ANALYSIS & PDE

Volume 11 No. 1 2018

ANTONIO SICONOLFI AND ALFONSO SORRENTINO

GLOBAL RESULTS FOR EIKONAL HAMILTON–JACOBI EQUATIONS ON NETWORKS

msp

ANALYSIS AND PDE
Vol. 11, No. 1, 2018
dx.doi.org/10.2140/apde.2018.11.171

GLOBAL RESULTS FOR EIKONAL HAMILTON–JACOBI EQUATIONS ON NETWORKS

ANTONIO SICONOLFI AND ALFONSO SORRENTINO

We study a one-parameter family of eikonal Hamilton–Jacobi equations on an embedded network, and prove that there exists a unique critical value for which the corresponding equation admits global solutions, in a suitable viscosity sense. Such a solution is identified, via a Hopf–Lax-type formula, once an admissible trace is assigned on an *intrinsic boundary*. The salient point of our method is to associate to the network an *abstract graph*, encoding all of the information on the complexity of the network, and to relate the differential equation to a *discrete functional equation* on the graph. Comparison principles and representation formulae are proven in the supercritical case as well.

1. Introduction

Over the last few years there has been an increasing interest in the study of the Hamilton–Jacobi equation on networks and related questions. These problems, in fact, involve a number of subtle theoretical issues and have a great impact in the applications in various fields, for example, to data transmission, traffic management problems, etc. While locally — i.e., on each branch of the network (*arcs*) — the study reduces to the analysis of 1-dimensional problems, the main difficulties arise in matching together the information “converging” at the *juncture* of two or more arcs, and relating the *local* analysis at a juncture with the *global* structure/topology of the network.

In this article, we provide a thorough discussion of the above issues in the case of eikonal-type Hamilton–Jacobi equations on embedded networks (in $\mathbb{R}^n$ or on a Riemannian manifold, see Remark 3.1). We show that there exists a unique critical value for which the corresponding equation admits global solutions, and extend most of the results known in the continuous setting for the critical and supercritical cases. More specifically: we determine a uniqueness set (the *Aubry set*) for global solutions and provide -1-Lax-type representation formulae; we study critical subsolutions, their properties and constraints, and show the existence of C^1 critical subsolutions; we describe -1-Lax representation formulae for maximal supercritical subsolutions. See the Main Theorem in Section 4 for a more detailed description.

The main rationale behind our approach consists in neatly distinguishing between the local problem on the arcs and the global analysis on the network. While the former can be solved by means of (classical) 1-dimensional viscosity techniques, the latter is definitely more engaging.

MSC2010: primary 35F21, 35R02; secondary 35B51, 49L25.
Keywords: Hamilton–Jacobi equation, embedded networks, graphs, viscosity solutions, viscosity subsolutions, comparison principle, discrete functional equation on graphs, Hopf–Lax formula, discrete weak KAM theory.

Our novel idea is to tackle it by associating to the network an *abstract graph*, encoding all of the information on the complexity of the network, and to relate the problem to a *discrete functional equation* on the graph. This allows us to pursue a global analysis of the equation — that goes beyond what happens at a single juncture — as well as to prove uniqueness and comparison principles in a simpler way. To the best of our knowledge, this is the first time that comparison-type results are obtained in the network setting by completely bypassing the difficulties involved in the Crandall–Lions doubling variable method, in favor of a more direct analysis of a discrete equation.

In addition to this, by exploiting the simple geometry of the abstract graph we are able to identify an intrinsic boundary — the *Aubry set* — on which admissible traces can be assigned in order to get unique critical solutions on the whole network; these solutions can be represented by means of Hopf–Lax-type formulae. In the supercritical case we get existence and uniqueness of solutions, on any open subset of the network, continuously extending admissible data prescribed on the complement.

Let us point out that the problem of formulating boundary problems on the network and accordingly determining "natural" subsets on which to assign boundary data is a subtle issue, yet not well settled in the literature; we believe that our approach helps clarify this matter, at least in the class of equations that we are considering.

The notions of viscosity solution and subsolution that we adopt are very natural in this setting (see Definitions 3.6 and 3.7). More specifically, the tests we use at vertices are classical in viscosity solutions theory and consist in (unilateral) state-constraint-type boundary conditions, introduced by Soner [1986] to study control problems with constraints. In this regard, the notion of solution requires that at each vertex the state-constraint condition holds for at least one arc ending there: it does not require other mixing conditions (on the vertices) between equations defined on different incident arcs.

Very recently, the same notion of solution has been also considered by Lions and Souganidis [2016] to deal with 1-dimensional junction-type problems for nonconvex discounted Hamilton–Jacobi equations and study its well-posedness (i.e., comparison principle and existence). Global solutions on networks, however, are not therein studied.

As far as subsolutions are concerned, we only ask that they are continuous on the network and are (viscosity) subsolutions to the equation on the interior of each arc; no extra conditions are required on vertices. These assumptions are the minimal requirements that one needs to ask and, at a first sight, it might seem surprising that they are sufficient to develop a significant global theory. However, the validity of this approach is supported, among other things, by the fact that the notion of solutions can be recovered in terms of a maximal subsolution attaining a specific value at a given point (vertex or internal point); see Theorem 7.1.

We also wish to point out that our hypotheses, both on the topology of network and the Hamiltonians, are very general. As far as the network is concerned, we only ask it to be made up by finite arcs and connected; hence, it may well include multiple connections between different vertices, as well as the presence of loops.

The Hamiltonians are assumed continuous in both variables, quasiconvex and coercive in the first-order variable on any arc. Hamiltonians on different arcs are independent from each other and no compatibility conditions at the vertices are required. See Section 3B for more details.

We are confident that this very same set of ideas can be successfully applied to a broad range of other problems, for example, to the study of the *discounted* Hamilton–Jacobi equation on networks or to prove *homogenization* results for the Hamilton–Jacobi equation on periodic networks (also known as *topological crystals*). We plan to address these and other questions in a future work (in preparation).

1A. ***Previous related literature.*** There is a huge amount of literature related to differential equations on networks, or other nonregular geometric structures (ramified/stratified spaces), in various contexts: hyperbolic problems, traffic flows, evolutionary equations, (regional) control problems, Hamilton–Jacobi equations, etc. An exhaustive description of all of these areas would go well beyond the aims of this paper; we mention a few noteworthy papers, [Achdou et al. 2013; Barles et al. 2013; 2014; Bressan and Hong 2007; Camilli and Marchi 2013; Camilli et al. 2013; Davini et al. 2016; Galise et al. 2015; Garavello and Piccoli 2006; Imbert and Monneau 2016; 2017; Imbert et al. 2013; Lions and Souganidis 2016; Pokornyi and Borovskikh 2004; Rao et al. 2014; Schieborn and Camilli 2013; Soner 1986].

A model similar to ours has been previously considered by Schieborn and Camilli [2013], however, just in the supercritical case and under some restriction on the topology of the network. In comparison with their hypothesis, we do not require continuity of the Hamiltonians at the vertices (and accordingly, no mixed conditions on the test functions at the vertices) and we do not ask a priori existence of a regular strict subsolution.

Other relevant recent contributions are [Lions and Souganidis 2016] (which we have already mentioned above) and [Imbert and Monneau 2017]. In particular, the latter is a substantial work — whose point of view and techniques are rather different from ours — in which Imbert and Monneau attempt to recover the doubling variable method to their setting, by introducing an extra parameter (the flux limiter) and a companion equation (the junction condition) and by using special vertex test functions. See also other related works by the same authors and collaborators [Galise et al. 2015; Imbert et al. 2013; Imbert and Monneau 2016].

Our analysis of the discrete functional equation is based on ideas and techniques inspired by the so-called *weak KAM theory*, first developed by Fathi [2008] for the study of Tonelli Hamiltonian systems on closed manifolds; see also [Sorrentino 2015]. Developing a similar approach in the discrete setting is very natural and has been already exploited in several other works. In [Bernard and Buffoni 2006; 2007], for example, a discretization of weak KAM theory was applied to investigate the properties of optimal transport maps; a more systematic development of a discrete weak KAM theory for *cost functions* was described by Zavidovique [2010; 2012]; see also [Davini et al. 2016]. In particular, [Zavidovique 2012] shares ideas similar to ours, although our setting has the peculiarity of this interplay between the discrete structure and the embedded network.

From a more dynamical systems point of view, a discrete analogue of Aubry–Mather theory and weak KAM theory was also discussed in [Gomes 2005]; see [Su and Thieullen 2015] for a recent related work.

1B. ***Organization of the article.*** The article is organized as follows.

In Section 2, we provide a brief introduction to some topics in graph theory that will be needed in the following.

In Section 3, we describe our setting and the main objects involved in our analysis. More specifically: in Section 3A we introduce the concept of embedded network and its properties; in Section 3B we define Hamiltonians on a network and we detail which hypotheses we will be imposing thereafter; see (Hγ1)–(Hγ4). Finally, in Section 3C we introduce the eikonal Hamilton–Jacobi equation on a network ($\mathcal{H}Ja$) and provide suitable notions for viscosity solutions and subsolutions (see Definitions 3.6 and 3.7).

Section 4 provides a statement of our main results (see the Main Theorem) and an outline of the strategy of the proof, in order to guide the reader through Section 5 (local part), Section 6 (global part) and Section 7 (from global to local part).

2. Preliminaries on graph theory

We recall some basic material on the theory of abstract graphs and on functions defined on them. For a more detailed presentation of these and other related topics, we refer the interested reader, for instance, to [Sunada 2013].

2A. ***Abstract graphs.*** A (abstract) graph $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E})$ is an ordered pair of disjoint sets $\boldsymbol{V}$ and $\boldsymbol{E}$, which are called, respectively, *vertices* and (directed) *edges*, plus two functions

$$\mathrm{o} : \boldsymbol{E} \to \boldsymbol{V}$$

and

$$\begin{aligned} {}^{-} : \boldsymbol{E} &\to \boldsymbol{E}, \\ e &\mapsto \bar{e}, \end{aligned}$$

with the latter assumed to be a fixed-point-free involution, namely satisfying

$$\bar{e} \neq e \quad \text{and} \quad \bar{\bar{e}} = e \quad \text{for any } e \in \boldsymbol{E}.$$

We give the following geometric picture of the setting: $\mathrm{o}(e)$ is the *origin* (initial vertex) of e and $\bar{e}$ is its *reversed* edge, namely the same edge but with the opposite orientation. Analogously we define

$$\mathrm{t}(e) = \mathrm{o}(\bar{e}),$$

the *terminal* vertex of e. The following compatibility condition holds true:

$$\mathrm{t}(\bar{e}) = \mathrm{o}(\bar{\bar{e}}) = \mathrm{o}(e).$$

We say that e links $\mathrm{o}(e)$ to $\mathrm{t}(e)$; observe that it might well happen that $\mathrm{o}(e) = \mathrm{t}(e)$, and in this case e will be called a *loop*. An edge is also said to be incident on $\mathrm{o}(e)$ and $\mathrm{t}(e)$. Two vertices are called *adjacent* if there is an edge linking them or, in other terms, if there is an edge incident on both of them.

We say that the graph is *finite* if the set $\boldsymbol{E}$, and consequently $\boldsymbol{V}$, has a finite number of elements. We denote by $|\boldsymbol{V}|$ and $|\boldsymbol{E}|$ the number of vertices and edges.

We define a *path* to be a finite sequence of concatenated edges, namely $\xi = (e_1, \dots, e_M) = (e_i)_{i=1}^M$ satisfying

$$\mathrm{t}(e_j) = \mathrm{o}(e_{j+1}) \quad \text{for any } j = 1, \dots, M-1.$$

We set $\mathrm{o}(\xi) = \mathrm{o}(e_1)$ and $\mathrm{t}(\xi) = \mathrm{t}(e_M)$ and call them the initial and final vertices of the path. We say that ξ links $\mathrm{o}(\xi)$ to $\mathrm{t}(\xi)$; we also say that ξ is incident on some vertex if there is some edge composing the path incident on it.

Given two paths ξ and η, we say that ξ is contained in η, mathematically $\xi \subset \eta$, if the edges of ξ make up a subset of the edges of η. If such a subset is proper, we say that ξ is *properly contained* in η. If $\mathrm{t}(\xi) = \mathrm{o}(\eta)$, we denote by $\xi \cup \eta$ the path obtained via concatenation of ξ and η.

We call a path a *loop* or a *cycle* if $\mathrm{o}(\xi) = \mathrm{t}(\xi)$. A path without repetition of vertices except possibly the initial and terminal ones will be called *simple*; in other terms $\xi = (e_i)_1^M$ is simple if

$$\mathrm{t}(e_i) = \mathrm{t}(e_j) \implies i = j,$$

or if there are no cycles properly contained in ξ. Note that there are finitely many simple paths in a finite graph.

A graph is called *connected* if any two vertices are linked by some path. All of the graphs we will consider hereafter are understood to be connected and finite. Observe that the connectedness assumption implies that the map o (and hence t) is surjective.

Given $x \in \boldsymbol{V}$, we set

$$\boldsymbol{E}_x = \{e \in \boldsymbol{E} \mid \mathrm{o}(e) = x\}, \tag{1}$$

which we call $\boldsymbol{E}_x$, the *star centered at* x; it should be considered as a sort of tangent space to the graph at x. The cardinality of $\boldsymbol{E}_x$ is called the *degree* (or *valence*) of the vertex x.

2B. *Functions on graphs.* In the following we will be interested in functions defined on abstract graphs. It is useful to introduce the following notions:

- We define the 0-*cochain group* $\mathfrak{C}^0(\boldsymbol{X}, \mathbb{R})$ as the space of functions from $\boldsymbol{V}$ to $\mathbb{R}$.
- We define the 1-*cochain group* $\mathfrak{C}^1(\boldsymbol{X}, \mathbb{R})$ as the space of functions from $\boldsymbol{E}$ to $\mathbb{R}$, with the compatibility condition $\omega(\bar{e}) = -\omega(e)$. This space plays the role of 1-forms on the graph. From now on we will indicate the reverse edge $\bar{e}$ by $-e$ and we will consider the pairing $\langle \omega, e\rangle := \omega(e)$.

The relation between $\mathfrak{C}^0(\boldsymbol{X}, \mathbb{R})$ and $\mathfrak{C}^1(\boldsymbol{X}, \mathbb{R})$ can be expressed in terms of the so-called *coboundary operator*, or *differential*, $\mathrm{d} : \mathfrak{C}^0(\boldsymbol{X}, \mathbb{R}) \to \mathfrak{C}^1(\boldsymbol{X}, \mathbb{R})$, which is defined for any $f \in \mathfrak{C}^0(\boldsymbol{X}, \mathbb{R})$ and $e \in \boldsymbol{E}$ as

$$\mathrm{d}f(e) := f(\mathrm{t}(e)) - f(\mathrm{o}(e)).$$

We can embed these spaces with the standard topology. A notion of convergence on the cochain spaces is given via

$$f_n \to f \iff f_n(x) \to f(x) \quad \text{for any } x \in \boldsymbol{V},$$
$$\omega_n \to \omega \iff \omega_n(e) \to \omega(e) \quad \text{for any } e \in \boldsymbol{E}.$$

A sequence f_n is said to be *equibounded* if

$$|f_n(x)| \le \beta \quad \text{for any } x \in \boldsymbol{V} \text{ and some } \beta > 0;$$

similarly ω_n is said equibounded if

$$|\langle \omega_n, e\rangle| \le \beta \quad \text{for any } e \in \boldsymbol{E} \text{ and some } \beta > 0.$$

It is clear that any equibounded sequences f_n and ω_n are convergent, up to subsequences.

We directly deduce from the above definitions:

Proposition 2.1. *Let f_n and f be in $\mathfrak{C}^0(\boldsymbol{X}, \mathbb{R})$:*

(i) *If $f_n \to f$, then $\mathrm{d}f_n \to \mathrm{d}f$.*

(ii) *If $\mathrm{d}f_n$ is equibounded and the sequence $f_n(x_0)$ is bounded for some vertex x_0, then f_n is convergent, up to subsequences.*

3. Setting

In this section we first explain our setting, namely what is an *embedded network* and what we mean by *Hamiltonian* on a network. Then we introduce the class of *Hamilton–Jacobi equations* on a network we are interested in, and specify the notions of solutions and subsolutions.

3A. ***Embedded networks.*** An *embedded network*, or *continuous graph*, is a subset $\Gamma \subset \mathbb{R}^N$ of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma([0, 1]) \subset \mathbb{R}^N,$$

where $\mathcal{E}$ is a finite collection of regular (i.e., C^1 with nonvanishing derivative) simple oriented curves, called *arcs* of the network, that we assume, without any loss of generality, to be parametrized on [0, 1]. We denote by $\mathcal{E}^*$ the subset of arcs γ which are closed, namely with $\gamma(0) = \gamma(1)$.

Remark 3.1. Our setting can be easily extended to the case in which Γ is embedded in a Riemannian manifold (M, g), for example by means of Nash embedding theorem [1956]. Moreover, the results are independent of the chosen parametrizations of the arcs. In this regard, one could also choose a more intrinsic approach and consider arcs as 1-dimensional submanifolds, and the whole network as a stratified space. Hereafter we do not adopt this point of view.

Observe that on the support of any arc γ, we also consider the inverse parametrization defined as

$$\tilde{\gamma}(s) = \gamma(1 - s) \quad \text{for } s \in [0, 1].$$

We call $\tilde{\gamma}$ the *inverse arc* of γ. We assume

$$\gamma((0, 1)) \cap \gamma'([0, 1]) = \varnothing \quad \text{whenever } \gamma \neq \gamma' \text{ and } \gamma \neq \tilde{\gamma}'. \tag{2}$$

We call *vertices* the initial and terminal points of the arcs, and denote by $\boldsymbol{V}$ the sets of all such vertices. Note that (2) implies

$$\gamma((0, 1)) \cap \boldsymbol{V} = \varnothing \quad \text{for any } \gamma \in \mathcal{E}.$$

We assume that the network is connected; namely given two vertices there is a finite concatenation of arcs linking them.

The network Γ inherits a *geodesic distance*, denoted by d_Γ, from the Euclidean metric of $\mathbb{R}^N$. Hence, hereafter the notions of continuity and Lipschitz continuity, when referring to functions defined on Γ, must be understood with respect to such distance (which is indeed equivalent to the Euclidean one) and the induced topology.

We can also consider a *differential structure* on Γ by defining the tangent space at any $x \in \Gamma \setminus \boldsymbol{V}$ as

$$T_\Gamma(x) = \{\lambda\, \dot\gamma(t) \mid \lambda \in \mathbb{R},\ \gamma \in \mathcal{E},\ t \in (0,1) \text{ and } x = \gamma(t)\}$$

and the cotangent space $T^*_\Gamma(x)$ as the dual space $(T_\Gamma(x))^*$; namely, it is the set of linear functionals $p : T_\Gamma(x) \to \mathbb{R}$.

We will say that a function $f : \Gamma \to \mathbb{R}$ is of class $C^1(\Gamma \setminus V)$ if it is continuous in Γ and

$$t \mapsto f(\gamma(t)) \text{ is of class } C^1 \text{ in } (0,1) \text{ for any } \gamma \in \mathcal{E}.$$

For such a function we define $D_\Gamma f(x)$, where $x = \gamma(t_0)$ for some $\gamma \in \mathcal{E}$ and $t_0 \in (0,1)$, as the unique covector in $T^*_\Gamma(x)$ satisfying

$$(D_\Gamma f(x), \dot\gamma(t_0)) = \frac{d}{dt} f(\gamma(t))|_{t=t_0},$$

where $(\,\cdot\,,\cdot\,)$ denotes the pairing between covectors and vectors.

Notice that this definition is invariant for a change of parametrization from γ to $\tilde\gamma$.

We can associate to any continuous network Γ an abstract graph $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E})$ with the same vertices as the network and edges corresponding to the arcs. More precisely, we consider an abstract set $\boldsymbol{E}$ with a bijection

$$\Psi : \boldsymbol{E} \to \mathcal{E}. \tag{3}$$

This induces maps $o : \boldsymbol{E} \to \boldsymbol{V}$ and $^- : \boldsymbol{E} \to \boldsymbol{E}$ via

$$\mathrm{o}(e) = \Psi(e)(0) \quad \text{and} \quad \bar e = \Psi^{-1}(\widetilde{\Psi(e)})$$

satisfying the properties in the definition of graph. Intuitively, in the passage from the embedded network to the underlying abstract graph $\boldsymbol{X}$, the arcs become *immaterial* edges.

3B. *Hamiltonians on networks.* A Hamiltonian on a network Γ is a collection of Hamiltonians $\mathcal{H} = \{H_\gamma\}_{\gamma\in\mathcal{E}}$, where

$$H_\gamma : [0,1] \times \mathbb{R} \to \mathbb{R},$$
$$(s,p) \mapsto H_\gamma(s,p),$$

satisfies

$$H_{\tilde\gamma}(s,p) = H_\gamma(1-s,-p) \quad \text{for any } \gamma \in \mathcal{E}. \tag{4}$$

Notice that we are not assuming any periodicity on H_γ when γ is a closed curve.

We require any H_γ to be

(Hγ1) continuous in (s,p);

(Hγ2) coercive in p;

(Hγ3) quasiconvex in p; i.e., for every $a \in \mathbb{R}$ the set $\{p \in \mathbb{R} \mid H_\gamma(x, p) \le a\}$ is convex (provided it is nonempty). Moreover, we assume that

$$\operatorname{Int}\big(\{p \mid H_\gamma(x, p) \le a\}\big) = \{p \mid H_\gamma(x, p) < a\} \quad \text{for any } a \in \mathbb{R},$$

where $\operatorname{Int}(\,\cdot\,)$ denotes the interior of a set.

We point out that, throughout the paper, the term (sub-)solution to Hamilton–Jacobi equations involving the H_γ, must be understood in the viscosity sense; see for example [Bardi and Capuzzo-Dolcetta 1997; Barles 1994] for a comprehensive treatment of viscosity solutions theory.

We set for any $\gamma \in \mathcal{E}$

$$a_\gamma := \max_{s \in [0,1]} \min_{p \in \mathbb{R}} H_\gamma(s, p), \tag{5}$$

$$c_\gamma := \min\{a : H_\gamma = a \text{ admits periodic subsolutions}\}. \tag{6}$$

By periodic subsolution, we mean subsolution to the equation in $(0, 1)$ taking the same value at the endpoints.

Remark 3.2. The definition of c_γ is indeed well-posed. In fact, given $\gamma \in \mathcal{E}$, because of the compactness of $[0, 1]$, we can choose a large enough to have

$$H(s, 0) \le a \quad \text{for any } s \in (0, 1).$$

This shows that any constant function is a subsolution and, consequently, the set in the definition of c_γ is nonempty. It is also bounded from below since for $a < a_\gamma$ the corresponding equation does not admit subsolutions and, therefore, it does not admit periodic ones. Finally, by basic stability properties in viscosity solution theory, there exists a periodic subsolution at the level c_γ, which justifies the minimum appearing in the definition.

We will essentially use c_γ for $\gamma \in \mathcal{E}^*$, but in principle the definition and the above considerations hold for any γ.

We stress that

$$a_\gamma \le c_\gamma \quad \text{for any } \gamma \in \mathcal{E}.$$

We further define

$$a_0 := \max\Big\{ \max_{\gamma \in \mathcal{E} \setminus \mathcal{E}^*} a_\gamma,\ \max_{\gamma \in \mathcal{E}^*} c_\gamma \Big\}. \tag{7}$$

We require a further condition:

(Hγ4) Given any $\gamma \in \mathcal{E}$ with $a_\gamma = a_0$, the map $s \mapsto \min_{p \in \mathbb{R}} H_\gamma(s, p)$ is constant in $[0, 1]$.

Remark 3.3. The main role of (Hγ4) is to ensure uniqueness of solutions to the Dirichlet problem associated to the equation $H_\gamma = a_\gamma$, at least for the γ with $a_\gamma = a_0$. The uniqueness property for such kind of problems holds in general when the equation admits a strict subsolution, which is not the case at the level a_γ. The relevant consequence of condition (Hγ4) is that the family of subsolutions to $H_\gamma = a_\gamma$ reduces to a singleton, up to additive constants; see Proposition 5.3.

Finally, condition (Hγ4) is automatically satisfied if the H_γ are independent of the state variable.

3C. ***The eikonal Hamilton–Jacobi equation on networks.*** We define a notion of subsolution and solution to an equation of the form

$$\mathcal{H}(x, Du) = a \quad \text{on } \Gamma, \tag{$\mathcal{H}Ja$}$$

where $a \in \mathbb{R}$. This notation synthetically indicates the family (for γ varying in $\mathcal{E}$) of Hamilton–Jacobi equations

$$H_\gamma(s, (u \circ \gamma)') = a \quad \text{on } (0, 1). \tag{$HJ_\gamma a$}$$

We start by recalling some terminology of viscosity solutions theory.

Definition 3.4. Given a continuous function w in $[0, 1]$ and a function $\varphi \in C^1([0, 1]$, we say that:

- φ is *subtangent* to w at $s \in (0, 1)$ if

$$w = \varphi \quad \text{at } s \qquad \text{and} \qquad w \geq \varphi \quad \text{in } (s - \delta, s + \delta) \text{ for some } \delta > 0.$$

  The notion of *supertangent* is given by just replacing "$\geq$" by "$\leq$" in the above formula.

- φ is a *constrained subtangent* to w at 1 if

$$w = \varphi \quad \text{at } 1 \qquad \text{and} \qquad w \geq \varphi \quad \text{in } (1 - \delta, 1) \text{ for some } \delta > 0.$$

  A similar notion, with obvious adaptations, can be given at $t = 0$.

Definition 3.5. Given a continuous function w in $[0, 1]$ and a point $s_0 \in \{0, 1\}$, we say that w satisfies *the state-constraint boundary condition* for $(HJ_\gamma a)$ at s_0 if

$$H_\gamma(s_0, \varphi'(s_0)) \geq a$$

for any φ that is a constrained C^1 subtangent to w at s_0.

Definition 3.6. We say that $u : \Gamma \to \mathbb{R}$ is a *subsolution* to $(\mathcal{H}Ja)$ if

(i) it is continuous on Γ;

(ii) $s \mapsto u(\gamma(s))$ is a subsolution to $(HJ_\gamma a)$ in $(0, 1)$ for any $\gamma \in \mathcal{E}$.

We say that u is *solution* to $(\mathcal{H}Ja)$ if

(i) it is continuous;

(ii) $s \mapsto u(\gamma(s))$ is a solution of $(HJ_\gamma a)$ in $(0, 1)$ for any $\gamma \in \mathcal{E}$;

(iii) for every vertex x there is at least one arc γ, having x as terminal point, such that $u(\gamma(s))$ satisfies the state-constraint boundary condition for $(HJ_\gamma a)$ at $s = 1$.

Compare also this definition with the one in [Lions and Souganidis 2016]. As far as we know, the idea of imposing a supersolution condition on just one arc incident to a given vertex first appeared in [Schieborn and Camilli 2013].

We do not provide a notion of supersolution. This could be done straightforwardly but we will not need it in the remainder of the paper.

Definition 3.7. Given an open (in the relative topology) subset $\Gamma' \subset \Gamma$, we say that a continuous function $u : \Gamma \to \mathbb{R}$ is *solution to* $(\mathcal{H}Ja)$ *in* Γ' if for any $x \in \Gamma' \setminus \boldsymbol{V},\ x = \gamma(s_0)$ with $\gamma \in \mathcal{E},\ s_0 \in (0, 1)$, the usual viscosity solution condition holds true for $u \circ \gamma$ at s_0. If instead $x \in \Gamma' \cap \boldsymbol{V}$, we require condition (iii) in Definition 3.6 to hold.

Remark 3.8. The definition of (sub-)solutions on Γ requires $u \circ \gamma$ to be a (sub-)solution of the corresponding equation in $(0, 1)$ on any arc γ. If, in particular γ is a closed curve, we must have in addition $u(\gamma(0)) = u(\gamma(1))$. This explains why on any arc $\gamma \in \mathcal{E}^*$ we are solely interested in periodic (sub-)solutions, namely (sub-)solutions in $(0, 1)$ taking the same value at 0 and 1. This also explains the role of c_γ.

Remark 3.9. Let us point out that if the network is *augmented* by changing the status of a finite number of intermediate points of arcs in Γ, which become new vertices, then the notion of solution to $(\mathcal{H}Ja)$ is not affected. More specifically, if a function is a solution with respect to the original network, then it is still a solution for the augmented one; the converse property holds as well. This issue will be discussed more in detail in Remark 5.16.

Given a continuous function u defined in $[0, 1]$, it is apparent that a C^1 function φ is supertangent (resp. subtangent) to u at $s_0 \in (0, 1)$ if and only if $\tilde{\varphi}(s) := \varphi(1 - s)$ is supertangent (resp. subtangent) to $s \mapsto u(1 - s)$ at $1 - s_0$. Taking into account (4), we derive the following result.

Proposition 3.10. *Given an arc* γ, *a function* $u(s)$ *is a subsolution* (*resp. solution*) *to* $(HJ_\gamma a)$ *if and only if* $s \mapsto u(1 - s)$ *is a subsolution* (*resp. solution*) *to the same equation with* $H_{\tilde{\gamma}}$ *in place of* H_γ.

It is not difficult to see that Lipschitz-continuity of subsolutions on any arc, coming from the coercivity condition in (Hγ2), implies Lipschitz-continuity in Γ with respect to the geodesic distance. We provide a proof in the Appendix for the reader's convenience.

Proposition 3.11. *The family of subsolutions to* $(HJ_\gamma a)$, *provided it is not empty*, *is equi-Lipschitz continuous on* Γ *with respect to the geodesic distance* d_Γ.

We derive from the previous result, plus basic properties of viscosity solutions, the existence of the maximal subsolution attaining a given value at a given point of the network.

Proposition 3.12. *Let a be such that the equation* $(\mathcal{H}Ja)$ *admits subsolution in* Γ. *Given* $y \in \Gamma,\ \alpha \in \mathbb{R}$, *the function*

$$w(x) = \max\{u(x) \mid \textit{subsolution to } (\mathcal{H}Ja) \textit{ with } u(y) = \alpha\}$$

is still a subsolution.

4. Main results and strategy of the proof

The remainder of the article consists of the proof of our results on existence, uniqueness and regularity of global (sub-)solutions to the eikonal Hamilton–Jacobi equation on Γ.

Main Theorem. *Let* Γ *be an embedded network (finite, connected, possibly including loops and more arcs connecting two vertices) and let* $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E})$ *be the underlying abstract graph. Let us consider a Hamiltonian* $\mathcal{H} = \{H_\gamma\}_{\gamma \in \mathcal{E}}$ *on the network, satisfying conditions* (Hγ1)–(Hγ4) *for any* $\gamma \in \mathcal{E}$ *and let* a_0 *denote the value defined in* (7).

I. *Global solutions*:

(i) (*existence*) *There exists a unique value* $c = c(\mathcal{H}) \geq a_0$ — *called the* ***Mañé critical value*** — *for which the equation* $\mathcal{H}(x, Du) = c$ *admits global solutions. In particular, these solutions are Lipschitz continuous on* Γ.

(ii) (*uniqueness*) *There exists a* ***uniqueness set*** $\mathcal{A}_{\boldsymbol{X}} = \mathcal{A}_{\boldsymbol{X}}(\mathcal{H}) \subseteq \boldsymbol{V}$, *called the* (***projected***) ***Aubry set*** *of* $\mathcal{H}$, *such that the following holds. Let* $S_c : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ *be the function defined in* (34); *then, given any* ***admissible trace*** g *on* $\mathcal{A}_{\boldsymbol{X}}$, *i.e., a function* $g : \mathcal{A}_{\boldsymbol{X}} \to \mathbb{R}$ *such that for every* $x, y \in \mathcal{A}_{\boldsymbol{X}}$

$$g(x) - g(y) \leq S_c(y, x),$$

there exists a unique global solution $u \in C(\Gamma, \mathbb{R})$ *to* $\mathcal{H}(x, Du) = c$ *agreeing with* g *on* $\mathcal{A}_{\boldsymbol{X}}$. *Conversely, for any solution* u *to* $\mathcal{H}(x, Du) = c$, *the function* $g = u|_{\mathcal{A}_{\boldsymbol{X}}}$ *gives rise to an admissible trace on* $\mathcal{A}_{\boldsymbol{X}}$.

(iii) (*Hopf–Lax-type formula* 1) *Let* $g : \mathcal{A}_{\boldsymbol{X}} \to \mathbb{R}$ *be an admissible trace and* $u \in C(\Gamma, \mathbb{R})$ *the corresponding solution to* $\mathcal{H}(x, Du) = c$. *Then, on the support of any arc* $\gamma \in \mathcal{E}$, u *is given by*

$$u(\gamma(s)) = \min\{\boldsymbol{A}, \boldsymbol{B}\},$$

where

$$\boldsymbol{A} := \min\{g(y) + S_c(y, \gamma(0)) \mid y \in \mathcal{A}_{\boldsymbol{X}}\} + \int_0^s \sigma_c^+(t)\, dt,$$
$$\boldsymbol{B} := \min\{g(y) + S_c(y, \gamma(1)) \mid y \in \mathcal{A}_{\boldsymbol{X}}\} - \int_s^1 \sigma_c^-(t)\, dt,$$

with $s \in [0, 1]$ *and* σ_c^+, σ_c^- *defined as in* (8), (9) *with* H_γ *in place of* H.

(iv) (*Hopf–Lax-type formula* 2): *Let* Γ' *be a closed subset of* Γ *with*

$$\Gamma' \cap \gamma([0, 1]) \neq \varnothing \quad \textit{for any } \gamma \textit{ with } \Psi^{-1}(\gamma) \in \mathcal{A}^*_{\boldsymbol{X}}.$$

For any ***admissible trace*** g *on* Γ', *in the sense of* (65) *with* c *in place of* a, *there exists a unique solution* $u \in C(\Gamma, \mathbb{R})$ *to* $\mathcal{H}(x, Du) = c$ *agreeing with* g *on* Γ', *which is given by*

$$u(x) = \min\{g(y) + S_c^\Gamma(y, x) \mid y \in \Gamma'\},$$

where $S_c^\Gamma(\cdot, \cdot)$ *denotes the intrinsic (semi-)distance defined in* (63).

II. *Subsolutions*:

(i) (*maximal subsolutions*) *For* $a \geq c$ *and* $y \in \Gamma$, *the maximal subsolution to* ($\mathcal{H}Ja$) *taking an assigned value at* y *is a solution in* $\Gamma \setminus \{y\}$.

(ii) (*PDE characterization of the Aubry set*) *Let* $\mathcal{A}_\Gamma = \mathcal{A}_\Gamma(\mathcal{H}) \subset \Gamma$ *be the Aubry set on the network, as defined in* (49). *The maximal subsolution to* ($\mathcal{H}Jc$) *taking a given value at a point* $y \in \Gamma$ *is a critical solution on the whole network if and only if* $y \in \mathcal{A}_\Gamma$.

(iii) (*regularity of critical subsolutions*) *Any subsolution* $v : \Gamma \to \mathbb{R}$ *to* $\mathcal{H}(x, Du) = c$ *is of class* $C^1(\Gamma \setminus \boldsymbol{V})$ *and they all possess the same differential on* $\mathcal{A}_\Gamma \setminus \boldsymbol{V}$. *More specifically, if* $x_0 \in \mathcal{A}_\Gamma$ *and* $x_0 = \gamma(s_0)$ *for some* $\gamma \in \mathcal{E}$ *and* $s_0 \in (0, 1)$, *then its differential at* x_0 *is uniquely determined by the relation*

$$(D_\Gamma v(x_0), \dot\gamma(s_0)) = \sigma_c^+(s_0),$$

where σ_c^+ *was defined in* (8), *and therefore*

$$v(\gamma(s)) = v(\gamma(0)) + \int_0^s \sigma_c^+(t)\,dt \quad \textit{for any } s \in [0, 1].$$

We infer from this that any pair of critical subsolutions differs by a constant on the support of γ.

(iv) (*existence of* C^1 *critical subsolutions*) *Given a function* $g : \boldsymbol{V} \to \mathbb{R}$ *such that*

$$g(x) - g(y) \le S_c(y, x) \quad \textit{for all } x, y \in \boldsymbol{V},$$

there exists a critical subsolution v *on* Γ, *with* $v = g$ *on* $\boldsymbol{V}$, *which is of class* C^1 *on* $\Gamma \setminus \boldsymbol{V}$. *In addition, there exists a critical subsolution* v *of class* $C^1(\Gamma \setminus \boldsymbol{V})$ *satisfying*

$$H_\gamma(s, Dv(\gamma(s))) < c$$

for all $s \in (0, 1)$ *and* $\gamma \in \mathcal{E}$ *with* $\gamma((0, 1)) \cap \mathcal{A}_\Gamma = \varnothing$.

(v) (*Hopf–Lax formula for maximal supercritical subsolutions* 1) *Let* $a > c$ *and* $\boldsymbol{V}' \subset \boldsymbol{V}$. *For any* $g : \boldsymbol{V}' \to \mathbb{R}$ *satisfying*

$$g(x) - g(y) \le S_a(y, x) \quad \textit{for all } x, y \in \boldsymbol{V}',$$

where $S_a(\cdot\,,\cdot)$ *was defined in* (34), *there exists a unique solution* u *to* $\mathcal{H}(x, Du) = a$ *in* $\Gamma \setminus \boldsymbol{V}'$ *agreeing with* g *on* $\boldsymbol{V}'$; *in addition,* u *is also a subsolution to* $\mathcal{H}(x, Du) = a$ *on the whole of* Γ. *In particular*, *on the support of any arc* $\gamma \in \mathcal{E}$, u *is given by*

$$u(\gamma(s)) = \min\{\boldsymbol{C},\ \boldsymbol{D}\},$$

where

$$\boldsymbol{C} := \tilde g(\gamma(0)) + \int_0^s \sigma_a^+(t)\,dt,$$

$$\boldsymbol{D} := \tilde g(\gamma(1)) - \int_s^1 \sigma_a^-(t)\,dt,$$

$$\tilde g(x) := \begin{cases} g(x) & \textit{if } x \in \boldsymbol{V}', \\ \min\{g(y) + S_a(y, x) \mid y \in \boldsymbol{V}'\} & \textit{if } x \notin \boldsymbol{V}', \end{cases}$$

with $s \in [0, 1]$ *and* σ_a^+, σ_a^- *defined as in* (8), (9).

(vi) (*Hopf–Lax formula for maximal supercritical subsolutions* 2) *Let* $a > c$ *and* Γ' *be a closed subset of* Γ. *Let* g *be an admissible trace on* Γ', *in the sense of* (65), *then there exists a unique solution* $u \in C(\Gamma, \mathbb{R})$ *to* ($\mathcal{H}Ja$) *on* $\Gamma \setminus \Gamma'$ *agreeing with* g *on* Γ', *which is given by*

$$u(x) = \min\{g(y) + S_a^\Gamma(y, x) \mid y \in \Gamma'\},$$

where $S_a^\Gamma(\cdot, \cdot)$ *denotes the intrinsic* (*semi-*)*distance defined in* (63).

4A. ***Organization of the remaining sections and proof of the Main Theorem.*** For the sake of clarity, we provide here an outline of the forthcoming discussion and of the main steps involved in the proof.

In Section 5, we focus on the *local* problem on each arc of the network. Namely, for each $\gamma \in \mathcal{E}$ we study the existence of (sub-)solutions to the 1-dimensional eikonal Hamilton–Jacobi equation ($HJ_\gamma a$) with boundary conditions. In particular:

- We show that under suitable *admissibility conditions* on the boundary data, see (17), there exists a unique solution and we provide a representation formula (Proposition 5.5).
- We derive a characterization of condition (iii) in Definition 3.6 in terms of this representation formula (Proposition 5.6).

In Section 6 we concentrate on the global aspects of the problem:

- We introduce a *discrete functional equation* ($\mathcal{D}FEa$) on the abstract graph $\boldsymbol{X}$ and provide the corresponding notions of solutions and subsolutions. The crucial result linking solutions to this equation and solutions to ($\mathcal{D}FEa$) is proven in Proposition 6.2.
- In (30) we define the *Mañé critical value* $c(\mathcal{H})$. We first prove that this is the unique value for which solutions to the discrete functional equation may exist (Proposition 6.5), and then that the critical equation ($\mathcal{D}FEc$) indeed admits solutions (Theorem 6.16).
- In (39) and (40) we define the *Aubry set* $\mathcal{A}_{\boldsymbol{X}}^*$ and the *projected Aubry set* $\mathcal{A}_{\boldsymbol{X}}$, which are nonempty (Lemma 6.20). We prove in Theorem 6.21 that $\mathcal{A}_{\boldsymbol{X}}$ is a uniqueness set and provide a Hopf–Lax-type representation formula for the solutions to ($\mathcal{D}FEc$) in terms of its values on $\mathcal{A}_{\boldsymbol{X}}$.

The supercritical case will be discussed in parallel to the critical one (see Propositions 6.3 and 6.6 and Theorem 6.23).

Finally, in Section 7 we switch our attention back to the immersed network:

- We prove in Theorem 7.1 that the notion of solution can be recovered in terms of maximal subsolution attaining a specific value at a given point.
- We introduce the analogue of the Aubry set on the network, we show in Theorem 7.5 that all critical subsolutions are of class C^1 on it and they all have the same differential on this set.
- We show the existence of C^1 critical subsolutions that are strict outside of the Aubry set (Theorem 7.6).
- We provide representation formulae and uniqueness results with traces that are not necessarily defined on vertices (Theorem 7.9).

For the reader's convenience, we provide here some references to the proof of each claim.

Proof of the Main Theorem.

(I) (i) Existence follows from Theorem 6.16 and Proposition 6.2; Lipschitz continuity follows from Proposition 3.11.
(ii) This part is obtained by combining Proposition 6.2 and Theorem 6.21.
(iii) This representation formula is proved in Proposition 5.5.
(iv) See Theorem 7.9(i).

(II) (i) See Theorem 7.1.
(ii) See Proposition 7.4.
(iii) See Theorem 7.5.
(iv) See Theorem 7.6.
(v) These results are obtained by combining Propositions 6.3 and 6.6 and Theorem 6.23 and using the representation formula in Proposition 5.5.
(vi) See Theorem 7.9(ii). □

5. Local part: the eikonal Hamilton–Jacobi equation with boundary conditions on arcs

In this section we focus on a single arc γ and study the family of equations $(HJ_\gamma a)$ in (0, 1), plus suitable boundary conditions. We assume

$$a \geq a_0 = \max\Big\{ \max_{\gamma\in\mathcal{E}\setminus\mathcal{E}^*} a_\gamma,\ \max_{\gamma\in\mathcal{E}^*} c_\gamma \Big\}.$$

Our aim is to find admissible conditions on boundary data at $s = 0$ and $s = 1$ to get solutions of the corresponding Dirichlet problem, to show uniqueness of such solutions and, finally, to provide a characterization of maximal subsolutions taking a given value at $s = 0$ via state-constraint boundary conditions.

We need specific results when γ is a closed curve because in this case we are solely interested in periodic (sub-)solutions, as explained in Remark 3.8. We address the issue in Section 5C. In Subsections 5A and 5B we will not distinguish between γ closed or not, and provide an unified presentation of the material.

The results are not new; we write down nevertheless the 1-dimensional representation formulae, which are easy to handle and allow a direct and simplified treatment of the matter. We recall that, due to coercivity and quasiconvexity assumptions, all subsolutions to $(HJ_\gamma a)$ are Lipschitz-continuous in [0, 1], and, in addition the notion of viscosity and a.e. subsolution are equivalent. Also notice that the subsolution property is not affected by addition of constants.

To ease notation, we write $H(s, p)$ instead of $H_\gamma(s, p)$, and accordingly we consider equation $(HJ_\gamma a)$ with H in place of H_γ. We recall that the assumptions (Hγ1)–(Hγ4) are in force.

5A. ***Setting of the local problem.*** We set, for $s \in [0, 1]$,

$$\sigma_a^+(s) = \max\{p \mid H(s, p) = a\}, \tag{8}$$

$$\sigma_a^-(s) = \min\{p \mid H(s, p) = a\}. \tag{9}$$

If $a > a_\gamma$, we have by (Hγ3)

$$(\sigma_a^-(s), \sigma_a^+(s)) = \{p \mid H(s, p) < a\} \quad \text{for } s \in [0, 1]. \tag{10}$$

We deduce from assumption (Hγ4) that if $a_\gamma = a_0$

$$\sigma_{a_\gamma}^+(s) = \sigma_{a_\gamma}^-(s) \quad \text{for any } s \in [0, 1]. \tag{11}$$

Proposition 5.1. *The functions $s \mapsto \sigma_a^+(s)$ and $s \mapsto \sigma_a^-(s)$ are continuous in* $[0, 1]$ *for any $a \geq a_\gamma$.*

Proof. It follows directly from the continuity and the coercivity of H that the function $s \mapsto \sigma_{a_\gamma}^+(s) = \sigma_{a_\gamma}^-(s)$ is continuous. If $a > a_\gamma$, the assertion follows from the fact that $\sigma_a^+(s)$ and $\sigma_a^-(s)$ are univocally determined for any s by the conditions $H(s, \sigma_a^+(s)) = H(s, \sigma_a^-(s)) = a$ and, respectively, $\sigma_a^+(s) > \sigma_{a_\gamma}^+(s)$ or $\sigma_a^-(s) < \sigma_{a_\gamma}^+(s)$. □

Notice that

$$u \text{ subsolution } \implies \sigma^-(s) \leq u'(s) \leq \sigma^+(s) \text{ for a.e. } s. \tag{12}$$

We introduce four relevant functions:

$$s \mapsto \int_0^s \sigma_a^+(t)\, dt, \tag{13}$$

$$s \mapsto \int_0^s \sigma_a^-(t)\, dt, \tag{14}$$

$$s \mapsto -\int_s^1 \sigma_a^-(t)\, dt, \tag{15}$$

$$s \mapsto -\int_s^1 \sigma_a^+(t)\, dt. \tag{16}$$

Remark 5.2. According to (12), the function in (13) is the maximal (sub-)solution to $(HJ_\gamma a)$ vanishing at $s = 0$, and the one in (14) the minimal (sub-)solution vanishing at $s = 0$. Analogously, the function defined in (15) is the maximal (sub-)solution vanishing at $s = 1$, and the one in (16) the minimal (sub-)solution vanishing at $s = 1$. All of these functions are of class C^1 because of Proposition 5.1.

We remark that when we write *maximal (sub-)solution* and the like, we mean it is maximal in the class of subsolution to $(HJ_\gamma a)$ with a given property and it is, in addition, a solution to the equation.

If $a = a_\gamma$, it follows from (11) that all of the above functions coincide up to an additive constant. We can state the following result.

Proposition 5.3. *The* (*sub-*)*solution to* $(HJ_\gamma a)$, *with $a = a_\gamma$, is unique up to additive constants.*

From the properties of the solutions in (13) and (14), we directly derive a necessary condition (*admissibility condition*) that two boundary data at 0 and 1 must satisfy in order to correspond to the values at the endpoints of a subsolution to $(HJ_\gamma a)$.

Lemma 5.4. *Assume that there is a subsolution to $(HJ_\gamma a)$ taking the values α and β at* 0 *and* 1. *Then*

$$\int_0^1 \sigma_a^-(t)\, dt \leq \beta - \alpha \leq \int_0^1 \sigma_a^+(t)\, dt. \tag{17}$$

The above condition is actually also sufficient:

Proposition 5.5. *Given boundary data* α, β *satisfying* (17), *the function* w,

$$s \mapsto w(s) := \min\left\{\alpha + \int_0^s \sigma_a^+(t)\,dt,\ \beta - \int_s^1 \sigma_a^-(t)\,dt\right\}, \tag{18}$$

is the unique solution to $(HJ_\gamma a)$ *taking the values* α *at* $s = 0$ *and* β *at* $s = 1$.

The proof is in the Appendix.

5B. *Maximal subsolutions.* The main result of this section is:

Proposition 5.6. *Assume that* w *is a solution in* (0, 1) *to* $(HJ_\gamma a)$ *for* $a \geq a_\gamma$, *continuously extended up to the boundary. If*

$$H(1, \varphi'(1)) \geq a \quad \textit{for any } C^1 \textit{ supertangent } \varphi \textit{ to } w \textit{ constrained to } [0, 1], \tag{19}$$

then w *is the maximal* (*sub-*)*solution taking the value* $w(0)$ *at* 0. *Namely*,

$$w(s) = w(0) + \int_0^s \sigma_a^+(t)\,dt \quad \textit{for } s \in [0, 1]. \tag{20}$$

Conversely, *if a solution* w *is of the form* (20), *then condition* (19) *holds true.*

The proof is in the Appendix.

We fix $s_0 \in (0, 1)$. By slightly generalizing the formulae provided in the previous result and arguing separately in the two subintervals $[0, s_0]$ and $[s_0, 1]$, we get:

Corollary 5.7. *Let* $s_0 \in (0, 1)$. *For any* $\alpha \in \mathbb{R}$, *the function*

$$s \mapsto \begin{cases} \alpha - \int_s^{s_0} \sigma_a^-(t)\,dt & \textit{for } s \leq s_0, \\ \alpha + \int_{s_0}^s \sigma_a^+(t)\,dt & \textit{for } s > s_0 \end{cases}$$

is the maximal subsolution to $(HJ_\gamma a)$ *taking the value* α *at* s_0. *It is*, *in addition*, *a solution in* $(0, 1) \setminus \{s_0\}$, *but the solution property fails at* s_0, *unless* $a = a_\gamma$.

Remark 5.8. In light of Proposition 3.10 and Remark 5.2, it is apparent that the maximal solution to $H(s, -u') = a$ vanishing at $s = 0$ is given by

$$s \mapsto -\int_{1-s}^1 \sigma_a^-(t)\,dt.$$

This function satisfies the state-constraint boundary condition at $s = 1$.

5C. *Closed arcs.* In this subsection we assume that γ is a closed curve. Keeping in mind Remark 3.8, we aim to show the existence of a periodic (sub-)solution for any a or, in other terms, that periodic boundary conditions at $s = 0$ and $s = 1$ are admissible in the sense of (17).

Recall that $a \geq a_0 \geq c_\gamma$. We derive further information in the case where $a = a_0 = c_\gamma$. We will exploit the existence of periodic subsolutions at the level c_γ in (0, 1), say, to fix ideas, vanishing at 0 and 1, as pointed out in Remark 3.2. These periodic subsolutions are *sandwiched* in between the function in (13) and the one in (14), according to Remark 5.2. We derive:

Lemma 5.9. *We have*

$$\int_0^1 \sigma_a^-(t)\,dt \le 0 \le \int_0^1 \sigma_a^+(t)\,dt, \tag{21}$$

and both the inequalities are strict if $a > c_\gamma$.

This, in view of (17), in turn implies:

Corollary 5.10. *There are periodic solutions to $(HJ_\gamma a)$ in $(0, 1)$.*

Moreover:

Proposition 5.11.
$$\min\left\{-\int_0^1 \sigma_{c_\gamma}^-(t)\,dt,\ \int_0^1 \sigma_{c_\gamma}^+(t)\,dt\right\} = 0.$$

The proof is in the Appendix.

From the previous result plus Proposition 5.6 and Remark 5.8, we derive the following.

Corollary 5.12. *Let $a = c_\gamma$ and $\alpha \in \mathbb{R}$; then, either the maximal solution to $H = a$ taking the value α at $s = 0$ or the maximal solution to $H(s, -u') = a$ taking the value α at $s = 0$ is periodic.*

In the final result of the section we provide a characterization for the maximal periodic subsolution taking a given value at $s_0 \in (0, 1)$. This corresponds, in the case of nonclosed arcs, to Corollary 5.7.

Corollary 5.13. *Let $s_0 \in (0, 1)$ and $\alpha \in \mathbb{R}$. We set*

$$\beta = \min\left\{-\int_0^{s_0} \sigma_a^-(t)\,dt,\ \int_{s_0}^1 \sigma_a^+(t)\,dt\right\}.$$

(i) *The maximal periodic subsolution to $(HJ_\gamma a)$ taking the value α at s_0, denoted by u, is uniquely determined by the condition of being solution of the equation in $(0, s_0)$ and $(s_0, 1)$ taking the values α at s_0 and $\alpha + \beta$ at 0 and 1.*

(ii) *If $\beta = -\int_0^{s_0} \sigma_a^-(t)\,dt$, then*

$$u(s) = \alpha - \int_s^{s_0} \sigma_a^-(t)\,dt \quad \text{for } s \in [0, s_0]. \tag{22}$$

If instead $\beta = \int_{s_0}^1 \sigma^+ a(t)\,dt$, then

$$u(s) = \alpha + \int_{s_0}^{s} \sigma_a^+(t)\,dt \quad \text{for } s \in [s_0, 1]. \tag{23}$$

The proof is in the Appendix.

5D. *From local to global.* The subsequent step in our analysis will be to transfer the Hamilton–Jacobi equation from Γ to the underlying graph $\boldsymbol{X}$, where it will take the form of a discrete functional equation. In doing this, the relevant information we derive from the above study is the value at $s = 1$ of the maximal solution to $H = a$ vanishing at $s = 0$. It is given, in accordance with Proposition 5.6, by

$$\int_0^1 \sigma_a^+(t)\,dt.$$

Therefore, if $\gamma = \Psi(e)$ and $a \geq a_\gamma$, we define

$$\sigma_a(e) := \int_0^1 \sigma_a^+(t)\,dt. \tag{24}$$

(recall that $a \geq a_0 \geq c_\gamma$).

Accordingly, we have

$$\sigma_a(-e) := -\int_0^1 \sigma_a^-(t)\,dt. \tag{25}$$

If e is a loop, or equivalently $\gamma = \Psi(e)$ a closed curve, we summarize the information gathered in Propositions 5.9 and 5.11 as follows:

Proposition 5.14. *If e is a loop then $\sigma_a(e) > 0$ for $a > c_\gamma$ and*

$$\min\{\sigma_{c_\gamma}(e),\ \sigma_{c_\gamma}(-e)\} = 0.$$

Moreover, we directly deduce from the definition of σ_a and (10) that:

Lemma 5.15. *The function*

$$a \mapsto \sigma_a(e)$$

is continuous and strictly increasing in $[a_\gamma, +\infty)$.

Remark 5.16. As already announced in Remark 3.9, we conclude this section with a remark on the invariance of the definition of solution to $(\mathcal{H}Ja)$ with respect to the addition of extra vertices to the network (*augmented network*). We discuss this issue in the case of a single extra vertex $x_0 = \gamma(s_0)$ for some $s_0 \in (0, 1)$ and γ a nonclosed arc.

We first prove that a solution u on Γ is also a solution for the augmented network. According to Proposition 5.5,

$$u(x_0) = \min\left\{u(\gamma(0)) + \int_0^{s_0} \sigma_a^+(t)\,dt,\ u(\gamma(1)) - \int_s^1 \sigma_a^-(t)\,dt\right\}.$$

If $u(x_0)$ is equal to the first term in the parentheses, then, by Proposition 5.6, u satisfies the state-constraint boundary condition with respect to the arc $\gamma|_{[0,s_0]}$, having the new vertex x_0 as terminal point. Whereas, if $u(x_0)$ equals the second term in the above formula, then the same property holds true for the arc $\tilde{\gamma}|_{[0,1-s_0]}$. This shows the claim.

To prove the converse, we start with a solution v to $(\mathcal{H}Ja)$ on the augmented network, with x_0 as the extra vertex, and consider the arcs $\gamma_1 = \gamma|_{[0,s_0]}$ and $\gamma_2 = \gamma|_{[s_0,1]}$, both parametrized on [0, 1]. The point is to show that the function $v \circ \gamma$ is a solution to $(HJ_\gamma a)$ in (0, 1). It is apparently a subsolution in the whole interval and a solution in $(0, 1) \setminus \{s_0\}$. It also satisfies the state-constraint boundary condition at $s = 1$ either for the arc γ_1 or for $\tilde{\gamma}_2$. Since any subtangent to $v \circ \gamma$ at s_0 is a constrained subtangent at $s = 1$ for both γ_1 and $\tilde{\gamma}_2$, we deduce the supersolution property for $v \circ \gamma$ at s_0.

Arguing along the same lines, one can also check that the forthcoming notions of critical value and Aubry set are not affected by additions of new vertices.

6. Global part: the discrete functional equation on the abstract graph

In this section we push our analysis beyond the local existence of solutions to $(HJ_\gamma a)$ on each arc γ, and study the global existence of solutions to $(\mathcal{H}Ja)$ on the whole network Γ.

Let us start by noticing that if we consider $\boldsymbol{V}$, the set of vertices of Γ, it is easy to check that any solution w to $(\mathcal{H}Ja)$ has a well-defined *trace* $u = w|_V$ on $\boldsymbol{V}$, simply because of the continuity assumption. The following uniqueness result is straightforward. We provide a proof in the Appendix for reader's convenience.

Proposition 6.1. *Let u be a function defined on $\boldsymbol{V}$. Then there exists at most one solution to $(\mathcal{H}Ja)$ on Γ agreeing with u on $\boldsymbol{V}$.*

A converse property is by far more interesting, namely to find conditions on a function defined on $\boldsymbol{V}$ in order to (uniquely) extend it on the whole network as solution to $(\mathcal{H}Ja)$.

This issue — which is profoundly related to the global structure of the network — will be carefully addressed in this section.

More precisely, we study the problem of the admissibility, with respect to the fullref $(\mathcal{H}Ja)$, of a trace $g : \boldsymbol{V} \to \mathbb{R}$ defined on the global network and characterize all traces g that can be continuously extended to solutions to $(\mathcal{H}Ja)$ on the whole of Γ as solutions to an appropriate *discrete functional equation* on the underlying abstract graph $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E})$.

6A. ***The discrete functional equation.*** Given $a \ge a_0$, the cochain $\sigma_a \in \mathfrak{C}^1(\boldsymbol{X}, \mathbb{R})$ is defined as in (24), where $e = \Psi^{-1}(\gamma)$ and Ψ has been defined in (3).

If we recall the admissibility condition introduced in (17) plus (24) and (25), it is clear that the trace on $\boldsymbol{V}$ of a function $g : \Gamma \to \mathbb{R}$ admissible for the equations on any arc satisfies

$$-\sigma_a(-e) \le \mathrm{d}g(e) = g(\mathrm{t}(e)) - g(\mathrm{o}(e)) \le \sigma_a(e) \quad \text{for any } e \in \boldsymbol{E}, \tag{26}$$

which in particular implies

$$g(x) \le \min_{e \in \boldsymbol{E}_x} \big(g(\mathrm{t}(e)) + \sigma_a(-e)\big) \quad \text{for } x \in \boldsymbol{V},$$

where $\boldsymbol{E}_x$ denotes the star centered at x, as defined in (1).

Inspired by this, we introduce the following *discrete functional equation*:

$$u(x) = \min_{e \in \boldsymbol{E}_x} \big(u(\mathrm{t}(e)) + \sigma_a(-e)\big) \quad \text{for } x \in \boldsymbol{V}. \tag{$\mathcal{D}FEa$}$$

Observe that the formulation of the discrete problem takes somehow into account the backward character of viscosity solutions.

A function v is a solution to $(\mathcal{D}FEa)$ in some subset $\boldsymbol{V}'$ of $\boldsymbol{V}$ if $(\mathcal{D}FEa)$ holds true with v in place of u and $x \in \boldsymbol{V}'$.

A function $u : \boldsymbol{V} \to \mathbb{R}$ is a *subsolution* to $(\mathcal{D}FEa)$ if

$$u(x) \le \min_{e \in \boldsymbol{E}_x} \big(u(\mathrm{t}(e)) + \sigma_a(-e)\big) \quad \text{for } x \in \boldsymbol{V} \tag{27}$$

or, equivalently, if for each $e \in \boldsymbol{E}$ we have

$$du(e) \le \sigma_a(e), \tag{28}$$

which is equivalent to asking that $u(\mathrm{t}(e)) \le u(\mathrm{o}(e)) + \sigma_a(e)$ for each $e \in \boldsymbol{E}$.

A subsolution is qualified as *strict at* $x_0 \in \boldsymbol{V}$ if a strict inequality prevails in (27) when x is replaced by x_0. We say that u is *strict on a set* $A \subseteq \boldsymbol{V}$ if it is strict at every $x \in A$. We say that u is *strict* if it is strict everywhere on $\boldsymbol{V}$.

It is apparent that the property of being a solution or a subsolution is not affected by the addition of constants.

Our goal is to prove the existence of a solution to ($\mathcal{D}FEa$) (see Theorem 6.16). In fact, there is a crucial relation between the functional equation ($\mathcal{D}FEa$) and ($\mathcal{H}Ja$):

Proposition 6.2. *Given* $a \ge a_0$:

(i) *Any solution to* ($\mathcal{D}FEa$) *in* $\boldsymbol{V}$ *can be* (*uniquely*) *extended to a solution of* ($\mathcal{H}Ja$) *in* Γ; *conversely the trace on* $\boldsymbol{V}$ *of any solution of* ($\mathcal{H}Ja$) *in* Γ *is a solution to* ($\mathcal{D}FEa$).

(ii) *Any subsolution to* ($\mathcal{D}FEa$) *in* $\boldsymbol{V}$ *can be extended to a subsolution of* ($\mathcal{H}Ja$) *in* Γ; *conversely the trace on* $\boldsymbol{V}$ *of any subsolution of* ($\mathcal{H}Ja$) *in* Γ *is a subsolution to* ($\mathcal{D}FEa$).

Proof. Assume that u solves ($\mathcal{D}FEa$). Let x and y be two adjacent vertices, and e an edge with initial vertex x and final vertex y. We set $\gamma = \Psi(e)$ and consequently $\tilde{\gamma} = \Psi(-e)$; then $\gamma(0) = \tilde{\gamma}(1) = x$ and $\gamma(1) = \tilde{\gamma}(0) = y$. By the very definition of (sub-)solution to ($\mathcal{D}FEa$), we have

$$\begin{aligned} &u(\gamma(1)) - u(\gamma(0)) \le \sigma_a(e), \\ &u(\gamma(1)) - u(\gamma(0)) = u(\tilde{\gamma}(0)) - u(\tilde{\gamma}(1)) \ge -\sigma_a(-e). \end{aligned}$$

Taking into account (17), we derive that the values $u(\gamma(0))$ and $u(\gamma(1))$ are admissible for ($HJ_\gamma a$) in $(0, 1)$. We therefore deduce from Proposition 5.5 that there is a unique solution, say $w : [0, 1] \to \mathbb{R}$, to ($HJ_\gamma a$) taking precisely these values at the boundary. We define

$$v(z) = w(\gamma^{-1}(z)) \quad \text{for } z \in \gamma((0, 1)).$$

Since $\gamma((0, 1)) = \tilde{\gamma}((0, 1))$, one needs to check that this definition is well-posed, performing the same construction for $\tilde{\gamma}$, but this is a direct consequence of Proposition 3.10.

So far, we have successfully checked conditions (i) and (ii) in the definition of solution to ($\mathcal{H}Ja$) (see Definition 3.6). It is left to show (iii). Since u is a solution to ($\mathcal{D}FEa$), for any $x \in \boldsymbol{V}$ there is an edge e_0 with x as terminal vertex such that

$$u(x) - u(\mathrm{o}(e_0)) = \sigma_a(e_0).$$

Taking into account (24) and Proposition 5.6, for $\gamma = \Psi(e_0)$, we deduce that $v \circ \gamma$ actually satisfies the state-constraint boundary condition in (iii) with respect to ($HJ_\gamma a$).

Conversely, let u be a real function on $\boldsymbol{V}$ which is the trace on Γ of a solution to ($\mathcal{H}Ja$). It follows from the compatibility condition (17), and the notations (24)–(25), that u is a subsolution to ($\mathcal{D}FEa$); i.e.,

$$u(x) \le \min_{e \in \boldsymbol{E}_x} \big(u(\mathrm{t}(e)) + \sigma_a(-e) \big) \quad \text{for } x \in \boldsymbol{V}. \tag{29}$$

In order to show that it is a solution to ($\mathcal{D}FEa$), we need to prove that equality holds in (29) for every $x \in \boldsymbol{V}$. In fact, since u is the trace of a solution to ($\mathcal{H}Ja$), it follows from condition (iii) in Definition 3.6, that for every vertex x there is at least one arc γ having x as terminal point such that $u(\gamma(s))$ satisfies the state-constraint boundary condition for ($HJ_\gamma a$) at $s = 1$. In particular, in light of Proposition 5.6, see (24), this implies that there exists e with $\mathrm{t}(e) = x$, or in other terms $-e \in \boldsymbol{E}_x$, such that

$$u(x) - u(\mathrm{o}(e)) = \sigma_a(e)$$

or equivalently

$$u(x) = u(\mathrm{t}(-e)) + \sigma_a(e).$$

Hence, equality holds in (29), and this completes the proof of item (i). Item (ii) can be proven arguing along the same lines. □

The same argument as in the above proof allows also showing the following:

Proposition 6.3. *Given $a \ge a_0$ and $\boldsymbol{V}' \subset \boldsymbol{V}$, a function $u : \boldsymbol{V} \to \mathbb{R}$ which is a subsolution to ($\mathcal{D}FEa$) in $\boldsymbol{V}$ and solution in $\boldsymbol{V} \setminus \boldsymbol{V}'$ can be (uniquely) extended to a function $v : \Gamma \to \mathbb{R}$ which is a subsolution of ($\mathcal{H}Ja$) in Γ and a solution in $\Gamma \setminus \boldsymbol{V}'$. Conversely, the trace on $\boldsymbol{V}$ of a function $v : \Gamma \to \mathbb{R}$ which is a subsolution to ($\mathcal{H}Ja$) in Γ and a solution in $\Gamma \setminus \boldsymbol{V}'$ is a subsolution to ($\mathcal{D}FEa$) in $\boldsymbol{V}$ and a solution in $\boldsymbol{V} \setminus \boldsymbol{V}'$.*

6B. ***Existence of solutions to ($\mathcal{D}FEa$) and critical value.*** We want to introduce a notion of *critical value* for ($\mathcal{D}FEa$) and prove the existence of solutions.

Let us start by proving the following stability properties of solutions and subsolutions.

Proposition 6.4. (i) *Let a_n be a sequence in $\mathbb{R}$ converging to some a. Let u_n be subsolution to ($\mathcal{D}FEa_n$) for every n, with $u_n(x_0)$ bounded for some $x_0 \in \boldsymbol{V}$; then u_n converges, up to subsequences, to a subsolution to ($\mathcal{D}FEa$).*

(ii) *Let v_n be a sequence of solutions to ($\mathcal{D}FEa$) for some $a \in \mathbb{R}$, with $v_n(x_0)$ bounded for some $x_0 \in \boldsymbol{V}$; then v_n converges, up to a subsequence, to a solution to ($\mathcal{D}FEa$).*

Proof. Owing to the definition of subsolution and Lemma 5.15, we see that

$$\langle \mathrm{d}u_n, e \rangle \le \sigma_b(e) \quad \text{for every } e \in \boldsymbol{E},$$

where $b = \sup a_n$. This implies that the $\mathrm{d}u_n$ are equibounded. We therefore get, exploiting the boundedness assumption on x_0 and Proposition 2.1(ii), that u_n is convergent, up to subsequences, to some u. By Lemma 5.15 we have

$$u(\mathrm{t}(e)) - u(\mathrm{o}(e)) - \sigma_a(e) = \lim_n \big(u_n(\mathrm{t}(e)) - u_n(\mathrm{o}(e)) - \sigma_{a_n}(e) \big) \le 0$$

for any e, showing that u is a subsolution to ($\mathcal{D}FEa$).

Let now v_n be a sequence of solutions to ($\mathcal{D}FEa$); because of the previous point, v_n converges, up to subsequences, to a subsolution v of the same equation. It is left to show that v is indeed a solution. Given $x \in \boldsymbol{V}$, we find $e_n \in \boldsymbol{E}_x$ with

$$v_n(\mathrm{t}(e_n)) - v_n(x) - \sigma_a(-e_n) = 0.$$

Since the edges are finite, we deduce that there exists $e_0 \in \boldsymbol{E}_x$ such that

$$e_n = e_0 \quad \text{for infinitely many } n.$$

Up to extracting a subsequence, passing to the limit as n goes to infinity, we obtain

$$v(\mathrm{t}(e_0)) - v(x) - \sigma_a(-e_0) = 0. \qquad \square$$

We define the *critical value* for ($\mathcal{D}FEa$) (also called the *Mañé critical value*) as

$$c = c(\mathcal{H}) := \min\{a \geq a_0 \mid (\mathcal{D}FEa) \text{ admits subsolutions}\}. \tag{30}$$

First of all, notice that it is well-defined. In fact, because of the coercivity of the H_γ, we know σ_a is strictly positive for every e, when a is large enough, so that any constant function is a subsolution to ($\mathcal{D}FEa$). This shows that c is finite. Note the minimum in the definition of c is justified by Proposition 6.4, showing the existence of critical subsolutions (namely, subsolutions to ($\mathcal{D}FEa$) with $a = c$).

The relevance of the critical value is apparent from the following result.

Proposition 6.5. *If there exists a solution to* ($\mathcal{D}FEa$), *then* $a = c$.

Proof. Clearly $a \geq c$, since every solution is also a subsolution. If $a > c$, then there exists a strict subsolution u to ($\mathcal{D}FEa$). Let us assume, by contradiction, that there exists also a solution v. Let x_0 be point at which $u - v$ achieves its maximum; then

$$v(x_0) - v(\mathrm{t}(e)) \leq u(x_0) - u(\mathrm{t}(e)) \quad \text{for any } e \in \boldsymbol{E}_{x_0}. \tag{31}$$

By the very definition of solution applied to v, there is $e_0 \in \boldsymbol{E}_{x_0}$ such that

$$v(x_0) = v(\mathrm{t}(e_0)) + \sigma_a(-e_0).$$

We derive, taking into account (31),

$$u(x_0) \geq u(\mathrm{t}(e_0)) + \sigma_a(-e_0),$$

which is in contrast with the very definition of strict subsolution. $\square$

We further deduce a uniqueness result in the supercritical case.

Proposition 6.6. *Let $a > c$ and $\boldsymbol{V}' \subset \boldsymbol{V}$. For any given function u defined on $\boldsymbol{V}'$ there is at most one solution v of* ($\mathcal{D}FEa$) *in $\boldsymbol{V} \setminus \boldsymbol{V}'$ agreeing with u on $\boldsymbol{V}'$.*

Proof. Assume by contradiction that there are two distinct solutions u_1 and u_2 both satisfying the statement. Since $a > c$, we know that there is a strict subsolution w to ($\mathcal{D}FEa$). Therefore, given $\lambda \in (0, 1)$ we have

$$\lambda\, w(x) + (1-\lambda)\, u_1(x) < \min_{e \in \boldsymbol{E}_x} \big(\lambda\, w(\mathrm{t}(e)) + (1-\lambda)\, u_1(\mathrm{t}(e)) + \sigma_a(-e)\big) \tag{32}$$

for any $x \in \boldsymbol{V} \backslash \boldsymbol{V}'$. Up to interchanging the roles of u_1 and u_2, we can assume that $\max_{\boldsymbol{V}}(u_1 - u_2) > 0$, so that any maximizer is outside $\boldsymbol{V}'$. For λ sufficiently close to 0, we still have that $[\lambda\, w + (1-\lambda)\, u_1] - u_2$ achieves its maximum in $\boldsymbol{V} \backslash \boldsymbol{V}'$. Let x_0 be one of these points of maximum; then, for every $e \in \boldsymbol{E}_{x_0}$ we have

$$[\lambda\, w(x_0) + (1-\lambda)\, u_1(x_0)] - u_2(x_0) \geq [\lambda\, w(\mathrm{t}(e)) + (1-\lambda)\, u_1(\mathrm{t}(e))] - u_2(\mathrm{t}(e))$$

or

$$u_2(x_0) \leq u_2(\mathrm{t}(e)) + \lambda\, w(x_0) + (1-\lambda)\, u_1(x_0) - \lambda\, w(\mathrm{t}(e)) - (1-\lambda)\, u_1(\mathrm{t}(e)).$$

Using (32) we can deduce

$$u_2(x_0) < \min_{e \in \boldsymbol{E}_{x_0}} \big(u_2(\mathrm{t}(e)) - \sigma_a(-e)\big)$$

in contrast with $x_0 \notin \boldsymbol{V}'$ and u_2 being solution to ($\mathcal{D}FEa$) in $\boldsymbol{V} \backslash \boldsymbol{V}'$. □

Given $a \geq a_0$, we define for any path $\xi = (e_1, \ldots, e_M) = (e_i)_{i=1}^M$,

$$\sigma_a(\xi) = \sum_{i=1}^{M} \sigma_a(e_i), \tag{33}$$

and

$$S_a(x, y) := \inf\{\sigma_a(\xi) \mid \xi \text{ is a path linking } x \text{ to } y\}. \tag{34}$$

The following triangle inequality is a direct consequence of the definition:

$$S_a(x, y) \leq S_a(x, z) + S_a(z, y) \quad \text{for any } x, y, z \text{ in } \boldsymbol{V}. \tag{35}$$

The next result starts unveiling the major role of cycles in the forthcoming analysis.

Lemma 6.7. *$S_a \not\equiv -\infty$ if and only if*

$$\sigma_a(\xi) \geq 0 \quad \textit{for any cycle } \xi,$$

which is equivalent to saying that $S_a(x, x) \geq 0$ for any $x \in \boldsymbol{V}$.

Proof. If $\sigma_a(\xi) < 0$ for some cycle ξ, then going through it several times, we deduce that $S_a \equiv -\infty$. Conversely, if $\sigma_a(\xi) \geq 0$ for any cycle ξ, then

$$S_a(x, x) \geq 0 \quad \text{for any } x \in \boldsymbol{V}$$

and therefore $S_a \not\equiv -\infty$. □

From the very definition of subsolution we derive the following result.

Proposition 6.8. *A function u is a subsolution to ($\mathcal{D}FEa$) if and only if*

$$u(x) - u(y) \leq S_a(y, x) \quad \textit{for any } x, y \in \boldsymbol{V}.$$

Proof. It follows easily from the definitions of subsolution in (28) and σ_a in (33) that

$$u(x) - u(y) \leq \sigma_a(\xi) \quad \text{for any path } \xi \text{ linking } y \text{ to } x.$$

Taking the minimum over all such paths, we get the inequality in the statement. The converse is trivial, observing that

$$S_a(\mathrm{o}(e), \mathrm{t}(e)) \le \sigma_a(e) \quad \text{for every } e \in \boldsymbol{E}. \qquad \square$$

The previous result implies:

Corollary 6.9. *If $a \ge c$ then $S_a \not\equiv -\infty$.*

Moreover:

Corollary 6.10. *Given $a \ge c$ and x, y in $\boldsymbol{V}$, there exists a simple path η with $\mathrm{o}(\eta) = x$ and $\mathrm{t}(\eta) = y$ such that $\sigma_a(\eta) = S_a(x, y)$.*

Proof. Let $\xi = (e_i)_{i=1}^M$ be any path linking x to y. If ξ is not simple there are indices $k > j$ such that $\mathrm{t}(e_i) = \mathrm{t}(e_j)$. We assume, to ease notation, that $k < M$; the case $k = M$ can be treated with straightforward modifications.

We have that $(e_i)_{i=j+1}^k$ is a cycle and the paths $(e_i)_{i=1}^j$ and $(e_i)_{i=k+1}^M$ are concatenated. We get, according to Lemma 6.7, that

$$\sigma_a(\xi) = \sigma_a((e_i)_{i=1}^j) + \sigma_a((e_i)_{i=j+1}^k) + \sigma_a((e_i)_{i=k+1}^M) \ge \sigma_a((e_i)_{i=1}^j) + \sigma_a((e_i)_{i=k+1}^M)$$

and $(e_i)_{i=1}^j \cup (e_i)_{i=k+1}^M$ is still a path linking x to y. By iterating the above procedure, we remove all cycles properly contained in ξ and end up with a simple curve ξ_0 with $\mathrm{o}(\xi_0) = x$, $\mathrm{t}(\xi_0) = y$ and $\sigma_a(\xi_0) \le \sigma_a(\xi)$. This shows that $S_a(x, y)$ can be realized as the infimum of simple paths from x to y. Since there are finitely many such paths, we get the assertion. $\square$

The condition in Corollary 6.9 is actually necessary and sufficient, as shown by the next result. In the proof we will use a form of the basic *Bellman optimality principle* adapted to our frame. It can be stated as follows: if $\xi = (e_i)_{i=1}^M$ is a path with

$$\sigma_a(\xi) = S_a(\mathrm{o}(e), \mathrm{t}(e))$$

and $1 \le j < k \le M$, then $\eta := (e_i)_{i=j}^k$ satisfies $\sigma_a(\eta) = S_a(\mathrm{o}(e_j), \mathrm{t}(e_k))$.

Proposition 6.11. *Assume $S_a \not\equiv -\infty$. Given $y \in \boldsymbol{V}$, the function $u = S_a(y, \cdot)$ is a solution to ($\mathcal{D}$FEa) in $\boldsymbol{V} \setminus \{y\}$ and a subsolution to ($\mathcal{D}$FEa) in $\boldsymbol{V}$.*

Proof. The subsolution property comes from Proposition 6.8 and the triangle inequality (35). We proceed by showing that u is a solution in $\boldsymbol{V} \setminus \{y\}$. Let $x \neq y$; then, by Corollary 6.10, there is a path $\xi = (e_i)_{i=1}^M$ linking y to x with

$$\sigma_a(\xi) = S_a(y, x).$$

By the Bellman optimality principle, the path $\eta := (e_i)_{i=1}^{M-1}$ satisfies

$$\sigma_a(\eta) = S_a(y, \mathrm{t}(\eta)) = u(\mathrm{t}(\eta)).$$

Consequently

$$u(x) = \sigma_a(\eta) + \sigma_a(e_M) = u(\mathrm{t}(\eta)) + \sigma_a(e_M)$$

with $-e_M \in \boldsymbol{E}_x$. Hence

$$u(x) - u(\mathrm{t}(-e_M)) = u(x) - u(\mathrm{t}(\eta)) = \sigma_a(e_M). \qquad \square$$

Using Proposition 6.8 and the triangle inequality (35), we also obtain

Corollary 6.12. *The function*

$$x \mapsto -S_c(x, y)$$

is a critical subsolution for any fixed $y \in \boldsymbol{V}$.

Combining Corollary 6.9 and Proposition 6.11 we get:

Corollary 6.13. $S_a \not\equiv -\infty$ *if and only if* $a \geq c$.

We further have:

Proposition 6.14. *Given* $y \in \boldsymbol{V}$, *the function* $x \mapsto S_a(y, x)$ *is a solution to* ($\mathcal{D}FEa$) *if and only if there exists a cycle* ξ *incident on* y *with* $\sigma_a(\xi) = 0$.

Proof. ($\Longrightarrow$) We will prove in Proposition 6.15 a more general property, namely that if the equation ($\mathcal{D}FEa$) admits a solution, then there is a cycle ξ with $\sigma_a(\xi) = 0$.

($\Longleftarrow$) Assume the existence of a cycle, say $\xi = (e_i)_{i=1}^M$, with $\sigma_a(\xi) = 0$ incident on y. Up to relabelling the e_i, we can set $y = \mathrm{o}(\xi) = \mathrm{t}(\xi)$. We claim that $u := S_a(y, \cdot)$ is a solution on the whole of $\boldsymbol{V}$. By Proposition 6.11, it is enough to prove the assertion at y. We have

$$0 \leq S_a(y, y) = u(y) \leq \sigma_a(e_M) + S_a(y, \mathrm{o}(e_M)) \leq \sigma_a(\xi),$$

and since $\sigma_a(\xi) = 0$, all the inequalities in the above formula must indeed be equalities; in particular

$$u(y) - u(\mathrm{t}(-e_M)) - \sigma_a(e_M) = u(y) - S_a(y, \mathrm{o}(e_M)) - \sigma_a(e_M) = 0$$

with $-e_M \in \boldsymbol{E}_y$. This proves the claim. □

As announced, we complete the above proof by showing:

Proposition 6.15. *If the equation* ($\mathcal{D}FEa$) *admits a solution, then there is a cycle* ξ *with* $\sigma_a(\xi) = 0$.

Proof. Let us assume that v is a solution to ($\mathcal{D}FEa$). Take any $x \in \boldsymbol{V}$; by the definition of solution, we can find an edge e with terminal vertex x such that

$$v(x) - v(\mathrm{o}(e)) = \sigma_a(e).$$

By iterating backward the procedure, we can construct for any M a path $\xi = (e_i)_{i=1}^M$ such that

$$v(\mathrm{t}(e_j)) - v(\mathrm{o}(e_k)) = \sigma_a((e_i)_{i=k}^j) \quad \text{for any } j \geq k. \tag{36}$$

Since the graph is finite, taking M large enough, we have that for suitable indices $j > k$, the path $(e_i)_{i=k}^j$ is a cycle, so that $v(\mathrm{t}(e_j)) - v(\mathrm{o}(e_k)) = 0$, and the relation (36) provides the assertion. □

The argument of the next proof is reminiscent of the one used for the existence of critical solutions of Hamilton–Jacobi equations in compact manifolds; see [Fathi and Siconolfi 2005].

Theorem 6.16. *The critical equation* ($\mathcal{D}FEc$) *admits solutions.*

Proof. We break up the argument according to whether $c = a_0$ or $c > a_0$. Let us first discuss the first instance. If in addition $c = a_\gamma$ for some arc γ, and we set $e = \Psi^{-1}(\gamma)$, then we get from (11), (24), (25) that

$$\sigma_c(e \cup (-e)) = 0.$$

If instead $a_0 = c_\gamma$ for some closed arc γ of the network, then $e = \Psi^{-1}(\gamma)$ is a loop and we obtain, by Proposition 5.14,

$$\sigma_c(e) = 0 \quad \text{or} \quad \sigma_c(-e) = 0.$$

In both cases, we infer the existence of a critical solution in light of Proposition 6.14.

We proceed considering the case $c > a_0$. Let us assume by contradiction that there are no critical solutions. For any $y \in \boldsymbol{V}$, setting $u_y = S_c(y, \cdot)$, we can therefore find by Proposition 6.11 a positive constant δ_y with

$$\max_{e \in \boldsymbol{E}_y} \big(u_y(y) - u_y(\mathrm{t}(e)) - \sigma_c(-e)\big) = -\delta_y. \tag{37}$$

We define $u = \sum_y \lambda_y \, u_y$, where the λ_y are positive coefficients summing to 1, and set

$$\delta = \min_y \lambda_y \, \delta_y.$$

Exploiting that all the u_y are subsolutions on the whole of $\boldsymbol{V}$ and using (37), we conclude that for any $e \in \boldsymbol{E}$

$$\begin{aligned} u(\mathrm{t}(e)) - u(\mathrm{o}(e)) - \sigma_c(e) &= \sum_{y \neq \mathrm{t}(e)} \lambda_y \big(u_y(\mathrm{t}(e)) - u_y(\mathrm{o}(e)) - \sigma_c(e)\big) + \lambda_{\mathrm{t}(e)} \big(u_{\mathrm{t}(e)}(\mathrm{t}(e)) - u_{\mathrm{t}(e)}(\mathrm{o}(e)) - \sigma_c(e)\big) \\ &\leq -\lambda_{\mathrm{t}(e)} \, \delta_{\mathrm{t}(e)} \leq -\delta. \end{aligned} \tag{38}$$

Owing to Lemma 5.15 and the fact that $c > a_0$, there is $a_0 < b < c$ with

$$\sigma_b(e) > \sigma_c(e) - \delta \quad \text{for every } e \in \boldsymbol{E};$$

then we deduce from (38) that

$$u(\mathrm{t}(e)) - u(\mathrm{o}(e)) - \sigma_b(e) \leq 0 \quad \text{for every } e.$$

This proves that u is a subsolution to ($\mathcal{D}FEa$) with $a = b$, which is impossible because $b < c$. Therefore the maximum in (37) must be 0 for some y_0, which in turn implies that $S_c(y_0, \cdot)$ is a critical solution, as was claimed. □

Remark 6.17. Let u be a solution to ($\mathcal{D}FEc$). Let e be a loop with $\mathrm{o}(e) = \mathrm{t}(e) = x$, and $\gamma = \Psi(e)$ is hence a closed curve. If $c < c_\gamma$, then, according to Proposition 5.14

$$0 = u(\mathrm{o}(e)) - u(\mathrm{t}(e)) < \sigma_c(e), \quad 0 = u(\mathrm{o}(-e)) - u(\mathrm{t}(-e)) < \sigma_c(-e),$$

which shows that neither e nor $-e$ realizes

$$\min_{e \in \boldsymbol{E}_x} \big(u(\mathrm{t}(e)) + \sigma_a(-e)\big).$$

This in turn implies that the edge e, and consequently $-e$, can be removed from the edges of $\boldsymbol{X}$ without affecting the status of solution for u or any other critical solution.

Things are different if $c = c_\gamma$ because in this case, see Proposition 5.14,

$$0 = \min\{\sigma_c(e), \sigma_c(-e)\} = u(\mathrm{o}(e)) - u(\mathrm{t}(e)) = u(\mathrm{o}(-e)) - u(\mathrm{t}(-e)).$$

6C. ***The Aubry set and some structural properties of solutions.*** Inspired by what was discussed in the previous subsection, we introduce the following definition.

Definition 6.18. The *Aubry set* is defined as

$$\mathcal{A}^*_{\boldsymbol{X}} = \mathcal{A}^*_{\boldsymbol{X}}(\mathcal{H}) = \{e \in \boldsymbol{E} \mid \text{belonging to some cycle with } \sigma_c(\xi) = 0\}. \tag{39}$$

The *projected Aubry set* is given by

$$\mathcal{A}_{\boldsymbol{X}} = \mathcal{A}_{\boldsymbol{X}}(\mathcal{H}) = \{y \in \boldsymbol{V} \mid \exists\, \xi \text{ cycle incident on } y \text{ with } \sigma_c(\xi) = 0\}. \tag{40}$$

The projected Aubry set is partitioned into *static classes*, defined as the equivalence classes with respect to the relation

$$S_c(x, y) + S_c(y, x) = 0.$$

Equivalently x and y belong to the same static class if there is a cycle ξ with $\sigma_c(\xi) = 0$ incident on both of them; in particular, the whole cycle ξ is then contained in this static class.

Remark 6.19. Clearly, $x \in \mathcal{A}_{\boldsymbol{X}}$ if and only if $x = \mathrm{o}(e) = \mathrm{t}(e')$ for some e, e' in $\mathcal{A}^*_{\boldsymbol{X}}$; moreover, if $e \in \mathcal{A}^*_{\boldsymbol{X}}$, then $\mathrm{o}(e)$ and $\mathrm{t}(e)$ belong to $\mathcal{A}_{\boldsymbol{X}}$. The converse of this last property is not true because, for instance, if $e \in \mathcal{A}^*_{\boldsymbol{X}}$ then $-e$ might not belong to $\mathcal{A}^*_{\boldsymbol{X}}$. It is also possible to have a pair of adjacent vertices belonging to different static classes of $\mathcal{A}_{\boldsymbol{X}}$ linked by an edge not in $\mathcal{A}^*_{\boldsymbol{X}}$, or even vertices of the same static classes linked by multiple edges not all belonging to $\mathcal{A}^*_{\boldsymbol{X}}$.

We immediately derive from Proposition 6.15 and Theorem 6.16 the following result.

Lemma 6.20. *The Aubry sets are nonempty. Moreover*,

$$\mathcal{A}_{\boldsymbol{X}} = \{y \in \boldsymbol{V} \mid S_c(y, y) = 0\} = \{y \in \boldsymbol{V} \mid S_c(y, \cdot) \textit{ is a solution to } (\mathcal{D}FEc)\}.$$

We have a structural result on critical solutions. By admissible trace g on $\boldsymbol{V}' \subset \boldsymbol{V}$ (for the critical equation), we mean a function satisfying

$$g(x) - g(y) \le S_c(y, x) \quad \text{for any } x,\ y \text{ in } \boldsymbol{V}'. \tag{41}$$

Theorem 6.21. *Given an admissible trace g on $\mathcal{A}_{\boldsymbol{X}}$, the unique solution to* $(\mathcal{D}FEc)$ *taking the value g on $\mathcal{A}_{\boldsymbol{X}}$ is*

$$v(x) := \min\{g(y) + S_c(y, x) \mid y \in \mathcal{A}_{\boldsymbol{X}}\}. \tag{42}$$

In particular, $\mathcal{A}_{\boldsymbol{X}}$ *represents a uniqueness set for the equation.*

Proof. Taking into account (41) and the fact that $S_c(y, y) = 0$ for any $y \in \mathcal{A}_{\boldsymbol{X}}$, we deduce that g and v coincide on $\mathcal{A}_{\boldsymbol{X}}$. The function v is a critical solution, since it is the pointwise minimum of a finite family of solutions. This property can be easily derived from the definition of solution.

Assume now that w is another solution agreeing with g on $\mathcal{A}_{\boldsymbol{X}}$. Given any $x \in \boldsymbol{V}$, we construct, arguing as in Proposition 6.15, a path $\xi = (e_i)_{i=1}^M$ with $\mathrm{t}(\xi) = x$ and such that

$$w(\mathrm{t}(e_j)) - w(\mathrm{o}(e_k)) = \sigma_c((e_i)_{i=k}^j) \quad \text{for any } j \geq k.$$

If M is sufficiently large, there must exist $j_0 \geq k_0$ such that $(e_i)_{i=k_0}^{j_0}$ is a cycle. We deduce that there are $y \in \mathcal{A}_{\boldsymbol{X}}$ and a path η linking y to x with

$$w(x) = w(y) + \sigma_c(\eta) \geq g(y) + S_c(y, x) \geq v(x).$$

Since the converse inequality holds true by Proposition 6.8, we get $w(x) = v(x)$. □

We record for later use an immediate consequence of the above result:

Corollary 6.22. *Given $\boldsymbol{V}' \subset \mathcal{A}_{\boldsymbol{X}}$, and an admissible trace g on it, the function*

$$v(x) := \min\{g(y) + S_c(y, x) \mid y \in \boldsymbol{V}'\} \tag{43}$$

is the maximal solution to ($\mathcal{D}$FEc) taking the value g on $\boldsymbol{V}'$.

We can also derive a representation formula for solutions at $a > c$ in some subset of $\boldsymbol{V}$. To help in understanding the next statement, we recall that $S_a(x, x) > 0$ for any $x \in \boldsymbol{V}$ whenever $a > c$.

Theorem 6.23. *Let $a > c$ and $\boldsymbol{V}' \subset \boldsymbol{V}$. Let g be a function defined on $\boldsymbol{V}'$ satisfying* (41) *with S_a in place of S_c. Then the function*

$$v(x) = \begin{cases} g(x) & \text{if } x \in \boldsymbol{V}', \\ \min\{g(y) + S_a(y, x) \mid y \in \boldsymbol{V}'\} & \text{if } x \notin \boldsymbol{V}' \end{cases}$$

is the unique solution to ($\mathcal{D}$FEa) in $\boldsymbol{V} \setminus \boldsymbol{V}'$ agreeing with g on $\boldsymbol{V}'$. It is in addition a subsolution on the whole of $\boldsymbol{V}$.

Proof. We claim that

$$v(z) - v(x) \leq S_a(x, z) \quad \text{for any } z, x \text{ in } \boldsymbol{V}. \tag{44}$$

The property is true by assumption if both z and x are in $\boldsymbol{V}'$; if instead z and y are in $\boldsymbol{V} \setminus \boldsymbol{V}'$ we have

$$v(z) - v(x) \leq g(y) + S_a(y, z) - g(y) - S_a(y, x) \leq S_a(x, z),$$

where $y \in \boldsymbol{V}'$ is optimal for $v(x)$ and we have exploited the triangle inequality (35). If $z \notin \boldsymbol{V}'$ and $x \in \boldsymbol{V}'$, then (44) directly comes from the very definition of v. Finally, if $z \in \boldsymbol{V}'$ and $x \notin \boldsymbol{V}'$, we denote by y an optimal element in $\boldsymbol{V}'$ and use the triangle inequality to write

$$v(z) - v(x) = g(z) - g(y) - S_a(y, x) \leq S_a(y, z) - S_a(y, x) \leq S_a(x, z).$$

This concludes the proof of claim (44) and therefore shows, according to Proposition 6.8, that v is a subsolution in $\boldsymbol{V}$. Taking into account that $S_a(y, \cdot)$ is a solution in $\boldsymbol{V} \setminus \boldsymbol{V}'$, we also get, arguing as in Theorem 6.21, that v is a solution in $\boldsymbol{V} \setminus \boldsymbol{V}'$. Uniqueness follows from Proposition 6.6. □

7. Back to the network

In this section we switch our attention back to the network Γ, or in other terms, we give again visibility, besides the vertices, to the interior points of the arcs. We combine the global information gathered on the abstract graph with the outputs of the local analysis on the arcs of the network. We define an appropriate notion of Aubry set and provide a PDE characterization of its points.

Exploiting the richer (differentiable) structure of Γ, we establish, on the basis of our findings in the previous section, some regularity properties for critical subsolutions and solutions. This will generalize what is known for the continuous case in the framework of weak KAM theory; see for example [Fathi 2008]. Finally, we give specific uniqueness results and representation formulae for solutions on the network.

7A. ***Subsolutions and solutions on*** $\boldsymbol{\Gamma}$**.** The next result shows, as pointed out already in the Introduction, how the notion of solution to $(\mathcal{H}Ja)$ can be recovered from the notion of subsolution. The relevance of the issue is that the latter just requires the usual subsolution property on any arc and continuity at the junctures. The argument significantly illustrates the interplay between the immersed network and underlying abstract graph.

Theorem 7.1. *Let $a \geq c$ and $y \in \Gamma$; then the maximal subsolution to $(\mathcal{H}Ja)$ attaining a given value at y is a solution in $\Gamma \setminus \{y\}$.*

Proof. We can assume $y \in \Gamma \setminus \boldsymbol{V}$; otherwise the assertion is a consequence of Propositions 6.8 and 6.11 and Proposition 6.3 with $\boldsymbol{V}' = \{y\}$. It is not restrictive to take 0 as the value assigned at y. We therefore denote by v the maximal subsolution vanishing at y; see Proposition 3.12. We select $\gamma \in \mathcal{E}$ such that $y = \gamma(s_0)$ for some $s_0 \in (0, 1)$, and set $e = \Psi^{-1}(\gamma)$. We first assume that γ is not a closed arc. Since v must be in particular a subsolution in the arc γ, we have by Corollary 5.7

$$v(\gamma(1)) \leq \int_{s_0}^{1} \sigma_a^+(t)\, dt =: \beta,$$
$$v(\gamma(0)) \leq -\int_{0}^{s_0} \sigma_a^-(t)\, dt =: \alpha,$$

where σ_a^+, σ_a^- are defined as in (8), (9). The maximal admissible trace g, in the sense of (41), on $\boldsymbol{V}' := \{\mathrm{o}(e), \mathrm{t}(e)\}$ dominated by α at $\mathrm{o}(e) = \gamma(0)$, and β at $\mathrm{t}(e) = \gamma(1)$, is

$$\alpha^* := \min\{\alpha,\ \beta + S_a(\mathrm{t}(e), \mathrm{o}(e))\},$$
$$\beta^* := \min\{\beta,\ \alpha + S_a(\mathrm{o}(e), \mathrm{t}(e))\}.$$

According to Proposition 6.8, Theorem 6.23 and Corollary 6.22, the function $w : \boldsymbol{V} \to \mathbb{R}$, defined as

$$w(x) = \begin{cases} \alpha^* & \text{if } x = \mathrm{o}(e), \\ \beta^* & \text{if } x = \mathrm{t}(e) \\ \min\{\alpha^* + S_a(\mathrm{o}(e), x),\ \beta^* + S_a(\mathrm{t}(e), x)\} & \text{if } x \neq \mathrm{o}(e) \text{ and } x \neq \mathrm{t}(e), \end{cases}$$

is the maximal subsolution to $(\mathcal{D}FEa)$ on $\boldsymbol{V}$ agreeing with α^* and β^* at the vertices of e. It is in addition a solution in $\boldsymbol{V} \setminus \{\gamma(0), \gamma(1)\}$. By Proposition 6.3 it can thus be extended to a subsolution of $(\mathcal{H}Ja)$

in Γ, denoted by $\overline{w}$, which is in addition a solution in $\Gamma \setminus \{\gamma(0), \gamma(1)\}$. The function $\overline{w}$ is the maximal subsolution to $(\mathcal{H}Ja)$ taking the values α^* and β^* on the vertices of γ, but it does not necessarily vanish at y. We have in any case

$$v \le \overline{w} \quad \text{in } \Gamma. \tag{45}$$

To complete the proof, we need to suitably adjust $\overline{w}$ inside γ in order to attain the value 0 at y. To this end, we proceed by showing that the boundary data α^*, 0 and 0, β^* are admissible, in the sense of (17), for $(HJ_\gamma a)$ restricted to the subintervals $[0, s_0]$ and $[s_0, 1]$, respectively. In fact,

$$\alpha^* \le \alpha = -\int_0^{s_0} \sigma_a^-(t)\, dt, \tag{46}$$

and if a strict inequality prevails in the above formula, we get

$$\alpha^* = \int_{s_0}^1 \sigma_a^+(t)\, dt + S_a(\mathrm{t}(e), \mathrm{o}(e)). \tag{47}$$

Let us consider a cycle in $\boldsymbol{X}$ of the form $\xi \cup e$, where ξ is a path linking $\mathrm{t}(e)$ to $\mathrm{o}(e)$ with $\sigma_a(\xi) = S_a(\mathrm{t}(e), \mathrm{o}(e))$; see Corollary 6.10. Then $\sigma_a(\xi \cup e) \ge 0$ and consequently $S_a(\mathrm{t}(e), \mathrm{o}(e)) \ge -\sigma_a(e)$. By plugging this relation into (47) and recalling the definition of $\sigma_a(e)$, we get

$$\alpha^* \ge \int_{s_0}^1 \sigma_a^+(t)\, dt - \int_0^1 \sigma_a^+(t)\, dt = -\int_0^{s_0} \sigma_a^+(t)\, dt. \tag{48}$$

By combining (46) and (48) we have

$$\int_0^{s_0} \sigma_a^-(t)\, dt \le -\alpha^* \le \int_0^{s_0} \sigma_a^+(t)\, dt,$$

proving the claimed admissibility property in $[0, s_0]$. A straightforward modification of the previous argument shows the same in $[s_0, 1]$. Thus, there exists a function u on $\gamma([0, 1])$ uniquely determined by requiring $u \circ \gamma$ to be a solution to $(HJ_\gamma a)$ in $(0, s_0)$ and $(s_0, 1)$, and in addition taking the values α^*, 0 and β^* at $\gamma(0)$, y and $\gamma(1)$, respectively. This is also the maximal subsolution of $(HJ_\gamma a)$ in $(0, 1)$ taking such values at the boundary points and at $s = s_0$. The function

$$\overline{\overline{w}}(x) = \begin{cases} \overline{w} & \text{in } \Gamma \setminus \gamma([0, 1]), \\ u & \text{in } \gamma([0, 1]) \end{cases}$$

is a subsolution to $(\mathcal{H}Ja)$ in Γ and by the maximality property of u on γ and (45),

$$v \le \overline{\overline{w}} \quad \text{in } \Gamma,$$

which immediately implies $v = \overline{\overline{w}}$.

The function v is by construction a solution to $(\mathcal{H}Ja)$ in $\Gamma \setminus \{\gamma(0), y, \gamma(1)\}$. Moreover, taking into account Remark 5.2 and Proposition 5.6 applied to the subinterval $[0, s_0]$, we see that if $\overline{w}(\gamma(0)) = \alpha$ then $\overline{w}$ satisfies condition (iii) in the definition of solution to $(\mathcal{H}Ja)$ at $\gamma(0)$ with respect to the arc $\tilde{\gamma}$. If instead $\overline{w}(\mathrm{o}(e)) = \alpha + S_a(\mathrm{t}(e), \mathrm{o}(e))$ then again condition (iii) of the definition of solution is satisfied with respect to some arc different from γ, $\tilde{\gamma}$ because of Propositions 6.11 and 6.3. Similarly, we prove that v is a solution at $\gamma(1)$. This concludes the proof if γ is not a closed arc.

If instead γ is a closed arc, then we indicate by w the maximal periodic subsolution of $(HJ_\gamma a)$ in $(0, 1)$ vanishing at $s = s_0$; see Corollary 5.13. Arguing as in the first part of the proof, we see that the maximal subsolution v to $(\mathcal{H}Ja)$ in Γ vanishing at y is given by

$$v(x) = \begin{cases} w(\gamma^{-1}(x)) & \text{in } \gamma([0, 1]), \\ w(\gamma(0)) + S_a(\gamma(0), x) & \text{in } \Gamma \setminus \gamma([0, 1]). \end{cases}$$

Taking into account the representation formulae for w provided in item (ii) of Corollary 5.13 and arguing again as in the first part of the proof, we show that v is a solution to $(\mathcal{H}Ja)$ in $\Gamma \setminus \{y\}$, as was claimed. □

7B. *Aubry set in* Γ. We define the Aubry set $\mathcal{A}_\Gamma$ on the network as

$$\mathcal{A}_\Gamma := \big\{x \in \mathbb{R}^N \;\big|\; x = \Psi(e)(t) \text{ for some } e \in \mathcal{A}^*_{\boldsymbol{X}},\ t \in [0, 1]\big\}. \tag{49}$$

One could also consider a lift of $\mathcal{A}_\Gamma$ to the tangent bundle $T\Gamma$, as in the continuous case. For example, this could be useful to study the analogues in this setting of Mather's measures, Mather sets, minimal average actions, etc. (see for example [Fathi 2008; Sorrentino 2015] for precise definitions); this discussion, however, would go beyond our current objectives, so we decided to postpone it to a future investigation.

Remark 7.2. We point out for later use that the support of an arc γ belongs to $\mathcal{A}_\Gamma$ if and only if $\gamma = \Psi(e)$ and at least one between e or $-e$ is in $\mathcal{A}^*_{\boldsymbol{X}}$.

The first lemma regards subsolutions to the critical equation on $\boldsymbol{X}$. Briefly, it says that — analogously to what happens in the continuous case, see [Fathi 2008] — the differential of a critical subsolution is prescribed on the Aubry set and that critical subsolutions are never strict on the Aubry set. On the other hand, it is always possible to find critical subsolutions that are strict outside the Aubry set. This will be used in the next subsection to obtain the same results on networks. See Theorems 7.5 and 7.6.

Lemma 7.3. *Given a subsolution u to $(\mathcal{D}FEc)$, one has*

$$\langle \mathrm{d}u, e\rangle = \sigma_c(e) \quad \textit{for any } e \in \mathcal{A}^*_{\boldsymbol{X}}. \tag{50}$$

Furthermore, there exists a subsolution w to $(\mathcal{D}FEc)$ with

$$\langle \mathrm{d}w, e\rangle < \sigma_c(e) \quad \textit{for any } e \in \boldsymbol{E} \setminus \mathcal{A}^*_{\boldsymbol{X}}. \tag{51}$$

Proof. Let u be a critical subsolution and assume for purposes of contradiction that

$$\langle \mathrm{d}u, \bar{e}\rangle < \sigma_c(\bar{e}) \quad \text{for some } \bar{e} \in \mathcal{A}^*_{\boldsymbol{X}}.$$

By the very definition of Aubry set, we can find a cycle $\xi = (e_i)_{i=1}^M$ such that $\bar{e} = e_j$ for some $j = 1, \ldots, M$ and $\sigma_c(\xi) = 0$. Taking into account that u is a subsolution, we have

$$\langle \mathrm{d}u, e_i\rangle \le \sigma_c(e_i) \quad \text{for } i \ne j \qquad \text{and} \qquad \langle \mathrm{d}u, e_j\rangle < \sigma_c(e_j).$$

This implies

$$0 = \sum_i \langle \mathrm{d}u, e_i \rangle < \sum_i \sigma_c(e_i) = \sigma_c(\xi) = 0,$$

which is impossible. We pass to the second part of the statement. We start constructing for any $e_0 \in \boldsymbol{E} \setminus \mathcal{A}^*_{\boldsymbol{X}}$ a critical subsolution u_{e_0} with

$$\langle \, du_{e_0}, e_0 \rangle < \sigma_c(e_0). \tag{52}$$

The argument will be organized taking into account the classification of edges in $\mathcal{A}^*_{\boldsymbol{X}}$ provided in Remark 6.19. If $\mathrm{t}(e_0) \notin \mathcal{A}_{\boldsymbol{X}}$, then we set $u_{e_0} = S_c(\mathrm{t}(e_0), \cdot)$; according to Lemma 6.20, u_{e_0} is not a critical solution at $\mathrm{t}(e_0)$ which implies (52). If $\mathrm{t}(e_0) \in \mathcal{A}_{\boldsymbol{X}}$, we consider the critical subsolutions $S_c(\mathrm{t}(e_0), \cdot)$ and $-S_c(\cdot\,, \mathrm{t}(e_0))$; see Proposition 6.11 and Corollary 6.12. Taking into account the characterization of $\mathcal{A}_{\boldsymbol{X}}$ given in Lemma 6.20, we have

$$\begin{aligned} -S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) &= \quad S_c(\mathrm{t}(e_0), \mathrm{t}(e_0)) - S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) \quad \le \sigma_c(e_0), \\ S_c(\mathrm{o}(e_0), \mathrm{t}(e_0)) &= -S_c(\mathrm{t}(e_0), \mathrm{t}(e_0)) + S_c(\mathrm{o}(e_0), \mathrm{t}(e_0)) \le \sigma_c(e_0). \end{aligned}$$

If equality prevails in both above formulae, we get

$$S_c(\mathrm{o}(e_0), \mathrm{t}(e_0)) + S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) = 0,$$

which is possible if and only if both $\mathrm{o}(e_0)$ and $\mathrm{t}(e_0)$ are in the Aubry set and belong to the same static class. If this is not the case, we satisfy (52) up to choosing u_{e_0} equal to $S_c(\mathrm{t}(e_0), \cdot)$ or $-S_c(\cdot\,, \mathrm{t}(e_0))$. If instead the two vertices are in the same static class, we claim that

$$S_c(\mathrm{t}(e_0), \mathrm{t}(e_0)) - S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) = -S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) < \sigma_c(e_0). \tag{53}$$

In fact, we know, by the very definition of static class, that there is a path ξ linking $\mathrm{t}(e_0)$ to $\mathrm{o}(e_0)$ with all the edges belonging to $\mathcal{A}^*_{\boldsymbol{X}}$. Therefore, using Lemma 6.20 and the first part of the statement that we have just proven, applied to the critical subsolution $-S_c(\cdot\,, \mathrm{o}(e_0))$, we have that

$$S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) = -S_c(\mathrm{o}(e_0), \mathrm{o}(e_0)) + S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) = \sigma_c(\xi).$$

Were (53) false, we should further have

$$0 = -S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) + S_c(\mathrm{t}(e_0), \mathrm{o}(e_0)) = \sigma_c(\xi \cup e_0)$$

and consequently $e_0 \in \mathcal{A}^*_{\boldsymbol{X}}$, which is impossible. Formula (52) is therefore satisfied with $u_{e_0} = S_c(\mathrm{t}(e_0), \cdot)$. This completes the proof of (52).

We conclude arguing along the same lines as Theorem 6.16. Given $e \in \boldsymbol{E} \setminus \mathcal{A}^*_{\boldsymbol{X}}$, we denote by u_e a critical subsolution satisfying (52) with e in place of e_0. We choose positive constants λ_e for $e \in \boldsymbol{E} \setminus \mathcal{A}^*_{\boldsymbol{X}}$, summing to 1, and define a critical subsolution via

$$w = \sum_{e \in \boldsymbol{E} \setminus \mathcal{A}^*_{\boldsymbol{X}}} \lambda_e \, u_e.$$

Given $e_0 \in \boldsymbol{E} \setminus \mathcal{A}^*_{\boldsymbol{X}}$, we have

$$\langle \mathrm{d}w, e_0 \rangle = \sum_{e \neq e_0} \lambda_e \langle \mathrm{d}u_e, e_0 \rangle + \lambda_{e_0} \langle \mathrm{d}u_{e_0}, e_0 \rangle < \sigma_c(e_0),$$

as we wished to prove. □

We derive a PDE characterization of points in the Aubry set, generalizing a property of the continuous case.

Proposition 7.4. *The maximal subsolution to* ($\mathcal{H}Jc$) *taking a given value at a point* $y \in \Gamma$ *is a critical solution on the whole network if and only if* $y \in \mathcal{A}_\Gamma$.

Proof. If $y \in \boldsymbol{V}$, the assertion comes from Lemma 6.20; we can then assume from now on that $y \in \Gamma \setminus \boldsymbol{V}$. We prescribe, without loss of generality, the value 0 at y, and denote by v the maximal subsolution vanishing at y; see Proposition 3.12. We denote by γ an arc whose support contains y.

We first assume that γ is not a closed curve. Taking into account Theorem 7.1, it is enough to show that v is a solution at y if and only if $y \in \mathcal{A}_\Gamma$. Looking at the proof of Theorem 7.1 (we adopt the same notations), we see that the solution property at y is in turn equivalent to the following: the solution of ($HJ_\gamma c$) in (0, 1) taking the values $v(\gamma(0))$, $v(\gamma(1))$ at 0, 1, respectively, vanishes at $s = s_0$. In light of Proposition 5.5, this boils down to showing

$$\min\{v(\gamma(0)) + \boldsymbol{A},\ v(\gamma(1)) - \boldsymbol{B}\} = 0, \tag{54}$$

where σ_c^+, σ_c^- are defined as in (8), (9), respectively, and

$$\boldsymbol{A} = \int_0^{s_0} \sigma_c^+(t)\, dt, \quad \boldsymbol{B} = \int_{s_0}^1 \sigma_c^-(t)\, dt.$$

Taking into account the proof of Theorem 7.1, we know that

$$v(\gamma(0)) = \min\{-\boldsymbol{D},\ \boldsymbol{C} + S_c(\gamma(1), \gamma(0))\}, \tag{55}$$

$$v(\gamma(1)) = \min\{\boldsymbol{C},\ -\boldsymbol{D} + S_c(\gamma(0), \gamma(1))\}, \tag{56}$$

where

$$\boldsymbol{C} = \int_{s_0}^1 \sigma_c^+(t)\, dt, \quad \boldsymbol{D} = \int_0^{s_0} \sigma_c^-(t)\, dt.$$

Then

$$v(\gamma(0)) + \boldsymbol{A} = \begin{cases} \int_0^{s_0} [\sigma_c^+(t) - \sigma_c^-(t)]\, dt & \text{if } v(\gamma(0)) = -\boldsymbol{D}, \\ \int_0^1 \sigma_c^+(t)\, dt + S_c(\gamma(1), \gamma(0)) & \text{if } v(\gamma(0)) = \boldsymbol{C} + S_c(\gamma(1), \gamma(0)) \end{cases} \tag{57}$$

and

$$v(\gamma(1)) - \boldsymbol{B} = \begin{cases} \int_{s_0}^1 [\sigma_c^+(t) - \sigma_c^-(t)]\, dt & \text{if } v(\gamma(1)) = \boldsymbol{C}, \\ -\int_0^1 \sigma_c^-(t)\, dt + S_c(\gamma(0), \gamma(1)) & \text{if } v(\gamma(1)) = -\boldsymbol{D} + S_c(\gamma(0), \gamma(1)). \end{cases} \tag{58}$$

Exploiting the property that $\sigma_c(\xi) \geq 0$ for any cycle ξ in $\boldsymbol{X}$, we see that

$$S_c(\gamma(0), \gamma(1)) \geq -\sigma_c(-e) = \int_0^1 \sigma_c^-(t)\, dt,$$
$$S_c(\gamma(1), \gamma(0)) \geq -\sigma_c(e) = -\int_0^1 \sigma_c^+(t)\, dt.$$

Equality holds in the first formula if and only if there is a cycle ξ with $-e \subset \xi$ and $\sigma_c(\xi) = 0$, and in the second one if and only if there a cycle η with $e \subset \xi$ and $\sigma_c(\eta) = 0$. We in addition have that

$$\int_0^{s_0} [\sigma_c^+(t) - \sigma_c^-(t)]\, dt = 0 \quad \text{or} \quad \int_{s_0}^1 [\sigma_c^+(t) - \sigma_c^-(t)]\, dt = 0$$

if and only if $c = a_\gamma$, and in this case both e and $-e$ belong to $\mathcal{A}^*_{\boldsymbol{X}}$. In light of the above remarks, (57) and (58), we conclude that (54) holds if and only if $y \in \mathcal{A}_\Gamma$.

This concludes the proof when γ is not a closed arc. The argument for γ a closed arc goes along the same lines just adapting the representation formulae for solutions of $(HJ_\gamma c)$ and taking into account Corollary 5.13. □

7C. *Regularity results for critical subsolutions.* We state and prove the main regularity results of this section. They can be considered as a generalization to the network setting of the results in [Fathi and Siconolfi 2004].

Theorem 7.5. *Any critical subsolution $u : \Gamma \to \mathbb{R}$ is of class C^1 in $\mathcal{A}_\Gamma \setminus \boldsymbol{V}$, and all such subsolutions possess the same differential in $\mathcal{A}_\Gamma \setminus \boldsymbol{V}$.*

Proof. Let u be a critical subsolution on Γ and $\gamma = \Psi(e)$ an arc with $e \in \mathcal{A}^*_{\boldsymbol{X}}$. According to Lemma 7.3 and formula (50),

$$u(\gamma(1)) - u(\gamma(0)) = \sigma_c(e).$$

Therefore $u \circ \gamma$ is the maximal subsolution taking the value $u(\gamma(0))$ at $s = 0$ and, according to Proposition 5.6, has the form

$$u(\gamma(s)) = \int_0^s \sigma_c^+(t)\, dt,$$

where σ_c^+ is as in (8) with H_γ in place of H and c in place of a. We deduce that $s \mapsto u(\gamma(s))$ is of class C^1 for $t \in (0, 1)$ and for any $x = \gamma(t_0)$, with $t_0 \in (0, 1)$, the differential $D_\Gamma u(x)$ is uniquely determined among the elements of $T^*_\Gamma(x)$ by the condition

$$(D_\Gamma u(x), \dot\gamma(t_0)) = \frac{d}{dt} u(\gamma(t))|_{t=t_0} = \sigma_c^+(t_0). \qquad \square$$

Moreover:

Theorem 7.6. *For any critical subsolution w on $\boldsymbol{X}$, there exists a critical subsolution u on Γ, with $w = u$ on $\boldsymbol{V}$, which is of class C^1 in $\Gamma \setminus \boldsymbol{V}$. There exists in addition a critical subsolution v on Γ of class $C^1(\Gamma \setminus \boldsymbol{V})$ satisfying*

$$\mathcal{H}(x, D_\Gamma v(x)) < c \quad \textit{for } x \in \Gamma \setminus (\mathcal{A}_\Gamma \cup \boldsymbol{V}).$$

Proof. Let w be a critical subsolution in $\boldsymbol{X}$. Given any arc $\gamma = \Psi(e)$, we know, see Proposition 6.2, that $w(\gamma(0))$ and $w(\gamma(1))$ satisfy the compatibility condition (17), so that

$$w(\gamma(0)) + \int_0^1 \sigma_c^-(t)\,dt \le w(\gamma(1)) \le w(\gamma(0)) + \int_0^1 \sigma_c^+(t)\,dt, \tag{59}$$

where σ_c^+, σ_c^- are defined as in (8), (9) with H_γ, c in place of H, a, respectively. We can therefore find $\lambda \in [0, 1]$ with

$$w(\gamma(1)) = w(\gamma(0)) + \int_0^1 [\lambda\,\sigma_c^-(t) + (1-\lambda)\,\sigma_c^+(t)]\,dt, \tag{60}$$

and the function

$$s \mapsto w(\gamma(0)) + \int_0^s [\lambda\,\sigma_c^-(t) + (1-\lambda)\,\sigma_c^+(t)]\,dt \tag{61}$$

is a subsolution of class C^1 to $H_\gamma = c$ in (0, 1) taking the values $w(\gamma(0))$ and $w(\gamma(1))$ at $s = 0$ and $s = 1$, respectively. This shows the first part of the assertion.

As far as the second claim is concerned, we proceed by taking a critical subsolution w satisfying (51). This implies that strict inequalities prevail in formula (59) whenever $\gamma = \Psi(e)$ with e, $-e$ not in $\mathcal{A}^*_{\boldsymbol{X}}$. The λ appearing in (60) can be consequently taken in (0, 1), so that the function defined in (61) is a strict subsolution to $H_\gamma = c$. This concludes the proof in light of Remark 7.2. □

Remark 7.7. Notice that if we apply the procedure of the first part of the previous result starting with a critical solution rather than a critical subsolution, then the property of being a solution could be possibly false for the regularized function.

7D. ***Representation formulae and uniqueness results on the network.*** In this section, we want to provide representation formulae and uniqueness results with traces that are not necessarily defined on vertices, but on a general subset of the network Γ. To this aim, we extend S_a, for $a \ge c$, from $\boldsymbol{V}$ to the whole of Γ defining a semidistance intrinsically related to $\mathcal{H}$ and the level a. This is basically the same object introduced in [Schieborn and Camilli 2013]. We do not develop here any further the metric point of view, but just use it to establish an admissibility condition for data assigned on subsets of Γ, and provide representation formulae.

Given a portion of arc $\gamma|_{[s_1,s_2]}$, for $0 \le s_1 \le s_2 \le 1$, we define

$$\ell_a(\gamma|_{[s_1,s_2]}) = \int_{s_1}^{s_2} (\sigma_a^+)^\gamma(t)\,dt,$$

where $(\sigma_a^+)^\gamma$ is defined as in (8). We get in particular, for the whole arc, the relation

$$\ell_a(\gamma) = \sigma_a(\Psi^{-1}(\gamma)) \quad \text{for any } \gamma \in \mathcal{E}. \tag{62}$$

We define ℓ_a for a curve on Γ given by a finite number of concatenated arcs or portions of arcs as the sum of the lengths of the arcs or portion of arcs making it up. We introduce the related geodesic (semi-)distance on Γ via

$$S_a^\Gamma(x, y) = \min\{\ell_a(\xi) \mid \xi \text{ a union of concatenated arcs linking } x \text{ to } y\}. \tag{63}$$

We deduce from the results on σ_a and (62) the following lemma.

Lemma 7.8. (i) *If $x \neq y$ are in $\boldsymbol{V}$, then $S_a(x, y) = S_a^{\Gamma}(x, y)$.*

(ii) *If ξ is a closed curve on Γ, then $\ell_a(\xi) \geq 0$.*

It is easy to check that the maximal subsolution v to ($\mathcal{DFE}a$) vanishing at $y \in \Gamma$ given in Theorem 7.1 and Proposition 7.4 is

$$v(x) = S_a^{\Gamma}(y, x) \quad \text{for any } a \geq c, \;\; x \in \Gamma.$$

We derive, taking also into account Proposition 6.8, that for a continuous function $u : \Gamma \to \mathbb{R}$, the condition

$$u(x) - u(y) \leq S_a^{\Gamma}(y, x) \quad \text{for any pair } x, y \text{ in } \Gamma' \tag{64}$$

is necessary and sufficient for being a subsolution to ($\mathcal{H}Ja$). Given a function g defined on a subset Γ' of Γ, we therefore introduce the following admissibility condition for ($\mathcal{DFE}a$):

$$g(x) - g(y) \leq S_a^{\Gamma}(x, y) \quad \text{for any } x, y \text{ in } \Gamma'. \tag{65}$$

We give in the next theorem a couple of examples of uniqueness results for solutions to ($\mathcal{DFE}a$), and corresponding representation formulae, one can obtain prescribing values on subsets not necessarily contained in $\boldsymbol{V}$. Further results are reachable along the same lines. Similar formulae, even if for subsets of vertices and just in the supercritical case, have been already obtained in [Schieborn and Camilli 2013].

Theorem 7.9. *Let Γ' be a closed subset of Γ and g an admissible trace defined on it, in the sense of* (65). *We set*

$$v(x) = \min\{g(y) + S_a^{\Gamma}(y, x) \mid y \in \Gamma'\}.$$

(i) *Critical case: if $a = c$ and $\Gamma' \subset \mathcal{A}_\Gamma$ with*

$$\Gamma' \cap \gamma([0, 1]) \neq \varnothing \quad \text{for any } \gamma \text{ with } \Psi^{-1}(\gamma) \in \mathcal{A}^*_{\boldsymbol{X}}, \tag{66}$$

then v is the unique solution in Γ to $\mathcal{H}(x, Du) = c$ agreeing with g on Γ'.

(ii) *Supercritical case: if $a > c$, then v is uniquely characterized by the properties of being in $\boldsymbol{C}(\Gamma, \mathbb{R})$, being a solution of ($\mathcal{H}Ja$) in $\Gamma \setminus \Gamma'$, and agreeing with g on Γ'.*

Proof. The solution property of v in both cases, in Γ and $\Gamma \setminus \Gamma'$ respectively, follows directly from being a subsolution in Γ, in light of (64), and satisfying the subtangent test as a minimum of solutions, in Γ and $\Gamma \setminus \Gamma'$ respectively. In addition v is the maximal solution (in Γ or $\Gamma \setminus \Gamma'$) agreeing with g on Γ' in light of Theorem 7.1, Proposition 7.4, and the admissibility condition (65).

Now, assume u to be another solution taking the value g on Γ'; by adapting the backward procedure explained in Proposition 6.15 and Theorem 6.21, we construct, for any $x \in \Gamma \setminus \Gamma'$, a curve ξ made up by concatenated arcs or portion of arcs starting at some point $y \in \Gamma'$ and ending at x with

$$u(x) = g(y) + \ell_a(\xi) \geq v(x).$$

In the critical case, condition (66) plays a crucial role for this. The maximality property of v then implies that equality must hold in the above formula. □

Appendix

Proof of Proposition 3.11. Taking into account that for any $\gamma \in \mathcal{E}$ (which is a finite set) $w \circ \gamma$ is Lipschitz-continuous in [0, 1], thanks to the coercivity condition (Hγ2), we deduce that there exists $L > 0$ such that, for any given subsolution w,

$$|w(\gamma(s_2)) - w(\gamma(s_1))| \le L\, \ell(\gamma|_{[s_1,s_2]}) \quad \text{for all } \gamma \in \mathcal{E}, \text{ and } s_1 \le s_2 \in [0,1]; \tag{67}$$

hereafter ℓ indicates the Euclidean length of curves in $\mathbb{R}^N$.

We proceed by considering x and y in Γ and a finite sequence of concatenated arcs $\gamma_1, \dots, \gamma_M$, for some index M, that realize the geodesic distance $d_\Gamma(x, y)$. More specifically, we assume that $x = \gamma_1(t_x)$, $y = \gamma_M(t_y)$ with t_x, t_y in [0, 1] and that

$$d_\Gamma(x, y) = \ell(\gamma_1|_{[t_x,1]}) + \sum_{i=2}^{M-1} \ell(\gamma_i) + \ell(\gamma_M|_{[0,t_y]}).$$

In the remainder of the proof we assume that $M > 2$ in order to ease the notation (the other cases can be treated analogously).

We deduce from (67) that

$$\begin{aligned}|w(y) - w(x)| &\le |w(\gamma_1(1)) - w_1(\gamma_1(t_x))| + \sum_{i=2}^{M-1} |w(\gamma_i(1)) - w(\gamma_i(0))| + |w(\gamma_M(t_y)) - w_1(\gamma_M(0))| \\ &\le L\left[\ell(\gamma_1|_{[t_x,1]}) + \sum_{i=2}^{M-1} \ell(\gamma_i) + \ell(\gamma_M|_{[0,t_y]})\right] = L\, d_\Gamma(x, y).\end{aligned}$$

□

Proof of Proposition 5.5. We denote by w the function appearing in the statement. If $a = a_\gamma$, the assertion comes from (11) and Proposition 5.3. Instead, if $a > a_\gamma$, the function w is an a.e. subsolution, being the minimum of two C^1 (sub-)solutions. Using a basic property in viscosity solutions theory, it is also a supersolution, as a minimum of supersolutions. Moreover, $w(0) = \alpha$ and $w(1) = \beta$ hold thanks to (17).

Finally, the function $s \mapsto \int_0^s \sigma^+_{a_\gamma}$ is a strict subsolution to ($HJ_\gamma a$), and this implies by an argument going back to [Ishii 1987] that the Dirichlet problem with admissible data α, β is uniquely solved. □

Proof of Proposition 5.6. If $a = a_\gamma$, then, as already pointed out in Proposition 5.3, the solution is unique up to additive constants; hence it is automatically given by (20) once the value $w(0)$ is assigned.

Therefore, from now on we can assume that $a > a_\gamma$. By Proposition 5.5,

$$w(s) = \min\left\{w(0) + \int_0^s \sigma_a^+(t)\,dt,\ w(1) - \int_s^1 \sigma_a^-(t)\,dt\right\} \quad \text{for any } s.$$

We claim that if

$$w(s_0) = w(1) - \int_{s_0}^1 \sigma_a^-(t)\,dt \tag{68}$$

for some $s_0 \in (0, 1)$, then

$$w(s) = w(1) - \int_s^1 \sigma_a^-(t)\,dt \quad \text{for any } s \in (s_0, 1].$$

Assume by contradiction that there exists $s_1 > s_0$ such that

$$w(0)+\int_0^{s_1}\sigma_a^+(t)\,dt = w(0)+\int_0^{s_0}\sigma_a^+(t)\,dt+\int_{s_0}^{s_1}\sigma_a^+(t)\,dt < w(1)-\int_{s_1}^{1}\sigma^-(t)\,dt;$$

this implies

$$w(0)+\int_0^{s_0}\sigma_a^+(t)\,dt < w(1)-\int_{s_1}^{1}\sigma_a^-(t)\,dt-\int_{s_0}^{s_1}\sigma_a^+(t)\,dt. \tag{69}$$

It is apparent that

$$\int_{s_0}^{s_1}\sigma_a^+(t)\,dt > \int_{s_0}^{s_1}\sigma_a^-(t)\,dt$$

and we can consequently deduce from (69) that

$$w(0)+\int_0^{s_0}\sigma_a^+(t)\,dt < w(1)-\int_{s_1}^{1}\sigma_a^-(t)\,dt-\int_{s_0}^{s_1}\sigma_a^-(t)\,dt = w(1)-\int_{s_0}^{1}\sigma^-(t)\,dt,$$

in contrast with (68). We assume, for purposes of contradiction, that (68) holds true for some $s_0 \in (0, 1)$. Since $a > a_\gamma$, we can take p_0 with $H(1, p_0) < a$. If w is not of the form (20), then, owing to the previous claim, we can fix s_0 in such a way that

$$w(s) = w(1)-\int_s^1\sigma_a^-(t)\,dt \quad\text{and}\quad H(s, p_0) < a$$

for $s \in [s_0, 1]$. This implies

$$\varphi(s) := w(1)+p_0(s-1) \le w(1)-\int_s^1\sigma_a^-(t)\,dt = w(s)$$

for $s \in [s_0, 1]$, and consequently φ is a constrained subtangent to w at 1 with

$$H(1, \varphi'(1)) = H(1, p_0) < 1,$$

contradicting (19). We deduce that w is of the form (20) showing the first part of the assertion.

Conversely, if w is of the form (20), then it is of class C^1 in $(0, 1)$ with $w'(s) = \sigma_a^+(s)$. Consequently, any constrained subtangent φ at $t = 1$ must satisfy

$$w(1)-\int_s^1\varphi'\,dt = \varphi(s) \le w(s) = w(1)-\int_s^1\sigma_a^+\,dt$$

for s sufficiently close to 1. This implies

$$\int_s^1\varphi'\,dt \ge \int_s^1\sigma_a^+\,dt$$

and shows the existence of a sequence s_n contained in $(0, 1)$ and converging to 1 as n goes to infinity, with $\varphi'(s_n) \ge \sigma_a^+(s_n)$. Passing to the limit as n goes to infinity, we get $\varphi'(1) \ge \sigma_a^+(1)$. We deduce from this the inequality (19) and conclude the proof. □

Proof of Proposition 5.11. If $a = c_\gamma = a_\gamma$ then the integrals in (21) coincide in light of (11); then they must both vanish, and this shows the assertion. Assume now that $c_\gamma > a_\gamma$ and also assume for purposes of contradiction that strict inequalities prevail instead in (21). Then, we can find $\lambda \in (0, 1)$ with

$$\int_0^1 [\lambda\, \sigma^+_{c_\gamma}(t) + (1-\lambda)\, \sigma^-_{c_\gamma}(t)]\, dt = 0.$$

Taking into account that $\sigma^+_{c_\gamma}(t) > \sigma^-_{c_\gamma}(t)$ for any t, this implies

$$s \mapsto \int_0^s [\lambda\, \sigma^+_{c_\gamma}(t) + (1-\lambda)\, \sigma^-_{c_\gamma}(t)]\, dt$$

is a strict periodic subsolution to $H = c_\gamma$. This is impossible by the very definition of c_γ. □

Proof of Corollary 5.13. The unique point to check is that the values $\alpha + \beta$ at $s = 0$ and α at $s = s_0$ are admissible, in the sense of (17), for $(HJ_\gamma a)$ in $(0, s_0)$, and the same holds true in $(s_0, 1)$ for the values α at $s = s_0$ and $\alpha + \beta$ at $s = 1$. The argument is the same for the two subintervals. We therefore focus on $(s_0, 1)$.

If $u(1) - u(s_0) = \beta = \int_{s_0}^1 \sigma^+_a(t)\, dt$, the compatibility property is immediate and the solution in $(s_0, 1)$ is given by (23), as asserted in item (ii) of the statement. Let us instead assume

$$u(1) - u(s_0) = \beta = -\int_0^{s_0} \sigma^-_a(t)\, dt < \int_{s_0}^1 \sigma^+_a(t)\, dt. \tag{70}$$

We have by Lemma 5.9, $\int_0^1 \sigma^-_a(t)\, dt \le 0$ and consequently

$$u(1) - u(s_0) \ge \int_{s_0}^1 \sigma^-_a(t)\, dt.$$

The last inequality plus (70) shows the claimed admissibility property. □

Proof of Proposition 6.1. Let w be a solution to $(\mathcal{H}J a)$ with trace u on $\boldsymbol{V}$. By the very definition of solution, given any arc γ, we know $w \circ \gamma$ is a solution to $H_\gamma = a$ in $(0, 1)$ taking the values $u(\gamma(0))$ and $u(\gamma(1))$ at 0 and 1, respectively. This implies that such boundary values are admissible with respect to H_γ, in the sense of formula (17) with H_γ in place of H. By the uniqueness property showcased in Proposition 5.5, the values of w on the support of γ are therefore uniquely determined by $u(\gamma(0))$, $u(\gamma(1))$ and H_γ. Since the arc γ has been arbitrarily chosen, we can hence conclude the asserted uniqueness. □

Acknowledgments

This work has been supported by the INdAM-GNAMPA research project "Fenomeni asintotici e omogeneizzazione". Siconolfi acknowledges the Progetto Ateneo 2015, Rome *La Sapienza* University: Asintotica e omogeneizzazione di dinamiche Hamiltoniane". Sorrentino acknowledges the PRIN-2012-74FYK7 grant "Variational and perturbative aspects of nonlinear differential problems". The authors are grateful to the anonymous referees for their valuable comments and suggestions.

References

[Achdou et al. 2013] Y. Achdou, F. Camilli, A. Cutrì, and N. Tchou, "Hamilton–Jacobi equations constrained on networks", *NoDEA Nonlinear Differential Equations Appl.* **20**:3 (2013), 413–445. MR Zbl

[Bardi and Capuzzo-Dolcetta 1997] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations*, Birkhäuser, Boston, 1997. MR Zbl

[Barles 1994] G. Barles, *Solutions de viscosité des équations de Hamilton–Jacobi*, Mathématiques & Applications **17**, Springer, 1994. MR Zbl

[Barles et al. 2013] G. Barles, A. Briani, and E. Chasseigne, "A Bellman approach for two-domains optimal control problems in $\mathbb{R}^N$", *ESAIM Control Optim. Calc. Var.* **19**:3 (2013), 710–739. MR Zbl

[Barles et al. 2014] G. Barles, A. Briani, and E. Chasseigne, "A Bellman approach for regional optimal control problems in $\mathbb{R}^N$", *SIAM J. Control Optim.* **52**:3 (2014), 1712–1744. MR Zbl

[Bernard and Buffoni 2006] P. Bernard and B. Buffoni, "The Monge problem for supercritical Mañé potentials on compact manifolds", *Adv. Math.* **207**:2 (2006), 691–706. MR Zbl

[Bernard and Buffoni 2007] P. Bernard and B. Buffoni, "Optimal mass transportation and Mather theory", *J. Eur. Math. Soc. (JEMS)* **9**:1 (2007), 85–121. MR Zbl

[Bressan and Hong 2007] A. Bressan and Y. Hong, "Optimal control problems on stratified domains", *Netw. Heterog. Media* **2**:2 (2007), 313–331. MR Zbl

[Camilli and Marchi 2013] F. Camilli and C. Marchi, "A comparison among various notions of viscosity solution for Hamilton–Jacobi equations on networks", *J. Math. Anal. Appl.* **407**:1 (2013), 112–118. MR Zbl

[Camilli et al. 2013] F. Camilli, D. Schieborn, and C. Marchi, "Eikonal equations on ramified spaces", *Interfaces Free Bound.* **15**:1 (2013), 121–140. MR Zbl

[Davini et al. 2016] A. Davini, A. Fathi, R. Iturriaga, and M. Zavidovique, "Convergence of the solutions of the discounted equation: the discrete case", *Math. Z.* **284**:3-4 (2016), 1021–1034. MR Zbl

[Fathi 2008] A. Fathi, "Weak KAM Theorem in Lagrangian dynamics", lecture notes, 2008, https://www.math.u-bordeaux.fr/~pthieull/Recherche/KamFaible/Publications/Fathi2008_01.pdf.

[Fathi and Siconolfi 2004] A. Fathi and A. Siconolfi, "Existence of C^1 critical subsolutions of the Hamilton–Jacobi equation", *Invent. Math.* **155**:2 (2004), 363–388. MR Zbl

[Fathi and Siconolfi 2005] A. Fathi and A. Siconolfi, "PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians", *Calc. Var. Partial Differential Equations* **22**:2 (2005), 185–228. MR Zbl

[Galise et al. 2015] G. Galise, C. Imbert, and R. Monneau, "A junction condition by specified homogenization and application to traffic lights", *Anal. PDE* **8**:8 (2015), 1891–1929. MR Zbl

[Garavello and Piccoli 2006] M. Garavello and B. Piccoli, *Traffic flow on networks: conservation laws models*, AIMS Series on Applied Mathematics **1**, American Institute of Mathematical Sciences, Springfield, MO, 2006. MR Zbl

[Gomes 2005] D. A. Gomes, "Viscosity solution methods and the discrete Aubry–Mather problem", *Discrete Contin. Dyn. Syst.* **13**:1 (2005), 103–116. MR Zbl

[Imbert and Monneau 2016] C. Imbert and R. Monneau, "Quasi-convex Hamilton–Jacobi equations posed on junctions: the multi-dimensional case", preprint, 2016. arXiv

[Imbert and Monneau 2017] C. Imbert and R. Monneau, "Flux-limited solutions for quasi-convex Hamilton–Jacobi equations on networks", *Ann. Sci. Éc. Norm. Supér.* (4) **50**:2 (2017), 357–448. MR Zbl

[Imbert et al. 2013] C. Imbert, R. Monneau, and H. Zidani, "A Hamilton–Jacobi approach to junction problems and application to traffic flows", *ESAIM Control Optim. Calc. Var.* **19**:1 (2013), 129–166. MR Zbl

[Ishii 1987] H. Ishii, "A simple, direct proof of uniqueness for solutions of the Hamilton–Jacobi equations of eikonal type", *Proc. Amer. Math. Soc.* **100**:2 (1987), 247–251. MR Zbl

[Lions and Souganidis 2016] P.-L. Lions and P. Souganidis, "Viscosity solutions for junctions: well posedness and stability", *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27**:4 (2016), 535–545. MR Zbl

[Nash 1956] J. Nash, "The imbedding problem for Riemannian manifolds", *Ann. of Math.* (2) **63** (1956), 20–63. MR Zbl

[Pokornyi and Borovskikh 2004] Y. V. Pokornyi and A. V. Borovskikh, "Differential equations on networks (geometric graphs)", *J. Math. Sci.* (*N. Y.*) **119**:6 (2004), 691–718. MR Zbl

[Rao et al. 2014] Z. Rao, A. Siconolfi, and H. Zidani, "Transmission conditions on interfaces for Hamilton–Jacobi–Bellman equations", *J. Differential Equations* **257**:11 (2014), 3978–4014. MR Zbl

[Schieborn and Camilli 2013] D. Schieborn and F. Camilli, "Viscosity solutions of Eikonal equations on topological networks", *Calc. Var. Partial Differential Equations* **46**:3-4 (2013), 671–686. MR Zbl

[Soner 1986] H. M. Soner, "Optimal control with state-space constraint, I", *SIAM J. Control Optim.* **24**:3 (1986), 552–561. MR Zbl

[Sorrentino 2015] A. Sorrentino, *Action-minimizing methods in Hamiltonian dynamics: an introduction to Aubry–Mather theory*, Mathematical Notes **50**, Princeton University Press, 2015. MR Zbl

[Su and Thieullen 2015] X. Su and P. Thieullen, "Convergence of discrete Aubry–Mather model in the continuous limit", preprint, 2015. arXiv

[Sunada 2013] T. Sunada, *Topological crystallography: with a view towards discrete geometric analysis*, Surveys and Tutorials in the Applied Mathematical Sciences **6**, Springer, 2013. MR Zbl

[Zavidovique 2010] M. Zavidovique, "Existence of $C^{1,1}$ critical subsolutions in discrete weak KAM theory", *J. Mod. Dyn.* **4**:4 (2010), 693–714. MR Zbl

[Zavidovique 2012] M. Zavidovique, "Strict sub-solutions and Mañé potential in discrete weak KAM theory", *Comment. Math. Helv.* **87**:1 (2012), 1–39. MR Zbl

Received 5 Dec 2016. Revised 25 May 2017. Accepted 10 Aug 2017.

ANTONIO SICONOLFI: siconolfi@mat.uniroma1.it
Department of Mathematics, Sapienza Università di Roma, Rome, Italy

ALFONSO SORRENTINO: sorrentino@mat.uniroma2.it
Department of Mathematics, Università degli Studi di Roma "Tor Vergata", Rome, Italy

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2018 is US $275/year for the electronic version, and $480/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 11 No. 1 2018

2157-5045(2018)11:1;1-T