TRISTAN C. COLLINS AND CONNOR MOONEY

DIMENSION OF THE MINIMUM SET FOR THE REAL AND COMPLEX MONGE–AMPÈRE EQUATIONS IN CRITICAL SOBOLEV SPACES

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DIMENSION OF THE MINIMUM SET FOR THE REAL AND COMPLEX MONGE–AMPÈRE EQUATIONS IN CRITICAL SOBOLEV SPACES

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We prove that the zero set of a nonnegative plurisubharmonic function that solves $\det(\partial\bar{\partial}u) \geq 1$ in $\mathbb{C}^n$ and is in $W^{2,n(n-k)/k}$ contains no analytic subvariety of dimension k or larger. Along the way we prove an analogous result for the real Monge–Ampère equation, which is also new. These results are sharp in view of well-known examples of Pogorelov and Błocki. As an application, in the real case we extend interior regularity results to the case that u lies in a critical Sobolev space (or more generally, certain Sobolev–Orlicz spaces).

1. Introduction

In this paper we investigate the dimension of the singular set for the real and complex Monge–Ampère equations, assuming critical Sobolev regularity.

We first discuss the real case. It is well known that convex (e.g., viscosity) solutions to $\det D^2 u = 1$ are not always classical solutions. Pogorelov constructed examples in dimension $n \geq 3$ of the form

$$u(x', x_n) = |x'|^{2-\frac{2}{n}} f(x_n)$$

that solve $\det D^2 u = 1$ in $|x_n| < \rho$ for some $\rho > 0$ and some smooth, positive f. This example is $C^{1,\alpha}$ for $\alpha \leq 1 - \frac{2}{n}$, and $W^{2,p}$ for $p < \frac{n(n-1)}{2}$. Furthermore, this solution is not strictly convex, and it vanishes on $\{x' = 0\}$.

On the other hand, it is known that strictly convex solutions are smooth. The proof of this fact is closely related to the solution of the Dirichlet problem, which has a long history, beginning with work of Pogorelov [1971a; 1971b; 1973; 1978], Cheng and Yau [1976; 1977] and Calabi [1958]. Cheng and Yau [1976] solved the Minkowski problem on the sphere, and in [Cheng and Yau 1977] proved the existence of solutions to the Dirichlet problem which are smooth in the interior and Lipschitz up to the boundary. P. L. Lions [1983; 1985] gave an independent proof of this result. Caffarelli, Nirenberg and Spruck [Caffarelli et al. 1984] and Krylov [1983] established the existence of solutions smooth up to the boundary, provided the boundary data are $C^{3,1}$. Trudinger and Wang [2008] proved optimal boundary regularity results, where the optimality comes from earlier examples of Wang [1996].

Remark 1.1. In the case $n = 2$ it is a classical result of Alexandroff [1942] that solutions to $\det D^2 u \geq 1$ are strictly convex.

In view of the above discussion, to show interior regularity for $\det D^2 u = 1$ it is enough to show strict convexity. (We remark that interior *estimates* generally depend on the modulus of strict convexity). Urbas [1988] showed strict convexity when u is in $C^{1,\alpha}$ for $\alpha > 1 - \frac{2}{n}$, or in $W^{2,p}$ for $p > \frac{n(n-1)}{2}$. (Note that for these values of p, $W^{2,p}$ embeds into $C^{1,\alpha}$ for $\alpha > 1 - \frac{2}{n-1}$, so neither result implies the other). Caffarelli [1990b] showed that if $1 \le \det D^2 u < \Lambda$ and u is not strictly convex, then the graph of u contains an affine set with no interior extremal points, and if $\det D^2 u \ge 1$, then the dimension of any affine set in the graph of u is strictly smaller than $\frac{n}{2}$ [Caffarelli 1993]; see also [Mooney 2015]. These results led to interior $C^{2,\alpha}$ and $W^{2,q}$ estimates for solutions with linear boundary data, when $\det D^2 u$ is strictly positive and C^α, resp. C^0 [Caffarelli 1990a]. Finally, in [Mooney 2015] the second author showed that if $\det D^2 u \ge 1$, then u is strictly convex away from a set of Hausdorff $(n-1)$-dimensional measure zero, and that this is optimal by example (even when $\det D^2 u = 1$).

In view of the Pogorelov example, the $C^{1,\alpha}$ hypothesis in [Urbas 1988] is sharp, and the $W^{2,p}$ hypothesis is nearly sharp. In this paper we show interior regularity for the borderline case $p = \frac{n(n-1)}{2}$. Our result in the real case is:

Theorem 1.1. *Assume that u is a convex solution to $\det D^2 u \ge 1$ in $B_1 \subset \mathbb{R}^n$, and let $0 < k < \frac{n}{2}$. If $u \in W^{2,p}(B_1)$ for some $p \ge \frac{n}{2k}(n-k)$, then the dimension of the set where u agrees with a tangent plane is at most $k-1$.*

Remark 1.2. In particular, if $u \in W^{2,\frac{n(n-1)}{2}}$, then it is strictly convex. We in fact show that u is strictly convex if Δu lies in Orlicz spaces that are slightly weaker than $L^{\frac{n(n-1)}{2}}$ (see Section 3), strengthening the result from [Urbas 1988]. Our result is sharp in view of the Pogorelov example.

As a consequence, we can extend interior estimates to the borderline case $p = \frac{n(n-1)}{2}$ (see Section 5). Interior estimates of this kind are often important in geometric applications, where one does not control the boundary data.

Remark 1.3. There are analogues of the Pogorelov example that vanish on sets of dimension k for any $k < \frac{n}{2}$ and are not in $W^{2,\frac{n}{2k}(n-k)}$ [Caffarelli 1993]. These show that Theorem 1.1 is also sharp in the case $k > 1$.

We now discuss the complex case. Like in the real case, there exist singular Pogorelov-type examples of the form

$$u(z', z_n) = |z'|^{2-\frac{2}{n}} f(z_n)$$

for $n \ge 2$, such that $\det(\partial\bar\partial u)$ is a strictly positive polynomial [Błocki 1999].

Remark 1.4. In fact, there are analogues of this example that vanish on sets of complex dimension k for any $k < n$. Furthermore, these singular examples are global.

Less is known about interior regularity for the complex Monge–Ampère equation $\det(\partial\bar\partial u) = 1$. Błocki and Dinew [2011] showed that if $u \in W^{2,p}$ for some $p > n(n-1)$, then u is smooth. This result relies on an important estimate of Kołodziej [1996]. The same result is true provided Δu is bounded; see, e.g., [Wang 2012]. In this case the point is that the operator becomes uniformly elliptic, and by its concavity an important $C^{2,\alpha}$ estimate of Evans and Krylov applies; see, e.g., [Caffarelli and Cabré 1995]. Thus far,

there does not seem to be a geometric condition analogous to strict convexity that guarantees interior regularity.

However, if u is nonnegative then something can be said about analytic structures in the minimum set. A classical theorem of Harvey and Wells [1973] says that the minimum set of a smooth, strictly plurisubharmonic function is contained in a C^1, totally real submanifold. Dinew and Dinew [2016] recently showed that if $\det(\partial\bar{\partial}u)$ has a positive lower bound and $u \in C^{1,\alpha}$ for $\alpha > 1 - \frac{2k}{n}$, or C^β for $\beta > 2 - \frac{2k}{n}$ in the case $k > \frac{n}{2}$, then the minimum set of u contains no analytic subvarieties of dimension k or larger. We investigate the same situation assuming Sobolev regularity. In the complex case, our main result is:

Theorem 1.2. *Assume that u is a nonnegative plurisubharmonic function satisfying $\det(\partial\bar{\partial}u) \geq 1$ in the viscosity sense in $B_1 \subset \mathbb{C}^n$, and let $0 < k < n$. If $\Delta u \in L^p$ for some $p \geq \frac{n}{k}(n-k)$, then the zero set $\{u = 0\}$ contains no analytic subvarieties of dimension k or larger.*

Remark 1.5. We recall that $\det(\partial\bar{\partial}u) \geq 1$ in the viscosity sense if u is continuous in B_1, and whenever a quadratic polynomial P of the form $A_{\bar{j}i}z_i\bar{z}_j + \mathrm{Re}(Q(z_1, \ldots, z_n))$ such that A is a positive semidefinite Hermitian matrix touches u from above in B_1, then $\det(\partial\bar{\partial}P) \geq 1$.

Remark 1.6. Since $W^{2,\frac{n}{k}(n-k)}$ embeds into $C^{1,1-\frac{2k}{n-k}}$, this result is different from that in [Dinew and Dinew 2016]. It is sharp in view of Pogorelov-type examples.

Remark 1.7. It is not known whether all singularities of solutions to $\det(\partial\bar{\partial}u) = 1$ arise as analytic subvarieties, or that they occur on a complex analogue of the agreement set with a tangent plane. Thus, Theorem 1.2 does not immediately imply smoothness of solutions to $\det(\partial\bar{\partial}u) = 1$ when $u \in W^{2,n(n-1)}$ (unlike in the real case).

The critical Sobolev spaces arise naturally in geometric applications. For example, in complex dimension 2 the L^2 norm of the Laplacian is a scale invariant, monotone quantity whose concentration controls, at least qualitatively, the regularity of functions with Monge–Ampère mass bounded below. In this sense, Theorem 1.2 can be seen as a step toward understanding the regularity and compactness properties of sequences of (quasi)-PSH functions with lower bounds for the Monge–Ampère mass, which arise frequently in Kähler geometry.

The proof of Theorem 1.1 relies on two key observations. The first is that u grows at least like $\mathrm{dist}^{2-\frac{2k}{n}}$ away from a zero set of dimension k. The second is that the $W^{2,\frac{n}{2k}(n-k)}$ norm is invariant under the rescalings that fix the k-dimensional zero set, and preserve functions with this growth. By combining these observations with some convex analysis, we show that the mass of $(\Delta u)^{\frac{n}{2k}(n-k)}$ is at least some fixed positive constant in each dyadic annulus around the zero set.

In the complex case the strategy is similar, but an important difficulty is that we don't have convexity. We overcome this in two ways. First, using subharmonicity along complex lines, we can say that u grows at a certain rate from its zero set at many points. Second, we use a dichotomy argument: either the mass of $|D^2u|^{\frac{n(n-k)}{k}}$ is at least a small constant in an annulus around the zero set, or it is very large and concentrates close to the zero set. Using that the $W^{2,\frac{n(n-k)}{k}}$ norm is bounded, we can rule out the second case and proceed as before.

The paper is organized as follows. In Section 2 we prove some estimates from convex analysis that are useful in the real case. We then prove an analogue in the general setting that is useful in the complex case. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2. Finally, in Section 5 we give some applications of Theorem 1.1 to interior estimates for the real Monge–Ampère equation.

2. Preliminaries

Here we prove some useful functional inequalities. The first inequality is from convex analysis. This will be used to prove Theorem 1.1. We then prove a certain analogue in the general setting. This will be used to prove Theorem 1.2.

Estimate from convex analysis.

Lemma 2.1. *Let $n \geq 2$ and let w be a nonnegative convex function on $B_2 \subset \mathbb{R}^n$, with $w(0) = 0$ and $\sup_{\partial B_1} w \geq 1$. Then there is some positive constant $c(n)$ such that*

$$\int_{B_2 \setminus B_1} \Delta w \, dx > c(n).$$

Proof. By integration by parts, we have

$$\int_{B_2 \setminus B_1} \Delta w \, dx = \int_{\partial B_2} \partial_r w \, ds - \int_{\partial B_1} \partial_r w \, ds,$$

where ∂_r denotes radial derivative. By convexity, $\partial_r w$ is increasing on radial lines. We conclude that

$$\int_{B_2 \setminus B_1} \Delta w \, dx \geq \frac{1}{2} \int_{\partial B_2} \partial_r w \, ds.$$

Assume that the maximum of w on ∂B_1 is achieved at e_n. By convexity, $w \geq 1$ in $B_2 \cap \{x_n \geq 1\}$; hence $\partial_r w > \frac{1}{2}$ on $\partial B_2 \cap \{x_n \geq 1\}$. Since $\partial_r w \geq 0$, the conclusion follows.   □

As a consequence, the Sobolev regularity of a convex function whose maximum on ∂B_r grows like r^q is no better than that of r^q:

Lemma 2.2. *Assume that w is a nonnegative convex function on $B_1 \subset \mathbb{R}^n$ ($n \geq 2$) such that $w(0) = 0$ and $\sup_{\partial B_r} w \geq r^q$ for some $q \in [1, 2)$ and all $r < 1$. Then*

$$\int_{B_1 \setminus B_r} (\Delta w)^{\frac{n}{2-q}} \, dx \geq c(n, q) |\log r|$$

for some $c(n, q) > 0$ and all $r \in \left(0, \frac{1}{2}\right)$.

Remark 2.3. We take $q \geq 1$ since convex functions are locally Lipschitz.

Proof. Fix $\rho < \frac{1}{2}$ and let $w_\rho(x) = \rho^{-q} w(\rho x)$. Note that the $L^{\frac{n}{2-q}}$ norm of Δw is invariant under such rescalings. We conclude from this observation and Lemma 2.1 that

$$\int_{B_{2\rho} \setminus B_\rho} (\Delta w)^{\frac{n}{2-q}} \, dx = \int_{B_2 \setminus B_1} (\Delta w_\rho)^{\frac{n}{2-q}} \, dx \geq c(n, q).$$

The estimate follows by summing this inequality over dyadic annuli.   □

Remark 2.4. One can refine this estimate to Orlicz norms. Let $F : [0, \infty) \to [0, \infty)$ be a convex function with $F(0) = 0$. By Lemma 2.1 we have

$$\frac{1}{|B_{2r} \setminus B_r|} \int_{B_{2r} \setminus B_r} \Delta u \, dx > c(n) r^{q-2}.$$

Using Jensen's inequality and summing over dyadic annuli, we obtain

$$\int_{B_1} F(\Delta u / \lambda) \, dx \geq \sum_{k=1}^{\infty} |B_{2^{-k}} \setminus B_{2^{-k-1}}| F\big(c(n) 2^{-k(q-2)} / \lambda\big).$$

In particular, the Orlicz norm $\|\Delta u\|_{L^F(B_1)}$ is equal to ∞ if

$$\int_1^{\infty} t^{-\frac{n}{2-q}} F(t) \frac{dt}{t} = \infty.$$

Examples $F(t)$ that agree with $t^{\frac{n}{2-q}} |\log t|^{-p}$ for t large and $0 \leq p \leq 1$ satisfy this condition, and give weaker norms than $L^{\frac{n}{2-q}}$. For a reference on Orlicz spaces, see, e.g., [Simon 2011].

Estimate without convex analysis. The following estimate is a certain analogue of Lemma 2.1, in the general setting.

Lemma 2.5. *Let w be a nonnegative function on $B_2 \subset \mathbb{R}^n$ with $w(0) = 0$, and let $p > \frac{n}{2}$. Then there exists $c_0 > 0$ depending on n, p such that for all $\epsilon \in (0, 1)$, there exists some $\delta(\epsilon, n, p)$ such that either*

$$\left(\int_{B_2 \setminus B_\epsilon} |D^2 w|^p \, dx \right)^{\frac{1}{p}} \geq \delta \sup_{\partial B_1} w,$$

or

$$\left(\int_{B_{2\epsilon}} |D^2 w|^p \, dx \right)^{\frac{1}{p}} \geq c_0 \epsilon^{\frac{n}{p} - 2} \sup_{\partial B_1} w.$$

Proof. After multiplying by a constant we may assume that $\sup_{\partial B_1} w = 1$. Assume that the first case is not satisfied. Then by the Sobolev–Poincaré and Morrey inequalities we have

$$\|w - l\|_{L^\infty(B_2 \setminus B_\epsilon)} < C(n, p, \epsilon) \, \delta$$

for some linear function l. Take δ so small that the right side is less than $\frac{1}{8}$.

By the hypotheses on w, we have $l(0) > \frac{1}{2}$. Indeed, after a rotation we have $l(e_n) > \frac{7}{8}$ and $l(-2e_n) \geq -\frac{1}{8}$.

Let $\tilde{w}(x) = (w - l)(\epsilon x)$. Then $|\tilde{w}| < \frac{1}{8}$ in $B_2 \setminus B_1$, and furthermore, $\tilde{w}(0) < -\frac{1}{2}$. It follows again from standard embeddings that

$$\left(\int_{B_2} |D^2 \tilde{w}|^p \, dx \right)^{\frac{1}{p}} > c_0(n, p).$$

Scaling back, we obtain the desired inequality. $\qquad\square$

3. Proof of Theorem 1.1

We recall some estimates on the geometry of solutions to $\det D^2 u \geq 1$. The first says that the volume of sublevel sets grows at most as fast as for the paraboloids with Hessian determinant 1:

Lemma 3.1. *Assume* $\det D^2 u \geq 1$ *in a convex subset of* $\mathbb{R}^n$ *containing* 0, *with* $u \geq 0$ *and* $u(0) = 0$. *Then*

$$|\{u < h\}| < C(n)h^{\frac{n}{2}}$$

for all $h > 0$.

The proof follows from the affine invariance of the Monge–Ampère equation and a quadratic barrier; see, e.g., [Mooney 2015], Lemma 2.2.

Using Lemma 3.1 we can quantify how quickly u grows from a singularity. Below we fix $n \geq 3$ and $0 < k < \frac{n}{2}$, and we write $(x, y) \in \mathbb{R}^n$ with $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$.

Lemma 3.2. *Assume that* $\det D^2 u \geq 1$ *in* $\{|x| < 1\} \cap \{|y| < 1\} \subset \mathbb{R}^n$, *with* $u \geq 0$ *and* $u = 0$ *on* $\{x = 0\}$. *Then for all* $r < 1$ *we have*

$$\inf_y \sup_{|x|=r} u(x, y) > c(n)r^{2-\frac{2k}{n}}.$$

Proof. Take $c = c(n)$ small and assume by way of contradiction that for some r_0 the conclusion is false. Set $h = cr_0^{2-\frac{2k}{n}}$. Then for some y_0 we have

$$\{y = y_0\} \cap \left\{|x| < (c^{-1}h)^{\frac{n}{2}\frac{1}{n-k}}\right\} \subset \{u < h\}.$$

Since $\{u < h\}$ is convex, it contains the convex hull of the set on the left and $\pm e_n$. We conclude that

$$|\{u < h\}| \geq \tilde{c}(n)(c^{-1}h)^{\frac{n}{2}},$$

which contradicts Lemma 3.1 for c small. $\qquad \square$

The main theorem follows from the growth established in Lemma 3.2 and the convex analysis estimate Lemma 2.2.

Proof of Theorem 1.1. Assume that u agrees with a tangent plane on a set of dimension k. After subtracting the tangent plane, translating and rescaling, we may assume that $u \geq 0$ on $\{|x| < 1\} \cap \{|y| < 1\}$, and that $u = 0$ on $\{x = 0\}$. By Lemma 3.2, we also have that

$$\inf_{\{|y|<1\}} \sup_{|x|=r} u(x, y) \geq c(n)r^{2-\frac{2k}{n}}.$$

Apply Lemma 2.2 on the slices $\{y = \text{const.}\}$, taking $q = 2 - \frac{2k}{n}$ and replacing n by $n - k$, and integrate in y to conclude that

$$\int_{\{|y|<1\} \cap \{r<|x|<1\}} (\Delta u)^{\frac{n}{2k}(n-k)} \, dx \geq c(n, k)|\log r|.$$

Taking $r \to 0$ completes the proof. $\qquad \square$

Remark 3.3. By Remark 2.4, one obtains the same result if Δu is in the (weaker) Orlicz space L^F for any convex $F : [0, \infty) \to [0, \infty)$ satisfying $F(0) = 0$ and

$$\int_1^\infty t^{-\frac{n(n-k)}{2k}} F(t) \frac{dt}{t} = \infty. \tag{1}$$

4. Proof of Theorem 1.2

We first prove an analogue of Lemma 3.2. We fix $n \geq 2$ and $0 < k < n$, and we use coordinates $(z, w) \in \mathbb{C}^n$ with $z \in \mathbb{C}^{n-k}$ and $w \in \mathbb{C}^k$.

Lemma 4.1. *Assume that* $\det(\partial\bar\partial u) \geq 1$ *in* $\{|z| < 1\} \cap \{|w| < 1\} \subset \mathbb{C}^n$ *in the viscosity sense, with* $u \geq 0$ *and* $u = 0$ *on* $\{z = 0\}$. *Then for all* $r < 1$ *we have*

$$\sup_{|w| < \frac{1}{4}} \sup_{|z| = r} u(z, w) \geq c(n) r^{2 - \frac{2k}{n}}.$$

Proof. Take $c = c(n)$ small and assume by way of contradiction that for some r_0 the conclusion is false. Let $h = c r_0^{2 - \frac{2k}{n}}$. Since u is subharmonic on the slices $\{w = \text{const.}\}$, by the maximum principle we have

$$\{|w| < \tfrac{1}{4}\} \cap \{|z| < (c^{-1}h)^{\frac{n}{2(n-k)}}\} \subset \{u < h\}.$$

(Note that the volume of the set on the left is much larger than h^n for c small.) The proof then proceeds as in the real case. For c small, the convex quadratics $Q_t = 2h(16|w|^2 + (c^{-1}h)^{-\frac{n}{n-k}}|z|^2) + t$ are supersolutions that lie strictly above u on $\partial(\{|w| < \frac{1}{4}\} \cap \{|z| < r_0\})$ for $t \geq 0$. For some $t \geq 0$, Q_t touches u from above somewhere inside this set, contradicting that u is a viscosity subsolution. $\square$

Proof of Theorem 1.2. Assume that the minimum set of u contains an analytic subvariety of dimension k. After a biholomorphic transformation and a rescaling, we may assume that $u \geq 0$ on $\{|z| < 1\} \cap \{|w| < 1\}$ and $u = 0$ on $\{z = 0\}$ (see, e.g., [Dinew and Dinew 2016, Theorem 32] for details) and that

$$\|u\|_{W^{2, \frac{n(n-k)}{k}}(\{|w|<1\}\cap\{|z|<1\})} = C_0 < \infty.$$

(Here we used elliptic theory: Δu controls $D^2 u$ in L^p for $1 < p < \infty$.)

For any $r < \frac{1}{2}$ we define

$$u_r(z, w) = \frac{1}{r^{2 - \frac{2k}{n}}} u(rz, w).$$

We claim that there exist $\epsilon, \delta > 0$ small depending on n, k, C_0 (but not r) such that

$$\int_{\{|w|<1\}\cap\{\epsilon<|z|<2\}} |D_z^2 u_r|^{\frac{n(n-k)}{k}} |dz| |dw| > \delta. \tag{2}$$

Here D_z^2 denotes the Hessian in the z-variable. We first indicate how to complete the proof given the claim. The invariance of this norm under the rescalings used to obtain u_r gives that

$$\int_{\{|w|<1\}\cap\{(\frac{\epsilon}{2})r<|z|<r\}} |D^2 u|^{\frac{n(n-k)}{k}} |dz| |dw| > \delta$$

for all $r < 1$. By summing this over the annuli $\{|w| < 1\} \cap \{ \left(\frac{\epsilon}{2}\right)^{k+1} < |z| < \left(\frac{\epsilon}{2}\right)^k \}$ we eventually contradict the upper bound on the $W^{2, \frac{n(n-k)}{k}}$ norm of u.

We now prove the claim. By Lemma 4.1, there exists some $(z_0, w_0) \in \{|z| = 1\} \cap \{|w| < \frac{1}{4}\}$ with $u_r(z_0, w_0) \geq c(n) > 0$. Let

$$M(w) = u_r(z_0, w).$$

Since $M(w)$ is positive and subharmonic, we have by the mean value inequality that

$$\int_{\{|w|<1\}} M(w)\,|dw| > c(n) > 0. \tag{3}$$

By Lemma 2.5, for all ϵ small, there exists $\delta(n, k, \epsilon)$ such that either

$$\left(\int_{\{\epsilon < |z| < 2\}} |D_z^2 u_r(z,w)|^{\frac{n(n-k)}{k}}\,|dz| \right)^{\frac{k}{n(n-k)}} \geq \delta M(w)$$

or

$$\left(\int_{\{|z|<2\}} |D_z^2 u_r(z,w)|^{\frac{n(n-k)}{k}}\,|dz| \right)^{\frac{k}{n(n-k)}} \geq c(n,k)\epsilon^{-\frac{2(n-k)}{n}} M(w).$$

Let A_ϵ be the set of w such that the first case holds. We conclude from the scale-invariance of the norm we consider that

$$C_0 \geq c(n,k) \int_{A_\epsilon^c} \left(\int_{\{|z|<2\}} |D_z^2 u_r|^{\frac{n(n-k)}{k}}\,|dz| \right)^{\frac{k}{n(n-k)}} |dw| \geq c(n,k)\epsilon^{-\frac{2(n-k)}{n}} \int_{A_\epsilon^c} M(w)\,|dw|.$$

By taking $\epsilon(n, k, C_0)$ small, we conclude that the mass of $M(w)$ in A_ϵ^c is less than its mass in A_ϵ. We conclude from the estimate (3) that

$$\left(\int_{\{|w|<1\}\cap\{\epsilon<|z|<2\}} |D_z^2 u_r|^{\frac{n(n-k)}{k}}\,|dz||dw| \right)^{\frac{k}{n(n-k)}} \geq c(n,k)\delta \int_{A_\epsilon} M(w)\,|dw| \geq c(n,k)\delta(n,k,C_0). \quad \square$$

Remark 4.2. To justify the above computations, on the slices $\{w = \text{const.}\}$ we use that, for almost every w in $\{|w| < 1\}$, the restrictions $f_w(z) := u(z, w)$ are in $W^{2, \frac{n(n-k)}{k}}(\{|z| < 1\})$ with $D_z^2 f_w = D_z^2 u(\cdot, w)$. To see this, let $\{u^j\}$ be a sequence of smooth functions approximating u in $W^{2, \frac{n(n-k)}{k}}(B_1)$, and apply the Fubini theorem to $|u^j - u| + |\nabla u^j - \nabla u| + |D^2 u^j - D^2 u|$.

Remark 4.3. Theorem 1.2 actually implies a slightly more general result. Namely, if u is plurisubharmonic on B_1 and satisfies $\det \partial\bar\partial u \geq 1$, and $\Delta u \in L^p$ for some $p \geq \frac{n}{k}(n-k)$, then u cannot be pluriharmonic when restricted to any analytic set of dimension greater than or equal to k. This follows from Theorem 1.2 and the proof of Theorem 35 in [Dinew and Dinew 2016].

5. Applications

As a consequence of Theorem 1.1 we obtain interior estimates for the real Monge–Ampère equation depending on the $W^{2,p}$ norm of the solution for any $p \geq \frac{n(n-1)}{2}$. This extends a result of Urbas [1988] to the equality case $p = \frac{n(n-1)}{2}$.

Remark 5.1. In fact, we obtain interior estimates depending on certain Orlicz norms that are slightly weaker than $L^{\frac{n(n-1)}{2}}$.

We recall the definition of sections of a convex function. Let u be a convex function on $B_1 \subset \mathbb{R}^n$. If l is a supporting linear function to u at $x \in B_1$, we set

$$S_h^l(x) = \{u < l + h\} \cap B_1.$$

Lemma 5.2. *Assume that* $\det D^2 u \geq 1$ *in* $B_1 \subset \mathbb{R}^n$, *and that* $\|u\|_{W^{2,p}(B_1)} < C_0$ *for some* $p \geq \frac{n(n-1)}{2}$. *Then there exists* $h_0 > 0$ *depending only on* n, p *and* C_0 *such that*

$$S_{h_0}^l(x) \Subset B_1$$

for all $x \in B_{\frac{1}{2}}$ *and supporting linear functions* l *at* x.

Proof. The result follows from a standard compactness argument using the closedness of the condition $\det D^2 u \geq 1$ under uniform convergence, the lower semicontinuity of the $W^{2,p}$ norm under weak convergence and Theorem 1.1. □

Remark 5.3. The conclusion is the same if $\|\Delta u\|_{L^F(B_1)} < C_0$ for some F satisfying condition (1) for $k = 1$, and in addition, e.g., $\|u\|_{W^{2,2}(B_1)} < C_0$. The argument is by compactness again, but one has to work harder to extract a limit whose Hessian has bounded Orlicz norm. Rather than using weak $W^{2,2}$ convergence of a subsequence $\{u_k\}$, invoke the Banach–Saks theorem and use the strong convergence in $W^{2,2}$ of Cesàro means $\frac{1}{N} \sum_{k=1}^N u_k$. The convexity of F then implies that the Hessian of the limit has bounded Orlicz norm.

(In order to use Banach–Saks we need control of Δu in L^p for some $p > 1$, which does not follow from bounded Orlicz norm. This is the reason for the second condition).

Interior, e.g., $C^{2,\alpha}$ estimates and $W^{2,q}$ estimates in terms of $\|u\|_{W^{2,n(n-1)/2}(B_1)}$ follow, where the estimates also depend on n, α and the C^α norm (resp. n, q and the modulus of continuity) of $\det D^2 u$.

Indeed, by Lemma 5.2 we have $S_{h_0}^l(x) \Subset B_1$ for some universal h_0 and all $x \in B_{\frac{1}{2}}$. Since $\det D^2 u$ also has an upper bound (depending on the C^α norm or modulus of continuity of $\det D^2 u$), we have the lower volume bound $|S_{h_0}^l(x)| > ch_0^{\frac{n}{2}}$ for compactly contained sections [Caffarelli 1990b]. Combining this with the diameter estimate $\mathrm{diam}(S_{h_0}^l(x)) < 2$, we see that the eigenvalues of the affine transformations normalizing these sections (taking B_1 to their John ellipsoids) are bounded above and below by positive universal constants. The estimates follow by applying Caffarelli's results [1990a] in the normalized sections, scaling back, and doing a covering argument.

Acknowledgements

References

[Alexandroff 1942] A. Alexandroff, "Smoothness of the convex surface of bounded Gaussian curvature", *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **36** (1942), 195–199. In Russian. MR Zbl

[Błocki 1999] Z. Błocki, "On the regularity of the complex Monge–Ampère operator", pp. 181–189 in *Complex geometric analysis in Pohang* (Pohang, 1997), Contemp. Math. **222**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Błocki and Dinew 2011] Z. Błocki and S. Dinew, "A local regularity of the complex Monge–Ampère equation", *Math. Ann.* **351**:2 (2011), 411–416. MR Zbl

[Caffarelli 1990a] L. A. Caffarelli, "Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation", *Ann. of Math.* (2) **131**:1 (1990), 135–150. MR Zbl

[Caffarelli 1990b] L. A. Caffarelli, "A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity", *Ann. of Math.* (2) **131**:1 (1990), 129–134. MR Zbl

[Caffarelli 1993] L. A. Caffarelli, "A note on the degeneracy of convex solutions to Monge Ampère equation", *Comm. Partial Differential Equations* **18**:7-8 (1993), 1213–1217. MR Zbl

[Caffarelli and Cabré 1995] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications **43**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Caffarelli et al. 1984] L. Caffarelli, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations, I: Monge–Ampère equation", *Comm. Pure Appl. Math.* **37**:3 (1984), 369–402. MR Zbl

[Calabi 1958] E. Calabi, "Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens", *Michigan Math. J.* **5** (1958), 105–126. MR Zbl

[Cheng and Yau 1976] S. Y. Cheng and S. T. Yau, "On the regularity of the solution of the n-dimensional Minkowski problem", *Comm. Pure Appl. Math.* **29**:5 (1976), 495–516. MR Zbl

[Cheng and Yau 1977] S. Y. Cheng and S. T. Yau, "On the regularity of the Monge–Ampère equation $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x, u)$", *Comm. Pure Appl. Math.* **30**:1 (1977), 41–68. MR Zbl

[Dinew and Dinew 2016] S. Dinew and Ż. Dinew, "The minimum sets and free boundaries of strictly plurisubharmonic functions", *Calc. Var. Partial Differential Equations* **55**:6 (2016), art. id. 148. MR Zbl

[Harvey and Wells 1973] F. R. Harvey and R. O. Wells, Jr., "Zero sets of non-negative strictly plurisubharmonic functions", *Math. Ann.* **201** (1973), 165–170. MR Zbl

[Kołodziej 1996] S. Kołodziej, "Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator", *Ann. Polon. Math.* **65**:1 (1996), 11–21. MR Zbl

[Krylov 1983] N. V. Krylov, "Boundedly inhomogeneous elliptic and parabolic equations in a domain", *Izv. Akad. Nauk SSSR Ser. Mat.* **47**:1 (1983), 75–108. In Russian; translated in *Math. USSR-Izv.* **22**:1 (1984), 67–97. MR Zbl

[Lions 1983] P.-L. Lions, "Sur les équations de Monge–Ampère, I", *Manuscripta Math.* **41**:1-3 (1983), 1–43. MR Zbl

[Lions 1985] P.-L. Lions, "Sur les équations de Monge–Ampère", *Arch. Rational Mech. Anal.* **89**:2 (1985), 93–122. MR Zbl

[Mooney 2015] C. Mooney, "Partial regularity for singular solutions to the Monge–Ampère equation", *Comm. Pure Appl. Math.* **68**:6 (2015), 1066–1084. MR Zbl

[Pogorelov 1971a] A. V. Pogorelov, "The Dirichlet problem for the multidimensional analogue of the Monge–Ampère equation", *Dokl. Akad. Nauk SSSR* **201** (1971), 790–793. In Russian; translated in *Sov. Math., Dokl.* **12** (1971), 1727–1731. MR Zbl

[Pogorelov 1971b] A. V. Pogorelov, "The regularity of the generalized solutions of the equation $\det(\partial^2 u/\partial x^i \partial x^j) = \varphi(x^1, x^2, \dots, x^n) > 0$", *Dokl. Akad. Nauk SSSR* **200** (1971), 534–537. In Russian; translated in *Sov. Math., Dokl.* **12** (1971), 1436–1440. MR Zbl

[Pogorelov 1973] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Translations of Mathematical Monographs **35**, Amer. Math. Soc., Providence, RI, 1973. MR Zbl

[Pogorelov 1978] A. V. Pogorelov, *The Minkowski multidimensional problem*, Scripta Series in Mathematics **5**, V. H. Winston & Sons, Washington, D.C., 1978. MR Zbl

[Simon 2011] B. Simon, *Convexity: an analytic viewpoint*, Cambridge Tracts in Mathematics **187**, Cambridge University Press, 2011. MR Zbl

[Trudinger and Wang 2008] N. S. Trudinger and X.-J. Wang, "Boundary regularity for the Monge–Ampère and affine maximal surface equations", *Ann. of Math.* (2) **167**:3 (2008), 993–1028. MR Zbl

[Urbas 1988] J. I. E. Urbas, "Regularity of generalized solutions of Monge–Ampère equations", *Math. Z.* **197**:3 (1988), 365–393. MR Zbl

[Wang 1996] X.-J. Wang, "Regularity for Monge–Ampère equation near the boundary", *Analysis* **16**:1 (1996), 101–107. MR Zbl

[Wang 2012] Y. Wang, "On the $C^{2,\alpha}$-regularity of the complex Monge–Ampère equation", *Math. Res. Lett.* **19**:4 (2012), 939–946. MR Zbl

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