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EXISTENCE THEOREMS OF THE FRACTIONAL YAMABE PROBLEM

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Let X be an asymptotically hyperbolic manifold and M its conformal infinity. This paper is devoted to deducing several existence results of the fractional Yamabe problem on M under various geometric assumptions on X and M. Firstly, we handle when the boundary M has a point at which the mean curvature is negative. Secondly, we re-encounter the case when M has zero mean curvature and satisfies one of the following conditions: nonumbilic, umbilic and a component of the covariant derivative of the Ricci tensor on $\overline{X}$ is negative, or umbilic and nonlocally conformally flat. As a result, we replace the geometric restrictions given by González and Qing (2013) and González and Wang (2017) with simpler ones. Also, inspired by Marques (2007) and Almaraz (2010), we study lower-dimensional manifolds. Finally, the situation when X is Poincaré–Einstein and M is either locally conformally flat or 2-dimensional is covered under a certain condition on a Green's function of the fractional conformal Laplacian.

1. Introduction and the main results

Given $n \in \mathbb{N}$, let X^{n+1} be an $(n+1)$-dimensional smooth manifold with smooth boundary M^n. A function ρ in X is called a *defining function* of the boundary M in X if $\rho > 0$ in X and $\rho = 0$, $d\rho \neq 0$ on M. A metric g^+ in X is *conformally compact* if there exists a boundary-defining function ρ such that the conformal metric $\bar{g} := \rho^2 g^+$ extends to M and the closure $(\overline{X}, \bar{g})$ of X is compact. This induces the conformal class $[\hat{h}]$ of the metric $\hat{h} := \bar{g}|_M$, which is referred to as the *conformal infinity* of (X, g^+). A manifold (X, g^+) is called *asymptotically hyperbolic* if g^+ is conformally compact and $|d\rho|_{\bar{g}} \to 1$ as $\rho \to 0$. Also if (X, g^+) is conformally compact and Einstein, then it is said to be *Poincaré–Einstein* or *conformally compact Einstein*. All Poincaré–Einstein manifolds can be shown to be asymptotically hyperbolic.

Suppose an asymptotically hyperbolic manifold (X, g^+) with the conformal infinity $(M^n, [\hat{h}])$ is given. Also, for any $\gamma \in (0, 1)$, let $P^\gamma_{\hat{h}} = P^\gamma[g^+, \hat{h}]$ be the *fractional conformal Laplacian* whose principle symbol is equal to that of $(-\Delta_{\hat{h}})^\gamma$; see [Mazzeo and Melrose 1987; Joshi and Sá Barreto 2000; Graham and Zworski 2003; Chang and González 2011; González and Qing 2013] for its precise definition. In this article, we are interested in finding a conformal metric $\hat{h}$ on M with constant *fractional scalar curvature* $Q^\gamma_{\hat{h}} := P^\gamma_{\hat{h}}(1)$. This problem is called the *fractional Yamabe problem* or the *γ-Yamabe problem*, and it was introduced and investigated by González and Qing [2013] and González and Wang [2017]. By imposing some restrictions on the dimension and geometric behavior of the manifold, the authors obtained existence results when M is nonumbilic or M is umbilic but not locally conformally flat. Here we relieve the hypotheses made in [González and Qing 2013; González and Wang 2017] and examine when the bubble (see (1-13) below for its precise definition) cannot be used as an appropriate test function.

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As its name alludes, the fractional conformal Laplacian $P^{\gamma}_{\hat{h}}$ has the *conformal covariance property*: it holds that

$$P^{\gamma}_{\hat{h}_w}(u) = w^{-\frac{n+2\gamma}{n-2\gamma}} P^{\gamma}_{\hat{h}}(wu) \tag{1-1}$$

for a conformal change of the metric $\hat{h}_w = w^{4/(n-2\gamma)}\hat{h}$. Hence the fractional Yamabe problem can be formulated as looking for a positive solution of the nonlocal equation

$$P^{\gamma}_{\hat{h}} u = c u^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on } M \tag{1-2}$$

for some $c \in \mathbb{R}$ provided $n > 2\gamma$. On the other hand, if (X, g^+) is Poincaré–Einstein, then $P^{\gamma}_{\hat{h}}$ and $Q^{\gamma}_{\hat{h}}$ with $\gamma = 1$ precisely match with the classical *conformal Laplacian* $L_{\hat{h}}$ and a constant multiple of the scalar curvature $R[\hat{h}]$ on $(M, \hat{h})$:

$$P^1_{\hat{h}} = L_{\hat{h}} := -\Delta_{\hat{h}} + \frac{n-2}{4(n-1)} R[\hat{h}] \quad \text{and} \quad Q^1_{\hat{h}} = \frac{n-2}{4(n-1)} R[\hat{h}]. \tag{1-3}$$

If $\gamma = 2$, they coincide with the Paneitz operator [2008] and Branson's Q-curvature [1985] (see [Graham and Zworski 2003, Proposition 4.3] for its proof). Hence, in this case, the 1- and 2-Yamabe problems are reduced to the classical Yamabe problem and the Q-curvature problem, respectively.

Thanks to the efforts of various mathematicians, a complete solution of the Yamabe problem is known. After Yamabe [1960] raised the problem and suggested an outline of the proof, Trudinger [1968] first obtained a least energy solution to (1-2) under the setting that the scalar curvature of $(M, \hat{h})$ is nonpositive. Successively, Aubin [1976] examined the case when $n \geq 6$ and M is nonlocally conformally flat, and Schoen [1984] gave an affirmative answer when $n = 3, 4, 5$ or M is locally conformally flat by using the positive mass theorem [Schoen and Yau 1979a; 1979b; 1988]. Lee and Parker [1987] provided a new proof which unified the local proof of Aubin and the global proof of Schoen, introducing the notion of the conformal normal coordinates.

There also have been lots of results on the Q-curvature problem ($\gamma = 2$) for 4-dimensional manifolds $(M^4, [\hat{h}])$. By the Chern–Gauss–Bonnet formula, the *total Q-curvature*

$$k_P := \int_{M^4} Q^2_{\hat{h}}\, dv_{\hat{h}},$$

where $dv_{\hat{h}}$ is the volume form of $(M, \hat{h})$, is a conformal invariant. Gursky [1999] proved that if a manifold M^4 has the positive Yamabe constant $\Lambda^1(M, [\hat{h}]) > 0$, see (1-10), and satisfies $k_P \geq 0$, then its Paneitz operator $P^2_{\hat{h}}$ has the properties

$$\ker P^2_{\hat{h}} = \mathbb{R} \quad \text{and} \quad P^2_{\hat{h}} \geq 0. \tag{1-4}$$

Also Chang and Yang [1995] proved that any compact 4-manifold such that (1-4) and $k_P < 8\pi^2$ hold has a solution to

$$P^2_{\hat{h}} u + 2Q^2_{\hat{h}} u = 2ce^{4u} \quad \text{on } M,\ c \in \mathbb{R},$$

where $Q^2_{\hat{h}}$ is the Q-curvature. This result was generalized by Djadli and Malchiodi [2008] where only $\ker P^2_{\hat{h}} = \mathbb{R}$ and $k_P \neq 8m\pi^2$ for all $m \in \mathbb{N}$ are demanded. For other dimensions than 4, Gursky and

Malchiodi [2015] recently discovered the strong maximum principle of $P^2_{\hat{h}}$ for manifolds M^n ($n \geq 5$) with nonnegative scalar curvature and semipositive Q-curvature. Motivated by this result, Hang and Yang developed the existence theory of (1-2) for a general class of manifolds M^n, including ones such that $\Lambda^1(M, [\hat{h}]) > 0$ and there exists $\hat{h}' \in [\hat{h}]$ with $Q^2_{\hat{h}'} > 0$, provided $n \geq 5$ [Hang and Yang 2015; 2016b] or $n = 3$ [Hang and Yang 2004; 2015; 2016a]. In [Hang and Yang 2016b], the positive mass theorem for the Paneitz operator [Humbert and Raulot 2009; Gursky and Malchiodi 2015] was used to construct a test function. We also point out that a solution to (1-2) was obtained in [Qing and Raske 2006] for a locally conformally flat manifold ($n \geq 5$) with positive Yamabe constant and Poincaré exponent less than $(n-4)/2$.

In addition, when $\gamma = \frac{1}{2}$, the fractional Yamabe problem has a deep relationship with the boundary Yamabe problem proposed by Cherrier [1984] and Escobar [1992a], which can be regarded as a generalization of the Riemann mapping theorem: It asks if a compact manifold $\overline{X}$ with boundary is conformally equivalent to one of zero scalar curvature whose boundary M has constant mean curvature. It was solved by the series of works by Escobar [1992a; 1996], Marques [2005; 2007] and Almaraz [2010] who used a minimization argument. See also [Chen 2009; Mayer and Ndiaye 2015a], in which different approaches are pursued. It is worthwhile to mention that there is another type of boundary Yamabe problem also suggested by Escobar [1992b]: find a conformal metric such that the scalar curvature of X is constant and the boundary M is minimal. It was further studied by Brendle and Chen [2014] and Mayer and Ndiaye [2015b].

Chang and González [2011] observed that the fractional conformal Laplacian, defined through scattering theory (see, e.g., [Mazzeo and Melrose 1987; Joshi and Sá Barreto 2000; Graham and Zworski 2003]), can be described in terms of Dirichlet–Neumann operators; see also [Case and Chang 2016]. Specifically, (1-2) has an equivalent extension problem, which is degenerate elliptic but local.

Theorem A. *Suppose that $n > 2\gamma$, $\gamma \in (0,1)$, and (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$. Assume also that ρ is a defining function associated to M such that $|d\rho|_{\bar{g}} = 1$ near M (such ρ is called geodesic), and $\bar{g} = \rho^2 g^+$ is a metric of the compact manifold $\overline{X}$. In addition, we let the mean curvature H on $(M, \hat{h}) \subset (\overline{X}, \bar{g})$ be 0 if $\gamma \in \left(\frac{1}{2}, 1\right)$, and set*

$$E(\rho) = \rho^{-1-s}(-\Delta_{g^+} - s(n-s))\rho^{n-s} \quad \text{in } X, \tag{1-5}$$

where $s := n/2 + \gamma$. It can be shown that (1-5) is reduced to

$$E(\rho) = \frac{n-2\gamma}{4n}\left[R[\bar{g}] - (n(n+1) + R[g^+])\rho^{-2}\right]\rho^{1-2\gamma} \quad \text{near } M, \tag{1-6}$$

where $R[\bar{g}]$ and $R[g^+]$ are the scalar curvature of $(\overline{X}, \bar{g})$ and (X, g^+), respectively.

(1) *If a positive function U satisfies*

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^{1-2\gamma}\nabla U) + E(\rho)U = 0 & \text{in } (X, \bar{g}), \\ U = u & \text{on } M \end{cases} \tag{1-7}$$

and

$$\partial^\gamma_\nu U := -\kappa_\gamma \left(\lim_{\rho \to 0+} \rho^{1-2\gamma}\frac{\partial U}{\partial \rho}\right) = \begin{cases} c u^{\frac{n+2\gamma}{n-2\gamma}} & \text{for } \gamma \in (0,1) \setminus \left\{\frac{1}{2}\right\}, \\ c u^{\frac{n+2\gamma}{n-2\gamma}} - \left(\frac{1}{2}(n-1)\right)Hu & \text{for } \gamma = \left\{\frac{1}{2}\right\} \end{cases} \tag{1-8}$$

on M, then u solves (1-2). Here $\kappa_\gamma > 0$ is the constant whose explicit value is given in (1-23) below and ν stands for the outward unit normal vector with respect to the boundary M.

(2) *Assume further that the first L^2-eigenvalue $\lambda_1(-\Delta_{g^+})$ of the Laplace-Beltrami operator $-\Delta_{g^+}$ satisfies*

$$\lambda_1(-\Delta_{g^+}) > \tfrac{1}{4}n^2 - \gamma^2. \tag{1-9}$$

Then there is a special defining function ρ^* such that $E(\rho^*) = 0$ in X and $\rho^*(\rho) = \rho(1 + O(\rho^{2\gamma}))$ near M. Furthermore the function $\widetilde{U} := (\rho/\rho^*)^{(n-2\gamma)/2}U$ solves a degenerate elliptic equation of pure divergent form

$$\begin{cases} -\operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla\widetilde{U}) = 0 & \text{in } (X, \bar{g}^*), \\ \partial_\nu^\gamma \widetilde{U} = -\kappa_\gamma\left(\lim\limits_{\rho^*\to 0+}(\rho^*)^{1-2\gamma}\dfrac{\partial \widetilde{U}}{\partial \rho^*}\right) = P_{\hat{h}}^\gamma u - Q_{\hat{h}}^\gamma u = cu^{\frac{n+2\gamma}{n-2\gamma}} - Q_{\hat{h}}^\gamma u & \text{on } M, \end{cases}$$

where $\bar{g}^* := (\rho^*)^2 g^+$ and $Q_{\hat{h}}^\gamma$ is the fractional scalar curvature.

Notice that in order to seek a solution of (1-2), it is natural to introduce the γ-*Yamabe functional*

$$I_{\hat{h}}^\gamma[u] = \frac{\int_M uP_{\hat{h}}^\gamma u\, dv_{\hat{h}}}{\left(\int_M |u|^{\frac{2n}{n-2\gamma}}\, dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}} \quad \text{for } u \in H^\gamma(M)\setminus\{0\}, \tag{1-10}$$

where $H^\gamma(M)$ denotes the standard fractional Sobolev space, and its infimum $\Lambda^\gamma(M, [\hat{h}])$, called the γ-*Yamabe constant*. By the previous theorem and the energy inequality due to Case [2017, Theorem 1.1], it follows under the assumption (1-9) that if one defines the functionals

$$\bar{I}_{\hat{h}}^\gamma[U] = \frac{\kappa_\gamma \int_X (\rho^{1-2\gamma}|\nabla U|_{\bar{g}}^2 + E(\rho)U^2)\, dv_{\bar{g}}}{\left(\int_M |U|^{\frac{2n}{n-2\gamma}}\, dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}}, \quad \tilde{I}_{\hat{h}}^\gamma[U] = \frac{\kappa_\gamma \int_X (\rho^*)^{1-2\gamma}|\nabla U|_{\bar{g}}^2\, dv_{\bar{g}} + \int_M Q_{\hat{h}}^\gamma U^2\, dv_{\hat{h}}}{\left(\int_M |U|^{\frac{2n}{n-2\gamma}}\, dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}} \tag{1-11}$$

for each element U of the weighted Sobolev space $W^{1,2}(X, \rho^{1-2\gamma})$ such that $U \neq 0$ on M (in view of (1-8), a suitable modification is necessary if $\gamma = \frac{1}{2}$), and values

$$\bar{\Lambda}^\gamma(X, [\hat{h}]) = \inf\{\bar{I}_{\hat{h}}^\gamma[U] : U \in W^{1,2}(X, \rho^{1-2\gamma}), U \neq 0 \text{ on } M\},$$
$$\tilde{\Lambda}^\gamma(X, [\hat{h}]) = \inf\{\tilde{I}_{\hat{h}}^\gamma[U] : U \in W^{1,2}(X, \rho^{1-2\gamma}), U \neq 0 \text{ on } M\},$$

then

$$\Lambda^\gamma(M, [\hat{h}]) = \bar{\Lambda}^\gamma(X, [\hat{h}]) = \tilde{\Lambda}^\gamma(X, [\hat{h}]) > -\infty.$$

Besides it was shown in [González and Qing 2013] that the sign of c in (1-2) is the same as that of $\Lambda^\gamma(M, [\hat{h}])$, as in the local case $\gamma = 1$.

On the other hand, the Sobolev trace inequality

$$\left(\int_{\mathbb{R}^n} |U(\bar{x}, 0)|^{\frac{2n}{n-2\gamma}}\, d\bar{x}\right)^{\frac{n-2\gamma}{n}} \leq S_{n,\gamma} \int_0^\infty \int_{\mathbb{R}^n} x_{n+1}^{1-2\gamma}|\nabla U(\bar{x}, x_n)|^2\, d\bar{x}\, dx_{n+1} \tag{1-12}$$

is true for all functions U which belong to the homogeneous weighted Sobolev space $D^{1,2}(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2\gamma})$. In addition, the equality is attained by $U = cW_{\lambda,\sigma}$ for any $c \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$, where $W_{\lambda,\sigma}$ are the *bubbles* defined as

$$\begin{aligned} W_{\lambda,\sigma}(\bar{x}, x_{n+1}) &= p_{n,\gamma} \int_{\mathbb{R}^n} \frac{x_{n+1}^{2\gamma}}{(|\bar{x}-\bar{y}|^2 + x_{n+1}^2)^{\frac{n+2\gamma}{2}}} w_{\lambda,\sigma}(\bar{y})\, d\bar{y} \\ &= g_{n,\gamma} \int_{\mathbb{R}^n} \frac{1}{(|\bar{x}-\bar{y}|^2 + x_{n+1}^2)^{\frac{n-2\gamma}{2}}} w_{\lambda,\sigma}^{\frac{n+2\gamma}{n-2\gamma}}(\bar{y})\, d\bar{y} \end{aligned} \tag{1-13}$$

with

$$w_{\lambda,\sigma}(\bar{x}) := \alpha_{n,\gamma} \left(\frac{\lambda}{\lambda^2 + |\bar{x}-\sigma|^2} \right)^{\frac{n-2\gamma}{2}} = W_{\lambda,\sigma}(\bar{x}, 0). \tag{1-14}$$

The values of the positive numbers $p_{n,\gamma}$, $g_{n,\gamma}$ and $\alpha_{n,\gamma}$ can be found in (1-23). Particularly, it holds that

$$\begin{cases} -\operatorname{div}(x_{n+1}^{1-2\gamma} \nabla W_{\lambda,\sigma}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^\gamma_\nu W_{\lambda,\sigma} = -\kappa_\gamma \left(\lim\limits_{x_{n+1}\to 0+} x_{n+1}^{1-2\gamma} \dfrac{\partial W_{\lambda,\sigma}}{\partial x_{n+1}} \right) = (-\Delta)^\gamma w_{\lambda,\sigma} = w_{\lambda,\sigma}^{\frac{n+2\gamma}{n-2\gamma}} & \text{on } \mathbb{R}^n. \end{cases} \tag{1-15}$$

(In light of the equation that $W_{\lambda,\sigma}$ solves, we say that $W_{\lambda,\sigma}$ is γ-*harmonic*. Refer to [Caffarelli and Silvestre 2007]. For future use, let $W_\lambda = W_{\lambda,0}$ and $w_\lambda = w_{\lambda,0}$.) Moreover, if $S_{n,\gamma} > 0$ denotes the best constant one can achieve in (1-12) and $(\mathbb{S}^n, [g_c])$ is the standard unit n-dimensional sphere, then

$$\Lambda^\gamma(\mathbb{S}^n, [g_c]) = S_{n,\gamma}^{-1} \kappa_\gamma = \left(\int_{\mathbb{R}^n} w_{\lambda,\sigma}^{\frac{2n}{n-2\gamma}}\, d\bar{x} \right)^{\frac{2\gamma}{n}}. \tag{1-16}$$

Related to this fact, we have the following compactness result.

Proposition B. *Let $n > 2\gamma$, $\gamma \in (0,1)$ and (X^{n+1}, g^+) be an asymptotically hyperbolic manifold with the conformal infinity $(M^n, [\hat{h}])$. Also, assume that (1-9) is true. Then*

$$-\infty < \Lambda^\gamma(M, [\hat{h}]) \le \Lambda^\gamma(\mathbb{S}^n, [g_c]), \tag{1-17}$$

and the fractional Yamabe problem (1-7)–(1-8) has a positive solution if the strict inequality holds.

Refer to [González and Qing 2013, Sections 5 and 6] for its proof. Moreover since (1-17) automatically holds if the γ-Yamabe constant $\Lambda^\gamma(M, [\hat{h}])$ is negative or 0, *we assume that* $\Lambda^\gamma(M, [\hat{h}]) > 0$ *from now on*.

The purpose of this paper is to construct a proper nonzero test function $\Phi \in W^{1,2}(X, \rho^{1-2\gamma})$ such that $0 < \bar{I}^\gamma_{\hat{h}}[\Phi] < \Lambda^\gamma(\mathbb{S}^n, [g_c])$ when $\gamma \in (0,1)$, (X^{n+1}, g^+) is an asymptotically hyperbolic manifold, (1-9) holds and

- M^n has a point where the mean curvature H is negative, $n \ge 2$ and $\gamma \in \left(0, \frac{1}{2}\right)$; or
- M^n is the nonumbilic boundary of X^{n+1}, $n \ge 4$ and assumption (1-18) holds; or
- M^n is the umbilic boundary of X^{n+1}, the covariant derivative $R_{\rho\rho;\rho}[\bar{g}]$ of the Ricci tensor $R_{\rho\rho}[\bar{g}]$ on $(\bar{X}, \bar{g})$ is negative at a certain point of M, $n > 3 + 2\gamma$ and hypothesis (1-18) holds (where ρ is the geodesic defining function associated to $(M, \hat{h})$ and $\bar{g} = \rho^2 g^+$); or

- M^n is the umbilic but nonlocally conformally flat boundary of X^{n+1}, $n > 4+2\gamma$ and condition (1-19) is satisfied; or
- X^{n+1} is Poincaré–Einstein, either M^n is $n \geq 3$ and locally conformally flat or $n = 2$, the expansion (1-21) of the Green's function $G(\cdot\,, y)$ holds in a neighborhood of an arbitrarily chosen point $y \in M$ and the constant-order term A of $G(\cdot\,, y)$ is positive.

Once it is achieved, Proposition B will imply the existence of a positive solution to (1-2) automatically. The natural candidate for a positive test function is certainly the standard bubble, possibly truncated. Indeed, this is a good choice for the first and third cases mentioned above. Nevertheless, to cover lower-dimensional manifolds or locally conformally flat boundaries, it is necessary to find more accurate test functions than the truncated bubbles; cf. [González and Qing 2013; González and Wang 2017]. To take into account the second and fourth situations, we shall add a correction term on the bubble by adapting the idea of Marques [2007] and Almaraz [2010]. For the fifth case, we will construct an appropriate test function by utilizing the Green's function $G(\cdot\,, y)$. In the local situation $\gamma = 1$, such an approach was successfully applied by Schoen [1984]. His idea was later extended by Escobar [1992a] in the work on the boundary Yamabe problem, which has close relationship to the fractional Yamabe problem with $\gamma = \frac{1}{2}$, as discussed.

Let π be the second fundamental form of $(M, \hat{h}) \subset (\overline{X}, \bar{g})$. The boundary M is called *umbilic* if the tensor $T := \pi - H\bar{g}$ vanishes on M. Also M is *nonumbilic* if it possesses a point at which $T \neq 0$. Our first main result reads as follows:

Theorem 1.1. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold, $(M, [\hat{h}])$ is its conformal infinity and (1-9) holds. Assume also that ρ is a geodesic defining function of $(M, \hat{h})$ and $\bar{g} = \rho^2 g^+ = d\rho^2 \oplus h_\rho$ near $M = \{\rho = 0\}$. If either*

- *$n \geq 2$, $\gamma \in \left(0, \frac{1}{2}\right)$ and M^n has a point at which the mean curvature H is negative; or*
- *$n \geq 4$, $\gamma \in (0, 1)$, M^n is the nonumbilic boundary of X^{n+1} and*

$$R[g^+] + n(n+1) = o(\rho^2) \quad \text{as } \rho \to 0 \text{ uniformly on } M, \tag{1-18}$$

then the γ-Yamabe problem is solvable — namely, (1-2) has a positive solution.

Remark 1.2. (1) As pointed out in [González and Qing 2013], we are only permitted to change the metric on the conformal infinity M. Once the boundary metric $\hat{h}$ is fixed, the geodesic boundary-defining function ρ and a compact metric $\bar{g}$ on X are automatically determined by the relations $|d\rho|_{\rho^2 g^+} = 1$ and $\bar{g} = \rho^2 g^+$. This is a huge difference between the fractional Yamabe problem (especially, with $\gamma = \frac{1}{2}$) and the boundary Yamabe problem, in that one has a freedom of conformal change of the metric in the whole manifold X when he/she is concerned with the boundary Yamabe problem.

Due to this reason, while it is possible to make the "extrinsic" metric H vanish at a point by a conformal change in the boundary Yamabe problem, one cannot do the same thing in the setting of the fractional Yamabe problem. This forced us to separate the cases in the statement of Theorem 1.1.

(2) As a particular consequence of the previous discussion, the Ricci tensor $R_{\rho\rho}[\bar{g}](y)$ of $(X, \bar{g})$ evaluated at a point y on M is governed by $\hat{h}$ and (1-18) (see Lemma 2.4). In the boundary Yamabe problem, Escobar [1992a] could choose a metric in X such that $R_{ij}[\hat{h}](y) = 0$ and $R_{\rho\rho}[\bar{g}](y) = 0$ simultaneously.

Moreover, by putting (1-6) and (1-18) together, we get

$$E(\rho) = \frac{n-2\gamma}{4n} R[\bar{g}]\, \rho^{1-2\gamma} + o(\rho^{1-2\gamma}) \quad \text{near } M.$$

Hence, on account of the energy expansion, (1-18) is the very condition that makes the boundary Yamabe problem and the $\frac{1}{2}$-Yamabe problem identical modulo the remainder. Refer to Subsections 2C and 2D.

(3) The sign of the mean curvature at a fixed point on M and (1-18) are "intrinsic" curvature conditions of an asymptotically hyperbolic manifold in the sense that these properties are independent of the choice of a representative of the class $[\hat{h}]$. Refer to Lemma 2.1 below for its proof. Also Lemma 2.3 claims that (1-18) implies $H = 0$ on M.

(4) Note also that $2 + 2\gamma \in \mathbb{N}$ and $\gamma \in (0, 1)$ if and only if $\gamma = \frac{1}{2}$, and the boundary Yamabe problem on nonumbilic manifolds in dimension $n = 2 + 2\gamma = 3$ was covered in [Marques 2007]. We expect that the strategy suggested in that paper can be applied for $\frac{1}{2}$-Yamabe problem in the same setting.

We next consider the case when the boundary M is umbilic but either $R_{\rho\rho;\rho}[\bar{g}] < 0$ at some point on M or it is nonlocally conformally flat.

Theorem 1.3. *Suppose that (X^{n+1}, g^+) is an asymptotic hyperbolic manifold such that (1-9) holds and the boundary $(M^n, [\hat{h}])$ is umbilic. If either*

- $n > 3 + 2\gamma$, $\gamma \in (0, 1)$, *that is, either $n \geq 5$ and $\gamma \in (0, 1)$ or $n = 4$ and $\gamma \in \left(0, \frac{1}{2}\right)$, the tensor $R_{\rho\rho;\rho}[\bar{g}]$ is negative at a certain point of M and (1-18) is valid; or*
- $n > 4 + 2\gamma$, $\gamma \in (0, 1)$, *that is, either $n \geq 6$ and $\gamma \in (0, 1)$ or $n = 5$ and $\gamma \in \left(0, \frac{1}{2}\right)$, there is a point $y \in M$ such that the Weyl tensor $W[\hat{h}]$ on M is nonzero at y and*

$$\begin{cases} R[g^+] + n(n+1) = o(\rho^4), \\ \partial_{\bar{x}}^m (R[g^+] + n(n+1)) = o(\rho^2) \quad (m = 1, 2), \\ \partial_{\rho}^m (R[g^+] + n(n+1)) = o(\rho^2) \quad (m = 1, 2) \end{cases} \tag{1-19}$$

  as $\rho \to 0$ uniformly on M,

then the γ-Yamabe problem is solvable. Here $\bar{x}$ is a coordinate on M.

Remark 1.4. (1) As we will see later, the main order of the energy for the fractional Yamabe problem (1-2) is ϵ^4 on an umbilic but nonlocally conformal flat boundary M, while it is ϵ^2 on a nonumbilic boundary; see (2-11), (2-14), (3-14) and (3-16). This explains why the necessary decay rate of $R[g^+] + n(n+1)$ to 0 as $\rho \to 0$ in Theorem 1.3 should be ρ^2-times as fast as that in Theorem 1.1.

On the other hand, (1-19) is responsible for determining all the values of quantities which emerge in the coefficient of ϵ^4 in the energy (such as $R_{,ii}[\bar{g}](y)$ and $R_{NN,ii}[\bar{g}](y)$ — see Lemma 3.2) and making the term $(n(n+1) + R[g^+])\rho^{-2}$ in $E(\rho)$ to be ignorable.

(2) Owing to Lemmas 2.1 and 2.3, condition (1-19) is again intrinsic and sufficient to deduce that $H = 0$ on M. Moreover every Poincaré–Einstein manifold satisfies (1-19).

In [González and Wang 2017, Lemma 2.3], it is proved that the sign of the tensor $R_{\rho\rho;\rho}[\bar{g}]$ at a fixed point on M is intrinsic.

(3) It is notable that $4+2\gamma \in \mathbb{N}$ and $\gamma \in (0,1)$ if and only if $\gamma = \frac{1}{2}$, and the boundary Yamabe problem for $n = 4+2\gamma = 5$ was studied in [Almaraz 2010]. We believe that Theorem 1.3 can be extended to the case $\gamma = \frac{1}{2}$, $n = 5$, $W[\hat{h}] \neq 0$ on M and (1-19) is valid.

In order to describe the last result, we first have to take into account of the existence of a Green's function under our setting.

Proposition 1.5. *Suppose that all the hypotheses of Theorem A hold true, including (1-9), and $H = 0$ on M. In addition, assume further that $\Lambda^\gamma(M,[\hat{h}]) > 0$. Then for each $y \in M$, there exists a Green's function $G(x,y)$ on $\bar{X} \setminus \{y\}$ which satisfies*

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^{1-2\gamma}\nabla G(\cdot,y)) + E(\rho)\, G(\cdot,y) = 0 & \text{in } (X,\bar{g}), \\ \partial_\nu^\gamma G(\cdot,y) = \delta_y & \text{on } (M,\hat{h}) \end{cases} \tag{1-20}$$

in the distribution sense, where δ_y is the Dirac measure at y. The function G is unique and positive on $\bar{X}$.

The proof is postponed until Section 4A. The readers may compare the above result with [Guillarmou and Qing 2010]. Based on standard elliptic regularity and the facts that if $(X,\bar{g})$ is the Poincaré half-plane $(\mathbb{R}^{n+1}_+, x_{n+1}^{-2}\,dx)$, then

$$G(x,\bar{y}) = \frac{g_{n,\gamma}}{|(\bar{x}-\bar{y},x_{n+1})|^{n-2\gamma}} \quad \text{for all } (\bar{x},x_{n+1}) \in \mathbb{R}^{n+1}_+ \text{ and } \bar{y} \in \mathbb{R}^n,$$

and that the compactified metric $\bar{g}$ on $\bar{X}$ of a Poincaré–Einstein manifold (X,g^+) can be assumed to be Euclidean up to order $|x|^n$ in its coordinate $x \in \mathbb{R}^{n+1}_+$ (refer to Lemma 4.3 below), we expect the following.

Conjecture 1.6 (expansion of the Green's function). Assume that $\gamma \in (0,1)$, $n > 2\gamma$ and (X^{n+1},g^+) is Poincaré–Einstein. Also, suppose that $\Lambda^\gamma(M,[\hat{h}]) > 0$ and that either $(M^n,[\hat{h}])$ has $n \geq 3$ and is locally conformally flat or $n = 2$. Fix any $y \in M$. Then there exists a local coordinate x of the compact manifold $(\bar{X},\bar{g})$ around y (identified with $0 \in \mathbb{R}^n$) defined in a small closed neighborhood $\mathcal{N} \subset \overline{\mathbb{R}}^{n+1}_+$ of 0 such that

$$G(x,0) = g_{n,\gamma}|x|^{-(n-2\gamma)} + A + \Psi(x) \quad \text{for } x \in \mathcal{N}. \tag{1-21}$$

Here $g_{n,\gamma} > 0$ is a number that appeared in (1-13), $A \in \mathbb{R}$ and Ψ is a function in $\mathcal{N}$ satisfying

$$|\Psi(x)| \leq C|x|^{\min\{1,2\gamma\}} \quad \text{and} \quad |\nabla\Psi(x)| \leq C|x|^{\min\{0,2\gamma-1\}} \quad \text{for } x \in \mathcal{N} \tag{1-22}$$

for some constant $C > 0$.

Now we can state our third main theorem.

Theorem 1.7. *Suppose that $\gamma \in (0,1)$, $n > 2\gamma$ and (X^{n+1}, g^+) is a Poincaré–Einstein manifold with conformal infinity $(M^n, [\hat{h}])$. Let ρ be a geodesic defining function for $(M, \hat{h})$ and $\bar{g} = \rho^2 g^+$. If (1-9) holds, Conjecture 1.6 is valid, $A > 0$, and either M^n has $n \geq 3$ and is locally conformally flat or $n = 2$, then the fractional Yamabe problem is solvable.*

Remark 1.8. (1) Let us set a 2-tensor

$$F = \rho(\mathrm{Ric}[g^+] + ng^+) \quad \text{in } X,$$

which is identically 0 if (X, g^+) is Poincaré–Einstein. As a matter of fact, if M is locally conformally flat, the only property of the tensor F necessary to derive Theorem 1.7 is that $\partial_\rho^m F|_{\rho=0} = 0$ for $m = 0, \dots, n-1$ (refer to Lemma 4.3). We guess that (1-21) and (1-22) are still valid under this assumption. Similarly, for the case $n = 2$, the assumption $\partial_\rho^m F|_{\rho=0} = 0$ for $m = 0, 1$ would suffice.

(2) Since (X^{n+1}, g^+) is Poincaré–Einstein, the second fundamental form on M is trivial. Particularly, the mean curvature H on M vanishes and M is umbilic.

(3) Suppose that we are in the local case $\gamma = 1$, and either $n \geq 7$ or M is locally conformally flat. Then, as shown in [Lee and Parker 1987, Lemma 6.4], the expansion (1-21) is valid. Furthermore, the classical positive mass theorem of Schoen and Yau [1979a; 1979b; 1988] states that $A \geq 0$, and the positivity condition $A > 0$ holds if and only if $(M, \hat{h})$ is not conformally diffeomorphic to the standard sphere $\mathbb{S}^n$. Determining the sign of A at each point $y \in M$ is a still natural problem for $\gamma \in (0,1)$. However, it is difficult to perform, because A may be a nonlocal quantity, namely, one depending on the whole geometry of (X, g^+) and $(M, [\hat{h}])$.

This paper is organized as follows: In Section 2, we establish Theorem 1.1 by intensifying the ideas of Marques [2007] and González and Qing [2013]. Section 3 provides the proof of Theorem 1.3, which further develops the approach of Almaraz [2010] and González and Wang [2017]. In Section 4, Theorem 1.7 is achieved, which can be understood as a sort of generalization of the results of Schoen [1984] and Escobar [1992a]. In particular, Section 4A is devoted to investigating the existence and some qualitative properties of a Green's function (i.e., Proposition 1.5). Then we are concerned with the case that M is locally conformally flat (in Section 4B) and 2-dimensional (in Section 4C). Finally, we examine the asymptotic behavior of the bubble $W_{1,0}$ near infinity in Appendix A, and compute some integrations regarding $W_{1,0}$, which are needed in the energy expansions in Appendix B.

Notation. • The Einstein convention is used throughout the paper. The indices i, j, k and l always take values from 1 to n, and a and b range over values from 1 to $n+1$.

• For a tensor T, notations $T_{;a}$ and $T_{,a}$ indicate covariant differentiation and partial differentiation of T, respectively.

• For a tensor T and a number $q \in \mathbb{N}$, we use

$$\mathrm{Sym}_{i_1 \cdots i_q} T_{i_1 \cdots i_q} = \frac{1}{q!} \sum_{\sigma \in S_q} T_{i_{\sigma(1)} \cdots i_{\sigma(q)}},$$

where S_q is the group of all permutations of q elements.

- We let $N = n+1$. Also, for $x \in \mathbb{R}^N_+ := \{(x_1, \dots, x_n, x_N) \in \mathbb{R}^N : x_N > 0\}$, we write $\bar{x} = (x_1, \dots, x_n, 0) \in \partial\mathbb{R}^N_+ \simeq \mathbb{R}^n$ and $r = |\bar{x}|$.
- For $n > 2\gamma$, we set $p = (n+2\gamma)/(n-2\gamma)$.
- For any $\varrho > 0$, let $B^n(0,\varrho)$ and $B^N_+(0,\varrho)$ be the n-dimensional ball and the N-dimensional upper half-ball centered at 0 whose radius is ϱ, respectively.
- $|\mathbb{S}^{n-1}|$ is the surface area of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.
- For any $t \in \mathbb{R}$, let $t_+ = \max\{0, t\} \geq 0$ and $t_- = \max\{0, -t\} \geq 0$ so that $t = t_+ - t_-$.
- For $\gamma \in (0,1)$, the space $H^\gamma(M)$ is the completion of $C^\infty(M)$ with respect to the norm which one obtains by pulling back

$$u \in C^\infty_c(\mathbb{R}^n) \mapsto \left(\int_{\mathbb{R}^n} u^2 \, d\bar{x} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(\bar{x}) - u(\bar{y})|^2}{|\bar{x}-\bar{y}|^{n+2\gamma}} \, d\bar{x}\, d\bar{y} \right)^{\frac{1}{2}}$$

to M through coordinate charts.
- The space $D^{1,2}(\mathbb{R}^N_+, x_N^{1-2\gamma})$ denotes the completion of $C^\infty_c(\overline{\mathbb{R}}^N_+)$ with respect to the norm

$$U \mapsto \left(\int_{\mathbb{R}^N_+} x_N^{1-2\gamma} |\nabla U|^2 \, dx \right)^{\frac{1}{2}},$$

and the space $W^{1,2}(X, \rho^{1-2\gamma})$ denotes the completion of $C^\infty_c(\overline{X})$ with respect to the norm

$$U \mapsto \left(\int_X \rho^{1-2\gamma} (|\nabla U|^2_{\bar{g}} + U^2) \, dv_{\bar{g}} \right)^{\frac{1}{2}}.$$

In light of Theorem A, $W^{1,2}(X, \rho^{1-2\gamma})$ is the natural functional space for the fractional Yamabe problem.
- The following positive constants are given in (1-8), (1-13) and (1-14):

$$\begin{aligned} \kappa_\gamma &= \frac{\Gamma(\gamma)}{2^{1-2\gamma}\Gamma(1-\gamma)}, \quad & p_{n,\gamma} &= \frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(\gamma)}, \\ g_{n,\gamma} &= \frac{\Gamma\left(\frac{n-2\gamma}{2}\right)}{\pi^{\frac{n}{2}} 2^{2\gamma}\Gamma(\gamma)}, & \alpha_{n,\gamma} &= 2^{\frac{n-2\gamma}{2}} \left(\frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\Gamma\frac{1}{2}(n-2\gamma)} \right)^{\frac{n-2\gamma}{4\gamma}}. \end{aligned} \tag{1-23}$$

- $C > 0$ is a generic constant which may vary from line to line.

2. Nonminimal and nonumbilic conformal infinities

2A. *Geometric background.* We begin this section by proving that the sign of the mean curvature, (1-18) and nonumbilicity of a point on M are intrinsic conditions.

Lemma 2.1. *Suppose that (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$. Moreover, let ρ and $\tilde{\rho}$ be the geodesic boundary-defining functions associated to two representatives $\hat{h}$ and $\tilde{h}$ of the class $[\hat{h}]$, respectively. We also define $\bar{g} = \rho^2 g^+$ and $\tilde{g} := \tilde{\rho}^2 g^+$, denote by*

$\pi = -\bar{g}_{,N}/2$ and $\tilde{\pi}$ the second fundamental forms of $(M, \hat{h}) \subset (\bar{X}, \bar{g})$ and $(M, \tilde{h}) \subset (\bar{X}, \tilde{g})$, respectively, and set $H = \bar{g}^{ij}\pi_{ij}/n$ and $\tilde{H} = \tilde{g}^{ij}\tilde{\pi}_{ij}/n$. Then we have

$$C^{-1} \le \frac{\tilde{\rho}}{\rho} \le C \quad \text{in } X \qquad \text{and} \qquad H = \frac{\tilde{\rho}}{\rho}\bigg|_{\rho=0} \tilde{H} \quad \text{on } M \tag{2-1}$$

for some $C > 1$. Furthermore if $H = 0$ on M, then

$$\pi = \frac{\rho}{\tilde{\rho}}\bigg|_{\rho=0} \tilde{\pi} \quad \text{on } M. \tag{2-2}$$

Proof. The assertion on H in (2-1) is proved in [González and Qing 2013, Lemma 2.3]. For the first inequality in (2-1), it suffices to observe that $\tilde{\rho}/\rho$ is bounded above and bounded away from 0 near M. Indeed, this follows from the fact that

$$\tilde{h} = \tilde{g}|_M = \tilde{\rho}^2 g^+|_M = \left(\frac{\tilde{\rho}}{\rho}\right)^2 \bar{g}|_M = \left(\frac{\tilde{\rho}}{\rho}\right)^2 \hat{h} \quad \text{on } M.$$

Let us define tensors $T = \pi - H\bar{g}$ and $\widetilde{T} = \tilde{\pi} - \tilde{H}\tilde{g}$ on M. Then we see from [Escobar 1992b, Proposition 1.2] that

$$\tilde{\pi} = \widetilde{T} = \frac{\tilde{\rho}}{\rho} T = \frac{\tilde{\rho}}{\rho}\pi \quad \text{on } M$$

provided $H = 0$ on M, which confirms (2-2). □

Given any fixed point $y \in M$, let $\bar{x} = (x_1, \dots, x_n)$ be normal coordinates on M at y (identified with 0) and $x_N = \rho$. In other words, let $x = (\bar{x}, x_N)$ be *Fermi coordinates*. The following lemma provides the expansion of the metric $\bar{g}$ near $y = 0$. See [Escobar 1992a, Lemma 3.1] for its proof.

Lemma 2.2. *Let $(\bar{X}, \bar{g})$ be a compact manifold with boundary $(M, \hat{h})$ and $y \in M$. Then, in terms of Fermi coordinates around y, it holds that*

$$\sqrt{|\bar{g}|}(x) = 1 - nHx_N + \tfrac{1}{2}\big(n^2H^2 - \|\pi\|^2 - R_{NN}[\bar{g}]\big)x_N^2 - H_{,i}x_ix_N - \tfrac{1}{6}R_{ij}[\hat{h}]x_ix_j + O(|x|^3)$$

and

$$\bar{g}^{ij}(x) = \delta_{ij} + 2\pi_{ij}x_N + \tfrac{1}{3}R_{ikjl}[\hat{h}]x_kx_l + \bar{g}^{ij}{}_{,Nk}x_Nx_k + \big(3\pi_{ik}\pi_{kj} + R_{iNjN}[\bar{g}]\big)x_N^2 + O(|x|^3)$$

near y (identified with a small half-ball $B^N_+(0, 2\eta_0)$ near 0 in $\mathbb{R}^N_+$). Here $\|\pi\|^2 = \hat{h}^{ik}\hat{h}^{jl}\pi_{ij}\pi_{kl}$ is the square of the norm of the second fundamental form π on $(M, \hat{h}) \subset (\bar{X}, \bar{g})$, $R_{ikjl}[\hat{h}]$ is a component of the Riemannian curvature tensor on M, $R_{iNjN}[\bar{g}]$ is that of the Riemannian curvature tensor in X, $R_{ij}[\hat{h}] = R_{ikjk}[\hat{h}]$ and $R_{NN}[\bar{g}] = R_{iNiN}[\bar{g}]$. Every tensor in the expansions is computed at $y = 0$.

Now notice that the transformation law of the scalar curvature [Escobar 1992a, (1.1)] implies

$$R[g^+] + n(n+1) = 2n\frac{\partial_\rho\sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}}\rho + R[\bar{g}]\rho^2. \tag{2-3}$$

It readily shows that (1-18) and (1-19) indicate $H = 0$ on M.

Lemma 2.3. *Suppose that (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$. If $R[g^+] + n(n+1) = o(\rho)$ as $\rho \to 0$, then $H = 0$ on M.*

Proof. Fix any $y \in M$. By (2-3), we have

$$o(1) = 2n \frac{\partial_\rho \sqrt{|\bar{g}|(y)}}{\sqrt{|\bar{g}|(y)}} + R[\bar{g}](y)\rho + o(1) = -2n^2 H(y) + o(1)$$

as a point tends to y. This implies $H(y) = 0$, and therefore the assertion follows. □

We next select a good background metric on X under the validity of hypothesis (1-18).

Lemma 2.4. *Let (X, g^+) be an asymptotically hyperbolic manifold such that condition (1-18) holds. Then the conformal infinity $(M, [\hat{h}])$ admits a representative $\hat{h} \in [\hat{h}]$, the geodesic boundary-defining function ρ and the metric $\bar{g} = \rho^2 g^+$ satisfying*

$$H = 0 \quad \text{on } M, \qquad R_{ij}[\hat{h}](y) = 0 \qquad \text{and} \qquad R_{\rho\rho}[\bar{g}](y) = \frac{1-2n}{2(n-1)}\|\pi(y)\|^2 \tag{2-4}$$

for a fixed point $y \in M$.

Proof. According to [Lee and Parker 1987, Theorem 5.2], one may choose a representative $\hat{h}$ of the conformal class $[\hat{h}]$ such that $R_{ij}[\hat{h}](y) = 0$. Additionally Lemmas 2.3 and 2.1 ensure that $H = 0$ on M for any $\hat{h} \in [\hat{h}]$. Hence assumption (1-18) can be interpreted as

$$\begin{aligned} o(1) = 2n \frac{\partial_\rho \sqrt{|\bar{g}|}}{\rho\sqrt{|\bar{g}|}} + R[\bar{g}] &= \frac{n}{\rho}\bar{g}^{ab}\bar{g}_{ab,\rho} + R[\bar{g}] = n(\bar{g}^{ab}{}_{,\rho}\,\bar{g}_{ab,\rho} + \bar{g}^{ab}\bar{g}_{ab,\rho\rho}) + R[\bar{g}] + o(1) \\ &= -2n(R_{\rho\rho}[\bar{g}] + \|\pi\|^2) + \big(2R_{\rho\rho}[\bar{g}] + \|\pi\|^2 + R[\hat{h}] - H^2\big) + o(1) \end{aligned}$$

as $\rho \to 0$, where we used $H = 0$ on M for the third equality and the Gauss–Codazzi equation for the fourth equality; see the proof of Lemmas 3.1 and 3.2 of [Escobar 1992a]. Taking the limit to $y \in M$, we get

$$0 = 2(1-n)R_{\rho\rho}[\bar{g}](y) + (1-2n)\|\pi(y)\|^2.$$

The third equality of (2-4) is its direct consequence. □

Lastly, we recall the function E in (1-5) and (1-6). In a collar neighborhood of M where $\rho = x_N$, it can be seen that

$$E(x_N) = \frac{n-2\gamma}{4n}\big[R[\bar{g}] - (n(n+1) + R[g^+])x_N^{-2}\big]x_N^{1-2\gamma} = -\tfrac{1}{2}(n-2\gamma)\frac{\partial_N\sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}}x_N^{-2\gamma}, \tag{2-5}$$

where the second equality holds because of (2-3).

2B. ***Nonminimal conformal infinity.*** Let $y \in M$ be a point identified with $0 \in \mathbb{R}^n$ such that $H(y) < 0$ and $B_+^N(0, 2\eta_0) \subset \mathbb{R}_+^N$ be its neighborhood which appeared in Lemma 2.2. Also, we select any smooth radial cut-off function $\psi \in C_c^\infty(\mathbb{R}_+^N)$ such that $\psi = 1$ in $B_+^N(0, \eta_0)$ and 0 in $\mathbb{R}_+^N \setminus B_+^N(0, 2\eta_0)$. In this subsection, we shall show that $\bar{I}_{\hat{h}}^\gamma[\psi W_\epsilon] < \Lambda^\gamma(\mathbb{S}^n, [g_c])$ for any $n \geq 2$ and $\gamma \in \big(0, \frac{1}{2}\big)$, where $W_\epsilon = W_{\epsilon,0}$ as before.

Before starting the computation, let us make one useful observation: Assume that $n > m + 2\gamma$ for a certain $m \in \mathbb{N}$. Then we get from (A-3) and (A-4) that

$$\int_{B_+^N(0,\eta_0)} x_N^{1-2\gamma}|x|^{m+1}|\nabla W_\epsilon|^2\,dx = \eta_0^{1-\zeta}\int_{B_+^N(0,\eta_0)} x_N^{1-2\gamma}|x|^{m+\zeta}|\nabla W_\epsilon|^2\,dx = O(\epsilon^{m+\zeta}) = o(\epsilon^m) \quad (2\text{-}6)$$

by choosing a small number $\zeta > 0$ such that $n > m + 2\gamma + \zeta$.

Proposition 2.5. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$ and $y \in M$ is a point such that $H(y) < 0$. Then for any $\epsilon > 0$ small, $n \geq 2$ and $\gamma \in \left(0, \frac{1}{2}\right)$, we have*

$$\bar{I}_{\hat{h}}^{\gamma}[\psi W_\epsilon] \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) + \epsilon \underbrace{\frac{2n^2-2n+1-4\gamma^2}{2(1-2\gamma)}\,\frac{\kappa_\gamma\int_{\mathbb{R}_+^N} x_N^{2-2\gamma}|\nabla W_1|^2\,dx}{\left(\int_{\mathbb{R}^n} w_1^{p+1}\,dx\right)^{\frac{n-2\gamma}{n}}}}_{>0} H(y) + o(\epsilon) < \Lambda^\gamma(\mathbb{S}^n, [g_c]), \quad (2\text{-}7)$$

where $\bar{I}_{\hat{h}}^{\gamma}$ is the γ-Yamabe functional given in (1-11), and $\Lambda^\gamma(\mathbb{S}^n, [g_c])$ and κ_γ are positive constants introduced in (1-16) and (1-23).

Proof. Since the proof is essentially the same as that of [Choi and Kim 2017, Proposition 6.1], we briefly sketch it. By Lemma 2.2 and (2-6), we discover

$$\begin{aligned}\int_{B_+^N(0,\eta_0)} & x_N^{1-2\gamma}|\nabla W_\epsilon|_{\bar{g}}^2\,dv_{\bar{g}} \\ &= \int_{B_+^N(0,\eta_0)} x_N^{1-2\gamma}|\nabla W_1|^2\,dx + \epsilon H\left(2\int_{\mathbb{R}_+^N} x_N^{2-2\gamma}|\nabla_{\bar{x}} W_1|^2\,dx - n\int_{\mathbb{R}_+^N} x_N^{2-2\gamma}|\nabla W_1|^2\,dx\right) + o(\epsilon)\end{aligned}$$

and

$$\int_M (\psi W_\epsilon)^{p+1}\,dv_{\hat{h}} = \int_{B^n(0,\eta_0)} w_\epsilon^{p+1}(1 + O(|\bar{x}|^2))\,d\bar{x} + O(\epsilon^n) = \int_{\mathbb{R}^n} w_1^{p+1}\,dx + o(\epsilon).$$

Moreover, according to Lemma 2.2 and (2-5), we have

$$\int_{B_+^N(0,\eta_0)} E(x_N)W_\epsilon^2\,dv_{\bar{g}} = \tfrac{1}{2}n(n-2\gamma)\epsilon H\int_{\mathbb{R}_+^N} x_N^{-2\gamma}W_1^2\,dx + o(\epsilon).$$

Thus the above estimates and Lemma B.3 confirm (2-7). □

Unlike the other existence results to be discussed later, we need to assume that $\gamma \in \left(0, \frac{1}{2}\right)$ for Proposition 2.5. Such a restriction is necessary in two reasons: First of all, $\gamma \in \left(0, \frac{1}{2}\right)$ is necessary for the function $x_N^{-2\gamma}W_1^2$ to be integrable in $\mathbb{R}_+^N$. Secondly the mean curvature H should vanish for $\gamma \in \left(\frac{1}{2}, 1\right)$ to guarantee the validity of the extension theorem (Theorem A).

2C. *Nonumbilic conformal infinity: higher-dimensional cases.* We fix a nonumbilic point $y = 0 \in M$. Let also $B_+^N(0, 2\eta_0) \subset \mathbb{R}_+^N$ be a small neighborhood of 0 and $\psi \in C_c^\infty(B_+^N(0, 2\eta_0))$ a cut-off function chosen in the previous subsection.

Lemma 2.6. *Let $J^{\gamma}_{\hat{h}}$ be the energy functional defined as*

$$J^{\gamma}_{\hat{h}}[U;X] = \int_X (\rho^{1-2\gamma}|\nabla U|^2_{\bar{g}} + E(\rho)U^2)\,dv_{\bar{g}} \quad \textit{for any } U \in W^{1,2}(X,\rho^{1-2\gamma}). \tag{2-8}$$

Assume also that (2-4) holds. Then for any $\epsilon > 0$ small, $n > 2+2\gamma$ and $\gamma \in (0,1)$, it is valid that

$$\begin{aligned} J^{\gamma}_{\hat{h}}[\psi W_\epsilon; B^N_+(0,\eta_0)] &= \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma}|\nabla W_1|^2\,dx \\ &\quad + \epsilon^2\|\pi\|^2\big[-\tfrac{1}{2}(1+b)\mathcal{F}_2 + \tfrac{1}{n}(3+b)\mathcal{F}_3 + \tfrac{1}{2}(n-2\gamma)(1+b)\mathcal{F}_1\big] + o(\epsilon^2), \end{aligned} \tag{2-9}$$

where $b := (1-2n)/(2n-2)$, $\|\pi\|$ is the norm of the second fundamental form at $y = 0 \in M$, and the values $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are given in Lemma B.4.

Proof. We borrow the argument presented in [González and Qing 2013, Theorem 1.5]. According to Lemma 2.2 and (2-4), it holds that

$$\sqrt{|\bar{g}|}(\bar{x}, x_N) = 1 - \tfrac{1}{2}(1+b)\|\pi\|^2 x_N^2 + O(|(\bar{x}, x_N)|^3) \quad \text{in } B^N_+(0,\eta_0). \tag{2-10}$$

Hence we obtain from (2-6) that

$$\begin{aligned} &\int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma}|\nabla W_\epsilon|^2_{\bar{g}}\,dv_{\bar{g}} \\ &\quad = \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}|\nabla W_\epsilon|^2\,dx + \epsilon^2\bigg[(3\pi_{ik}\pi_{kj} + R_{iNjN}[\bar{g}])\int_{\mathbb{R}^N_+} x_N^{3-2\gamma}\partial_i W_1 \partial_j W_1\,dx \\ &\qquad\qquad - \tfrac{1}{2}(1+b)\|\pi\|^2 \int_{\mathbb{R}^N_+} x_N^{3-2\gamma}|\nabla W_1|^2\,dx\bigg] + o(\epsilon^2). \end{aligned}$$

Also, by means of (2-5) and (2-10),

$$E(x_N) = \tfrac{1}{2}(n-2\gamma)(1+b)\|\pi\|^2 x_N^{1-2\gamma} + O(|x|^2 x_N^{-2\gamma})$$

for $x_N \geq 0$ small, so

$$\int_{B^N_+(0,\eta_0)} E(x_N)W_\epsilon^2\,dv_{\bar{g}} = \epsilon^2 \tfrac{1}{2}(n-2\gamma)(1+b)\|\pi\|^2 \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} W_1^2\,dx + o(\epsilon^2).$$

Collecting every calculation, we discover (2-9). □

The previous lemma ensures the existence of a positive solution to (1-2) for nonumbilic conformal infinity M^n with $n \in \mathbb{N}$ sufficiently high.

Corollary 2.7. *Assume that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold and $\hat{h}$ is the representative of the conformal infinity M found in Lemma 2.2. If $n > 2+2\gamma$ and $\gamma \in (0,1)$, we have*

$$\bar{I}^{\gamma}_{\hat{h}}[\psi W_\epsilon] \leq \Lambda^\gamma(\mathbb{S}^n,[g_c]) - \epsilon^2 \mathcal{C}'(n,\gamma)\,\Lambda^\gamma(\mathbb{S}^n,[g_c])^{-\frac{n-2\gamma}{2\gamma}}\kappa_\gamma\,|\mathbb{S}^{n-1}|A_3 B_2\|\pi\|^2 + o(\epsilon^2), \tag{2-11}$$

where the positive constants $\Lambda^\gamma(\mathbb{S}^n,[g_c])$, κ_γ, A_3 and B_2 are introduced in (1-16), (1-23) and (B-3), respectively, and $\mathcal{C}'(n,\gamma)$ is the number given by

$$\mathcal{C}'(n,\gamma)=\frac{3n^2+n(16\gamma^2-22)+20(1-\gamma^2)}{8n(n-1)(1-\gamma^2)}. \tag{2-12}$$

Proof. Estimate (2-11) comes from Lemmas 2.6 and B.4 and the computations made in the proof of [González and Qing 2013, Theorem 1.5]. The details are left to the reader. □

By (2-2), we still have that $\pi\neq 0$ at $y\in M$, even after picking a new representative of the conformal infinity. Furthermore, the number $\mathcal{C}'(n,\gamma)$ is positive when $n\geq 4$ for $\gamma>\sqrt{\frac{5}{11}}\simeq 0.674$, $n\geq 5$ for $\gamma>\frac{1}{2}$, $n\geq 6$ for $\gamma>\sqrt{\frac{1}{19}}\simeq 0.229$ and $n\geq 7$ for any $\gamma>0$. Hence, in this regime, one is able to deduce the existence of a positive solution of (1-2) by testing the truncated standard bubble into the γ-Yamabe functional.

2D. *Nonumbilic conformal infinity: lower-dimensional cases.* We recall the nonumbilic point $y\in M$ identified with the origin of $\mathbb{R}^N_+$, the small number $\eta_0>0$ and the cut-off function $\psi\in C^\infty_c(\mathbb{R}^N_+)$. Furthermore, we introduce

$$\Psi_\epsilon(\bar{x},x_N)=M_1\pi_{ij}x_ix_jx_Nr^{-1}\partial_rW_\epsilon=\epsilon\cdot\epsilon^{-\frac{n-2\gamma}{2}}\Psi_1(\epsilon^{-1}\bar{x},\epsilon^{-1}x_N) \tag{2-13}$$

for each $\epsilon>0$, where $M_1\in\mathbb{R}$ is a number to be determined later, the π_{ij} are the coefficients of the second fundamental form at y and $r=|\bar{x}|$. Our ansatz to deal with lower-dimensional cases is defined by

$$\Phi_\epsilon:=\psi(W_\epsilon+\Psi_\epsilon)\quad\text{in } X.$$

The definition of Φ_ϵ is inspired by [Marques 2007].

The main objective of this subsection is to prove:

Proposition 2.8. *Suppose that (X^{n+1},g^+) is an asymptotically hyperbolic manifold and $\hat{h}$ is the representative of the conformal infinity M satisfying (2-4). If $n>2+2\gamma$ and $\gamma\in(0,1)$, we have*

$$\bar{I}^\gamma_{\hat{h}}[\Phi_\epsilon]\leq\Lambda^\gamma(\mathbb{S}^n,[g_c])-\epsilon^2\mathcal{C}(n,\gamma)\,\Lambda^\gamma(\mathbb{S}^n,[g_c])^{-\frac{n-2\gamma}{2\gamma}}\kappa_\gamma\,|\mathbb{S}^{n-1}|A_3B_2\|\pi\|^2+o(\epsilon^2), \tag{2-14}$$

where $\mathcal{C}(n,\gamma)$ is the number defined by

$$\mathcal{C}(n,\gamma)=\frac{3n^2+n(16\gamma^2-22)+20(1-\gamma^2)}{8n(n-1)(1-\gamma^2)}+\frac{16(n-1)(1-\gamma^2)}{n(3n^2+n(2-8\gamma^2)+4\gamma^2-4)}.$$

It can be checked that $\mathcal{C}(n,\gamma)>0$ whenever $n\geq 4$ and $\gamma\in(0,1)$. Thus the above proposition, along with Proposition 2.5, justifies the statement of Theorem 1.1. We have $\mathcal{C}(3,\gamma)>0$ for $\gamma>\frac{1}{2}$, but it also holds that $n>2+2\gamma>3$. Therefore we get no result for $n=3$.

Proof of Proposition 2.8. The proof consists of three steps.

Step 1: energy in the half-ball $B^N_+(0,\eta_0)$. Since $\psi=1$ in $B^N_+(0,\eta_0)$, we discover

$$\begin{aligned}&J^\gamma_{\hat{h}}[\psi(W_\epsilon+\Psi_\epsilon);B^N_+(0,\eta_0)]\\&=J^\gamma_{\hat{h}}[\psi W_\epsilon;B^N_+(0,\eta_0)]+2\int_{B^N_+(0,\eta_0)}x_N^{1-2\gamma}\langle\nabla W_\epsilon,\nabla\Psi_\epsilon\rangle_{\bar{g}}\,dv_{\bar{g}}+\int_{\mathbb{R}^N_+}x_N^{1-2\gamma}|\nabla\Psi_\epsilon|^2\,dx+o(\epsilon^2),\end{aligned} \tag{2-15}$$

where the functional $J^{\gamma}_{\hat{h}}$ is defined in (2-8). Moreover, we note from Lemma 2.2 that the mean curvature $H = \pi_{ii}/n$ vanishes at the origin, which yields

$$\begin{aligned}\int_{B^N_+(0,\eta_0)} & x_N^{1-2\gamma}\nabla W_\epsilon\cdot\nabla\Psi_\epsilon\,dx\\ &=\epsilon M_1\int_{B^N_+(0,\eta_0/\epsilon)} x_N^{2-2\gamma}\pi_{ij}x_ix_j\big[2r^{-2}(\partial_r W_1)^2+r\partial_r(r^{-1}\partial_r W_1)\big]\,dx\\ &\qquad +\epsilon M_1\int_{B^N_+(0,\eta_0/\epsilon)} x_N^{1-2\gamma}\pi_{ij}x_ix_jr^{-1}(\partial_N W_1)\big[(\partial_r W_1)+x_N(\partial_{Nr} W_1)\big]\,dx=0. \quad (2\text{-}16)\end{aligned}$$

Hence we obtain from the definition (2-13) of Ψ_ϵ and (2-16) that

$$\begin{aligned}2\int_{B^N_+(0,\eta_0)} & x_N^{1-2\gamma}\langle\nabla W_\epsilon,\nabla\Psi_\epsilon\rangle_{\bar g}\,dv_{\bar g}\\ &=2\int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma}\nabla W_\epsilon\cdot\nabla\Psi_\epsilon\,dx+4\pi_{ij}\int_{\mathbb{R}^N_+} x_N^{2-2\gamma}\partial_i W_\epsilon\,\partial_j\Psi_\epsilon\,dx+o(\epsilon^2)\\ &=\epsilon^2 4M_1\pi_{ij}\int_{\mathbb{R}^N_+} x_N^{3-2\gamma}x_i\big[2\pi_{jk}x_kr^{-2}(\partial_r W_1)^2+\pi_{kl}x_kx_lx_jr^{-2}(\partial_r W_1)\,\partial_r(r^{-1}\partial_r W_1)\big]\,dx+o(\epsilon^2)\\ &=\epsilon^2 4M_1\left[\frac{2}{n}\mathcal{F}_3+\frac{2}{n(n+2)}(-\mathcal{F}_3+\mathcal{F}_4)\right]\|\pi\|^2+o(\epsilon^2)\\ &=\epsilon^2\left(\frac{4}{n}\right)M_1|\mathbb{S}^{n-1}|A_3B_2\|\pi\|^2+o(\epsilon^2), \quad (2\text{-}17)\end{aligned}$$

where the constants $\mathcal{F}_3,\mathcal{F}_4$ as well as $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_5,\dots,\mathcal{F}_8$, are defined in Lemma B.4. In a similar fashion, it can be found that

$$\begin{aligned}\int_{\mathbb{R}^N_+} x_N^{1-2\gamma}|\nabla\Psi_\epsilon|^2\,dx &=\epsilon^2\frac{2M_1^2}{n(n+2)}(\mathcal{F}_3-2\mathcal{F}_4+\mathcal{F}_5+\mathcal{F}_6+2\mathcal{F}_7+\mathcal{F}_8)\|\pi\|^2+o(\epsilon^2)\\ &=\epsilon^2\,\frac{3n^2+2n(1-4\gamma^2)-4(1-\gamma^2)}{4n(n-1)(1-\gamma^2)}\,M_1^2|\mathbb{S}^{n-1}|A_3B_2\|\pi\|^2+o(\epsilon^2). \quad (2\text{-}18)\end{aligned}$$

Step 2: energy in the half-annulus $B^N_+(0,2\eta_0)\setminus B^N_+(0,\eta_0)$. According to (A-1), (A-3) and (A-4), see (2-6), it holds that

$$J^{\gamma}_{\hat{h}}[\psi(W_\epsilon+\Psi_\epsilon);X\setminus B^N_+(0,\eta_0)]=o(\epsilon^2). \quad (2\text{-}19)$$

Consequently, one deduces from (2-15), (2-17)–(2-19) and Lemma B.4 that

$$J^{\gamma}_{\hat{h}}[\psi(W_\epsilon+\Psi_\epsilon);X]\le\int_{\mathbb{R}^N_+} x_N^{1-2\gamma}|\nabla W_1|^2\,dx-\epsilon^2\mathcal{C}(n,\gamma)|\mathbb{S}^{n-1}|A_3B_2\|\pi\|^2+o(\epsilon^2) \quad (2\text{-}20)$$

by choosing the optimal $M_1\in\mathbb{R}$.

Step 3: completion of the proof. Lemma 2.2 and the fact that $\Psi_\epsilon=0$ on M tell us that

$$\int_M|\psi(W_\epsilon+\Psi_\epsilon)|^{p+1}\,dv_{\hat h}=\int_{B^n(0,2\eta_0)}(\psi w_\epsilon)^{p+1}(1+O(|\bar x|^3))\,d\bar x\ge\int_{\mathbb{R}^n} w_1^{p+1}\,d\bar x+o(\epsilon^2). \quad (2\text{-}21)$$

Combining (2-20) and (2-21) gives estimate (2-14). □

3. Umbilic conformal infinities

3A. *Geometric background.* For a fixed point $y \in M$ identified with $0 \in \mathbb{R}^n$, let $\bar{x} = (x_1, \dots, x_n)$ be the normal coordinate on M at y and $x_N = \rho$. The following expansion of the metric is borrowed from [Marques 2005].

Lemma 3.1. *Let $(\overline{X}, \bar{g})$ be a compact manifold with boundary $(M, \hat{h})$ and $y \in M$ such that $\pi = \pi_{;i} = \pi_{;ij} = \pi_{;ijk} = 0$, $R_{ij}[\hat{h}] = 0$ and $R_{NN}[\bar{g}] = 0$ at y. Then, in terms of Fermi coordinates around y, it holds that*

$$\begin{aligned}\sqrt{|\bar{g}|}(\bar{x}, x_N) = 1 &- \tfrac{1}{12} R_{ij;k}[\hat{h}] x_i x_j x_k - \tfrac{1}{2} R_{NN;i}[\bar{g}] x_N^2 x_i - \tfrac{1}{6} R_{NN;N}[\bar{g}] x_N^3 \\ &- \tfrac{1}{20}\big(\tfrac{1}{2} R_{ij;kl}[\hat{h}] + \tfrac{1}{9} R_{miqj}[\hat{h}] R_{mkql}[\hat{h}]\big) x_i x_j x_k x_l - \tfrac{1}{4} R_{NN;ij}[\bar{g}] x_N^2 x_i x_j \\ &- \tfrac{1}{6} R_{NN;Ni}[\bar{g}] x_N^3 x_i - \tfrac{1}{24}\big[R_{NN;NN}[\bar{g}] + 2(R_{iNjN}[\bar{g}])^2\big] x_N^4 + O(|(\bar{x}, x_N)|^5) \qquad (3\text{-}1)\end{aligned}$$

and

$$\begin{aligned}\bar{g}^{ij}(\bar{x}, x_N) = \delta_{ij} &+ \tfrac{1}{3} R_{ikjl}[\hat{h}] x_k x_l + R_{iNjN}[\bar{g}] x_N^2 + \tfrac{1}{6} R_{ikjl;m}[\hat{h}] x_k x_l x_m + R_{iNjN;k}[\bar{g}] x_N^2 x_k \\ &+ \tfrac{1}{3} R_{iNjN;N}[\bar{g}] x_N^3 + \big(\tfrac{1}{20} R_{ikjl;mq}[\hat{h}] + \tfrac{1}{15} R_{iksl}[\hat{h}] R_{jmsq}[\hat{h}]\big) x_k x_l x_m x_q \\ &+ \big(\tfrac{1}{2} R_{iNjN;kl}[\bar{g}] + \tfrac{1}{3} \mathrm{Sym}_{ij}(R_{iksl}[\hat{h}] R_{sNjN}[\bar{g}])\big) x_N^2 x_k x_l + \tfrac{1}{3} R_{iNjN;kN}[\bar{g}] x_N^3 x_k \\ &+ \tfrac{1}{12}(R_{iNjN;NN}[\bar{g}] + 8 R_{iNsN}[\bar{g}] R_{sNjN}[\bar{g}]) x_N^4 + O(|(\bar{x}, x_N)|^5) \qquad (3\text{-}2)\end{aligned}$$

near y (identified with a small half-ball $B_+^N(0, 2\eta_0)$ near 0 in $\mathbb{R}_+^N$). Here all tensors are computed at y and the indices m, q and s run from 1 to n as well.

To treat umbilic but nonlocally conformally flat boundaries, we also need the following extension of Lemma 2.4.

Lemma 3.2. *For $n \geq 3$, let (X^{n+1}, g^+) be an asymptotically hyperbolic manifold such that the conformal infinity $(M^n, [\hat{h}])$ is umbilic and (1-19) holds. For a fixed point $y \in M$, there exist a representative $\hat{h}$ of the class $[\hat{h}]$, the geodesic boundary-defining function ρ ($= x_N$ near M) and the metric $\bar{g} = \rho^2 g^+$ such that*

(1) $R_{ij;k}[\hat{h}](y) + R_{jk;i}[\hat{h}](y) + R_{ki;j}[\hat{h}](y) = 0$,

(2) $\mathrm{Sym}_{ijkl}\big(R_{ij;kl}[\hat{h}] + \tfrac{2}{9} R_{miqj}[\hat{h}] R_{mkql}[\hat{h}]\big)(y) = 0$,

(3) $\pi = 0$ *on* M, $\quad R_{NN;N}[\bar{g}](y) = R_{aN}[\bar{g}](y) = 0$,

(4) $R_{;ii}[\bar{g}](y) = -\dfrac{n\|W\|^2}{6(n-1)}$, $\quad R_{NN;ii}[\bar{g}](y) = -\dfrac{\|W\|^2}{12(n-1)}$, $\quad R_{iNjN}[\bar{g}](y) = R_{ij}[\bar{g}](y)$,

(5) $R_{NN;NN}[\bar{g}](y) = \dfrac{3}{2n} R_{;NN}[\bar{g}](y) - 2(R_{ij}[\bar{g}](y))^2$,

(6) $R_{iNjN;ij}[\bar{g}](y) = \dfrac{3-n}{2n} R_{;NN}[\bar{g}](y) - (R_{ij}[\bar{g}](y))^2 - \dfrac{\|W\|^2}{12(n-1)}$

if normal coordinates around $y \in (M, \hat{h})$ are assumed. Here $\|W\|$ is the norm of the Weyl tensor of $(M, \hat{h})$ at y.

Note that the first partial derivatives of $\hat{h}$ and the Christoffel symbols $\Gamma_{ij}^k[\hat{h}] = \Gamma_{ij}^k[\bar{g}]$ at y vanish. Also a simple computation utilizing $\pi = 0$ on M shows that $\Gamma_{aa}^b[\bar{g}] = \Gamma_{bN}^a[\bar{g}] = 0$ on M.

Proof of Lemma 3.2. Theorem 5.2 of [Lee and Parker 1987] guarantees the existence of a representative $\hat{h} \in [\hat{h}]$ on M such that (1), (2) and $R_{ij}[\hat{h}](y) = 0$ hold. Furthermore, [Escobar 1992b, Proposition 1.2] shows that umbilicity is preserved under the conformal transformation, and so $\pi = 0$ on M. The proof of the remaining identities in (3)–(6) is presented in two steps.

Step 1: By differentiating (2-3) in x_N and using the assumption that $\partial_N(R[g^+] + n(n+1)) = o(x_N^2)$ as $x_N \to 0$, see (1-19), we obtain

$$o(1) = n\left[\frac{\partial_N|\bar{g}|}{|\bar{g}|x_N^2} + \frac{\partial_{NN}|\bar{g}|}{|\bar{g}|x_N} - \frac{(\partial_N|\bar{g}|)^2}{|\bar{g}|^2 x_N}\right] + \frac{2R[\bar{g}]}{x_N} + R_{,N}[\bar{g}] \quad \text{as } x_N \to 0. \tag{3-3}$$

Also, since we supposed that the mean curvature H vanishes on the umbilic boundary M, we get from (2-4) that $R_{NN}[\bar{g}](y) = \pi(y) = 0$. This in turn gives that $|\bar{g}|(y) = 1$ and $\partial_N|\bar{g}|(y) = \partial_{NN}|\bar{g}|(y) = R[\bar{g}](y) = 0$. Consequently, by taking the limit to y in (3-3), we find that

$$\begin{aligned} 0 &= n\left[\tfrac{1}{2}\partial_{NNN}|\bar{g}|(y) + \partial_{NNN}|\bar{g}|(y) - 0\right] + 2R_{,N}[\bar{g}](y) + R_{,N}[\bar{g}](y) \\ &= n\partial_{NNN}|\bar{g}|(y) + 2R_{,N}[\bar{g}](y). \end{aligned} \tag{3-4}$$

Now we observe from Lemma 3.1 that $\partial_{NNN}|\bar{g}|(y) = -2R_{NN;N}[\bar{g}](y)$. In addition, by the second Bianchi identity, the Codazzi equation and the fact that $\pi = 0$ on M, one can achieve

$$\begin{aligned} R_{,N}[\bar{g}] = R_{;N}[\bar{g}] &= 2R_{NN;N}[\bar{g}] + R_{ijij;N}[\bar{g}] = 2R_{NN;N}[\bar{g}] + (R_{ijiN;j}[\bar{g}] - R_{ijjN;i}[\bar{g}]) \\ &= 2R_{NN;N}[\bar{g}] + 2(\pi_{ii;jj} - \pi_{ij;ij}) = 2R_{NN;N}[\bar{g}] \end{aligned} \tag{3-5}$$

and

$$R_{iN}[\bar{g}] = \pi_{jj;i} - \pi_{ij;j} = 0$$

at $y \in M$. Combining (3-4) and (3-5), we get

$$0 = (2-n)R_{NN;N}[\bar{g}](y).$$

Since $n \geq 3$, it follows that $R_{NN;N}[\bar{g}](y) = 0$, as we wanted.

Step 2: It is well known that $R_{,ii}[\hat{h}](y) = R_{;ii}[\hat{h}](y) = -\frac{1}{6}\|W(y)\|^2$ in the normal coordinate around $y \in M$. Therefore the Gauss–Codazzi equation and the fact that $H = \pi = 0$ on M imply

$$R_{,ii}[\bar{g}](y) = 2R_{NN,ii}[\bar{g}](y) - \tfrac{1}{6}\|W(y)\|^2 \quad \text{and} \quad R_{iNjN}[\bar{g}](y) = R_{ij}[\bar{g}](y). \tag{3-6}$$

Moreover, since $\Delta_{\bar{x}}(R[g^+] + n(n+1)) = o(x_N^2)$ near $y \in \overline{X}$, refer to (1-19), by differentiating (2-3) in x_i twice, dividing the result by x_N^2 and then taking the limit to y, one obtains

$$R_{,ii}[\bar{g}](y) = 2nR_{NN,ii}[\bar{g}](y). \tag{3-7}$$

As a result, putting (3-7) into (3-6) and applying the relations at y

$$R_{;ii}[\bar{g}] = R_{,ii}[\bar{g}] \quad \text{and} \quad R_{NN;ii}[\bar{g}] = R_{NN,ii}[\bar{g}] - 2(\partial_i\Gamma_{iN}^a[\bar{g}])R_{aN}[\bar{g}] \underset{\text{by (3)}}{=} R_{NN,ii}[\bar{g}]$$

allow one to find (4).

On the other hand, arguing as before but using the hypothesis that $\partial_{NN}(R[g^+]+n(n+1)) = o(x_N^2)$ near $y \in \bar{X}$ at this time, one derives equalities

$$3R_{,NN}[\bar{g}](y) = -n\,\partial_{NNNN}|\bar{g}|(y) = 2n\big[R_{NN;NN}[\bar{g}](y) + 2(R_{iNjN}[\bar{g}](y))^2\big].$$

Because $R_{;NN}[\bar{g}](y) = R_{,NN}[\bar{g}](y)$, it is identical to (5). Hence the contracted second Bianchi identity, the Ricci identity and (3)–(5) give

$$\begin{aligned} R_{;NN}[\bar{g}] &= 2R_{iN;iN}[\bar{g}] + 2R_{NN;NN}[\bar{g}] = 2\big[R_{iN;Ni}[\bar{g}] + (R_{ij}[\bar{g}])^2 - (R_{aN}[\bar{g}])^2\big] + 2R_{NN;NN}[\bar{g}] \\ &= 2(R_{iN;Ni}[\bar{g}] + (R_{ij}[\bar{g}])^2) + \big(\tfrac{3}{n}R_{;NN}[\bar{g}] - 4(R_{ij}[\bar{g}])^2\big) \end{aligned}$$

at y. Now assertion (6) directly follows from the above equality and

$$\begin{aligned} R_{iN;Ni}[\bar{g}](y) &= R_{Njij;Ni}[\bar{g}](y) \\ &= -R_{iNjN;ij}[\bar{g}](y) + R_{NN;ii}[\bar{g}](y) = -R_{iNjN;ij}[\bar{g}](y) - \frac{\|W(y)\|^2}{12(n-1)}. \end{aligned}$$ □

3B. *Umbilic conformal infinity having the property $R_{\rho\rho;\rho}[\bar{g}] < 0$.* Like the previous section, we fix a smooth radial cut-off function $\psi \in C_c^\infty(\mathbb{R}^N_+)$ such that $\psi = 1$ in $B^N_+(0,\eta_0)$ and 0 in $\mathbb{R}^N_+ \setminus B^N_+(0,2\eta_0)$. Also, assume that $W_\epsilon = W_{\epsilon,0}$ denotes the bubble defined in (1-13). Let $y \in M$ be any fixed point identified with $0 \in \mathbb{R}^n$.

Lemma 3.3. *Suppose $J^\gamma_{\hat{h}}$ is the functional given in (2-8). If (2-4) is valid and $\pi = 0$ on M, then*

$$J^\gamma_{\hat{h}}[\psi W_\epsilon; B^N_+(0,\eta_0)] = \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma}|\nabla W_1|^2\,dx + \epsilon^3 R_{NN;N}[\bar{g}](y)\big[\tfrac{1}{4}(n-2\gamma)\mathcal{F}_1' - \tfrac{1}{6}\mathcal{F}_2' + \tfrac{1}{3n}\mathcal{F}_3'\big] + o(\epsilon^3) \tag{3-8}$$

for any $\epsilon > 0$ small, $n > 3+2\gamma$ and $\gamma \in (0,1)$. Here the values $\mathcal{F}_1'$, $\mathcal{F}_2'$ and $\mathcal{F}_3'$ are given in Lemma B.5.

Proof. Since $H = R_{NN}[\bar{g}] = 0$ at y and the bubbles W_ϵ depend only on the variables $|\bar{x}|$ and x_N, we have

$$\begin{aligned} &\int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma}|\nabla W_\epsilon|^2_{\bar{g}}\,dv_{\bar{g}} \\ &\quad= \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma}|\nabla W_1|^2\,dx \\ &\qquad + \epsilon^3 R_{NN;N}[\bar{g}](y)\bigg(\frac{1}{3n}\int_{\mathbb{R}^N_+} x_N^{4-2\gamma}|\nabla_{\bar{x}} W_1|^2\,dx - \frac{1}{6}\int_{\mathbb{R}^N_+} x_N^{4-2\gamma}|\nabla W_1|^2\,dx\bigg) + o(\epsilon^3). \end{aligned} \tag{3-9}$$

In addition, utilizing (2-5) and (3-1), we obtain

$$E(x_N) = \tfrac{1}{2}(n-2\gamma)\big(R_{NN;i}[\bar{g}](y)x_i + \tfrac{1}{2}R_{NN;N}[\bar{g}](y)x_N\big)x_N^{1-2\gamma} + O(|x|^2 x_N^{1-2\gamma})$$

for $x_N \geq 0$ small enough. Therefore

$$\int_{B^N_+(0,\eta_0)} E(x_N)W_\epsilon^2\,dv_{\bar{g}} = \epsilon^3 R_{NN;N}[\bar{g}](y)\Big(\frac{n-2\gamma}{4}\Big)\int_{\mathbb{R}^N_+} x_N^{2-2\gamma}W_1^2\,dx + o(\epsilon^3). \tag{3-10}$$

Combining (3-9) and (3-10), we deduce (3-8). □

As a consequence of the previous lemma, we obtain the following result.

Proposition 3.4. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold with umbilic conformal infinity $(M, [\hat{h}])$. If (2-4) is valid and $y \in M$ is a point such that $R_{NN;N}(y) < 0$, then for any $\epsilon > 0$ small and $n > 3 + 2\gamma$, we have*

$$\bar{I}^{\gamma}_{\hat{h}}[\psi W_\epsilon] \le \Lambda^{\gamma}(\mathbb{S}^n, [g_c]) + \epsilon^3 \underbrace{\frac{4n^2-12n+9-4\gamma^2}{24n(3-2\gamma)} \frac{\kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{4-2\gamma} |\nabla W_1|^2 \, dx}{\left(\int_{\mathbb{R}^n} w_1^{p+1} \, dx\right)^{\frac{n-2\gamma}{n}}}}_{>0} R_{NN;N}(y) + o(\epsilon^3) < \Lambda^{\gamma}(\mathbb{S}^n, [g_c]),$$

where $\bar{I}^{\gamma}_{\hat{h}}$ is the γ-Yamabe functional given in (1-11), and $\Lambda^{\gamma}(\mathbb{S}^n, [g_c])$ and κ_γ are the positive constants introduced in (1-16) and (1-23), respectively.

Proof. By (A-1), (A-3) and (A-4), see (2-6), it is true that

$$J^{\gamma}_{\hat{h}}[\psi W_\epsilon; X \setminus B^N_+(0, \eta_0)] = o(\epsilon^3).$$

Moreover, we infer from (3-1) and radial symmetry of the function ψw_ϵ in $\mathbb{R}^n$ that

$$\int_M (\psi W_\epsilon)^{p+1} \, dv_{\hat{h}} = \int_{\mathbb{R}^n} w_1^{p+1} \, d\bar{x} + o(\epsilon^3).$$

Hence Lemmas 3.3 and B.5 give the desired estimate. $\square$

3C. *Umbilic nonlocally conformally flat conformal infinity.* We now study the case when the boundary M is umbilic, nonlocally conformally flat and (1-19) holds. In view of Lemma 3.2(3), the tensor $R_{NN;N}[\bar{g}]$ has no role and one needs to expand the energy up to one higher order in ϵ.

Lemma 3.5. *Let $y = 0 \in M$ be any fixed point and $J^{\gamma}_{\hat{h}}$ the functional given in (2-8). If (2-4) and Lemma 3.2(1)–(6) are valid, then*

$$\begin{aligned} &J^{\gamma}_{\hat{h}}[\psi W_\epsilon; B^N_+(0, \eta_0)] \\ &= \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 \, dx + \epsilon^4 \Bigg[\frac{\|W\|^2}{4n} \left(\frac{\mathcal{F}_5''}{12(n-1)} - \frac{\mathcal{F}_6''}{2(n-1)(n+2)} - \frac{(n-2\gamma)\mathcal{F}_4''}{12(n-1)} \right) \\ &\quad + \frac{R_{;NN}[\bar{g}]}{2} \left(-\frac{\mathcal{F}_2''}{8n} + \frac{\mathcal{F}_3''}{4n^2} - \frac{(n-3)\mathcal{F}_6''}{n^2(n+2)} + \frac{(n-2\gamma)\mathcal{F}_1''}{4n} \right) + \frac{(R_{ij}[\bar{g}])^2}{n} \left(\frac{\mathcal{F}_3''}{2} - \frac{\mathcal{F}_6''}{n+2} \right) \Bigg] + o(\epsilon^4) \end{aligned} \tag{3-11}$$

for any $\epsilon > 0$ small, $n > 4 + 2\gamma$ and $\gamma \in (0, 1)$. Here the tensors are computed at y and the values $\mathcal{F}_1'', \dots, \mathcal{F}_6''$ are given in Lemma B.6.

Proof. Step 1: estimate on the second- and third-order terms. To begin with, we ascertain that

$$J^{\gamma}_{\hat{h}}[\psi W_\epsilon; B^N_+(0, \eta_0)] = \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 \, dx + O(\epsilon^4). \tag{3-12}$$

In fact, thanks to (1-19), (2-5) and $R[\bar g](y) = R_{,N}[\bar g](y) = 0$, it holds that

$$\begin{aligned}
&\int_{B^N_+(0,\eta_0)} E(x_N) W_\epsilon^2 \, dv_{\bar g} \\
&\quad = \int_{B^N_+(0,\eta_0)} E(x_N) W_\epsilon^2 \, dx + O\Big(\epsilon^{4+\zeta} \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} W_1^2 |x|^{4+\zeta} \, dx\Big) \\
&\quad = \epsilon^2 \Big(\frac{n-2\gamma}{4n}\Big) \int_{B^N_+(0,\eta_0/\epsilon)} x_N^{1-2\gamma} \big(R[\bar g](y) + \epsilon R_{,a}[\bar g](y) x_a + \tfrac{1}{2}\epsilon^2 R_{,ab}[\bar g](y) x_a x_b\big) W_1^2 \, dx + o(\epsilon^4) \\
&\quad = \epsilon^4 \Big(\frac{n-2\gamma}{4n}\Big) \cdot \Big[\frac{1}{2n} R_{;ii}[\bar g](y) \, \mathcal{F}_4'' + \frac{1}{2} R_{;NN}[\bar g](y) \, \mathcal{F}_1''\Big] + o(\epsilon^4), \qquad (3\text{-}13)
\end{aligned}$$

where $\zeta > 0$ is a sufficiently small number. Because $R_{NN;N}[\bar g](y) = 0$ by Lemma 3.2(3), we see from (3-9) and (3-13) that estimate (3-12) is true.

Step 2: estimate on the fourth-order terms. Let $\sqrt{|\bar g|}^{(4)}$ and $(\bar g^{ij})^{(4)}$ be the fourth-order terms in the expansions (3-1) and (3-2) of $\sqrt{|\bar g|}$ and $\bar g^{ij}$. In view of (2-6), Lemma 3.2(2) and [Brendle 2008, Corollary 29], one can show that

$$\begin{aligned}
&\int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|^2 \sqrt{|\bar g|}^{(4)} \, dx \\
&\qquad = -\epsilon^4 \big[\tfrac{1}{4n} R_{NN;ii}[\bar g](y) \, \mathcal{F}_5'' + \tfrac{1}{24} (R_{NN;NN}[\bar g](y) + 2(R_{iNjN}[\bar g](y))^2) \mathcal{F}_2''\big] + o(\epsilon^4)
\end{aligned}$$

and

$$\begin{aligned}
\int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} (\bar g^{ij})^{(4)} \partial_i W_\epsilon \partial_j W_\epsilon \, dx = \epsilon^4 \Big[&\frac{1}{2n(n+2)} (R_{NN;ii}[\bar g](y) + 2R_{iNjN;ij}[\bar g](y)) \mathcal{F}_6'' \\
&+ \frac{1}{12n} (R_{NN;NN}[\bar g](y) + 8(R_{iNjN}[\bar g](y))^2) \mathcal{F}_3''\Big] + o(\epsilon^4);
\end{aligned}$$

see [González and Wang 2017, Section 4]. Therefore (2-4), (3-9) and Lemma 3.2(4)–(6) yield

$$\begin{aligned}
&\int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|^2_{\bar g} \, dv_{\bar g} \\
&= \int_{B^N_+(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 \, dx + \epsilon^4 \Big[\frac{\|W\|^2}{8n(n-1)} \Big(\frac{\mathcal{F}_5''}{6} - \frac{\mathcal{F}_6''}{n+2}\Big) + \frac{R_{;NN}[\bar g]}{2n} \Big(-\frac{\mathcal{F}_2''}{8} + \frac{\mathcal{F}_3''}{4n} - \frac{(n-3)\mathcal{F}_6''}{n(n+2)}\Big) \\
&\qquad + \frac{(R_{ij}[\bar g])^2}{n} \Big(\frac{\mathcal{F}_3''}{2} - \frac{\mathcal{F}_6''}{n+2}\Big)\Big] + o(\epsilon^4).
\end{aligned}$$

Now (3-13) and the previous estimate lead us to (3-11). □

Corollary 3.6. *Assume that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold, $\hat h$ is the representative of the conformal infinity M in Lemma 3.1 and $\bar I^\gamma_{\hat h}$ is the γ-Yamabe functional in (1-11). If $n > 4 + 2\gamma$, $\gamma \in (0, 1)$ and Lemma 3.2(1)–(6) hold, we have*

$$\begin{aligned}
&\bar I^\gamma_{\hat h}[\psi W_\epsilon] \le \Lambda^\gamma(\mathbb{S}^n, [g_c]) + \epsilon^4 \Lambda^\gamma(\mathbb{S}^n, [g_c])^{-\frac{n-2\gamma}{2\gamma}} \kappa_\gamma \, |\mathbb{S}^{n-1}| A_3 B_2 \\
&\qquad \times \big(-\|W\|^2 \mathcal{D}_1'(n,\gamma) + R_{;NN}[\bar g] \mathcal{D}_2'(n,\gamma) - (R_{ij}[\bar g])^2 \mathcal{D}_3'(n,\gamma)\big) + o(\epsilon^4), \qquad (3\text{-}14)
\end{aligned}$$

where $\Lambda^\gamma(\mathbb{S}^n,[g_c])$, κ_γ, A_3 *and* B_2 *are the positive constants introduced in* (1-16), (1-23) *and* (B-3), *respectively. Furthermore*

$$\mathcal{D}_1'(n,\gamma)=\frac{15n^4-120n^3+20n^2(17-2\gamma^2)-80n(5-2\gamma^2)+48(4-5\gamma^2+\gamma^4)}{480n(n-1)(n-4)(n-4-2\gamma)(n-4+2\gamma)(1-\gamma^2)}>0,$$
$$\mathcal{D}_2'(n,\gamma)=0 \quad \text{and} \quad \mathcal{D}_3'(n,\gamma)=\frac{5n^2-4n(13-2\gamma^2)+28(4-\gamma^2)}{5n(n-4)(n-4-2\gamma)(n-4+2\gamma)}. \tag{3-15}$$

Proof. By Lemmas 3.1 and 3.2(1)–(2), it holds that

$$\begin{aligned}\int_M(\psi W_\epsilon)^{p+1}\,dv_{\hat h} &= \int_{B^n(0,\eta_0)} w_\epsilon^{p+1}\big[1-\tfrac{1}{40}\big(R_{ij,kl}[\hat h]+\tfrac{2}{9}R_{miqj}[\hat h]R_{mkql}[\hat h]\big)x_ix_jx_kx_l+O(|\bar x|^5)\big]\,d\bar x+O(\epsilon^n)\\ &=\int_{\mathbb{R}^n} w_1^{p+1}\,d\bar x+o(\epsilon^4).\end{aligned}$$

Thus the conclusion follows from an easy estimate,

$$J^\gamma_{\hat h}[\psi W_\epsilon; X\setminus B^N_+(0,\eta_0)]=o(\epsilon^4)$$

with Lemmas 3.5 and B.6 at once. □

It is interesting to see that the quantity $R_{;NN}[\bar g](y)$ does not contribute to the existence of a least energy solution, since the coefficient of $R_{;NN}[\bar g](y)$, denoted by $\mathcal{D}_2'(n,\gamma)$, is always zero for any n and γ. Such a phenomenon has been already observed in the boundary Yamabe problem [Marques 2005]. We also note that the number $\mathcal{D}_3'(n,\gamma)$ has a nonnegative sign in some situations: when $n=7$ and $\gamma\in\big[\frac12,1\big)$, or $n\ge 8$ and $\gamma\in(0,1)$. In order to cover lower-dimensional cases, we need a more refined test function.

Let $y\in M$ be a point such that $W[\hat h](y)\neq 0$. Motivated by [Almaraz 2010], we define functions

$$\widetilde\Psi_\epsilon=\Psi_\epsilon(\bar x,x_N)=M_2R_{iNjN}[\bar g]x_ix_jx_N^2r^{-1}\partial_rW_\epsilon=\epsilon^2\cdot\epsilon^{-\frac{n-2\gamma}{2}}\widetilde\Psi_1(\epsilon^{-1}\bar x,\epsilon^{-1}x_N)$$

for some $M_2\in\mathbb{R}$ and

$$\widetilde\Phi_\epsilon:=\psi(W_\epsilon+\widetilde\Psi_\epsilon)\quad\text{in } X.$$

Proposition 3.7. *Suppose that* (X^{n+1},g^+) *is an asymptotically hyperbolic manifold. Moreover* $\hat h$ *is the representative of the conformal infinity* M *satisfying* (2-4) *and Lemma 3.2*(1)–(6). *If* $n>4+2\gamma$ *and* $\gamma\in(0,1)$, *we have*

$$\begin{aligned}\bar I^\gamma_{\hat h}[\widetilde\Phi_\epsilon]\le \Lambda^\gamma(\mathbb{S}^n,[g_c])+\epsilon^4\Lambda^\gamma(\mathbb{S}^n,[g_c])^{-\frac{n-2\gamma}{2\gamma}}\kappa_\gamma\,|\mathbb{S}^{n-1}|A_3B_2&\\ \times\big(-\|W\|^2\mathcal{D}_1(n,\gamma)+R_{;NN}[\bar g]\mathcal{D}_2(n,\gamma)-(R_{ij}[\bar g])^2\mathcal{D}_3(n,\gamma)\big)+o(\epsilon^4),&\end{aligned}\tag{3-16}$$

where

$$\mathcal{D}_1(n,\gamma)=\mathcal{D}_1'(n,\gamma),\qquad \mathcal{D}_2(n,\gamma)=0,$$

see (3-15) *for the definition of the positive constant* $\mathcal{D}_1'(n,\gamma)$, *and*

$$\mathcal{D}_3(n,\gamma)=\frac{25n^3-20n^2(9-\gamma^2)+100n(4-\gamma^2)-16(4-\gamma^2)^2}{5n(n-4-2\gamma)(n-4+2\gamma)\big(5n^2-4n(1+\gamma^2)-8(4-\gamma^2)\big)}.$$

Proof. Since $R_{NN}[\bar{g}](y) = 0$, we obtain

$$J^{\gamma}_{\hat{h}}[\widetilde{\Phi}_\epsilon; B^N_+(0,\eta_0)]$$
$$= J^{\gamma}_{\hat{h}}[\psi W_\epsilon; B^N_+(0,\eta_0)] + 2\int_{\mathbb{R}^N_+} x_N^{1-2\gamma}(\bar{g}^{ij} - \delta^{ij})\,\partial_i W_\epsilon \partial_j \widetilde{\Psi}_\epsilon\,dx + \int_{\mathbb{R}^N_+} x_N^{1-2\gamma}|\nabla\widetilde{\Psi}_\epsilon|^2\,dx + o(\epsilon^4). \quad (3\text{-}17)$$

Also a tedious computation with Lemmas 3.1 and 3.2(4) reveals that the second term of the right-hand side of (3-17) is equal to

$$\tfrac{2}{3}R_{ikjl}[\hat{h}]\int_{\mathbb{R}^N_+} x_N^{1-2\gamma}x_kx_l\,\partial_i W_\epsilon\partial_j\widetilde{\Psi}_\epsilon\,dx + 2R_{iNjN}[\bar{g}]\int_{\mathbb{R}^N_+} x_N^{3-2\gamma}\partial_i W_\epsilon\partial_j\widetilde{\Psi}_\epsilon\,dx + o(\epsilon^4)$$
$$= 0 + \epsilon^4 4M_2\left[\frac{1}{n}\mathcal{F}_3'' + \frac{1}{n(n+2)}(-\mathcal{F}_3'' + \mathcal{F}_7'')\right](R_{ij}[\bar{g}])^2 + o(\epsilon^4),$$

and it holds that

$$\int_{\mathbb{R}^N_+} x_N^{1-2\gamma}|\nabla\widetilde{\Psi}_\epsilon|^2\,dx = \epsilon^4\frac{2M_2^2}{n(n+2)}\big(\mathcal{F}_3'' - 2\mathcal{F}_7'' + \mathcal{F}_8'' + 4\mathcal{F}_6'' + 4\mathcal{F}_9'' + \mathcal{F}_{10}''\big)(R_{ij}[\bar{g}])^2 + o(\epsilon^4);$$

see (2-17) and (2-18). Here the constants $\mathcal{F}_1'', \dots, \mathcal{F}_{10}''$ are defined in Lemma B.6.

On the other hand, we have

$$J^{\gamma}_{\hat{h}}[\widetilde{\Phi}_\epsilon; X \setminus B^N_+(0,\eta_0)] = o(\epsilon^4),$$

and since $\widetilde{\Psi}_\epsilon = 0$ on M, the integral of $|\widetilde{\Phi}_\epsilon|^{p+1}$ over the boundary M does not contribute to the fourth-order term in the right-hand side of (3-16). By combining all information, employing Lemma B.6 and selecting the optimal $M_2 \in \mathbb{R}$, we complete the proof. □

One can verify that $\mathcal{D}_3(n,\gamma) > 0$ whenever $n > 4 + 2\gamma$ and $\gamma \in (0,1)$. Consequently we deduce Theorem 1.3 from Propositions 3.7 and 3.4.

4. Locally conformally flat or 2-dimensional conformal infinities

4A. ***Analysis of the Green's function.*** In this subsection, we prove Proposition 1.5. By Theorem A, solvability of problem (1-20) for each $y \in M$ is equivalent to the existence of a solution G^* to the equation

$$\begin{cases} -\operatorname{div}_{\bar{g}^*}\big((\rho^*)^{1-2\gamma}\nabla G^*(\cdot,y)\big) = 0 & \text{in } (X,\bar{g}^*),\\ \partial^{\gamma}_{\nu}G^*(\cdot,y) = \delta_y - Q^{\gamma}_{\hat{h}}G^*(\cdot,y) & \text{on } (M,\hat{h}),\end{cases}$$

and we have $|\bar{g}^*_{iN}| + |\bar{g}^*_{NN} - 1| = O(\rho^{2\gamma})$. We also recall [González and Qing 2013, Corollary 4.3] which states that if $\Lambda^{\gamma}(M,[\hat{h}]) > 0$, then M admits a metric $\hat{h}_0 \in [\hat{h}]$ such that $Q^{\gamma}_{\hat{h}_0} > 0$ on M. Thanks to the following lemma, it suffices to show Proposition 1.5 for $\hat{h}_0 \in [\hat{h}]$.

Lemma 4.1. *Let (X, g^+) be any conformally compact Einstein manifold with conformal infinity $(M,[\hat{h}])$, ρ the geodesic defining function of M in X and $\bar{g} = \rho^2 g^+$. For any positive smooth function w on M, define a new metric $\hat{h}_w = w^{4/(n-2\gamma)}\hat{h}$, denote the corresponding geodesic boundary-defining function* *by ρ_w and set $\bar{g}_w = \rho_w^2 g^+$. Suppose that $G = G(x, y)$ solves (1-20). Then the function*

$$G_w(x, y) := \left(\frac{\rho(x)}{\rho_w(x)}\right)^{\frac{n-2\gamma}{2}} w^{\frac{n+2\gamma}{n-2\gamma}}(y)\, G(x, y) \quad \text{for } (x, y) \in \overline{X} \times M,\ x \neq y$$

again satisfies (1-20) with $(\bar{g}_w, \hat{h}_w)$ and ρ_w substituted for $(\bar{g}, \hat{h})$ and ρ, respectively.

Proof. By (1-5), the first equality in (1-20) is re-expressed as

$$L_{\bar{g}}(\rho^{\frac{1-2\gamma}{2}} G(\cdot, y)) + \left(\gamma^2 - \tfrac{1}{4}\right)\rho^{-(\frac{3+2\gamma}{2})} G(\cdot, y) = 0 \quad \text{in } (X, \bar{g}), \tag{4-1}$$

where $L_{\bar{g}}$ is the conformal Laplacian in $(X, \bar{g})$ defined in (1-3). Therefore one observes from (1-1) that G_w is a solution of (4-1) if $\bar{g}$ and ρ are replaced with $\bar{g}_w$ and ρ_w, respectively. Also, since $w = (\rho_w/\rho)^{(n-2\gamma)/2}$ on M, we see

$$\begin{aligned} \partial_\nu^\gamma G_w(\cdot, y) = P^\gamma_{\hat{h}_w} G_w(\cdot, y) &= w^{\frac{n+2\gamma}{n-2\gamma}}(y)\, P^\gamma_{w^{\frac{4}{n-2\gamma}}\hat{h}}((\rho/\rho_w)^{\frac{n-2\gamma}{2}} G(\cdot, y)) \\ &= w^{\frac{n+2\gamma}{n-2\gamma}}(y)\, P^\gamma_{w^{\frac{4}{n-2\gamma}}\hat{h}}(w^{-1} G(\cdot, y)) = w^{\frac{n+2\gamma}{n-2\gamma}}(y)\, w^{-\frac{n+2\gamma}{n-2\gamma}} P^\gamma_{\hat{h}}(G(\cdot, y)) \\ &= w^{\frac{n+2\gamma}{n-2\gamma}}(y)\, w^{-\frac{n+2\gamma}{n-2\gamma}} \partial_\nu^\gamma(G(\cdot, y)) = w^{\frac{n+2\gamma}{n-2\gamma}}(y)\, w^{-\frac{n+2\gamma}{n-2\gamma}} \delta_y = \delta_y \quad \text{on } M, \end{aligned}$$

where we have applied Theorem A and (1-1) for the first, fourth and fifth equalities. □

For brevity, we write $\hat{h} = \hat{h}_0$, $\bar{g} = \bar{g}^*$, $\rho = \rho^*$ and $G = G^*$ here and henceforth. Further, recalling that $Q^\gamma_{\hat{h}} > 0$ on M, let us define a norm

$$\|U\|_{\mathcal{W}^{1,q}(X,\rho^{1-2\gamma})} = \left(\int_X \rho^{1-2\gamma} |\nabla U|_{\bar{g}}^q \, dv_{\bar{g}} + \int_M Q^\gamma_{\hat{h}} U^q \, dv_{\hat{h}}\right)^{\frac{1}{q}}$$

for any $q \geq 1$ and set a space $\mathcal{W}^{1,q}(X, \rho^{1-2\gamma})$ as the completion of $C_c^\infty(\overline{X})$ with respect to the above norm.

Given any bounded Radon measure f (such as the Dirac measures), a function $U \in \mathcal{W}^{1,q}(X, \rho^{1-2\gamma})$ is said to be a *weak solution* of

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^{1-2\gamma} \nabla U) = 0 & \text{in } (X, \bar{g}), \\ \partial_\nu^\gamma U + Q^\gamma_{\hat{h}} U = f & \text{on } (M, \hat{h}) \end{cases} \tag{4-2}$$

if it satisfies that

$$\int_X \rho^{1-2\gamma} \langle \nabla U, \nabla \Psi \rangle_{\bar{g}} \, dv_{\bar{g}} + \int_M Q^\gamma_{\hat{h}} U \Psi \, dv_{\hat{h}} = \int_M f \Psi \tag{4-3}$$

for any $\Psi \in C^1(\overline{X})$.

The $\mathcal{W}^{1,2}(X, \rho^{1-2\gamma})$-norm is equivalent to the standard weighted Sobolev norm $\|U\|_{W^{1,2}(X,\rho^{1-2\gamma})}$; see [Choi and Kim 2017, Lemma 3.1]. Thus for any fixed $f \in (H^\gamma(M))^*$, the existence and uniqueness of a solution $U \in W^{1,2}(X, \rho^{1-2\gamma})$ to (4-2) are guaranteed by the Riesz representation theorem.

Lemma 4.2. *Assume that $n > 2\gamma$, $f \in (H^{\gamma}(M))^*$ and $1 \le \alpha < \min\{\frac{n}{n-2\gamma}, \frac{2n+2}{2n+1}\}$. Then there exists a constant $C = C(\bar{X}, g^+, \rho, n, \gamma, \alpha)$ such that*

$$\|U\|_{\mathcal{W}^{1,\alpha}(X,\rho^{1-2\gamma})} \le C\|f\|_{L^1(M)} \tag{4-4}$$

for a weak solution $U \in W^{1,2}(X, \rho^{1-2\gamma})$ to (4-2). As a result, if f is the Dirac measure δ_y at $y \in M$, then (4-2) has a unique nonnegative weak solution $G(\cdot, y) \in \mathcal{W}^{1,\alpha}(X, \rho^{1-2\gamma})$.

Proof. Step 1: We are going to verify estimate (4-4) by suitably modifying the argument in [Brézis and Strauss 1973, Section 5]. To this aim, we consider the *formal adjoint* of (4-2): Given any $h_0 \in L^q(M)$ and $H_1, \dots, H_N \in L^q(X, \rho^{1-2\gamma})$ for some $q > \max\{\frac{n}{2\gamma}, 2(n+1)\}$, we study a function V such that

$$\int_X \rho^{1-2\gamma}\langle\nabla V, \nabla\Psi\rangle_{\bar{g}}\, dv_{\bar{g}} + \int_M Q^{\gamma}_{\hat{h}} V\Psi\, dv_{\hat{h}} = \int_M h_0\Psi\, dv_{\hat{h}} + \sum_{a=1}^{N}\int_X \rho^{1-2\gamma} H_a\partial_a\Psi\, dv_{\bar{g}} \tag{4-5}$$

for any $\Psi \in C^1(\bar{X})$. Indeed, by the Lax–Milgram theorem, (4-5) has a unique solution $V \in W^{1,2}(X, \rho^{1-2\gamma})$. Moreover, employing Moser's iteration technique, we observe that V satisfies

$$\|V\|_{L^\infty(M)} + \|V\|_{L^\infty(X)} \le C\left(\|h_0\|_{L^q(M)} + \sum_{a=1}^{N}\|H_a\|_{L^q(X,\rho^{1-2\gamma})}\right). \tag{4-6}$$

Therefore taking $\Psi = V$ in (4-3) and U in (4-5) respectively (which is allowed thanks to the density argument) and then employing (4-6), we find

$$\begin{aligned}\int_M Uh_0\, dv_{\hat{h}} + \sum_{a=1}^{N}\int_X \rho^{1-2\gamma}\partial_a U H_a\, dv_{\bar{g}} &= \int_M fV\, dv_{\hat{h}} \le \|f\|_{L^1(M)}\|V\|_{L^\infty(M)} \\ &\le C\|f\|_{L^1(M)}\left(\|h_0\|_{L^q(M)} + \sum_{a=1}^{N}\|H_a\|_{L^q(X,\rho^{1-2\gamma})}\right).\end{aligned}$$

This implies the validity of (4-4) with $\alpha = q'$, where q' designates the Hölder conjugate of q.

Step 2: Assume now that $f = \delta_y$ for some $y \in M$. Then one is capable of constructing a sequence $\{f_m\}_{m\in\mathbb{N}} \subset C^1(M)$ with an approximation to the identity or a mollifier so that $f_m \ge 0$ on M,

$$\sup_{m\in\mathbb{N}}\|f_m\|_{L^1(M)} \le C, \quad f_m \to 0 \text{ in } C^1_{\mathrm{loc}}(M\setminus\{y\}) \quad \text{and} \quad f_m \rightharpoonup \delta_y \text{ in the distributional sense.}$$

Denote by $\{U_m\}_{m\in\mathbb{N}} \subset W^{1,2}(X, \rho^{1-2\gamma})$ a sequence of the corresponding weak solutions to (4-2). By (4-4) and elliptic regularity, there exist a function $G(\cdot, y)$ and a number $\varepsilon_0 \in (0, 1)$ such that $U_m \rightharpoonup G(\cdot, y)$ weakly in $\mathcal{W}^{1,\alpha}(X, \rho^{1-2\gamma})$ and $U_m \to G(\cdot, y)$ in $C^{\varepsilon_0}_{\mathrm{loc}}(\bar{X}\setminus\{y\})$. It is a simple task to confirm that $G(\cdot, y)$ satisfies (4-3).

Also, putting $(U_m)_- \in W^{1,2}(X, \rho^{1-2\gamma})$ into (4-3) yields $U_m \ge 0$ in X, which in turn gives $G(\cdot, y) \ge 0$ in X. Finally, the uniqueness of $G(\cdot, y)$ comes as a consequence of (4-4). □

Completion of the proof of Proposition 1.5. The existence and nonnegativity of the Green's function G is deduced in the previous lemma. Owing to Hopf's lemma, see [González and Qing 2013, Theorem 3.5], G is positive on the compact manifold $\bar{X}$. Recall that the coercivity of (4-3) implies the uniqueness of G. □

4B. *Locally conformally flat case.* This subsection is devoted to provide the proof of Theorem 1.7, which treats locally conformally flat conformal infinities M.

Pick any point $y \in M$. Since it is supposed to be locally conformally flat, we can assume that y is the origin in $\mathbb{R}^n$ and identify a neighborhood $\mathcal{U}$ of y in M with a Euclidean ball $B^n(0, \varrho_1)$ for some $\varrho_1 > 0$ small, namely, $\hat{h}_{ij} = \delta_{ij}$ in $\mathcal{U} = B^n(0, \varrho_1)$. Write x_N to denote the geodesic defining function ρ for the boundary M near y. Then we have smooth symmetric n-tensors $h^{(1)}, \ldots, h^{(n-1)}$ on $B^n(0, \varrho_1)$ such that

$$\bar{g} = h_{x_N} \oplus dx_N^2, \quad \text{where } (h_{x_N})_{ij}(\bar{x}, x_N) = \delta_{ij} + \sum_{m=1}^{n-1} h_{ij}^{(m)}(\bar{x}) x_N^m + O(x_N^n), \tag{4-7}$$

for $(\bar{x}, x_N) \in \mathcal{R}^N(\varrho_1, \varrho_2) := B^n(0, \varrho_1) \times [0, \varrho_2) \subset \overline{X}$, where $\varrho_2 > 0$ is a number small enough. In fact, as the next lemma indicates, the local conformal flatness on M and the assumption that X is Poincaré–Einstein together imply that all low-order tensors $h^{(m)}$ should vanish. In particular, the second fundamental form $h^{(1)}$ on M (up to a constant factor) is 0, which implies Remark 1.8(2).

Lemma 4.3. *If (X, g^+) is Poincaré–Einstein, we have $h^{(m)} = 0$ in (4-7) for each $m = 1, \ldots, n-1$.*

Proof. Follow the argument of [Graham 2000], which starts from the paragraph after (2.4). Due to the condition $\hat{h}_{ij} = \delta_{ij}$, the right-hand side of (2.6) in that paper becomes 0, from which one can deduce the result. □

Therefore (4-7) is reduced to

$$\bar{g}_{ij}(\bar{x}, x_N) = \delta_{ij} + O(x_N^n) \quad \text{and} \quad |\bar{g}| = 1 + O(x_N^n) \quad \text{for } (\bar{x}, x_N) \in \mathcal{R}^N(\varrho_1, \varrho_2) \subset \overline{X}. \tag{4-8}$$

Choose any smooth function $\chi : [0, \infty) \to [0, 1]$ such that $\chi(t) = 1$ for $0 \le t \le 1$ and $\chi(t) = 0$ for $t \ge 2$. Recall the bubble W_ϵ defined in (1-13) and (1-14), the Green's function $G(\cdot, 0)$, its regular part Ψ given in (1-21), and the numbers $\alpha_{n,\gamma}$ and $g_{n,\gamma}$ given in (1-23). Then we construct a nonnegative, continuous and piecewise smooth function $\Phi_{\epsilon,\varrho_0}$ on $\overline{X}$ by

$$\Phi_{\epsilon,\varrho_0}(x) = \begin{cases} W_\epsilon(x) & \text{if } x \in X \cap B^N(0, \varrho_0), \\ V_{\epsilon,\varrho_0}(x)\big(G(x,0) - \chi_{\varrho_0}(x)\Psi(x)\big) & \text{if } x \in X \cap (B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0)), \\ V_{\epsilon,\varrho_0}(x) G(x,0) & \text{if } x \in X \setminus B^N(0, 2\varrho_0), \end{cases} \tag{4-9}$$

where $0 < \epsilon \ll \varrho_0 \le \min\{\varrho_1, \varrho_2\}/5$ sufficiently small, $\chi_{\varrho_0}(x) := \chi(|x|/\varrho_0)$ and

$$V_{\epsilon,\varrho_0}(x) := \left[\alpha_{n,\gamma}\left(\frac{\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{n-2\gamma}}\right) + \chi_{\varrho_0}(x)\left(W_\epsilon(x) - \alpha_{n,\gamma}\frac{\epsilon^{\frac{n-2\gamma}{2}}}{|x|^{n-2\gamma}}\right)\right] \cdot (g_{n,\gamma}\, \varrho_0^{-(n-2\gamma)} + A)^{-1}. \tag{4-10}$$

We remark that the main block V_{ϵ,ϱ_0} of the test function $\Phi_{\epsilon,\varrho_0}$ is different from the function W in (4.2) of [Escobar 1992a], but they share common characteristics such as decay properties, as proved in the next lemma.

Lemma 4.4. *There are constants $C, \eta_1, \eta_2 > 0$ depending only on n and γ such that*

$$|V_{\epsilon,\varrho_0}(x)| \le C\epsilon^{\frac{n-2\gamma}{2}} \quad \textit{for any } x \in X \setminus B^N(0, \varrho_0) \tag{4-11}$$

and

$$|\nabla_{\bar{x}} V_{\epsilon,\varrho_0}(x)| \le C\varrho_0^{-\eta_1} \epsilon^{\frac{n-2\gamma+2\eta_2}{2}} \quad \textit{and} \quad |\partial_N V_{\epsilon,\varrho_0}(x)| \le C\rho_0^{-\eta_1}(\epsilon^{\frac{n-2\gamma+2\eta_2}{2}} + x_N^{2\gamma-1}\epsilon^{\frac{n+2\gamma}{2}}) \tag{4-12}$$

for $x = (\bar{x}, x_N) \in X \cap (B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0))$. *Also we have* $\nabla V_{\epsilon,\varrho_0} = 0$ *in* $X \setminus B^N(0, 2\varrho_0)$.

Proof. We observe from (A-1) and (4-10) that

$$|V_{\epsilon,\varrho_0}(x)| \le C\varrho_0^{n-2\gamma}\left[\left(\frac{\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{n-2\gamma}}\right) + \left|W_\epsilon(x) - \alpha_{n,\gamma}\frac{\epsilon^{\frac{n-2\gamma}{2}}}{|x|^{n-2\gamma}}\right|\right] \le C\left(\epsilon^{\frac{n-2\gamma}{2}} + \frac{\epsilon^{\frac{n-2\gamma+2\vartheta_1}{2}}}{\varrho_0^{\vartheta_1}}\right) \le C\epsilon^{\frac{n-2\gamma}{2}}$$

for all $\varrho_0 \le |x| \le 2\varrho_0$ and some $\vartheta_1 \in (0, 1)$, so (4-11) follows. One can derive (4-12) by making the use of (A-1), (A-3) and (A-4). We leave the details to the reader. □

Now we assert the following proposition, which suffices to conclude that the fractional Yamabe problem is solvable in this case.

Proposition 4.5. *For* $n > 2\gamma$ *and* $\gamma \in (0, 1)$, *let* (X^{n+1}, g^+) *be a Poincaré–Einstein manifold with conformal infinity* $(M^n, [\hat{h}])$ *such that* (1-9) *has the validity. Assume also that* M *is locally conformally flat. If* Conjecture 1.6 *holds and* $A > 0$, *then*

$$0 < \bar{I}^\gamma_{\hat{h}}[\Phi_{\epsilon,\varrho_0}] < \Lambda^\gamma(\mathbb{S}^n, [g_c]),$$

where $\bar{I}^\gamma_{\hat{h}}$ *is the* γ*-Yamabe functional defined in* (1-11), *and* $\Lambda^\gamma(\mathbb{S}^n, [g_c]) > 0$ *is the constant defined in* (1-16).

Proof. The proof is divided into three steps.

<u>Step 1</u>: estimation in $X \cap B^N(0, \varrho_0)$. Applying (1-15), (1-16), (4-8), (A-3), (A-4), Lemma A.2 and integrating by parts, we obtain

$$\begin{aligned}\kappa_\gamma \int_{X\cap B^N(0,\varrho_0)} & x_N^{1-2\gamma}|\nabla W_\epsilon|^2_{\bar{g}}\, dv_{\bar{g}} \\ &\le \Lambda^\gamma(\mathbb{S}^n, [g_c])\left(\int_{B^n(0,\varrho_0)} w_\epsilon^{p+1}\, d\bar{x}\right)^{\frac{n-2\gamma}{n}} \\ &\quad + \kappa_\gamma \int_{X\cap\partial B^N(0,\varrho_0)} x_N^{1-2\gamma} W_\epsilon \frac{\partial W_\epsilon}{\partial \nu}\, dS + \underbrace{O\left(\int_{B^n(0,\varrho_0)} x_N^{n+1-2\gamma}|\nabla W_\epsilon|^2\, d\bar{x}\right)}_{=O(\varrho_0^{2\gamma}\epsilon^{n-2\gamma})},\end{aligned} \tag{4-13}$$

where ν is the outward unit normal vector and dS is the Euclidean surface measure. On the other hand, if we write $g^+ = x_N^{-2}(dx_N^2 + h_{x_N})$, then

$$E(x_N) = -\tfrac{1}{4}(n-2\gamma)x_N^{-2\gamma}\operatorname{tr}(h_{x_N}^{-1}\partial_N h_{x_N}) = O(x_N^{n-1-2\gamma}) \tag{4-14}$$

in $X \cap B^N(0, 2\varrho_0)$; see (2-5). Therefore

$$\kappa_\gamma \int_{X\cap B^N(0,\varrho_0)} E(x_N) W_\epsilon^2\, dv_{\bar{g}} = O(\varrho_0^{2\gamma}\epsilon^{n-2\gamma}). \tag{4-15}$$

Step 2: estimation in $X \setminus B^N(0, \varrho_0)$. By its own definition (4-9) of the test function $\Phi_{\epsilon,\varrho_0}$, its energy on $\overline{X}$ can be evaluated as

$$\begin{aligned}
&\int_{X\setminus B^N(0,\varrho_0)} (\rho^{1-2\gamma}|\nabla\Phi_{\epsilon,\varrho_0}|_{\bar g}^2 + E(\rho)\Phi_{\epsilon,\varrho_0}^2)\,dv_{\bar g} \\
&\quad = \int_{X\setminus B^N(0,\varrho_0)} \big(\rho^{1-2\gamma}\langle\nabla(V_{\epsilon,\varrho_0}^2 G), \nabla G\rangle_{\bar g} + E(\rho)V_{\epsilon,\varrho_0}^2 G^2 + \rho^{1-2\gamma}|\nabla V_{\epsilon,\varrho_0}|^2 (G-\chi_{\varrho_0}\Psi)^2\big)\,dv_{\bar g} \\
&\qquad + \int_{X\cap(B^N(0,2\varrho_0)\setminus B^N(0,\varrho_0))} \rho^{1-2\gamma}\big(\tfrac12\big\langle\nabla V_{\epsilon,\varrho_0}^2, \nabla(-2G\chi_{\varrho_0}\Psi + \chi_{\varrho_0}^2\Psi^2)\big\rangle_{\bar g}\big)\,dv_{\bar g} \\
&\qquad\quad + \int_{X\cap(B^N(0,2\varrho_0)\setminus B^N(0,\varrho_0))} \rho^{1-2\gamma} V_{\epsilon,\varrho_0}^2 (|\nabla(\chi_{\varrho_0}\Psi)|^2 - 2\langle\nabla G, \nabla(\chi_{\varrho_0}\Psi)\rangle_{\bar g})\,dv_{\bar g} \\
&\qquad\qquad + \int_{X\cap(B^N(0,2\varrho_0)\setminus B^N(0,\varrho_0))} E(\rho) V_{\epsilon,\varrho_0}^2 (\chi_{\varrho_0}^2\Psi^2 - 2G\chi_{\varrho_0}\Psi)\,dv_{\bar g},
\end{aligned}$$

where $G = G(\cdot\,, 0)$. From (1-20), (1-22), (4-14) and Lemma 4.4, we see that

$$\begin{aligned}
&\kappa_\gamma \int_{X\setminus B^N(0,\varrho_0)} (\rho^{1-2\gamma}|\nabla\Phi_{\epsilon,\varrho_0}|_{\bar g}^2 + E(\rho)\Phi_{\epsilon,\varrho_0}^2)\,dv_{\bar g} \\
&\quad \le -\kappa_\gamma \int_{X\cap\partial B^N(0,\varrho_0)} x_N^{1-2\gamma} V_{\epsilon,\varrho_0}^2 G\,\frac{\partial G}{\partial\nu}(1+O(x_N^n))\,dS + C\epsilon^{n-2\gamma+2\eta_2}\varrho_0^{-(n-2\gamma-2+2\eta_1)} \\
&\qquad + C\epsilon^{n-2\gamma+\eta_2}\varrho_0^{\min\{1,2\gamma\}+1-\eta_1} + C\epsilon^n\varrho_0^{\min\{1,2\gamma\}+2\gamma-\eta_1} + C\epsilon^{n-2\gamma}\varrho_0^{\min\{1,2\gamma\}},
\end{aligned} \tag{4-16}$$

where $C > 0$ depends only on n, γ, ϱ_1 and ϱ_2. For instance, we have

$$\begin{aligned}
&\int_{X\setminus B^N(0,\varrho_0)} \rho^{1-2\gamma}|\nabla V_{\epsilon,\varrho_0}|^2 (G-\chi_{\varrho_0}\Psi)^2\,dv_{\bar g} \\
&\quad \le C\varrho_0^{-2\eta_1} \int_{B^N(0,2\varrho_0)\setminus B^N(0,\varrho_0)} x_N^{1-2\gamma}(\epsilon^{n-2\gamma+2\eta_2} + x_N^{2(2\gamma-1)}\epsilon^{n+2\gamma})\cdot\left(\frac{1}{|x|^{2(n-2\gamma)}}+1\right)dx \\
&\quad \le C\big(\epsilon^{n-2\gamma+2\eta_2}\varrho_0^{-(n-2\gamma-2+2\eta_1)} + \epsilon^{n+2\gamma}\varrho_0^{-n+6\gamma}|\log\varrho_0|\big) \le C\epsilon^{n-2\gamma+2\eta_2}\varrho_0^{-(n-2\gamma-2+2\eta_1)}
\end{aligned}$$

for $0 < \epsilon \ll \varrho_0$ small. The other terms can be managed in a similar manner.

Step 3: conclusion. By combining (4-13), (4-15) and (4-16), we deduce

$$\begin{aligned}
&\kappa_\gamma \int_X (\rho^{1-2\gamma}|\nabla\Phi_{\epsilon,\varrho_0}|_{\bar g}^2 + E(\rho)\Phi_{\epsilon,\varrho_0}^2)\,dv_{\bar g} \\
&\quad \le \Lambda^\gamma(\mathbb{S}^n,[g_c])\left(\int_{B^n(0,\varrho_0)} w_\epsilon^{p+1}\,d\bar x\right)^{\frac{n-2\gamma}{n}} + \kappa_\gamma \int_{X\cap\partial B^N(0,\varrho_0)} \underbrace{x_N^{1-2\gamma}\left(W_\epsilon\frac{\partial W_\epsilon}{\partial\nu} - V_{\epsilon,\varrho_0}^2 G\,\frac{\partial G}{\partial\nu}\right)}_{=:I}\,dS \\
&\qquad + C\epsilon^{n-2\gamma}\varrho_0^{\min\{1,2\gamma\}}.
\end{aligned} \tag{4-17}$$

Let us compute the integral of I over the boundary $X \cap \partial B^N(0, \varrho_0)$ in the right-hand side of (4-17). Because of Lemma A.1 and (1-22), one has

$$\begin{aligned}\frac{\partial W_\epsilon}{\partial \nu} - V_{\epsilon,\varrho_0}\frac{\partial G}{\partial \nu} &\le -\frac{\alpha_{n,\gamma}(n-2\gamma)\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{n-2\gamma+1}} + (g_{n,\gamma}\,\varrho_0^{-(n-2\gamma)} + A)^{-1}\frac{\alpha_{n,\gamma}\, g_{n,\gamma}\,(n-2\gamma)\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{2(n-2\gamma)+1}} \\ &\qquad + C\epsilon^{\frac{n-2\gamma}{2}+\vartheta_1}\varrho_0^{-(n-2\gamma+1+\vartheta_1)} + C\epsilon^{\frac{n-2\gamma}{2}}\varrho_0^{\min\{0,2\gamma-1\}} \\ &\le -\alpha_{n,\gamma}\, g_{n,\gamma}^{-1}\,(n-2\gamma)A\frac{\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0} + C\epsilon^{\frac{n-2\gamma}{2}}\varrho_0^{\min\{0,2\gamma-1\}} + C\epsilon^{\frac{n-2\gamma}{2}+\vartheta_1}\varrho_0^{-(n-2\gamma+1+\vartheta_1)}\end{aligned}$$

on $\{|x| = \varrho_0\}$ for some $\vartheta_1 \in (0,1)$. Therefore using the fact that $W_1(x) \ge \frac{1}{2}\alpha_{n,\gamma}\epsilon^{\frac{n-2\gamma}{2}}\varrho_0^{-(n-2\gamma)}$ on $\{|x| = \varrho_0\}$, we discover

$$\begin{aligned}\int_{X\cap\partial B^N(0,\varrho_0)} I\, dS &= \int_{X\cap\partial B^N(0,\varrho_0)} x_N^{1-2\gamma}\left[W_\epsilon\left(\frac{\partial W_\epsilon}{\partial\nu} - V_{\epsilon,\varrho_0}\frac{\partial G}{\partial \nu}\right) - V_{\epsilon,\varrho_0}^2\frac{\partial G}{\partial\nu}\Psi\right] dS \\ &\le -\frac{\alpha_{n,\gamma}^2}{g_{n,\gamma}}\left(\frac{n-2\gamma}{4}\right)\left(\int_{\partial B^N(0,1)} x_N^{1-2\gamma}\, dS\right)A\epsilon^{n-2\gamma} + C\epsilon^{n-2\gamma}\varrho_0^{\min\{1,2\gamma\}} \\ &\qquad + C\epsilon^{n-2\gamma+\vartheta_1}\varrho_0^{-(n-2\gamma+\vartheta_1)}.\end{aligned}$$

Now the previous estimate, (4-17), (1-16) and the assumption $A > 0$ yield that

$$\begin{aligned}\bar{I}_{\hat{h}}^{\gamma}[\Phi_{\epsilon,\varrho_0}] &\le \Lambda^\gamma(\mathbb{S}^n,[g_c]) - \frac{\alpha_{n,\gamma}^2}{g_{n,\gamma}}\left(\frac{S_{n,\gamma}}{\kappa_\gamma}\right)^{\frac{n-2\gamma}{2\gamma}}\left(\tfrac{1}{8}(n-2\gamma)\right)\cdot\frac{|\mathbb{S}^{n-1}|}{2}B\left(1-\gamma,\tfrac{1}{2}n\right)\cdot A\epsilon^{n-2\gamma} \\ &\qquad + C\epsilon^{n-2\gamma}\varrho_0^{\min\{1,2\gamma\}} + C\epsilon^{n-2\gamma+\vartheta_1}\varrho_0^{-(n-2\gamma+\vartheta_1)} \\ &< \Lambda^\gamma(\mathbb{S}^n,[g_c]),\end{aligned}$$

where B is the beta function. Additionally the last strict inequality holds for $0 < \epsilon \ll \varrho_0$ small enough. □

4C. ***2-dimensional case.*** We are now led to treat the case when $(M, [\hat{h}])$ is a 2-dimensional closed manifold.

Fix an arbitrary point $p \in M$ and let $\bar{x} = (x_1, x_2)$ be normal coordinates at p. Since X is Poincaré–Einstein, it holds that $h^{(1)} = 0$ in (4-7), whence we have

$$\bar{g}_{ij}(\bar{x}, x_N) = \delta_{ij} + O(|x|^2) \quad\text{and}\quad |\bar{g}| = 1 + O(|x|^2) \quad \text{for } (\bar{x}, x_N) \in \mathcal{R}^N(\varrho_1,\varrho_2) \subset \bar{X}, \tag{4-18}$$

where the rectangle $\mathcal{R}^N(\varrho_1,\varrho_2)$ is defined in the line following (4-7).

With Proposition B in the Introduction, the next result will give the validity of Theorem 1.7 if $n = 2$.

Proposition 4.6. *For $\gamma \in (0,1)$, let (X^3, g^+) be a Poincaré–Einstein manifold with conformal infinity $(M^2, [\hat{h}])$ such that (1-9) holds. If Conjecture 1.6 holds and $A > 0$, then*

$$0 < \bar{I}_{\hat{h}}^{\gamma}[\Phi_{\epsilon,\varrho_0}] < \Lambda^\gamma(\mathbb{S}^2,[g_c])$$

for the test function $\Phi_{\epsilon,\varrho_0}$ introduced in (4-9).

Proof. We compute the error in $X \cap B^N_+(0, \varrho_0)$ due to the metric. As in (4-13) and (4-15), one has

$$\int_{X\cap B^N(0,\varrho_0)} x_N^{1-2\gamma}|\nabla W_\epsilon|_{\bar g}^2\, dv_{\bar g} = \int_{X\cap B^N(0,\varrho_0)} x_N^{1-2\gamma}|\nabla W_\epsilon|^2\, dx + \underbrace{O\left(\int_{X\cap B^N(0,\varrho_0)} x_N^{1-2\gamma}|x|^2|\nabla W_\epsilon|^2\, dx\right)}_{=O(\varrho_0^{2\gamma}\epsilon^{2-2\gamma})}$$

and

$$\int_{X\cap B^N(0,\varrho_0)} E(x_N)W_\epsilon^2\, dv_{\bar g} = O\left(\int_{X\cap B^N(0,\varrho_0)} x_N^{1-2\gamma} W_\epsilon^2\, dx\right) = O(\varrho_0^{2\gamma}\epsilon^{2-2\gamma})$$

from (4-18). Therefore the error arising from the metric is ignorable, and the same argument in proof of Proposition 4.5 works. □

Appendix A: Expansion of the standard bubble $W_{1,0}$ near infinity

This appendix is devoted to finding expansions of the function $W_1 = W_{1,0}$, defined in (1-13), and its derivatives near infinity. Specifically we improve [Choi and Kim 2017, Lemma A.2] by pursuing a new approach based on conformal properties of W_1.

For the functions W_1 and $x \cdot \nabla W_1$, we have:

Lemma A.1. *Suppose that $n > 2\gamma$ and $\gamma \in (0, 1)$. For any fixed large number $R_0 > 0$, we have*

$$\left|W_1(x) - \frac{\alpha_{n,\gamma}}{|x|^{n-2\gamma}}\right| + \left|x\cdot\nabla W_1(x) + \frac{\alpha_{n,\gamma}(n-2\gamma)}{|x|^{n-2\gamma}}\right| \le \frac{C}{|x|^{n-2\gamma+\vartheta_1}} \tag{A-1}$$

for $|x| \ge R_0$, where numbers $\vartheta_1 \in (0, 1)$ and $C > 0$ rely only on n, γ and R_0.

Proof. Given any function F in $\mathbb{R}^N_+$, let F^* be its fractional Kelvin transform defined as

$$F^*(x) = \frac{1}{|x|^{n-2\gamma}} F\left(\frac{x}{|x|^2}\right) \quad \text{for } x \in \mathbb{R}^N_+.$$

Then it is known that $W_1^* = W_1$. Let us claim that $(x\cdot\nabla W_1)^*(0) = -\alpha_{n,\gamma}(n-2\gamma)$ and $(x\cdot\nabla W_1)^*$ is C^∞ in the $\bar x$-variable and Hölder continuous in the x_N-variable. Since

$$x_N^{2-2\gamma}\partial_{NN}W_1 = -(1-2\gamma)x_N^{1-2\gamma}\partial_N W_1 - x_N^{2-2\gamma}\Delta_{\bar x}W_1 \quad \text{in } \mathbb{R}^N_+,$$

we have

$$\begin{cases} -\operatorname{div}\left(x_N^{1-2\gamma}\nabla(x\cdot\nabla W_1)\right) = 0 & \text{in } \mathbb{R}^N_+,\\ \partial_\nu^\gamma(x\cdot\nabla W_1) = \sum_{i=1}^n x_i\partial_{x_i}\partial_\nu^\gamma W_1 + \partial_\nu^\gamma W_1 - \lim_{x_N\to 0} x_N^{2-2\gamma}\partial_{NN}W_1 & \\ \qquad\qquad\qquad = p\sum_{i=1}^n x_i\partial_{x_i}(w_1^p) + 2\gamma w_1^p & \text{on } \mathbb{R}^n. \end{cases}$$

Employing [Fall and Weth 2012, Proposition 2.6; Caffarelli and Silvestre 2007] and doing some computations, we obtain that

$$\begin{cases} -\operatorname{div}(x_N^{1-2\gamma}\nabla(x\cdot\nabla W_1)^*) = 0 & \text{in } \mathbb{R}^N_+,\\ \partial_\nu^\gamma(x\cdot\nabla W_1)^* = (-\Delta)^\gamma(x\cdot\nabla W_1)^* = \alpha_{n,\gamma}^p\left(\dfrac{2\gamma|\bar x|^2 - n}{(1+|\bar x|^2)^{\frac{n+2\gamma+2}{2}}}\right) & \text{on } \mathbb{R}^n. \end{cases}$$

Therefore $(x\cdot\nabla W_1)^*$ has regularity stated above, and according to Green's representation formula,

$$(x\cdot\nabla W_1)^*(0)=\alpha^p_{n,\gamma}g_{n,\gamma}\int_{\mathbb{R}^n}\frac{1}{|\bar y|^{n-2\gamma}}\left(\frac{2\gamma|\bar y|^2-n}{(1+|\bar y|^2)^{\frac{n+2\gamma+2}{2}}}\right)d\bar y=-\alpha_{n,\gamma}(n-2\gamma).$$

This proves the assertion.

Now we can check (A-1) with the above observations. By standard elliptic theory, there exist constants $c_1,\dots,c_N>0$ such that

$$\left|W_1^*(x)-\alpha_{n,\gamma}\right|+\left|(x\cdot\nabla W_1)^*(x)+\alpha_{n,\gamma}(n-2\gamma)\right|\le\sum_{i=1}^n c_i|x_i|+c_Nx_N^{\vartheta_1}\tag{A-2}$$

for any $|x|\le R_0^{-1}$ and some $\vartheta_1\in(0,1)$. Hence, by taking the Kelvin transform in (A-2), we see that the desired inequality (A-1) is valid for all $|x|\ge R_0$. □

Additionally we have the following decay estimate of the derivatives of W_1.

Lemma A.2. *Assume that $n>2\gamma$ and $\gamma\in(0,1)$. For any fixed large number $R_0>0$, there exist constants $C>0$ and $\vartheta_2\in(0,\min\{1,2\gamma\})$ depending only on n, γ and R_0 such that*

$$\left|\nabla_{\bar x}W_1(x)+\frac{\alpha_{n,\gamma}(n-2\gamma)\bar x}{|x|^{n-2\gamma+2}}\right|\le\frac{C}{|x|^{n-2\gamma+1+\vartheta_2}}\tag{A-3}$$

and

$$\left|\partial_NW_1(x)+\frac{\alpha_{n,\gamma}(n-2\gamma)x_N}{|x|^{n-2\gamma+2}}\right|\le C\left(\frac{1}{|x|^{n-2\gamma+2}}+\frac{x_N^{2\gamma-1}}{|x|^{n+2\gamma}}\right)\tag{A-4}$$

for $|x|\ge R_0$.

Proof. The precise values of the constants $p_{n,\gamma}$, $\alpha_{n,\gamma}$ and κ_γ, which will appear during the proof, are found in (1-23).

Step 1: By (1-13), (1-14) and Taylor's theorem, it holds that

$$\begin{aligned}\partial_iW_1(x)&=p_{n,\gamma}\int_{\mathbb{R}^n}\frac{1}{(|\bar y|^2+1)^{\frac{n+2\gamma}{2}}}\,\partial_iw_1(\bar x-x_N\bar y)\,d\bar y\\&=p_{n,\gamma}\int_{\mathbb{R}^n}\frac{1}{(|\bar y|^2+1)^{\frac{n+2\gamma}{2}}}\left[\partial_iw_1(-x_N\bar y)+\partial_{ij}w_1(-x_N\bar y)x_j+O(|\bar x|^2)\right]d\bar y\\&=p_{n,\gamma}\int_{\mathbb{R}^n}\frac{1}{(|\bar y|^2+1)^{\frac{n+2\gamma}{2}}}\left[\partial_{ii}w_1(0)x_i+O((x_N|\bar y|)^{\vartheta_2}|\bar x|)+O(|\bar x|^2)\right]d\bar y\\&=-\alpha_{n,\gamma}(n-2\gamma)x_i+O(|x|^{1+\vartheta_2})\end{aligned}$$

for $|x|\le R_0^{-1}$. Here we also used the facts that the $C^2(\mathbb{R}^n)$-norm of w_1 and the $C^{\vartheta_2}(\mathbb{R}^n)$-norm of $\partial_{ij}w_1$ are bounded for some $\vartheta_2\in(0,\min\{1,2\gamma\})$. On the other hand, the uniqueness of the γ-harmonic extension yields that $(\partial_iW_1)^*=\partial_iW_1$ for $i=1,\dots,n$. Therefore

$$\left|\partial_iW_1(x)+\frac{\alpha_{n,\gamma}(n-2\gamma)x_i}{|x|^{n-2\gamma+2}}\right|=\left|(\partial_iW_1)^*(x)+\alpha_{n,\gamma}(n-2\gamma)x_i^*\right|\le C(|x|^{1+\vartheta_2})^*\le\frac{C}{|x|^{n+2\gamma+1+\vartheta_2}}$$

for $|x|\ge R_0$, which is the desired inequality (A-3).

Step 2: If $\gamma = \frac{1}{2}$, it is known that

$$W_1(\bar{x}, x_N) = \alpha_{n,\frac{1}{2}} \left(\frac{1}{|\bar{x}|^2 + (x_n + 1)^2} \right)^{\frac{n-1}{2}} \quad \text{for all } (\bar{x}, x_N) \in \mathbb{R}^N_+,$$

from which (A-4) follows. Therefore it is sufficient to consider when $\gamma \in (0, 1) \setminus \{\frac{1}{2}\}$. In light of duality [Caffarelli and Silvestre 2007, Section 2.3], we have

$$\begin{cases} -\operatorname{div}(x_N^{1-2(1-\gamma)} \nabla (x_N^{1-2\gamma} \partial_N W_1)) = 0 & \text{in } \mathbb{R}^N_+, \\ x_N^{1-2\gamma} \partial_N W_1 = -\kappa_\gamma^{-1} w_1^p & \text{on } \mathbb{R}^n. \end{cases}$$

Hence if we define

$$F^{**}(x) = \frac{1}{|x|^{n-2(1-\gamma)}} F\left(\frac{x}{|x|^2} \right) \quad \text{for } x \in \mathbb{R}^N_+$$

for an arbitrary function F in $\mathbb{R}^N_+$, then

$$\begin{cases} -\operatorname{div}\big(x_N^{1-2(1-\gamma)} \nabla (x_N^{1-2\gamma} \partial_N W_1)^{**}\big) = 0 & \text{in } \mathbb{R}^N_+, \\ (x_N^{1-2\gamma} \partial_N W_1)^{**} = -\alpha_{n,\gamma}^p \kappa_\gamma^{-1} \dfrac{|\bar{x}|^2}{(1 + |\bar{x}|^2)^{\frac{n+2\gamma}{2}}} & \text{on } \mathbb{R}^n. \end{cases}$$

This implies

$$\begin{aligned} (x_N^{1-2\gamma} \partial_N W_1)^{**}(\bar{x}, x_N) &= -\alpha_{n,\gamma}^p \kappa_\gamma^{-1} p_{n,1-\gamma}\, x_N^{2-2\gamma} \int_{\mathbb{R}^n} \frac{1}{|\bar{y}|^{n-2\gamma}} \frac{1}{(1+|\bar{y}|^2)^{\frac{n+2\gamma}{2}}} \, d\bar{y} + O(x_N^{2-2\gamma}|x| + |x|^2) \\ &= -\alpha_{n,\gamma}(n-2\gamma) x_N^{2-2\gamma} + O(x_N^{2-2\gamma}|x| + |x|^2) \end{aligned} \tag{A-5}$$

for all $|x| \le R_0^{-1}$. Accordingly, we have

$$\left| x_N^{1-2\gamma} \partial_N W_1(x) + \frac{\alpha_{n,\gamma}(n-2\gamma) x_N^{2-2\gamma}}{|x|^{n-2\gamma+2}} \right| \le C \left(\frac{x_N^{2-2\gamma}}{|x|^{n-2\gamma+3}} + \frac{1}{|x|^{n+2\gamma}} \right)$$

for $|x| \ge R_0$. Dividing the both sides by $x_N^{1-2\gamma}$ finishes the proof of (A-4). □

Appendix B: Some integrations regarding the standard bubble $W_{1,0}$ on $\mathbb{R}^N_+$

The following lemmas are due to González and Qing [2013, Section 7] and the authors [Kim et al. 2015, Section 4.3].

Lemma B.1. *Suppose that $n > 4\gamma - 1$. For each $x_N > 0$ fixed, let $\widehat{W}_1(\xi, x_N)$ be the Fourier transform of $W_1(\bar{x}, x_N)$ with respect to the variable $\bar{x} \in \mathbb{R}^n$. In addition, we use K_γ to signify the modified Bessel function of the second kind of order γ. Then we have*

$$\widehat{W}_1(\xi, x_N) = \hat{w}_1(\xi)\, \varphi(|\xi| x_N) \quad \textit{for all } \xi \in \mathbb{R}^n \textit{ and } x_N > 0,$$

where $\varphi(t) = d_1 t^\gamma K_\gamma(t)$ is the solution to

$$\phi''(t) + \frac{1-2\gamma}{t}\phi'(t) - \phi(t) = 0, \quad \phi(0) = 1 \textit{ and } \phi(\infty) = 0, \tag{B-1}$$

and $\hat{w}_1(t) := \hat{w}_1(|\xi|) = d_2|\xi|^{-\gamma} K_\gamma(|\xi|)$ *solves*

$$\phi''(t) + \frac{1+2\gamma}{t}\phi'(t) - \phi(t) = 0 \quad \text{and} \quad \lim_{t\to 0} t^{2\gamma}\phi(t) + \lim_{t\to\infty} t^{\gamma+\frac{1}{2}} e^t \phi(t) \le C \tag{B-2}$$

for some $C > 0$*. The numbers* $d_1, d_2 > 0$ *depend only on* n *and* γ*.*

Lemma B.2. *Let*

$$\begin{aligned} A_\alpha &= \int_0^\infty t^{\alpha-2\gamma}\varphi^2(t)\,dt, & B_\alpha &= \int_0^\infty t^{-\alpha+2\gamma}\hat{w}_1^2(t)t^{n-1}\,dt, \\ A'_\alpha &= \int_0^\infty t^{\alpha-2\gamma}\varphi(t)\,\varphi'(t)\,dt, & B'_\alpha &= \int_0^\infty t^{-\alpha+2\gamma}\hat{w}_1(t)\,\hat{w}'_1(t)t^{n-1}\,dt, \\ A''_\alpha &= \int_0^\infty t^{\alpha-2\gamma}(\varphi'(t))^2\,dt, & B''_\alpha &= \int_0^\infty t^{-\alpha+2\gamma}(\hat{w}'_1(t))^2 t^{n-1}\,dt \end{aligned} \tag{B-3}$$

for $\alpha \in \mathbb{N}\cup\{0\}$*. Then*

$$A_\alpha = \left(\frac{\alpha+2}{\alpha+1}\right)\cdot\left[\left(\frac{\alpha+1}{2}\right)^2 - \gamma^2\right]^{-1} A_{\alpha+2} = -\left(\frac{\alpha+1}{2}-\gamma\right)^{-1} A'_{\alpha+1} = \left(\frac{\alpha+1}{2}-\gamma\right)\left(\frac{\alpha-1}{2}+\gamma\right)^{-1} A''_\alpha$$

for α *odd*, $\alpha \ge 1$ *and*

$$\begin{aligned} B_\alpha &= \frac{4(n-\alpha+1)B_{\alpha-2}}{(n-\alpha)(n+2\gamma-\alpha)(n-2\gamma-\alpha)} = -\frac{2B'_{\alpha-1}}{n+2\gamma-\alpha}, \\ B_{\alpha-2} &= \frac{(n-2\gamma-\alpha)B''_{\alpha-2}}{n+2\gamma-\alpha+2} \end{aligned}$$

for α *even*, $\alpha \ge 2$*.*

Proof. Apply (B-1), (B-2) and the identity

$$\int_0^\infty t^{\alpha-1}u(t)\,u'(t)\,dt = -\frac{\alpha-1}{2}\int_0^\infty t^{\alpha-2}u(t)^2\,dt,$$

which holds for any $\alpha > 1$ and $u \in C^1(\mathbb{R})$ decaying sufficiently fast. □

Utilizing the above lemmas, we compute some integrals regarding the standard bubble W_1 and its derivatives. The next identities are necessary in the energy expansion when nonminimal conformal infinities are considered. See Section 2B.

Lemma B.3. *Suppose that* $n \ge 2$ *and* $\gamma \in \left(0, \frac{1}{2}\right)$*. Then*

$$\int_{\mathbb{R}^N_+} x_N^{2-2\gamma}|\nabla W_1|^2\,dx = \frac{4}{1+2\gamma}\int_{\mathbb{R}^N_+} x_N^{2-2\gamma}(\partial_r W_1)^2\,dx = \frac{1-2\gamma}{2}\int_{\mathbb{R}^N_+} x_N^{-2\gamma} W_1^2\,dx < \infty.$$

Proof. Refer to [Choi and Kim 2017, Lemma 6.3]. □

The following is used in the energy expansion for the nonumbilic case. Refer to Sections 2C and 2D.

Lemma B.4. *For $n > 2+2\gamma$, it holds that*

$$\mathcal{F}_1 := \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} W_1^2\, dx = \frac{3}{2(1-\gamma^2)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_2 := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} |\nabla W_1|^2\, dx = \frac{3}{1+\gamma} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_3 := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} (\partial_r W_1)^2\, dx = |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_4 := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r(\partial_r W_1)(\partial_{rr} W_1)\, dx = -\tfrac{1}{2} n |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_5 := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r^2 (\partial_{rr} W_1)^2\, dx = \frac{5n^3 - 4n(1+\gamma^2) + 4(1-4\gamma^2)}{20(n-1)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_6 := \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} r^2 (\partial_r W_1)^2\, dx = \frac{(n+2)(3n^2-6n+4-4\gamma^2)}{8(n-1)(1-\gamma^2)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_7 := \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} r^2 (\partial_r W_1)(\partial_{r x_N} W_1)\, dx = -\frac{(n+2)(3n^2-6n+4-4\gamma^2)}{8(n-1)(1+\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_8 := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r^2 (\partial_{r x_N} W_1)^2\, dx = \frac{(2-\gamma)(5n^3 - 4n(2-2\gamma+\gamma^2) + 8(1-\gamma-2\gamma^2))}{20(n-1)(1+\gamma)} |\mathbb{S}^{n-1}| A_3 B_2.$$

Here $r = |\bar{x}|$, and the positive constants A_3 and B_2 are defined by (B-3).

Proof. The values $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_6$ were computed in [González and Qing 2013; Kim et al. 2015], so it suffices to consider the others.

Step 1: calculation of $\mathcal{F}_4$. Integration by parts gives

$$\begin{aligned}\mathcal{F}_4 &= \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r(\partial_r W_1)(\partial_{rr} W_1)\, dx = |\mathbb{S}^{n-1}| \int_0^\infty x_N^{3-2\gamma} \left(\frac{1}{2} \int_0^\infty r^n \partial_r (\partial_r W_1)^2\, dr \right) dx_N \\ &= |\mathbb{S}^{n-1}| \int_0^\infty x_N^{3-2\gamma} \left(-\frac{n}{2} \int_0^\infty r^{n-1} (\partial_r W_1)^2\, dr \right) dx_N = -\frac{n}{2} \mathcal{F}_3 = -\frac{n}{2} |\mathbb{S}^{n-1}| A_3 B_2.\end{aligned}$$

Step 2: calculation of $\mathcal{F}_5$. Since $\Delta_{\bar{x}} W_1 = W_1'' + (n-1) r^{-1} W_1'$ (where $'$ stands for the differentiation in r), it holds that

$$\int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r^2 (\Delta_{\bar{x}} W_1)^2\, dx = \mathcal{F}_5 + 2(n-1)\mathcal{F}_4 + (n-1)^2 \mathcal{F}_3. \tag{B-4}$$

By the Plancherel theorem, Lemma B.1 and the relation

$$\begin{aligned}&\Delta_\xi (|\xi|^2 \hat{w}_1(|\xi|) \varphi(|\xi| x_N)) \\ &= 2n \hat{w}_1 \varphi + (n+2-2\gamma)|\xi| \hat{w}_1' \varphi + (n+2+2\gamma)|\xi| \hat{w}_1 \varphi' x_N + |\xi|^2 \hat{w}_1 \varphi + 2|\xi|^2 \hat{w}_1' \varphi' x_N + |\xi|^2 \hat{w}_1 \varphi x_N^2,\end{aligned}$$

where the variable of $\hat{w}_1$ and $\hat{w}_1'$ is $|\xi|$, that of φ and φ' is $|\xi|x_N$, and $'$ represents the differentiation with respect to the radial variable $|\xi|$, we see

$$\begin{aligned}\int_{\mathbb{R}^N_+} & x_N^{3-2\gamma} r^2 (\Delta_{\bar{x}} W_1)^2\, dx \\ &= \int_0^\infty x_N^{3-2\gamma} \int_{\mathbb{R}^n} (-\Delta_\xi)\big(|\xi|^2 \hat{w}_1(|\xi|)\varphi(|\xi|x_N)\big) \cdot \big(|\xi|^2 \hat{w}_1(|\xi|)\varphi(|\xi|x_N)\big)\, d\xi\, dx_N \\ &= |\mathbb{S}^{n-1}| \big[2nA_3B_2 + (n+2-2\gamma)A_3B_1' + (n+2+2\gamma)A_4'B_2 + A_3B_0 + 2A_4'B_1' + A_5B_2\big].\end{aligned}$$

Therefore Lemma B.2 implies

$$\int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r^2 (\Delta_{\bar{x}} W_1)^2\, dx = \frac{5n^3 - 20n^2 + 4n(9-\gamma^2) - 16(1+\gamma^2)}{20(n-1)} |\mathbb{S}^{n-1}| A_3 B_2.$$

Now (B-4) and the information on $\mathcal{F}_3$ and $\mathcal{F}_4$ yield the desired estimate for $\mathcal{F}_5$.

Step 3: calculation of $\mathcal{F}_7$ and $\mathcal{F}_8$. Since the basic strategy is similar to Step 2, we will just sketch the proof. We observe

$$\begin{aligned}\mathcal{F}_7 &= \frac{1}{2}\int_0^\infty x_N^{2-2\gamma} \partial_N \left(\int_{\mathbb{R}^n} r^2 (\partial_r W_1)^2\, d\bar{x}\right) dx_N = \frac{1}{2}\int_0^\infty x_N^{2-2\gamma} \partial_N \left(\sum_{i=1}^n \int_{\mathbb{R}^n} |\bar{x}|^2 (\partial_{x_i} W_1)^2\, d\bar{x}\right) dx_N \\ &= \frac{1}{2}\int_0^\infty x_N^{2-2\gamma} \underbrace{\partial_N \left(\sum_{i=1}^n \int_{\mathbb{R}^n} (-\Delta_\xi)\big(\xi_i \hat{w}_1(|\xi|)\varphi(|\xi|x_N)\big) \cdot \big(\xi_i \hat{w}_1(|\xi|)\varphi(|\xi|x_N)\big)\, d\xi\right)}_{=(I)} dx_N.\end{aligned}$$

Owing to Lemmas B.1, B.2 and the expansion

$$\begin{aligned}(I) = -(n+1)&\int_{\mathbb{R}^n} \partial_N \big(|\xi|(\hat{w}_1\hat{w}_1')(|\xi|)\, \varphi^2(|\xi|x_N) + |\xi|\hat{w}_1^2(|\xi|)\, (\varphi\varphi')(|\xi|x_N)x_N\big)\, d\xi \\ &- \int_{\mathbb{R}^n} \partial_N \big(|\xi|^2(\hat{w}_1\hat{w}_1'')(|\xi|)\, \varphi^2(|\xi|x_N) + 2|\xi|^2(\hat{w}_1\hat{w}_1')(|\xi|)\, (\varphi\varphi')(|\xi|x_N)x_N\big)\, d\xi \\ &- \int_{\mathbb{R}^n} \partial_N \big(|\xi|^2\hat{w}_1^2(|\xi|)\, (\varphi\varphi'')(|\xi|x_N)x_N^2\big)\, d\xi,\end{aligned}$$

one can compute the integral $\mathcal{F}_7 = \frac{1}{2}\int_0^\infty x_N^{2-2\gamma}\, (I)\, dx_N$ to get its value given in the statement of the lemma. Moreover,

$$\begin{aligned}\mathcal{F}_8 &= \int_0^\infty x_N^{3-2\gamma} \left(\int_{\mathbb{R}^n} |\bar{x}|^2 |\nabla_{\bar{x}}(\partial_N W_1)|^2\, d\bar{x}\right) dx_N \\ &= \int_0^\infty x_N^{3-2\gamma} \left(\sum_{i=1}^n \int_{\mathbb{R}^n} (-\Delta_\xi)(\xi_i \partial_N \widehat{W}_1) \cdot (\xi_i \partial_N \widehat{W}_1)\, d\xi\right) dx_N.\end{aligned}$$

The rightmost term is computable with Lemmas B.1 and B.2. □

The next lemmas list the values of several integrals which are needed in the energy expansion for the umbilic case (see Sections 3B and 3C).

Lemma B.5. *For $n > 3+2\gamma$, let*

$$\mathcal{F}_1' := \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} W_1^2\, dx, \quad \mathcal{F}_2' := \int_{\mathbb{R}^N_+} x_N^{4-2\gamma} |\nabla W_1|^2\, dx \quad \text{and} \quad \mathcal{F}_3' := \int_{\mathbb{R}^N_+} x_N^{4-2\gamma} (\partial_r W_1)^2\, dx,$$

where $r = |\bar{x}|$. Then

$$\mathcal{F}_2' = \frac{3(3-2\gamma)}{2} \mathcal{F}_1' = \frac{8}{3+2\gamma} \mathcal{F}_3'.$$

Proof. One can argue as in [González and Qing 2013, Lemma 7.2] or [Choi and Kim 2017, Lemma 6.3]. □

Lemma B.6. *For $n > 4+2\gamma$, we have*

$$\mathcal{F}_1'' := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} W_1^2\, dx = \frac{4(n-3)}{(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_2'' := \int_{\mathbb{R}^N_+} x_N^{5-2\gamma} |\nabla W_1|^2\, dx = \frac{16(n-3)(2-\gamma)}{(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_3'' := \int_{\mathbb{R}^N_+} x_N^{5-2\gamma} (\partial_r W_1)^2\, dx = \frac{16(n-3)(4-\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_4'' := \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} r^2 W_1^2\, dx = \frac{n(3n^2-18n+28-4\gamma^2)}{2(n-4)(n-4-2\gamma)(n-4+2\gamma)(1-\gamma^2)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_5'' := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r^2 |\nabla W_1|^2\, dx = \frac{n(3n^2+2n(-7+2\gamma)-4(-4+3\gamma+\gamma^2))}{(n-4)(n-4-2\gamma)(n-4+2\gamma)(1+\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_6'' := \int_{\mathbb{R}^N_+} x_N^{3-2\gamma} r^2 (\partial_r W_1)^2\, dx = \frac{(n+2)(5n^2-20n+16-4\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_7'' := \int_{\mathbb{R}^N_+} x_N^{5-2\gamma} r (\partial_r W_1)(\partial_{rr} W_1)\, dx = -\frac{8n(n-3)(4-\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_8'' := \int_{\mathbb{R}^N_+} x_N^{5-2\gamma} r^2 (\partial_{rr} W_1)^2\, dx = \frac{4(4-\gamma^2)(7n^3-14n^2-4n(5+\gamma^2)+4-16\gamma^2)}{35(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

$$\mathcal{F}_9'' := \int_{\mathbb{R}^N_+} x_N^{4-2\gamma} r^2 (\partial_r W_1)(\partial_{r x_N} W_1)\, dx = -\frac{(n+2)(2-\gamma)(5n^2-20n+16-4\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,$$

and

$$\begin{aligned}\mathcal{F}_{10}'' &:= \int_{\mathbb{R}^N_+} x_N^{5-2\gamma} r^2 (\partial_{r x_N} W_1)^2\, dx \\ &= \frac{4(2-\gamma)(3-\gamma)(7n^3-14n^2-4n(6-2\gamma+\gamma^2)+8(2-3\gamma-2\gamma^2))}{35(n-4)(n-4-2\gamma)(n-4+2\gamma)} |\mathbb{S}^{n-1}| A_3 B_2,\end{aligned}$$

where $r = |\bar{x}|$, and the positive constants A_3 and B_2 are defined by (B-3).

Proof. The proof is analogous to those of Lemma B.4 and [Kim et al. 2015, Lemma 4.4], so we skip it. □

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Note added in the proof

(1) During the submission process, [Mayer and Ndiaye 2017] was posted on the arXiv. It proposes a proof of Theorem 1.7 without the positivity assumption on the constant A. In particular, they computed the expansion of a Green's function (compare our Conjecture 1.6 and their Corollary 6.1) and applied the Bahri–Coron-type topological argument in order to bypass the issue on A.

(2) Recently, Remarks 1.2(4) and 1.4(3) were confirmed affirmatively by the first author of this paper [Kim 2017].

(3) Suppose that $n \in \mathbb{N}$ and $\gamma \in (0, 1)$ satisfy $\mathcal{C}'(n, \gamma) > 0$, where $\mathcal{C}'(n, \gamma)$ is the quantity defined in (2-12). Moreover assume that $(M^n, [\hat{h}])$ is the conformal infinity of an asymptotic hyperbolic manifold (X, g^+) such that (1-9) and (1-18) hold, and the second fundamental form π never vanishes on M. Then the solution set of (1-2) (with $c > 0$) is compact in $C^2(M)$, as shown in [Kim et al. $\geq$ 2018].

References

[Almaraz 2010] S. d. M. Almaraz, "An existence theorem of conformal scalar-flat metrics on manifolds with boundary", *Pacific J. Math.* **248**:1 (2010), 1–22. MR Zbl

[Aubin 1976] T. Aubin, "Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire", *J. Math. Pures Appl.* (9) **55**:3 (1976), 269–296. MR Zbl

[Branson 1985] T. P. Branson, "Differential operators canonically associated to a conformal structure", *Math. Scand.* **57**:2 (1985), 293–345. MR Zbl

[Brendle 2008] S. Brendle, "Blow-up phenomena for the Yamabe equation", *J. Amer. Math. Soc.* **21**:4 (2008), 951–979. MR Zbl

[Brendle and Chen 2014] S. Brendle and S.-Y. S. Chen, "An existence theorem for the Yamabe problem on manifolds with boundary", *J. Eur. Math. Soc.* (*JEMS*) **16**:5 (2014), 991–1016. MR Zbl

[Brézis and Strauss 1973] H. Brézis and W. A. Strauss, "Semi-linear second-order elliptic equations in L^1", *J. Math. Soc. Japan* **25** (1973), 565–590. MR Zbl

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:7-9 (2007), 1245–1260. MR Zbl

[Case 2017] J. S. Case, "Some energy inequalities involving fractional GJMS operators", *Anal. PDE* **10**:2 (2017), 253–280. MR Zbl

[Case and Chang 2016] J. S. Case and S.-Y. A. Chang, "On fractional GJMS operators", *Comm. Pure Appl. Math.* **69**:6 (2016), 1017–1061. MR Zbl

[Chang and González 2011] S.-Y. A. Chang and M. d. M. González, "Fractional Laplacian in conformal geometry", *Adv. Math.* **226**:2 (2011), 1410–1432. MR Zbl

[Chang and Yang 1995] S.-Y. A. Chang and P. C. Yang, "Extremal metrics of zeta function determinants on 4-manifolds", *Ann. of Math.* (2) **142**:1 (1995), 171–212. MR Zbl

[Chen 2009] S.-y. S. Chen, "Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions", preprint, 2009. arXiv

[Cherrier 1984] P. Cherrier, "Problèmes de Neumann non linéaires sur les variétés riemanniennes", *J. Funct. Anal.* **57**:2 (1984), 154–206. MR Zbl

[Choi and Kim 2017] W. Choi and S. Kim, "On perturbations of the fractional Yamabe problem", *Calc. Var. Partial Differential Equations* **56**:1 (2017), art. id. 14. MR Zbl

[Djadli and Malchiodi 2008] Z. Djadli and A. Malchiodi, "Existence of conformal metrics with constant Q-curvature", *Ann. of Math.* (2) **168**:3 (2008), 813–858. MR Zbl

[Escobar 1992a] J. F. Escobar, "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary", *Ann. of Math.* (2) **136**:1 (1992), 1–50. MR Zbl

[Escobar 1992b] J. F. Escobar, "The Yamabe problem on manifolds with boundary", *J. Differential Geom.* **35**:1 (1992), 21–84. MR Zbl

[Escobar 1996] J. F. Escobar, "Conformal metrics with prescribed mean curvature on the boundary", *Calc. Var. Partial Differential Equations* **4**:6 (1996), 559–592. MR Zbl

[Fall and Weth 2012] M. M. Fall and T. Weth, "Nonexistence results for a class of fractional elliptic boundary value problems", *J. Funct. Anal.* **263**:8 (2012), 2205–2227. MR Zbl

[González and Qing 2013] M. d. M. González and J. Qing, "Fractional conformal Laplacians and fractional Yamabe problems", *Anal. PDE* **6**:7 (2013), 1535–1576. MR Zbl

[González and Wang 2017] M. d. M. González and M. Wang, "Further results on the fractional Yamabe problem: the umbilic case", *J. Geom. Anal.* (online publication March 2017).

[Graham 2000] C. R. Graham, "Volume and area renormalizations for conformally compact Einstein metrics", pp. 31–42 in *The Proceedings of the 19th Winter School "Geometry and Physics"* (Srní, 1999), edited by J. Slovák and M. Cadek, Rend. Circ. Mat. Palermo (2) Suppl. **63**, Circolo Matematico di Palermo, Palermo, 2000. MR Zbl

[Graham and Zworski 2003] C. R. Graham and M. Zworski, "Scattering matrix in conformal geometry", *Invent. Math.* **152**:1 (2003), 89–118. MR Zbl

[Guillarmou and Qing 2010] C. Guillarmou and J. Qing, "Spectral characterization of Poincaré–Einstein manifolds with infinity of positive Yamabe type", *Int. Math. Res. Not.* **2010**:9 (2010), 1720–1740. MR Zbl

[Gursky 1999] M. J. Gursky, "The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE", *Comm. Math. Phys.* **207**:1 (1999), 131–143. MR Zbl

[Gursky and Malchiodi 2015] M. J. Gursky and A. Malchiodi, "A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature", *J. Eur. Math. Soc.* (*JEMS*) **17**:9 (2015), 2137–2173. MR Zbl

[Hang and Yang 2004] F. Hang and P. C. Yang, "The Sobolev inequality for Paneitz operator on three manifolds", *Calc. Var. Partial Differential Equations* **21**:1 (2004), 57–83. MR Zbl

[Hang and Yang 2015] F. Hang and P. C. Yang, "Sign of Green's function of Paneitz operators and the Q curvature", *Int. Math. Res. Not.* **2015**:19 (2015), 9775–9791. MR Zbl

[Hang and Yang 2016a] F. Hang and P. C. Yang, "Q curvature on a class of 3-manifolds", *Comm. Pure Appl. Math.* **69**:4 (2016), 734–744. MR Zbl

[Hang and Yang 2016b] F. Hang and P. C. Yang, "Q-curvature on a class of manifolds with dimension at least 5", *Comm. Pure Appl. Math.* **69**:8 (2016), 1452–1491. MR Zbl

[Humbert and Raulot 2009] E. Humbert and S. Raulot, "Positive mass theorem for the Paneitz–Branson operator", *Calc. Var. Partial Differential Equations* **36**:4 (2009), 525–531. MR Zbl

[Joshi and Sá Barreto 2000] M. S. Joshi and A. Sá Barreto, "Inverse scattering on asymptotically hyperbolic manifolds", *Acta Math.* **184**:1 (2000), 41–86. MR Zbl

[Kim 2017] S. Kim, "Conformal metrics with prescribed fractional scalar curvature on conformal infinities with positive fractional Yamabe constants", preprint, 2017. arXiv

[Kim et al. 2015] S. Kim, M. Musso, and J. Wei, "A non-compactness result on the fractional Yamabe problem in large dimensions", preprint, 2015. arXiv

[Kim et al. $\geq$ 2018] S. Kim, M. Musso, and J. Wei, "A compactness theorem of the fractional Yamabe problem, I: The non-umbilic conformal infinity", preprint.

[Lee and Parker 1987] J. M. Lee and T. H. Parker, "The Yamabe problem", *Bull. Amer. Math. Soc.* (*N.S.*) **17**:1 (1987), 37–91. MR Zbl

[Marques 2005] F. C. Marques, "Existence results for the Yamabe problem on manifolds with boundary", *Indiana Univ. Math. J.* **54**:6 (2005), 1599–1620. MR Zbl

[Marques 2007] F. C. Marques, "Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary", *Comm. Anal. Geom.* **15**:2 (2007), 381–405. MR Zbl

[Mayer and Ndiaye 2015a] M. Mayer and C. B. Ndiaye, "Barycenter technique and the Riemann mapping problem of Escobar", preprint, 2015. arXiv

[Mayer and Ndiaye 2015b] M. Mayer and C. B. Ndiaye, "Proof of the remaining cases of the Yamabe boundary problem", preprint, 2015. arXiv

[Mayer and Ndiaye 2017] M. Mayer and C. B. Ndiaye, "Fractional Yamabe problem on locally flat conformal infinities of Poincaré–Einstein manifolds", preprint, 2017. arXiv

[Mazzeo and Melrose 1987] R. R. Mazzeo and R. B. Melrose, "Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature", *J. Funct. Anal.* **75**:2 (1987), 260–310. MR Zbl

[Paneitz 2008] S. M. Paneitz, "A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary)", *SIGMA Symmetry Integrability Geom. Methods Appl.* **4** (2008), art. id. 036. MR Zbl

[Qing and Raske 2006] J. Qing and D. Raske, "On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds", *Int. Math. Res. Not.* **2006** (2006), art. id. 94172. MR Zbl

[Schoen 1984] R. Schoen, "Conformal deformation of a Riemannian metric to constant scalar curvature", *J. Differential Geom.* **20**:2 (1984), 479–495. MR Zbl

[Schoen and Yau 1979a] R. Schoen and S. T. Yau, "On the proof of the positive mass conjecture in general relativity", *Comm. Math. Phys.* **65**:1 (1979), 45–76. MR Zbl

[Schoen and Yau 1979b] R. Schoen and S.-T. Yau, "Proof of the positive action conjecture in quantum relativity", *Phys. Rev. Lett.* **42**:9 (1979), 547–548.

[Schoen and Yau 1988] R. Schoen and S.-T. Yau, "Conformally flat manifolds, Kleinian groups and scalar curvature", *Invent. Math.* **92**:1 (1988), 47–71. MR Zbl

[Trudinger 1968] N. S. Trudinger, "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds", *Ann. Scuola Norm. Sup. Pisa* (3) **22** (1968), 265–274. MR Zbl

[Yamabe 1960] H. Yamabe, "On a deformation of Riemannian structures on compact manifolds", *Osaka Math. J.* **12**:1 (1960), 21–37. MR Zbl

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