliptic. A different problem is the definition of the strata. It seems that, because of [Albano et al. 2016] and the present result, the way of defining the strata has to be changed in the statement of the conjecture. The authors have no solution to this problem right now.

The necessary part of the conjecture, i.e., the nonanalytic hypoellipticity in the presence of nonsymplectic strata, is, as far as we know, still an open problem, although it might be of limited interest if the definition of the stratification is changed.

In [Albano et al. 2016] it was shown that the sufficient part of the above conjecture is false by exhibiting an operator with a single symplectic stratum, defined according to Conjecture 1.1, of dimension 4 (and codimension 4) and proving that the operator is not analytic hypoelliptic. Actually its Gevrey regularity has been completely characterized.

It is not difficult to exhibit, based on [Albano et al. 2016], examples of sums of squares having a single symplectic stratum Σ defined according to Conjecture 1.1, and such that $\operatorname{codim} \Sigma = 2\nu$, $2 \le \nu \le n-2$, for which a proof analogous to that of the same paper implies the nonanalytic hypoellipticity of the operator. Here is an example.

Let us define $x' = (x_1, \dots, x_\nu)$ and $x'' = (x_{\nu+1}, \dots, x_n)$, so that $x = (x', x'')$, $x \in \mathbb{R}^n$, where ν satisfies the above conditions. Define $x'_\nu = (x_1, \dots, x_{\nu-1})$, $x''_\nu = (x_{\nu+2}, \dots, x_n)$ and set

$$|D_{x'_\nu}|^2 + [x'_\nu]_r^2 |D_{x''}|^2 + D_{x_\nu}^2 + x_\nu^{2(p-1)} D_{x_{\nu+1}}^2 + x_\nu^{2(q-1)} |D_{x''_\nu}|^2,$$

where $[x'_\nu]_r^2 = \sum_{j=1}^{\nu-1} x_j^{2(r_j-1)}$, $|D_{x''}|^2 = \sum_{j=\nu+1}^{n} D_{x_j}^2$, and analogously for $|D_{x'_\nu}|^2$, $|D_{x''_\nu}|^2$, with the condition that $1 < \min r_j \le \max r_j < p < q$.

A number of papers have been written both in the case of a single stratum and of a more complex stratification. The meaning of the word "stratum" is henceforth that defined in Conjecture 1.1. If we are in the presence of a single symplectic stratum of codimension 2, then the conjecture has been proved true by Métivier [1981], Ōkaji [1985], Cordaro and Hanges [2009], and Albano and Bove [2013]. The papers by Métivier, Ōkaji and Albano and Bove include also a higher codimension single symplectic stratum, provided additional assumptions are satisfied.

We say that an operator exhibits nested strata, always according to Conjecture 1.1, if the associated stratification has at least two strata Σ_1, Σ_2 such that $\Sigma_1 \neq \Sigma_2$ and $\Sigma_1 \cap \bar{\Sigma}_2 \neq \varnothing$. By definition this implies that $\Sigma_1 \subset \partial\Sigma_2$ and, in particular, that the dimension of Σ_1 is smaller than that of Σ_2.

Theorem 2.1 proved below implies that the conjecture does not hold, in general, when there are several symplectic nested Poisson–Treves strata.

The conjecture fails even if the characteristic manifold has codimension 2, as in the case we are going to examine. This is actually proved in the remainder of the present paper.

Thus we can state:

Theorem 1.3. *Let us consider the class of all sums of squares with analytic coefficients such that the associated stratification near a point $\rho_0 \in \operatorname{Char}(P)$ has not a single stratum. Then the sufficient part of Conjecture 1.1 is false even in the case of a characteristic manifold of codimension* 2.

We remark that if the characteristic variety is a real analytic manifold of dimension 2, and the Treves strata are symplectic, then we may have only a single symplectic stratum of Treves type — obviously coinciding with the characteristic manifold.

If the characteristic variety is a manifold of codimension 2 as well as of dimension 2, by the results quoted above one deduces that the operator is analytic hypoelliptic. If, on the other hand, the codimension of the characteristic manifold is larger than 2, we do not think that analytic hypoellipticity ensues and thus Conjecture 1.1 would be contradicted.

To clarify the above sentence let us consider the operator

$$Q(x,D) = D_1^2 + D_2^2 + x_1^2 D_3^2 + x_2^4 D_3^2 + x_2^2 x_3^2 D_3^2.$$

It is easily seen that $\operatorname{Char}(Q) = \{(x,\xi) \mid x_i = \xi_i = 0,\ i = 1,2,\ \xi_3 \neq 0\}$, which has dimension 2 and codimension 4. $\operatorname{Char}(Q)$ is a single symplectic stratum for Q and it is not too difficult to prove, either by using the subelliptic estimate (see Section 4 below and formula (4-2)) or the method described in [Bove and Mughetti 2016], that Q is Gevrey $\frac{4}{3}$-hypoelliptic (see Definition 2.2 for a definition of the Gevrey classes).

We think that the Gevrey $\frac{4}{3}$-regularity is optimal for the operator Q, but the proof of optimality is however an open problem. The difficulty of the proof is due to the following fact: in the examples of [Albano et al. 2016] and (2-1) of this paper, as well as in Q, there are strata which do not appear in Treves stratification. The Hamilton bicharacteristics associated to these phantom strata either project injectively on the base space, like in the case of [Albano et al. 2016] and (2-1), or project injectively onto the fibers of the cotangent bundle. The latter is the case for Q. In this sense the operator Q shares this difficulty with the Métivier operator, [1981] where optimality is very hard to prove.

We would also like to recall that due to [Bove et al. 2013] the conjecture does not hold for sums of squares of complex vector fields.

Here is the structure of the paper. The result is stated in Section 2. Section 3 is devoted to the proof of the optimality of the s_ℓ Gevrey regularity. We construct a solution to $Pu = g$, where g is real analytic, which is not better than Gevrey s_ℓ. In Section 4 we prove that every solution to $Pu = f$ is Gevrey s_ℓ, if $f \in G^{s_\ell}$. This is done using the subelliptic estimate for the operator.

2. Statement of the result

Let $\ell, r, p, q \in \mathbb{N}$, $1 < r < p < q$, and $x = (x_1, \dots, x_4) \in \mathbb{R}^4$. The objective of this section is to state the optimal Gevrey regularity for the operator

$$P(x, D) = D_1^2 + x_1^{2(\ell+r-1)}(D_3^2 + D_4^2) + x_1^{2\ell}[D_2^2 + x_2^{2(p-1)} D_3^2 + x_2^{2(q-1)} D_4^2]. \tag{2-1}$$

Hörmander's condition is satisfied by P and thus P is C^∞ hypoelliptic.

The characteristic manifold of P is the real analytic manifold

$$\mathrm{Char}(P) = \{(x, \xi) \in T^*\mathbb{R}^4 \setminus \{0\} \mid \xi_1 = 0,\ x_1 = 0,\ \xi_2^2 + \xi_3^2 + \xi_4^2 > 0\}. \tag{2-2}$$

According to Treves' conjecture one has to look at the strata associated with P.

The stratification associated with P is made up of two symplectic strata Σ_1 and Σ_2:

$$\Sigma_1 = \{(0, x_2, x_3, x_4; 0, \xi_2, \xi_3, \xi_4) \mid \xi_2^2 + x_2^2 > 0,\ \xi_2^2 + \xi_3^2 + \xi_4^2 > 0\} \tag{α}$$

at depth $\ell + 1$.[1] Σ_1 is a symplectic stratum and the restriction of the symplectic form to it has rank 6.

$$\Sigma_2 = \{(0, 0, x_3, x_4; 0, 0, \xi_3, \xi_4) \mid \xi_3^2 + \xi_4^2 > 0\} \tag{β}$$

at depth $\ell + r$. This is also a symplectic stratum and the restriction of the symplectic form to it has rank 4.

We point out that the above stratification does not depend on the choice of the indices p and q.

According to the conjecture we would expect local real analyticity near the origin for the distribution solutions u of $Pu = f$, with a real analytic right-hand side.

We are ready to state the theorem that is proved in the next two sections of the paper.

Theorem 2.1. *Let*

$$\frac{1}{s_\ell} = \frac{\ell+1}{\ell+r} + \frac{r-1}{\ell+r}\,\frac{p-1}{q-1}.$$

Then P is locally Gevrey s_ℓ-hypoelliptic and not better near the origin.

We recall here the definition of the Gevrey classes:

Definition 2.2. If Ω is an open subset of $\mathbb{R}^n$ and $s \geq 1$ we denote by $G^s(\Omega)$ the class of all functions $u \in C^\infty(\Omega)$ such that for every compact set $K \Subset \Omega$ there is a positive constant $C = C_K$ such that

$$|\partial_x^\alpha u(x)| \leq C^{|\alpha|+1} \alpha!^s \quad \text{for every } x \in K.$$

[1] This means that all the Poisson brackets of the (symbols of) the vector fields of length less than $\ell + 1$ are identically zero on the characteristic manifold and that there is at least one bracket of length $\ell + 1$ which is nonzero.

We observe that $G^1(\Omega)$ coincides with the class of all real analytic functions in Ω.

As a consequence of Theorem 2.1 we have:

Corollary 2.3. *The operator P is analytic hypoelliptic if and only if $p = q$.*

The proof of the corollary is contained in Section 4.

Moreover from Theorem 2.1 we deduce that Theorem 1.3 holds since the operator P above has a stratification made of two nested symplectic strata.

Remark 2.4. The geometric interpretation of the above result is not known. We believe that a different definition of the associated stratification should be given, allowing the existence of an additional stratum for P.

More precisely the missing stratum seems to be

$$\widetilde{\Sigma} = \{(0, 0, x_3, x_4; 0, 0, 0, \xi_4) \mid \xi_4 \neq 0\},$$

which can be seen as the set where ξ_3 vanishes in Σ_2.

$\widetilde{\Sigma}$ is not symplectic; its Hamilton curves are the x_3-lines and this fact gives us a lead as to why, in the following proof of Theorem 2.1, we may conclude that the operator P is not analytic hypoelliptic. We shall come back on this further on.

We would also like to observe that the point $(0, e_4)$ is the only interesting characteristic point where we have a lack of analytic hypoellipticity. In fact the operator P is microlocally analytic hypoelliptic at all points in $\Sigma_1 \setminus \Sigma_2$, as well as at points in Σ_2 where $\xi_3 \neq 0$. This can be proved via L^2 (microlocalized) estimates of the type of (but easier than those) used in Section 4.

It is also worth noting that if we accept, without any proof or other justification, that the stratification associated to P is made of (the connected components of) $\Sigma_1 \setminus \Sigma_2$, $\Sigma_2 \setminus \widetilde{\Sigma}$ and $\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is given above, then all points of the first two "strata" are points of analytic hypoellipticity and the strata are symplectic, while the nonanalytic hypoellipticity comes in at points of the nonsymplectic stratum $\widetilde{\Sigma}$.[2]

3. Proof of Theorem 2.1

In this section we prove the optimality of the Gevrey regularity in Theorem 2.1.

We construct a solution to the equation $Pu = f$, for a real analytic function f, which is not Gevrey s for any $s < s_\ell$ and is defined in a neighborhood of the origin.

In fact we look for a function $u(x, y, t)$ defined in $\widetilde{U} \times [1, +\infty[\subset \mathbb{R}_x \times \mathbb{R}_y \times [1, +\infty[$, where $\widetilde{U}$ denotes a neighborhood of the origin in $\mathbb{R}^2_{x,y}$, and such that

$$P(x, D)A(u) = g, \tag{3-1}$$

where

$$A(u)(x) = \int_1^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} u(\rho^{\frac{1}{\ell+r}} x_1, x_2, \rho)\, d\rho, \tag{3-2}$$

and

$$\theta = \frac{1}{s_\ell}.$$

[2]We are indebted to the referees for pointing out to us the need of such a remark.

The function $z(\rho)$ is to be determined. Here we assume that $x \in U$, a suitable neighborhood of the origin in $\mathbb{R}^4$ whose size will ultimately depend on $z(\rho)$. Furthermore $g \in C^\omega(U)$.

We have

$$P(x,D)A(u)(x) = \int_1^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[-\rho^{\frac{2}{\ell+r}} \partial_{x_1}^2 u - x_1^{2(\ell+r-1)} (z(\rho))^2 \rho^{2\theta} u + x_1^{2(\ell+r-1)} \rho^2 u \\ + x_1^{2\ell} \big(-\partial_{x_2}^2 u - x_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta} u + x_2^{2(q-1)} \rho^2 u \big) \Big] \, d\rho.$$

Rewriting the right-hand side of the above relation in terms of the variable $y_1 = \rho^{\frac{1}{\ell+r}} x_1$, we obtain

$$P(x,D)A(u)(x) = \int_1^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \\ \times \Big[-\rho^{\frac{2}{\ell+r}} \big(\partial_1^2 - y_1^{2(\ell+r-1)} (1 - (z(\rho))^2 \rho^{2(\theta-1)}) \big) u \\ + \rho^{-\frac{2\ell}{\ell+r}} y_1^{2\ell} \big[\big(-\partial_2^2 - x_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta} + x_2^{2(q-1)} \rho^2 \big) \big] u \Big]_{y_1 = \rho^{1/(\ell+r)} x_1} \, d\rho.$$

We point out that

$$\theta - 1 < 0.$$

Choose

$$u(y_1, x_2, \rho) = u_1(y_1, \rho) u_2(x_2, \rho), \tag{3-3}$$

where u_j, $j = 1, 2$, will be chosen later. Plugging this into the above formula yields

$$P(x,D)A(u)(x) \\ = \int_1^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \\ \times \Big[-\rho^{\frac{2}{\ell+r}} u_2(x_2,\rho) \big(\partial_1^2 - y_1^{2(\ell+r-1)} (1 - (z(\rho))^2 \rho^{2(\theta-1)}) \big) u_1(y_1,\rho) \\ + \rho^{-\frac{2\ell}{\ell+r}} y_1^{2\ell} u_1(y_1,\rho) \big[\big(-\partial_2^2 - x_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta} + x_2^{2(q-1)} \rho^2 \big) \big] u_2(x_2,\rho) \Big]_{y_1 = \rho^{1/(\ell+r)} x_1} \, d\rho. \tag{3-4}$$

We want to determine u_1, u_2 so that $P(x,D)A(u)(x) = 0$. In particular

$$-\rho^{\frac{2}{\ell+r}} u_2(x_2,\rho) \big(\partial_1^2 - y_1^{2(\ell+r-1)} (1 - (z(\rho))^2 \rho^{2(\theta-1)}) \big) u_1(y_1,\rho) \\ + \rho^{-\frac{2\ell}{\ell+r}} y_1^{2\ell} u_1(y_1,\rho) \big[\big(-\partial_2^2 - x_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta} + x_2^{2(q-1)} \rho^2 \big) \big] u_2(x_2,\rho) = 0 \tag{3-5}$$

for ρ large.

Let us start by considering the operator in the x_2-variable:

$$\rho^{2\frac{\ell+1}{\ell+r}} L(x_2, \partial_{x_2}) = -\partial_2^2 - x_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta} + x_2^{2(q-1)} \rho^2.$$

Performing the dilation

$$x_2 = y_2 \rho^{-\mu},$$

where

$$\mu = \frac{r-1}{\ell+r} \frac{1}{q-1}, \tag{3-6}$$

we obtain

$$L(y_2, \partial_{y_2}) = -\rho^{2\left(\frac{r-1}{\ell+r}\frac{1}{q-1}-\frac{\ell+1}{\ell+r}\right)}\partial_{y_2}^2 - z(\rho)^2 y_2^{2(p-1)} + y_2^{2(q-1)}. \tag{3-7}$$

Set

$$h = \rho^{\frac{r-1}{\ell+r}\frac{1}{q-1}-\frac{\ell+1}{\ell+r}} = \rho^{-\varkappa}. \tag{3-8}$$

Observe that the exponent above is negative, since $r-1 < (\ell+1)(q-1)$ if and only if $r-q < \ell(q-1)$, which is obviously true for any value of ℓ as $r < q$ by assumption. Thus for large ρ, we know h tends to zero and hence we have to study the semiclassical stationary Schrödinger operator

$$L(y_2, \partial_{y_2}) = -h^2\partial_{y_2}^2 - z(h)^2 y_2^{2(p-1)} + y_2^{2(q-1)} \tag{3-9}$$

exhibiting a double-well potential. We point out that the dilation (3-6) has been chosen in such a way to get rid of the parameter ρ, i.e., h, from the double-well potential.

Let us make the following ansatz: the quantity $z(h)$ above is positive and such that there is an $h_0 > 0$ for which

$$0 < \inf_{0<h<h_0} z(h) < +\infty.$$

We shall return to this ansatz and show that it is actually compatible with our findings.

We may further dilate the operator in (3-9) in such a way that the quantity $z(h)$ appears as a coefficient of the second derivative, modulo a multiplying factor. Set

$$y_2 = (z(h))^{\frac{1}{q-p}} y.$$

Note that the above dilation is well defined because of our ansatz when $p < q$. If, on the other hand $p = q$ the whole construction is not needed, since $s_\ell = 1$ then.

Thus (3-9) becomes

$$L(y, \partial_y) = z^{2\frac{q-1}{q-p}}\left[-(z^{-\frac{q}{q-p}}h)^2\partial_y^2 - y^{2(p-1)} + y^{2(q-1)}\right]. \tag{3-10}$$

Let

$$\hat{\gamma} = -\frac{q-p}{q-1}\left(\frac{p-1}{q-1}\right)^{\frac{p-1}{q-p}} < 0$$

denote the minimum of the potential $-y^{2(p-1)} + y^{2(q-1)}$. Then

$$-(z^{-\frac{q}{q-p}}h)^2\partial_y^2 - y^{2(p-1)} + y^{2(q-1)} - \hat{\gamma} \tag{3-11}$$

has a discrete spectrum made of simple positive eigenvalues accumulating at infinity; see, e.g., [Berezin and Shubin 1991]. Hence the eigenvalues are real analytic functions of the parameter $hz^{-\frac{q}{q-p}} > 0$.

At this point we might choose to select the ground state of (3-11). This would allow us to treat only the case of an even eigenfunction with well-known Agmon estimates. However, we would like to emphasize the fact that the Gevrey regularity we find is a consequence of the nature of the spectrum of the operator in (3-11) and that, in particular, any eigenvalue allows us to conclude the Gevrey regularity of the solution to (3-1).

There is a price to pay for this generality: we cannot a priori use the fact that the associated eigenfunction is symmetric and positive — which is true for the fundamental eigenstate.

Denote by $E = E(h(z(h))^{-\frac{q}{q-p}})$ one of the energy levels of (3-11). Also let $u_2(y,h)$ denote the corresponding eigenfunction, i.e.,

$$\big[-(z^{-\frac{q}{q-p}}h)^2\partial_y^2 - y^{2(p-1)} + y^{2(q-1)} - \hat{\gamma}\big]u_2 = Eu_2, \tag{3-12}$$

or, changing back the variables

$$\big[-\partial_{x_2}^2 - z^2\rho^{2\theta}x_2^{2(p-1)} + x_2^{2(q-1)}\rho^2 - \hat{\gamma}z^{2\frac{q-1}{q-p}}\rho^{2\frac{\ell+1}{\ell+r}}\big]u_2 = Ez^{2\frac{q-1}{q-p}}\rho^{2\frac{\ell+1}{\ell+r}}u_2. \tag{3-13}$$

The operator in (3-11) has a symmetric nonnegative double-well potential with two nondegenerate minima and which is unbounded at infinity. Theorem 1.1 in [Simon 1983] asserts:

Theorem 3.1. *For every eigenvalue $E(\mu)$ in the spectrum of*

$$-\mu^2\partial_y^2 - y^{2(p-1)} + y^{2(q-1)} - \hat{\gamma},$$

we have

$$\lim_{\mu\to 0+}\frac{E(\mu)}{\mu} = e^* > 0. \tag{3-14}$$

As a consequence we may continue the function $E(\mu)$ at zero by setting $E(0) = 0$. Thus $E(\mu)$ is differentiable for $0 \le \mu$.

Furthermore we have the following.

Lemma 3.2. *For every $h_0 > 0$, we have that $\partial_h E(h)$ exists and is bounded for $0 \le h \le h_0$.*

Proof. This is basically proved by deriving in this case the Feynman–Hellmann formula expressing the derivative with respect to h of the eigenvalues in terms of the associated eigenfunctions and the derivative of the Hamiltonian; see, e.g., the proof of Lemma 3.1 in [Albano et al. 2016]. □

Let us now go back to equation (3-5). Neglecting the factor $\rho^{\frac{2}{\ell+r}}$ and writing everything as a function of h, we obtain

$$\big[\partial_1^2 - y_1^{2(\ell+r-1)}(1-(z(h))^2h^{2\frac{1-\theta}{\varkappa}})\big]u_1(y_1,h) - y_1^{2\ell}z(h)^{2\frac{q-1}{q-p}}(E(hz(h)^{-\frac{q}{q-p}}) + \hat{\gamma})u_1(y_1,h) = 0, \tag{3-15}$$

where, as specified above, E is an eigenvalue of the operator (3-9) and h is small.

We want to show that, for small h, we can find a bounded positive function $z = z(h)$ such that (3-15) has a nontrivial kernel, which will be made of rapidly decreasing eigenfunctions corresponding to the eigenvalue zero.

First set

$$\tau(h) = 1 - (z(h))^2h^{2\frac{1-\theta}{\varkappa}}.$$

Note that, due to our ansatz, τ is a positive number if h is suitably small. We are thus entitled to perform the dilation

$$t_1 = y_1\tau^{\frac{1}{2(\ell+r)}},$$

so that the differential operator in (3-15) becomes

$$\tau^{\frac{1}{\ell+r}}\bigl[-\partial_{t_1}^2+t_1^{2(\ell+r-1)}+t_1^{2\ell}z(h)^{2\frac{q-1}{q-p}}\tau^{-\frac{\ell+1}{\ell+r}}\bigl(E(hz(h)^{-\frac{q}{q-p}})+\hat{\gamma}\bigr)\bigr]. \tag{3-16}$$

Proposition 3.3. *There exists a positive number σ such that the operator*

$$-\partial_t^2+t^{2(\ell+r-1)}-\sigma t^{2\ell} \tag{3-17}$$

has a nontrivial kernel.

Proof. The proof is just an analysis of the behavior of the eigenvalues of the operator as functions of the parameter σ; see, for instance, [Mughetti 2014; 2015]. First of all we remark that the function $t\mapsto t^{2(\ell+r-1)}-\sigma t^{2\ell}$ has a (negative) minimum

$$-\frac{r-1}{\ell+r-1}\left(\frac{\ell}{\ell+r-1}\right)^{\frac{\ell}{r-1}}\sigma^{\frac{\ell+r-1}{r-1}}=\sigma^{\frac{\ell+r-1}{r-1}}\hat{\mu}.$$

Performing the dilation

$$t=t_1\sigma^{\frac{1}{2(r-1)}},$$

the operator becomes

$$-\sigma^{\frac{-1}{r-1}}\partial_{t_1}^2+\sigma^{\frac{\ell+r-1}{r-1}}(t_1^{2(\ell+r-1)}-t_1^{2\ell}),$$

which can be written as

$$\sigma^{\frac{\ell+r-1}{r-1}}\bigl[(-\sigma^{-\frac{\ell+r}{r-1}}\partial_{t_1}^2+t_1^{2(\ell+r-1)}-t_1^{2\ell}-\hat{\mu})+\hat{\mu}\bigr].$$

Now the operator in parentheses is again a Schrödinger operator with a double-well positive potential and hence it has a positive discrete spectrum accumulating at infinity, by [Berezin and Shubin 1991].

Denote by $\lambda(\sigma)$ one such eigenvalue, so that the eigenvalues of the operator above are

$$\sigma^{\frac{\ell+r-1}{r-1}}(\lambda(\sigma)+\hat{\mu}).$$

If $\sigma\to+\infty$, then, by Theorem 3.1, $\lambda(\sigma)\to 0+$, so that the expression above is negative. On the other hand it is obvious that for $\sigma=0$ the eigenvalues of the operator (3-17) are positive. Furthermore they are simple, whatever the value of σ is, and thus they are also continuously dependent on σ. We conclude hence that there is a value $\bar{\sigma}$ of σ for which $\lambda(\bar{\sigma})=0$. This proves the nontriviality of the kernel. □

Going back to (3-16), we see that, in order to solve (3-15) it is enough to choose one such value σ given by Proposition 3.3 and then solve, with respect to z, the equation

$$-\bar{\sigma}=z(h)^{2\frac{q-1}{q-p}}\tau^{-\frac{\ell+1}{\ell+r}}\bigl(E(hz(h)^{-\frac{q}{q-p}})+\hat{\gamma}\bigr). \tag{3-18}$$

Set

$$\tilde{z}=\left(\frac{-\bar{\sigma}}{\hat{\gamma}}\right)^{\frac{q-p}{2(q-1)}}>0.$$

Observe that the values $h=0$, $z(0)=\tilde{z}$ verify equation (3-18), since $\tau(0)=1$.

Proposition 3.4. *There is an* $h_0 > 0$ *such that* (3-18) *implicitly defines a function* $z \in C([0, h_0[) \cap C^\omega(]0, h_0[)$. *Moreover* $\lim_{h\to 0+} z(h) = \tilde{z}$.

Proof. The proof is analogous to the proof of Proposition 3.1 in [Albano et al. 2016] and we just sketch it.

Consider the function

$$f(h,z) = z^{2\frac{q-1}{q-p}} \tau^{-\frac{\ell+1}{\ell+r}} (E(hz^{-\frac{q}{q-p}}) + \hat{\gamma}) + \bar{\sigma}.$$

Let us compute the derivative with respect to z of the above function in the interval $[0, h_0[\times]\tilde{z}-\delta, \tilde{z}+\delta[$, where δ is a small positive number:

$$\begin{aligned}\partial_z f(h,z) &= 2\frac{q-1}{q-p} z^{2\frac{q-1}{q-p}-1} \tau^{-\frac{\ell+1}{\ell+r}} (E(hz^{-\frac{q}{q-p}}) + \hat{\gamma}) \\ &\quad + 2\frac{\ell+1}{\ell+r} z^{2\frac{q-1}{q-p}+1} h^{2\frac{1-\theta}{\varkappa}} \tau^{-\frac{\ell+1}{\ell+r}-1} (E(hz^{-\frac{q}{q-p}}) + \hat{\gamma}) - \frac{q}{q-p} z^{\frac{q-2}{q-p}-1} h \tau^{-\frac{\ell+1}{\ell+r}} E'(hz^{-\frac{q}{q-p}}).\end{aligned}$$

In view of Lemma 3.2, the derivative above is strictly negative if $(h,z) \in [0, h_0[\times]\tilde{z}-\delta, \tilde{z}+\delta[$ for a suitable choice of small h_0, δ. Note that $f(0,\tilde{z}) = 0$.

Because of the definition of $\tilde{z}$ and the definition of f, we have $f(h, \tilde{z}-\delta) > 0$, $f(h, \tilde{z}+\delta) < 0$ possibly taking a smaller h_0, δ for $0 \le h \le h_0$. Since f is continuous and strictly decreasing on the h-lines, there is a unique zero of the equation $f(h, z(h)) = 0$ with $z(h) \in [\tilde{z}-\delta, \tilde{z}+\delta]$ for $0 \le h \le h_0$.

For positive h, trivially $z(h)$ is real analytic. Let us show that $z(h) \in C([0, h_0[)$. Arguing by contradiction, assume that $z(h) \not\to \tilde{z}$ for $h \to 0+$. Then there is a sequence $h_k \to 0+$ such that $z(h_k) \to \hat{z} \ne \tilde{z}$. Then $0 = f(h_k, z(h_k)) \to f(0, \hat{z})$, which is false since $\tilde{z}$ is the only zero of $f(0,z) = 0$.

The conclusion follows. □

We state also a couple of lemmas that will be needed in the sequel.

Write $V(x) = x^{2(\ell+r-1)} - x^{2\ell} - \hat{\mu}$ and $Q_h = -h^2\partial_x^2 + V$. We have:

Lemma 3.5. *For every* $h_0 > 0$ *and every* $v \in \mathscr{S}(\mathbb{R})$ *the following a priori inequality holds*:

$$h^2\|v''\| + \|Vv\| \le C(\|Q_h v\| + h\|v\|) \tag{3-19}$$

for a positive constant C *independent of* $h \in]0, h_0[$.

Lemma 3.6. *Let* $v(x,h)$ *denote the* $L^2(\mathbb{R})$ *normalized eigenfunction of* Q_h *corresponding to* $E(h)$. *Then* v *is rapidly decreasing with respect to* x *and satisfies the estimates*

$$|v^{(j)}(x,h)| \le C_j h^{-(j+1)/2} \tag{3-20}$$

for $x \in \mathbb{R}$, $C_j > 0$ *independent of* $0 < h < h_0$, $j = 0, 1, 2$, *with* h_0 *suitably small.*

Lemmas 3.5 and 3.6 are rather standard and, for a proof, we refer for instance to the appendix of [Albano et al. 2016].

We can now go back to (3-4). With the choice above of both u_1, u_2 and $z = z(\rho)$, we can satisfy (3-5), provided $\rho \ge \rho_0$, with ρ_0 large enough. Furthermore we also remark that the formal operation of taking derivatives under the integral sign is completely legitimate, due to Lemma 3.6, since a power singularity at infinity does not affect the convergence of the integral. We have then that

$$
\begin{aligned}
&P(x,D)A(u)(x)\\
&=\int_1^{\rho_0} e^{-i\rho x_4+x_3 z(\rho)\rho^\theta-\rho^\theta}\\
&\qquad\times\Big[-\rho^{\frac{2}{\ell+r}}u_2(x_2,\rho)\big(\partial_1^2-y_1^{2(\ell+r-1)}(1-(z(\rho))^2\rho^{2(\theta-1)})\big)u_1(y_1,\rho)\\
&\qquad\qquad+\rho^{-\frac{2\ell}{\ell+r}}y_1^{2\ell}u_1(y_1,\rho)\big[\big(-\partial_2^2-x_2^{2(p-1)}(z(\rho))^2\rho^{2\theta}+x_2^{2(q-1)}\rho^2\big)\big]u_2(x_2,\rho)\Big]_{y_1=\rho^{1/(\ell+r)}x_1}d\rho.
\end{aligned}
\tag{3-21}
$$

Here we used that for $\rho\ge\rho_0$, (3-5) vanishes and we are left with an integral over a finite interval whose upper endpoint depends on the problem data. This defines a real analytic function, $g(x)$.

We need now to check that the growth rates of u_1 and u_2 do not affect the behavior of the integral (3-2) where u has been replaced by the right-hand side of (3-3).

Both u_1 and u_2 are eigenfunctions of the same kind of Schrödinger-type operator with different expressions of Planck's constant: u_2 is an eigenfunction of the operator in (3-11), while u_1 is an eigenfunction of the operator (3-15) where $z(h)$ has been determined according to Proposition 3.3. It is not difficult to see that the two equations are similar, so that discussing one of them is enough.

Let us focus on (3-11). We have to discuss u_2 in a classically forbidden region, i.e., where h is small, which corresponds to large values of ρ, since x is in a neighborhood of the origin. More precisely we need an estimate of the form (3-24), i.e., a bound from below of $u_2(0,\hbar)$. This type of tunneling estimate could be deduced from the results of Helffer and Sjöstrand [1984]; see also [Helffer 1988, Section 2.3]. Another way of deriving such an estimate as a consequence of [Helffer and Sjöstrand 1984] uses [Martinez 1987].

In the present particular case, we may easily reduce the problem of a pointwise estimate to the problem of an L^2 estimate and we actually use a bound, given by Zworski [2012], for the L^2 norm of u_2 in a "forbidden region":

Theorem 3.7 [Zworski 2012, Theorem 7.7]. *Let U be a neighborhood of the origin in $\mathbb{R}$. There exist positive constants $C,\hbar_0$ such that, for $0<\hbar\le\hbar_0$,*

$$\|u_2\|_{L^2(U)}\ge e^{-\frac{C}{\hbar}}\|u_2\|_{L^2(\mathbb{R})}. \tag{3-22}$$

Here

$$\hbar=\frac{h}{z(h)^{\frac{q}{q-p}}}, \tag{3-23}$$

and we note that $\hbar$ is small if and only if h is small, because $z(h)$ is bounded away from zero when h tends to zero.

The Schrödinger operators we deal with have a symmetric potential, so that their eigenfunctions are either even or odd functions with respect to the variable x. The argument is analogous to that in [Albano et al. 2016] and we just sketch it for completeness.

Case of even eigenfunctions u_1, u_2. We may assume that

$$\|u_2\|_{L^2(\mathbb{R})} = 1, \quad u_2(0,\hbar) > 0,$$

since $u_2'(0,\hbar) = 0$ because of its parity and $u_2(0,\hbar) = 0$ would imply that u_2, being a solution of (3-12), is identically zero. A similar conclusion holds for u_1.

Moreover, by (3-12), $\partial_y^2 u_2(0,\hbar) > 0$.

Denote by $x_0 = x_0(\hbar)$ the first positive zero of $V(x) - E(\hbar) = x^{2(q-1)} - x^{2(p-1)} - \hat{\gamma} - E(\hbar)$. Note that u_2 is strictly positive in the interval $0 \le x \le x_0$.

By (3-12), u_2 is strictly convex for $0 \le x \le x_0$ and has its minimum at the origin and its maximum at x_0.

Define $\varphi = \partial_x u_2/u_2$. We have $\varphi > 0$ if $0 < y \le x_0$. Then, writing φ' for $\partial_y \varphi$,

$$\varphi' = \frac{V-E}{\hbar^2} - \varphi^2.$$

The function φ has a maximum in the interval $]0, x_0[$ because $\varphi'(0) > 0$ and $\varphi'(x_0) = -\varphi^2(x_0) < 0$. Denote by $\bar{x}$ the point where the maximum is attained: it lies in the interior of the interval $[0, x_0]$. Moreover we get

$$\varphi(\bar{x}) = \frac{(V(\bar{x}) - E(\hbar))^{1/2}}{\hbar}.$$

From the definition of φ we obtain

$$u_2(0,\hbar) = e^{-\int_0^{x_0}\varphi(s)\,ds} u_2(x_0,\hbar) \ge e^{-x_0\varphi(\bar{x})} \frac{1}{\sqrt{2x_0}} \|u_2\|_{L^2([-x_0,x_0])} \ge \frac{1}{\sqrt{2x_0}} e^{-\frac{(-\hat{\gamma})^{1/2}}{\hbar}} e^{-\frac{C}{\hbar}}. \tag{3-24}$$

Here we used Theorem 3.7, as well as the facts that $x_0 < 1$, $E(\hbar) > 0$ and u_2 is normalized. We remark that $\liminf x_0(\hbar) > 0$ when $\hbar > 0$.

We are now in a position to conclude the proof of Theorem 2.1 for an even function u_2. We recall that

$$\hbar = \mathscr{O}(\rho^{\frac{r-1}{\ell+r}\frac{1}{q-1} - \frac{\ell+1}{\ell+r}}) = \mathscr{O}(\rho^{-\varkappa}).$$

Note that

$$A(u) = A_0(u) + A_1(u),$$

where A_0 is defined as the integral in (3-2) over the interval $[1, \rho_0]$, while $A_1(u)$ is the integral in (3-2) over the half-line $[\rho_0, +\infty[$. It is very easy to show that $A_0(u)$, as well as the right-hand side of (3-21), are real analytic functions of x, so that $PA_1(u) = P[A(u) - A_0(u)] \in C^\omega$.

We now compute, assuming that both u_1 and u_2 are even,

$$(-D_{x_4})^k A_1(u)(0) = \int_{\rho_0}^{+\infty} e^{-\rho^\theta} \rho^k u_1(0) u_2(0,\rho)\,d\rho \ge u_1(0) C \int_{\rho_0}^{+\infty} e^{-\rho^\theta - C_1\rho^\varkappa} \rho^k\,d\rho \ge C_2^{k+1} k!^{s_\ell}.$$

The last inequality above holds since, observing that $\varkappa < \theta$,

$$\int_{\rho_0}^{+\infty} e^{-\rho^\theta - C_1\rho^\varkappa} \rho^k \, d\rho \geq C_1 \int_{\rho_0}^{+\infty} e^{-c\rho^\theta} \rho^k \, d\rho = -C_1 \int_0^{\rho_0} e^{-c\rho^\theta} \rho^k \, d\rho + C_2^{k+1} k!^{s_\ell}$$
$$\geq C_2^{k+1} k!^{s_\ell} \left(1 - C_1 C_2^{-(k+1)} \rho_0 e^{-c\rho_0^\theta} \frac{\rho_0^k}{k!^{s_\ell}}\right) \geq C_3^{k+1} k!^{s_\ell},$$

if k is suitably large and C_3 is suitable and positive.

Case when u_1 is even and u_2 is odd. We may assume that

$$\|u_2\|_{L^2(\mathbb{R})} = 1, \quad u_2'(0, \hbar) > 0.$$

Moreover, due to the parity, $u_2''(0, \hbar) = 0$. Arguing as above we obtain that u_2' is positive in $[0, x_0]$. Set

$$\varphi = \frac{u_2''}{u_2'}.$$

Arguing as above we deduce

$$u_2'(0, \hbar) \geq e^{-\frac{(-\hat{\nu})^{1/2}}{\hbar}} u_2'(x_0, \hbar) \geq \frac{1}{\sqrt{2x_0}} e^{-\frac{(-\hat{\nu})^{1/2}}{\hbar}} \|u_2'\|_{L^2([-x_0, x_0])}.$$

Since

$$\|u_2\|_{L^2([-x_0, x_0])} \leq x_0 \|u_2'\|_{L^2([-x_0, x_0])},$$

we get

$$u_2'(0, \hbar) \geq \frac{1}{x_0\sqrt{2x_0}} e^{-\frac{(-\hat{\nu})^{1/2}}{\hbar}} \|u_2\|_{L^2([-x_0, x_0])}.$$

Using Theorem 3.7 as before we can conclude exactly as in the case of an even eigenfunction.

To finish the proof of Theorem 2.1 we recall that Lemma 3.6 implies that the integral in the definition of $A(\partial_{x_2} u)$ is absolutely convergent, so that, arguing as before, we have

$$(-D_{x_4})^k A_1(\partial_{x_2} u)(0) = \int_{\rho_0}^{+\infty} e^{-\rho^\theta} \rho^k u_1(0)\, \partial_{x_2} u_2(0, \rho) \, d\rho$$
$$\geq u_1(0)\, C \int_{\rho_0}^{+\infty} e^{-\rho^\theta - C_1 \rho^\varkappa} \rho^k \, d\rho \geq C_2^{k+1} k!^{s_\ell},$$

again provided k is suitably large.

The other cases, when u_1 is odd and u_2 is even or odd, are treated analogously and we skip them.

This concludes the proof of the optimality part of Theorem 2.1.

4. Proof of Theorem 2.1 (continued)

In this section we prove that the operator P is Gevrey s_ℓ-hypoelliptic. We also point out that the proof given here works when $p = q$, yielding analytic hypoellipticity.

It is useful to establish notation for the vector fields defining P:

$$P(x,D) = D_1^2 + x_1^{2\ell} D_2^2 + x_1^{2(\ell+r-1)} D_3^2 + x_1^{2(\ell+r-1)} D_4^2 + x_1^{2\ell} x_2^{2(p-1)} D_3^2 + x_1^{2\ell} x_2^{2(q-1)} D_4^2$$
$$= \sum_{j=1}^{6} X_j(x,D)^2. \tag{4-1}$$

We note that, using commutators of the fields up to the length $\ell + r$, we generate the ambient space.

The basic idea for the proof is to use the subelliptic estimate; see, e.g., [Jerison 1986] or [Bolley et al. 1982] for a proof of the inequality

$$\|u\|^2_{\frac{1}{\ell+r}} + \sum_{j=1}^{6} \|X_j(x,D)u\|^2 \le C\big(\langle P(x,D)u, u\rangle + \|u\|^2\big), \tag{4-2}$$

where C is a positive constant, $\|\cdot\|_{\frac{1}{\ell+r}}$ is the Sobolev norm of order $\frac{1}{\ell+r}$ and $u \in C_0^\infty(\mathbb{R}^4)$.

A further remark is that we may assume $\xi_4 \ge 1$: in fact denoting by ψ a cutoff function such that $\psi \ge 0$, $\psi(\xi_4) = 1$ if $\xi_4 \ge 2$ and $\psi(\xi_4) = 0$ if $\xi_4 \le 1$, we may apply $\psi(D_4)$ to the equation $Pu = f$, getting $P\psi u = \psi f$, since ψ commutes with P. On the other hand, $\psi f \in G^s$ if $f \in G^s$, for $s \ge 1$, and we are interested in the microlocal Gevrey regularity of u at the point $(0, e_4)$. We write u instead of ψu.

The proof below uses the estimate (4-2) in the following way. Since P is C^∞ hypoelliptic, we may assume the function u to be smooth. If we want to show that it belongs to a Gevrey class we have to bound its derivatives, or, which is easier using (4-2), the L^2 norm of its derivatives by suitable factorials (see Definition 2.2).

To do that, we start with the quantity $X_j \varphi D_4^N u$, where X_j is one of the vector fields appearing in the operator, φ is a cutoff — in general a microlocal cutoff — which is discussed below, and N is an arbitrarily large natural number. If we succeed in bounding $\|X_j \varphi D_4^N u\|$ with $C^{N+1} N!^s$, then we may deduce that u belongs to the Gevrey class G^s on the domain of the cutoff φ.

Thus, feeding the quantity $\varphi D_4^N u$ on the left-hand side of (4-2), we have to estimate the right-hand side: there we have an error term that usually is easy to absorb on the left, but also a term where P appears. In particular we have to treat the term $\langle P\varphi D_4^N u, \varphi D_4^N u\rangle$. Since we are assuming Pu is real analytic or Gevrey s, it is evident that commuting P past the cutoff will lead us to a term $\langle \varphi D_4^N Pu, \varphi D_4^N u\rangle$. This is good, since Pu has analytic estimates and the right factor of the scalar product can be absorbed on the left-hand side of the inequality, like the error term. Unfortunately there is also the commutator $\langle [P, \varphi D_4^N]u, \varphi D_4^N u\rangle$.

Now the commutator either gives vector fields applied to $\varphi D_4^n u$, which are easily absorbed on the left, or gives derivatives of the cutoff and of the coefficients of the vector fields. The derivatives of the cutoff are fine, provided we use suitably chosen cutoffs, see below for this, but the derivatives of the coefficients of the vector fields are more difficult to handle. Actually either one is able to extract another vector field from them — which seldom occurs — or the best resource available is the subelliptic term: we may lower the exponent N by the subellipticity and use the subelliptic part of the estimate to start over with a lower N and derivatives of φ replacing φ. The process terminates when N is completely used and the derivatives of the cutoff give us the final estimate.

Let us start by denoting by $\varphi_N = \varphi_N(x_3, x_4)$, $\chi_N = \chi_N(\xi_4)$ cutoffs of Ehrenpreis type, i.e., $\varphi_N \in C_0^\infty(\mathbb{R}^2)$, $\chi_N \in C^\infty(\mathbb{R})$, with $\varphi_N = 1$ near the origin, $\chi_N = 0$ for $x < 3$ and $\chi_N = 1$ for $x > 4$, and φ_N, χ_N nonnegative.

Ehrenpreis-type functions have the property that $|\partial^k \varphi_N(x)| \le C_\varphi^{k+1} N^k$ for $k \le RN$, $R > R_0$, and C_φ independent of N. See, e.g., [Hörmander 1971, Lemma 2.2] for the definition as well as a construction of such type of functions.

As sketched above we want to estimate the quantity $\|X_j \varphi_N D_4^N u\|$, $j = 1, \dots, 6$, so that getting an estimate of the form $\|X_j \varphi_N D_4^N u\| \le C^{N+1} N^{s_\ell N}$ will be enough to conclude that $u \in G^{s_\ell}$ microlocally at $(0, e_4)$.

As a preliminary remark we point out that if $Pu = f$, $f \in G^{s_\ell}(\Omega)$, then we may assume that $u \in C^\infty(\Omega)$ and u has compact support with respect to the variables x_1, x_2. In fact, if $\theta = \theta(x_1, x_2) \in G^{s_\ell} \cap C_0^\infty$ and is identically equal to 1 in a neighborhood of the origin, we obtain, multiplying the equation $Pu = f$ by θ, that $P(\theta u) = \theta f - [P, \theta]u$ and the commutator term is identically zero in a neighborhood of the origin in the (x_1, x_2)-plane; i.e., it is in G^{s_ℓ}, since u is in G^{s_ℓ} outside of the characteristic manifold. We write u instead of θu.

Now

$$\|X_j \varphi_N D_4^N u\| \le \|X_j \varphi_N (1 - \chi_N(N^{-1} D_4)) D_4^N u\| + \|X_j \varphi_N \chi_N(N^{-1} D_4) D_4^N u\|. \tag{4-3}$$

Consider the first summand above. Since $(1 - \chi_N)\psi$ has support for $1 \le \xi_4 \le 4N$, we deduce immediately a bound of the first summand:

$$\|X_j \varphi_N (1 - \chi_N(N^{-1} D_4)) D_4^N u\| \le C^{N+1} N^N,$$

where C denotes a positive constant independent of N, but depending on u. This means a real analytic growth rate for u. It is enough then to bound the second summand in (4-3).

To do this we plug the quantity $\varphi_N \chi_N D_4^N u$ into (4-2) and, as a consequence, we obtain

$$\begin{aligned}
&\|X_j \varphi_N \chi_N(N^{-1} D_4) D_4^N u\|^2 \\
&\qquad \le C\big(\langle P \varphi_N \chi_N(N^{-1} D_4) D_4^N u, \varphi_N \chi_N(N^{-1} D_4) D_4^N u\rangle + \|\varphi_N \chi_N(N^{-1} D_4) D_4^N u\|^2\big)
\end{aligned}$$

Our main concern is the estimate of the scalar product in the next-to-last line of the above formula. We have

$$\begin{aligned}
&\langle P \varphi_N \chi_N(N^{-1} D_4) D_4^N u, \varphi_N \chi_N(N^{-1} D_4) D_4^N u\rangle \\
&\qquad = \langle \varphi_N \chi_N(N^{-1} D_4) D_4^N Pu, \varphi_N \chi_N(N^{-1} D_4) D_4^N u\rangle \\
&\qquad\qquad + \sum_{j=1}^{6} \langle [X_j^2, \varphi_N] \chi_N(N^{-1} D_4) D_4^N u, \varphi_N \chi_N(N^{-1} D_4) D_4^N u\rangle.
\end{aligned}$$

The first term in the right-hand side of the above relation poses no problem: in fact $Pu = f \in G^{s_\ell}$ and thus the scalar product is easily estimated by $C^{N+1} N!^{s_\ell}$, while the right factor can be absorbed on the left.

As for the summands containing a commutator,

$$\begin{aligned}\big\langle [X_j^2, \varphi_N]\chi_N(N^{-1}D_4)D_4^N u, \varphi_N\chi_N(N^{-1}D_4)D_4^N u\big\rangle \\ = 2\big\langle [X_j, \varphi_N]\chi_N(N^{-1}D_4)D_4^N u, X_j\varphi_N\chi_N(N^{-1}D_4)D_4^N u\big\rangle \\ -\big\langle N^{-1}[X_j, [X_j, \varphi_N]]\chi_N(N^{-1}D_4)D_4^N u, N\varphi_N\chi_N(N^{-1}D_4)D_4^N u\big\rangle. \quad (4\text{-}4)\end{aligned}$$

Here we multiplied and divided by N the factors of the second scalar product to compensate for the second derivative landing on φ_N because of the double commutator. The naïve idea behind this is that one derivative of φ_N is worth N, since φ_N is an Ehrenpreis-type cutoff function.

We are going to examine the terms with a single commutator first. Both X_1, X_2 commute with φ_N at this moment, since φ_N depends on x_3 and x_4, even though we shall see shortly that this is not going to be true any longer, for both X_1 and X_2 will have a nonzero commutator with the coefficients of the vector fields. Moreover

$$[X_j, \varphi_N]\chi_N(N^{-1}D_4)D_4^N u = x_1^{\ell+r-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u \quad (4\text{-}5)$$

for $j = 3, 4$. Here we just denote by φ_N' a (self-adjoint) derivative with respect to x_3 or x_4, since a more precise notation would only burden the exposition. Furthermore we have

$$[X_5, \varphi_N]\chi_N(N^{-1}D_4)D_4^N u = x_1^{\ell}x_2^{p-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u$$

and

$$[X_6, \varphi_N]\chi_N(N^{-1}D_4)D_4^N u = x_1^{\ell}x_2^{q-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u.$$

Let us consider the terms corresponding to $j = 3, 4$ first:

$$\begin{aligned}2\big|\big\langle x_1^{\ell+r-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u, X_j\varphi_N\chi_N(N^{-1}D_4)D_4^N u\big\rangle\big| \\ \le \delta\|X_j\varphi_N\chi_N(N^{-1}D_4)D_4^N u\|^2 + \frac{1}{\delta}\|x_1^{\ell+r-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u\|^2, \quad (4\text{-}6)\end{aligned}$$

where δ is a positive number so small to allow us to absorb the first summand in the right-hand side of (4-6) on the left of the subelliptic estimate.

In order to be able to apply again the subelliptic estimate to the second summand above we need to use the formula

$$\varphi_N' D_4^N = \sum_{j=0}^{N-1}(-1)^j D_4\varphi_N^{(j+1)}D_4^{N-j-1} + (-1)^N\varphi_N^{(N+1)}. \quad (4\text{-}7)$$

Thus, since $\chi_N(N^{-1}D_4)$ commutes with D_4^N,

$$\begin{aligned}\|x_1^{\ell+r-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u\| \\ \le \sum_{j=0}^{N-1}\|X_4\varphi_N^{(j+1)}\chi_N(N^{-1}D_4)D_4^{N-j-1}u\| + \|\varphi_N^{(N+1)}\chi_N(N^{-1}D_4)u\|, \quad (4\text{-}8)\end{aligned}$$

where we used the fact that the field X_4 can be reconstructed using the factor $x_1^{\ell+r-1}$ and just "pulling back" one x_4-derivative. A completely analogous treatment leads to an analogous conclusion when $j = 6$:

$$\begin{aligned}\|[X_6,\varphi_N]\chi_N(N^{-1}D_4)D_4^N u\| &= \|x_1^\ell x_2^{q-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u\| \\ &\le \sum_{j=0}^{N-1}\|X_6\varphi_N^{(j+1)}\chi_N(N^{-1}D_4)D_4^{N-j-1}u\|+\|\varphi_N^{(N+1)}\chi_N(N^{-1}D_4)u\|.\end{aligned} \tag{4-9}$$

Furthermore it is clear that the terms on the right of inequalities (4-8) and (4-9) yield a real analytic growth estimate: in fact a typical term in the sum loses $j+1$ derivatives of u with respect to x_4 while the cutoff function φ_N picks up $j+1$ derivatives. Using the properties of φ_N we see that, arguing inductively, this gives analytic growth with respect to N. Same argument for the last terms in (4-8) and (4-9).

We are thus left with the commutator term for $j = 5$ in (4-4):

$$\begin{aligned}&2\left|\langle x_1^\ell x_2^{p-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u, X_5\varphi_N\chi_N(N^{-1}D_4)D_4^N u\rangle\right| \\ &\qquad \le \delta\|X_5\varphi_N\chi_N(N^{-1}D_4)D_4^N u\|^2+\frac{1}{\delta}\|x_1^\ell x_2^{p-1}\varphi_N'\chi_N(N^{-1}D_4)D_4^N u\|^2.\end{aligned} \tag{4-10}$$

Here, again, δ is chosen so that the first term in the right-hand side above can be absorbed on the left of the subelliptic estimate, as before. We just need to be concerned with the second term. Contrary to what has been done before, pulling back one derivative is of no help, since $p < q$ and the derivative with respect to x_4 is the only derivative available here. Note that if $p = q$ then we may act at this point as we did for $j = 3, 4, 6$, obtaining analytic growth estimates. So let us go on assuming that $p < q$.

Hence we have to resort to the subelliptic part of the subelliptic estimate, i.e., the $1/(\ell + r)$–Sobolev norm. To do this we pull back $D_4^{1/(\ell+r)}$. This is well defined since $\xi_4 > 1$, but is a pseudodifferential operator, and its commutator with φ_N needs some care.

We actually have the following lemmas. Let $\omega_N \in C^\infty(\mathbb{R})$ be an Ehrenpreis-type cutoff such that $\omega_N = 1$ for $x > 2$ and $\omega_N = 0$ for $x < 1$, ω_N nonnegative and such that $\omega_N\chi_N = \chi_N$. Then we have:

Lemma 4.1. *Let* $0 < \theta < 1$. *Then*

$$[\omega_N(N^{-1}D)D^\theta,\varphi_N(x)]\chi_N(N^{-1}D)D^{N-\theta}=\sum_{k=1}^N a_{N,k}(x,D)\chi_N(N^{-1}D)D^N, \tag{4-11}$$

where $a_{N,k}$ *is a pseudodifferential operator of order* $-k$ *such that*

$$|\partial_\xi^\alpha a_{N,k}(x,\xi)|\le C_a^{k+1}N^{k+\alpha}\xi^{-k-\alpha},\quad 1\le k\le N,\ \alpha\le N. \tag{4-12}$$

Corollary 4.2. *For* $1 \le k \le N-1$ *in* (4-11) *we have that*

$$a_{N,k}(x,D)\chi_N(N^{-1}D)D^N=\frac{\theta(\theta-1)\cdots(\theta-k+1)}{k!}D_x^k\varphi_N(x)\chi_N(N^{-1}D)D^{N-k}. \tag{4-13}$$

For the proofs we refer to [Albano et al. 2016, Appendix B].

Applying Corollary 4.2, we find that

$$
\begin{aligned}
\|x_1^{\ell} x_2^{p-1} \varphi_N' \chi_N(N^{-1}D_4) D_4^N u\| &= \|x_1^{\ell} x_2^{p-1} \varphi_N' \omega_N(N^{-1}D_4) D_4^{\frac{1}{\ell+r}} \chi_N(N^{-1}D_4) D_4^{N-\frac{1}{\ell+r}} u\| \\
&\le c_0 \|x_1^{\ell} x_2^{p-1} \varphi_N' \chi_N(N^{-1}D_4) D_4^{N-\frac{1}{\ell+r}} u\|_{\frac{1}{\ell+r}} \\
&\quad + \sum_{k=1}^{N-1} c_k \|x_1^{\ell} x_2^{p-1} \varphi_N^{(k+1)} \chi_N(N^{-1}D_4) D_4^{N-k} u\| \\
&\quad + c_N \|x_1^{\ell} x_2^{p-1} a_{N,N}(x,D) \chi_N(N^{-1}D_4) D_4^N u\|. \qquad (4\text{-}14)
\end{aligned}
$$

Here the constants c_j, $j = 0, 1, \dots, N$, are bounded independently of N by some absolute constant.

The last term in the right-hand side of (4-14) has analytic growth, because $a_{N,N}$ has order $-N$, so that it balances the N-th derivative on u, and is bounded by $C_a^{N+1} N^N$, according to (4-12). Thus we may forget about it because it gives better estimates than those we are going to get.

The first term on the right-hand side of (4-14) can be resubjected to the subelliptic estimate — Sobolev part — and treated as we just did. This means that again we have to consider the commutator of the vector fields in the operator P with $x_1^{\ell} x_2^{p-1} \varphi_N'$, since D_4 commutes with P. Hence the quantity we have to estimate is

$$
\|[X_j, x_1^{\ell} x_2^{p-1} \varphi_N'] \chi_N(N^{-1}D_4) D_4^{N-\frac{1}{\ell+r}} u\|;
$$

see (4-4), (4-5). As we said before, the commutators with X_1, X_2 are no longer zero, because of the monomials $x_1^{\ell} x_2^{p-1}$. Note that commuting with X_1 has just the effect of lowering the exponent ℓ by one unit, while commuting with X_2 lowers the exponent $p-1$ by one unit and increases the exponent ℓ by ℓ units.

On the other hand commuting with X_3, X_4 and X_6 ignores these types of monomials and allows us to treat the above quantity exactly as we did before, yielding analytic-type growth estimates. Note that the monomials can be estimated by a constant, since we are in a neighborhood of the origin in the (x_1, x_2)-plane.

Finally commuting with X_5 is done again as before, but its outcome is to double the exponents of the monomial above, increase the number of derivatives of φ_N and lower the x_4-derivatives on u by $1/(\ell+r)$-th of a unit. In other words, the first term on the right-hand side of (4-14) at the second application of the subelliptic estimate would be

$$
\|x_1^{2\ell} x_2^{2(p-1)} \varphi_N'' \chi_N(N^{-1}D_4) D_4^{N-\frac{2}{\ell+r}} u\|_{\frac{1}{\ell+r}}.
$$

We use the same argument for every summand in the sum in (4-14), except that here we have to pull back a $D_4^{1/(\ell+r)}$ once more in order to use the subelliptic estimate employing Lemma 4.1 as well as Corollary 4.2.

Hence applying the subelliptic estimate — Sobolev part — alters the exponents in the monomials $x_1^{\ell} x_2^{p-1}$, lowers the x_4-derivatives on u, and increases the derivatives of φ_N.

When the exponent of x_2 becomes greater than or equal to $q-1$ or the exponent of x_1 becomes greater than or equal to $\ell+r-1$, whichever comes first, we do not need to use the Sobolev part of the subelliptic estimate anymore. At that point we are able to pull back a whole x_4-derivative, reconstructing the field X_6 in the first case or X_4 in the second. The powers of x_1 and x_2 are decreased in the first case, but only the power of x_1 is decreased in the second case.

Before writing the product of such an iteration, we discuss also the terms from (4-4) containing a double commutator. For $j=3,4$ we have that

$$\begin{aligned}\big|\big\langle N^{-1}x_1^{\ell+r-1}\varphi_N''\chi_N(N^{-1}D_4)D_4^N u, Nx_1^{\ell+r-1}\varphi_N\chi_N(N^{-1}D_4)D_4^N u\big\rangle\big| \\ \le \tfrac{1}{2}\|N^{-1}x_1^{\ell+r-1}\varphi_N''\chi_N(N^{-1}D_4)D_4^N u\|^2+\tfrac{1}{2}\|Nx_1^{\ell+r-1}\varphi_N\chi_N(N^{-1}D_4)D_4^N u\|^2.\end{aligned}$$

Each of the summands in the right-hand side is then treated as we did before with the terms involving a single commutator. Note that $N^{-1}\varphi_N''$ counts as a first derivative and so does $N\varphi_N$. The same argument holds for $j=5,6$.

For $j=1,\dots,6$, we iterate this procedure. The following notation is useful.

Let us denote by a_j the number of times we take a commutator, according to the procedure outlined above, with X_j. If, as we saw before, $j=3,4,6$, the net result is a decrease by one unit of the derivatives of u and a parallel increase by one unit of the derivatives of φ_N. No Sobolev part of the subelliptic estimate is used in this case.

If $j=1,2,5$, the situation is more complicated. As we said, the vector field X_5 contributes a monomial $x_1^\ell x_2^{p-1}$ and the Sobolev part will decrease the derivatives on u by $1/(\ell+r)$. However we do not always need the Sobolev part of the estimate. Let us call α, β the number of times we do not apply it, but are able to reconstruct the vector field X_4 — using $x_1^{\ell+r-1}$ — or X_6 — using the monomial $x_1^\ell x_2^{p-1}$. Of course this has an effect also on the number of derivatives on u, which lose $\alpha+\beta$ units instead of just $(\alpha+\beta)/(\ell+r)$.

Finally we must take into account all terms deriving from the application of Lemma 4.1 and Corollary 4.2. Let us denote by h the number of times we apply them and by k_j, $j=1,\dots,h$, the summation indices in (4-11). As we saw, these also have an effect on the derivatives landing on u and φ_N.

Hence we wind up with the estimate

$$\begin{aligned}&\|X_j\varphi_N\chi_N(N^{-1}D_4)D_4^N u\|\\ &\lesssim\sum\Big\|x_1^{a_5\ell-\alpha(\ell+r-1)-\beta\ell-a_1}x_2^{a_5(p-1)-\beta(q-1)-a_2}\varphi_N^{(1+a_3+a_4+a_5+a_6+k_1+\dots+k_h)}\\ &\qquad\qquad\cdot\chi_N(N^{-1}D_4)D_4^{N-k_1-\dots-k_h-\frac{a_1+a_2+a_5}{\ell+r}-a_3-a_4-a_6-(\alpha+\beta)\frac{\ell+r-1}{\ell+r}}u\Big\|\\ &\quad+\sum N^{-(a_3+a_4+a_5+a_6)}\Big\|x_1^{a_5\ell-\alpha(\ell+r-1)-\beta\ell-a_1}x_2^{a_5(p-1)-\beta(q-1)-a_2}\varphi_N^{(1+2(a_3+a_4+a_5+a_6)+k_1+\dots+k_h)}\\ &\qquad\qquad\cdot\chi_N(N^{-1}D_4)D_4^{N-k_1-\dots-k_h-\frac{a_2+a_5}{r}-a_3-a_4-a_6-b\frac{r-1}{r}}u\Big\|\\ &\quad+\sum N^{a_3+a_4+a_5+a_6}\Big\|x_1^{a_5\ell-\alpha(\ell+r-1)-\beta\ell-a_1}x_2^{a_5(p-1)-b(q-1)-a_2}\varphi_N^{(1+k_1+\dots+k_h)}\\ &\qquad\qquad\cdot\chi_N(N^{-1}D_4)D_4^{N-k_1-\dots-k_h-\frac{a_2+a_5}{r}-a_3-a_4-a_6-b\frac{r-1}{r}}u\Big\|,\end{aligned}\tag{4-15}$$

where each sum is taken on the indices $a_1, a_2, \dots, a_6, k_1, \dots, k_h$ such that

$$0 \le N - k_1 - \cdots - k_h - \frac{a_1 + a_2 + a_5}{\ell + r} - a_3 - a_4 - a_6 - (\alpha + \beta)\frac{\ell + r - 1}{\ell + r} < 1, \tag{4-16}$$

and the indices satisfy the conditions

$$0 \le a_5(p-1) - \beta(q-1) - a_2 < q - 1 \tag{4-17}$$

and

$$0 \le a_5 \ell - a_1 - \alpha(\ell + r - 1) - \beta \ell < \ell. \tag{4-18}$$

A few explanations are in order. Note that (4-16) expresses the fact that the number of derivatives on u is not negative, but is less than 1 at the end. Conditions (4-17) and (4-18) express the fact that the exponent of x_2 is less than $q-1$ and that the exponent of x_1 is less than ℓ, i.e., when we cannot pull back a derivative and need to apply the Sobolev part of the subelliptic estimate when doing the iteration.

Finally we observe that (4-15) has three sums on the right-hand side. The first is the sum coming from the single commutator terms in (4-4), while the others come from the double commutator terms in (4-4). Note that in the last one the power of N is positive, but φ_N has less derivatives than in the second sum. The balance though is the same in both terms.

We have to evaluate the supremum of the right-hand side in (4-15). From the second condition,

$$\frac{a_5(p-1) - a_2}{q-1} - 1 < \beta \le \frac{a_5(p-1) - a_2}{q-1},$$

while from the third we get

$$\frac{a_5\ell - a_1 - \frac{a_5(p-1)-a_2}{q-1}\ell - \ell}{\ell + r - 1} < \alpha < \frac{a_5\ell - a_1 - \frac{a_5(p-1)-a_2}{q-1}\ell + \ell}{\ell + r - 1},$$

so that from the first condition we deduce

$$N - 1 - \frac{\ell}{\ell + r} < \sum_{j=1}^{h} k_j + a_2\gamma_2 + a_3 + a_4 + a_5\frac{1}{s_\ell} + a_6 < N + 1 + \frac{\ell - 1}{\ell + r},$$

where

$$\gamma_2 = \frac{1}{\ell + r} - \frac{r-1}{(\ell + r)(q-1)} > 0.$$

In order to estimate (4-15) we need to compute

$$\max\left\{\sum_{j=1}^{h} k_j + a_3 + a_4 + a_5 + a_6\right\},$$

where the maximum is on all indices verifying conditions (4-16)–(4-18). More precisely, the three sums in (4-15) give the same contribution, because when an index is missing among the derivatives of φ_N it is found at the exponent of the factor N and, vice versa, when it appears twice, it appears with a negative sign at the exponent of N.

It is then clear that the maximum is $\sim N^{Ns_\ell}$ so that we finally get

$$\|X_j \varphi_N \chi_N(N^{-1}D_4) D_4^N u\| \le C^{N+1} N!^{s_\ell},$$

and this achieves the proof of the theorem, since the above inequality shows that the function u is in the Gevrey class G^{s_ℓ} in the support of φ_N, $N \in \mathbb{N}$.

Note that if $p = q$, a much simpler proof — which means without any use of the Sobolev part of the subelliptic estimate — yields analytic regularity.

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