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CONSTRUCTION OF TWO-BUBBLE SOLUTIONS
FOR THE ENERGY-CRITICAL NLS

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We construct pure two-bubbles for the energy-critical focusing nonlinear Schrödinger equation in space dimension $N \geq 7$. The constructed solution is global in (at least) one time direction and approaches a superposition of two stationary states both centered at the origin, with the ratio of their length scales converging to 0. One of the bubbles develops at scale 1, whereas the length scale of the other converges to 0 at rate $|t|^{-\frac{2}{N-6}}$. The phases of the two bubbles form the right angle.

1. Introduction

Setting of the problem. We consider the Schrödinger equation with the focusing energy-critical power nonlinearity given by

$$i\,\partial_t u(t, x) + \Delta u(t, x) + f(u(t, x)) = 0, \quad f(z) := |z|^{\frac{4}{N-2}} z, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^N. \tag{1-1}$$

This equation can be studied in space dimension $N \geq 3$, but we will restrict our attention to the case $N \geq 7$.

The *energy functional* associated with this equation is defined for $u_0 \in \dot{H}^1(\mathbb{R}^N; \mathbb{C})$ by the formula

$$E(u_0) := \int_{\mathbb{R}^N} \tfrac{1}{2} |\nabla u_0(x)|^2 - F(u_0(x)) \, dx,$$

where

$$F(z) := \frac{N-2}{2N} |z|^{\frac{2N}{N-2}}.$$

Note that $E(u_0)$ is well-defined due to the Sobolev embedding theorem. The differential of E is $\mathrm{D}E(u_0) = -\Delta u_0 - f(u_0)$; hence we have the following Hamiltonian form of (1-1):

$$\partial_t u(t) = -i\,\mathrm{D}E(u(t)).$$

Equation (1-1) is locally well-posed in the space $\dot{H}^1(\mathbb{R}^N)$, as was proved by Cazenave and Weissler [1990]; see also a complete review of Cauchy theory in [Kenig and Merle 2006] for $N \in \{3, 4, 5\}$ and [Killip and Visan 2010] for $N \geq 6$. By "well-posed" we mean that for any initial data $u_0 \in \dot{H}^1(\mathbb{R}^N)$ there exists $\tau > 0$ and a linear subspace $S \subset C([t_0 - \tau, t_0 + \tau]; \dot{H}^1(\mathbb{R}^N))$ such that there exists a unique weak solution $u(t) \in S$ of (1-1) satisfying $u(t_0) = u_0$, and that this solution is continuous with respect to the initial data. By standard arguments, there exists a maximal time of existence (T_-, T_+), $-\infty \leq T_- < t_0 < T_+ \leq +\infty$, and a unique solution $u \in C((T_-, T_+); \dot{H}^1(\mathbb{R}^N))$. Moreover, if $u_0 \in X^1 :=$

$\dot{H}^2(\mathbb{R}^N) \cap \dot{H}^1(\mathbb{R}^N)$, then $u \in C((T_-, T_+); X^1)$. If $T_+ < +\infty$, then $u(t)$ leaves every compact subset of $\dot{H}^1(\mathbb{R}^N)$ as t approaches T_+. A crucial property of the solutions of (1-1) is that the energy E is a conservation law. If $u_0 \in L^2$, then the mass $\|u(t)\|_{L^2}^2$ is another conservation law, but we will never use this fact.

In this paper, we always assume that the initial data are radially symmetric. This symmetry is preserved by the flow. We denote by $\mathcal{E}$ the space of radially symmetric functions in $\dot{H}^1(\mathbb{R}^N; \mathbb{C})$.

For a function $v \in \mathcal{E}$, we define

$$v_\lambda(x) := \frac{1}{\lambda^{\frac{N-2}{2N}}} v\left(\frac{x}{\lambda}\right).$$

A change of variables shows that

$$E((u_0)_\lambda) = E(u_0).$$

Equation (1-1) is invariant under the same scaling: if $u(t)$ is a solution of (1-1) and $\lambda > 0$, then $t \mapsto u(t_0 + \lambda^{-2}t)_\lambda$ is also a solution with initial data $(u_0)_\lambda$ at time $t = 0$. This is why (1-1) is called *energy-critical*.

The solutions of the corresponding *defocusing* equation exist globally and scatter. This was proved by Bourgain [1999] and Tao [2005] for radial solutions, and by Colliander, Keel, Staffilani, Takaoka and Tao [Colliander et al. 2008], Ryckman and Visan [2007], and Visan [2007] for nonradial data.

The study of the dynamical behavior of solutions of the focusing equation (1-1) for large initial data was initiated by Kenig and Merle [2006]. In this case, an important role is played by the family of stationary solutions $u(t) \equiv e^{i\theta} W_\lambda$, where

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}.$$

The functions $e^{i\theta} W_\lambda$ are called *ground states* or *bubbles* (of energy). They are the only radially symmetric solutions of the critical elliptic problem

$$-\Delta u - f(u) = 0.$$

The ground states achieve the optimal constant in the critical Sobolev inequality, which was proved by Aubin [1976] and Talenti [1976]. They are the "mountain passes" for the potential energy.

Kenig and Merle [2006] exhibited the special role of the ground states $e^{i\theta} W_\lambda$ as the *threshold elements* for nonlinear dynamics of the solutions of (1-1) in space dimensions $N = 3, 4, 5$ for radial data. They proved the so-called *threshold conjecture* by completely classifying the dynamical behavior of solutions $u(t)$ of (1-1) such that $E(u(t)) < E(W)$. An analogous result in higher dimensions, for nonradial data, was obtained by Killip and Visan [2010].

A much stronger statement about the dynamics of solutions is the *soliton resolution conjecture*, which predicts that a bounded (in an appropriate sense) solution decomposes asymptotically into a sum of energy bubbles at different scales and a radiation term (a solution of the linear Schrödinger equation). This was proved for the radial energy-critical wave equation in dimension $N = 3$ by Duyckaerts, Kenig and Merle [Duyckaerts et al. 2013]; see also [Duyckaerts et al. 2017] for the nonradial case. For (1-1) this problem is completely open.

Solutions slightly above the ground state energy threshold were studied by Ortoleva and Perelman [2013] in dimension $N = 3$; see also [Perelman 2014] for the closely related critical equivariant Schrödinger map equation with values in the sphere. They constructed global solutions which stay close to $e^{i\theta} W_\lambda$ in the energy space, with λ converging to 0 as time t goes to $+\infty$. These solutions decompose into a concentrating bubble and a radiation term, in accordance with the soliton resolution conjecture. The works of Ortoleva and Perelman follow the approach developed by Krieger, Schlag and Tataru [Krieger et al. 2008; 2009] for wave equations. For the Schrödinger maps, following a different approach, Merle, Rodnianski and Raphaël [Merle et al. 2013] obtained blow-up solutions which are stable relative to a set of finite codimension in some space which contains the bubble.

On the classification side, it is unknown whether the soliton resolution conjecture holds even with an additional assumption that the solution remains close to the family of the ground states. In the mass-critical case and for a solution blowing up in finite time, this was proved by Merle and Raphaël [2004; 2005]; see also [Fan 2016].

Main results. In view of the soliton resolution conjecture, solutions which exhibit no dispersion in one or both time directions play a distinguished role. One obvious example of such solutions are the static solutions $e^{i\theta} W_\lambda$. In this paper, we consider the simplest nontrivial case; namely we construct global radial solutions which approach, in the energy space, a sum of two bubbles. The ratio of the scales at which these bubbles develop tends to 0.

Theorem 1. *There exists a solution* $u : (-\infty, T_0] \to \mathcal{E}$ *of* (1-1) *such that*

$$\lim_{t \to -\infty} \left\| u(t) - (-i W + W_{\frac{1}{\kappa}(\kappa|t|)^{-2/(N-6)}}) \right\|_{\mathcal{E}} = 0,$$

where κ *is an explicit constant.*

Remark 1.1. For the value of κ, see (3-4).

Remark 1.2. More precisely, we will prove that

$$\left\| u(t) - (-i W + W_{\frac{1}{\kappa}(\kappa|t|)^{-2/(N-6)}}) \right\|_{\mathcal{E}} \leq C_1 |t|^{-\frac{1}{2(N-6)}}$$

for some constant $C_1 > 0$.

Remark 1.3. We construct here *pure* two-bubbles; that is, the solution approaches a superposition of two stationary states, with no energy transformed into radiation. By the conservation of energy and the decoupling of the two bubbles, we necessarily have $E(u(t)) = 2E(W)$. Pure one-bubbles cannot concentrate and are completely classified; see [Duyckaerts and Merle 2009].

Remark 1.4. For energy-critical wave equations, similar objects were constructed in [Jendrej 2016].

Remark 1.5. In dimension $N = 6$ one can expect an analogous result, with an exponential concentration rate.

Remark 1.6. In higher dimension, fast dispersion or dissipation sometimes excludes the possibility of a concentration of a bubble of energy for solutions which belong to a small neighborhood of a bubble. This

was proved in [Collot et al. 2017] in the case of the critical heat equation; Perelman addressed the case for the Schrödinger equation in a lecture given at an IHES seminar in July 2016. We prove here that once we leave a small neighborhood of a bubble, concentration of a bubble of energy is possible in arbitrarily high dimension.

A similar phenomenon was observed by Martel and Raphaël [2015] for the mass-critical NLS.

Remark 1.7. We expect that the phases of the two bubbles forming the right angle is the only configuration in which a two-bubble can form.

Outline of the proof. The overall structure is similar to the earlier work of the author on the critical wave equations [Jendrej 2016]. We build a sequence $u_n : [T_n, T_0] \to \mathcal{E}$ of solutions of (1-1) with $T_n \to -\infty$ and $u_n(t)$ close to a two-bubble solution for $t \in [T_n, T_0]$. Taking a weak limit finishes the proof. This type of argument goes back to the works of Merle [1990] and Martel [2005]. The heart of the analysis is to obtain uniform energy bounds for the sequence u_n. This is achieved by means of a bootstrap argument, which can be resumed as follows.

We study solutions of (1-1) close to a sum of two bubbles:

$$u(t) = e^{i\zeta(t)} W_\mu + e^{i\theta(t)} W_{\lambda(t)} + g(t).$$

One should think of $\zeta(t)$ as being close to $-\frac{\pi}{2}$, $\mu(t) \simeq 1$, $\theta(t) \sim 0$, $\lambda(t) \ll 1$ and $\|g(t)\|_\mathcal{E} \ll 1$. In order to specify the values of the modulation parameters, we impose the orthogonality conditions, which make disappear terms linear in g in the modulation equations. There is essentially a unique choice of such orthogonality conditions. In Lemma 3.1 we establish bounds on the evolution of the modulation parameters under some bootstrap assumptions. The goal is to improve these bounds, thus closing the bootstrap. The essential point is to improve the estimate of g, which is the infinite-dimensional part. The novelty of this paper is to use the energy conservation to deal with this. Namely, the energy of the initial data is chosen close to $2E(W)$ and is conserved by the flow. It turns out that if we control the modulation parameters sufficiently well, we can improve the bound on $\|g\|_\mathcal{E}$ by simply expanding the formula for $E(u)$ and using coercivity of the energy near a ground state; see Step 3 of the proof of Proposition 4.4.

It remains to control the modulation parameters. Note that the interaction between the two bubbles appears explicitly in the modulation equation for $\lambda'(t)$; see (3-11). In fact, the configuration of the two bubbles (phases forming the right angle) is chosen so as to *maximize* the size of the term appearing in (3-11) and leading to the growth of the parameter λ. The critical part of the proof consists in improving the bound (3-7) on $\theta(t)$. To this end, we add a localized virial correction to $\theta(t)$ to cancel the main quadratic, which is $K(t)$ in the modulation equation (3-12). Note that the size of the term

$$\frac{K(t)}{\lambda(t)^2 \|W\|_{L^2}^2}$$

in (3-12) is $O(|t|^{-\frac{N-5}{N-6}})$. Adding the virial correction allows us to gain a small constant on the right-hand side of (3-12), which is decisive for closing the bootstrap.

Finally, in order to deal with the linear instabilities of the flow, we use a classical topological argument based on the Brouwer fixed point theorem.

Notation. For $z = x + iy \in \mathbb{C}$ we define $\Re(z) = x$ and $\Im(z) = y$. For two functions $v, w \in L^2(\mathbb{R}^N, \mathbb{C})$ we define

$$\langle v, w \rangle := \Re \int_{\mathbb{R}^N} \overline{v(x)} \cdot w(x) \, dx.$$

In this paper all the functions are radially symmetric. We write $L^2 := L^2_{\mathrm{rad}}(\mathbb{R}^N; \mathbb{C})$ and $\mathcal{E} := \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^N; \mathbb{C})$. We will think of them as of *real* vector spaces. We define $X^1 := \mathcal{E} \cap \dot{H}^2(\mathbb{R}^N)$.

2. Variational estimates

Linearization near a ground state. Recall that for $z \in \mathbb{C}$ we define

$$f(z) := |z|^{\frac{4}{N-2}} z \quad \text{and} \quad F(z) := \frac{N-2}{2N} |z|^{\frac{2N}{N-2}}.$$

For $z \in \mathbb{C}$ we define the $\mathbb{R}$-linear function $f'(z) : \mathbb{C} \to \mathbb{C}$ by

$$f'(z) z_1 := |z|^{\frac{4}{N-2}} \left(z_1 + \frac{4}{N-2} z \, \Re(z^{-1} z_1) \right)$$

(with the convention $f'(0) z_1 = 0$). It is easy to check that for any $z_1, z_2, z \in \mathbb{C}$,

$$\Re\big(\bar{z}_2 (f'(z) z_1)\big) = \Re\big(\bar{z}_1 (f'(z) z_2)\big) = \Re\big((\overline{f'(z) z_2}) z_1\big). \tag{2-1}$$

Integrating this identity on $\mathbb{R}^N$ we see that for a complex-valued function $u(x)$ the operator $g \mapsto f'(u) g$ is symmetric with respect to the real L^2 scalar product. We define

$$|f'(z)| := \frac{N+2}{N-2} |z|^{\frac{4}{N-2}},$$

which is the norm of $f'(z)$ as a linear map up to a constant. For $u : \mathbb{R}^N \to \mathbb{C}$ we define $\|f'(u)\|_{L^p} := \big(\int_{\mathbb{R}^N} |f'(u(x))|^p \, dx\big)^{\frac{1}{p}}$ for $1 \leq p < +\infty$ and $\|f'(u)\|_{L^\infty} := \sup_{x \in \mathbb{R}^N} |f'(u(x))|$.

Lemma 2.1. *Let $N \geq 7$. For $z_1, z_2, z_3 \in \mathbb{C}$,*

$$|f'(z_1 + z_2) - f'(z_1)| \lesssim |f'(z_2)|, \quad |f'(z_1 + z_2) - f'(z_1)| \lesssim |z_1|^{-\frac{N-6}{N-2}} |z_2| \quad \text{if } z_1 \neq 0, \tag{2-2}$$

$$|f(z_1 + z_2) - f(z_1)| \lesssim |f'(z_1)| |z_2| + |f(z_2)|, \tag{2-3}$$

$$\begin{aligned} |f(z_1 + z_2) - f(z_1) - f'(z_1) z_2| &\lesssim f(|z_2|), \\ |f(z_1 + z_2) - f(z_1) - f'(z_1) z_2| &\lesssim |z_1|^{-\frac{N-6}{N-2}} |z_2|^2 \end{aligned} \quad \text{if } z_1 \neq 0, \tag{2-4}$$

$$|F(z_1 + z_2) - F(z_1) - \Re(\overline{f(z_1)} z_2)| \lesssim |f'(z_1)| |z_2|^2 + F(z_2), \tag{2-5}$$

$$|F(z_1 + z_2) - F(z_1) - \Re(\overline{f(z_1)} z_2) - \Re(\overline{f'(z_1) z_2} z_2)| \lesssim F(z_2). \tag{2-6}$$

Remark 2.2. In (2-2), $|f'(z_1 + z_2) - f'(z_1)|$ denotes the norm of $f'(z_1 + z_2) - f'(z_1)$ as an $\mathbb{R}$-linear map.

Remark 2.3. Note that (2-4) implies $f'(z)$ is the derivative (in the real sense) of f at z; in particular, f is a C^1 function.

Proof. All the bounds are immediate if $|z_2| \geq \frac{1}{2}|z_1|$; hence we can assume $|z_2| < \frac{1}{2}|z_1|$, in particular $z_1 \neq 0$.

The formulas $f'(z_1)z_2 = f(z_1)f'(1)(z_1^{-1}z_2)$ and $f'(z_1+z_2)z_3 = f(z_1)f'(1+z_1^{-1}z_2)(z_1^{-1}z_3)$ allow us to reduce the proof to the case $z_1 = 1$. For $|z| < \frac{1}{2}$, the mappings $F(1+z)$, $f(1+z)$ and $f'(1+z)$ are real-analytic with respect to z and the required bounds follow by writing standard asymptotic expansions. $\square$

We denote by $Z_{\theta,\lambda} := i\Delta + if'(e^{i\theta}W_\lambda)$ the linearization of $i\Delta u + if(u)$ near $u = e^{i\theta}W_\lambda$. In order to express $Z_{\theta,\lambda}$ in a more explicit way, we introduce the following notation:

$$V^+ := -\frac{N+2}{N-2}W^{\frac{4}{N-2}}, \quad V^- := -W^{\frac{4}{N-2}}, \quad L^+ := -\Delta + V^+, \quad L^- := -\Delta + V^-.$$

We also introduce the generators of the $\dot{H}^1$-critical and the L^2-critical scaling. For a function $v : \mathbb{R}^N \to \mathbb{C}$ we define

$$\Lambda v := -\frac{\partial}{\partial \lambda}\Big|_{\lambda=1}(v_\lambda) = \left(\frac{N-2}{2} + x \cdot \nabla\right)v,$$

$$\Lambda_0 v := -\frac{\partial}{\partial \lambda}\Big|_{\lambda=1}\left(\frac{1}{\lambda}v_\lambda\right) = \left(\frac{N}{2} + x \cdot \nabla\right)v.$$

It is known that for all $g \in \mathcal{E}$ we have $\langle g, L^- g \rangle \geq 0$ and $\ker L^- = \mathrm{span}(W)$. The operator L^+ has one simple strictly negative eigenvalue and, restricting to radially symmetric functions, $\ker L^+ = \mathrm{span}(\Lambda W)$; see for instance [Nakanishi and Roy 2016].

For future reference, we provide here the values of some integrals involving W and ΛW:

$$\int_{\mathbb{R}^N} W^2 \, dx = \frac{1}{2}(N(N-2))^{\frac{N}{2}} B\left(\frac{N-4}{2}, \frac{N}{2}\right), \tag{2-7}$$

$$\int_{\mathbb{R}^N} W^{\frac{N+2}{N-2}} \, dx = \frac{1}{N}(N(N-2))^{\frac{N}{2}}, \tag{2-8}$$

$$-\frac{N+2}{N-2}\int_{\mathbb{R}^N} W^{\frac{4}{N-2}}\Lambda W \, dx = \frac{N-2}{2N}(N(N-2))^{\frac{N}{2}}. \tag{2-9}$$

For the first integral, we use the formula $B(x,y) = \int_0^{+\infty} t^{x-1}(1+t)^{-x-y} \, dt$. For the second, we write $W^{\frac{N+2}{N-2}} = -\Delta W$ and we integrate by parts. For the last integral, we write

$$-\frac{N+2}{N-2}W^{\frac{4}{N-2}}\Lambda W = V^+\Lambda W = \Delta\Lambda W$$

and we integrate by parts.

Using the definition of f', one can check that if $g_1 = \Re g$ and $g_2 = \Im g$, then

$$Z_{\theta,\lambda}(e^{i\theta}g_\lambda) = \frac{e^{i\theta}}{\lambda^2}(L^- g_2 - iL^+ g_1)_\lambda.$$

In particular, we obtain

$$Z_{\theta,\lambda}(ie^{i\theta}W_\lambda) = \frac{e^{i\theta}}{\lambda^2}(L^- W)_\lambda = 0,$$

$$Z_{\theta,\lambda}(e^{i\theta}\Lambda W_\lambda) = \frac{e^{i\theta}}{\lambda^2}(-iL^+\Lambda W)_\lambda = 0.$$

This can also be seen by differentiating $i\Delta(e^{i\theta}W_\lambda) + if(e^{i\theta}W_\lambda)$ with respect to θ and λ.

Consider now the operator $Z_{\theta,\lambda}^*$. We claim that $\{e^{i\theta}W_\lambda, i\,e^{i\theta}\Lambda W_\lambda\} \subset \ker Z_{\theta,\lambda}^*$. Indeed, we have

$$\langle e^{i\theta}W_\lambda, Z_{\theta,\lambda}(e^{i\theta}g_\lambda)\rangle = \left\langle e^{i\theta}W_\lambda, \frac{e^{i\theta}}{\lambda^2}(L^-g_2 - iL^+g_1)_\lambda\right\rangle$$
$$= \langle W, L^-g_2\rangle = \langle L^-W, g_2\rangle = 0, \tag{2-10}$$

$$\langle i\,e^{i\theta}\Lambda W_\lambda, Z_{\theta,\lambda}(e^{i\theta}g_\lambda)\rangle = \left\langle i\,e^{i\theta}\Lambda W_\lambda, \frac{e^{i\theta}}{\lambda^2}(L^-g_2 - iL^+g_1)_\lambda\right\rangle$$
$$= -\langle \Lambda W, L^+g_1\rangle = -\langle L^+\Lambda W, g_1\rangle = 0. \tag{2-11}$$

One can show that there exist real functions $\mathcal{Y}^{(1)}, \mathcal{Y}^{(2)} \in \mathcal{S}$ and a real number $\nu > 0$ such that

$$L^+\mathcal{Y}^{(1)} = -\nu\mathcal{Y}^{(2)}, \quad L^-\mathcal{Y}^{(2)} = \nu\mathcal{Y}^{(1)} \tag{2-12}$$

(the proof given in [Duyckaerts and Merle 2009, Section 7] for $N = 5$ works in any dimension $N \geq 5$). We can assume that $\|\mathcal{Y}^{(1)}\|_{L^2} = \|\mathcal{Y}^{(2)}\|_{L^2} = 1$. We define

$$\alpha_{\theta,\lambda}^+ := \frac{e^{i\theta}}{\lambda^2}(\mathcal{Y}_\lambda^{(2)} + i\mathcal{Y}_\lambda^{(1)}), \quad \alpha_{\theta,\lambda}^- := \frac{e^{i\theta}}{\lambda^2}(\mathcal{Y}_\lambda^{(2)} - i\mathcal{Y}_\lambda^{(1)}). \tag{2-13}$$

For $g = g_1 + ig_2$ we have

$$\langle \alpha_{\theta,\lambda}^+, e^{i\theta}g_\lambda\rangle = \langle \mathcal{Y}^{(2)}, g_1\rangle + \langle \mathcal{Y}^{(1)}, g_2\rangle \quad \text{and} \quad \langle \alpha_{\theta,\lambda}^-, e^{i\theta}g_\lambda\rangle = \langle \mathcal{Y}^{(2)}, g_1\rangle - \langle \mathcal{Y}^{(1)}, g_2\rangle.$$

Note that

$$\langle W, \mathcal{Y}^{(1)}\rangle = \frac{1}{\nu}\langle W, L^-\mathcal{Y}^{(2)}\rangle = \frac{1}{\nu}\langle L^-W, \mathcal{Y}^{(2)}\rangle = 0,$$
$$\langle \Lambda W, \mathcal{Y}^{(2)}\rangle = -\frac{1}{\nu}\langle \Lambda W, L^+\mathcal{Y}^{(1)}\rangle = -\frac{1}{\nu}\langle L^+(\Lambda W), \mathcal{Y}^{(1)}\rangle = 0.$$

It follows that

$$\langle \alpha_{\theta,\lambda}^+, i\,e^{i\theta}W_\lambda\rangle = \langle \alpha_{\theta,\lambda}^-, i\,e^{i\theta}W_\lambda\rangle = 0, \tag{2-14}$$
$$\langle \alpha_{\theta,\lambda}^+, e^{i\theta}\Lambda W_\lambda\rangle = \langle \alpha_{\theta,\lambda}^-, e^{i\theta}\Lambda W_\lambda\rangle = 0. \tag{2-15}$$

Since $\mathcal{Y}^{(2)} \neq W$, we also have

$$\langle \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}\rangle = \frac{1}{\nu}\langle \mathcal{Y}^{(2)}, L^-\mathcal{Y}^{(2)}\rangle > 0. \tag{2-16}$$

We claim that $\alpha_{\theta,\lambda}^+$ and $\alpha_{\theta,\lambda}^-$ are eigenfunctions of $Z_{\theta,\lambda}^*$, with eigenvalues $\frac{\nu}{\lambda^2}$ and $-\frac{\nu}{\lambda^2}$ respectively. Indeed, we have

$$\langle \alpha_{\theta,\lambda}^+, Z_{\theta,\lambda}(e^{i\theta}g_\lambda)\rangle = \left\langle \alpha_{\theta,\lambda}^+, \frac{e^{i\theta}}{\lambda^2}(L^-g_2 - iL^+g_1)_\lambda\right\rangle$$
$$= \frac{1}{\lambda^2}\big(\langle \mathcal{Y}^{(2)}, L^-g_2\rangle - \langle \mathcal{Y}^{(1)}, L^+g_1\rangle\big) = \frac{1}{\lambda^2}\big(\langle L^-\mathcal{Y}^{(2)}, g_2\rangle - \langle L^+\mathcal{Y}^{(1)}, g_1\rangle\big)$$
$$= \frac{\nu}{\lambda^2}\big(\langle \mathcal{Y}^{(1)}, g_2\rangle + \langle \mathcal{Y}^{(2)}, g_1\rangle\big) = \frac{\nu}{\lambda^2}\langle \alpha_{\theta,\lambda}^+, e^{i\theta}g_\lambda\rangle. \tag{2-17}$$

Similarly, $\langle \alpha_{\theta,\lambda}^-, Z_{\theta,\lambda}(e^{i\theta}g_\lambda)\rangle = -\frac{\nu}{\lambda^2}\langle \alpha_{\theta,\lambda}^-, e^{i\theta}g_\lambda\rangle$.

Coercivity of the energy near a two-bubble. We consider $u \in \mathcal{E}$ of the form $u = e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g$ with

$$\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta| + \lambda + \|g\|_{\mathcal{E}} \ll 1.$$

Moreover, we will assume that g satisfies

$$\langle i e^{i\zeta} \Lambda W_\mu, g \rangle = \langle -e^{i\zeta} W_\mu, g \rangle = \langle i e^{i\theta} \Lambda W_\lambda, g \rangle = \langle -e^{i\theta} W_\lambda, g \rangle = 0. \tag{2-18}$$

This choice of the orthogonality conditions is dictated by the kernel of $Z^*_{\theta,\lambda}$; see (2-10) and (2-11). In this section this has little importance, but will be crucial in the sequel.

When ζ, μ, θ, λ and g are known from the context, we define

$$a_1^+ := \langle \alpha^+_{\zeta,\mu}, g \rangle, \quad a_1^- := \langle \alpha^-_{\zeta,\mu}, g \rangle, \quad a_2^+ := \langle \alpha^+_{\theta,\lambda}, g \rangle, \quad a_2^- := \langle \alpha^-_{\theta,\lambda}, g \rangle.$$

Our objective to prove the following result.

Proposition 2.4. *There exist constants $\eta, C_0, C > 0$ depending only on N such that for all $u \in \mathcal{E}$ of the form $u = e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g$, with $\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta| + \lambda + \|g\|_{\mathcal{E}} \leq \eta$ and g verifying (2-18), we have*

$$|E(u) - 2E(W)| \leq C \left(\left(\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta| + \lambda\right) \lambda^{\frac{N-2}{2}} + \|g\|_{\mathcal{E}}^2 \right), \tag{2-19}$$

$$\|g\|_{\mathcal{E}}^2 + C_0 \theta \lambda^{\frac{N-2}{2}} \leq C \left(\lambda^{\frac{N-2}{2}} \left(\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta|^3 + \lambda\right) + E(u) - 2E(W) + \sum_{j=1,2} \left((a_j^+)^2 + (a_j^-)^2\right) \right). \tag{2-20}$$

The scheme of the proof is the following. Inequality (2-6) yields the Taylor expansion of the energy:

$$\left| E(u) - E(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - \langle DE(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda), g \rangle - \tfrac{1}{2} \langle D^2 E(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) g, g \rangle \right| \lesssim \|g\|_{\mathcal{E}}^{\frac{2N}{N-2}}. \tag{2-21}$$

We just have to compute all the terms with a sufficiently high precision. We split this computation into a few lemmas.

Lemma 2.5. *Let ζ, μ, θ, λ be as in Proposition 2.4. Then*

$$\left| E(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - 2E(W) - \tfrac{1}{N} (N(N-2))^{\frac{N}{2}} \theta \lambda^{\frac{N-2}{2}} \right| \leq C \lambda^{\frac{N-2}{2}} \left(\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta|^3 + \lambda \right), \tag{2-22}$$

with a constant C depending only on N.

Proof. Expanding the energy we find

$$E(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) = E(e^{i\zeta} W_\mu) + E(e^{i\theta} W_\lambda) + \Re \int_{\mathbb{R}^N} e^{i(\zeta - \theta)} \nabla(W_\mu) \cdot \nabla(W_\lambda) \, dx$$
$$- \int_{\mathbb{R}^N} \left(F(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - F(e^{i\zeta} W_\mu) - F(e^{i\theta} W_\lambda) \right) dx. \tag{2-23}$$

By scaling invariance, $E(e^{i\zeta} W_\mu) + E(e^{i\theta} W_\lambda) = 2E(W)$. Integrating by parts we get

$$\Re \int_{\mathbb{R}^N} e^{i(\zeta - \theta)} \nabla(W_\mu) \cdot \nabla(W_\lambda) \, dx = -\Re \int_{\mathbb{R}^N} \overline{e^{i\theta} W_\lambda} \Delta(e^{i\zeta} W_\mu) \, dx = \Re \int_{\mathbb{R}^N} \overline{e^{i\theta} W_\lambda} \cdot f(e^{i\zeta} W_\mu) \, dx.$$

Hence (2-23) yields

$$E(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)$$
$$= 2E(W) - \int_{\mathbb{R}^N} \left(F(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - F(e^{i\zeta}W_\mu) - F(e^{i\theta}W_\lambda) - \Re(\overline{e^{i\theta}W_\lambda} \cdot f(e^{i\zeta}W_\mu)) \right) dx. \quad (2\text{-}24)$$

In the region $|x| \geq \sqrt{\lambda}$, using (2-5) with $z_1 = e^{i\zeta}W_\mu$ and $z_2 = e^{i\theta}W_\lambda$, we obtain

$$\left| F(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - F(e^{i\zeta}W_\mu) - F(e^{i\theta}W_\lambda) - \Re(\overline{e^{i\theta}W_\lambda} \cdot f(e^{i\zeta}W_\mu)) \right| \lesssim W_\lambda^2,$$

and we see that

$$\int_{|x|\geq\sqrt{\lambda}} W_\lambda^2 = \lambda^2 \int_{|x|\geq 1/\sqrt{\lambda}} W^2 \, dx \lesssim \lambda^2 \int_{1/\sqrt{\lambda}}^{+\infty} r^{-2N+4} r^{N-1} \, dr = \lambda^{2 + \frac{N-4}{2}} = \lambda^{\frac{N}{2}}.$$

In the region $|x| \leq \sqrt{\lambda}$ the last term in (2-24) is negligible, because $|\Re(\overline{e^{i\theta}W_\lambda} \cdot f(e^{i\zeta}W_\mu))| \lesssim W_\lambda$ and $\int_{|x|\leq\sqrt{\lambda}} W_\lambda \, dx \lesssim \lambda^{\frac{N+2}{2}} \int_0^{1/\sqrt{\lambda}} r^{-N+2} r^{N-1} \, dr \sim \lambda^{\frac{N}{2}}$. Similarly, the term $F(e^{i\zeta}W_\mu)$ is negligible. Using (2-5) with $z_1 = e^{i\theta}W_\lambda$ and $z_2 = e^{i\zeta}W_\mu$, we obtain

$$\left| F(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - F(e^{i\theta}W_\lambda) - \Re(\overline{e^{i\zeta}W_\mu} \cdot f(e^{i\theta}W_\lambda)) \right| \lesssim W_\lambda^{\frac{4}{N-2}},$$

and we see that

$$\int_{|x|\leq\sqrt{\lambda}} W_\lambda^{\frac{4}{N-2}} \, dx = \lambda^{N-2} \int_{|x|\leq 1/\sqrt{\lambda}} W^{\frac{4}{N-2}} \, dx \lesssim \lambda^{N-2} \int_{1/\sqrt{\lambda}}^{+\infty} r^{-4} r^{N-1} \, dr = \lambda^{N-2-\frac{N-4}{2}} = \lambda^{\frac{N}{2}}.$$

In order to complete the proof of (2-22), we thus need to check that

$$\left| -\int_{|x|\leq\sqrt{\lambda}} \Re(\overline{e^{i\zeta}W_\mu} \cdot f(e^{i\theta}W_\lambda)) \, dx - \frac{1}{N}(N(N-2))^{\frac{N}{2}} \theta \lambda^{\frac{N-2}{2}} \right|$$
$$\lesssim \lambda^{\frac{N-2}{2}} \left(\left| \zeta + \frac{\pi}{2} \right| + |\mu - 1| + |\theta|^3 + \lambda \right). \quad (2\text{-}25)$$

The following holds:

$$\left| \int_{|x|\leq\sqrt{\lambda}} \Re(\overline{e^{i\zeta}W_\mu} \cdot f(e^{i\theta}W_\lambda)) \, dx - \Re(e^{i(\zeta-\theta)}) \int_{\mathbb{R}^N} W_\lambda^{\frac{N+2}{N-2}} \, dx \right|$$
$$\lesssim \int_{|x|\leq\sqrt{\lambda}} |W_\mu - 1| W_\lambda^{\frac{N+2}{N-2}} \, dx + \int_{|x|\geq\sqrt{\lambda}} W_\lambda^{\frac{N+2}{N-2}} \, dx$$
$$\lesssim (|\mu - 1| + \lambda) \int_{|x|\leq\sqrt{\lambda}} W_\lambda^{\frac{N+2}{N-2}} + \int_{|x|\geq\sqrt{\lambda}} W_\lambda^{\frac{N+2}{N-2}} \, dx$$
$$\lesssim (|\mu - 1| + \lambda) \lambda^{\frac{N-2}{2}} + \lambda^{\frac{N-2}{2}} \int_{|x|\geq 1/\sqrt{\lambda}} W^{\frac{N+2}{N-2}} \, dx \lesssim (|\mu - 1| + \lambda) \lambda^{\frac{N-2}{2}}$$

and

$$\int_{\mathbb{R}^N} W_\lambda^{\frac{N+2}{N-2}} \, dx = \lambda^{\frac{N-2}{2}} \int_{\mathbb{R}^N} W^{\frac{N+2}{N-2}} \, dx = \frac{1}{N}(N(N-2))^{\frac{N}{2}} \lambda^{\frac{N-2}{2}}.$$

We have

$$|\Re(-i\,\mathrm{e}^{-i\theta}) + \theta| = |\Im(\mathrm{e}^{-i\theta}) + \theta| \lesssim |\theta|^3$$

and, using (2-8),

$$|\mathrm{e}^{i(\zeta-\theta)} + i\,\mathrm{e}^{-i\theta}| = |\mathrm{e}^{i\zeta} + i| \leq |\zeta + \tfrac{\pi}{2}|;$$

hence

$$|\Re(\mathrm{e}^{i(\zeta-\theta)}) + \theta| \lesssim |\theta|^3 + |\zeta + \tfrac{\pi}{2}| \lesssim |t|^{-\frac{3}{N-6}}. \tag{2-26}$$

The bound (2-25) follows now from (2-26), which finishes the proof. $\qquad\square$

Lemma 2.6. *Under the assumptions of Proposition 2.4, we have*

$$\left|\langle DE(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda), g\rangle\right| \lesssim \|g\|_\varepsilon \cdot \lambda^{\frac{N+2}{4}}. \tag{2-27}$$

Proof. Using the fact that $DE(\mathrm{e}^{i\zeta}W_\mu) = DE(\mathrm{e}^{i\theta}W_\lambda) = 0$, (2-27) is seen to be equivalent to

$$\left|\langle f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda), g\rangle\right| \lesssim \|g\|_\varepsilon \cdot \lambda^{\frac{N+2}{4}}.$$

By the Sobolev inequality, it suffices to check that

$$\left\| f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda)\right\|_{L^{2N/(N+2)}} \lesssim \lambda^{\frac{N+2}{4}}.$$

As usual, we consider separately the regions $|x| \leq \sqrt{\lambda}$ and $|x| \geq \sqrt{\lambda}$. In the first region we have $W_\mu \lesssim W_\lambda$; hence (2-3) with $z_1 = W_\lambda$ and $z_2 = W_\mu$ yields

$$\left| f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda)\right| \lesssim W_\lambda^{\frac{4}{N-2}}W_\mu + W_\mu^{\frac{N+2}{N-2}} \lesssim W_\lambda^{\frac{4}{N-2}}W_\mu \lesssim W_\lambda^{\frac{4}{N-2}}.$$

By a change of variable we obtain

$$\|W_\lambda^{\frac{4}{N-2}}\|_{L^{2N/(N+2)}(|x|\leq\sqrt{\lambda})} = \lambda^{N\cdot\frac{N+2}{2N} - \frac{N-2}{2}\cdot\frac{4}{N-2}}\|W^{\frac{4}{N-2}}\|_{L^{2N/(N+2)}(|x|\leq 1/\sqrt{\lambda})}$$

$$\lesssim \lambda^{\frac{N-2}{2}}\left(\int_0^{1/\sqrt{\lambda}} r^{-4\frac{2N}{N+2}} r^{N-1}\,\mathrm{d}r\right)^{\frac{N+2}{2N}}$$

$$\sim \lambda^{\frac{N-2}{2} - \frac{(N-6)N}{2(N+2)}\cdot\frac{N+2}{2N}} = \lambda^{\frac{N+2}{4}}.$$

In the region $|x| \geq \sqrt{\lambda}$ we have $W_\lambda \lesssim W_\mu$; hence (2-3) with $z_1 = W_\mu$ and $z_2 = W_\lambda$ yields

$$\left| f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda)\right| \lesssim W_\mu^{\frac{4}{N-2}}W_\lambda + W_\lambda^{\frac{N+2}{N-2}} \lesssim W_\mu^{\frac{4}{N-2}}W_\lambda \lesssim W_\lambda,$$

and we have

$$\|W_\lambda\|_{L^{2N/(N+2)}(|x|\geq\sqrt{\lambda})} = \lambda^2\|W\|_{L^{2N/(N+2)}(|x|\geq 1/\sqrt{\lambda})}$$

$$\lesssim \lambda^2\left(\int_{1/\sqrt{\lambda}}^{+\infty} r^{-(N-2)\cdot\frac{2N}{N+2}} r^{N-1}\,\mathrm{d}r\right)^{\frac{N+2}{2N}} \sim \lambda^{2+\frac{(N-6)N}{2(N+2)}\cdot\frac{N+2}{2N}} = \lambda^{\frac{N+2}{4}}. \quad\square$$

We now examine coercivity of the quadratic part in (2-21).

Lemma 2.7. *There exist constants $c, C > 0$ such that*

- *for any real-valued radial $g \in \mathcal{E}$,*

$$\langle g, L^+ g \rangle \geq c \int_{\mathbb{R}^N} |\nabla g|^2 \, dx - C \left(\langle W, g \rangle^2 + \langle \mathcal{Y}^{(2)}, g \rangle^2 \right), \tag{2-28}$$

$$\langle g, L^- g \rangle \geq c \int_{\mathbb{R}^N} |\nabla g|^2 \, dx - C \langle \Lambda W, g \rangle^2, \tag{2-29}$$

- *if $r_1 > 0$ is large enough, then for any real-valued radial $g \in \mathcal{E}$,*

$$(1 - 2c) \int_{|x| \leq r_1} |\nabla g|^2 \, dx + c \int_{|x| \geq r_1} |\nabla g|^2 \, dx + \int_{\mathbb{R}^N} V^+ |g|^2 \, dx \geq -C \left(\langle W, g \rangle^2 + \langle \mathcal{Y}^{(2)}, g \rangle^2 \right), \tag{2-30}$$

$$(1 - 2c) \int_{|x| \leq r_1} |\nabla g|^2 \, dx + c \int_{|x| \geq r_1} |\nabla g|^2 \, dx + \int_{\mathbb{R}^N} V^- |g|^2 \, dx \geq -C \langle \Lambda W, g \rangle^2, \tag{2-31}$$

- *if $r_2 > 0$ is small enough, then for any real-valued radial $g \in \mathcal{E}$,*

$$(1 - 2c) \int_{|x| \geq r_2} |\nabla g|^2 \, dx + c \int_{|x| \leq r_2} |\nabla g|^2 \, dx + \int_{\mathbb{R}^N} V^+ |g|^2 \, dx \geq -C \left(\langle W, g \rangle^2 + \langle \mathcal{Y}^{(2)}, g \rangle^2 \right), \tag{2-32}$$

$$(1 - 2c) \int_{|x| \geq r_2} |\nabla g|^2 \, dx + c \int_{|x| \leq r_2} |\nabla g|^2 \, dx + \int_{\mathbb{R}^N} V^- |g|^2 \, dx \geq -C \langle \Lambda W, g \rangle^2. \tag{2-33}$$

Proof. In the proofs of (2-28) and (2-29) we repeat with minor modifications the arguments of Nakanishi and Roy [2016]. We include them for the reader's convenience.

Let us show that

$$g \in \mathcal{E}, \quad \langle \mathcal{Y}^{(2)}, g \rangle = 0 \quad \implies \quad \langle g, L^+, g \rangle \geq 0. \tag{2-34}$$

Suppose the contrary. Let $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and consider $ag + b\mathcal{Y}^{(1)} \in \mathcal{E}$. Since $\mathcal{Y}^{(2)} \neq W$, (2-12) yields

$$\langle \mathcal{Y}^{(1)}, L^+ \mathcal{Y}^{(1)} \rangle = -\nu \langle \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)} \rangle = -\langle L^- \mathcal{Y}^{(2)}, \mathcal{Y}^{(2)} \rangle < 0, \tag{2-35}$$

so we obtain

$$\langle ag + bY^{(1)}, L^+ (ag + b\mathcal{Y}^{(1)}) \rangle = a^2 \langle g, L^+ g \rangle + 2ab \langle g, L^+ \mathcal{Y}^{(1)} \rangle + b^2 \langle \mathcal{Y}^{(1)}, L^+ \mathcal{Y}^{(1)} \rangle$$
$$= a^2 \langle g, L^+ g \rangle - 2ab\nu \langle g, \mathcal{Y}^{(2)} \rangle + b^2 \langle \mathcal{Y}^{(1)}, L^+ \mathcal{Y}^{(1)} \rangle < 0.$$

This is impossible, because L^+ has only one negative direction. This proves (2-34).

Suppose (2-28) fails. Then there exists a sequence $g_n \in \mathcal{E}$ such that $\|g_n\|_{\mathcal{E}} = 1$ and

$$\langle g_n, L^+ g_n \rangle \leq c_n - C_n \left(\langle W, g_n \rangle^2 + \langle \mathcal{Y}^{(2)}, g \rangle^2 \right), \quad c_n \to 0, \quad C_n \to +\infty. \tag{2-36}$$

Upon extracting a subsequence, we can assume that $g_n \rightharpoonup g \in \mathcal{E}$. Since $|\langle g_n, L^+ g_n \rangle| \lesssim \|g_n\|_{\mathcal{E}}^2 = 1$, from (2-36) we immediately get $\langle W, g \rangle = \langle \mathcal{Y}^{(2)}, g \rangle = 0$. Also, by standard arguments $\langle g_n, V^+ g_n \rangle \to \langle g, V^+ g \rangle$. Hence by the Fatou property

$$\langle g, L^+ g \rangle \leq \liminf_n \langle g_n, L^+, g_n \rangle \leq \liminf_n c_n = 0.$$

Thus g is a minimizer for the quadratic form associated with L^+ on the hyperplane orthogonal to $\mathcal{Y}^{(2)}$. This implies $\langle h, L^+ g\rangle = 0$ for all $h \in \mathcal{E}$ such that $\langle \mathcal{Y}^{(2)}, h\rangle = 0$. But we also have $\langle \mathcal{Y}^{(1)}, L^+ g\rangle = \langle L^+ \mathcal{Y}^{(1)}, g\rangle = -\nu\langle \mathcal{Y}^{(2)}, g\rangle = 0$ and $\langle \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}\rangle \neq 0$, see (2-35), so we obtain $\langle h, L^+ g\rangle = 0$ for all $h \in \mathcal{E}$. Hence $g = \Lambda W$. But $\langle W, \Lambda W\rangle = -\|W\|_{L^2}^2 \neq 0$, so we get a contradiction. This proves (2-28).

The proof of (2-29) is similar. We obtain that the weak limit g is a minimizer for the quadratic form associated with L^- (without constraints); hence $g = W$, which is incompatible with the orthogonality condition.

Once we have (2-28) and (2-29), the bounds (2-30)–(2-33) follow by repeating the proof of Lemma 2.1 in [Jendrej 2015]. $\qquad\square$

We now use this lemma to study the linearization around $e^{i\theta} W_\lambda$ for a complex-valued perturbation g.

Proposition 2.8. *There exist constants $c, C > 0$ such that for any $\theta \in \mathbb{R}$ and $\lambda > 0$,*

- *for any complex-valued radial $g \in \mathcal{E}$,*

$$\int_{\mathbb{R}^N} |\nabla g|^2 \, dx - \Re \int_{\mathbb{R}^N} \bar{g} \cdot f'(e^{i\theta} W_\lambda) g \, dx$$
$$\geq c \int_{\mathbb{R}^N} |\nabla g|^2 \, dx - C\big(\langle \lambda^{-2} e^{i\theta} W_\lambda, g\rangle^2 + \langle \lambda^{-2} i e^{i\theta} \Lambda W_\lambda, g\rangle^2 + \langle \alpha_{\theta,\lambda}^+, g\rangle^2 + \langle \alpha_{\theta,\lambda}^-, g\rangle^2\big), \quad (2\text{-}37)$$

- *if $r_1 > 0$ is large enough, then for any complex-valued radial $g \in \mathcal{E}$,*

$$(1-2c)\int_{|x|\leq r_1} |\nabla g|^2 \, dx + c\int_{|x|\geq r_1} |\nabla g|^2 \, dx - \Re \int_{\mathbb{R}^N} \bar{g} \cdot f'(e^{i\theta} W_\lambda) g \, dx$$
$$\geq -C\big(\langle \lambda^{-2} e^{i\theta} W_\lambda, g\rangle^2 + \langle \lambda^{-2} i e^{i\theta} \Lambda W_\lambda, g\rangle^2 + \langle \alpha_{\theta,\lambda}^+, g\rangle^2 + \langle \alpha_{\theta,\lambda}^-, g\rangle^2\big), \quad (2\text{-}38)$$

- *if $r_2 > 0$ is small enough, then for any complex-valued radial $g \in \mathcal{E}$,*

$$(1-2c)\int_{|x|\geq r_2} |\nabla g|^2 \, dx + c\int_{|x|\leq r_2} |\nabla g|^2 \, dx - \Re \int_{\mathbb{R}^N} \bar{g} \cdot f'(e^{i\theta} W_\lambda) g \, dx$$
$$\geq -C\big(\langle \lambda^{-2} e^{i\theta} W_\lambda, g\rangle^2 + \langle \lambda^{-2} i e^{i\theta} \Lambda W_\lambda, g\rangle^2 + \langle \alpha_{\theta,\lambda}^+, g\rangle^2 + \langle \alpha_{\theta,\lambda}^-, g\rangle^2\big). \quad (2\text{-}39)$$

Remark 2.9. Note that the scalar products on the right-hand side of these estimates are the ones which appear in the orthogonality conditions (2-18). For the definition of $\alpha_{\theta,\lambda}^\pm$, see (2-13).

Proof. Without loss of generality we can assume that $\theta = 0$ and $\lambda = 1$. Let $g = g_1 + i g_2$. Observe that

$$-f'(W)(g_1 + i g_2) = -W^{\frac{4}{N-2}}(g_1 + i g_2) - \frac{4}{N-2} W^{\frac{4}{N-2}} g_1 = V^+ g_1 + i V^- g_2,$$

which gives

$$-\Re \int_{\mathbb{R}^N} \bar{g} \cdot f'(W) g \, dx = \int_{\mathbb{R}^N} V^+ g_1^2 \, dx + \int_{\mathbb{R}^N} V^- g_2^2 \, dx.$$

Also, $\langle W, g\rangle = \langle W, g_1\rangle$ and $\langle i \Lambda W, g\rangle = \langle \Lambda W, g_2\rangle$. We have $\mathcal{Y}^{(2)} = \frac{1}{2}(\alpha_{0,1}^+ + \alpha_{0,1}^-)$, so

$$\langle \mathcal{Y}^{(2)}, g_1\rangle^2 = \langle \mathcal{Y}^{(2)}, g\rangle^2 \leq \frac{1}{2}\big(\langle \alpha_{0,1}^+, g\rangle^2 + \langle \alpha_{0,1}^-, g\rangle^2\big).$$

Applying (2-28) with $g = g_1$ and (2-29) with $g = g_2$ we obtain (2-37). The proofs of (2-38) and (2-39) are similar. $\qquad\square$

One consequence of the last proposition is the coercivity near a sum of two bubbles at different scales:

Lemma 2.10. *There exist $\eta, C > 0$ such that if $\lambda \leq \eta\mu$, then for all $g \in \mathcal{E}$ satisfying* (2-18),

$$\frac{1}{C}\|g\|_{\mathcal{E}}^2 \leq \frac{1}{2}\langle D^2 E(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g, g\rangle + 2\big((a_1^+)^2 + (a_1^-)^2 + (a_2^+)^2 + (a_2^-)^2\big) \leq C\|g\|_{\mathcal{E}}^2.$$

Proof. It is essentially the same as the proof of [Jendrej 2015, Lemma 3.5]. □

Proof of Proposition 2.4. Bound (2-19) follows immediately from (2-21), Lemmas 2.5, 2.6 and 2.10 and the triangle inequality.

For any $c > 0$ we have $\|g\|_{\mathcal{E}}^{\frac{2N}{N-2}} \leq c\|g\|_{\mathcal{E}}^2$ if η is chosen small enough; hence (2-21) and Lemmas 2.5 and 2.6 yield

$$\left|E(u) - 2E(W) - \frac{1}{N}(N(N-2))^{\frac{N}{2}}\theta\lambda^{\frac{N-2}{2}} - \frac{1}{2}\langle D^2 E(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g, g\rangle\right|$$
$$\leq C\left(\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta|^3 + \lambda\right)\lambda^{\frac{N-2}{2}} + c\|g\|_{\mathcal{E}}^2;$$

hence

$$\frac{1}{N}(N(N-2))^{\frac{N}{2}}\theta\lambda^{\frac{N-2}{2}} + \frac{1}{2}\langle D^2 E(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g, g\rangle$$
$$\leq E(u) - 2E(W) + C\left(\left|\zeta + \tfrac{\pi}{2}\right| + |\mu - 1| + |\theta|^3 + \lambda\right)\lambda^{\frac{N-2}{2}} + c\|g\|_{\mathcal{E}}^2.$$

Choosing c small enough and invoking Lemma 2.10 finishes the proof of (2-20). □

3. Modulation

Bounds on the modulation parameters. We study solutions of the form

$$u(t) = e^{i\zeta(t)}W_{\mu(t)} + e^{i\theta(t)}W_{\lambda(t)} + g(t), \tag{3-1}$$

with

$$|\mu(t) - 1| \ll 1, \quad \left|\zeta(t) + \tfrac{\pi}{2}\right| \ll 1, \quad \lambda(t) \ll 1, \quad |\theta(t)| \ll 1 \quad \text{and} \quad \|g\|_{\mathcal{E}} \ll 1. \tag{3-2}$$

We will often omit the time variable and write ζ for $\zeta(t)$ etc.

Differentiating (3-1) in time we obtain

$$\partial_t u = \zeta' i e^{i\zeta}W_\mu - \frac{\mu'}{\mu}e^{i\zeta}\Lambda W_\mu + \theta' i e^{i\theta}W_\lambda - \frac{\lambda'}{\lambda}\Lambda W_\lambda + \partial_t g.$$

On the other hand, using $\Delta(W_\mu) + f(W_\mu) = \Delta(W_\lambda) + f(W_\lambda) = 0$ we get

$$i\,\Delta u + if(u) = i\,\Delta g + i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda)\big);$$

hence (1-1) yields

$$\partial_t g = i\,\Delta g + i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda)\big)$$
$$- \zeta' i e^{i\zeta}W_\mu + \frac{\mu'}{\mu}e^{i\zeta}\Lambda W_\mu - \theta' i e^{i\theta}W_\lambda + \frac{\lambda'}{\lambda}e^{i\theta}\Lambda W_\lambda. \tag{3-3}$$

Since we work with nonclassical solutions, it is worth pointing out that the equation above should be understood as a notational simplification. Any computation involving $g(t)$ could be rewritten in terms of

$u(t)$ and the modulation parameters ζ, μ, θ, λ. Most of the time we only use the fact that (3-3) holds in the weak sense, but later we will also need to compute the time derivative of a quadratic form in $g(t)$, in which case the rigorous meaning of the computation is less clear.

We impose the orthogonality conditions (2-18). By standard arguments using the implicit function theorem, they uniquely determine the modulation parameters.

We need precise bootstrap assumptions about the parameters quantifying (3-2). In order to formulate them, set

$$\kappa := \left(\frac{N-6}{N \cdot B\left(\frac{N-4}{2}, \frac{N}{2}\right)} \right)^{\frac{2}{N-4}}. \tag{3-4}$$

Lemma 3.1. *Let $c > 0$ be an arbitrarily small constant. Let $T_0 < 0$ with $|T_0|$ large enough* (*depending on c*) *and $T < T_1 \le T_0$. Suppose that for $T \le t \le T_1$ we have*

$$\left| \zeta(t) + \tfrac{\pi}{2} \right| \le |t|^{-\frac{3}{N-6}}, \tag{3-5}$$

$$|\mu(t) - 1| \le |t|^{-\frac{3}{N-6}}, \tag{3-6}$$

$$|\theta(t)| \le |t|^{-\frac{1}{N-6}}, \tag{3-7}$$

$$\left| \lambda(t) - \tfrac{1}{\kappa}(\kappa|t|)^{-\frac{2}{N-6}} \right| \le |t|^{-\frac{5}{2(N-6)}}, \tag{3-8}$$

$$\|g\|_{\varepsilon} \le |t|^{-\frac{N-1}{2(N-6)}}. \tag{3-9}$$

Then

$$|\zeta'(t)| \le c|t|^{-\frac{N-3}{N-6}}, \tag{3-10}$$

$$|\mu'(t)| \le c|t|^{-\frac{N-3}{N-6}},$$

$$\left| \lambda'(t) - \frac{2\kappa^{\frac{N-4}{2}}}{N-6} \lambda(t)^{\frac{N-4}{2}} \right| \le c|t|^{-\frac{2N-7}{2(N-6)}}, \tag{3-11}$$

$$\left| \theta'(t) + \frac{(N-2)\kappa^{\frac{N-4}{2}}}{N-6} \theta(t)\lambda(t)^{\frac{N-6}{2}} - \frac{K(t)}{\lambda(t)^2\|W\|_{L^2}^2} \right| \le c|t|^{-\frac{N-5}{N-6}} \tag{3-12}$$

for $T \le t \le T_1$, where

$$K := -\langle e^{i\theta}\Lambda W_\lambda, \, f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\rangle. \tag{3-13}$$

Remark 3.2. We will not really use (3-8), but only the fact that $\lambda(t) \sim |t|^{-\frac{2}{N-6}}$.

Proof. We use the usual method of differentiating the orthogonality conditions in time, which will yield a linear system of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} \mu^2\zeta' \\ \mu\mu' \\ \lambda^2\theta' \\ \lambda\lambda' \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix}.$$

Here, the coefficients M_{ij} and B_i depend on g, ζ, μ, θ and λ. We will now compute all these coefficients and prove appropriate bounds.

<u>First row</u>. Differentiating $\langle i e^{i\zeta} \Lambda W_\mu, g \rangle = 0$ and using (3-3) we obtain

$$0 = \frac{d}{dt} \langle i e^{i\zeta} \Lambda W_\mu, g \rangle = -\zeta' \langle e^{i\zeta} \Lambda W_\mu, g \rangle - \frac{\mu'}{\mu} \langle i e^{i\zeta} \Lambda \Lambda W_\mu, g \rangle + \langle i e^{i\zeta} \Lambda W_\mu, \partial_t g \rangle$$

$$= \zeta' \big(-\langle i e^{i\zeta} \Lambda W_\mu, i e^{i\zeta} W_\mu \rangle - \langle e^{i\zeta} \Lambda W_\mu, g \rangle \big) + \frac{\mu'}{\mu} \big(\langle i e^{i\zeta} \Lambda W_\mu, e^{i\zeta} \Lambda W_\mu \rangle - \langle i e^{i\zeta} \Lambda \Lambda W_\mu, g \rangle \big)$$

$$+ \theta' \langle i e^{i\zeta} \Lambda W_\mu, -i e^{i\theta} W_\lambda \rangle + \frac{\lambda'}{\lambda} \langle i e^{i\zeta} \Lambda W_\mu, e^{i\theta} \Lambda W_\lambda \rangle$$

$$+ \big\langle i e^{i\zeta} \Lambda W_\mu, i \Delta g + i \big(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \big) \big\rangle.$$

Note that $\langle -\Lambda W_\mu, W_\mu \rangle = \|W_\mu\|_{L^2}^2 = \mu^2 \|W\|_{L^2}^2$; hence we get

$$M_{11} = \mu^{-2} \big(-\langle i e^{i\zeta} \Lambda W_\mu, i e^{i\zeta} W_\mu \rangle - \langle e^{i\zeta} \Lambda W_\mu, g \rangle \big) = \|W\|_{L^2}^2 + O(\|g\|_{\varepsilon}) = \|W\|_{L^2}^2 + O(|t|^{-\frac{N-1}{2(N-6)}}),$$

$$M_{12} = \mu^{-2} \big(\langle i e^{i\zeta} \Lambda W_\mu, e^{i\zeta} \Lambda W_\mu \rangle - \langle i e^{i\zeta} \Lambda \Lambda W_\mu, g \rangle \big) = O(\|g\|_{\varepsilon}) = O(|t|^{-\frac{N-1}{2(N-6)}}),$$

$$M_{13} = \lambda^{-2} \langle i e^{i\zeta} \Lambda W_\mu, -i e^{i\theta} W_\lambda \rangle = O(1),$$

$$M_{14} = \lambda^{-2} \langle i e^{i\zeta} \Lambda W_\mu, e^{i\theta} \Lambda W_\lambda \rangle = O(1).$$

Let us consider the term

$$B_1 = -\big\langle i e^{i\zeta} \Lambda W_\mu, i \Delta g + i \big(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \big) \big\rangle.$$

From (2-11) (with θ replaced by ζ and λ replaced by μ) we obtain

$$B_1 = -\mu^{-2} \big\langle i e^{i\zeta} \Lambda W_\mu, i \big(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu) g \big) \big\rangle$$

$$= -\mu^{-2} \big\langle e^{i\zeta} \Lambda W_\mu, \big(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu) g \big) \big\rangle.$$

First we show that

$$\big| \big\langle e^{i\zeta} \Lambda W_\mu, f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) g \big\rangle \big| \lesssim \|g\|_{\varepsilon}^2. \quad (3\text{-}14)$$

Note that (3-5) and (3-7) imply $|e^{i\zeta} W_\mu + e^{i\theta} W_\lambda| \gtrsim W_\mu$; hence (2-4) with $z_1 = e^{i\zeta} W_\mu + e^{i\theta} W_\lambda$ and $z_2 = g$ yields

$$\big| f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) g \big| \lesssim W_\mu^{-\frac{N-6}{N-2}} |g|^2.$$

Using the fact that $|\Lambda W| \lesssim W$ and the Hölder inequality, we arrive at (3-14).

Next we show that

$$\big| \big\langle e^{i\zeta} \Lambda W_\mu, f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \big\rangle \big| \lesssim \lambda^{\frac{N-2}{2}}. \quad (3\text{-}15)$$

Using (2-3) we get

$$\big| f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \big| \lesssim |W_\mu|^{\frac{4}{N-2}} W_\lambda + |f(W_\lambda)|.$$

The second term is easy. We have $f(W) \in L^1$ and we check that $\|f(W_\lambda)\|_{L^1} \sim \lambda^{\frac{N-2}{2}}$ by a change of variable. Consider the first term. In the region $|x| \leq 1$ we write

$$\|W_\lambda\|_{L^1(|x| \leq 1)} = \lambda^{\frac{N+2}{2}} \|W\|_{L^1(|x| \leq \lambda^{-1})} \lesssim \lambda^{\frac{N+2}{2}} \int_0^{\lambda^{-1}} r^{-N+2} r^{N-1} \, dr \sim \lambda^{\frac{N-2}{2}}.$$

As for $|x| \geq 1$, we notice that $\|W_\lambda\|_{L^\infty(|x| \geq 1)} \lesssim \lambda^{\frac{N-2}{2}}$ and $|\Lambda W_\mu||W_\mu|^{\frac{4}{N-2}}$ is bounded in L^1.

Finally, we show that

$$\left|\langle e^{i\zeta} \Lambda W_\mu, (f'(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu))g \rangle\right| \lesssim \lambda^{\frac{N-2}{4}} \|g\|_\mathcal{E}. \tag{3-16}$$

In the region $|x| \leq \sqrt{\lambda}$ it suffices to use the bound

$$|f'(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu)| \lesssim W_\lambda^{\frac{4}{N-2}}$$

and the fact that

$$\|W_\lambda^{\frac{4}{N-2}}\|_{L^{2N/(N+2)}(|x| \leq \sqrt{\lambda})} = \lambda^{\frac{N-2}{2}} \|W^{\frac{4}{N-2}}\|_{L^{2N/(N+2)}(|x| \leq \lambda^{-1/2})} \lesssim \lambda^{\frac{N+2}{4}},$$

where the last inequality follows from $W^{\frac{4}{N-2}}(x) \lesssim |x|^{-4}$. In the region $|x| \geq \sqrt{\lambda}$ we use Hölder and the fact that

$$\|W_\lambda\|_{L^{2N/N-2}(|x| \geq \sqrt{\lambda})} = \|W\|_{L^{2N/(N-2)}(|x| \geq \lambda^{-1/2})} \lesssim \left(\int_{\lambda^{-1/2}}^{+\infty} r^{-2N} r^{N-1} \, dr\right)^{\frac{N-2}{2N}} \lesssim \lambda^{\frac{N-2}{4}}.$$

Taking the sum of (3-14), (3-15), (3-16) and using (3-8), (3-9) we obtain

$$|B_1| \lesssim |t|^{-\frac{N-2}{N-6}}. \tag{3-17}$$

<u>Second row.</u> Differentiating $\langle -e^{i\zeta} W_\mu, g \rangle = 0$, we obtain

$$0 = \frac{d}{dt}\langle -e^{i\zeta} W_\mu, g \rangle = -\zeta'\langle i e^{i\zeta} W_\mu, g \rangle + \frac{\mu'}{\mu}\langle e^{i\zeta} \Lambda W_\mu, g \rangle - \langle e^{i\zeta} W_\mu, \partial_t g \rangle$$

$$= \zeta'\left(\langle e^{i\zeta} W_\mu, i e^{i\zeta} W_\mu \rangle - \langle i e^{i\zeta} \Lambda W_\mu, g \rangle\right) + \frac{\mu'}{\mu}\left(-\langle e^{i\zeta} W_\mu, e^{i\zeta} \Lambda W_\mu \rangle + \langle e^{i\zeta} \Lambda W_\mu, g \rangle\right)$$

$$+ \theta'\langle e^{i\zeta} W_\mu, i e^{i\theta} W_\lambda \rangle + \frac{\lambda'}{\lambda}\langle -e^{i\zeta} W_\mu, e^{i\theta} \Lambda W_\lambda \rangle$$

$$- \langle e^{i\zeta} W_\mu, i \Delta g + i\left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda)\right)\rangle,$$

which yields

$$M_{21} = \mu^{-2}\left(\langle e^{i\zeta} W_\mu, i e^{i\zeta} W_\mu \rangle - \langle i e^{i\zeta} W_\mu, g \rangle\right) = O(\|g\|_\mathcal{E}),$$

$$M_{22} = \mu^{-2}\left(-\langle e^{i\zeta} W_\mu, e^{i\zeta} \Lambda W_\mu \rangle + \langle e^{i\zeta} \Lambda W_\mu, g \rangle\right) = \|W\|_{L^2}^2 + O(\|g\|_\mathcal{E}),$$

$$M_{23} = \lambda^{-2}\langle e^{i\zeta} W_\mu, i e^{i\theta} W_\lambda \rangle = O(1),$$

$$M_{24} = \lambda^{-2}\langle -e^{i\zeta} W_\mu, e^{i\theta} \Lambda W_\lambda \rangle = O(1).$$

Consider now the term

$$B_2 = \langle e^{i\zeta} W_\mu, i \Delta g + i\left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda)\right)\rangle$$

$$= \langle e^{i\zeta} W_\mu, i\left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu)g\right)\rangle,$$

where the second equality follows from (2-10). The proof of (3-17) yields

$$|B_2| \lesssim |t|^{-\frac{N-2}{N-6}}.$$

<u>Third row.</u> Differentiating $\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, g \rangle = 0$ we obtain

$$
\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, g\rangle &= -\theta'\langle \mathrm{e}^{i\theta}\Lambda W_\lambda, g\rangle - \frac{\lambda'}{\lambda}\langle i\,\mathrm{e}^{i\theta}\Lambda\Lambda W_\lambda, g\rangle + \langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, \partial_t g\rangle \\
&= \zeta'\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, -i\,\mathrm{e}^{i\zeta}W_\mu\rangle + \frac{\mu'}{\mu}\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, \mathrm{e}^{i\zeta}\Lambda W_\mu\rangle \\
&\quad + \theta'\big(\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, -i\,\mathrm{e}^{i\theta}W_\lambda\rangle - \langle \mathrm{e}^{i\theta}\Lambda W_\lambda, g\rangle\big) + \frac{\lambda'}{\lambda}\big(\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, \mathrm{e}^{i\theta}\Lambda W_\lambda\rangle - \langle i\,\mathrm{e}^{i\theta}\Lambda\Lambda W_\lambda, g\rangle\big) \\
&\quad + \langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, i\,\Delta g + i\big(f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda + g) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda)\big)\rangle,
\end{aligned}
$$

which yields

$$
\begin{aligned}
M_{31} &= \mu^{-2}\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, -i\,\mathrm{e}^{i\zeta}W_\mu\rangle = O(\lambda^2) = O(|t|^{-\frac{4}{N-6}}), \\
M_{32} &= \mu^{-2}\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, \mathrm{e}^{i\zeta}\Lambda W_\mu\rangle = O(\lambda^2) = O(|t|^{-\frac{4}{N-6}}), \\
M_{33} &= \lambda^{-2}\big(\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, -i\,\mathrm{e}^{i\theta}W_\lambda\rangle - \langle \mathrm{e}^{i\theta}\Lambda W_\lambda, g\rangle\big) = \|W\|_{L^2}^2 + O(\|g\|_{\mathcal{E}}) = \|W\|_{L^2}^2 + O(|t|^{-\frac{N-1}{2(N-6)}}), \\
M_{34} &= \lambda^{-2}\big(\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, \mathrm{e}^{i\theta}\Lambda W_\lambda\rangle - \langle i\,\mathrm{e}^{i\theta}\Lambda\Lambda W_\lambda, g\rangle\big) = O(\|g\|_{\mathcal{E}}) = O(|t|^{-\frac{N-1}{2(N-6)}}).
\end{aligned}
$$

Let us consider the term

$$
\begin{aligned}
B_3 &= -\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, i\,\Delta g + i\big(f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda + g) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda)\big)\rangle \\
&= -\langle i\,\mathrm{e}^{i\theta}\Lambda W_\lambda, i\big(f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda + g) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda) - f'(\mathrm{e}^{i\theta}W_\lambda)g\big)\rangle \\
&= -\langle \mathrm{e}^{i\theta}\Lambda W_\lambda, f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda + g) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda) - f'(\mathrm{e}^{i\theta}W_\lambda)g\rangle,
\end{aligned}
$$

where the second equality follows from (2-11). Comparing this formula with (3-13) we obtain

$$
\begin{aligned}
B_3 - K = &-\langle \mathrm{e}^{i\theta}\Lambda W_\lambda, f(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f(\mathrm{e}^{i\zeta}W_\mu) - f(\mathrm{e}^{i\theta}W_\lambda)\rangle \\
&-\langle \mathrm{e}^{i\theta}\Lambda W_\lambda, \big(f'(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f'(\mathrm{e}^{i\theta}W_\lambda)\big)g\rangle. \quad (3\text{-}18)
\end{aligned}
$$

First we treat the second line by showing that

$$\big|\langle \mathrm{e}^{i\theta}\Lambda W_\lambda, \big(f'(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f'(\mathrm{e}^{i\theta}W_\lambda)\big)g\rangle\big| \lesssim \lambda^{\frac{N}{4}}\|g\|_{\mathcal{E}}. \quad (3\text{-}19)$$

We consider separately $|x| \leq \lambda^\gamma$ and $|x| \geq \lambda^\gamma$ with $\gamma = \frac{N-4}{2(N-2)}$. In the region $|x| \leq \lambda^\gamma$ we use the bound

$$|f'(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f'(\mathrm{e}^{i\theta}W_\lambda)| \lesssim W_\lambda^{-\frac{N-6}{N-2}}W_\mu.$$

It implies

$$|\mathrm{e}^{i\theta}\Lambda W_\lambda|\big|\big(f'(\mathrm{e}^{i\zeta}W_\mu + \mathrm{e}^{i\theta}W_\lambda) - f'(\mathrm{e}^{i\theta}W_\lambda)\big)g\big| \lesssim W_\lambda^{\frac{4}{N-2}}|g|$$

pointwise and it suffices to see that

$$\|W_\lambda^{\frac{4}{N-2}}\|_{L^{2N/(N+2)}(|x|\le\lambda^\gamma)} = \lambda^{\frac{N-2}{2}}\|W^{\frac{4}{N-2}}\|_{L^{2N/(N+2)}(|x|\le\lambda^{\gamma-1})} \lesssim \lambda^{\frac{N-2}{2}}\left(\int_0^{\lambda^{\gamma-1}} r^{-4\frac{2N}{N+2}} r^{N-1}\,\mathrm{d}r\right)^{\frac{N+2}{2N}}$$

$$\lesssim \lambda^{\frac{N-2}{2}}\lambda^{(\gamma-1)\frac{N(N-6)}{N+2}\cdot\frac{N+2}{2N}} = \lambda^{\frac{N-2}{2}-\frac{N(N-6)}{4(N-2)}} = \lambda^{\frac{N^2-2N+8}{4(N-2)}} \ll \lambda^{\frac{N}{4}}.$$

In the region $|x| \ge \lambda^\gamma$ we have

$$\|\Lambda W_\lambda\|_{L^{2N/(N-2)}(|x|\ge\lambda^\gamma)} \lesssim \|W_\lambda\|_{L^{2N/(N-2)}(|x|\ge\lambda^\gamma)} = \|W\|_{L^{2N/(N-2)}(|x|\ge\lambda^{\gamma-1})}$$

$$\lesssim \left(\int_{\lambda^{\gamma-1}}^{+\infty} r^{-2N} r^{N-1}\,\mathrm{d}r\right)^{\frac{N-2}{2N}} \lesssim \lambda^{(1-\gamma)N\frac{N-2}{2N}} = \lambda^{\frac{N}{4}},$$

which yields the required bound by Hölder.

We are left with the first line in (3-18). We will prove that

$$\left|\langle e^{i\theta}\Lambda W_\lambda, f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda)\rangle - \frac{(N-2)\kappa^{\frac{N-4}{2}}\|W\|_{L^2}^2}{N-6}\theta\lambda^{\frac{N-2}{2}}\right| \lesssim |t|^{-\frac{N}{N-6}}. \quad (3\text{-}20)$$

For this, we first check that

$$\left|\langle e^{i\theta}\Lambda W_\lambda, f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\rangle\right| \lesssim \lambda^{\frac{N}{2}}. \quad (3\text{-}21)$$

In the region $|x| \ge \sqrt{\lambda}$ we have $W_\lambda \lesssim W_\mu$, which implies

$$\left|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\right| \lesssim W_\lambda^{\frac{4}{N-2}}W_\mu;$$

hence the required bound follows from $|\Lambda W| \lesssim W$ and

$$\|W_\lambda^{\frac{N+2}{N-2}}\|_{L^1(|x|\ge\sqrt\lambda)} \lesssim \lambda^{\frac{N-2}{2}}\int_{\lambda^{-1/2}}^{+\infty} r^{-N-2}r^{N-1}\,\mathrm{d}r \sim \lambda^{\frac{N}{2}}.$$

In the region $|x| \le \sqrt{\lambda}$ we have $W_\mu \lesssim W_\lambda$, which implies

$$\left|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\right| \lesssim W_\mu^{\frac{N+2}{N-2}};$$

hence the required bound follows from

$$\|W_\lambda\|_{L^1(|x|\le\sqrt\lambda)} \lesssim \lambda^{\frac{N+2}{2}}\int_0^{\lambda^{-1/2}} r^{-N+2}r^{N-1}\,\mathrm{d}r \sim \lambda^{\frac{N}{2}}.$$

Finally, we need to check that

$$\left|\langle e^{i\theta}\Lambda W_\lambda, f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\rangle - \frac{(N-2)\kappa^{\frac{N-4}{2}}\|W\|_{L^2}^2}{N-6}\theta\lambda^{\frac{N-2}{2}}\right| \lesssim |t|^{-\frac{N}{N-6}}. \quad (3\text{-}22)$$

The definition of $f'(z)$ yields

$$f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu) = W_\mu W_\lambda^{\frac{4}{N-2}}\left(e^{i\zeta} + \frac{4}{N-2}e^{i\theta}\Re(e^{i(\zeta-\theta)})\right); \quad (3\text{-}23)$$

hence

$$\langle e^{i\theta}\Lambda W_\lambda, f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\rangle = \frac{N+2}{N-2}\Re(e^{i(\zeta-\theta)})\int W_\mu W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx. \tag{3-24}$$

Since

$$\left|\int W_\mu W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx\right| \lesssim \lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N-2}{N-6}},$$

we obtain

$$\left|\frac{N+2}{N-2}\Re(e^{i(\zeta-\theta)})\int W_\mu W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx + \frac{N+2}{N-2}\theta\int W_\mu W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx\right| \lesssim |t|^{-\frac{N+1}{N-6}} \ll |t|^{-\frac{N}{N-6}}. \tag{3-25}$$

Next, we prove that

$$\left|\int W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx - \int W_\mu W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx\right| \lesssim \lambda^{\frac{N}{2}} + |\mu-1|\lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N}{N-6}}. \tag{3-26}$$

Indeed, in the region $|x| \geq \sqrt{\lambda}$ both terms satisfy the bound. In the region $|x| \leq \sqrt{\lambda}$ we have

$$|W_\mu - \mu^{-\frac{N-2}{2}}| \lesssim |x|^2 \lesssim \lambda \quad \text{and} \quad |\mu^{-\frac{N-2}{2}} - 1| \lesssim |\mu-1|,$$

from which (3-26) follows.

From (2-9) and (2-7) we get

$$\frac{N+2}{N-2}\int W_\lambda^{\frac{4}{N-2}}\Lambda W_\lambda \, dx = -\frac{(N-2)\kappa^{\frac{N-4}{2}}\|W\|_{L^2}^2}{N-6}\lambda^{\frac{N-2}{2}},$$

and (3-22) follows from (3-24)–(3-26).

From (3-18)–(3-20) and the triangle inequality we infer

$$\left|B_3 - K + \frac{(N-2)\kappa^{\frac{N-4}{2}}\|W\|_{L^2}^2}{N-6}\theta\lambda^{\frac{N-2}{2}}\right| \lesssim |t|^{-\frac{N}{N-6}} + |t|^{-\frac{N}{2(N-6)}}\|g\|_{\mathcal{E}} \lesssim |t|^{-\frac{2N-1}{2(N-6)}}.$$

In particular, since $|\theta\lambda^{\frac{N-2}{2}}| \leq |t|^{-\frac{N-1}{N-6}}$, we have

$$|B_3| \lesssim |t|^{-\frac{N-1}{N-6}} + |t|^{-\frac{N}{N-6}} + |t|^{-\frac{N}{2(N-6)}}\|g\|_{\mathcal{E}} + \|g\|_{\mathcal{E}}^2 \lesssim |t|^{-\frac{N-1}{N-6}} + C_0^2|t|^{-\frac{N}{N-6}} \lesssim |t|^{-\frac{N-1}{N-6}}.$$

<u>Fourth row.</u> Differentiating $\langle -e^{i\theta}W_\lambda, g\rangle = 0$ we obtain

$$0 = \frac{d}{dt}\langle -e^{i\theta}W_\lambda, g\rangle = -\theta'\langle i e^{i\theta}W_\lambda, g\rangle + \frac{\lambda'}{\lambda}\langle e^{i\theta}\Lambda W_\lambda, g\rangle - \langle e^{i\theta}W_\lambda, \partial_t g\rangle$$

$$= \zeta'\langle i e^{i\theta}W_\lambda, i e^{i\zeta}W_\mu\rangle - \frac{\mu'}{\mu}\langle e^{i\theta}W_\lambda, e^{i\zeta}\Lambda W_\mu\rangle$$

$$+ \theta'\big(\langle e^{i\theta}W_\lambda, i e^{i\theta}W_\lambda\rangle - \langle i e^{i\theta}W_\lambda, g\rangle\big) + \frac{\lambda'}{\lambda}\big(\langle -e^{i\theta}W_\lambda, e^{i\theta}\Lambda W_\lambda\rangle + \langle e^{i\theta}\Lambda W_\lambda, g\rangle\big)$$

$$- \langle e^{i\theta}W_\lambda, i\,\Delta g + i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda)\big)\rangle,$$

which yields

$$M_{41} = \mu^{-2}\langle i\,e^{i\theta}W_\lambda, i\,e^{i\zeta}W_\mu\rangle = O(\lambda^2) = O(|t|^{-\frac{4}{N-6}}),$$

$$M_{42} = \mu^{-2}\langle e^{i\theta}W_\lambda, e^{i\zeta}\Lambda W_\mu\rangle = O(\lambda^2) = O(|t|^{-\frac{4}{N-6}}),$$

$$M_{43} = \lambda^{-2}\big(\langle e^{i\theta}W_\lambda, i\,e^{i\theta}W_\lambda\rangle - \langle i\,e^{i\theta}W_\lambda, g\rangle\big) = O(\|g\|_\varepsilon) = O(|t|^{-\frac{N-1}{2(N-6)}}),$$

$$M_{44} = \lambda^{-2}\big(\langle -e^{i\theta}W_\lambda, e^{i\theta}\Lambda W_\lambda\rangle + \langle e^{i\theta}\Lambda W_\lambda, g\rangle\big) = \|W\|_{L^2}^2 + O(\|g\|_\varepsilon) = \|W\|_{L^2}^2 + O(|t|^{-\frac{N-1}{2(N-6)}}).$$

Let us consider the term

$$\begin{aligned}B_4 &= \big\langle e^{i\theta}W_\lambda, i\,\Delta g + i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda)\big)\big\rangle\\ &= \big\langle e^{i\theta}W_\lambda, i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)g\big)\big\rangle,\end{aligned}$$

where the last equality follows from (2-10).

First we show that

$$\big|\big\langle e^{i\theta}W_\lambda, i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\big)\big\rangle\big| \lesssim \|g\|_\varepsilon^2. \quad (3\text{-}27)$$

Note that (3-5) and (3-7) imply $|e^{i\zeta}W_\mu + e^{i\theta}W_\lambda| \gtrsim W_\lambda$; hence (2-4) with $z_1 = e^{i\zeta}W_\mu + e^{i\theta}W_\lambda$ and $z_2 = g$ yields

$$\big|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\big| \lesssim W_\lambda^{-\frac{N-6}{N-2}}|g|^2.$$

Using the fact that $|\Lambda W| \lesssim W$ and the Hölder inequality we arrive at (3-27).

The proof of (3-19) yields

$$\big|\big\langle e^{i\theta}W_\lambda, i\big(f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)\big)g\big\rangle\big| \lesssim \lambda^{\frac{N}{4}}\|g\|_\varepsilon. \quad (3\text{-}28)$$

The proof of (3-21) yields

$$\big|\big\langle e^{i\theta}W_\lambda, i\big(f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\big)\big\rangle\big| \lesssim \lambda^{\frac{N}{2}}. \quad (3\text{-}29)$$

Finally, we show that

$$\left|\big\langle e^{i\theta}W_\lambda, i f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\big\rangle - \frac{2\kappa^{\frac{N-4}{2}}\|W\|_{L^2}^2}{N-6}\lambda^{\frac{N-2}{2}}\right| \lesssim |t|^{-\frac{N}{N-6}}. \quad (3\text{-}30)$$

Using again (3-23) we get

$$\big\langle e^{i\theta}W_\lambda, i f'(e^{i\theta}W_\lambda)(e^{i\zeta}W_\mu)\big\rangle = \Re(i\,e^{i(\zeta-\theta)})\int W_\mu W_\lambda^{\frac{N+2}{N-2}}\,\mathrm{d}x. \quad (3\text{-}31)$$

We have $|\Re(e^{-i\theta}) - 1| \lesssim |\theta|^2 \le |t|^{-\frac{2}{N-6}}$ and $|i\,e^{i(\zeta-\theta)} - e^{-i\theta}| = |e^{i\zeta} + i| \lesssim |\zeta| \le |t|^{-\frac{3}{N-6}}$; hence

$$|\Re(i\,e^{i(\zeta-\theta)}) - 1| \lesssim |t|^{-\frac{2}{N-6}}.$$

Since $\left| \int W_\mu W_\lambda^{\frac{N+2}{N-2}} \, dx \right| \lesssim \lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N-2}{N-6}}$, we obtain

$$\left| \Re(i \, e^{i(\zeta-\theta)}) \int W_\mu W_\lambda^{\frac{N+2}{N-2}} \, dx - \int W_\mu W_\lambda^{\frac{N+2}{N-2}} \, dx \right| \lesssim |t|^{-\frac{N}{N-6}}. \tag{3-32}$$

The proof of (3-26) yields

$$\left| \int W_\lambda^{\frac{N+2}{N-2}} \, dx - \int W_\mu W_\lambda^{\frac{N+2}{N-2}} \, dx \right| \lesssim \lambda^{\frac{N}{2}} + |\mu - 1| \lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N}{N-6}}. \tag{3-33}$$

From (2-8) we get

$$\int W_\lambda^{\frac{N+2}{N-2}} \, dx = \frac{2\kappa^{\frac{N-4}{2}} \|W\|_{L^2}^2}{N-6} \lambda^{\frac{N-2}{2}};$$

hence (3-30) follows from (3-31)–(3-33).

From (3-27)–(3-30) and the triangle inequality we obtain

$$\left| B_4 - \frac{2\kappa^{\frac{N-4}{2}} \|W\|_{L^2}^2}{N-6} \lambda(t)^{\frac{N-2}{2}} \right| \lesssim |t|^{-\frac{N}{N-6}} + \|g\|_{\mathcal{E}}^2,$$

in particular

$$|B_4| \lesssim |t|^{-\frac{N-2}{N-6}} + \|g\|_{\mathcal{E}}^2 \lesssim |t|^{-\frac{N-2}{N-6}}. \qquad \square$$

Remark 3.3. A computation similar to the proof of (3-14) shows that $|K| \lesssim \|g\|_{\mathcal{E}}^2 \le |t|^{-\frac{N-1}{N-6}}$, so we obtain the following simple consequence of Lemma 3.1:

$$|\zeta'(t)| + \left| \frac{\mu'(t)}{\mu(t)} \right| + |\theta'(t)| + \left| \frac{\lambda'(t)}{\lambda(t)} \right| \lesssim |t|^{-1} \tag{3-34}$$

(for the last term, this bound is sharp).

Control of the stable and unstable component. An important step is to control the stable and unstable components $a_1^{\pm}(t) = \langle \alpha_{\zeta(t),\mu(t)}^{\pm}, g(t) \rangle$ and $a_2^{\pm}(t) = \langle \alpha_{\theta(t),\lambda(t)}^{\pm}, g(t) \rangle$. Recall that $\nu > 0$ is the positive eigenvalue of the linearized flow; see (2-12).

Lemma 3.4. *Under assumptions of Lemma 3.1, for $t \in [T, T_1]$ we have*

$$\left| \frac{d}{dt} a_1^+(t) - \frac{\nu}{\mu(t)^2} a_1^+(t) \right| \le \frac{c}{\mu(t)^2} |t|^{-\frac{N}{2(N-6)}}, \tag{3-35}$$

$$\left| \frac{d}{dt} a_1^-(t) + \frac{\nu}{\mu(t)^2} a_1^-(t) \right| \le \frac{c}{\mu(t)^2} |t|^{-\frac{N}{2(N-6)}},$$

$$\left| \frac{d}{dt} a_2^+(t) - \frac{\nu}{\lambda(t)^2} a_2^+(t) \right| \le \frac{c}{\lambda(t)^2} |t|^{-\frac{N}{2(N-6)}}, \tag{3-36}$$

$$\left| \frac{d}{dt} a_2^-(t) + \frac{\nu}{\lambda(t)^2} a_2^-(t) \right| \le \frac{c}{\lambda(t)^2} |t|^{-\frac{N}{2(N-6)}}, \tag{3-37}$$

with $c \to 0$ as $|T_0| \to +\infty$.

Proof. We will give a proof of (3-35) and (3-36), the other two inequalities being analogous.

Applying the chain rule to the formula $a_1^+(t) = \langle \alpha_{\zeta(t),\mu(t)}^+, g(t) \rangle$ and using the definition of $\alpha_{\zeta,\mu}^+$ we obtain

$$\frac{d}{dt} a_1^+ = -\frac{\mu'}{\mu} \left\langle \frac{e^{i\zeta}}{\mu^2} (\Lambda_{-1} \mathcal{Y}_\mu^{(2)} + i\Lambda_{-1} \mathcal{Y}_\mu^{(1)}), g \right\rangle + \zeta' \left\langle \frac{e^{i\zeta}}{\mu^2} (i\mathcal{Y}_\mu^{(2)} - \mathcal{Y}_\mu^{(1)}), g \right\rangle + \langle \alpha_{\zeta,\mu}^+, \partial_t g \rangle.$$

Thanks to (3-34) and (3-9), the size of the first two terms is $\lesssim |t|^{-1} |t|^{-\frac{N-1}{2(N-6)}} = |t|^{-\frac{3N-13}{2(N-6)}} \ll |t|^{-\frac{N}{2(N-6)}}$. We are left with the third term, and we expand $\partial_t g$ according to (3-3).

Let us consider, one by one, the contributions of the four terms in the second line of (3-3):

(1) The term $\langle \alpha_{\zeta,\mu}^+, -\zeta' i e^{i\zeta} W_\mu \rangle$ is equal to 0 thanks to (2-14).

(2) The term $\langle \alpha_{\zeta,\mu}^+, \frac{\mu'}{\mu} e^{i\zeta} \Lambda W_\mu \rangle$ is equal to 0 thanks to (2-15).

(3) Consider the term $\langle \alpha_{\zeta,\mu}^+, -\theta' i e^{i\theta} W_\lambda \rangle$. We have $\|\alpha_{\zeta,\mu}^+\|_{\dot{H}^1} \lesssim 1$; hence

$$|\langle \alpha_{\zeta,\mu}^+, -\theta' i e^{i\theta} W_\lambda \rangle| \lesssim |\theta'| \|\alpha_{\zeta,\mu}^+\|_{\dot{H}^1} \|W_\lambda\|_{\dot{H}^{-1}} \lesssim |\theta'| \lambda^2,$$

and (3-34) yields $|\theta'| \lambda^2 \lesssim |t|^{-1} |t|^{-\frac{4}{N-6}} = |t|^{-\frac{N-2}{N-6}} \ll |t|^{-\frac{N}{2(N-6)}}$.

(4) The term $\langle \alpha_{\zeta,\mu}^+, \frac{\lambda'}{\lambda} e^{i\theta} \Lambda W_\lambda \rangle$ is treated as the previous one, using $\left|\frac{\lambda'}{\lambda}\right| \lesssim |t|^{-1}$ instead of $|\theta'| \lesssim |t|^{-1}$.

Let us finally consider the contribution of the first line of (3-3). We have

$$i\Delta g + i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \right)$$
$$= Z_{\zeta,\mu} g + i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu) g \right).$$

From (2-17) we obtain $\langle \alpha_{\zeta,\mu}^+, Z_{\zeta,\mu} g \rangle = \frac{\nu}{\mu^2} a_1^+$; hence we need to show that

$$\left| \langle \alpha_{\zeta,\mu}^+, i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu) g \right) \rangle \right| \ll |t|^{-\frac{N}{2(N-6)}}.$$

The proof of (3-17) yields the bound $|t|^{-\frac{N-2}{N-6}} \ll |t|^{-\frac{N}{2(N-6)}}$.

We turn to the proof of (3-36). Applying the chain rule to the formula $a_2^+(t) = \langle \alpha_{\zeta(t),\mu(t)}^+, g(t) \rangle$ and using the definition of $\alpha_{\theta,\lambda}^+$ we obtain

$$\frac{d}{dt} a_2^+ = -\frac{\lambda'}{\lambda} \left\langle \frac{e^{i\theta}}{\lambda^2} (\Lambda_{-1} \mathcal{Y}_\lambda^{(2)} + i\Lambda_{-1} \mathcal{Y}_\lambda^{(1)}), g \right\rangle + \theta' \left\langle \frac{e^{i\theta}}{\lambda^2} (i\mathcal{Y}_\lambda^{(2)} - \mathcal{Y}_\lambda^{(1)}), g \right\rangle + \langle \alpha_{\theta,\lambda}^+, \partial_t g \rangle.$$

The first two terms are treated as in the case of a_1^+. In the third term, we expand $\partial_t g$ using (3-3). Let us consider, one by one, the contributions of the four terms in the second line of (3-3):

(1) In order to bound the term $\langle \alpha_{\theta,\lambda}^+, -\zeta' i e^{i\zeta} W_\mu \rangle$, notice that

$$\|\alpha_{\theta,\lambda}^+\|_{L^1} \lesssim \int_{\mathbb{R}^N} \frac{1}{\lambda^2} (|\mathcal{Y}_\lambda^{(1)}| + |\mathcal{Y}_\lambda^{(2)}|) \, dx \lesssim \lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N-2}{N-6}} \ll |t|^{-\frac{N}{2(N-6)}}.$$

This is sufficient since $\|-\zeta' i e^{i\zeta} W_\mu\|_{L^\infty} \lesssim 1$.

(2) The term $\langle \alpha_{\theta,\lambda}^+, \frac{\mu'}{\mu} e^{i\zeta} \Lambda W_\mu \rangle$ is analogous.

(3) The term $\langle \alpha_{\theta,\lambda}^+, -\theta' i e^{i\theta} W_\lambda \rangle$ is equal to 0 thanks to (2-14).

(4) The term $\langle \alpha_{\theta,\lambda}^+, \frac{\lambda'}{\lambda} e^{i\theta} \Lambda W_\lambda \rangle$ is equal to 0 thanks to (2-15).

Let us finally consider the contribution of the first line of (3-3). We have

$$i \Delta g + i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \right)$$
$$= Z_{\theta,\lambda} g + i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\theta} W_\lambda) g \right).$$

From (2-17) we obtain $\langle \alpha_{\theta,\lambda}^+, Z_{\theta,\lambda} g \rangle = \frac{v}{\lambda^2} a_2^+$; hence we need to show that

$$\lambda^2 \left| \langle \alpha_{\theta,\lambda}^+, i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) - f'(e^{i\theta} W_\lambda) g \right) \rangle \right| \ll |t|^{-\frac{N}{2(N-6)}}. \quad (3\text{-}38)$$

The proof of (3-19) yields

$$\lambda^2 \left| \langle \alpha_{\theta,\lambda}^+, i \left(f'(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f'(e^{i\theta} W_\lambda) \right) g \rangle \right| \lesssim \lambda^{\frac{N}{4}} \|g\|_{\mathcal{E}} \lesssim |t|^{-\frac{N}{2(N-6)} - \frac{N-1}{2(N-6)}} \ll |t|^{-\frac{N}{2(N-6)}}. \quad (3\text{-}39)$$

The proof of (3-27) yields

$$\lambda^2 \left| \langle \alpha_{\theta,\lambda}^+, i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda + g) - f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f'(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) g \right) \rangle \right|$$
$$\lesssim \|g\|_{\mathcal{E}}^2 \ll |t|^{-\frac{N}{2(N-6)}}. \quad (3\text{-}40)$$

Using (2-3) we get

$$\left\| f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \right\|_{L^\infty} \lesssim \|W_\lambda^{\frac{4}{N-2}} W_\mu\|_{L^\infty} \lesssim \frac{1}{\lambda^2}.$$

By a change of variable, $\|\lambda^2 \alpha_{\theta,\lambda}^+\|_{L^1} \lesssim \lambda^{\frac{N+2}{2}}$; hence

$$\lambda^2 \left| \langle \alpha_{\theta,\lambda}^+, i \left(f(e^{i\zeta} W_\mu + e^{i\theta} W_\lambda) - f(e^{i\zeta} W_\mu) - f(e^{i\theta} W_\lambda) \right) \rangle \right| \lesssim \lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N-2}{N-6}} \ll |t|^{-\frac{N}{2(N-6)}}. \quad (3\text{-}41)$$

Taking the sum of (3-39)–(3-41) and using the triangle inequality, we obtain (3-38). $\qquad \square$

4. Bootstrap

We turn to the heart of the proof, which consists in establishing bootstrap estimates. We consider a solution $u(t)$, decomposed according to (3-1), (3-2) and (2-18). The initial data at time $T \le T_0$ is chosen as follows.

Lemma 4.1. *There exists $T_0 < 0$ such that for all $T \le T_0$ and for all λ^0, a_1^0, a_2^0 satisfying*

$$\left| \lambda^0 - \frac{1}{\kappa} (\kappa|T|)^{-\frac{2}{N-6}} \right| \le \frac{1}{2} |T|^{-\frac{5}{2(N-6)}}, \quad |a_1^0| \le \frac{1}{2} |T|^{-\frac{N}{2(N-6)}}, \quad |a_2^0| \le \frac{1}{2} |T|^{-\frac{N}{2(N-6)}}, \quad (4\text{-}1)$$

there exists $g^0 \in X^1$ satisfying

$$\langle \Lambda W, g^0 \rangle = \langle i W, g^0 \rangle = \langle i \Lambda W_{\lambda^0}, g^0 \rangle = \langle -W_{\lambda^0}, g^0 \rangle = 0, \quad (4\text{-}2)$$

$$\langle \alpha_{-\frac{\pi}{2},1}^-, g^0 \rangle = 0, \quad \langle \alpha_{-\frac{\pi}{2},1}^+, g^0 \rangle = a_1^0, \quad \langle \alpha_{0,\lambda^0}^-, g^0 \rangle = 0, \quad \langle \alpha_{0,\lambda^0}^+, g^0 \rangle = a_2^0, \quad (4\text{-}3)$$

$$\|g^0\|_{\mathcal{E}} \lesssim |T|^{-\frac{N}{2(N-6)}}. \quad (4\text{-}4)$$

This g^0 is continuous for the X^1 topology with respect to λ^0, a_1^0 and a_2^0.

Remark 4.2. For the continuity, we just claim that the function g^0 constructed in the proof is continuous with respect to λ^0, a_1^0 and a_2^0. Clearly, g^0 is not uniquely determined by (4-2)–(4-4).

Remark 4.3. Condition (4-2) is exactly (2-18) with $(\zeta, \mu, \theta, \lambda) = \left(-\frac{\pi}{2}, 1, 0, \lambda^0\right)$. Hence, if we consider the solution $u(t)$ of (1-1) with initial data $u(T) = -iW + W_{\lambda^0} + g^0$ and decompose it according to (3-1), then $g(T) = g^0$ and the initial values of the modulation parameters are $(\zeta(T), \mu(T), \theta(T), \lambda(T)) = \left(-\frac{\pi}{2}, 1, 0, \lambda^0\right)$.

Proof. We consider functions of the form

$$g^0 = a_1^+ i\alpha_{-\frac{\pi}{2},1}^- - a_1^- i\alpha_{-\frac{\pi}{2},1}^+ + b_1 W + c_1(-i\Lambda W) + a_2^+ (\lambda^0)^2 i\alpha_{0,\lambda^0}^- - a_2^- (\lambda^0)^2 i\alpha_{0,\lambda^0}^+ + b_2 i W_{\lambda^0} + c_2 \Lambda W_{\lambda^0},$$

with a_1^+, a_1^-, b_1, c_1, a_2^+, a_2^-, b_2, c_2 being real numbers. Let $\Phi : \mathbb{R}^8 \to \mathbb{R}^8$ be the linear map defined as

$$\Phi(a_1^+, a_1^-, b_1, c_1, a_2^+, a_2^-, b_2, c_2) := \left(\langle \alpha_{-\frac{\pi}{2},1}^+, g^0 \rangle, \langle \alpha_{-\frac{\pi}{2},1}^-, g^0 \rangle, \langle \Lambda W, g^0 \rangle, \langle iW, g^0 \rangle, \right.$$
$$\left. \langle \alpha_{0,\lambda^0}^+, g^0 \rangle, \langle \alpha_{0,\lambda^0}^-, g^0 \rangle, \langle (\lambda^0)^{-2} i\Lambda W_{\lambda^0}, g^0 \rangle, \langle -(\lambda^0)^{-2} W_{\lambda^0}, g^0 \rangle \right).$$

Using (2-14)–(2-16) and the fact that λ^0 is small we obtain that the matrix of Φ is strictly diagonally dominant, which implies the result. $\qquad\square$

In the remaining part of this section, we will analyze solutions $u(t)$ of (1-1) with the initial data $u(T) = -iW + W_{\lambda^0} + g^0$, where g^0 is given by the previous lemma.

Proposition 4.4. *There exists $T_0 < 0$ with the following property. Let $T < T_1 < T_0$ and let λ^0, a_1^0, a_2^0 satisfy (4-1). Let $g^0 \in X^1$ be given by Lemma 4.1 and consider the solution $u(t)$ of (1-1) with the initial data $u(T) = -iW + W_{\lambda^0} + g^0$. Suppose that $u(t)$ exists on the time interval $[T, T_1]$, that for $t \in [T, T_1]$ conditions (3-5)–(3-9) hold, and moreover that*

$$|a_1^+(t)| \le |t|^{-\frac{N}{2(N-6)}}, \quad |a_2^+(t)| \le |t|^{-\frac{N}{2(N-6)}}. \tag{4-5}$$

Then for $t \in [T, T_1]$,

$$\left| \zeta(t) + \tfrac{\pi}{2} \right| \le \tfrac{1}{2} |t|^{-\frac{3}{N-6}}, \tag{4-6}$$

$$|\mu(t) - 1| \le \tfrac{1}{2} |t|^{-\frac{3}{N-6}}, \tag{4-7}$$

$$|\theta(t)| \le \tfrac{1}{2} |t|^{-\frac{1}{N-6}}, \tag{4-8}$$

$$\|g(t)\|_\varepsilon \le \tfrac{1}{2} |t|^{-\frac{N-1}{2(N-6)}}. \tag{4-9}$$

Before we give a proof, we need a little preparation.

A virial-type correction. The delicate part of the proof of Proposition 4.4 will be to control $\theta(t)$. For this, we will need to use a virial functional, which we now define.

Lemma 4.5. *For any $c > 0$ and $R > 0$ there exists a radial function $q(x) = q_{c,R}(x) \in C^{3,1}(\mathbb{R}^N)$ with the following properties:*

(P1) $q(x) = \tfrac{1}{2}|x|^2$ for $|x| \le R$.

(P2) *There exists $\widetilde{R} > 0$ (depending on c and R) such that $q(x) \equiv \mathrm{const}$ for $|x| \geq \widetilde{R}$.*

(P3) $|\nabla q(x)| \lesssim |x|$ *and* $|\Delta q(x)| \lesssim 1$ *for all* $x \in \mathbb{R}^N$, *with constants independent of c and R.*

(P4) $\sum_{1 \leq j,k \leq N} (\partial_{x_j x_k} q(x)) \bar{v}_j v_k \geq -c \sum_{j=1}^N |v_j|^2$ *for all* $x \in \mathbb{R}^N$, $v_j \in \mathbb{C}$.

(P5) $\Delta^2 q(x) \leq c \cdot |x|^{-2}$, *for all* $x \in \mathbb{R}^N$.

Remark 4.6. We require $C^{3,1}$ regularity in order not to worry about boundary terms in Pohozaev identities; see the proof of (4-13).

Proof. It suffices to prove the result for $R = 1$ since the function $q_R(x) := R^2 q\left(\frac{x}{R}\right)$ satisfies the listed properties if and only if $q(x)$ does.

Let r denote the radial coordinate. Define $q_0(x)$ by the formula

$$q_0(r) := \begin{cases} \frac{1}{2}r^2, & r \leq 1, \\ \dfrac{N(N-2)r}{(N-1)(N-3)} - \dfrac{N}{2(N-4)} + \dfrac{N}{2(N-3)(N-4)r^{N-4}} - \dfrac{1}{2(N-1)r^{N-2}}, & r \geq 1. \end{cases}$$

A direct computation shows that for $r > 1$ we have

$$q_0'(r) = \frac{N(N-2)}{(N-1)(N-3)} - \frac{N}{2(N-3)r^{N-3}} + \frac{N-2}{2(N-1)r^{N-1}},$$

$$q_0''(r) = \frac{N}{2r^{N-2}} - \frac{N-2}{2r^N} > 0 \quad (\text{so } q_0(x) \text{ is convex}),$$

$$q_0'''(r) = \frac{N(N-2)}{2}\left(-\frac{1}{r^{N-1}} + \frac{1}{r^{N+1}}\right),$$

$$\Delta^2 q_0(r) = -N(N-2)r^{-3} < 0.$$

In particular,

$$\lim_{r \to 1^+} \left(q_0(r), q_0'(r), q_0''(r), q_0'''(r)\right) = \left(\tfrac{1}{2}, 1, 1, 0\right).$$

Hence $q_0 \in C^{3,1}$ and it satisfies all the listed properties except for (P2). We correct it as follows.

Let $e_j(r) := (1/j!)r^j \cdot \chi(r)$ for $j \in \{1, 2, 3\}$, where $\chi(r)$ is the standard cut-off function:

$$\chi \in C^\infty((0, +\infty), \mathbb{R}), \qquad \chi(r) = 1 \quad \text{for } r \leq 1, \qquad \chi(r) = 0 \quad \text{for } r \geq 2.$$

Let $R_0 \gg 1$. We define

$$q(r) := \begin{cases} q_0(r), & r \leq R_0, \\ q_0(R_0) + \sum_{j=1}^3 q_0^{(j)}(R_0) \cdot R_0^j \cdot e_j(-1 + R_0^{-1}r), & r \geq R_0. \end{cases}$$

Note that $q_0'(R_0) \sim 1$, $q_0''(R_0) \sim R_0^{-N+2}$ and $q_0'''(R_0) \sim R_0^{-N+1}$. It is clear that $q(x) \in C^{3,1}(\mathbb{R}^N)$. Property (P1) holds since $R_0 > 1$. By the definition of the functions e_j we have $q(r) = q_0(R_0) = \mathrm{const}$ for $r \geq 3R_0$; hence (P2) holds with $\widetilde{R} = 3R_0$. From the definition of $q(r)$ we get $|q'(r)| \lesssim |q_0'(R_0)| \lesssim r$ and $|q''(r)| \lesssim |q_0''(R_0)| \lesssim R_0^{-N+2} \lesssim 1$ for $r \geq R_0$, with a constant independent of R_0, which implies (P3). Similarly, $|\partial_{x_i x_j} q(x)| \lesssim R_0^{-1}$ for $|x| \geq R_0$, which implies (P4) if R_0 is large enough. Finally $|\Delta^2 q(x)| \lesssim R_0^{-3}$ for $|x| \geq R_0$ and $\Delta^2 q(x) = 0$ for $|x| \geq 3R_0$. This proves (P5) if R_0 is large enough. $\square$

In the sequel $q(x)$ always denotes a function of class $C^{3,1}(\mathbb{R}^N)$ verifying (P1)–(P5) with sufficiently small c and sufficiently large R.

For $\lambda > 0$ we define the operators $A(\lambda)$ and $A_0(\lambda)$ as

$$[A(\lambda)h](x) := \frac{N-2}{2N\lambda^2}\Delta q\left(\frac{x}{\lambda}\right)h(x) + \frac{1}{\lambda}\nabla q\left(\frac{x}{\lambda}\right) \cdot \nabla h(x),$$

$$[A_0(\lambda)h](x) := \frac{1}{2\lambda^2}\Delta q\left(\frac{x}{\lambda}\right)h(x) + \frac{1}{\lambda}\nabla q\left(\frac{x}{\lambda}\right) \cdot \nabla h(x).$$

Combining these definitions with the fact that $q(x)$ is an approximation of $\frac{1}{2}|x|^2$ we see that $A(\lambda)$ and $A_0(\lambda)$ are approximations (in a sense not yet made precise) of $\lambda^{-2}\Lambda$ and $\lambda^{-2}\Lambda_0$ respectively. We will write A and A_0 instead of $A(1)$ and $A_0(1)$ respectively. Note the following scale-change formulas, which follow directly from the definitions:

$$\text{for all } h \in \mathcal{E}, \quad A(\lambda)(h_\lambda) = \lambda^{-2}(Ah)_\lambda, \quad A_0(\lambda)(h_\lambda) = \lambda^{-2}(A_0h)_\lambda. \tag{4-10}$$

Lemma 4.7. *The operators $A(\lambda)$ and $A_0(\lambda)$ have the following properties*:

- *For $\lambda > 0$, the families $\{A(\lambda)\}$, $\{A_0(\lambda)\}$, $\{\lambda\partial_\lambda A(\lambda)\}$, $\{\lambda\partial_\lambda A_0(\lambda)\}$ are bounded in $\mathscr{L}(\mathcal{E}; \dot{H}^{-1})$ and the families $\{\lambda A(\lambda)\}$, $\{\lambda A_0(\lambda)\}$ are bounded in $\mathscr{L}(\mathcal{E}; L^2)$, with the bound depending on the choice of the function $q(x)$,*

- *For all complex-valued $h_1, h_2 \in X^1(\mathbb{R}^N)$ and $\lambda > 0$,*

$$\langle A(\lambda)h_1, f(h_1 + h_2) - f(h_1) - f'(h_1)h_2 \rangle = -\langle A(\lambda)h_2, f(h_1 + h_2) - f(h_1) \rangle, \tag{4-11}$$

$$\langle h_1, A_0(\lambda)h_2 \rangle = -\langle A_0(\lambda)h_1, h_2 \rangle, \quad \text{and hence } iA_0(\lambda) \text{ is a symmetric operator.} \tag{4-12}$$

- *For any $c_0 > 0$, if we choose c in Lemma 4.5 small enough, then for all $h \in X^1$,*

$$\langle A_0(\lambda)h, \Delta h \rangle \leq \frac{c_0}{\lambda^2}\|h\|_{\mathcal{E}}^2 - \frac{1}{\lambda^2}\int_{|x| \leq R\lambda} |\nabla h(x)|^2 \, dx. \tag{4-13}$$

In dimension $N = 6$ and for real-valued functions, this was proved in [Jendrej 2016, Lemma 3.12]. Most arguments apply without change, but we provide here a full computation for the reader's convenience.

Proof. Since $\nabla q(x)$ and $\nabla^2 q(x)$ are continuous and of compact support, it is clear that A and A_0 are bounded operators $\mathcal{E} \to \dot{H}^{-1}$. From the invariance (4-10) we see that $A(\lambda)$ and $A_0(\lambda)$ have the same norms as A and A_0 respectively. For $\lambda A(\lambda)$, $\lambda A_0(\lambda)$, $\lambda\partial_\lambda A(\lambda)$ and $\lambda\partial_\lambda A_0(\lambda)$ the proof is similar. We compute

$$\partial_\lambda A(\lambda) = -\frac{N-2}{N\lambda^3}\Delta q\left(\frac{x}{\lambda}\right) - \frac{N-2}{2N\lambda^4}x \cdot \nabla\Delta q\left(\frac{x}{\lambda}\right) - \frac{1}{\lambda^3}x \cdot \nabla^2 q\left(\frac{x}{\lambda}\right) \cdot \nabla.$$

Since $\nabla q(x)$, $\nabla^2 q(x)$ and $\nabla^3 q(x)$ are continuous and of compact support, we get boundedness of $\partial_\lambda A(1)$, and boundedness $\{\lambda\partial_\lambda A(\lambda)\}$ follows by the scaling invariance.

In (4-11), we may assume without loss of generality that $\lambda = 1$. Notice that both sides are continuous with respect to the topology $\|h_1\|_{X^1} + \|h_2\|_{X^1}$. Indeed, A is continuous from X^1 to $\mathcal{E}$ and $(h_1, h_2) \mapsto \left(f(h_1 + h_2) - f(h_1) - f'(h_1)h_2, f(h_1 + h_2) - f(h_1)\right)$ is continuous from $\mathcal{E}$ to $\dot{H}^{-1}$ by

Sobolev and dual Sobolev. We may therefore assume that $h_1, h_2 \in C_0^\infty$. Observe that for any $h \in C_0^\infty$ we have $f(h)\bar{h} = \frac{2N}{N-2}F(h)$ and $\Re(f(h)\nabla\bar{h}) = \nabla(F(h))$; hence

$$\langle Ah, f(h) \rangle = \Re \int_{\mathbb{R}^N} \left(\frac{N-2}{2N} \Delta q \bar{h} + \nabla q \cdot \nabla \bar{h} \right) f(h)\, dx = \int_{\mathbb{R}^N} \Delta q \cdot F(h) + \nabla q \cdot \nabla(F(h))\, dx = 0.$$

Using this for $h = h_1 + h_2$ and for $h = h_1$, (4-11) is seen to be equivalent to

$$\langle Ah_2, f(h_1) \rangle + \langle Ah_1, f'(h_1)h_2 \rangle = 0. \tag{4-14}$$

Expanding the left side using the definition of A we obtain

$$\langle Ah_2, f(h_1) \rangle + \langle Ah_1, f'(h_1)h_2 \rangle = \Re \int_{\mathbb{R}^N} \frac{N-2}{2N} \Delta q \cdot \bar{h}_2 \cdot f(h_1) + \nabla q \cdot \nabla \bar{h}_2 \cdot f(h_1)\, dx$$
$$+ \Re \int_{\mathbb{R}^N} \frac{N-2}{2N} \Delta q \cdot \bar{h}_1 \cdot f'(h_1)h_2 + \nabla q \cdot \nabla \bar{h}_1 \cdot f'(h_1)h_2\, dx. \tag{4-15}$$

We have

$$\Re \int_{\mathbb{R}^N} \nabla q \cdot \nabla \bar{h}_2 \cdot f(h_1)\, dx = -\Re \int_{\mathbb{R}^N} \bar{h}_2 \cdot \Delta q \cdot f(h_1)\, dx - \Re \int_{\mathbb{R}^N} \bar{h}_2 \cdot \nabla q \cdot \nabla f(h_1)\, dx.$$

Using (2-1) and the fact that $f'(h_1)h_1 = \frac{N+2}{N-2} f(h_1)$ we get

$$\Re \int_{\mathbb{R}^N} \frac{N-2}{2N} \Delta q \cdot \bar{h}_1 \cdot f'(h_1)h_2\, dx = \Re \int_{\mathbb{R}^N} \frac{N-2}{2N} \bar{h}_2 \cdot \Delta q \cdot f'(h_1)h_1\, dx = \Re \int_{\mathbb{R}^N} \bar{h}_2 \cdot \frac{N+2}{2N} \Delta q \cdot f(h_1)\, dx.$$

Using (2-1) and the fact that $f'(h_1)\nabla h_1 = \nabla(f(h_1))$ we get

$$\Re \int_{\mathbb{R}^N} \nabla q \cdot \nabla \bar{h}_1 \cdot f'(h_1)h_2\, dx = \Re \int_{\mathbb{R}^N} \bar{h}_2 \cdot \nabla q \cdot f'(h_1)\nabla h_1\, dx = \Re \int_{\mathbb{R}^N} \bar{h}_2 \cdot \nabla q \cdot \nabla(f(h_1))\, dx.$$

Plugging the last three formulas into (4-15) we obtain

$$\langle Ah_2, f(h_1) \rangle + \langle Ah_1, f'(h_1)h_2 \rangle$$
$$= \left\langle h_2, \frac{N-2}{2N} \Delta q \cdot f(h_1) - \Delta q \cdot f(h_1) - \nabla q \cdot \nabla(f(h_1)) + \frac{N+2}{2N} \Delta q \cdot f(h_1) + \nabla q \cdot \nabla(f(h_1)) \right\rangle = \langle h_2, 0 \rangle = 0,$$

which proves (4-14).

Identity (4-12) follows by an integration by parts.

In (4-13), we can again assume that $\lambda = 1$ and $h \in C_0^\infty$ (we use the fact that $q \in C^{3,1}$, and hence $\Delta^2 q$ is bounded and of compact support). Inequality (4-13) follows easily from (P1), (P4) and (P5), once we check the following identity:

$$\Re \int_{\mathbb{R}^N} \Delta h \cdot \left(\frac{1}{2} \Delta q \cdot \bar{h} + \nabla q \cdot \nabla \bar{h} \right) dx = \frac{1}{4} \int_{\mathbb{R}^N} (\Delta^2 q)|h|^2\, dx - \int_{\mathbb{R}^N} \sum_{i,j=1}^N \partial_{ij} q\, \partial_i \bar{h}\, \partial_j h\, dx. \tag{4-16}$$

We can assume that $q \in C_0^\infty$, and (4-16) follows from integration by parts:

$$\Re \int_{\mathbb{R}^N} \tfrac{1}{2} \Delta h \cdot \Delta q \cdot \bar{h} + \Delta h \cdot \nabla q \cdot \nabla \bar{h} \, dx$$

$$= \Re \int_{\mathbb{R}^N} \sum_{j,k=1}^N \left(\tfrac{1}{2} \partial_{jj} h \cdot \partial_{kk} q \cdot \bar{h} + \partial_{jj} h \cdot \partial_k q \cdot \partial_k \bar{h} \right) dx$$

$$= \Re \int_{\mathbb{R}^N} -\tfrac{1}{2} \sum_{j,k} \partial_j h (\partial_{kk} q \, \partial_j \bar{h} + \partial_{jkk} q \cdot \bar{h}) + \sum_j \tfrac{1}{2} \partial_j (|\partial_j h|^2) \partial_j q + \sum_{j \neq k} \left(-\tfrac{1}{2} \partial_k |\partial_j h|^2 \partial_k q - \partial_{jk} q \, \partial_j \bar{h} \, \partial_k h \right) dx$$

$$= \Re \int_{\mathbb{R}^N} -\tfrac{1}{2} \sum_{j,k} (\partial_{kk} q |\partial_j h|^2 - \tfrac{1}{2} \partial_{jjkk} q \cdot |h|^2) - \tfrac{1}{2} \sum_j \partial_{jj} q |\partial_j h|^2 + \tfrac{1}{2} \sum_{j \neq k} \partial_{kk} q |\partial_j h|^2 - \sum_{j \neq k} \partial_{jk} q \, \partial_j \bar{h} \, \partial_k h \, dx$$

$$= \int_{\mathbb{R}^N} \tfrac{1}{4} \sum_{j,k} \partial_{jjkk} q \cdot |h|^2 - \sum_{j,k} \partial_{jk} q \, \partial_j \bar{h} \, \partial_k h \, dx. \qquad \square$$

Closing the bootstrap.

Proof of Proposition 4.4. We split the proof into three steps. First we prove (4-6) and (4-7). Then we use the virial functional and variational estimates to prove (4-8), with $\tfrac{1}{2}$ replaced by any strictly positive constant. To do this, we have to deal somehow with the term $\|W\|_{L^2}^{-2} K$ in the modulation equation (3-12). It involves terms quadratic in g, which is the critical size. However, it turns out that we can use a virial functional to absorb the essential part of K. Proving (4-8) is the most difficult step. Finally, (4-9) will follow from variational estimates.

Step 1. Integrating (3-10) on $[T, t]$ and using the fact that $\zeta(T) = -\tfrac{\pi}{2}$ we get

$$\left| \zeta(t) + \tfrac{\pi}{2} \right| = \left| \zeta(t) - \zeta(T) \right| = \left| \int_T^t \zeta'(\tau) \, d\tau \right| \le c \int_T^t |\tau|^{-\frac{N-3}{N-6}} \, d\tau \le c \cdot \frac{N-6}{3} |t|^{-\frac{3}{N-6}} \le \tfrac{1}{2} |t|^{-\frac{3}{N-6}},$$

provided that $c \le \frac{3}{2(N-6)}$. Recall that $c > 0$ can be made arbitrarily small by choosing $|T_0|$ large enough, in particular smaller than $\frac{3}{2(N-6)}$. The proof of (4-7) is similar.

Step 2. First, let us show that for $t \in [T, T_1]$ we have

$$|a_1^-(t)| < |t|^{-\frac{N}{2(N-6)}}, \quad |a_2^-(t)| < |t|^{-\frac{N}{2(N-6)}}. \tag{4-17}$$

This is verified initially; see (4-3). Suppose that $T_2 \in (T, T_1)$ is the last time for which (4-17) holds for $t \in [T, T_2)$. Let for example $a_1^-(T_2) = |T_2|^{-\frac{N}{2(N-6)}}$. But since $\|g(T_2)\|_{\mathcal{E}}^2 \lesssim |T_2|^{-\frac{N-1}{N-6}} \ll |T_2|^{-\frac{N}{2(N-6)}}$, (3-37) implies $\frac{d}{dt} a_1^-(T_2) < 0$, which contradicts the assumption that $a_1^-(t) < |T_2|^{-\frac{N}{2(N-6)}}$ for $t < T_2$. The proof of the other inequality is similar.

Let $c_0 > 0$. We will prove that if T_0 is chosen large enough (depending on c_0), then

$$|\theta(t)| \le c_0 |t|^{-\frac{1}{N-6}}, \quad \text{for } t \in [T, T_1]. \tag{4-18}$$

By the conservation of energy, (2-19) and (4-4) we have

$$|E(u) - 2E(W)| = |E(u(T)) - 2E(W)| \lesssim |T|^{-\frac{N}{N-6}} \le |t|^{-\frac{N}{N-6}};$$

hence (2-20) yields

$$\theta\lambda^{\frac{N-2}{2}} \lesssim |t|^{-\frac{N}{N-6}} \quad \Longrightarrow \quad \theta \lesssim |t|^{-\frac{N}{N-6}+\frac{N-2}{N-6}} = |t|^{-\frac{2}{N-6}} \ll |t|^{-\frac{1}{N-6}}. \tag{4-19}$$

It remains to prove that

$$\theta \geq -c_0|t|^{-\frac{1}{N-6}}. \tag{4-20}$$

To this end, we consider the real scalar function

$$\psi(t) := \theta(t) - \frac{1}{2\|W\|_{L^2}^2} \langle g(t), iA_0(\lambda(t))g(t)\rangle.$$

We will show that for $t \in [T, T_1]$ we have

$$\psi'(t) \geq -c_1|t|^{-\frac{N-5}{N-6}}, \tag{4-21}$$

with $c_1 > 0$ as small as we like, by eventually enlarging $|T_0|$.

From (4-19) we get $\theta\lambda^{-\frac{N-6}{2}} \ll |t|^{-\frac{N-5}{N-6}}$; hence, taking in Lemma 3.1, say, $c = \frac{1}{4}c_1$ and choosing $|T_0|$ large enough, (3-12) yields

$$\begin{aligned}
\psi' &\geq -\frac{(N-2)\kappa^{\frac{N-4}{2}}}{N-6}\theta\lambda^{\frac{N-6}{2}} + \frac{K}{\lambda^2\|W\|_{L^2}^2} - \frac{c_1}{4}|t|^{-\frac{N-5}{N-6}} + \frac{1}{2\|W\|_{L^2}^2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle \\
&\geq \frac{1}{\|W\|_{L^2}^2}\left(\frac{1}{\lambda^2}K - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle\right) - \frac{c_1}{2}|t|^{-\frac{N-5}{N-6}},
\end{aligned} \tag{4-22}$$

so we need to compute $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle$, up to terms of order $\ll |t|^{-\frac{N-5}{N-6}}$. In this proof, the sign $\simeq$ will mean "up to terms of order $\ll |t|^{-\frac{N-5}{N-6}}$ as $|T_0| \to +\infty$".

Since $iA_0(\lambda)$ is symmetric, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle = \frac{1}{2}\lambda'\langle g, i\partial_\lambda A_0(\lambda)g\rangle + \langle \partial_t g, iA_0(\lambda)g\rangle. \tag{4-23}$$

The first term is of size $\lesssim \left|\frac{\lambda'}{\lambda}\right|\cdot\|g\|_{\mathcal{E}}^2 \ll |t|^{-\frac{N-5}{N-6}}$, and hence is negligible. We expand $\partial_t g$ according to (3-3). Consider the terms in the second line of (3-3). It follows from (3-34) and the fact that $\|A_0(\lambda)g\|_{\dot{H}^{-1}} \lesssim \|g\|_{\mathcal{E}}$ that their contribution is $\lesssim |t|^{-1}\|g\|_{\mathcal{E}} \leq |t|^{-\frac{3N-13}{2(N-6)}} \ll |t|^{-\frac{N-5}{N-6}}$, and hence is negligible, so we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle \simeq \langle \Delta g + f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda), A_0(\lambda)g\rangle. \tag{4-24}$$

We now check that

$$\left|\langle f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda), A_0(\lambda)g\rangle\right| \ll |t|^{-\frac{N-5}{N-6}}. \tag{4-25}$$

The function $A_0(\lambda)g$ is supported in the ball of radius $\widetilde{R}\lambda$. In this region we have $W_\lambda \ll W_\mu$; hence (2-3) yields $|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f(e^{i\zeta}W_\mu) - f(e^{i\theta}W_\lambda)| \lesssim |W_\lambda|^{\frac{4}{N-2}}$. By a change of variable we obtain

$$\|W_\lambda^{\frac{4}{N-2}}\|_{L^2(|x|\leq\widetilde{R}\lambda)} = \lambda^{\frac{N-2}{2}}\|W^{\frac{4}{N-2}}\|_{L^2(|x|\leq\widetilde{R})} \lesssim |t|^{-\frac{N-2}{N-6}}.$$

By the first property in Lemma 4.7, $\|A_0(\lambda)g\|_{L^2} \lesssim \lambda^{-1}\|g\|_{\mathcal{E}} \lesssim |t|^{-\frac{N-5}{2(N-6)}}$; hence the Cauchy–Schwarz inequality implies (4-25) (with a large margin). By the triangle inequality, (4-24) and (4-25) yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle \simeq \langle \Delta g + f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda), A_0(\lambda)g\rangle.$$

We transform the right-hand side using (4-11), (4-13) and the fact that $A_0(\lambda)g = \frac{1}{N\lambda^2}\Delta q\left(\frac{\cdot}{\lambda}\right)g + A(\lambda)g$. Note that for any $c_2 > 0$ we have

$$\frac{c_0}{\lambda^2}\|g\|_{\mathcal{E}}^2 \le \frac{c_2}{2}|t|^{-\frac{N-5}{N-6}}$$

if we choose c_0 small enough; thus

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle \\
&\le c_2|t|^{-\frac{N-5}{N-6}} - \frac{1}{\lambda^2}\left(\int_{|x|\le R\lambda}|\nabla g|^2\,\mathrm{d}x - \langle f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda), \tfrac{1}{N}\Delta q\left(\tfrac{\cdot}{\lambda}\right)g\rangle\right) \\
&\quad - \langle A(\lambda)(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda), f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\rangle, \quad (4\text{-}26)
\end{aligned}$$

where c_2 can be made arbitrarily small. Consider the second line. We will check that

$$\left|\langle f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda), \tfrac{1}{N}\Delta q\left(\tfrac{\cdot}{\lambda}\right)g\rangle - \langle f'(e^{i\theta}W_\lambda)g, g\rangle\right| \ll |t|^{-\frac{N-1}{N-6}}. \quad (4\text{-}27)$$

Indeed, Δq is bounded; hence $\left\|\frac{1}{N}\Delta q\left(\frac{\cdot}{\lambda}\right)g\right\|_{L^{\frac{2N}{N-2}}} \lesssim \|g\|_{\mathcal{E}}$. By (2-4) we have

$$\left\|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\right\|_{L^{2N/(N+2)}} \lesssim \|g\|_{\mathcal{E}}^{\frac{N+2}{N-2}} \ll \|g\|_{\mathcal{E}}.$$

Now from (2-2) we obtain

$$\left\|\left(f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\theta}W_\lambda)\right)g\right\|_{L^{2N/(N+2)}(|x|\le \tilde{R}\lambda)} \lesssim \|f'(e^{i\zeta}W_\mu)\|_{L^{N/2}(|x|\le \tilde{R}\lambda)}\|g\|_{\mathcal{E}} \ll \|g\|_{\mathcal{E}}.$$

We have obtained

$$\left|\langle f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda), \tfrac{1}{N}\Delta q\left(\tfrac{\cdot}{\lambda}\right)g\rangle - \langle f'(e^{i\theta}W_\lambda)g, \tfrac{1}{N}\Delta q\left(\tfrac{\cdot}{\lambda}\right)g\rangle\right| \ll |t|^{-\frac{N-1}{N-6}}.$$

But $\frac{1}{N}\Delta q\left(\frac{x}{\lambda}\right) = 1$ for $|x| \le R\lambda$ and $\|f'(e^{i\theta}W_\lambda)\|_{L^{N/2}(|x|\ge R\lambda)} \ll 1$ for R large. This proves (4-27).

The bounds (4-5) and (4-17) together with (2-38) imply

$$\int_{|x|\le R\lambda}|\nabla g|^2\,\mathrm{d}x - \langle f'(e^{i\theta}W_\lambda))g, g\rangle \ge -c_3\|g\|_{\mathcal{E}}^2,$$

with c_3 as small as we like by enlarging R. Thus, we have obtained that the second line in (4-26) is $\le c_2|t|^{-\frac{N-5}{N-6}}$, with c_2 which can be made arbitrarily small.

We are left with the third line of (4-26). We will show that it equals $\frac{1}{\lambda^2}K$ up to negligible terms. The support of $A(\lambda)(e^{i\zeta}W_\mu)$ is contained in $|x| \le \tilde{R}\lambda$ and $\|A(\lambda)(e^{i\zeta}W_\mu)\|_{L^\infty} \lesssim \lambda^{-2}$; hence

$$\|A(\lambda)(e^{i\zeta}W_\mu)\|_{L^{2N/N-2}} \lesssim \left(\lambda^N \lambda^{-\frac{4N}{N-2}}\right)^{\frac{N-2}{2N}} = \lambda^{\frac{N-6}{2}} \sim |t|^{-1}.$$

From (2-4) and Hölder we have

$$\|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\|_{L^{2N/(N+2)}} \lesssim \|g\|_{\mathcal{E}}^{\frac{N+2}{N-2}} \ll |t|^{-\frac{1}{N-6}}.$$

Thus, in the third line of (4-26) we can replace $A(\lambda)(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)$ by $A(\lambda)(e^{i\theta}W_\lambda)$. Property (P3) implies $|AW - \Lambda W| \lesssim W$ pointwise, with a constant independent of c and R used in the definition of the function q. After rescaling and phase change we obtain $|A(\lambda)(e^{i\theta}W_\lambda) - \frac{1}{\lambda^2}e^{i\theta}\Lambda W_\lambda| \lesssim \frac{1}{\lambda^2}W_\lambda$. But $A(\lambda)W = \frac{1}{\lambda^2}\Lambda W_\lambda$ for $|x| \leq R\lambda$, so we obtain

$$\left|\left\langle A(\lambda)(e^{i\theta}W_\lambda) - \frac{1}{\lambda^2}e^{i\theta}\Lambda W_\lambda, f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\right\rangle\right|$$

$$\lesssim \frac{1}{\lambda^2}\int_{|x|\geq R\lambda} W_\lambda \cdot \left|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\right| \mathrm{d}x.$$

Since $|\zeta - \theta| \simeq \frac{\pi}{2}$, we have $|e^{i\zeta}W_\mu + e^{i\theta}W_\lambda| \gtrsim W_\lambda$; hence (2-4) yields

$$W_\lambda \cdot \left|f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\right| \lesssim W_\lambda^{\frac{4}{N-2}}|g|^2.$$

Integrating over $|x| \geq R\lambda$ and using Hölder we find

$$\left|\left\langle A(\lambda)(e^{i\theta}W_\lambda) - \frac{1}{\lambda^2}e^{i\theta}\Lambda W_\lambda, f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda + g) - f(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda) - f'(e^{i\zeta}W_\mu + e^{i\theta}W_\lambda)g\right\rangle\right|$$

$$\lesssim c_2|t|^{-\frac{N-5}{N-6}}, \quad \text{with } c_2 \text{ arbitrarily small as } R \to +\infty.$$

Resuming all the computations starting with (4-23), we have shown that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle g, iA_0(\lambda)g\rangle \leq \frac{c_1}{2}|t|^{-\frac{N-5}{N-6}} + \frac{1}{\lambda^2}K.$$

Hence (4-22) yields (4-21).

Since $\theta(T) = 0$, we have $|\theta(T)| \lesssim \|g(T)\|_{\mathcal{E}}^2 \ll |T|^{-\frac{1}{N-6}}$. Integrating (4-21) on $[T, t]$ we get $\psi(t) \gtrsim -c_1|t|^{-\frac{1}{N-6}}$. But

$$|\langle g(t), A_0(\lambda)g(t)\rangle| \lesssim \|g(t)\|_{\mathcal{E}}^2 \leq |t|^{-\frac{N-1}{N-6}} \ll |t|^{-\frac{1}{N-6}};$$

hence we obtain $\theta(t) \gtrsim -c_1|t|^{-\frac{1}{N-6}}$, which yields (4-20) if c_1 is chosen small enough. This finishes the proof of (4-8).

Step 3. From (2-20) we obtain $\|g\|_{\mathcal{E}}^2 + C_0\theta\lambda^{\frac{N-2}{2}} \leq C_1|t|^{-\frac{N}{N-6}}$; hence

$$\|g\|_{\mathcal{E}}^2 \leq -C_0\theta\lambda^{\frac{N-2}{2}} + C_1|t|^{-\frac{N}{N-6}} \leq \tfrac{1}{8}|t|^{-\frac{N-1}{N-6}} + C_1|t|^{-\frac{N}{N-6}},$$

provided that c_0 in (4-18) is small enough. This yields (4-9). □

Choice of the initial data by a topological argument. The bootstrap in Proposition 4.4 leaves out the control of $\lambda(t)$, $a_1^+(t)$ and $a_2^+(t)$. We will tackle this problem here.

Proposition 4.8. *Let $|T_0|$ be large enough. For all $T < T_0$ there exist λ^0, a_1^0, a_2^0 satisfying (4-1) such that the solution $u(t)$ with the initial data $u(T) = -iW + W_{\lambda^0} + g^0$ exists on the time interval $[T, T_0]$ and for $t \in [T, T_0]$ the bounds (4-6)–(4-9) and*

$$\left|\lambda(t) - \tfrac{1}{\kappa}(\kappa|t|)^{-\frac{2}{N-6}}\right| \le \tfrac{1}{2}|t|^{-\frac{5}{2(N-6)}}, \tag{4-28}$$

$$|a_1^+(t)| \le \tfrac{1}{2}|t|^{-\frac{N}{2(N-6)}}, \tag{4-29}$$

$$|a_2^+(t)| \le \tfrac{1}{2}|t|^{-\frac{N}{2(N-6)}} \tag{4-30}$$

hold.

The proof will be split into some lemmas. For $t \in [T, T_0]$, $\tilde{\lambda} > 0$, $\tilde{a}_1 \in \mathbb{R}$ and $\tilde{a}_2 \in \mathbb{R}$ we define

$$X_t(\tilde{\lambda}, \tilde{a}_1, \tilde{a}_2) := \left(\tfrac{1}{\kappa}(\kappa|t|)^{-\frac{2}{N-6}} + \tilde{\lambda}|t|^{-\frac{5}{2(N-6)}}, \tilde{a}_1|t|^{-\frac{N}{2(N-6)}}, \tilde{a}_2|t|^{-\frac{N}{2(N-6)}}\right).$$

We see that $\lambda(t)$, $a_1^+(t)$ and $a_2^+(t)$ satisfy (4-28)–(4-30) if and only if

$$X_t^{-1}\left(\lambda(t), a_1^+(t), a_2^+(t)\right) \in Q := \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^3.$$

Lemma 4.9. *Assume that $\lambda(t)$, $a_1^+(t)$ and $a_2^+(t)$ satisfy (3-11), (3-35) and (3-36) on the time interval $t \in (T_1, T_2)$ and that*

$$(p_0, p_1, p_2) := X_t^{-1}\left(\lambda(t), a_1^+(t), a_2^+(t)\right) \in Q \setminus \partial Q \quad \text{for all } t \in (T_1, T_2).$$

Then for all $t \in (T_1, T_2)$,

$$\left|p_0'(t) - \frac{2N-13}{2(N-6)}|t|^{-1}p_0(t)\right| \le c|t|^{-1}, \tag{4-31}$$

$$\left|p_1'(t) - \frac{\nu}{\mu(t)}p_1(t)\right| \le \frac{c}{\mu(t)}, \tag{4-32}$$

$$\left|p_2'(t) - \frac{\nu}{\lambda(t)}p_2(t)\right| \le \frac{c}{\lambda(t)}, \tag{4-33}$$

where $c > 0$ can be made arbitrarily small by taking T_0 large enough.

Proof. By the definition of $p_0(t)$ we have

$$\lambda(t) = \tfrac{1}{\kappa}(\kappa|t|)^{-\frac{2}{N-6}} + p_0(t)|t|^{-\frac{5}{2(N-6)}}. \tag{4-34}$$

Differentiating in time we obtain

$$\lambda'(t) = \frac{2}{N-6}(\kappa|t|)^{-\frac{N-4}{N-6}} + \frac{5}{2(N-6)}|t|^{-\frac{2N-7}{2(N-6)}}p_0(t) + |t|^{-\frac{5}{2(N-6)}}p_0'(t).$$

Applying the Newton formula $\left(\text{the binomial expansion with power } \frac{N-4}{2}\right)$ to (4-34) and using the fact that $|p_0| \lesssim 1$ we get

$$\lambda(t)^{\frac{N-4}{2}} = \kappa^{-\frac{N-4}{2}}(\kappa|t|)^{-\frac{N-4}{N-6}} + \frac{N-4}{2}\kappa^{-\frac{N-6}{2}}(\kappa|t|)^{-1}p_0(t)|t|^{-\frac{5}{2(N-6)}} + O(|t|^{-\frac{N-3}{N-6}}).$$

Thus

$$\lambda'(t) - \frac{2\kappa^{\frac{N-4}{2}}}{N-6}\lambda(t)^{\frac{N-4}{2}} = \left(\frac{5}{2(N-6)} - \frac{N-4}{N-6}\right)|t|^{-\frac{2N-7}{2(N-6)}}p_0(t) + |t|^{-\frac{5}{2(N-6)}}p_0'(t) + O(|t|^{-\frac{N-3}{N-6}}).$$

Using (3-11) and multiplying both sides by $|t|^{\frac{5}{2(N-6)}}$ we obtain (4-31).

We have $a_1^+(t) = |t|^{-\frac{N}{2(N-6)}}p_1(t)$, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}a_1^+ - \frac{\nu}{\mu}a_1^+ = |t|^{-\frac{N}{2(N-6)}}\left(p_1'(t) - \frac{\nu}{\mu}p_1(t)\right) + O(|t|^{-\frac{N}{2(N-6)}-1}),$$

so (3-35) implies (4-32). The proof of (4-33) is similar. $\qquad\square$

For $C > 1$, $j \in \{0, 1, 2\}$ and $p \in \mathbb{R}^3$ we define

$$V_j(C, p) := \{p + (r_0, r_1, r_2) : \mathrm{sign}(r_j) = \mathrm{sign}(p_j) \text{ and } \max_j |r_j| < C|r_j|\}.$$

Lemma 4.10. *Assume that* $\lambda(t)$, $a_1^+(t)$ *and* $a_2^+(t)$ *satisfy* (3-8), (3-11), (4-5), (3-35) *and* (3-36) *for* $t \in (T_1, T_2)$. *There exists a constant* $C > 0$, *depending on* T_1 *and* T_2, *such that if for some* $T_3 \in (T_1, T_2)$ *and* $j \in \{0, 1, 2\}$ *we have* $|p_j(T_3)| \geq \frac{1}{4}$, *then for all* $t \in (T_3, T_2)$ *we have* $p(t) \in V_j(C, p(T_3))$.

Proof. From the previous lemma we infer that there exist strictly positive constants c_1 and C_1, depending on T_1 and T_2, such that $|p_j'(t)| \leq C_1$ and

$$|p_j(t)| \geq \tfrac{1}{4} \implies |p_j'(t)| \geq c_1 \text{ and } \mathrm{sign}\, p_j'(t) = \mathrm{sign}\, p_j(t).$$

It is sufficient to take $C > \frac{C_1}{c_1}$. $\qquad\square$

Proof of Proposition 4.8. The proof proceeds by contradiction. Supposing that the result does not hold, we will construct a continuous retraction $\Phi : Q \to \partial Q$, $\Phi(p) = p$ for $p \in \partial Q$. It is a well-known fact from topology that such a function Φ does not exist.

Let $p^0 \in Q$. Take $(\lambda^0, \tilde{a}_1^0, \tilde{a}_2^0) = X_T(p^0)$ and let g^0 be given by Lemma 4.1. Let $u : [T, T_+) \to \mathcal{E}$ be the solution of (1-1) for the initial data $u(T) = -iW + W_{\lambda^0} + g^0$. We will say that the solution u is associated with $p^0 \in Q$.

Let T_2 be the infimum of the values of $t \in [T, T_+)$ such that (4-6), (4-7), (4-8), (4-9), (4-28), (4-29) or (4-30) does not hold. By our assumption that Proposition 4.8 is false, we have that T_2 exists and $T_2 < T_0$. Indeed, if all the listed conditions were satisfied for $t \in [T, T_+)$, then Corollary A.3 would imply $T_+ > T_0$; hence all the conditions would hold on $[T, T_0]$, which contradicts the assumption.

Set $p^1 := X_{T_2}^{-1}\big(\lambda(T_2), a_1^+(T_2), a_2^+(T_2)\big)$. By continuity $p^1 \in Q$, and we will show that in fact $p^1 \in \partial Q$. Indeed, by continuity of the flow, the assumptions of Proposition 4.4 are satisfied for $T_1 = T_2 + \tau$ for some $\tau > 0$. Hence (4-6)–(4-9) continue to hold on $[T_2, T_2 + \tau]$, so one of the conditions (4-28), (4-29) or (4-30) is violated somewhere on $[T_2, T_2 + \tau]$ for every $\tau > 0$. By continuity of the parameters with respect to time, this yields $p^1 \in \partial Q$.

We set

$$\Phi : Q \to \partial Q, \quad \Phi(p^0) := p^1.$$

It is immediate from the definition that $\Phi(p) = p$ for $p \in \partial Q$, and it remains to show that Φ is continuous.

Let $p^0 \in Q$, $\Phi(p^0) = p^1 \in \partial Q$ and $\varepsilon > 0$. Let C be the constant from Lemma 4.10 for $T_1 = T$ and $T_2 = T_0$. We will consider the case $p_0^1 = \frac{1}{2}$, the other cases being similar. It is clear that for $\delta > 0$ small enough $V_\delta := V_0\big(C, \frac{1}{2} - \delta, p_1^1, p_2^1\big) \cap \partial Q$ is an ε-neighborhood of p^1. Thus, by Lemma 4.10, in order to finish the proof it suffices to show that if $q^0 \in Q$ with $|q^0 - p^0|$ small enough, then the solution associated with q passes through V_δ.

If $p^0 = p^1 \in \partial Q$, this is obvious, since V_δ is in this case a neighborhood of p^0. In the case $p^0 \in Q \setminus \partial Q$, the solution associated with p^0 passes through V_δ before reaching ∂Q. Thus, by the continuous dependence on the initial data, the solution associated with q^0 passes through V_δ if $|q^0 - p^0|$ is small enough. □

Proof of Theorem 1. Let $T_0 < 0$ be given by Proposition 4.8 and let $T_0, T_1, T_2, \ldots$ be a decreasing sequence tending to $-\infty$. For $n \geq 1$, let u_n be the solution given by Proposition 4.8. Inequalities (4-6), (4-7), (4-8), (4-28) and (4-9) yield

$$\big\| u_n(t) - (-i W + W_{\frac{1}{\kappa}(\kappa|t|)^{-2/(N-6)}}) \big\|_{\mathcal{E}} \lesssim |t|^{-\frac{1}{2(N-6)}} \tag{4-35}$$

for all $t \in [T_n, T_0]$ and with a constant independent of n. Upon passing to a subsequence, we can assume that $u_n(T_0) \rightharpoonup u_0 \in \mathcal{E}$. Let u be the solution of (1-1) with the initial condition $u(T_0) = u_0$. Corollary A.4 implies u exists on the time interval $(-\infty, T_0]$ and for all $t \in (-\infty, T_0]$ we have $u_n(t) \rightharpoonup u(t)$. Passing to the weak limit in (4-35) finishes the proof. □

Appendix: Cauchy theory

Profile decomposition. We recall briefly the profile decomposition method of Bahouri and Gérard [1999] and Merle and Vega [1998]. In the case of the energy-critical defocusing NLS, the corresponding theory was developed by Keraani [2001]. For the focusing NLS in high dimensions, which is the case discussed in this paper, see [Killip and Visan 2010].

Proposition A.1 (Killip, Visan). *Let $u_{0,n}$ be a bounded sequence in $\mathcal{E}$. There exists a subsequence of $u_{0,n}$, still denoted $u_{0,n}$, such that there exist a family of solutions of the linear Schrödinger equation $U_L^j(t) = e^{it\Delta} U_0^j$ and a family of sequences of parameters t_n^j and λ_n^j satisfying the pseudo-orthogonality condition*

$$j \neq k \implies \lim_{n \to +\infty} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|t_n^j - t_n^k|}{\lambda_n^j} = +\infty$$

such that for all $J \geq 0$

$$u_{0,n} = \sum_{j=1}^{J} U_L^j \left(\frac{-t_n^j}{\lambda_n^j} \right)_{\lambda_n^j} + w_n^J, \tag{A-1}$$

with

$$\lim_{J \to +\infty} \limsup_{n \to +\infty} \| e^{it\Delta} w_n^J \|_{L_{t,x}^{2(N+2)/(N-2)}} = 0.$$

Moreover, for any $J \geq 0$,

$$\lim_{n \to +\infty} \left| \| u_{0,n} \|_{\mathcal{E}}^2 - \sum_{j=1}^{J} \| U_0^j \|_{\mathcal{E}}^2 - \| w_n^J \|_{\mathcal{E}}^2 \right| = 0. \qquad □$$

Formula (A-1) is called the linear profile decomposition. In the applications, we regard $u_{0,n}$ as a sequence of initial data of solutions u_n of (1-1). In order to approximate the solutions u_n, we introduce nonlinear profiles. The nonlinear profile U^j corresponding to the linear profile U_L^j is defined as the solution of (1-1) such that

$$\lim_{n \to +\infty} \left\| U^j\left(\frac{-t_n^j}{\lambda_n^j} \right) - U_L^j\left(\frac{-t_n^j}{\lambda_n^j} \right) \right\|_{\mathcal{E}} = 0.$$

The next proposition is a version of the result of Keraani for the focusing NLS. Its statement is very similar to Proposition 2.8 in [Duyckaerts et al. 2011].

Proposition A.2. *Let $u_{0,n}$ be a sequence in $\mathcal{E}$ with a linear profile decomposition (A-1) and let U^j : $(T_-(U^j), T_+(U^j)) \to \mathcal{E}$ be the nonlinear profiles. Let $\tau_n > 0$ be a sequence such that for all j and n*

$$\frac{\tau_n - t_n^j}{(\lambda_n^j)^2} < T_+(U^j), \qquad \limsup_{n \to +\infty} \|U^j\|_{L^{2(N+2)/(N-2)}(I \times \mathbb{R}^N)} < +\infty, \quad where \ I = \left[\frac{-t_n^j}{(\lambda_n^j)^2}, \frac{\tau_n - t_n^j}{(\lambda_n^j)^2} \right].$$

Let u_n be the solution of (1-1) with the initial data $u_n(0) = u_{0,n}$. Then, for n large, u_n exists on the time interval $[0, \tau_n]$, $\limsup_{n \to +\infty} \|u_n\|_{L^{2(N+2)/(N-2)}([0,\tau_n] \times \mathbb{R}^N)} < +\infty$ and for all $J \geq 0$,

$$u_n(t) = \sum_{j=1}^{J} U^j\left(\frac{t - t_n^j}{(\lambda_n^j)^2} \right)_{\lambda_n^j} + w_n^J(t) + r_n^J(t),$$

with

$$\lim_{J \to +\infty} \limsup_{n \to +\infty} \left(\|r_n^J\|_{L^{2(N+2)/(N-2)}([0,\tau_n] \times \mathbb{R}^N)} + \sup_{t \in [0,\tau_n]} \|r_n^J\|_{\mathcal{E}} \right) = 0.$$

Proof. See [Duyckaerts et al. 2011, proof of Proposition 2.8] and [Killip and Visan 2010, proof of Lemma 3.2]. □

Corollaries.

Corollary A.3. *There exists a constant $\eta > 0$ such that the following holds. Let $u : [t_0, T_+) \to \mathcal{E}$ be a maximal solution of (1-1) with $T_+ < +\infty$. Then for any compact set $K \subset \mathcal{E}$ there exists $\tau < T_+$ such that $\mathrm{dist}(u(t), K) > \eta$ for $t \in [\tau, T_+)$.*

Proof. See [Jendrej 2016, Corollary A.4]. □

Corollary A.4. *There exists a constant $\eta > 0$ such that the following holds. Let $K \subset \mathcal{E}$ be a compact set and let $u_n : [T_1, T_2] \to \mathcal{E}$ be a sequence of solutions of (1-1) such that*

$$\mathrm{dist}(u_n(t), K) \leq \eta \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [T_1, T_2].$$

Suppose that $u_n(T_1) \rightharpoonup u_0 \in \mathcal{E}$. Then the solution $u(t)$ of (1-1) with the initial condition $u(T_1) = u_0$ is defined for $t \in [T_1, T_2]$ and

$$u_n(t) \rightharpoonup u(t) \quad \text{for all } t \in [T_1, T_2].$$

Proof. See [Jendrej 2016, Corollary A.6]. □

Acknowledgments

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JACEK JENDREJ: jacek@math.uchicago.edu
Department of Mathematics, University of Chicago, Chicago, IL, United States

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