Analysis & PDE

Volume 11 No. 1 2018

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ANALYSIS AND PDE
Vol. 11, No. 1, 2018
dx.doi.org/10.2140/apde.2018.11.133

NODAL GEOMETRY, HEAT DIFFUSION AND BROWNIAN MOTION

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We use tools from n-dimensional Brownian motion in conjunction with the Feynman–Kac formulation of heat diffusion to study nodal geometry on a compact Riemannian manifold M. On one hand we extend a theorem of Lieb (1983) and prove that any Laplace nodal domain $\Omega_\lambda \subseteq M$ almost fully contains a ball of radius $\sim 1/\sqrt{\lambda_1(\Omega_\lambda)}$, and such a ball can be centred at any point of maximum of the Dirichlet ground state $\varphi_{\lambda_1(\Omega_\lambda)}$. This also gives a slight refinement of a result by Mangoubi (2008) concerning the inradius of nodal domains. On the other hand, we also prove that no nodal domain can be contained in a reasonably narrow tubular neighbourhood of unions of finitely many submanifolds inside M.

1. Introduction

We consider a compact n-dimensional smooth Riemannian manifold M, and the Laplacian (or the Laplace–Beltrami operator) $-\Delta$ on M. We use the analyst's sign convention; namely, $-\Delta$ is positive semidefinite. For an eigenvalue λ of $-\Delta$ and a corresponding eigenfunction φ_λ, recall that a nodal domain Ω_λ is a connected component of the complement of the nodal set $N_{\varphi_\lambda} := \{x \in M : \varphi_\lambda(x) = 0\}$. In this paper, we are interested in the asymptotic geometry of a nodal domain Ω_λ as $\lambda \to \infty$.

In this note we address the following two questions.

First, we start by discussing the problem of whether a nodal domain can be squeezed in a tubular neighbourhood around a certain subset $\Sigma \subseteq M$. A result of Steinerberger [2014, Theorem 2] states that for some constant $r_0 > 0$, a nodal domain Ω_λ cannot be contained in an $(r_0/\sqrt{\lambda})$-tubular neighbourhood of the hypersurface Σ, provided that Σ is sufficiently flat in the following sense: Σ must admit a unique metric projection in a wavelength (i.e., $\sim 1/\sqrt{\lambda}$) tubular neighbourhood. The proof involves the study of a heat process associated to the nodal domain, where one also uses estimates for Brownian motion and the Feynman–Kac formula.

We relax the conditions imposed on Σ. Our first result is a direct extension of [Steinerberger 2014, Theorem 2]. Before stating the result, we begin with the following definition:

Definition 1.1 (admissible collections). For each fixed eigenvalue λ, we consider a natural number $m_\lambda \in \mathbb{N}$ and a collection $\Sigma_\lambda := \bigcup_{i=1}^{m_\lambda} \Sigma_\lambda^i$, where Σ_λ^i is an embedded smooth submanifold (without boundary) of dimension k $(1 \leq k \leq n-1)$.

We call Σ_λ admissible up to a distance r if the following property is satisfied: for any $x \in M$ with $\operatorname{dist}(x, \Sigma_\lambda) \leq r$ there exists a unique index $1 \leq i_x(\lambda) \leq m_\lambda$ and a unique point $y \in \Sigma_\lambda^{i_x(\lambda)}$ realizing $\operatorname{dist}(x, \Sigma_\lambda)$, that is, $\operatorname{dist}(x, y) = \operatorname{dist}(x, \Sigma_\lambda)$.

MSC2010: primary 35P20, 53B20, 53Z05; secondary 35K05.
Keywords: Laplace eigenfunctions, nodal domains, Brownian motion.

We note that if Σ_λ consists of one submanifold which is admissible up to distance r, then Definition 1.1 means that r is smaller than the normal injectivity radius of Σ_λ. Moreover, if Σ_λ consists of more submanifolds, then these submanifolds must be disjoint and the distance between every two of them must be greater than r.

Let us also remark that, [Steinerberger 2014, Theorem 2] holds true when the hypersurface Σ is allowed to vary with respect to λ in a controlled way, which is made precise by Definition 1.1. With that clarification in place, our Theorem 1.2 is an extension of that result.

Theorem 1.2. *There is a constant r_0 depending only on (M, g) such that if a submanifold $\Sigma_\lambda \subset M$ is admissible up to distance $1/\sqrt{\lambda}$, then no nodal domain Ω_λ can be contained in an $(r_0/\sqrt{\lambda})$-tubular neighbourhood of Σ_λ.*

Further, it turns out that we can select Σ_λ to be a union of submanifolds of varying dimensions, having relaxed admissibility conditions.

Elaborating on this, we observe that getting entirely rid of the admissibility condition, as in Definition 1.1, allows situations where Σ^i_λ is dense in M, for example, $M = \mathbb{T}^2$ and Σ^1_λ being a generic geodesic. By assuming Σ^i_λ is compact, we avoid such situations. Also, since we are considering unions of surfaces, the restriction of "unique projection" of nearby points, as in Definition 1.1, makes no sense anymore, and one can see that the approach of the proof of Theorem 1.2 does not work.

First, for ease of presentation, we adopt the following notation.

Definition 1.3. Given a compact subset K of M, let $\psi_K(t, x)$ denote the probability that a particle undergoing a Brownian motion starting at the point x will reach K within time t.

We now introduce the following relaxed notion of admissibility.

Definition 1.4 (α-admissible collections). Let $0 < \alpha < 1$ be a constant. For each fixed eigenvalue λ, we consider a natural number $m_\lambda \in \mathbb{N}$ and a collection $\Sigma_\lambda := \bigcup_{i=1}^{m_\lambda} \Sigma^i_\lambda$, where Σ^i_λ is a compact embedded smooth submanifold (without boundary) of dimension k_i, $(1 \le k_i \le n-1)$. Denote the respective tubular neighbourhoods by $N_\varepsilon(\Sigma^i_\lambda) := \{x \in M : \operatorname{dist}(x, \Sigma^i_\lambda) < \varepsilon\}$, and let $N_\varepsilon(\Sigma_\lambda) = \bigcup_{i=1}^{m_\lambda} N_\varepsilon(\Sigma^i_\lambda)$.

We say that the collection Σ_λ is α-admissible if for each sufficiently small $\varepsilon > 0$ and each $x \in N_\varepsilon(\Sigma_\lambda)$

$$\psi_{\partial B(x,2\varepsilon) \setminus N_\varepsilon(\Sigma_\lambda)}(4\varepsilon^2, x) \ge \alpha\, \psi_{\partial B(x,2\varepsilon)}(4\varepsilon^2, x). \tag{1}$$

Intuitively, using the above implicit formulation via Brownian motion hitting probabilities, we wish to ensure that $N_\varepsilon(\Sigma_\lambda)$ does not occupy too large a proportion of each $B(x, 2\varepsilon)$ for $x \in N_\varepsilon(\Sigma_\lambda)$; see Figure 2.

In other words, we allow the family Σ_λ to intersect, but the intersections should not be "too dense". To illustrate the idea, let us for simplicity assume that $M = \mathbb{R}^n$ and let us suppose that each member Σ^i_λ of the collection Σ_λ is a line passing through the origin. If the collection of these lines gets sufficiently close together or in other words "dense", then no matter how small an $\varepsilon > 0$ we take, the tubular neighbourhood $N_\varepsilon(\Sigma_\lambda)$ will contain the ball $B(0, 2\varepsilon)$. In particular, the left-hand side of (1) is vanishing and so, there is no $\alpha > 0$ for which the collection Σ_λ is α-admissible. Clearly, in the above example, replacing the lines Σ^i_λ by linear subspaces of varying dimensions will deliver a similar example of a collection, which is not α-admissible.

Having this intuition in mind, we have the following result.

Theorem 1.5. *Given an α-admissible collection Σ_λ, there exists a constant C, independent of λ, such that $N_{C/\sqrt{\lambda}}(\Sigma_\lambda)$ cannot fully contain a nodal domain Ω_λ.*

Theorem 1.5 gives a strong indication as to the "thickness" or general shape of a nodal domain in many situations of practical interest. For example, in dimension 2, numerics show nodal domains to look like a tubular neighbourhood of a tree. We also note that our proof of Theorem 1.5 reveals a bit more information, but for aesthetic reasons, we prefer to state the theorem this way. Heuristically, the proof reveals that the nodal domain Ω_λ is thicker at the points where the eigenfunction φ_λ attains its maximum, or at points where

$$\varphi_\lambda(x) \geq \beta \max_{y\in\Omega_\lambda} |\varphi_\lambda(y)|$$

for a fixed constant $\beta > 0$.

Second, we study the problem of how large a ball one may inscribe in a nodal domain Ω_λ at a point where the eigenfunction achieves extremal values on Ω_λ. We show:

Theorem 1.6. *Let* $\dim M \geq 3$, $\varepsilon_0 > 0$ *be fixed and* $x_0 \in \Omega_\lambda$ *be such that* $|\varphi_\lambda(x_0)| = \max_{\Omega_\lambda} |\varphi_\lambda|$. *There exists* $r_0 = r_0(\varepsilon_0)$ *such that*

$$\frac{\mathrm{Vol}(B(x_0, r_0\lambda^{-1/2}) \cap \Omega_\lambda)}{\mathrm{Vol}(B(x_0, r_0\lambda^{-1/2}))} \geq 1 - \varepsilon_0. \tag{2}$$

A celebrated theorem of Lieb [1983] considers the case of a domain $\Omega \subset \mathbb{R}^n$ and states that there exists a point $x_0 \in \Omega$ where a ball of radius $C/\sqrt{\lambda_1(\Omega)}$ can almost be inscribed (in the sense of our Theorem 1.6). A further generalization was obtained in [Maz'ya and Shubin 2005] (see, in particular, Theorem 1.1 and Section 5.1 of that paper). However, the point x_0 was not specified. Physically, one expects that x_0 is close to the point where the first Dirichlet eigenfunction of Ω attains extremal values. This is in fact the essential statement of Theorem 1.6 above. Also, in this context, it is illuminating to compare the main theorem from [Croke and Derdziński 1987].

We reiterate that the proof of Theorem 1.6 uses estimates from [Grigor'yan and Saloff-Coste 2002], see (31), and a certain isocapacitary estimate, see (32), that work only in dimensions $n \geq 3$. As far as dimension $n = 2$ is concerned, it is known due to [Mangoubi 2008b, Theorem 1.2], see also [Hayman 1978], that any nodal domain has wavelength inradius; see further discussion on this at the beginning of Section 4.

As a corollary of Theorem 1.6, we derive the following:

Corollary 1.7. *Let M be a closed manifold of dimension $n \geq 3$, and $\Omega_\lambda \subseteq M$ be a nodal domain upon which the corresponding eigenfunction φ_λ is positive. Let x_0 be a point of maximum of φ_λ on Ω_λ. Then there exists a ball $B(x_0, C/\lambda^{\alpha(n)}) \subseteq \Omega_\lambda$ with $\alpha(n) = \frac{1}{4}(n-1) + \frac{1}{2n}$ and a constant $C = C(M, g)$.*

This recovers Theorem 1.5 of [Mangoubi 2008a], with the additional information that the ball of radius $C/\lambda^{\alpha(n)}$ is centred around the max point of the eigenfunction φ_λ (for more discussion on this, see Section 4). We also point out that using Theorem 1.6, the first author has established in [Georgiev 2016] using results from [Jakobson and Mangoubi 2009], the following inner radius bounds for real analytic manifolds:

Theorem 1.8 [Georgiev 2016]. *Let (M, g) be a real-analytic closed manifold of dimension at least* 3. *Let φ_λ be a Laplacian eigenfunction and Ω_λ be a nodal domain of φ_λ. Then, there exist constants c_1, c_2 depending only on (M, g) such that*

$$\frac{c_1}{\lambda} \le \operatorname{inrad}(\Omega_\lambda) \le \frac{c_2}{\sqrt{\lambda}}.$$

Moreover, if φ_λ is positive (*resp. negative*) *on Ω_λ, then a ball of this radius can be inscribed within a wavelength distance to a point where φ_λ achieves its maximum* (*resp. minimum*) *on Ω_λ.*

For another improvement of inner radius estimates in the smooth setting under certain conditional bounds on $\|\varphi_\lambda\|_{L^\infty(\Omega_\lambda)}$, see Theorem 1.7 of [Georgiev and Mukherjee 2016].

A few assorted remarks: as advertised, in Section 3 we address the problem of inscribing a nodal domain Ω_λ in a tubular neighbourhood around Σ. In this context, an interesting subcase one might also consider is Σ having conical singularities: at its singular points Σ looks locally like $\mathbb{R}^{n-1-k} \times \partial C^k$ for some $k = 1, \dots, n-1$, where ∂C^k denotes the boundary of a generalized cone, i.e., the cone generated by some open set $D \subseteq \mathbb{S}^{n-1}$.

In this situation a useful tool is an explicit heat kernel formula for generalized cones $C \subseteq \mathbb{R}^n$. One denotes the associated Dirichlet eigenfunctions and eigenvalues of the generating set D by m_j, l_j respectively. Using polar coordinates $x = \rho\theta$, $y = r\eta$, one has that the heat kernel of $P_C(t, x, y)$ of the generalized cone C is given by

$$P_C(t, x, y) = \frac{e^{-\frac{\rho^2+r^2}{2t}}}{t(\rho r)^{\frac{n}{2}+1}} \sum_{j=1}^{\infty} I_{\sqrt{l_j + (\frac{n}{2}-1)^2}}\left(\frac{\rho r}{t}\right) m_j(\theta)\, m_j(\eta), \tag{3}$$

where $I_\nu(z)$ denotes the modified Bessel function of order ν. For more on the formula (3) we refer to [Bañuelos and Smits 1997]. An even more general formula can be found in [Cheeger 1983].

The expression for $P_C(t, x, y)$ provides means for estimating $p_t(x)$ from below, as in Section 3. However, some features of the conical singularity (i.e., the eigenvalues and eigenfunctions l_j, m_j of the generating set D) enter explicitly in the estimate. Such considerations appear promising in discussing theorems of the following type, for example, and their higher-dimensional analogues; see also [Steinerberger 2014]:

Theorem 1.9 (Bers, Cheng). *Let $n = 2$. If $-\Delta u = \lambda u$, then any nodal set satisfies an interior cone condition with opening angle $\alpha \gtrsim \lambda^{-1/2}$.*

1A. ***Basic heuristics.*** We outline the main idea behind Theorems 1.2, 1.5 and 1.6.

First, one considers a point $x_0 \in \Omega_\lambda$ where the eigenfunction achieves a maximum on the nodal domain (without loss of generality we assume that the eigenfunction is positive on Ω_λ). One then considers the quantity $p(t, x_0)$, i.e., the probability that a Brownian motion started at x_0 escapes the nodal domain within time t.

The main strategy is to obtain two-sided bounds for $p(t, x_0)$.

On one hand, we have the Feynman–Kac formula (see Section 2A), which provides a straightforward upper bound only in terms of t (see (13) below).

On the other hand, depending on the context of the theorems above, we provide a lower bound for $p(t, x_0)$ in terms of some geometric data. To this end, we take advantage of various tools, some of which are: formulas for hitting probabilities of spheres and the parabolic scaling between the space and time variables, comparability of Brownian motions on manifolds with similar geometry (see Section 2B), bounds for hitting probabilities in terms of 2-capacity (see [Grigor'yan and Saloff-Coste 2002]), etc.

1B. ***Outline of the paper.*** In Section 2, we recall tools from n-dimensional Brownian motion and the Feynman–Kac formulation of heat diffusion, and discuss the parabolic scaling technique we referred to above. We include some background material on stochastic analysis on Riemannian manifolds, some of which (to our knowledge) is not widely known, but is important to our investigation. We also believe such results to be of independent interest to the community. Worthy of particular mention is Theorem 2.2, which roughly says that if the metric is perturbed slightly, hitting probabilities of compact sets by Brownian particles are also perturbed slightly. This allows us to apply Brownian motion formulae from $\mathbb{R}^n$ to compact manifolds, on small distance and time scales.

In Section 3, we begin by proving Theorem 1.2. As mentioned before, we then take the generalization one step further, by considering intersecting surfaces of different dimensions. Our main result in this direction is Theorem 1.5, which gives a quantitative lower bound on how "thin" or "narrow" a nodal domain can be.

In Section 4, we take up the investigation of inradius estimates of Ω_λ. As mentioned before, our main result in this direction is Theorem 1.6. We also establish Corollary 1.7.

2. Preliminaries: heat equation, Feynman–Kac and Bessel processes

2A. ***Feynman–Kac formula.*** We begin by stating a Feynman–Kac formula for open connected domains in compact manifolds for the heat equation with Dirichlet boundary conditions. Such formulas seem to be widely known in the community, but since we were unable to find out an explicit reference, we also indicate a line of proof.

Theorem 2.1. *Let M be a compact Riemannian manifold. For any open connected $\Omega \subset M$, $f \in L^2(\Omega)$, we have*

$$e^{t\Delta} f(x) = \mathbb{E}_x\big(f(\omega(t))\phi_\Omega(\omega, t)\big), \quad t > 0,\ x \in \Omega, \tag{4}$$

where $\omega(t)$ denotes an element of the probability space of Brownian motions starting at x, $\mathbb{E}_x$ is the expectation with regard to the measure on that probability space, and

$$\phi_\Omega(\omega, t) = \begin{cases} 1 & \text{if } \omega([0,t]) \subset \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

A proof of Theorem 2.1 can be constructed in three steps. First, one proves the corresponding statement when $\Omega = M$. This can be found, for example, in [Bär and Pfäffle 2011, Theorem 6.2]. One can then combine this with a barrier potential method to prove a corresponding statement for domains Ω with Lipschitz boundary. Lastly, the extension to domains with no regularity requirements on the boundary is achieved by a standard limiting argument. For details on the last two steps, see [Taylor 1996, Chapter 11, Section 3].

2B. ***Euclidean comparability of hitting probabilities.*** Implicit in many of our calculations is the following heuristic: if the metric is perturbed slightly, hitting probabilities of compact sets by Brownian particles are also perturbed slightly, provided one is looking at small distances r and at small time scales $t = O(r^2)$.

To describe the set up, let (M, g) be a compact Riemannian manifold and cover M by charts (U_k, ϕ_k) such that in these charts g is bi-Lipschitz to the Euclidean metric. Consider an open ball $B(p, r) \subset M$, where r is considered small, and in particular, smaller than the injectivity radius of M. Let $B(p, r)$ sit inside a chart (U, ϕ) and let $\phi(p) = q$ and $\phi(B(p, r)) = B(q, s) \subset \mathbb{R}^n$. Let K be a compact set inside $B(p, r)$ and let $K' := \phi(K) \subset B(q, s)$.

Now, let $\psi_K^M(T, p)$ denote the probability that a Brownian motion on (M, g) started at p and killed at a fixed time T hits K within time T. The probability $\psi_{K'}^e(t, q)$ is defined similarly for the standard Brownian motion in $\mathbb{R}^n$ started at q and killed at the same fixed time T. Now, we fix the time $T = cr^2$, where c is a constant. The following is the comparability result:

Theorem 2.2. *There exist constants c_1, c_2 depending only on c and M such that*

$$c_1 \psi_{K'}^e(T, q) \le \psi_K^M(T, p) \le c_2 \psi_{K'}^e(T, q). \tag{5}$$

The proof uses the concept of Martin capacity; see [Benjamini et al. 1995, Definition 2.1]:

Definition 2.3. Let Λ be a set and $\mathcal{B}$ a σ-field of subsets of Λ. Given a measurable function $F : \Lambda \times \Lambda \to [0, \infty]$ and a finite measure μ on $(\Lambda, \mathcal{B})$, the F-energy of μ is

$$I_F(\mu) = \int_\Lambda \int_\Lambda F(x, y)\, d\mu(x)\, d\mu(y).$$

The capacity of Λ in the kernel F is

$$\mathrm{Cap}_F(\Lambda) = \big[\inf_\mu I_F(\mu)\big]^{-1}, \tag{6}$$

where the infimum is over probability measures μ on $(\Lambda, \mathcal{B})$, and by convention, $\infty^{-1} = 0$.

Now we quote the following general result, which is Theorem 2.2 in [Benjamini et al. 1995].

Theorem 2.4. *Let $\{X_n\}$ be a transient Markov chain on the countable state space Y with initial state ρ and transition probabilities $p(x, y)$. For any subset Λ of Y, we have*

$$\tfrac{1}{2}\mathrm{Cap}_M(\Lambda) \le \mathbb{P}_\rho\{\exists n \ge 0 : X_n \in \Lambda\} \le \mathrm{Cap}_M(\Lambda), \tag{7}$$

where M is the Martin kernel $M(x, y) = G(x, y)/G(\rho, y)$, and $G(x, y)$ denotes the Green's function.

For the special case of Brownian motions, this reduces to (see Proposition 1.1 of [Benjamini et al. 1995] and Theorem 8.24 of [Mörters and Peres 2010]):

Theorem 2.5. *Let $\{B(t) : 0 \le t \le T\}$ be a transient Brownian motion in $\mathbb{R}^n$ starting from the point ρ, and $A \subset D$ be closed, where D is a bounded domain. Then,*

$$\tfrac{1}{2}\mathrm{Cap}_M(A) \le \mathbb{P}_\rho\{B(t) \in A \text{ for some } 0 < t \le T\} \le \mathrm{Cap}_M(A). \tag{8}$$

An inspection of the proofs reveals that they go through with basically no changes on a compact Riemannian manifold M, when the Brownian motion is killed at a fixed time $T = cr^2$, and the Martin kernel $M(x, y)$ is defined as $G(x, y)/G(\rho, y)$, with $G(x, y)$ being the "cut-off" Green's function defined as follows: if $h_M(t, x, y)$ is the heat kernel of M,

$$G(x, y) := \int_0^T h_M(t, x, y)\,dt.$$

Now, to state it formally, in our setting, we have

Theorem 2.6.
$$\tfrac{1}{2}\operatorname{Cap}_M(K) \le \psi_K^M(T, p) \le \operatorname{Cap}_M(K). \tag{9}$$

Now, let $h_{\mathbb{R}^n}(t, x, y)$ denote the heat kernel on $\mathbb{R}^n$. To prove Theorem 2.2, it suffices to show that for $y \in K$, and $y' = \phi(y) \in K'$, we have constants C_1, C_2 (depending on c and M) such that

$$C_1 \int_0^T h_{\mathbb{R}^n}(t, q, y')\,dt \le \int_0^T h_M(t, p, y)\,dt \le C_2 \int_0^T h_{\mathbb{R}^n}(t, q, y')\,dt. \tag{10}$$

In other words, we need to demonstrate comparability of Green's functions "cut off" at time $T = cr^2$. Recall that we have the following Gaussian two-sided heat kernel bounds on a compact manifold (see, for example, Theorem 5.3.4 of [Hsu 2002] for the lower bound and Theorem 4 of [Cheng et al. 1981] for the upper bound, also (4.27) of [Grigor'yan and Saloff-Coste 2002]): for all $(t, p, y) \in (0, 1) \times M \times M$, and positive constants c_1, c_2, c_3, c_4 depending only on the geometry of M,

$$\frac{c_3}{t^{\frac{n}{2}}} e^{\frac{-c_1 d(p,y)^2}{4t}} \le h_M(t, p, y) \le \frac{c_4}{t^{\frac{n}{2}}} e^{\frac{-c_2 d(p,y)^2}{4t}},$$

where d denotes the distance function on M. Then, using the comparability of the distance function on M with the Euclidean distance function (which comes via metric comparability in local charts), for establishing (10), it suffices to observe that for any positive constant c_5, we have

$$\int_0^{cr^2} t^{-\frac{n}{2}} e^{-\frac{c_5 r^2}{4t}}\,dt = \frac{2^{n-2}}{c_5^{\frac{n}{2}-1}} \frac{1}{r^{n-2}} \Gamma\left(\frac{n}{2} - 1, \frac{c_5}{4c}\right),$$

where $\Gamma(s, x)$ is the (upper) incomplete Gamma function. Since r is a small constant chosen independently of λ, we observe that C_1, C_2 are constants in (10) depending only on c, c_1, c_2, c_3, c_4, c_5, r and M, which finally proves (5).

Remark 2.7. Theorem 2.2 is implicit in [Steinerberger 2014], but it was not precisely stated or proved there. Since we are unable to find an explicit reference, here we have given a formal statement and indicated a proof. We believe that the statement of Theorem 2.2 will also be of independent interest for people interested in stochastic analysis on manifolds.

2C. ***Brownian motion on a manifold and Euclidean Bessel processes.*** Using the probabilistic formulation of the heat equation for the study of nodal geometry, we are largely inspired by the methods in [Steinerberger 2014]. Of course, such ideas have appeared in the literature before; for example, they are implicit in [Grieser and Jerison 1998]. Here we extend some ideas of Steinerberger with the help of tools from n-dimensional Brownian motion.

Given an open subset $V \subset M$, consider the solution $p_t(x)$ to the following diffusion process:

$$\begin{aligned} (\partial_t - \Delta) p_t(x) &= 0, \quad x \in V, \\ p_t(x) &= 1, \quad x \in \partial V, \\ p_0(x) &= 0, \quad x \in V. \end{aligned}$$

By the Feynman–Kac formula (see Section 2A), this diffusion process can be understood as the probability that a Brownian motion particle started in x will hit the boundary within time t. Now, fix an eigenfunction φ (corresponding to the eigenvalue λ) and a nodal domain Ω, so that $\varphi > 0$ on Ω without loss of generality. Calling Δ the Dirichlet Laplacian on Ω and setting $\Phi(t,x) := e^{t\Delta}\varphi(x)$, we see that Φ solves

$$\begin{aligned} (\partial_t - \Delta)\Phi(t,x) &= 0, && x \in \Omega, \\ \Phi(t,x) &= 0, && \text{on } \{\varphi = 0\}, \\ \Phi(0,x) &= \varphi(x), && x \in \Omega. \end{aligned} \tag{11}$$

Using the Feynman–Kac formula given by Theorem 2.1, we have,

$$e^{t\Delta} f(x) = \mathbb{E}_x(f(\omega(t))\phi_\Omega(\omega,t)), \quad t > 0, \tag{12}$$

where $\omega(t)$ denotes an element of the probability space of Brownian motions starting at x, $\mathbb{E}_x$ is the expectation with regard to the measure on that probability space, and

$$\phi_\Omega(\omega,t) = \begin{cases} 1 & \text{if } \omega([0,t]) \subset \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider a nodal domain Ω corresponding to the eigenfunction φ, and consider the heat flow (11). Let $x_0 \in \Omega$ such that $\varphi(x_0) = \|\varphi\|_{L^\infty(\Omega)}$. We use the following upper bound derived in [Steinerberger 2014]:

$$\begin{aligned} \Phi(t,x) = e^{-\lambda t}\varphi(x) &= \mathbb{E}_x\big(\varphi(\omega(t))\phi_\Omega(\omega,t)\big) \\ &\le \|\varphi\|_{L^\infty(\Omega)} \mathbb{E}_x(\phi_\Omega(\omega,t)) = \|\varphi\|_{L^\infty(\Omega)}(1 - p_t(x)). \end{aligned} \tag{13}$$

Setting $t = \lambda^{-1}$ and $x = x_0$, we see that the probability of the Brownian motion starting at an extremal point x_0 leaving Ω within time λ^{-1} is $\le 1 - e^{-1}$. A rough interpretation is that maximal points x are situated deeply into the nodal domain. Using the notation introduced in the Introduction, the last derived upper estimate translates to $\psi_{M\setminus\Omega}(\lambda^{-1}, x) \le 1 - e^{-1}$.

Now, we consider an m-dimensional Brownian motion of a particle starting at the origin in $\mathbb{R}^m$, and calculate the probability of the particle hitting a sphere $\{x \in \mathbb{R}^m : \|x\| \le r\}$ of radius r within time t. By a well-known formula first derived in [Kent 1980], we see that such a probability is given as

$$\mathbb{P}\Big(\sup_{0\le s\le t} \|B(s)\| \ge r\Big) = 1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2 t}{2r^2}}, \quad \nu > -1, \tag{14}$$

where $\nu = (m-2)/2$ is the "order" of the Bessel process, J_ν is the Bessel function of the first kind of order ν, and $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ is the sequence of positive zeros of J_ν.

Choose $x = x_0$, $t = \lambda^{-1}$, as before, and let $r = C^{1/2}\lambda^{-1/2}$, where C is a constant to be chosen later, independently of λ. Plugging this in (14) then reads as

$$\mathbb{P}\big(\sup_{0\le s\le\lambda^{-1}} \|B(s)\| \ge C\lambda^{-1/2}\big) = 1 - \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2}{2C}}, \quad \nu > -1. \tag{15}$$

We need to make a few comments about the asymptotic behaviour of $j_{\nu,k}$ here. For notational convenience, we write $\alpha_k \sim \beta_k$ as $k\to\infty$ if we have $\alpha_k/\beta_k \to 1$ as $k\to\infty$. The asymptotic expansion

$$j_{\nu,k} = \big(k + \tfrac{1}{2}\nu + \tfrac{1}{4}\big)\pi + o(1) \quad \text{as } k\to\infty, \tag{16}$$

given in [Watson 1944, p. 506], tells us that $j_{\nu,k} \sim k\pi$. Also, from p. 505 of the same paper, we have

$$J_{\nu+1}(j_{\nu,k}) \sim (-1)^{k-1}\frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{k}}. \tag{17}$$

These asymptotic estimates, in conjunction with (15), tell us that keeping ν bounded, and given a small $\eta > 0$, one can choose the constant C small enough (depending on η) such that

$$\mathbb{P}\big(\sup_{0\le s\le\lambda^{-1}} \|B(s)\| \ge C\lambda^{-1/2}\big) > 1-\eta. \tag{18}$$

This estimate plays a role in Section 3. In this context, see also Proposition 5.1.4 of [Hsu 2002].

3. Admissibility conditions and intersecting surfaces

Proof of Theorem 1.2. If φ_λ attains its maximum within Ω_λ at x_0, we already know from (13) that

$$\psi_{M\setminus\Omega_\lambda}\left(\frac{t_0}{\lambda}, x_0\right) \le 1 - e^{-t_0}. \tag{19}$$

By the admissibility condition on Σ_λ we know that x_0 has a unique metric projection on one and only one $\Sigma_\lambda^{i_{x_0}}$ from the collection Σ_λ.

Now, suppose the result is not true. Choose R, t_0 small such that Theorem 2.2 applies. Choosing r_0 sufficiently smaller than R, we can find a λ such that Ω_λ is contained in an $(r_0/\sqrt{\lambda})$-tubular neighbourhood of Σ_λ, denoted by $N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)$. From the remarks after Definition 1.1, it follows that $\Omega_\lambda \subseteq N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}})$.

We start a Brownian motion at x_0 and, roughly speaking, we see that locally the particle has freedom to wander in $n-k$ "bad directions", namely the directions normal to $\Sigma_\lambda^{i_{x_0}}$, before it hits $\partial\Omega_\lambda$. That means, we may consider a $(n-k)$-dimensional Brownian motion $B(t)$ starting at x_0; see Figure 1.

More formally, we choose a normal coordinate chart (U,ϕ) around x_0 adapted to $\Sigma_\lambda^{i_{x_0}}$, where the metric is comparable to the Euclidean metric. We have $\phi(\Sigma_\lambda^{i_{x_0}}) = \phi(U)\cap\{\mathbb{R}^k\times\{0\}^{n-k}\}$ and

$$\phi(N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{i_{x_0}})) = \phi(U)\cap\left\{\mathbb{R}^k\times\left[-\frac{r_0}{\sqrt{\lambda}},\frac{r_0}{\sqrt{\lambda}}\right]^{n-k}\right\}.$$

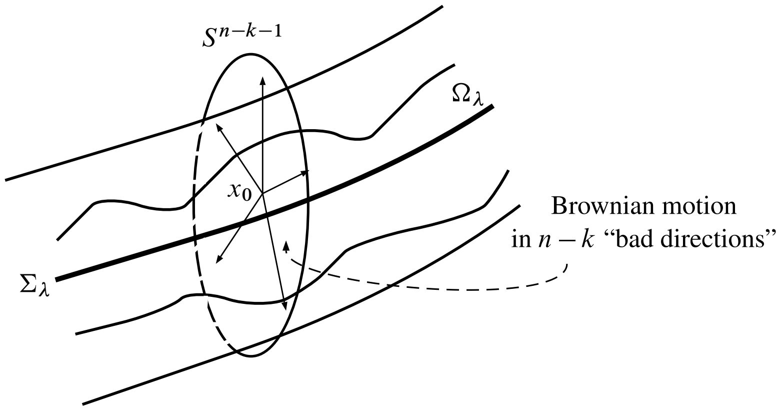

Figure 1. A Brownian motion at x_0.

We take a geodesic ball $B \subset U \subset M$ at x_0 of radius $R/\sqrt{\lambda}$. Using the hitting probability notation from Section 2 and monotonicity with respect to set inclusion we have

$$\psi_{M\setminus\Omega_\lambda}\left(\frac{t_0}{\lambda}, x_0\right) \geq \psi_{B\setminus\Omega_\lambda}\left(\frac{t_0}{\lambda}, x_0\right) \geq \psi_{B\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{ix_0})}\left(\frac{t_0}{\lambda}, x_0\right), \tag{20}$$

and the comparability lemma implies that, if $c = t_0/R^2$, then there exists a constant C, depending on c and M, such that

$$\psi_{B\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{ix_0})}\left(\frac{t_0}{\lambda}, x_0\right) \geq C\,\psi^e_{\phi(B\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{ix_0}))}\left(\frac{t_0}{\lambda}, \phi(x_0)\right), \tag{21}$$

where ψ^e denotes the hitting probability in Euclidean space. We define $N^e_{r_0\lambda^{-1/2}} := \phi(N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{ix_0}))$.

Let us consider the "solid cylinder" $S = B_{R/\sqrt{\lambda}} \times B_{r_0/\sqrt{\lambda}}$, a product of k and $(n-k)$-dimensional Euclidean balls centred at $\phi(x_0)$. S is clearly the largest cylinder contained in $N^e_{r_0\lambda^{-1/2}} \cap B$. We set $S = B_1 \times B_2$ for convenience. By monotonicity,

$$\psi^e_{\phi(B\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda^{ix_0}))}\left(\frac{t_0}{\lambda}, \phi(x_0)\right) \geq \psi^e_{B_1\times\partial B_2}\left(\frac{t_0}{\lambda}, \phi(x_0)\right). \tag{22}$$

If $B(t) = (B_1(t), \dots, B_n(t))$ is an n-dimensional Brownian motion, the components $B_i(t)$ are independent Brownian motions; see, for example, Chapter 2 of [Mörters and Peres 2010]. Denoting by $\mathcal{B}_k(t)$ and $\mathcal{B}_{n-k}(t)$ the projections of $B(t)$ onto the first k and last $n-k$ components respectively, it follows that

$$\begin{aligned}\psi^e_{B_1\times\partial B_2}\left(\frac{t_0}{\lambda}, \phi(x_0)\right) &\geq \mathbb{P}\left(\sup_{0\leq s\leq t_0\lambda^{-1}} \|\mathcal{B}_k(t)\| \leq \frac{R}{\sqrt{\lambda}}\right).\mathbb{P}\left(\sup_{0\leq s\leq t_0\lambda^{-1}} \|\mathcal{B}_{n-k}(t)\| \geq \frac{r_0}{\sqrt{\lambda}}\right)\\ &\geq c_k\mathbb{P}\left(\sup_{0\leq s\leq t_0\lambda^{-1}} \|\mathcal{B}_{n-k}(t)\| \geq \frac{r_0}{\sqrt{\lambda}}\right),\end{aligned}$$

where c_k is a constant depending on k and the ratio t_0/R^2, and can be calculated explicitly from (15).

Using the estimate in Section 2, we may take $r_0 \le R$ sufficiently small so that

$$\mathbb{P}\left(\sup_{0 \le s \le t_0 \lambda^{-1}} \|\mathcal{B}_{n-k}(t)\| \ge \frac{r_0}{\sqrt{\lambda}} \right) > 1 - \varepsilon, \tag{23}$$

where ε is sufficiently small. Keeping $c = t_0/R^2$ and (hence) C fixed, we take t_0 small enough and $r_0 \le R$ appropriately, so that (23) contradicts (20) and the fact that

$$\psi_{M\setminus\Omega_\lambda}(t_0\lambda^{-1}, x) \le 1 - e^{-t_0}. \qquad \square$$

Remark 3.1. Note that the constant r_0 above is independent of Σ_λ; in other words, the same constant r_0 will work for Theorem 1.2 as long as the surface is admissible up to a wavelength distance. Indeed, this results from the fact that r_0 depends only on the diffusion process associated to the Brownian motion, and is an inherent property of the manifold itself.

Now we address the generalizations of Theorem 1.2 for collections Σ_λ which are more complicated; namely, we assume Σ_λ is an α-admissible collection in the sense of Definition 1.4.

Proof of Theorem 1.5. By assumption, we have an α-admissible collection $\Sigma_\lambda := \bigcup_{i=1}^{m_\lambda} \Sigma_\lambda^i$.

Let us assume the contrary — if the statement is not true, we may select an arbitrarily small $r_0 > 0$ and find a corresponding inscribed nodal domain $\Omega_\lambda \subset N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)$.

As before, we choose a point $x_0 \in \Omega_\lambda$ such that

$$\varphi_\lambda(x_0) = \max_{x\in\Omega_\lambda} |\varphi_\lambda|.$$

Monotonicity of the hitting probability function $\psi_K(\cdot,\cdot)$ with respect to set inclusion in K, as well as the α-admissibility, imply that (see Figure 2)

$$\begin{aligned} \psi_{M\setminus\Omega_\lambda}(t, x_0) &\ge \psi_{B(x_0, 2r_0\lambda^{-1/2})\setminus\Omega_\lambda}(t, x_0) \\ &\ge \psi_{B(x_0, 2r_0\lambda^{-1/2})\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)}(t, x_0) \\ &= \psi_{\partial(B(x_0, 2r_0\lambda^{-1/2})\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda))}(t, x_0) \\ &\ge \psi_{\partial B(x_0, 2r_0\lambda^{-1/2})\setminus N_{r_0\lambda^{-1/2}}(\Sigma_\lambda)}(t, x_0) \\ &\ge \alpha\, \psi_{\partial B(x_0, 2r_0\lambda^{-1/2})}(t, x_0), \end{aligned} \tag{24}$$

where we introduce the constant $\alpha > 0$ coming from the α-admissibility condition. Moreover, following Definition 1.4 of α-admissibility, in (24) we also assume that the radius $r_0/\sqrt{\lambda}$ is sufficiently small and that $t := t_0/\lambda$ with $t_0 := 4r_0^2$.

The latter estimate (24) implies, in particular, that

$$\frac{\psi_{M\setminus\Omega_\lambda}(t, x_0)}{\psi_{M\setminus B(x_0, 2r_0\lambda^{-1/2})}(t, x_0)} = \frac{\psi_{M\setminus\Omega_\lambda}(t, x_0)}{\psi_{\partial B(x_0, 2r_0\lambda^{-1/2})}(t, x_0)} \ge \alpha. \tag{25}$$

We now observe that by setting $t = t_0/\lambda$ we still have the freedom to choose t_0. We show that we can select t_0 such that (25) is violated. To this end we observe that the upper bound (19) along with (15) and

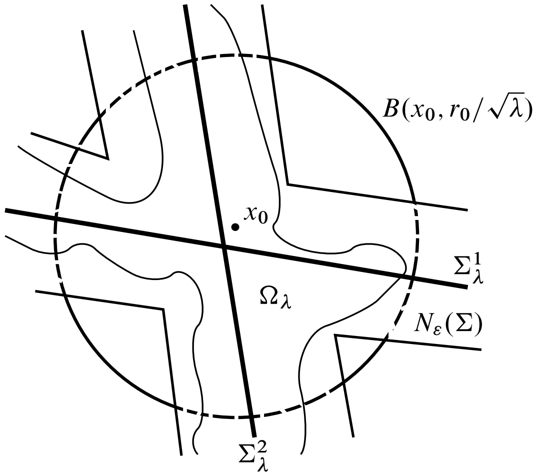

Figure 2. Nodal domain within a tubular neighbourhood of an admissible collection.

Theorem 2.2 give

$$
\begin{aligned}
\frac{\psi_{M\setminus\Omega_\lambda}(t_0/\lambda, x)}{\psi_{M\setminus B(x_0, 2r_0\lambda^{-1/2})}(t_0/\lambda, x)} &\lesssim (1-e^{-t_0})\left(1-\frac{1}{2^{\nu-1}\Gamma(\nu+1)}\sum_{k=1}^{\infty}\frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})}e^{-\frac{j_{\nu,k}^2 t_0}{2r_0^2}}\right)^{-1} \\
&= (1-e^{-t_0})\left(1-\frac{1}{2^{\nu-1}\Gamma(\nu+1)}\sum_{k=1}^{\infty}\frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})}e^{-2j_{\nu,k}^2}\right)^{-1} \\
&= (1-e^{-t_0})\widetilde{C}^{-1}.
\end{aligned} \tag{26}
$$

Now, we choose $t_0 = 4r_0^2$ small enough, so the last estimate yields a contradiction with (25). This proves the theorem. □

Remark 3.2. We wish to comment that in the above proof, it is not essential to look at the nodal domain only around the maximum point x_0. Given a predetermined positive constant β, choose a point $y \in \Omega_\lambda$ such that $\varphi_\lambda(y) \geq \beta\varphi_\lambda(x_0)$. Arguing similarly as in (13), we see that $\psi_{M\setminus\Omega_\lambda}(t, y) \leq 1-\beta e^{-t_0}$. Following the computations in (26), we get a constant r_0 (depending on β) such that $(1-\beta e^{-t_0})/\widetilde{C} < \alpha$, giving a contradiction. Also, it is clear that in Definitions 1.1 and 1.4, we do not actually need the submanifolds in the family Σ_λ to be smooth, and the proofs of Theorems 1.2 and 1.5 work with submanifolds of much lower regularity (for example, C^1 submanifolds).

4. Large ball at a max point

In this section we discuss the asymptotic thickness of nodal domains around extremal points of eigenfunctions. More precisely, let us consider a fixed nodal domain Ω_λ corresponding to the eigenfunction φ_λ. Let $x_0 \in \Omega_\lambda$ be such that

$$\varphi_\lambda(x_0) = \max_{x\in\Omega_\lambda} |\varphi_\lambda|. \tag{27}$$

In the case $\dim M = 2$, it was shown in Section 3 of [Mangoubi 2008b] that at such maximal points x_0 one can fully inscribe a large ball of wavelength radius (i.e $\sim 1/\sqrt{\lambda}$) into the nodal domain. In other words for Riemannian surfaces, one has that

$$\frac{C_1}{\sqrt{\lambda}} \le \operatorname{inrad}(\Omega_\lambda) \le \frac{C_2}{\sqrt{\lambda}}, \tag{28}$$

where C_i are constants depending only on M. Note that the proof for this case, as carried out in [Mangoubi 2008b] by following ideas in [Nazarov et al. 2005], makes use of essentially 2-dimensional tools (conformal coordinates and quasiconformality), which are not available in higher dimensions.

To our knowledge, in higher dimensions the sharpest known bounds on the inner radius of a nodal domain appear in [Mangoubi 2008a, Theorem 1.5] and state that

$$\frac{C_1}{\lambda^{\alpha(n)}} \le \operatorname{inrad}(\Omega_\lambda) \le \frac{C_2}{\sqrt{\lambda}}, \tag{29}$$

where $\alpha(n) := \frac{1}{4}(n-1) + \frac{1}{2n}$. A question of current investigation is whether the last lower bound on $\operatorname{inrad}(\Omega_\lambda)$ in higher dimensions is optimal.

Here we exploit heat equation and Brownian motion techniques to show that at least, one can expect to "almost" inscribe a large ball having radius to the order of $1/\sqrt{\lambda}$, in all dimensions. Now we prove Theorem 1.6:

Proof. We define $t' := t_0/\lambda$, and thus $\psi_{M\backslash\Omega_\lambda}(t', x) \le 1 - e^{-t_0}$, where t_0 is a small constant to be chosen suitably later.

Now, choosing t_0 small enough, and using monotonicity, we have

$$\psi_{B(x_0, r_0\lambda^{-1/2})\backslash\Omega_\lambda}(t, x_0) < \psi_{M\backslash\Omega_\lambda}(t, x_0) < \varepsilon. \tag{30}$$

For convenience, let us define $E_{r_0} := B(x_0, r_0\lambda^{-1/2})\backslash\Omega_\lambda$, a relatively compact set. Observe that Theorem 2.2 applies to open balls and compact subsets contained in open balls. To adapt to the setting of Theorem 2.2, choose a number $r_0' < r_0$ such that $B(x_0, r_0'\lambda^{-1/2})$ satisfies

$$\frac{\operatorname{Vol}\big(B(x_0, r_0\lambda^{-1/2}) \setminus B(x_0, r_0'\lambda^{-1/2})\big)}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))} < \varepsilon.$$

Call $E_{r_0'} := \overline{E_{r_0} \cap B(x_0, r_0'\lambda^{-1/2})}$. Observe that proving

$$\frac{\operatorname{Vol}(E_{r_0'})}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))} < \varepsilon$$

will imply

$$\frac{\operatorname{Vol}(E_{r_0})}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))} < 2\varepsilon,$$

which is what we want.

Now, we would like to compare the volumes of the two sets $E_{r_0'}$ and $B(x_0, r_0\lambda^{-1/2})$. Let $r = r_0/\sqrt{\lambda}$. Recall from [Grigor'yan and Saloff-Coste 2002, Remark 4.1] the following inequality:

$$c\,\frac{\operatorname{cap}(E_{r_0'})r^2}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))}e^{-C\frac{r^2}{t'}} \le \psi_{E_{r_0}}(t', x_0) < \varepsilon, \tag{31}$$

where $\operatorname{cap}(K)$ denotes the 2-capacity of the set $K \subset M$, and $0 < t' < 2r^2$; see also equation (3.20) of [Grigor'yan and Saloff-Coste 2002]. Recall that the 2-capacity of a set $K \subset M$ is defined as

$$\operatorname{cap}(K) = \inf_{\substack{\eta|_K \equiv 1\\ \eta \in C^\infty(M)}} \int_M |\nabla\eta|^2\, dM.$$

Formally, (31) holds on complete noncompact nonparabolic manifolds, which includes $\mathbb{R}^n$, $n \ge 3$. For bringing in our comparability result Theorem 2.2, we fix the ratio $t'/r^2 = \frac{1}{3}$, say, and then choose t_0 small enough that (30) still works. Now (31) applies, albeit with a new constant c as determined by the ratio t/r^2 and Theorem 2.2.

Now, to rewrite the capacity term in (31) in terms of volume, we bring in the following "isocapacitary inequality" [Maz'ya 2011, Section 2.2.3]:

$$\operatorname{cap}(E_{r_0}) \ge C' \operatorname{Vol}(E_{r_0})^{\frac{n-2}{n}}, \quad n \ge 3, \tag{32}$$

where C' is a constant depending only on the dimension n. We note that the isocapacitary inequality (in combination with a suitable Poincaré inequality) lies at the heart of the currently optimal inradius estimates, as derived in [Mangoubi 2008a].

Clearly, (31) and (32) together give

$$\left(\frac{\operatorname{Vol}(E_{r_0})}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))}\right)^{\frac{n-2}{n}} \lesssim \frac{\operatorname{cap}(E_{r_0})r^2}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))} \lesssim \psi_{E_{r_0}}(t, x) < \varepsilon. \tag{33}$$

The last inequalities contain constants depending only on M, so by taking ε even smaller we can arrange

$$\frac{\operatorname{Vol}(E_{r_0})}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))} < \varepsilon_0$$

for any initially given ε_0. □

Remark 4.1. We note that the heat equation method does not distinguish between a general domain and a nodal domain. This means that we cannot rule out the situation where $B(x_0, r_0/\sqrt{\lambda})\setminus\Omega_\lambda$ is a collection of "sharp spikes" entering into $B(x_0, r_0/\sqrt{\lambda})$. Indeed the probability of a Brownian particle hitting a spike, no matter how "thin" it is, or how far from x_0 it is, is always nonzero, a fact related to the infinite speed of propagation of heat diffusion. This is consistent with the heuristic discussed in [Hayman 1978; Lieb 1983].

Now we establish Corollary 1.7. First, we recall the following result, which gives a bound on the asymmetry between the volumes of positivity and negativity sets:

Theorem 4.2 [Mangoubi 2008a]. *Let B be a geodesic ball, so that $\left(\frac{1}{2}B \cap \{\varphi_\lambda = 0\}\right) \neq \varnothing$ with $\frac{1}{2}B$ denoting the concentric ball of half radius. Then*

$$\frac{\operatorname{Vol}(\{\varphi_\lambda > 0\} \cap B)}{\operatorname{Vol}(B)} \geq \frac{C}{\lambda^{\frac{n-1}{2}}}. \tag{34}$$

Proof of Corollary 1.7. It suffices to combine the estimate (33) with (34).

Let $r := r_0/\sqrt{\lambda}$ be the radius of the largest inscribed ball in the nodal domain at x_0. Noting that $\{\varphi_\lambda < 0\} \subseteq E_{r_0}$ and combining Theorem 4.2 for $B_{x_0}(2r)$ with (33), we get

$$\left(\frac{C}{\lambda^{\frac{n-1}{2}}}\right)^{\frac{n-2}{n}} \leq \left(\frac{\operatorname{Vol}(E_{r_0})}{\operatorname{Vol}(B(x_0, r_0\lambda^{-1/2}))}\right)^{\frac{n-2}{n}} \leq 1 - e^{-\sqrt{1/3}r_0^2}. \tag{35}$$

Expanding the right-hand side in Taylor series and rearranging finishes the proof. □

Remark 4.3. An inspection of the proof of Theorem 1.6 reveals that one can take $\varepsilon = r_0^{2n/(n-2)}$. In other words, the relative volume of the error set E_{r_0} decays as $r_0^{2n/(n-2)}$ as $r_0 \to 0$. This is slightly better than the scaling prescribed by Corollary 2 of [Lieb 1983].

Remark 4.4. There is a sizable literature around optimizing the fundamental frequency of the complement of an obstacle inside a domain; for example, see [Harrell et al. 2001]. As an explicit special case, consider a convex domain $\Omega \subset \mathbb{R}^n$ and a small ball $B \subseteq \Omega$. The question is to find possible placements of translate $x + B$ inside Ω such that $\lambda_1(\Omega \setminus (x + B))$ is maximized. For certain applications of Theorem 1.6 towards such questions, we refer to [Georgiev and Mukherjee 2017].

Acknowledgements

It is a pleasure to thank Stefan Steinerberger for his detailed comments on a draft version of this paper, as well as Dan Mangoubi for his comments and remarks. The authors further thank Yuval Peres and Itai Benjamini for advice regarding Martin capacity, as well as Steve Zelditch for discussions on the Feynman–Kac formula and Brownian motion. Thanks are also due to the anonymous referee for a substantial improvement in the final presentation. Lastly, the authors would also like to thank Werner Ballmann, and gratefully acknowledge the Max Planck Institute for Mathematics, Bonn for providing ideal working conditions.

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Received 11 Apr 2016. Revised 20 Jun 2017. Accepted 10 Aug 2017.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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