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HARDY–LITTLEWOOD INEQUALITIES ON COMPACT QUANTUM GROUPS OF KAC TYPE

SANG-GYUN YOUN

The Hardy–Littlewood inequality on the circle group $\mathbb{T}$ compares the L^p-norm of a function with a weighted ℓ^p-norm of its sequence of Fourier coefficients. The approach has recently been explored for compact homogeneous spaces and we study a natural analogue in the framework of compact quantum groups. In particular, in the case of the reduced group C^*-algebras and free quantum groups, we establish explicit $L^p-\ell^p$ inequalities through inherent information of the underlying quantum groups such as growth rates and the rapid decay property. Moreover, we show sharpness of the inequalities in a large class, including G a compact Lie group, $C_r^*(G)$ with G a polynomially growing discrete group and free quantum groups O_N^+, S_N^+.

1. Introduction

Hardy and Littlewood [1927] showed that there exists a constant C_p for each $1<p\le 2$ such that

$$\left(\sum_{n\in\mathbb{Z}}\frac{1}{(1+|n|)^{2-p}}|\hat{f}(n)|^p\right)^{\frac{1}{p}}\le C_p\|f\|_{L^p(\mathbb{T})} \tag{1-1}$$

for all $f\in L^p(\mathbb{T})$, where $(\hat{f}(n))_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of f.

This implies the multiplier

$$\mathcal{F}_w: L^p(\mathbb{T})\to\ell^p(\mathbb{Z}),\quad f\mapsto (w(n)\hat{f}(n))_{n\in\mathbb{Z}},\quad \text{with } w(n):=\frac{1}{(1+|n|)^{\frac{2-p}{p}}},$$

is bounded. Moreover, this is a stronger form of the Hardy–Littlewood–Sobolev embedding theorem

$$H_p^{\frac{1}{p}-\frac{1}{q}}(\mathbb{T})\subseteq L^q(\mathbb{T})\quad\text{for all } 1<p<q<\infty,$$

where $H_p^s(\mathbb{T}):=\{f\in L^p(\mathbb{T}):(1-\Delta)^{\frac{s}{2}}(f)\in L^p(\mathbb{T})\}$ is the Bessel potential space [Bényi and Oh 2013].

The Hardy–Littlewood inequality (1-1) has been studied on compact abelian groups by Hewitt and Ross [1974] and was recently extended to compact homogeneous manifolds by Akylzhanov, Nursultanov and Ruzhansky [Akylzhanov et al. 2015; 2016]. For compact Lie groups G with real dimension n, the

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2015 paper's result can be rephrased thus: for each $1 < p \le 2$, there exists a constant $C_p > 0$ such that

$$\Big(\sum_{\pi\in\mathrm{Irr}(G)} \frac{1}{(1+\kappa_\pi)^{\frac{n(2-p)}{2}}} n_\pi^{2-\frac{p}{2}} \|\hat{f}(\pi)\|_{\mathrm{HS}}^p\Big)^{\frac{1}{p}} \le C_p \|f\|_{L^p(G)} \quad \text{for all } f\in L^p(G). \tag{1-2}$$

Here, $\mathrm{Irr}(G)$ denotes a maximal family of mutually inequivalent irreducible unitary representations of G, $\|A\|_{\mathrm{HS}} := \mathrm{tr}(A^*A)^{\frac{1}{2}}$ and the Laplacian operator Δ on G satisfies $\Delta : \pi_{i,j} \mapsto -\kappa_\pi \pi_{i,j}$ for all $\pi = (\pi_{i,j})_{1\le i,j\le n_\pi} \in \mathrm{Irr}(G)$ and all $1 \le i, j \le n_\pi$.

The left-hand side of the inequality (1-2) can be shown to dominate a more familiar quantity, which is a natural weighted ℓ^p-norm of its sequence of Fourier coefficients:

$$\Big(\sum_{\pi\in\mathrm{Irr}(G)} \frac{1}{(1+\kappa_\pi)^{\frac{n(2-p)}{2}}} n_\pi \|\hat{f}(\pi)\|_{S^p_{n_\pi}}^p\Big)^{\frac{1}{p}} \le C_p \|f\|_{L^p(G)}. \tag{1-3}$$

Here, $\|A\|_{S^p_n} = \mathrm{tr}(|A|^p)^{\frac{1}{p}}$ is called the Schatten p-norm with respect to the (unnormalized) trace.

A notable point is that the Hardy–Littlewood inequalities on compact Lie groups (1-2) are determined by inherent geometric information, namely the real dimension and the natural length function on $\mathrm{Irr}(G)$. Indeed, $\pi \mapsto \sqrt{\kappa_\pi}$ is equivalent to the natural length $\|\cdot\|$ on $\mathrm{Irr}(G)$ (see Remark 6.1).

The main purpose of this paper is to establish new Hardy–Littlewood inequalities on compact quantum groups of Kac type by utilizing geometric information of the underlying quantum groups. As part of such efforts, we will present some explicit inequalities on important examples and such examples are listed as follows. The reduced group C^*-algebras $C_r^*(G)$ of discrete groups G, the free orthogonal quantum groups O_N^+ and the free permutation quantum groups S_N^+ are main targets. Of course, noncommutative L^p analysis on quantum groups is widely discussed from various perspectives [Caspers 2013; Franz et al. 2017; Junge et al. 2014; 2017; Wang 2017]. For the details of an operator algebraic approach to quantum groups themselves, see [Kustermans and Vaes 2000; 2003; Timmermann 2008; Woronowicz 1987].

In order to clarify our intention, let us show the main results of this paper on compact matrix quantum groups, which can be known to admit the natural length function $|\cdot| : \mathrm{Irr}(\mathbb{G}) \to \{0\}\cup\mathbb{N}$ (see Definition 3.3 and Proposition 3.4). The following inequalities are determined by inherent information of the underlying quantum groups, namely growth rates and the rapid decay property.

Theorem 1.1. *Let $\mathbb{G}$ be a compact matrix quantum group of Kac type and denote by $|\cdot|$ the natural length function on* $\mathrm{Irr}(\mathbb{G})$.

(1) *Let $\mathbb{G}$ have a polynomial growth with $\sum_{\alpha\in\mathrm{Irr}(\mathbb{G}):|\alpha|\le k} n_\alpha^2 \le (1+k)^\gamma$ and $\gamma > 0$. Then, for each $1 < p \le 2$, there exists a universal constant $K = K(p)$ such that*

$$\Big(\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \frac{1}{(1+|\alpha|)^{(2-p)\gamma}} n_\alpha \|\hat{f}(\alpha)\|_{S^p_{n_\alpha}}^p\Big)^{\frac{1}{p}} \le \Big(\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} \frac{1}{(1+|\alpha|)^{(2-p)\gamma}} n_\alpha^{2-\frac{p}{2}} \|\hat{f}(\alpha)\|_{\mathrm{HS}}^p\Big)^{\frac{1}{p}} \le K\|f\|_{L^p(\mathbb{G})} \tag{1-4}$$

for all $f \sim \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(\hat{f}(\alpha)u^\alpha) \in L^p(\mathbb{G})$.

(2) *Let $\widehat{\mathbb{G}}$ have the rapid decay property with universal constants $C, \beta > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{G})} \le C(1+k)^\beta \|f\|_{L^2(\mathbb{G})}$$

for all $f \in \mathrm{span}(\{u^{\alpha}_{i,j} : |\alpha| = k,\ 1 \le i, j \le n_\alpha\})$. *Define*

$$s_k := \sum_{\substack{\alpha\in \mathrm{Irr}(\mathbb{G})\\ |\alpha|=k}} n_\alpha^2.$$

Then, for each $1 < p \le 2$, *there exists a universal constant* $K = K(p)$ *such that*

$$\left(\sum_{k\ge 0}\sum_{\substack{\alpha\in \mathrm{Irr}(\mathbb{G})\\ |\alpha|=k}} \frac{1}{s_k^{\frac{(2-p)}{2}}(1+k)^{(2-p)(\beta+1)}} n_\alpha \|\hat{f}(\alpha)\|^p_{S^p_{n_\alpha}}\right)^{\frac{1}{p}} \le \left(\sum_{k\ge 0}\frac{1}{(1+k)^{(2-p)(\beta+1)}}\left(\sum_{\substack{\alpha\in \mathrm{Irr}(\mathbb{G})\\ |\alpha|=k}} n_\alpha \|\hat{f}(\alpha)\|^2_{\mathrm{HS}}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \le K\|f\|_{L^p(\mathbb{G})} \quad (1\text{-}5)$$

for all $f \sim \sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(\hat{f}(\alpha)u^\alpha) \in L^p(\mathbb{G})$.

In particular, it is known that the rapid decay property of $\mathbb{F}_N$ can be strengthened in a general holomorphic setting [Kemp and Speicher 2007]. The improved result is called the strong Haagerup inequality. Based on these data, it can be shown that we can improve the Hardy–Littlewood inequality on $C_r^*(\mathbb{F}_N)$ by focusing on holomorphic forms. Theorem 5.3 justifies the claim and it seems appropriate to call the improved result a "strong Hardy–Littlewood inequality".

A natural perspective on the Hardy–Littlewood inequalities on compact Lie groups is that a properly chosen weight function $w : \mathrm{Irr}(G) \to (0, \infty)$ makes the corresponding multiplier

$$\mathcal{F}_w : L^p(G) \to \ell^p(\widehat{G}), \quad \text{given by } f \mapsto (w(\pi)\hat{f}(\pi))_{\pi\in \mathrm{Irr}(G)},$$

bounded for each $1 < p \le 2$. Indeed, our newly derived Hardy–Littlewood inequalities on compact quantum groups will give a specific pair (r, s) whose corresponding multiplier $\mathcal{F}_{w_{r,s}}$ is bounded, where

$$w_{r,s}(\alpha) := \frac{1}{r^{|\alpha|}(1+|\alpha|)^s}.$$

Moreover, in Section 6, we will show that there is no better pair (r', s') in that $\mathcal{F}_{r',s'}$ is unbounded whenever (1) $r' < r$ or (2) $r' = r,\ s' < s$ if $\mathbb{G}$ is one of the following: G a compact Lie group, $C_r^*(G)$ with polynomially growing discrete groups or one of the free quantum groups O_N^+, S_N^+. See Theorem 6.6.

This approach is quite natural because it is strongly related to Sobolev embedding properties. We will explore how they are related in Sections 6 and 7B. Indeed, for $G = \mathbb{T}^d$, we have $\mathcal{F}_{w_{0,s}} : L^p(\mathbb{T}^d) \to \ell^p(\mathbb{Z}^d)$ is bounded if and only if

$$H_q^{\frac{ps}{2-p}\left(\frac{1}{q}-\frac{1}{r}\right)}(\mathbb{T}^d) \subseteq L^r(\mathbb{T}^d) \quad \text{for all } 1 < q < r < \infty,$$

where $H_p^s(\mathbb{T}^d)$ is the Bessel potential space.

Lastly, in Section 7, we present some remarks that follow from this approach. We show that many free quantum groups do not admit infinite (central) Sidon sets and give a Sobolev embedding theorem-type interpretation of our results to $C(G)$ with compact Lie groups and $C_r^*(G)$ with polynomially growing discrete groups G. Also, we present an explicit inequality on quantum torus $\mathbb{T}^d_\theta$, which is not a quantum group [Sołtan 2010].

2. Preliminaries

2A. ***Compact quantum groups.*** A compact quantum group $\mathbb{G}$ is given by a unital C^*-algebra A and a unital $*$-homomorphism $\Delta : A \to A \otimes_{\min} A$ satisfying

(1) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$;

(2) $\mathrm{span}\{\Delta(a)(b \otimes 1_A) : a, b \in A\}$ and $\mathrm{span}\{\Delta(a)(1_A \otimes b) : a, b \in A\}$ are dense in A.

Every compact quantum group admits a unique Haar state h on A such that

$$(h \otimes \mathrm{id})(\Delta(x)) = h(x)1_A = (\mathrm{id} \otimes h)(\Delta(x)) \quad \text{for all } x \in A.$$

A finite-dimensional representation of $\mathbb{G}$ is given by an element $u = (u_{i,j})_{1 \le i,j \le n} \in M_n(A)$ such that $\Delta(u_{i,j}) = \sum_{k=1}^n u_{i,k} \otimes u_{k,j}$ for all $1 \le i, j \le n$. We say that the representation u is unitary if $u^*u = uu^* = \mathrm{Id}_n \otimes 1_A \in M_n(A)$ and irreducible if $\{X \in M_n : Xu = uX\} = \mathbb{C} \cdot \mathrm{Id}_n$, where Id_n is the identity matrix in M_n.

We say that finite-dimensional unitary representations u_1 and u_2 are equivalent if there exists a unitary matrix X such that $u_1 X = X u_2$ and let $\{u^\alpha = (u^\alpha_{i,j})_{1 \le i,j \le n_\alpha}\}_{\alpha \in \mathrm{Irr}(\mathbb{G})}$ be a maximal family of mutually inequivalent finite-dimensional unitary irreducible representations of $\mathbb{G}$. It is well known that there is a unique positive invertible matrix $Q_\alpha \in M_{n_\alpha}$ for each $\alpha \in \mathrm{Irr}(\mathbb{G})$ such that $\mathrm{tr}(Q_\alpha) = \mathrm{tr}(Q_\alpha^{-1})$ and

$$h((u^\beta_{s,t})^* u^\alpha_{i,j}) = \frac{\delta_{\alpha,\beta}\delta_{j,t}(Q_\alpha^{-1})_{i,s}}{\mathrm{tr}(Q_\alpha)} \quad \text{for all } \alpha, \beta \in \mathrm{Irr}(\mathbb{G}),\ 1 \le i, j \le n_\alpha,\ 1 \le s, t \le n_\beta,$$

$$h(u^\beta_{s,t}(u^\alpha_{i,j})^*) = \frac{\delta_{\alpha,\beta}\delta_{i,s}(Q_\alpha)_{j,t}}{\mathrm{tr}(Q_\alpha)} \quad \text{for all } \alpha, \beta \in \mathrm{Irr}(\mathbb{G}),\ 1 \le i, j \le n_\alpha,\ 1 \le s, t \le n_\beta.$$

We say that $\mathbb{G}$ is of Kac type if $Q_\alpha = \mathrm{Id}_{n_\alpha} \in M_{n_\alpha}$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$. In this case, the Haar state h is tracial.

Lastly, we define $C_r(\mathbb{G})$ as the image of A in the GNS representation with respect to the Haar state h and $L^\infty(\mathbb{G}) := C_r(\mathbb{G})''$. The Haar state h has a normal faithful extension to $L^\infty(\mathbb{G})$.

2B. ***Noncommutative L^p-spaces.*** Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful tracial state ϕ. Note that the von Neumann algebra $\mathcal{M}$ admits the unique predual $\mathcal{M}_*$. We define $L^1(\mathcal{M}, \phi) := \mathcal{M}_*$ and $L^\infty(\mathcal{M}, \phi) := \mathcal{M}$, and then consider a contractive injection $j : \mathcal{M} \to \mathcal{M}_*$, given by $[j(x)](y) := h(yx)$ for all $y \in \mathcal{M}$. The map j has dense range.

Now $(\mathcal{M}, \mathcal{M}_*)$ is a compatible pair of Banach spaces and for all $1 < p < \infty$, we can define noncommutative L^p-space $L^p(\mathcal{M}, \phi) := (\mathcal{M}, \mathcal{M}_*)_{\frac{1}{p}}$, where $(\cdot, \cdot)_{\frac{1}{p}}$ is the complex interpolation space. For any $x \in L^\infty(\mathcal{M}, \phi)$, its L^p-norm, for all $1 \le p < \infty$, is given by

$$\|x\|_{L^p(\mathcal{M},\phi)} = \phi(|x|^p)^{\frac{1}{p}}.$$

In particular, for all $1 \le p \le \infty$, we denote by $L^p(\mathbb{G})$ the noncommutative L^p-space associated with the von Neumann algebra $L^\infty(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ of Kac type and the tracial Haar state h. Then the space of polynomials

$$\mathrm{Pol}(\mathbb{G}) := \mathrm{span}\big(\{u^\alpha_{i,j} : \alpha \in \mathrm{Irr}(\mathbb{G}),\ 1 \le i, j \le n_\alpha\}\big)$$

is dense in $C_r(\mathbb{G})$ and $L^p(\mathbb{G})$ for all $1 \le p < \infty$.

Under the assumption that $\mathbb{G}$ is of Kac type, for $1 \le p < \infty$,

$$\ell^p(\widehat{\mathbb{G}}) := \left\{ (A_\alpha)_{\alpha\in \mathrm{Irr}(\mathbb{G})} \in \prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} M_{n_\alpha} : \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(|A_\alpha|^p) < \infty \right\}$$

and the natural ℓ^p-norm of $(A_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})} \in \ell^p(\widehat{\mathbb{G}})$ is defined by

$$\|(A_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\|_{\ell^p(\widehat{\mathbb{G}})} := \left(\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(|A_\alpha|^p)\right)^{\frac{1}{p}} = \left(\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \|A_\alpha\|^p_{S^p_{n_\alpha}}\right)^{\frac{1}{p}}.$$

Also,

$$\ell^\infty(\widehat{\mathbb{G}}) := \left\{ (A_\alpha)_{\alpha\in \mathrm{Irr}(\mathbb{G})} \in \prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} M_{n_\alpha} : \sup_{\alpha\in\mathrm{Irr}(\mathbb{G})} \|A_\alpha\| < \infty \right\}$$

and the ℓ^∞-norm of $(A_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})} \in \ell^\infty(\widehat{\mathbb{G}})$ is defined by

$$\|(A_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}\|_{\ell^\infty(\widehat{\mathbb{G}})} := \sup_{\alpha\in\mathrm{Irr}(\mathbb{G})} \|A_\alpha\|.$$

It is known that $\ell^1(\widehat{\mathbb{G}}) = (\ell^\infty(\widehat{\mathbb{G}}))_*$ and $\ell^p(\widehat{\mathbb{G}}) = \big(\ell^\infty(\widehat{\mathbb{G}}), \ell^1(\widehat{\mathbb{G}})\big)_{\frac{1}{p}}$ for all $1 < p < \infty$. For details and more general framework of noncommutative L^p theory, see [Haagerup 1979; Pisier and Xu 2003; Xu 2007].

2C. ***Fourier analysis on compact quantum groups.*** For a compact quantum group $\mathbb{G}$, the Fourier transform $\mathcal{F} : L^1(\mathbb{G}) \to \ell^\infty(\widehat{\mathbb{G}})$, $\psi \mapsto \widehat{\psi}$, is defined by

$$(\widehat{\psi}(\alpha))_{i,j} := \psi((u^\alpha_{j,i})^*) \quad \text{for all } \alpha \in \mathrm{Irr}(\mathbb{G}),\ 1 \le i, j \le n_\alpha.$$

It is also known that $\mathcal{F}$ is an injective contractive map and it is an isometry from $L^2(\mathbb{G})$ onto $\ell^2(\widehat{\mathbb{G}})$ [Wang 2017, Propositions 3.1 and 3.2]. Then, by the interpolation theorem, we are able to induce the Hausdorff–Young inequality again; i.e., $\mathcal{F}$ is a contractive map from $L^p(\mathbb{G})$ into $\ell^{p'}(\widehat{\mathbb{G}})$ for each $1 \le p \le 2$, where p' is the conjugate of p.

We define the Fourier series of $f \in L^1(\mathbb{G})$ by $\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} d_\alpha \,\mathrm{tr}(\hat{f}(\alpha) Q_\alpha u^\alpha)$ and denote it by $f \sim \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} d_\alpha \,\mathrm{tr}(\hat{f}(\alpha) Q_\alpha u^\alpha)$. In particular, if $\mathbb{G}$ is of Kac type, the Fourier series is of the form $f \sim \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(\hat{f}(\alpha) u^\alpha)$.

2D. ***The reduced group C^*-algebras.*** The reduced group C^*-algebra $C^*_r(G)$ can be defined for all locally compact groups, but we only consider discrete groups in this paper since we aim to regard it as a compact quantum group.

Definition 2.1. Let G be a discrete group and define $\lambda_g \in B(\ell^2(G))$ for each $g \in G$ by

$$[(\lambda_g)(f)](x) := f(g^{-1}x) \quad \text{for all } x \in G.$$

Then the reduced group C^*-algebra $C^*_r(G)$ is defined as the norm-closure of the space $\mathrm{span}(\{\lambda_g : g \in G\})$ in $B(\ell^2(G))$. Moreover, we define a comultiplication $\Delta : C^*_r(G) \to C^*_r(G) \otimes_{\min} C^*_r(G)$ by $\lambda_g \mapsto \lambda_g \otimes \lambda_g$ for all $g \in G$. Then $(C^*_r(G), \Delta)$ forms a compact quantum group.

Note that, for $\mathbb{G} = (C^*_r(G), \Delta)$ of a discrete group G, it is of Kac type and $L^\infty(\mathbb{G})$ is nothing but the group von Neumann algebra $\mathrm{VN}(G)$ and $\mathrm{Irr}(\mathbb{G}) = \{\lambda_g\}_{g\in G}$ can be identified with G. In this case, we use the notation $L^p(\mathrm{VN}(G)) = L^p(\mathbb{G})$ conventionally. In particular, $L^1(\mathrm{VN}(G))$ is called the Fourier algebra, denoted by $A(G)$. It is well known that $A(G)$ embeds contractively into $C_0(G)$, so that $A(G)$ can be considered as a function space on G.

2E. ***Free quantum groups of Kac type.***

Definition 2.2 (free orthogonal quantum group [Wang 1995]). Let $N \geq 2$ and A be the universal unital C^*-algebra, which is generated by the N^2 self-adjoint elements $u_{i,j}$ with $1 \leq i, j \leq N$ satisfying the relations

$$\sum_{k=1}^{N} u_{k,i}u_{k,j} = \sum_{k=1}^{N} u_{i,k}u_{j,k} = \delta_{i,j} \quad \text{for all } 1 \leq i, j \leq N.$$

Also, we define a comultiplication $\Delta : A \to A \otimes_{\min} A$ by $u_{i,j} \mapsto \sum_{k=1}^{N} u_{i,k} \otimes u_{k,j}$. Then (A, Δ) forms a compact quantum group called the free orthogonal quantum group. We denote it by O_N^+.

Definition 2.3 (free permutation quantum group [Wang 1998]). Let $N \geq 2$ and A be the universal unital C^*-algebra generated by the N^2 self-adjoint elements $u_{i,j}$ with $1 \leq i, j \leq N$ satisfying the relations

$$u_{i,j}^2 = u_{i,j} = u_{i,j}^* \quad \text{and} \quad \sum_{k=1}^{N} u_{i,k} = \sum_{k=1}^{N} u_{k,j} = 1_A \quad \text{for all } 1 \leq i, j \leq N.$$

Also, we define a comultiplication $\Delta : A \to A \otimes_{\min} A$ by $u_{i,j} \mapsto \sum_{k=1}^{N} u_{i,k} \otimes u_{k,j}$. Then (A, Δ) forms a compact quantum group called the free permutation quantum group. We denote it by S_N^+.

These free quantum groups are of Kac type, so that the Haar states are tracial states. Also, for all $N \geq 2$, the families $\mathrm{Irr}(O_N^+)$ and $\mathrm{Irr}(S_{N+2}^+)$ can be identified with $\{0\} \cup \mathbb{N}$. Moreover,

$$\begin{aligned} n_k &= k+1 && \text{if } \mathbb{G} = O_2^+, \\ n_k &= 2k+1 && \text{if } \mathbb{G} = S_4^+, \\ n_k &\approx r_0^k && \text{if } \mathbb{G} = O_N^+ \text{ or } S_{N+2}^+ \text{ with } N \geq 3, \end{aligned}$$

where r_0 is the largest solution of the equation $X^2 - NX + 1 = 0$ [Banica and Vergnioux 2009].

2F. ***The noncommutative Marcinkiewicz interpolation theorem.*** The classical Marcinkiewicz interpolation theorem [Folland 1999, Theorem 6.28] has a natural noncommutative analogue for semifinite von Neumann algebras. Throughout this paper, we say that a map $T : L^p(M) \to L^q(N)$ is sublinear if for any $x, y \in L^p(M)$ and $\alpha \in \mathbb{C}$

$$|T(x+y)| \leq |T(x)| + |T(y)| \quad \text{and} \quad |T(\alpha x)| = |\alpha||T(x)|.$$

The following theorem is a special case of [Bekjan and Chen 2012, Theorem 2.1]. Denote by $L^0(M)$ the topological $*$-algebra of measurable operators with respect to (M, ϕ).

Theorem 2.4 (the noncommutative Marcinkiewicz interpolation theorem). *Let M and N be von Neumann algebras equipped with normal semifinite faithful traces ϕ and ψ respectively and let $1 \leq p_1 < p < p_2 < \infty$.*

Assume that a sublinear map $A : L^0(M) \to L^0(N)$ satisfies the following conditions: there exist $C_1, C_2 > 0$ such that for any $T_1 \in L^{p_1}(M)$, $T_2 \in L^{p_2}(M)$ and any $y > 0$,

$$\psi\big(1_{(y,\infty)}(|AT_1|)\big) \le \left(\frac{C_1}{y}\right)^{p_1} \|T_1\|^{p_1}_{L^{p_1}(M)}, \quad \psi\big(1_{(y,\infty)}(|AT_2|)\big) \le \left(\frac{C_2}{y}\right)^{p_2} \|T_2\|^{p_2}_{L^{p_2}(M)}. \tag{2-1}$$

Then $A : L^p(M) \to L^p(N)$ is a bounded map.

Proof. Choose a specific Orlicz function $\Phi(t) = t^p$. Then $p_\Phi = q_\Phi = p$ under the notation of [Bekjan and Chen 2012, Theorem 2.1]. □

If the sublinear operator A satisfies the left condition of the inequality (2-1), then we say that A is of weak type (p_1, p_1). Also, the boundedness of $A : L^p(M) \to L^p(N)$ implies A is of weak type (p, p).

Now denote the space of all functions on the discrete space $\mathrm{Irr}(\mathbb{G})$ by $c(\mathrm{Irr}(\mathbb{G}), \nu)$ with a positive measure ν. Then the theorem above is reformulated as follows:

Corollary 2.5. *Let $\mathbb{G}$ be a compact quantum group of Kac type and let $1 \le p_1 < p < p_2 < \infty$. Assume that $A : L^\infty(\mathbb{G}) \to c(\mathrm{Irr}(\mathbb{G}), \nu)$ is sublinear and satisfies the following conditions: there exist $C_1, C_2 > 0$ such that for any $T_1 \in L^{p_1}(\mathbb{G})$, $T_2 \in L^{p_2}(\mathbb{G})$ and any $y > 0$,*

$$\sum_{\alpha:\ |(AT_1)(\alpha)|\ge y} \nu(\alpha) \le \left(\frac{C_1}{y}\right)^{p_1} \|T_1\|^{p_1}_{L^{p_1}(\mathbb{G})}, \qquad \sum_{\alpha:\ |(AT_2)(\alpha)|\ge y} \nu(\alpha) \le \left(\frac{C_2}{y}\right)^{p_2} \|T_2\|^{p_2}_{L^{p_2}(\mathbb{G})}.$$

Then $A : L^p(\mathbb{G}) \to \ell^p(\mathrm{Irr}(\mathbb{G}), \nu)$ is a bounded map.

3. Paley-type inequalities

3A. *General approach.* In this subsection, a Paley-type inequality is derived for compact quantum groups of Kac type by employing fundamental techniques such as the Hausdorff–Young inequality, the Plancherel theorem and the noncommutative Marcinkiewicz interpolation theorem.

We prove the following theorem by adapting techniques used in [Akylzhanov et al. 2015].

Theorem 3.1. *Let $\mathbb{G}$ be a compact quantum group of Kac type and let $w : \mathrm{Irr}(\mathbb{G}) \to (0, \infty)$ be a function such that $C_w := \sup_{t>0}\{t \cdot \sum_{\alpha:\ w(\alpha)\ge t} n_\alpha^2\} < \infty$. Then, for each $1 < p \le 2$, there exists a universal constant $K = K(p) > 0$ such that*

$$\left(\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} w(\alpha)^{2-p} n_\alpha^{2-\frac{p}{2}} \|\hat{f}(\alpha)\|^p_{\mathrm{HS}}\right)^{\frac{1}{p}} \le K\|f\|_{L^p(\mathbb{G})} \tag{3-1}$$

for all $f \sim \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \operatorname{tr}(\hat{f}(\alpha)u^\alpha) \in L^p(\mathbb{G})$.

Proof. Put $\nu(\alpha) := w(\alpha)^2 n_\alpha^2$. We will show that the sublinear operator

$$A : L^1(\mathbb{G}) \to c(\mathrm{Irr}(\mathbb{G}), \nu), \quad f \mapsto \left(\frac{\|\hat{f}(\alpha)\|_{\mathrm{HS}}}{\sqrt{n_\alpha} w(\alpha)}\right)_{\alpha\in\mathrm{Irr}(\mathbb{G})},$$

is a well-defined bounded map from $L^p(\mathbb{G})$ into $\ell^p(\mathrm{Irr}(\mathbb{G}), \nu)$ for all $1 < p \le 2$.

First of all,

$$\sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} \|(Af)(\alpha)\|_{\mathrm{HS}}^2 \nu(\alpha) = \sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} n_\alpha \|\hat{f}(\alpha)\|_{\mathrm{HS}}^2 = \|f\|_{L^2(\mathbb{G})}^2.$$

This implies $A : L^2(\mathbb{G}) \to \ell^2(\mathrm{Irr}(\mathbb{G}), \nu)$ is an isometry.

Second, for all $y > 0$, since

$$\frac{\|\hat{f}(\alpha)\|_{\mathrm{HS}}}{\sqrt{n_\alpha}} = \left(\frac{\mathrm{tr}(\hat{f}(\alpha)^* \hat{f}(\alpha))}{n_\alpha}\right)^{\frac{1}{2}} \le \|\hat{f}(\alpha)\| \le \|f\|_{L^1(\mathbb{G})},$$

we have

$$\begin{aligned}
\sum_{\alpha:\ \|Af(\alpha)\|_{\mathrm{HS}}\ge y} \nu(\alpha) \le \sum_{\alpha:\ w(\alpha)\le \frac{\|f\|_1}{y}} w(\alpha)^2 n_\alpha^2 &= \sum_{\alpha:\ w(\alpha)\le \frac{\|f\|_1}{y}} \int_0^{w(\alpha)^2} n_\alpha^2\, dx \\
&= \int_0^{\left(\frac{\|f\|_1}{y}\right)^2} \left(\sum_{\alpha:\ x^{\frac{1}{2}} \le w(\alpha)\le \frac{\|f\|_1}{y}} n_\alpha^2 \right) dx \quad \text{(by the Fubini theorem)} \\
&= 2\int_0^{\frac{\|f\|_1}{y}} t \left(\sum_{\alpha:\ t\le w(\alpha)\le \frac{\|f\|_1}{y}} n_\alpha^2 \right) dt \quad \text{(by substituting } x \text{ to } t^2\text{)} \\
&\le 2C_w \frac{\|f\|_1}{y}.
\end{aligned}$$

This shows that A is of weak type (1, 1) with $C_1 = 2C_w$.

Now, by Corollary 2.5,

$$\left(\sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} w(\alpha)^{2-p} n_\alpha^{p(\frac{2}{p}-\frac{1}{2})} \|\hat{f}(\alpha)\|_{\mathrm{HS}}^p \right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(\mathbb{G})}. \qquad \square$$

The left-hand side of the inequality (3-1) dominates a more familiar quantity, which is a natural weighted ℓ^p-norm of the sequence of Fourier coefficients $(\hat{f}(\alpha))_{\alpha\in \mathrm{Irr}(\mathbb{G})}$. Recall that the natural noncommutative ℓ^p-norm on $\ell^\infty(\widehat{\mathbb{G}}) = \ell^\infty - \bigoplus_{\alpha\in \mathrm{Irr}(\mathbb{G})} M_{n_\alpha}$ is given by

$$\|(A_\alpha)_{\alpha\in \mathrm{Irr}(\mathbb{G})}\|_{\ell^p(\widehat{\mathbb{G}})} = \left(\sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} n_\alpha \|A_\alpha\|_{S^p_{n_\alpha}}^p \right)^{\frac{1}{p}}$$

under the condition where $\mathbb{G}$ is of Kac type.

Corollary 3.2. *Let $1 < p \le 2$ and w be a function which satisfies the condition of Theorem 3.1. Then we have that*

$$\left(\sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} w(\alpha)^{2-p} n_\alpha \|\hat{f}(\alpha)\|_{S^p_{n_\alpha}}^p \right)^{\frac{1}{p}} \le K \|f\|_{L^p(\mathbb{G})}$$

for all $f \sim \sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(\hat{f}(\alpha) u^\alpha) \in L^p(\mathbb{G})$.

Proof. First of all,

$$\operatorname{tr}(|\hat{f}(\alpha)|^p) = \|\hat{f}(\alpha)\|^p_{S^p_{n_\alpha}}.$$

Put $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Then $2 < r \leq \infty$ and

$$\operatorname{tr}(|\hat{f}(\alpha)|^p) \leq \|\hat{f}(\alpha)\|^p_{\mathrm{HS}} \, \|\mathrm{Id}_{n_\alpha}\|^p_{S^r_{n_\alpha}} = n_\alpha^{1-\frac{p}{2}} \|\hat{f}(\alpha)\|^p_{\mathrm{HS}}. \qquad \square$$

Now we discuss an important subclass of compact quantum groups, namely compact matrix quantum groups which admit the natural length function on $\mathrm{Irr}(\mathbb{G})$.

Definition 3.3. A compact matrix quantum group is given by a triple (A, Δ, u), where A is a unital C^*-algebra A, $\Delta : A \to A \otimes_{\min} A$ is a $*$-homomorphism and $u = (u_{i,j})_{1 \leq i,j \leq n} \in M_n(A)$ is a unitary such that

(1) $\Delta : u_{i,j} \mapsto \sum_{k=1}^n u_{i,k} \otimes u_{k,j}$,

(2) $\bar{u} = (u^*_{i,j})_{1 \leq i,j \leq n}$ is invertible in $M_n(A)$,

(3) $\{u_{i,j}\}_{1 \leq i,j \leq n}$ generates A as a C^*-algebra.

By definition, the free orthogonal quantum groups O_N^+ and the free permutation quantum groups S_N^+ are compact matrix quantum groups. Also, in the class of compact quantum groups, the subclass of compact matrix quantum groups is characterized by the following proposition. The conjugate $\bar{\alpha} \in \mathrm{Irr}(\mathbb{G})$ of $\alpha \in \mathrm{Irr}(\mathbb{G})$ is determined by $u^{\bar{\alpha}} := Q_\alpha^{\frac{1}{2}} \bar{u}^\alpha Q_\alpha^{-\frac{1}{2}}$.

Proposition 3.4 [Timmermann 2008]. *A compact quantum group is a compact matrix quantum group if and only if there exists a finite set $S := \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathrm{Irr}(\mathbb{G})$ such that any $\alpha \in \mathrm{Irr}(\mathbb{G})$ is contained in some iterated tensor product of elements $\alpha_1, \bar{\alpha}_1, \ldots, \alpha_n, \bar{\alpha}_n$ and the trivial representation.*

Then there is a natural way to define a length function on $\mathrm{Irr}(\mathbb{G})$ [Vergnioux 2007]. For nontrivial $\alpha \in \mathrm{Irr}(\mathbb{G})$, the natural length $|\alpha|$ is defined by

$$\min\{m \in \mathbb{N} : \exists \beta_1, \ldots, \beta_m \text{ such that } \alpha \subseteq \beta_1 \otimes \cdots \otimes \beta_m,\ \beta_j \in \{\alpha_k, \bar{\alpha}_k\}_{k=1}^n\}.$$

The length of the trivial representation is defined by 0.

Then it becomes possible to extract explicit inequalities from Theorem 3.1 and Corollary 3.2 by inserting geometric information of the underlying quantum groups, namely growth rates that are estimated by the quantities $b_k := \sum_{|\alpha| \leq k} n_\alpha^2$ [Banica and Vergnioux 2009].

Corollary 3.5. *Let a compact matrix quantum group $\mathbb{G}$ of Kac type satisfy*

$$b_k = \sum_{\substack{\alpha \in \mathrm{Irr}(\mathbb{G}) \\ |\alpha| \leq k}} n_\alpha^2 \leq C(1+k)^\gamma \quad \textit{for all } k \geq 0 \textit{ with } C, \gamma > 0$$

with respect to the natural length function. Then, for each $1 < p \leq 2$, there exists a universal constant $K = K(p)$ such that

$$\Bigg(\sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \frac{1}{(1+|\alpha|)^{(2-p)\gamma}} n_\alpha \|\hat{f}(\alpha)\|^p_{S^p_{n_\alpha}}\Bigg)^{\frac{1}{p}} \leq \Bigg(\sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \frac{1}{(1+|\alpha|)^{(2-p)\gamma}} n_\alpha^{2-\frac{p}{2}} \|\hat{f}(\alpha)\|^p_{\mathrm{HS}}\Bigg)^{\frac{1}{p}} \leq K \|f\|_{L^p(\mathbb{G})} \tag{3-2}$$

for all $f \sim \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} n_\alpha \operatorname{tr}(\hat{f}(\alpha) u^\alpha) \in L^p(\mathbb{G})$.

Proof. Consider the weight function $w(\alpha) := 1/(1+|\alpha|)^\gamma$. Then

$$\sup_{t>0}\left\{t\cdot \sum_{\alpha:|\alpha|\le t^{-1/\gamma}-1} n_\alpha^2\right\} = \sup_{0<t\le 1}\left\{t\cdot \sum_{\alpha:|\alpha|\le t^{-1/\gamma}-1} n_\alpha^2\right\} \le C \sup_{0<t\le 1} t\cdot (t^{-\frac{1}{\gamma}})^\gamma = C.$$

Now the conclusion is reached by Theorem 3.1 and Corollary 3.2. □

3B. ***A Paley-type inequality under the rapid decay property.*** In this subsection, we still assume that $\mathbb{G}$ is a compact matrix quantum group of Kac type. One of the major observations of this paper is that more detailed geometric information improves Theorem 3.1 and Corollary 3.2 in various "exponentially growing" cases. A more refined Paley-type inequality can be obtained under the condition that $\widehat{\mathbb{G}}$ has the rapid decay property in the sense of [Vergnioux 2007].

Definition 3.6 [Vergnioux 2007]. Let $\mathbb{G}$ be a compact matrix quantum group of Kac type. Then we say that $\widehat{\mathbb{G}}$ has the rapid decay property with respect to the natural length function on $\mathrm{Irr}(\mathbb{G})$ if there exist $C, \beta > 0$ such that

$$\Bigg\| \sum_{\substack{\alpha\in \mathrm{Irr}(\mathbb{G})\\ |\alpha|=k}} \sum_{i,j=1}^{n_\alpha} a_{i,j}^\alpha u_{i,j}^\alpha \Bigg\|_{L^\infty(\mathbb{G})} \le C(1+k)^\beta \Bigg\| \sum_{\substack{\alpha\in \mathrm{Irr}(\mathbb{G})\\ |\alpha|=k}} \sum_{i,j=1}^{n_\alpha} a_{i,j}^\alpha u_{i,j}^\alpha \Bigg\|_{L^2(\mathbb{G})} \tag{3-3}$$

for any $k \ge 0$ and scalars $a_{i,j}^\alpha \in \mathbb{C}$.

Notation. (1) When the natural length function on $\mathrm{Irr}(\mathbb{G})$ is given, we use the notation

$$S_k := \{\alpha \in \mathrm{Irr}(\mathbb{G}) : |\alpha| = k\} \quad \text{and} \quad s_k := \sum_{\alpha\in S_k} n_\alpha^2.$$

(2) We denote by p_k the orthogonal projection from $L^2(\mathbb{G})$ to the closure of

$$\mathrm{span}\big(\{u_{i,j}^\alpha : \alpha \in S_k,\ 1 \le i, j \le n_\alpha\}\big).$$

Proposition 3.7. *Suppose a compact matrix quantum group $\mathbb{G}$ is of Kac type and $\widehat{\mathbb{G}}$ has the rapid decay property with respect to the natural length function on* $\mathrm{Irr}(\mathbb{G})$ *and with inequality* (3-3). *Then we have*

$$\sup_{k\ge 0} \frac{\left(\sum_{\alpha\in \mathrm{Irr}(\mathbb{G}):|\alpha|=k} n_\alpha \|\hat{f}(\alpha)\|_{\mathrm{HS}}^2\right)^{\frac{1}{2}}}{(k+1)^\beta} \le C\|f\|_{L^1(\mathbb{G})} \quad \textit{for all } f \in L^1(\mathbb{G}). \tag{3-4}$$

Proof. Since $L^1(\mathbb{G})$ is isometrically embedded into the dual space $M(\mathbb{G}) := C_r(\mathbb{G})^*$ and $\mathrm{Pol}(\mathbb{G})$ is dense in $C_r(\mathbb{G})$, we have

$$\begin{aligned}\|f\|_{L^1(\mathbb{G})} &= \sup_{\substack{x\in \mathrm{Pol}(\mathbb{G})\\ \|x\|_{L^\infty(\mathbb{G})}\le 1}} \langle f, x\rangle_{L^1(\mathbb{G}),L^\infty(\mathbb{G})} = \sup_{\substack{x\in \mathrm{Pol}(\mathbb{G})\\ \|x\|_{L^\infty(\mathbb{G})}\le 1}} \langle f, x^*\rangle_{L^1(\mathbb{G}),L^\infty(\mathbb{G})} \\ &= \sup_{\substack{x\in \mathrm{Pol}(\mathbb{G})\\ \|x\|_{L^\infty(\mathbb{G})}\le 1}} \sum_{\alpha\in \mathrm{Irr}(\mathbb{G})} n_\alpha \,\mathrm{tr}(\hat{f}(\alpha)\hat{x}(\alpha)^*)\end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{\substack{x \in \mathrm{Pol}(\mathbb{G}) \\ \sum_{k\geq 0} C(k+1)^{\beta}\left(\sum_{\alpha:|\alpha|=k} n_\alpha \|\hat{x}(\alpha)\|_{\mathrm{HS}}^2\right)^{1/2} \leq 1}} \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} n_\alpha \operatorname{tr}(\hat{f}(\alpha)\hat{x}(\alpha)^*) \\
&\geq \sup_{k \geq 0} \sup_{\left(\sum_{\alpha:|\alpha|=k} n_\alpha \|\hat{x}(\alpha)\|_{\mathrm{HS}}^2\right)^{1/2} \leq 1} \sum_{\substack{\alpha \in \mathrm{Irr}(\mathbb{G}) \\ |\alpha|=k}} \frac{n_\alpha}{C(k+1)^{\beta}} \operatorname{tr}(\hat{f}(\alpha)\hat{x}(\alpha)^*) \\
&= \sup_{k \geq 0} \frac{\left(\sum_{\alpha \in \mathrm{Irr}(\mathbb{G}):|\alpha|=k} n_\alpha \|\hat{f}(\alpha)\|_{\mathrm{HS}}^2\right)^{\frac{1}{2}}}{C(k+1)^{\beta}}. \qquad (3\text{-}5)
\end{aligned}
$$

This completes the proof. □

Theorem 3.8. *Let a compact matrix quantum group* $\mathbb{G}$ *be of Kac type and* $\widehat{\mathbb{G}}$ *have the rapid decay property with respect to the natural length function on* $\mathrm{Irr}(\mathbb{G})$ *and with inequality* (3-3)*. Also, suppose that a weight function* $w : \{0\} \cup \mathbb{N} \to (0, \infty)$ *satisfies*

$$
C_w := \sup_{y>0} \left\{ y \cdot \sum_{k \geq 0 : \frac{(k+1)^{\beta}}{w(k)} \leq \frac{1}{y}} (k+1)^{2\beta} \right\} < \infty. \qquad (3\text{-}6)
$$

Then, for each $1 < p \leq 2$*, there exists a universal constant* $K = K(p) > 0$ *such that*

$$
\left(\sum_{k \geq 0} w(k)^{2-p} \left(\sum_{\substack{\alpha \in \mathrm{Irr}(\mathbb{G}) \\ |\alpha|=k}} n_\alpha \|\hat{f}(\alpha)\|_{\mathrm{HS}}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq K \|f\|_{L^p(\mathbb{G})} \qquad (3\text{-}7)
$$

for all $f \sim \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} n_\alpha \operatorname{tr}(\hat{f}(\alpha) u^\alpha) \in L^p(\mathbb{G})$.

Proof. Put $\nu(k) := w(k)^2$. We will show that the sublinear operator

$$
A : L^1(\mathbb{G}) \to c(\{0\} \cup \mathbb{N}, \nu), \quad f \mapsto \left(\frac{\|p_k(f)\|_{L^2(\mathbb{G})}}{w(k)} \right)_{k \geq 0},
$$

is a well-defined bounded map from $L^p(\mathbb{G})$ into $\ell^p(\{0\} \cup \mathbb{N}, \nu)$ for all $1 < p \leq 2$.

First of all,

$$
\sum_{k \geq 0} \left(\frac{\|p_k(f)\|_{L^2(\mathbb{G})}}{w(k)} \right)^2 \nu(k) = \sum_{k \geq 0} \|p_k(f)\|_{L^2(\mathbb{G})}^2 = \|f\|_{L^2(\mathbb{G})}^2.
$$

Therefore, $A : L^2(\mathbb{G}) \to \ell^2(\{0\} \cup \mathbb{N}, \nu)$ is an isometry.

Secondly, for all $y > 0$,

$$
\sum_{\substack{k \geq 0 \\ (Af)(k) > y}} \nu(k) \leq \sum_{k : \frac{w(k)}{(k+1)^{\beta}} < \frac{C\|f\|_{L^1(\mathbb{G})}}{y}} w(k)^2
$$

by Proposition 3.7.

Now put $\tilde{w}(k) := w(k)/(k+1)^{\beta}$. Then

$$\sum_{k:\tilde{w}(k)<\frac{C\|f\|_{L^1(\mathbb{G})}}{y}} \int_0^{\tilde{w}(k)^2} (k+1)^{2\beta}\,dx \le \int_0^{(\frac{C\|f\|_{L^1(\mathbb{G})}}{y})^2} \sum_{k:\sqrt{x}\le\tilde{w}(k)} (k+1)^{2\beta}\,dx$$

$$= 2\int_0^{\frac{C\|f\|_{L^1(\mathbb{G})}}{y}} t\cdot \sum_{k:t\le\tilde{w}(k)} (k+1)^{2\beta}\,dt \quad \text{(by substituting } x=t^2\text{)}$$

$$\le \frac{2C_w C\|f\|_{L^1(\mathbb{G})}}{y}.$$

Therefore, by Corollary 2.5, we can obtain

$$\Big(\sum_{k\ge 0} w(k)^{2-p}\|p_k(f)\|_{L^2(\mathbb{G})}^p\Big)^{\frac{1}{p}} \lesssim \|f\|_{L^p(\mathbb{G})}. \qquad \square$$

Corollary 3.9. *Let a compact matrix quantum group* $\mathbb{G}$ *be of Kac type and* $\widehat{\mathbb{G}}$ *have the rapid decay property with respect to the natural length function on* Irr($\mathbb{G}$) *and with inequality* (3-3). *Then, for each* $1 < p \le 2$, *there exists a universal constant* $K = K(p) > 0$ *such that*

$$\Bigg(\sum_{k\ge 0} \frac{1}{(1+k)^{(2-p)(\beta+1)}}\Big(\sum_{\substack{\alpha\in\mathrm{Irr}(\mathbb{G})\\|\alpha|=k}} n_\alpha\|\hat{f}(\alpha)\|_{\mathrm{HS}}^2\Big)^{\frac{p}{2}}\Bigg)^{\frac{1}{p}} \le K\|f\|_{L^p(\mathbb{G})} \tag{3-8}$$

for all $f \sim \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \operatorname{tr}(\hat{f}(\alpha)u^\alpha) \in L^p(\mathbb{G})$.

Proof. Take $w(k) := 1/(1+k)^{\beta+1}$. Then

$$C_w = \sup_{y>0}\Big\{ y\cdot \sum_{k\ge 0:(1+k)^{2\beta+1}\le\frac{1}{y}} (1+k)^{2\beta}\Big\}$$

$$\le \sup_{0<y\le 1}\Big\{ y\cdot\int_1^{(\frac{1}{y})^{1/(2\beta+1)}+1} t^{2\beta}\,dt\Big\} \le \sup_{0<y\le 1}\Big\{ y\cdot \frac{\big(2\cdot(\frac{1}{y})^{\frac{1}{2\beta+1}}\big)^{2\beta+1}}{2\beta+1}\Big\} = \frac{2^{\beta+1}}{2\beta+1} < \infty. \qquad \square$$

Corollary 3.10. *Let a compact matrix quantum group* $\mathbb{G}$ *be of Kac type and* $\widehat{\mathbb{G}}$ *have the rapid decay property with respect to the natural length function on* Irr($\mathbb{G}$) *and with inequality* (3-3). *Then, for each* $1 < p \le 2$, *there exists a universal constant* $K = K(p) > 0$ *such that*

$$\Bigg(\sum_{k\ge 0}\sum_{\substack{\alpha\in\mathrm{Irr}(\mathbb{G})\\|\alpha|=k}} \frac{1}{(1+|\alpha|)^{(2-p)(\beta+1)}\big(\sum_{\beta\in S_k} n_\beta^2\big)^{\frac{2-p}{2}}} n_\alpha\|\hat{f}(\alpha)\|_{S_{n_\alpha}^p}^p\Bigg)^{\frac{1}{p}} \le K\|f\|_{L^p(\mathbb{G})} \tag{3-9}$$

for all $f \sim \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \operatorname{tr}(\hat{f}(\alpha)u^\alpha) \in L^p(\mathbb{G})$.

Proof. Since $\frac{1}{p}=\frac{1}{2}+\frac{2-p}{2p}$ and $n_\alpha^{-\frac{1}{p}}\|\hat{f}(\alpha)\|_{S^p_{n_\alpha}}\leq n_\alpha^{-\frac{1}{2}}\|\hat{f}(\alpha)\|_{\mathrm{HS}}$, we have

$$\sum_{\alpha\in S_k} n_\alpha\|\hat{f}(\alpha)\|^p_{S^p_{n_\alpha}}\leq\sum_{\alpha\in S_k} n_\alpha^{2-\frac{p}{2}}\|\hat{f}(\alpha)\|^p_{\mathrm{HS}}=\big\|\big(n_\alpha^{\frac{1}{2}}\|\hat{f}(\alpha)\|_{\mathrm{HS}}\cdot n_\alpha^{\frac{2}{p}-1}\big)_{\alpha\in S_k}\big\|^p_{\ell^p(S_k)}$$

$$\leq\big\|\big(n_\alpha^{\frac{1}{2}}\|\hat{f}(\alpha)\|_{\mathrm{HS}}\big)_{\alpha\in S_k}\big\|^p_{\ell^2(S_k)}\cdot\Big(\sum_{\alpha\in S_k} n_\alpha^2\Big)^{\frac{2-p}{2}}=\Big(\sum_{\alpha\in S_k} n_\alpha^2\Big)^{\frac{2-p}{2}}\cdot\Big(\sum_{\alpha\in S_k} n_\alpha\|\hat{f}(\alpha)\|^2_{\mathrm{HS}}\Big)^{\frac{p}{2}}.$$

Then we can obtain the conclusion above. □

4. Hardy–Littlewood inequalities

This section is dedicated to establishing explicit Hardy–Littlewood inequalities for the main targets: the reduced group C^*-algebras $C^*_r(G)$ with finitely generated discrete groups G, the free orthogonal quantum groups O_N^+ and the free permutation quantum groups S_N^+.

4A. *The reduced group C^*-algebras $C^*_r(G)$.* In this subsection, we deal with finitely generated discrete groups G. As expected, we find clear evidence that the geometric information of the underlying group is important for understanding noncommutative L^p-spaces $L^p(\mathrm{VN}(G))$.

Definition 4.1. A discrete group with a fixed finite symmetric generating set S is said to be polynomially growing if there exist $C>0$ and $k>0$ such that

$$\#\{g\in G:|g|\leq n\}\leq C(1+n)^k\quad\text{for all } n\geq 0.$$

In this case, the polynomial growth rate k_0 is defined as the minimum of such k. Then k_0 becomes a natural number and independent of the choice of generating set S.

Theorem 4.2. (1) *Let G be a finitely generated discrete group which has the polynomial growth rate k_0. Then, for each $1<p\leq 2$, there exists a universal constant $K=K(p)$ such that*

$$\Big(\sum_{g\in G}\frac{1}{(1+|g|)^{(2-p)k_0}}|f(g)|^p\Big)^{\frac{1}{p}}\leq K\|\lambda(f)\|_{L^p(\mathrm{VN}(G))}\tag{4-1}$$

for all $\lambda(f)\sim\sum_{g\in G} f(g)\lambda_g\in L^p(\mathrm{VN}(G))$.

(2) *Let G be a finitely generated discrete group with*

$$b_k=\#\{g\in G:|g|\leq k\}\leq Cr^k\quad\textit{for all } k\geq 0,$$

where $|\cdot|$ is the natural length function with respect to a finite symmetric generating set S. Then, for each $1<p\leq 2$, there exists a universal constant $K=K(p,S)>0$ such that

$$\Big(\sum_{g\in G}\frac{1}{r^{(2-p)|g|}}|f(g)|^p\Big)^{\frac{1}{p}}\leq K\|\lambda(f)\|_{L^p(\mathrm{VN}(G))}\tag{4-2}$$

for all $\lambda(f)\sim\sum_{g\in G} f(g)\lambda_g\in L^p(\mathrm{VN}(G))$.

Proof. (1) This is clear from Corollary 3.5.

(2) Consider $w(g) := 1/r^{|g|}$. Then

$$\sup_{t>0}\Big\{t\cdot\sum_{|g|\le\log_r\left(\frac{1}{t}\right)}1\Big\} = \sup_{0<t\le1}\Big\{t\cdot\sum_{|g|\le\log_r\left(\frac{1}{t}\right)}1\Big\} \le C.$$

Then the conclusion is derived from Theorem 3.1 □

Remark 4.3. (1) For every finitely generated discrete group, there exist $C, r > 0$ such that $b_k \le Cr^k$ for all $k \ge 0$ by Fekete's subadditivity lemma. Therefore, Theorem 4.2 covers all finitely generated discrete groups.

(2) In fact, we shall see that (4-1) is sharp because of Theorem 6.6.

Although we can always find inequality (4-2) for every finitely generated discrete group, we can achieve a better result by adding more detailed geometric information of the underlying groups. Indeed, if we assume hyperbolicity of a group, then the inequality is considerably improved.

Theorem 4.4. *Let G be any nonelementary word hyperbolic group with $b_k \le Cr^k$ for all $k \ge 0$ with respect to a finite symmetric generating set S. Then, for each $1 < p \le 2$, there exists a universal constant $K = K(S, p)$ such that*

$$\Big(\sum_{g\in G}\frac{1}{r^{\frac{(2-p)|g|}{2}}(1+|g|)^{4-2p}}|f(g)|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{k\ge0}\frac{1}{(k+1)^{4-2p}}\Big(\sum_{\substack{g\in G\\|g|=k}}|f(g)|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \le K\|\lambda(f)\|_{L^p(\mathrm{VN}(G))} \tag{4-3}$$

for all $\lambda(f) \sim \sum_{g\in G} f(g)\lambda_g \in L^p(\mathrm{VN}(G))$.

Proof. The conclusion comes from Corollary 3.10 and [de la Harpe 1988]. □

4B. *Free quantum groups.* Let us begin with the investigation of "genuine" quantum examples: the free orthogonal quantum groups O_N^+ and the free permutation quantum groups S_{N+2}^+. Moreover, we will find a subclass of $L^p(O_N^+)$ where the Hardy–Littlewood inequalities (4-4) and (4-5) on O_N^+ become equivalence (4-7), as for the result of SU(2) [Akylzhanov et al. 2015, Theorem 2.10].

Theorem 4.5. (1) *Let $\mathbb{G}$ be the free orthogonal quantum group O_2^+ or the free permutation quantum group S_4^+. Then, for each $1 < p \le 2$, there exists a universal constant $K = K(p)$ such that*

$$\Big(\sum_{k\ge0}\frac{1}{(1+k)^{6-3p}}n_k\|\hat{f}(k)\|_{S^p_{n_k}}^p\Big)^{\frac{1}{p}} \le \Big(\sum_{k\ge0}\frac{1}{(1+k)^{6-3p}}n_k^{2-\frac{p}{2}}\|\hat{f}(k)\|_{\mathrm{HS}}^p\Big)^{\frac{1}{p}} \le K\|f\|_{L^p(\mathbb{G})} \tag{4-4}$$

for all $f \sim \sum_{k\ge0} n_k\,\mathrm{tr}(\hat{f}(k)u^k) \in L^p(\mathbb{G})$.

(2) *Let* $\mathbb{G}$ *be a free orthogonal quantum group* O_N^+ *or a free permutation quantum group* S_{N+2}^+ *with* $N \geq 3$. *Then, for each* $1 < p \leq 2$, *there exists a universal constant* $K = K(p)$ *such that*

$$\left(\sum_{k\geq 0} \frac{1}{r_0^{(2-p)k}(1+k)^{4-2p}} n_k \|\hat{f}(k)\|_{S^p_{n_k}}^p\right)^{\frac{1}{p}} \leq \left(\sum_{k\geq 0} \frac{1}{r_0^{(2-p)k}(1+k)^{4-2p}} n_k^{2-\frac{p}{2}} \|\hat{f}(k)\|_{\mathrm{HS}}^p\right)^{\frac{1}{p}} \leq K\|f\|_{L^p(\mathbb{G})} \tag{4-5}$$

for all $f \sim \sum_{k\geq 0} n_k \operatorname{tr}(\hat{f}(k)u^k) \in L^p(\mathbb{G})$, *where* $r_0 = \frac{1}{2}(N + \sqrt{N^2-4})$.

Proof. (1) In this case, $n_k = k+1$ (resp. $2k+1$) for all k. Thus, the conclusion comes from Corollary 3.5.

(2) It is known that $\widehat{O}_N^+$ and $\widehat{S}_{N+2}^+$ with $N \geq 3$ have the rapid decay property with $\beta = 1$ [Vergnioux 2007; Brannan 2013]. Also, $s_k = n_k^2 \approx r_0^{2k}$ for all $k \in \{0\} \cup \mathbb{N}$. Therefore, Corollaries 3.9 and 3.10 complete the proof. □

Remark 4.6. All results of this paper for S_N^+ can be extended to quantum automorphism group $\mathbb{G}_{\mathrm{aut}}(B, \psi)$ with a δ-trace ψ and $\dim(B) = N$ by repeating the same proofs. See [Brannan 2012a; 2013].

An important observation for the free orthogonal quantum groups O_N^+ is that the inequalities (4-4) and (4-5) become equivalences (4-7) under several assumptions. Essentially, this is based on the result of SU(2) [Akylzhanov et al. 2015, Theorem 2.10] and the following lemma moves the result to O_N^+.

Lemma 4.7. *Let* $\mathbb{G} = O_N^+$ *or* S_{N+2}^+ *with* $N \geq 2$ *and consider* $G = \mathrm{SU}(2)$ *or* $\mathrm{SO}(3)$ *for each case. Then, for* $f \sim \sum_{n\geq 0} c_n \chi_n^1 \in L^p(\mathbb{G})$, *the associated function* $\Phi(f) \sim \sum_{n\geq 0} c_n \chi_n^2 \in L^p(G)$ *has the same norm. More precisely,*

$$\|f\|_{L^p(\mathbb{G})} = \left\|\Phi(f) \sim \sum_{n\geq 0} c_n \chi_n^2\right\|_{L^p(G)} \quad \text{for all } 1 \leq p \leq \infty.$$

Here, $\chi_n^1 = \operatorname{tr}(u^n)$ *and* $\chi_n^2 = \operatorname{tr}(v^n)$, *where* u^n *and* v^n *are the* n*-th irreducible unitary representations of* $\mathbb{G}$ *and* G, *respectively.*

Proof. In the cases above, it is known that $\mathbb{G}$ and G share the same fusion rule. In [Wang 2017, Proposition 6.7], it was pointed out that the restricted map $\Phi|_{\mathrm{Pol}(\mathbb{G})}$ is a trace-preserving $*$-isomorphism. Now, for any $x = \sum_{k\geq 0} c_k \chi_k^1 \in \mathrm{Pol}(\mathbb{G})$ and $m \in \mathbb{N}$,

$$\begin{aligned} h((x^*x)^m) &= h\left(\left(\sum_{k,l\geq 0} \bar{c}_k c_l (\chi_k^1)^* \chi_l^1\right)^m\right) \\ &= \sum_{k_1,l_1,\ldots,k_m,l_m\geq 0} \overline{c_{k_1}\cdots c_{k_m}} c_{l_1}\cdots c_{l_m} h\big((\chi_{k_1}^1)^*\chi_{l_1}^1\cdots(\chi_{k_m}^1)^*\chi_{l_m}^1\big) \\ &= \sum_{k_1,l_1,\ldots,k_m,l_m\geq 0} \overline{c_{k_1}\cdots c_{k_m}} c_{l_1}\cdots c_{l_m} \int_G \bar{\chi}_{k_1}^2 \chi_{l_1}^2 \cdots \bar{\chi}_{k_m}^2 \chi_{l_m}^2 \\ &= \int_G \left(\sum_{k,l\geq 0} \bar{c}_k c_l \bar{\chi}_k^2 \chi_l^2\right)^m = \int_G |x'|^{2m}, \end{aligned}$$

where $x' = \sum_{k\geq 0} c_k \chi_k^2 \in \mathrm{Pol}(G)$. Then the Stone–Weierstrass theorem completes the proof. □

Corollary 4.8. *Let $N \geq 2$, $\frac{3}{2} < p \leq 2$ and fix $D > 0$. Also, assume $f \sim \sum_{k\geq 0} c_k \chi_k \in L^{\frac{3}{2}}(O_N^+)$ satisfies*

$$c_k \geq c_{k+1} \geq 0 \quad \text{and} \quad \sum_{m\geq k} \frac{c_m}{m+1} \leq D \cdot c_k \quad \text{for } k \geq 0. \tag{4-6}$$

Then we have

$$\|f\|_{L^p(O_N^+)} \approx \left(\sum_{k\geq 0} (1+k)^{2p-4} c_k^p\right)^{\frac{1}{p}}. \tag{4-7}$$

Proof. It is sufficient to combine the Lemma 4.7 and [Akylzhanov et al. 2015, Theorem 2.10]. □

5. A strong Hardy–Littlewood inequality

The studies of Hardy–Littlewood inequalities in [Akylzhanov et al. 2015; Hardy and Littlewood 1927; Hewitt and Ross 1974] dealt with general L^p-functions, but plenty of classical results of harmonic analysis on $\mathbb{T}$ show that a theorem on a function space can have a stronger form when restricted to a holomorphic setting [Kemp and Speicher 2007].

Evidence of these phenomena in the noncommutative setting is the strong Haagerup inequality on the reduced group C^*-algebras $C_r^*(\mathbb{F}_N)$. More precisely, it was shown that the rapid decay property can be strengthened in a general holomorphic setting [Kemp and Speicher 2007]. Such a phenomenon also occurs on the free unitary quantum groups [Brannan 2012b; 2012a].

Let $g_1, \ldots, g_N$ be canonical generators of $\mathbb{F}_N$ and denote by $\mathbb{F}_N^+$ the set of elements of the form $g_{i_1} g_{i_2} \cdots g_{i_m}$ with $m \in \{0\} \cup \mathbb{N}$ and $1 \leq i_k \leq N$ for all $1 \leq k \leq m$.

Theorem 5.1 (strong Haagerup inequality on $C_r^*(\mathbb{F}_N)$). *Consider a subset $E := \mathbb{F}_N^+$ and $E_k := \{g \in E : |g| = k\}$. Then, for any $k \in \{0\} \cup \mathbb{N}$, we have*

$$\Big\| \sum_{g\in E_k} f(g)\lambda_g \Big\|_{C_r^*(\mathbb{F}_N)} \leq \sqrt{e}\sqrt{k+1} \Big(\sum_{g\in E_k} |f(g)|^2\Big)^{\frac{1}{2}}.$$

Based on this information, we can modify the inequality (3-4) as follows.

Proposition 5.2. *Let $N \geq 2$. Then we have*

$$\|f\|_{A(\mathbb{F}_N)} \geq \frac{1}{\sqrt{e}} \sup_{k\geq 0} \frac{\left(\sum_{g\in E_k} |f(g)|^2\right)^{\frac{1}{2}}}{(k+1)^{\frac{1}{2}}} \quad \text{for all } f \in A(\mathbb{F}_N). \tag{5-1}$$

Proof. We can use the proof of Proposition 3.7 again in this case. The only difference is the improvement of $(1+k)^\beta$ to $(1+k)^{\frac{1}{2}}$ in inequality (3-5). Then we are able to reach a conclusion by restricting support of $x \in C_c(G)$ to $\mathbb{F}_N^+$ in the proof. □

Theorem 5.3. *Let $N \geq 2$. Then, for each $1 < p \leq 2$, there exists a universal constant $K = K(p) > 0$ such that*

$$\begin{aligned}\left(\sum_{g\in\mathbb{F}_N} \frac{1}{(1+|g|)^{\frac{3}{2}(2-p)} N^{\frac{(2-p)|g|}{2}}} |f(g)|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{k\geq 0} \frac{1}{(1+k)^{\frac{3}{2}(2-p)}} \Big(\sum_{\substack{g\in\mathbb{F}_N \\ |g|=k}} |f(g)|^2\Big)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\ &\leq K\|\lambda(f)\|_{L^p(\mathrm{VN}(\mathbb{F}_N))}\end{aligned} \tag{5-2}$$

for all $\lambda(f) \sim \sum_{g\in\mathbb{F}_N} f(g)\lambda_g \in L^p(\mathrm{VN}(\mathbb{F}_N))$ with $\mathrm{supp}(f) \subseteq \mathbb{F}_N^+$.

Proof. It can be obtained by repeating the proofs of Theorem 3.8 and Corollary 3.10. The only difference is that it the operator A should be replaced with

$$\lambda(f) \mapsto \left(\frac{\|f \cdot \chi_{E_k}\|_{\ell^2(\mathbb{F}_N)}}{w(k)} \right)_{k \geq 0}$$

and $(1+k)^\beta$ with $(1+k)^{\frac{1}{2}}$. Also, we choose a weight function w on $\{0\} \cup \mathbb{N}$ by $w(k) := 1/(1+k)^{\frac{3}{2}}$. Then we can get a new inequality for general $\lambda(f) \in L^p(\mathrm{VN}(\mathbb{F}_N))$, but our consideration is in the case $\mathrm{supp}(f) \subseteq \mathbb{F}_N^+$. □

6. Sharpness

Hardy–Littlewood inequalities discussed in Section 4 give a specific pair (r, s) such that the multiplier

$$\mathcal{F}_{w_{r,s}} : L^p(\mathbb{G}) \to \ell^p(\widehat{\mathbb{G}}), \quad f \mapsto (w_{r,s}(\alpha) \hat{f}(\alpha))_{\alpha \in \mathrm{Irr}(\mathbb{G})},$$

is bounded for each case, where $w_{r,s}(\alpha) = 1/r^{|\alpha|}(1+|\alpha|)^s$ with respect to the natural length $|\cdot|$ on $\mathrm{Irr}(\mathbb{G})$.

Here is the list of such pairs:

- $\left(0, \frac{n(2-p)}{p}\right)$ for G a compact Lie group,
- $\left(0, \frac{k_0(2-p)}{p}\right)$ for $C_r^*(G)$ with a polynomially growing discrete group G,
- $\left(0, \frac{3(2-p)}{p}\right)$ for O_2^+ or S_4^+ and
- $\left(r_0^{\frac{2-p}{p}}, \frac{2(2-p)}{p}\right)$ for O_N^+ or S_{N+2}^+ with $N \geq 3$.

Remark 6.1 [Lee and Youn 2017; Wallach 1973]. If G is a compact Lie group, then $\sqrt{\kappa_\pi}$ is equivalent to the natural length function $\|\cdot\|_S$ generated by the fundamental generating set S of $\mathrm{Irr}(G)$. Equivalently, $(1+\kappa_\pi)^{\frac{\beta}{2}} \approx (1+\|\pi\|_S)^\beta$.

In order to claim that the established inequalities are sharp, we will show that there is no (r', s') better than the given specific pair in that $\mathcal{F}_{w_{r',s'}}$ is unbounded whenever (1) $r' < r$ or (2) $r' = r,\ s' < s$.

This viewpoint is different from the spirit of [Akylzhanov et al. 2015, Theorem 2.10] or Corollary 4.8, which requires finding an equivalence in a subclass. However, our approach is quite natural since it is strongly related to the Sobolev embedding theorem. In this section and Section 7B, we will discuss how they are related. For example, $\mathcal{F}_{w_{0,s}} : L^p(\mathbb{T}^d) \to \ell^p(\mathbb{Z}^d)$ is bounded if and only if $H_p^s(\mathbb{T}^d) \subseteq L^{p'}(\mathbb{T}^d)$ and it is equivalent to

$$H_q^{\frac{ps}{2-p}\left(\frac{1}{q}-\frac{1}{r}\right)}(\mathbb{T}^d) \subseteq L^r(\mathbb{T}^d)$$

for all $1 < q < r < \infty$, where $H_p^s(\mathbb{T}^d)$ is the Bessel potential space.

In addition, this view has a definite advantage over looking for equivalence because we can cover a much larger class.

Our first strategy is to handle an ultracontractivity problem on $C(G)$ with compact Lie groups and $C_r^*(G)$ with polynomially growing discrete groups. In fact, ultracontractivity problems are strongly related to Sobolev embedding properties [Xiong 2016].

Let M be the von Neumann subalgebra generated by $\{\chi_\alpha\}_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ in $L^\infty(\mathbb{G})$ and consider $L^p(M)$, the noncommutative L^p-space associated with the restriction of the Haar state on M. Now suppose that $l:\mathrm{Irr}(\mathbb{G})\to(0,\infty)$ is a positive function and there exist $1<p<2$ and a universal constant $C>0$ such that

$$\left\| J(f)\sim\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{1}{l(\alpha)^{\frac{\beta}{p}}}c_\alpha\chi_\alpha\right\|_{L^{p'}(M)}\le C\left\| f\sim\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}c_\alpha\chi_\alpha\right\|_{L^p(M)}, \tag{6-1}$$

where J is a densely defined positive operator on $L^2(M)$ which maps χ_α to $1/l(\alpha)^{\frac{\beta}{p}}\chi_\alpha$ for all $\alpha\in\mathrm{Irr}(\mathbb{G})$, $1\le i,j\le n_\alpha$. Indeed, $J=K^*K$, where $K:\chi_\alpha\mapsto 1/l(\alpha)^{\frac{\beta}{2p}}\chi_\alpha$.

Now take $\phi(t):=t^{\frac{2\beta}{2-p}}$, $\psi(z):=z^{\frac{2\beta}{2-p}}$ and $L:=J^{-\frac{p}{2\beta}}$. Then [Xiong 2016, Theorem 1.1] suggests that there exists a universal constant $C'>0$ such that

$$\left\| e^{-tL}(f)\sim\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{c_\alpha}{e^{tl(\alpha)^{1/2}}}\chi_\alpha\right\|_{L^\infty(M)}\le C'\frac{\|f\|_{L^1(M)}}{t^{\frac{2\beta}{2-p}}} \tag{6-2}$$

for all $0<t<\infty$ and all $f\sim\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}c_\alpha\chi_\alpha\in L^1(M)$.

The next thing to do is to prove that the following result (6-3) is achieved by combining (6-2), Lemma 6.2 and Lemma 6.3 below:

$$\sup_{0<t<\infty}\left\{t^{\frac{2\beta}{2-p}}\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{e^{tl(\alpha)^{1/2}}}\right\}=:C<\infty, \tag{6-3}$$

if $\mathbb{G}$ is G a compact Lie group or $C_r^*(G)$ with a polynomially growing discrete group.

Lemma 6.2 [Dasgupta and Ruzhansky 2014, Lemma 4.1; Lee and Youn 2017, Proposition 5.7].

(1) *Let G be a compact Lie group with the dimension n. Then*

$$\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi^2}{(1+\kappa_\pi)^{\frac{s}{2}}}<\infty$$

if and only if $s>n$.

(2) *Let G be a finitely generated discrete group with the polynomial growth rate k_0. Then*

$$\sum_{g\in G}\frac{1}{(1+|g|)^s}<\infty$$

if and only if $s>k_0$.

Lemma 6.3. (1) *Let G be a compact Lie group. Then there exist probability measures $\{\nu_t\}_{t>0}$ such that $\hat{\nu}_t(\pi)=1/e^{t\kappa_\pi}\mathrm{Id}_{n_\pi}$ for all $\pi\in\mathrm{Irr}(G)$. Moreover, $\{\nu_t\}_{t>0}\subseteq L^1(G)$.*

(2) *Let G be a compact Lie group and let $f\sim\sum_{\pi\in\mathrm{Irr}(G)}n_\pi\,\mathrm{tr}(\hat{f}(\pi)\pi)\in L^\infty(G)$ be such that $\hat{f}(\pi)\ge 0$ for all $\pi\in\mathrm{Irr}(G)$. Then*

$$\|f\|_{L^\infty(G)}=\sum_{\pi\in\mathrm{Irr}(G)}n_\pi\,\mathrm{tr}(\hat{f}(\pi)).$$

(3) *Let G be a discrete group. Then the Fourier algebra $A(G)$ has a bounded approximate identity if and only if G is amenable. In this case, the bounded approximate identity can be chosen as positive and compactly supported functions on G.*

(4) *If G is an amenable discrete group, we have*

$$\Big\|\lambda(f)\sim\sum_{g\in G}f(g)\lambda_g\Big\|_{\mathrm{VN}(G)}=\sum_{g\in G}f(g)$$

for any positive function $f\in\ell^1(G)$.

Proof. (1) Since

$$\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi^2}{e^{t\kappa_\pi}}<\infty$$

by Lemma 6.2, we know that $\nu_t\in A(G)\subseteq C(G)\subseteq L^1(G)$. The family $\{\nu_t\}_{t>0}$ is called the heat semigroup of measures.

(2) Since $f\mapsto\mu_t*f$ is a contractive map on $L^\infty(G)$ for all $t>0$, where $*$ is the convolution product, we have

$$\|f\|_{L^\infty(G)}\geq\sup_{t>0}\Big\|f_t\sim\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi}{e^{t\kappa_\pi}}\mathrm{tr}(\hat f(\pi)\pi)\Big\|_{C(G)}.$$

Here, since

$$\sum_\pi\frac{n_\pi}{e^{t\kappa_\pi}}\mathrm{tr}(\hat f(\pi))\leq\|f\|_{L^1(G)}\sum_\pi\frac{n_\pi^2}{e^{t\kappa_\pi}}<\infty$$

by Lemma 6.2, the Fourier series of f_t uniformly converges to $f_t\in C(G)$. Therefore,

$$\|f\|_{L^\infty(G)}\geq\sup_{t>0}f_t(1)=\sup_{t>0}\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi}{e^{t\kappa_\pi}}\mathrm{tr}(\hat f(\pi))=\sum_{\pi\in\mathrm{Irr}(G)}n_\pi\,\mathrm{tr}(\hat f(\pi)).$$

The other direction is trivial.

(3) See [Runde 2002, Theorem 7.1.3] and its proof. We may also assume the compact supportness by considering $f_\epsilon:=f\cdot\chi_{\{g\in G:f(g)>\epsilon\}}$ for positive $f\in\ell^1(G)$.

(4) This is Kesten's condition, which is equivalent to amenability. □

Now we can prove that the claim is true.

Proposition 6.4. *Let $\mathbb{G}$ be G a compact Lie group or $C_r^*(G)$ with a polynomially growing discrete group. Also, suppose that the inequality* (6-1) *holds. Then*

$$\sup_{0<t<\infty}\Big\{t^{\frac{2\beta}{2-p}}\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{e^{tl(\alpha)^{1/2}}}\Big\}=:C<\infty. \tag{6-4}$$

Proof. For G a compact Lie group, by Lemma 6.3, for all $0<t<\infty$,

$$\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi^2}{e^{tl(\pi)^{1/2}}}=\sup_{r>0}\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi^2}{e^{tl(\pi)^{1/2}}e^{r\kappa_\pi}}=\sup_{r>0}\|e^{-tL}(\nu_r)\|_{L^\infty(G)}\leq\frac{C'}{t^{\frac{2\beta}{2-p}}}\sup_{r>0}\|\nu_r\|_{L^1(G)}\leq\frac{C'}{t^{\frac{2\beta}{2-p}}}.$$

Now, for G a polynomially growing discrete group, there exists a bounded approximate identity $(e_i)_i$ in $A(G)$ that consists of positive and compactly supported functions since polynomially growing discrete groups are always amenable. Then (6-2) implies

$$\sum_{g\in G}\frac{1}{e^{tl(g)^{1/2}}}=\sup_i\sum_{g\in G}\frac{e_i(g)}{e^{tl(g)^{1/2}}}=\sup_i\Big\|\sum_{g\in G}\frac{e_i(g)}{e^{tl(g)^{1/2}}}\lambda_g\Big\|_{C_r^*(G)}\le\frac{C''}{t^{\frac{2\beta}{2-p}}}$$

since $\lim_i e_i(g)=1$ for all $g\in G$. □

Proposition 6.4 allows us to extract a quantitative observation.

Proposition 6.5. *Let $\mathbb{G}$ be G a compact Lie group or $C_r^*(G)$ with a polynomially growing discrete group. Also, suppose that the inequality* (6-1) *holds. Then we have*

$$\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{l(\alpha)^{\frac{m}{2}}}<\infty\quad\textit{for all natural numbers } m>\frac{2\beta}{2-p}. \tag{6-5}$$

Proof. Choose $\gamma\in\big(\max\{\frac{2\beta}{2-p},m-1\},m\big)$. Then we have

$$\sup_{0<t\le1}\Big\{t^\gamma\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{e^{tl(\alpha)^{1/2}}}\Big\}=:C_0<\infty$$

from (6-4), so that

$$\int_t^1\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{e^{xl(\alpha)^{1/2}}}\,dx\le C_0\int_t^1\frac{1}{x^\gamma}\,dx$$

for all $0<t\le1$.

This implies

$$\sup_{0<t\le1}\Big\{t^{\gamma-1}\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{l(\alpha)^{\frac12}e^{tl(\alpha)^{1/2}}}\Big\}=:C_1<\infty,$$

so that we can inductively see that

$$\sup_{0<t\le1}\Big\{t^{\gamma-(m-1)}\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{l(\alpha)^{\frac{m-1}{2}}e^{tl(\alpha)^{1/2}}}\Big\}=:C_{m-1}<\infty.$$

Then there exist $D_1,D_2>0$ such that

$$\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha^2}{l(\alpha)^{\frac{m}{2}}e^{tl(\alpha)^{1/2}}}\le D_1t^{m-\gamma}+D_2\quad\text{for all } 0<t\le1$$

through similar reasoning.

Lastly, taking the limit $t\to0$ completes the proof. □

Theorem 6.6. *Let $1<p\le2$.*

(1) *Let G be a compact Lie group with dimension n. Then*

$$\Big(\sum_{\pi\in\mathrm{Irr}(G)}\frac{1}{(1+\kappa_\pi)^{\frac{s}{2}}}n_\pi\|\hat f(\pi)\|_{S_{n_\pi}^p}^p\Big)^{\frac1p}\lesssim\|f\|_{L^p(G)}$$

holds if and only if $s\ge n(2-p)$.

(2) *Let G be a finitely generated discrete group with polynomial growth rate k_0. Then*

$$\left(\sum_{g\in G}\frac{1}{(1+|g|)^s}|f(g)|^p\right)^{\frac{1}{p}}\lesssim\|\lambda(f)\|_{L^p(\mathrm{VN}(G))}$$

holds if and only if $s\geq k_0(2-p)$.

(3) *Let $\mathbb{G}$ be O_2^+ or S_4^+. Then*

$$\left(\sum_{k\geq 0}\frac{1}{(1+k)^s}n_k\|\hat{f}(k)\|_{S^p_{n_k}}^p\right)^{\frac{1}{p}}\lesssim\|f\|_{L^p(\mathbb{G})}$$

holds if and only if $s\geq 3(2-p)$.

(4) *Let $\mathbb{G}$ be O_N^+ or S_{N+2}^+ with $N\geq 3$. Then*

$$\left(\sum_{k\geq 0}\frac{1}{r_0^{(2-p)k}(1+k)^s}n_k\|\hat{f}(k)\|_{S^p_{n_k}}^p\right)^{\frac{1}{p}}\lesssim\|f\|_{L^p(\mathbb{G})}$$

holds if and only if $s\geq 4-2p$, where $r_0=\frac{1}{2}(N+\sqrt{N^2-4})$.

Proof. One direction is obtained from the discussed Hardy–Littlewood inequalities (1-3), (4-1), (4-4) and (4-5). To prove the converse direction, firstly, define $l(\alpha)$ by (1) $(1+\kappa_\pi)^{\frac{1}{2}}$ and (2) $1+|g|$ respectively. Then the assumed inequality

$$\left(\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})}\frac{n_\alpha}{l(\alpha)^s}\|\hat{f}(\alpha)\|_{S^p_{n_\alpha}}^p\right)^{\frac{1}{p}}\lesssim\|f\|_{L^p(\mathbb{G})}$$

implies the inequality (6-1) for $\beta=s$. Then, by Proposition 6.5 and Lemma 6.2, we can get (1) $2n\leq\frac{2\beta}{2-p}$, (2) $2k_0\leq\frac{2\beta}{2-p}$ respectively.

In (3) and (4), let $G=\mathrm{SU}(2)$ if $\mathbb{G}=O_N^+$ and $G=\mathrm{SO}(3)$ if $\mathbb{G}=S_{N+2}^+$ for each case. Also, denote by $\chi_k'\in\mathrm{Pol}(G)$ the character corresponding to $\chi_k\in\mathrm{Pol}(\mathbb{G})$, as in Lemma 4.7. Define $l(k):=1+k$.

First of all, in (3), for each $f\sim\sum_{k\geq 0}c_k\chi_k\in L^p(\mathbb{G})$, we have

$$\left\|\sum_{k\geq 0}(1+k)^{-\frac{s}{p}}c_k\chi_k\right\|_{L^{p'}(G)}\leq\left(\sum_{k\geq 0}\frac{1}{(1+k)^{s+p-2}}|c_k|^p\right)^{\frac{1}{p}}\lesssim\|f\|_{L^p(\mathbb{G})}=\left\|f'\sim\sum_{k\geq 0}c_k\chi_k'\right\|_{L^p(G)}$$

by Lemma 4.7 and the Hausdorff–Young inequality. On the other hand, in (4), for each $f\sim\sum_{k\geq 0}c_k\chi_k\in L^p(\mathbb{G})$, we have

$$\left\|\sum_{k\geq 0}(1+k)^{-\frac{s+2-p}{p}}c_k\chi_k\right\|_{L^{p'}(G)}\leq\left(\sum_{k\geq 0}\frac{1}{(1+k)^{s}}|c_k|^p\right)^{\frac{1}{p}}\lesssim\|f\|_{L^p(\mathbb{G})}=\left\|f'\sim\sum_{k\geq 0}c_k\chi_k'\right\|_{L^p(G)}$$

by similar arguments.

Now we can apply Proposition 6.5 and Lemma 6.2 for compact Lie groups again, so that (3) $s\geq 6-3p$ and (4) $s-p+2\geq 6-3p(\Leftrightarrow s\geq 4-2p)$ respectively. ☐

7. Some remarks about Sidon sets, the Sobolev embedding theorem and quantum torus

The methods of this paper, combined with a lacunarity result for compact quantum groups, give a Sobolev-embedding-theorem-type interpretation for G compact Lie groups and for $C_r^*(G)$ with polynomially growing groups. Also, we show an explicit inequality on the quantum torus $\mathbb{T}_\theta^d$.

7A. ***Sidon sets on compact quantum groups.*** The study of lacunarity, particularly Sidon sets, is one of the major subjects in harmonic analysis and recently the notion has been extended to the setting of compact quantum groups [Wang 2017].

Definition 7.1. Let $\mathbb{G}$ be a compact quantum group.

(1) A subset $E \subseteq \mathrm{Irr}(\mathbb{G})$ is called a Sidon set if there exists $K > 0$ such that

$$\|\hat{f}\|_{\ell^1(\widehat{\mathbb{G}})} \leq K \|f\|_{L^\infty(\mathbb{G})} \quad \text{for all } f \in \mathrm{Pol}_E(\mathbb{G}),$$

where $\mathrm{Pol}_E(\mathbb{G}) := \{f \in \mathrm{Pol}(\mathbb{G}) : \hat{f}(\alpha) = 0 \text{ for all } \alpha \notin E\}$.

(2) A subset $E \subseteq \mathrm{Irr}(\mathbb{G})$ is called a central Sidon set if there exists $K > 0$ such that

$$\|\hat{f}\|_{\ell^1(\widehat{\mathbb{G}})} \leq K \|f\|_{L^\infty(\mathbb{G})} \quad \text{for all } f = \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} c_\alpha \chi_\alpha \in \mathrm{Pol}_E(\mathbb{G}).$$

Let $\mathbb{G} = (A, \Delta)$ be of Kac type and $E \subseteq \mathrm{Irr}(\mathbb{G})$ be a central Sidon set. Then [Wang 2017, Proposition 6.4] implies there exists $\mu \in M(\mathbb{G}) = C_r(\mathbb{G})^*$ such that $\hat{\mu}(\alpha) = (\mu((u_{j,i}^\alpha)^*))_{1 \leq i,j \leq n_\alpha} = \mathrm{Id}_{n_\alpha}$ for all $\alpha \in E$. Since $\mathrm{Pol}(\mathbb{G})$ is dense in $C_r(\mathbb{G})$, Proposition 3.7 still holds for $\mu \in M(\mathbb{G})$.

Now, if $\mathbb{G}$ satisfies the assumptions of Proposition 3.7 and if $E \subseteq \mathrm{Irr}(\mathbb{G})$ is a central Sidon set, we get

$$\infty > \sup_{k \geq 0} \frac{\left(\sum_{\alpha \in E_k} n_\alpha^2\right)^{\frac{1}{2}}}{(1+k)^\beta} \geq \sup_{k \geq 0} \frac{|E_k|^{\frac{1}{2}} \min_{\alpha \in E_k} n_\alpha}{(1+k)^\beta},$$

where $E_k := \{\alpha \in E : |\alpha| = k\}$.

Thus, the conditions $|E| = \infty$ and $n_\alpha > r^{|\alpha|}$ for all $\alpha \in \mathrm{Irr}(\mathbb{G})$ with $r > 1$ cannot hold true simultaneously.

Remark 7.2. (1) The argument above shows that there is no infinite (central) Sidon set in U_N^+ with $N \geq 3$, which is not explained in [Wang 2017].

(2) Shortly after this research, the author of [Wang 2017] personally informed me of a simple idea to further explain the U_N^+ cases. Under the identification $\mathrm{Irr}(U_N^+) \cong \mathbb{F}_2^+$, the fact that

$$\|\chi_\alpha\|_4 = (1 + |\alpha|)^{\frac{1}{4}} \quad \text{for all } \alpha \in \mathbb{F}_2^+$$

implies there is no infinite $\Lambda(4)$ set, so that there is no infinite Sidon set on U_N^+ with $N \geq 2$.

7B. ***Sobolev embedding properties.*** The content of Section 6 can be interpreted in terms of Sobolev embedding properties by [Xiong 2016, Theorem 1.1].

For G a compact Lie group whose real dimension is n, the computations in Section 6 suggest that

$$\Big\|(1-\Delta)^{-\frac{\beta}{2}}(f)\sim\sum_{\pi\in\mathrm{Irr}(G)}\frac{n_\pi}{(1+\kappa_\pi)^{\frac{\beta}{2}}}\,\mathrm{tr}(\hat{f}(\pi)\pi)\Big\|_{L^{p'}(G)}\lesssim\|f\|_{L^p(G)}$$

if and only if $\beta\geq\frac{n(2-p)}{p}$ for each $1<p\leq 2$. Moreover, it is equivalent to

$$\big\|(1-\Delta)^{-\frac{\beta}{2}(\frac{1}{p}-\frac{1}{q})}(f)\big\|_{L^q(G)}\lesssim\|f\|_{L^p(G)}$$

if and only if $\beta\geq n$ for each $1<p<q<\infty$. If we define the space $H_p^s(G):=\{f\in L^p(G):(1-\Delta)^{\frac{s}{2}}(f)\in L^p(G)\}$ as an analogue of the Bessel potential space, then the result above is interpreted as

$$H_p^s(G)\subseteq L^q(G)\quad\text{if and only if}\quad s\geq n\big(\tfrac{1}{p}-\tfrac{1}{q}\big)\tag{7-1}$$

for each $1<p<q<\infty$.

On the other hand, if G is a finitely generated discrete group with polynomial growth rate k_0, then we can define an infinitesimal generator L on $C_r^*(G)$ by $\lambda_g\mapsto-|g|\lambda_g$ for all $g\in G$. Then we are able to induce the Sobolev embedding property of noncommutative spaces $L^p(\mathrm{VN}(G))$ as follows:

$$\big\|(1-L)^{-\beta(\frac{1}{p}-\frac{1}{q})}(\lambda(f))\big\|_{L^q(\mathrm{VN}(G))}\lesssim\|\lambda(f)\|_{L^p(\mathrm{VN}(G))}\quad\text{if and only if}\quad\beta\geq k_0\tag{7-2}$$

for each $1<p<q<\infty$.

The reader may consider another natural infinitesimal generator $L':\lambda_g\mapsto-|g|^2\lambda_g$, but the result is essentially the same when $(1-L)$ is replaced with $(1-L')^{\frac{1}{2}}$.

7C. *Hardy–Littlewood inequality on quantum torus.* The quantum torus $\mathbb{T}_\theta^d$ is a widely studied example of quantum space, but it is not a quantum group [Sołtan 2010]. Nevertheless, we can establish Hardy–Littlewood inequalities on $\mathbb{T}_\theta^d$, which is of the same form as the case for $\mathbb{T}^d$. A proof can be given by repeating the proof of Theorem 3.1. See [Xiong et al. 2015] for Fourier analysis on the quantum torus.

Remark 7.3. For a quantum torus $\mathbb{T}_\theta^d$, for each $1<p\leq 2$, we have

$$\bigg(\sum_{m\in\mathbb{Z}^d}\frac{1}{(1+\|m\|_1)^{d(2-p)}}|\hat{x}(m)|^p\bigg)^{\frac{1}{p}}\lesssim\|x\|_{L_p(\mathbb{T}_\theta^d)}.\tag{7-3}$$

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