ANALYSIS & PDE

Volume 11 No. 2 2018

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ANALYSIS AND PDE
Vol. 11, No. 2, 2018
dx.doi.org/10.2140/apde.2018.11.351

ASYMPTOTIC LIMITS AND STABILIZATION FOR THE 2D NONLINEAR MINDLIN–TIMOSHENKO SYSTEM

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Dedicated to Enrique Fernández-Cara on the occasion of his 60th birthday

We show how the so-called von Kármán model can be obtained as a singular limit of a Mindlin–Timoshenko system when the modulus of elasticity in shear k tends to infinity. This result gives a positive answer to a conjecture by Lagnese and Lions in 1988. Introducing damping mechanisms, we also show that the energy of solutions for this modified Mindlin–Timoshenko system decays exponentially, uniformly with respect to the parameter k. As $k \to \infty$, we obtain the damped von Kármán model with associated energy exponentially decaying to zero as well.

1. Introduction

The Mindlin–Timoshenko system of equations is a widely used and physically fairly complete mathematical model to describe the dynamics of a plate, taking into account transverse shear effects; see, e.g., [Lagnese and Lions 1988]. This model is used, for example, to model aircraft wings; see, for instance, [Doyle 1997]. To describe this model, let $\Omega \subset \mathbb{R}^2$ be an open bounded set whose boundary Γ is regular enough. Consider $\{\Gamma_0, \Gamma_1\}$ to be a partition of Γ. Let $T > 0$ be given and consider the cylinder $Q = \Omega \times (0, T)$, with lateral boundary $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_i = \Gamma_i \times (0, T)$, $i = 0, 1$. The two-dimensional Mindlin–Timoshenko system is

$$\begin{cases} \frac{1}{12}\rho h^3 \phi_{1tt} - L_1(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \\ \frac{1}{12}\rho h^3 \phi_{2tt} - L_2(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \\ \rho h \psi_{tt} - L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) = 0 & \text{in } Q, \\ \rho h \eta_{1tt} - L_4(\psi, \eta_1, \eta_2) = 0 & \text{in } Q, \\ \rho h \eta_{2tt} - L_5(\psi, \eta_1, \eta_2) = 0 & \text{in } Q. \end{cases} \tag{1-1}$$

We complete the system with the boundary conditions

$$\begin{gathered} \phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0, \\ \big\{\mathcal{B}_1(\phi_1, \phi_2), \mathcal{B}_2(\phi_1, \phi_2), \mathcal{B}_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2), \mathcal{B}_4(\eta_1, \eta_2), \mathcal{B}_5(\eta_1, \eta_2)\big\} = \{0, 0, 0, 0, 0\} \quad \text{on } \Sigma_1, \end{gathered} \tag{1-2}$$

Araruna was partially supported by INCTMat, CAPES, CNPq (Brazil) and MathAmSud COSIP. Braz e Silva was partially supported by CAPES and CNPq (Brazil). Queiroz-Souza was partially supported by CAPES (Brazil) and MathAmSud COSIP.
MSC2010: 35Q74, 74K20, 35B40.
Keywords: vibrating plates, Mindlin–Timoshenko system, von Kármán system, singular limit, uniform stabilization.

and initial data

$$\{\phi_1(\cdot,0),\phi_2(\cdot,0),\psi(\cdot,0),\eta_1(\cdot,0),\eta_2(\cdot,0)\}=\{\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}\}\quad \text{in } \Omega, \tag{1-3a}$$

$$\{\phi_{1t}(\cdot,0),\phi_{2t}(\cdot,0),\psi_t(\cdot,0),\eta_{1t}(\cdot,0),\eta_{2t}(\cdot,0)\}=\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}\quad \text{in } \Omega, \tag{1-3b}$$

where

$$\begin{aligned}
L_1(\phi_1,\phi_2,\psi)&=D\big(\phi_{1xx}+\tfrac{1}{2}(1-\mu)\phi_{1yy}+\tfrac{1}{2}(1+\mu)\phi_{2xy}\big)-k(\phi_1+\psi_x),\\
L_2(\phi_1,\phi_2,\psi)&=D\big(\phi_{2yy}+\tfrac{1}{2}(1-\mu)\phi_{2xx}+\tfrac{1}{2}(1+\mu)\phi_{1xy}\big)-k(\phi_2+\psi_y),\\
L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)&=k[(\psi_x+\phi_1)_x+(\psi_y+\phi_2)_y]+(N_1\psi_x+N_{12}\psi_y)_x+(N_2\psi_y+N_{12}\psi_x)_y,\\
L_4(\psi,\eta_1,\eta_2)Y&=N_{1x}+N_{12y},\\
L_5(\psi,\eta_1,\eta_2)&=N_{2y}+N_{12x},\\
\mathcal{B}_1(\phi_1,\phi_2)&=D\big[\nu_1\phi_{1x}+\mu\nu_1\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})\nu_2\big],\\
\mathcal{B}_2(\phi_1,\phi_2)&=D\big[\nu_2\phi_{2y}+\mu\nu_2\phi_{1x}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})\nu_1\big],\\
\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)&=k\left(\frac{\partial\psi}{\partial\nu}+\nu_1\phi_1+\nu_2\phi_2\right)+(\nu_1N_1+\nu_2N_{12})\psi_x+(\nu_2N_2+\nu_1N_{12})\psi_y,\\
\mathcal{B}_4(\psi,\eta_1,\eta_2)&=\nu_1N_1+\nu_2N_{12},\\
\mathcal{B}_5(\psi,\eta_1,\eta_2)&=\nu_2N_2+\nu_1N_{12},\\
N_1&=\frac{Eh}{1-\mu^2}\big(\eta_{1x}+\mu\eta_{2y}+\tfrac{1}{2}\psi_x^2+\tfrac{1}{2}\mu\psi_y^2\big),\\
N_2&=\frac{Eh}{1-\mu^2}\big(\eta_{2y}+\mu\eta_{1x}+\tfrac{1}{2}\psi_y^2+\tfrac{1}{2}\mu\psi_x^2\big),\\
N_{12}&=\frac{Eh}{2(1+\mu)}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y).
\end{aligned}$$

In system (1-1), subscripts mean partial derivatives. The vector $\nu=(\nu_1,\nu_2)$ represents the outward unit normal to Ω and $\frac{\partial}{\partial\nu}$ stands for the normal derivative. The unknowns are $\phi_1=\phi_1(x,y,t)$, $\phi_2=\phi_2(x,y,t)$, $\psi=\psi(x,y,t)$, $\eta_1=\eta_1(x,y,t)$, and $\eta_2=\eta_2(x,y,t)$. Physically, the functions ϕ_1 and ϕ_2 represent, respectively, the angles of rotation of the cross sections $x=\text{const.}$, $y=\text{const.}$ containing the filament which, when the plate is in equilibrium, is orthogonal to the middle surface at the point $(x,y,0)$. The function ψ is the vertical displacement, and η_1, η_2 are the in-plane displacement of the plate at time t of the cross section located at (x,y) units from the endpoint $(x,y)=(0,0)$. The positive constant h represents the thickness of the plate which, in this model, is considered to be small and uniform with respect to x. The constant ρ is the mass density per unit volume of the plate and the parameter k is the so-called modulus of elasticity in shear. The constant E is the Young's modulus and the constant μ, $0<\mu<\frac{1}{2}$, is the Poisson's ratio. The constant D is the modulus of flexural rigidity and is given by $D=Eh^3/(12(1-\mu^2))$. The constant k is given by the expression $k=\hat{k}Eh/(2(1+\mu))$, where $\hat{k}$ is a shear correction coefficient. For more details concerning the Mindlin–Timoshenko hypotheses and the governing equations see, for instance, [Lagnese and Lions 1988].

For the nonlinear system (1-1)–(1-3), Rahmani [2014] considered a plate reinforced by a thin stiffener on a portion of its boundary and modeled this junction through an approximate model where the stiffener has a role on its boundary conditions.

The linear version of system (1-1)–(1-3) is

$$\begin{cases} \frac{1}{12}\rho h^3\phi_{1tt} - L_1(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \\ \frac{1}{12}\rho h^3\phi_{2tt} - L_2(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \\ \rho h\psi_{tt} - \widetilde{L}_3(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \end{cases} \tag{1-4}$$

where L_1, L_2 are defined above and

$$\widetilde{L}_3(\phi_1, \phi_2, \psi) = k[(\psi_x + \phi_1)_x + (\psi_y + \phi_2)_y].$$

There are quite a few works on this system: Lagnese and Lions [1988] studied its well-posedness and analyzed its asymptotic limit when the parameter k tends to infinity. Lagnese [1989] studied problems of existence, uniqueness and some other important properties as the asymptotic behavior in time when some damping effects are considered. Chueshov and Lasiecka [2006] studied the dynamics for a class of Mindlin–Timoshenko plate models with nonlinear feedback forces and showed the existence of a compact global attractor for the system. Furthermore they studied its limiting properties when the shear modulus tends to infinity. Fernández Sare [2009] investigated system (1-4) with frictional dissipations acting on the equations for the rotation angles and proved that this system is not exponentially stable independent of any relations between the constants of the system. Moreover, he showed that the solution decays polynomially to zero, with rates that can be improved depending on the regularity of the initial data. Rahmani [2015] studied system (1-4) and obtained results similar to those in [Rahmani 2014] for the system (1-1)–(1-3).

If one assumes the filament of the plate to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and the resulting model is the so-called von Kármán system; see [Lagnese and Lions 1988]:

$$\begin{cases} \rho h\psi_{tt} - \frac{1}{12}\rho h^3\Delta\psi_{tt} + D\Delta^2\psi - (N_1\psi_x + N_{12}\psi_y)_x - (N_2\psi_y + N_{12}\psi_x)_y = 0 & \text{in } Q, \\ \rho h\eta_{1tt} - (N_{1x} + N_{12y}) = 0 & \text{in } Q, \\ \rho h\eta_{2tt} - (N_{2y} + N_{12x}) = 0 & \text{in } Q, \end{cases} \tag{1-5}$$

with boundary conditions

$$\begin{gathered} \psi = \frac{\partial\psi}{\partial\nu} = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0, \\ D[\Delta\psi + (1-\mu)(2\nu_1\nu_2\psi_{xy} - \nu_1^2\psi_{yy} - \nu_2^2\psi_{xx})] = 0 \quad \text{on } \Sigma_1, \\ D\left[\frac{\partial(\Delta\psi)}{\partial\nu} + (1-\mu)\frac{\partial}{\partial\tau}[(\nu_1^2 - \nu_2^2)\psi_{xy} + \nu_1\nu_2(\psi_{yy} - \psi_{xx})]\right] - \frac{1}{12}\rho h^3\frac{\partial\psi_{tt}}{\partial\nu} \\ -(\nu_1 N_1 + \nu_2 N_{12})\psi_x - (\nu_2 N_2 + \nu_1 N_{12})\psi_y = 0 \quad \text{on } \Sigma_1, \\ \nu_1 N_1 + \nu_2 N_{12} = 0 \quad \text{on } \Sigma_1, \\ \nu_2 N_2 + \nu_1 N_{12} = 0 \quad \text{on } \Sigma_1, \end{gathered} \tag{1-6}$$

and initial data

$$\{\psi(\cdot\,,0),\eta_1(\cdot\,,0),\eta_2(\cdot\,,0)\}=\{\psi_0,\eta_{10},\eta_{20}\}\quad\text{in }\Omega, \tag{1-7a}$$

$$\{\psi_t(\cdot\,,0),\eta_{1t}(\cdot\,,0),\eta_{2t}(\cdot\,,0)\}=\{\psi_1,\eta_{11},\eta_{21}\}\quad\text{in }\Omega. \tag{1-7b}$$

In (1-6), $\tau=(-\nu_2,\nu_1)$ is the tangent vector to Ω and $\partial/\partial\tau$ represents the tangential derivative. System (1-5)–(1-7) has been an object of study for many years. Let us mention some known results about this type of system. Lasiecka [1998] and Favini et al. [1996] studied well-posedness for this problem, as well as the regularity of its solution. Perla Menzala and Zuazua [1997] proved exponential decay rates for the energy of the system for a bounded smooth thermoelastic plate clamped on its boundary. A similar result was obtained by Kang [2013] for von Kármán equations with a memory term. Finally, for monotonic functions with certain growth properties at the origin and at infinity, Lagnese and Leuring [1991] showed that the one-dimensional von Kármán is uniformly asymptotically stable.

Neglecting the shear effects of the plate, obtaining system (1-5) is formally equivalent to considering the modulus of elasticity k tending to infinity in system (1-1), since k is inversely proportional to the shear angle. The present paper is devoted to analyzing the asymptotic limit of the nonlinear Mindlin–Timoshenko system (1-1) as $k\to\infty$. This problem was mentioned in [Lagnese and Lions 1988, p. 24], where it was conjectured that system (1-1) approaches, in some sense, the von Kármán system (1-5), as $k\to\infty$:

> One expects that, as $k\to\infty$, solutions of the system (1-1) will converge (in some sense) to solution of the von Kármán system (1-5). However, a rigorous proof of convergence is lacking and seems to be a difficult question.

In this direction, Lagnese and Lions [1988] proved (see also [Araruna and Zuazua 2008] for the one-dimensional case) that, in the linear case, the solution of the Mindlin–Timoshenko model (1-4) converges, as $k\to\infty$, towards to the solution of the Kirchhoff model (subject to appropriate boundary conditions)

$$\rho h\psi_{tt}-\tfrac{1}{12}\rho h^3\Delta\psi_{tt}+D\Delta^2\psi=0. \tag{1-8}$$

Later on, in [Araruna et al. 2010], the authors studied the one-dimensional nonlinear Mindlin–Timoshenko system with an extra fourth-order regularizing term

$$\begin{cases}\frac{1}{12}\rho h^3\phi_{tt}-D\phi_{xx}+k(\phi+\psi_x)=0 & \text{in } Q,\\ \rho h\psi_{tt}-k(\phi+\psi_x)_x-Eh\big[\psi_x\big(\eta_x+\frac12\psi_x^2\big)\big]_x+\frac1k\psi_{xxxx}=0 & \text{in } Q,\\ \rho h\eta_{tt}-Eh\big(\eta_x+\frac12\psi_x^2\big)_x=0 & \text{in } Q,\end{cases} \tag{1-9}$$

and showed that, as $k\to\infty$, the system (1-9) converges toward the one-dimensional von Kármám system

$$\begin{cases}\rho h\psi_{tt}-\frac{1}{12}\rho h^3\psi_{xxtt}+D\psi_{xxxx}-Eh\big[\psi_x\big(\eta_x+\frac12\psi_x^2\big)\big]_x=0 & \text{in } Q,\\ \rho h\eta_{tt}-Eh\big(\eta_x+\frac12\psi_x^2\big)_x=0 & \text{in } Q.\end{cases} \tag{1-10}$$

In the argument used in [Araruna et al. 2010], the use of the extra fourth-order regularizing term was indispensable, since it ensures the compactness of a family of solutions, as $k\to\infty$, allowing one to pass to the limit in the nonlinear term. Here, we study the nonlinear two-dimensional problem without any regularizing term. We prove that the Mindlin–Timoshenko system converges to the von Kármán one, therefore giving a positive answer for the 1988 Lagnese–Lions conjecture. We note that our argument here can be used for the one-dimensional case as well, ensuring the conjecture holds also in the one-dimensional case (as would be expected).

In the context of asymptotic limits, with respect to singular coefficients, Perla Menzala and Zuazua [Perla Menzala and Zuazua 1999] proved that the one-dimensional von Kármán system of equations approaches (weakly) to a nonlocal beam equation of Timoshenko type as a suitable parameter tends to zero. In [Perla Menzala and Zuazua 2000a], the authors considered a dynamical one-dimensional nonlinear von Kármán model depending on one parameter $\varepsilon > 0$ and studied its weak limit as $\varepsilon \to 0$. Furthermore, they proved that, depending on the type of boundary condition, the nonlinearity of the Timoshenko model may either vanish or may become a nonlinearity concentrated on the extremes of the beam. In [Perla Menzala and Zuazua 2000b], the full nonlinear dynamic von Kármán system of equations was considered and the authors showed how the so-called Timoshenko and Berger models for thin plates may be obtained as singular limits of the von Kármán system when a suitable parameter tends to zero. We also mention the work [Perla Menzala et al. 2002], where the authors obtained the stabilization of Berger–Timoshenko's equation as a limit of the uniform stabilization of the von Kármán system of beams and plates with respect to a singular parameter.

The second part of this work concerns stabilization. To our knowledge, exponential stability has not been investigated for the two-dimension nonlinear Mindlin–Timoshenko system, so we study decay properties of its solutions with both internal and boundary damping. More precisely, we show the following: adding appropriate damping terms, there is a uniform (with respect to k) rate of decay for the total energy of the solutions for (1-1) as $t \to \infty$. As a consequence of this analysis, we obtain a decay rate for the total energy of the solutions for the von Kármán system (as $t \to \infty$) as a singular limit of the uniform (with respect to k) decay rate of the energy of the Mindlin–Timoshenko system.

Let us mention some known results related to the stabilization. In the one-dimensional case, Araruna et al. [2010] showed the exponential stability of the nonlinear Mindlin–Timoshenko beam under internal damping. Stabilization results for the linear model were obtained in [Lagnese 1989; Kim and Renardy 1987] considering damping in both equations, and in [Alabau-Boussouira 2007] with a single nonlinear feedback control. In [Ammar-Khodja et al. 2003], the system is damped by a memory-type term. In the two-dimensional case, the uniform stabilization for linear Mindlin–Timoshenko model was studied in [Fernández Sare 2009] considering frictional dissipations acting on the equations for the rotations angle. Grobbelaar-Van Dalsen [2015] studied the polynomial decay rate of the Mindlin–Timoshenko plate model with thermal dissipation. Stabilization results were obtained in [Nicaise 2011] for the multidimensional case with nonconstant and nonsmooth coefficients, when the interior dissipation acts either on both equations or only on the elasticity equation. The stabilization of the von Kármán system, in the two-dimensional case, was studied in [Perla Menzala and Zuazua 1997], where the energy decreases along trajectories. Bradley and Lasiecka [1992] studied the local exponential stabilization for an unstructured perturbation and feedback controls. Kang [2013] proved the exponential decay for the nonlinear von Kármán system with memory.

This work is organized as follows. In Section 2, we rigorously study the behavior of the Mindlin–Timoshenko system towards the von Kármán system as $k \to \infty$. More precisely, we prove that solutions

$\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ of (1-1)–(1-3) converge to $\{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\}$ as $k \to \infty$, where $\{\psi, \eta_1, \eta_2\}$ solves system (1-5)–(1-7). In Sections 3 and 4 we prove that, adding appropriate damping terms (internal and boundary, respectively), one can prove a uniform (in k) exponential decay property for the solutions of (1-1)–(1-3). Finally, in Section 5, we briefly discuss some related issues and open problems.

2. Asymptotic limit

In this section, we study the asymptotic limit of the solutions for the nonlinear Mindlin–Timoshenko system (1-1)–(1-3) as $k \to \infty$. To study this problem, we consider the Hilbert space

$$\mathcal{X} = [H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)]^2 \times [W^{1,4}(\Omega) \cap H^1_{\Gamma_0}(\Omega)] \times L^2(\Omega) \times [H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)]^2, \tag{2-1}$$

where $H^1_{\Gamma_0}(\Omega) = \{\varphi : \varphi \in H^1(\Omega),\ \varphi = 0 \text{ on } \Gamma_0\}$.

The energy $E_k(t)$ of solutions is given by

$$E_k(t) = \tfrac{1}{2}\Big\{ \tfrac{1}{12}\rho h^3\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]+\rho h\big[|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\big]+k\big[|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2\big] \\ +F([b_{ij}],[b_{ij}])+D\Big[|\phi_{1x}|^2+|\phi_{2y}|^2+\tfrac{1}{2}(1-\mu)|\phi_{1y}+\phi_{2x}|^2+2\mu\int_\Omega(\phi_{1y}\phi_{2x})\,dx\,dy\Big]\Big\}, \tag{2-2}$$

where

$$b_{11} = \eta_{1x} + \tfrac{1}{2}\psi_x^2, \quad b_{22} = \eta_{2y} + \tfrac{1}{2}\psi_y^2, \quad b_{12} = b_{21} = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

and

$$F([b_{ij}]) = \frac{Eh}{1-\mu^2}\left\{\mu\begin{bmatrix} b_{11}+b_{22} & 0 \\ 0 & b_{11}+b_{22}\end{bmatrix} + c(1-\mu)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right\}.$$

Note that

$$(F([b_{ij}]),[b_{ij}])_{(L^2(\Omega))^4} \\ = \left(\frac{Eh}{1-\mu^2}\left\{\mu\begin{bmatrix} b_{11}+b_{22} & 0 \\ 0 & b_{11}+b_{22}\end{bmatrix} + c(1-\mu)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right\}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right) \\ = \frac{Eh}{1-\mu^2}\big\{\mu\big|\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big|^2+(1-\mu)|b_{11}|^2+(1-\mu)|b_{22}|^2+\tfrac{1}{2}(1-\mu)|\eta_{1y}+\eta_{2x}+\psi_x\psi_y|^2\big\} > 0$$

since $Eh/(1-\mu^2) > 0$ and $0 < \mu < 1$, which shows that F is positive definite. Moreover, we have by [Lagnese 1989, Lemma 2.1] that

$$D\Big[|\phi_{1x}|^2 + |\phi_{2y}|^2 + \tfrac{1}{2}(1-\mu)|\phi_{1y}+\phi_{2x}|^2 + 2\mu\int_\Omega(\phi_{1y}\phi_{2x}\,dx\,dy)\Big] \geq C\|\phi_1\|^2_{H^1(\Omega)} + \|\phi_2\|^2_{H^1(\Omega)}.$$

So, the energy is positive. Furthermore,

$$E_k(t) = E_k(0) \quad \forall\, t \geq 0. \tag{2-3}$$

The main result of this paper is to give a positive response to a conjecture from [Lagnese and Lions 1988]. Our result is as follows.

Theorem 2.1. *Let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be a solution of the system (1-1)–(1-3) with initial data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$ satisfying*

$$\phi_{10} + \psi_{0x} = 0 \quad \textit{and} \quad \phi_{20} + \psi_{0y} = 0 \quad \textit{in } \Omega. \tag{2-4}$$

Then, letting $k \to \infty$, one gets

$$\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\} \to \{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\} \quad \textit{weak} * \textit{ in } L^\infty(0, T, [H^1_{\Gamma_0}(\Omega)]^3 \times [L^2(\Omega)]^2),$$

where $\{\psi, \eta_1, \eta_2\}$ solves (1-5)–(1-7).

Remark 2.2. The variational formulation of system (1-5)–(1-7) is given by

$$\rho h \frac{d}{dt}(\psi_t, c) + \tfrac{1}{12}\rho h^3 \frac{d}{dt}(\nabla\psi_t, \nabla c) + \rho h \frac{d}{dt}(\eta_{1t}, d) + \rho h \frac{d}{dt}(\eta_{2t}, e) + (N_1\psi_x + N_{12}\psi_y, c_x)$$
$$+ (N_2\psi_y + N_{12}\psi_x, c_y) + (N_1^k, d_x) + (N_{12}^k, d_y) + (N_2^k, e_y) + (N_{12}^k, e_x) + D(\Delta\psi, \Delta c) = 0, \tag{2-5}$$

for all $\{c, d, e\} \in [H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)] \times [H^1_{\Gamma_0}(\Omega)]^2$ and the initial conditions (1-6). In equation (2-5), $(\cdot\,,\cdot)$ represents the inner product in $L^2(\Omega)$. Furthermore, the system (1-5)–(1-7) is conservative; that is, its energy

$$E(t) = \tfrac{1}{2}\bigg\{\rho h\big[|\psi_t|^2 + |\eta_{1t}|^2 + |\eta_{2t}|^2\big] + \tfrac{1}{12}\rho h^3 |\nabla\psi_t|^2 + D|\Delta\psi|^2$$
$$+ \frac{Eh}{1-\mu}\int_\Omega \big[\eta_{1x} + \tfrac{1}{2}\psi_x^2\big]^2 + \big[\eta_{2y} + \tfrac{1}{2}\psi_y^2\big]^2$$
$$+ \big[\eta_{1x} + \eta_{2y} + \tfrac{1}{2}|\nabla\psi|^2\big]^2 + \tfrac{1}{2}(1-\mu)\big[\eta_{1y} + \eta_{2x} + \psi_x\psi_y\big]^2\,dx\,dy\bigg\} \tag{2-6}$$

satisfies $E(t) = E(0)$ for all $t \in [0, T]$.

Proof of Theorem 2.1. For each $k > 0$ fixed, let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be the solution of system (1-1)–(1-3) with data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. Since the initial data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ satisfy the condition (2-4), one has, due to the conservation of energy (2-3),

$$E_k(t) \le C \quad \forall k > 0,\ \forall t > 0. \tag{2-7}$$

From now on, the letter C stands for a generic positive constant which may vary from line to line (unless otherwise stated). The estimate (2-7) implies that the sequences (in k)

$$(\phi_{1t}^k), \quad (\phi_{2t}^k), \quad (\psi_t^k), \quad (\eta_{1t}^k), \quad (\eta_{2t}^k), \quad \sqrt{k}(\phi_1^k + \psi_x^k), \quad \sqrt{k}(\phi_2^k + \psi_y^k), \quad (\phi_{1x}^k), \quad (\phi_{2y}^k),$$
$$(\phi_{1y}^k + \phi_{2x}^k), \quad \big(\eta_{1x}^k + \tfrac{1}{2}[\psi_x^k]^2\big), \quad \big(\eta_{2y}^k + \tfrac{1}{2}[\psi_y^k]^2\big), \quad \big(\eta_{1x}^k + \eta_{2y}^k + \tfrac{1}{2}[\nabla\psi^k]^2\big), \quad \big(\eta_{1y}^k + \eta_{2x}^k + \psi_x^k\psi_y^k\big)$$

are bounded in $L^\infty(0, T, L^2(\Omega))$. Furthermore,

$$[\phi_{1y}^k]_x = [\phi_{1x}^k]_y \in H^{-1}(\Omega) \quad \text{and} \quad [\phi_{2x}^k]_y = [\phi_{2y}^k]_x \in H^{-1}(\Omega),$$

since (ϕ_{1x}^k) and (ϕ_{2y}^k) are bounded in $L^\infty(0, T, L^2(\Omega))$. On the other hand,

$$[\phi_{1y}^k]_y = [\phi_{1y}^k + \phi_{2x}^k]_y - [\phi_{2x}^k]_y = [\phi_{1y}^k + \phi_{2x}^k]_y - [\phi_{2y}^k]_x \in H^{-1}(\Omega),$$

which implies that (ϕ^k_{1y}) is bounded in $L^\infty(0, T, L^2(\Omega))$. Similarly, one can show that (ϕ^k_{2x}) is bounded in $L^\infty(0, T, L^2(\Omega))$. Thus, the sequences (in k) (ϕ^k_1), (ϕ^k_2) and (ψ^k) are bounded in $L^\infty(0, T, H^1_{\Gamma_0}(0, L))$.

Since, for each k, we have η^k_{1t} and η^k_{2t} belong to $C^0([0, T], L^2(\Omega))$, we can write

$$\eta^k_1(t) = \eta_{10} + \int_0^t \eta^k_{1t}(s)\,ds \quad \text{and} \quad \eta^k_2(t) = \eta_{20} + \int_0^t \eta^k_{2t}(s)\,ds.$$

Therefore, since (η^k_{1t}) is bounded in $L^\infty(0, T, L^2(\Omega))$, the sequence (η^k_1) is bounded $L^\infty(0, T, L^2(\Omega))$. Indeed,

$$|\eta^k_1| = \left|\eta_{10} + \int_0^t \eta^k_{1t}\,ds\right| \le C + \int_0^t |\eta^k_{1t}|\,ds \le C.$$

Analogously, it follows that (η^k_2) is bounded in $L^\infty(0, T, L^2(\Omega))$. Therefore, the sequences (η^k_1), (η^k_2) are bounded in $L^\infty(0, T, L^2(\Omega))$. Extracting subsequences, without changing notation, one gets

$$\{\phi^k_1, \phi^k_2, \psi^k, \eta^k_1, \eta^k_2\} \to \{\phi_1, \phi_2, \psi, \eta_1, \eta_2\} \quad \text{weak} * \text{ in } L^\infty(0, T; [H^1_{\Gamma_0}(\Omega)]^3 \times [L^2(\Omega)]^2), \tag{2-8}$$

with

$$\phi_1 + \psi_x = 0 \quad \text{and} \quad \phi_2 + \psi_y = 0, \tag{2-9}$$

$$\{\phi^k_{1t}, \phi^k_{2t}, \psi^k_t, \eta^k_{1t}, \eta^k_{2t}\} \to \{\phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}\} \quad \text{weak} * \quad \text{in} \quad L^\infty(0, T; [L^2(\Omega)]^5), \tag{2-10}$$

$$\eta^k_{1x} + \tfrac{1}{2}[\psi^k_x]^2 \to \alpha \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)), \tag{2-11}$$

$$\eta^k_{2y} + \tfrac{1}{2}[\psi^k_y]^2 \to \beta \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)), \tag{2-12}$$

$$\eta^k_{1y} + \eta^k_{2x} + \psi^k_x\psi^k_y \to \gamma \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)). \tag{2-13}$$

Now, using a compactness theorem due to Aubin and Lions see [Simon 1987, Corollary 4], we obtain

$$\phi^k_1 \to \phi_1 \quad \text{strongly in } L^2(Q), \tag{2-14}$$

$$\phi^k_2 \to \phi_2 \quad \text{strongly in } L^2(Q). \tag{2-15}$$

Therefore, given $\varepsilon > 0$, for large enough k one has

$$|\psi^k_x + \phi_1| \le |\psi^k_x + \phi^k_1| + |\phi^k_1 - \phi_1| \le \frac{C}{\sqrt{k}} + \varepsilon.$$

Consequently,

$$\psi^k_x \to -\phi_1 \quad \text{in } L^2(Q). \tag{2-16}$$

On the other hand, we also have by the convergence (2-8) that

$$\psi^k_x \to \psi_x \quad \text{in } \mathcal{D}'(Q). \tag{2-17}$$

Combining (2-16) and (2-17), we obtain

$$\psi_x = -\phi_1.$$

In a similar way, we get

$$\psi_y = -\phi_2.$$

Therefore,

$$\psi^k \to \psi \quad \text{strongly in } L^\infty(0,T,H^1_{\Gamma_0}(\Omega)). \tag{2-18}$$

By the previous convergence we conclude that

$$[\psi^k_x]^2 \to [\psi_x]^2 \quad \text{strongly in } L^\infty(0,T,L^1(\Omega)), \tag{2-19}$$

$$[\psi^k_y]^2 \to [\psi_y]^2 \quad \text{strongly in } L^\infty(0,T,L^1(\Omega)). \tag{2-20}$$

On other hand, the sequences (η^k_1), (η^k_2) are bounded in $L^\infty(0,T,L^2(\Omega))$ and so

$$\eta^k_{1x} \to \eta_{1x} \quad \text{weak} * \text{ in } L^\infty(0,T,H^{-1}(\Omega)), \tag{2-21}$$

$$\eta^k_{2y} \to \eta_{2y} \quad \text{weak} * \text{ in } L^\infty(0,T,H^{-1}(\Omega)). \tag{2-22}$$

The same holds for (η^k_{1y}) and (η^k_{2x}). Combining the convergences (2-18)–(2-22), it follows that

$$\alpha = \eta_{1x} + \tfrac{1}{2}\psi_x^2, \quad \beta = \eta_{2y} + \tfrac{1}{2}\psi_y^2, \quad \gamma = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

$$N^k_1\psi^k_x + N^k_{12}\psi^k_y \to N_1\psi_x + N_{12}\psi_y \quad \text{weak} * \text{ in } L^\infty(0,T,L^2(\Omega)), \tag{2-23}$$

$$N^k_2\psi^k_y + N^k_{12}\psi^k_x \to N_2\psi_y + N_{12}\psi_x \quad \text{weak} * \text{ in } L^\infty(0,T,L^2(\Omega)). \tag{2-24}$$

For $\{a,b,c,d,e\} \in [H^1_{\Gamma_0}(\Omega)]^5$ satisfying

$$a + c_x = 0 \quad \text{and} \quad b + c_y = 0, \tag{2-25}$$

the variational formulation of problem (1-1)–(1-3) is

$$\begin{aligned}
&\tfrac{1}{12}\rho h^3 \frac{d}{dt}(\phi^k_{1t},a) + \tfrac{1}{12}\rho h^3 \frac{d}{dt}(\phi^k_{2t},b) + \rho h \frac{d}{dt}(\psi^k_t,c) + \rho h \frac{d}{dt}(\eta^k_{1t},d) + \rho h \frac{d}{dt}(\eta^k_{2t},e)\\
&+ D\big[(\phi^k_{1x},a_x) + \tfrac{1}{2}(1-\mu)(\phi^k_{1y},a_y) + \tfrac{1}{2}(1+\mu)(\phi^k_{2x},a_y) + (\phi^k_{2y},b_y) + \tfrac{1}{2}(1-\mu)(\phi^k_{2x},b_x) + \tfrac{1}{2}(1+\mu)(\phi^k_{1y},b_x)\big]\\
&+ (N^k_1\psi^k_x + N^k_{12}\psi^k_y, c_x) + (N^k_1,d_x) + (N^k_{12},d_y) + (N^k_2\psi^k_y + N^k_{12}\psi^k_x, c_y) + (N^k_2,e_y) + (N^k_{12},e_x) = 0.
\end{aligned} \tag{2-26}$$

Using convergences (2-8), (2-10)–(2-13), (2-23) and (2-24) in equation (2-26), and applying identities (2-9) and (2-25), one obtains the weak formulation of the system (1-5)–(1-7) given in (2-5). To finish the proof, it remains to identify the initial data of the limit system. In view of the convergences (2-8), (2-10), and classical compactness arguments, one has $\{\psi^k, \eta^k_1, \eta^k_2\} \to \{\psi, \eta_1, \eta_2\}$ in $C^0([0,T];[L^2(\Omega)]^3)$. Then,

$$\{\psi^k(\cdot,0), \eta^k_1(\cdot,0), \eta^k_2(\cdot,0)\} \to \{\psi(\cdot,0), \eta_1(\cdot,0), \eta_2(\cdot,0)\} \quad \text{in } [L^2(\Omega)]^3,$$

which combined with (1-3a) guarantees that $\{\psi(\cdot,0), \eta_1(\cdot,0), \eta_2(\cdot,0)\} = \{\psi_0, \eta_{10}, \eta_{20}\}$. In order to identify $\{\psi_t(\cdot,0), \eta_{1t}(\cdot,0), \eta_{2t}(\cdot,0)\}$, multiply both sides of (2-26) by the function $\theta_\delta \in H^1(0,T)$ defined by

$$\theta_\delta(t) = \begin{cases} -t/\delta + 1 & \text{if } t \in [0,\delta], \\ 0 & \text{if } t \in (\delta, T], \end{cases}$$

and integrate by parts to obtain

$$\begin{aligned}
&-\frac{\rho h^3}{12}(\phi_{11},a)+\frac{\rho h^3}{12\delta}\int_0^\delta(\phi_{1t}^k,a)\,dt-\frac{\rho h^3}{12}(\phi_{21},b)+\frac{\rho h^3}{12\delta}\int_0^\delta(\phi_{2t}^k,b)\,dt-\rho h(\psi_1,c)\\
&+\frac{\rho h}{\delta}\int_0^\delta(\psi_t^k,c)\,dt-\rho h(\eta_{11},d)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{1t}^k,d)\,dt-\rho h(\eta_{21},e)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{2t}^k,e)\,dt\\
&+\int_0^T D\Big[(\phi_{1x}^k,a_x)+\frac{1-\mu}{2}(\phi_{1y}^k,a_y)+\frac{1+\mu}{2}(\phi_{2x}^k,a_y)+(\phi_{2y}^k,b_y)+\frac{1-\mu}{2}(\phi_{2x}^k,b_x)+\frac{1+\mu}{2}(\phi_{1y}^k,b_x)\Big]\theta_\delta\,dt\\
&+\int_0^t(N_1^k\psi_x^k+N_{12}^k\psi_y^k,c_x)\theta_\delta\,dt+\int_0^T(N_2^k\psi_y^k+N_{12}^k\psi_x^k,c_y)\theta_\delta\,dt-\int_0^T(N_{1x}^k+N_{12y}^k,d)\theta_\delta\,dt\\
&-\int_0^T(N_{2y}^k+N_{12x}^k,e)\theta_\delta\,dt=0. \qquad (2\text{-}27)
\end{aligned}$$

Passing to the limit as $k\to\infty$ in the last equation, and using (2-8), (2-10)–(2-23), one obtains

$$\begin{aligned}
&-\frac{\rho h^3}{12}(\phi_{11},c_x)+\frac{\rho h^3}{12\delta}\int_0^\delta(\psi_{xt},c_x)\,dt+\frac{\rho h^3}{12}(\phi_{21},c_y)+\frac{\rho h^3}{12\delta}\int_0^\delta(\psi_{yt},c_y)\,dt\\
&-\rho h(\psi_1,c)+\frac{\rho h}{\delta}\int_0^\delta(\psi_t,c)\,dt-\rho h(\eta_{11},d)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{1t},d)\,dt\\
&-\rho h(\eta_{21},e)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{2t},e)\,dt+D\int_0^T(\Delta\psi,\,\Delta c)\theta_\delta\,dt+\int_0^t(N_1\psi_x+N_{12}\psi_y,c_x)\theta_\delta\,dt\\
&+\int_0^T(N_2\psi_y+N_{12}\psi_x,c_y)\theta_\delta\,dt-\int_0^T(N_{1x}+N_{12y},d)\theta_\delta\,dt-\int_0^T(N_{2y}+N_{12x},e)\theta_\delta\,dt=0.
\end{aligned}$$

On the other hand, multiplying (2-5) by θ_δ and integrating in time, we get the identity

$$\begin{aligned}
&-\tfrac{1}{12}\rho h^3(\Delta\psi_t(\cdot\,,0),c)-\rho h(\psi_t(\cdot\,,0),c)-\rho h(\eta_{1t}(\cdot\,,0),d)-\rho h(\eta_{2t},e)\\
&\qquad=-\tfrac{1}{12}\rho h^3(\phi_{11x}+\phi_{21y},c)-\rho h(\psi_1,c)-\rho h(\eta_{11},d)-\rho h(\eta_{21},e). \qquad (2\text{-}28)
\end{aligned}$$

Therefore, $\big(-\tfrac{1}{12}h^2\Delta\psi+\psi\big)_t(\cdot\,,0)=\psi_1+\tfrac{1}{12}h^2(\phi_{11x}+\phi_{21y})$, $\eta_{1t}(\cdot\,,0)=\eta_{11}$, and $\eta_{2t}(\cdot\,,0)=\eta_{21}$. $\square$

Remark 2.3. In order to fully identify the initial data of the solutions of the limit system (1-5)–(1-7) and, more precisely, to determine the initial data of ψ_t, an elliptic equation has to be solved. Namely, the initial datum for the velocity ψ_t in (1-7b) is determined by solving the elliptic equation

$$\psi_t(\cdot\,,0)\in H^1_{\Gamma_0}(\Omega):\quad \big(-\tfrac{1}{12}h^2\Delta\psi+\psi\big)_t(\cdot\,,0)=\psi_1+\tfrac{1}{12}h^2(\phi_{11x}+\phi_{21y})\quad\text{in }\Omega,$$

as the proof of the theorem showed. More precisely, this elliptic equation can be written in the variational form

$$\tfrac{1}{12}h^2(\nabla\psi_t(\cdot\,,0),\nabla c)+(\psi_t(\cdot\,,0),c)=(\psi_1,c)-\tfrac{1}{12}h^2(\phi_{11},c_x)-\tfrac{1}{12}h^2(\phi_{21},c_y),$$

where the terms ϕ_{11x} and ϕ_{21y} are not derived from ϕ_1 and ϕ_2, respectively, in the sense of transposition, but they are rather the linear mappings which, when acting on any element c of $H^1_{\Gamma_0}(\Omega)$, produce $-(\phi_{11},c_x)$ and $-(\phi_{21},c_y)$. The same can be said about $\Delta\psi_t(\cdot\,,0)$, yielding $-(\nabla\psi_t(\cdot\,,0),\nabla c)$.

3. Stability: internal feedback

In this section we analyze the plate model with hinged boundary conditions and in the presence of internal damping distributed all along the plate. Consider the system

$$\begin{cases} \frac{1}{12}\rho h^3\phi_{1tt} - L_1(\phi_1,\phi_2,\psi) + \phi_{1t} = 0 & \text{in } Q, \\ \frac{1}{12}\rho h^3\phi_{2tt} - L_2(\phi_1,\phi_2,\psi) + \phi_{2t} = 0 & \text{in } Q, \\ \rho h\psi_{tt} - L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \psi_t = 0 & \text{in } Q, \\ \rho h\eta_{1tt} - L_4(\psi,\eta_1,\eta_2) + \eta_{1t} = 0 & \text{in } Q, \\ \rho h\eta_{2tt} - L_5(\psi,\eta_1,\eta_2) + \eta_{2t} = 0 & \text{in } Q, \end{cases} \tag{3-1}$$

under boundary conditions (1-2) and initial data (1-3). The energy of solutions for (3-1), (1-2), (1-3) decreases in time. Indeed, the energy given by (2-2) obeys the energy dissipation law

$$\frac{d}{dt}E_k(t) = -\big(|\phi_{1t}(t)|^2 + |\phi_{2t}(t)|^2 + |\psi_t(t)|^2 + |\eta_{1t}(t)|^2 + |\eta_{2t}(t)|^2\big). \tag{3-2}$$

The aim of this section is to obtain exponential decay for the energy (2-6) associated to the solution of the von Kármán system

$$\begin{cases} \rho h\psi_{tt} - \frac{1}{12}\rho h^3\Delta\psi_{tt} + D\Delta^2\psi - [N_1\psi_x + N_{12}\psi_y]_x - [N_2\psi_y + N_{12}\psi_x]_y + \psi_t - \Delta\psi_t = 0 & \text{in } Q, \\ \rho h\eta_{1tt} - [N_{1x} + N_{12y}] + \eta_{1t} = 0 & \text{in } Q, \\ \rho h\eta_{2tt} - [N_{2y} + N_{12x}] + \eta_{2t} = 0 & \text{in } Q, \end{cases} \tag{3-3}$$

with boundary conditions (1-6) and initial data (1-7), as a limit (as $k \to \infty$) of the uniform stabilization of the dissipative Mindlin–Timoshenko system (3-1), (1-2), (1-3).

Analogously to the proof of Theorem 2.1, considering the initial data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$ satisfying (2-4), system (3-3) can be obtained as a limit, as $k \to \infty$, of system (3-1), (1-2), (1-3).

Since the energy $E_k(t)$ in (2-2) is a nonincreasing function, we will show that this energy decays exponentially (as $t \to \infty$) uniformly with respect to k. More precisely, the following result holds:

Theorem 3.1. *Let $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ be the solution of system (3-1), (1-2), (1-3) for data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. There exists a constant $\omega > 0$ such that*

$$E_k(t) \le 4E_k(0)e^{-\omega t/2} \quad \forall\, t \ge 0. \tag{3-4}$$

Remark 3.2. As a consequence of inequality (3-4), if the initial data satisfy (2-4), letting $k \to \infty$ one recovers the exponential decay of the energy $E(t)$ associated to system (3-3), which is given by (2-6). This is in agreement with the results from [Perla Menzala et al. 2002] in the sense that the same decay rate for the solutions of the von Kármám system was obtained.

Proof of Theorem 3.1. For each $k \ge 1$ fixed, let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be the solution of system (3-1), (1-2), (1-3) with data $\{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\} \in \mathcal{X}$. From now on in this proof, we will omit the index k of the solution to simplify the notation. For an arbitrary $\lambda > 0$, define the perturbed energy

$$G_\lambda(t) := E_k(t) + \lambda F(t), \tag{3-5}$$

where F is the functional

$$F(t)=\theta\big(\tfrac{1}{12}\rho h^3\phi_{1t},\phi_1\big)+\theta\big(\tfrac{1}{12}\rho h^3\phi_{2t},\phi_2\big)+\theta(\rho h\psi_t,\psi)+2\theta(\rho h\eta_{1t},\eta_1)+2\theta(\rho h\eta_{2t},\eta_2), \tag{3-6}$$

where $\theta>0$ is a constant to be chosen later on. Let us bound each term on the right-hand side of identity (3-6) by an expression involving the energy (2-2).

- Analysis of $\theta\big(\tfrac{1}{12}\rho h^3\phi_{1t},\phi_1\big)+\theta\big(\tfrac{1}{12}\rho h^3\phi_{2t},\phi_2\big)$: using the Poincaré inequality, one obtains

$$\begin{aligned}\theta\big(\tfrac{1}{12}\rho h^3\phi_{1t},\phi_1\big)&+\theta\big(\tfrac{1}{12}\rho h^3\phi_{2t},\phi_2\big)\\ &\le C\theta\bigg(\tfrac{1}{12}\rho h^3|\phi_{1t}|^2+\tfrac{1}{12}\rho h^3|\phi_{2t}|^2+|\phi_{1x}|^2+|\phi_{2y}|^2+|\phi_{1y}+\phi_{2x}|^2-2\int_\Omega\phi_{1x}\phi_{2y}\,dx\,dy\bigg)\\ &\le C\theta E_k(t).\end{aligned} \tag{3-7}$$

- Analysis of $\theta(\rho h\psi_t(t),\psi(t))$: using the Poincaré inequality again, one gets

$$\begin{aligned}\theta(\rho h\psi_t,\psi)&\le C\theta\big(\rho h|\psi_t|^2+|\psi_x|^2+|\psi_y|^2\big)\\ &\le C\theta\big(\rho h|\psi_t|^2+|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2+|\phi_1|^2+|\phi_2|^2\big)\\ &\le C\theta\big(\rho h|\psi_t|^2+|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2+|\phi_{1x}|^2+|\phi_{1y}|^2+|\phi_{2x}|^2+|\phi_{2y}|^2\big)\\ &\le C\theta E_k(t).\end{aligned} \tag{3-8}$$

- Analysis of $2\theta(\rho h\eta_{1t},\eta_1)+2\theta(\rho h\eta_{2t},\eta_2)$: one has

$$\begin{aligned}&2\theta(\rho h\eta_{1t},\eta_1)+2\theta(\rho h\eta_{2t},\eta_2)\\ &\le C\theta\big(\rho h|\eta_{1t}|^2+|\eta_{1x}|^2+|\eta_{1y}|^2+\rho h|\eta_{2t}|^2+|\eta_{2x}|^2+|\eta_{2y}|^2\big)\\ &\le C\theta\big(\rho h|\eta_{1t}|^2+\rho h|\eta_{2t}|^2+\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}|^2+|\eta_{2x}|^2+\tfrac12|\psi_x^2|^2+\tfrac12|\psi_y^2|^2\big)\\ &\le C\theta\bigg(\rho h|\eta_{1t}|^2+\rho h|\eta_{2t}|^2+\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}\,dx\,dy+|\nabla\psi|^2\bigg)\\ &\le C\theta E_k(t).\end{aligned} \tag{3-9}$$

According to the bounds (3-7)–(3-9), we conclude that

$$|F(t)|\le CE_k(t). \tag{3-10}$$

Now, using (3-5) and (3-10), one obtains

$$|G_\lambda(t)-E_k(t)|\le\lambda|F(t)|\le\lambda CE_k(t),$$

which is equivalent to

$$(1-\lambda C)E_k(t)\le G_\lambda(t)\le(1+\lambda C)E_k(t).$$

Taking $0<\lambda\le 1/(2C)$, one gets

$$\tfrac12 E_k(t)\le G_\lambda(t)\le 2E_k(t). \tag{3-11}$$

Differentiating the functional F and using the equations in (3-1), one obtains

$$\begin{aligned}\frac{d}{dt}F(t) = &-\theta D|\phi_{1x}|^2 - \tfrac{1}{2}(1-\mu)\theta D|\phi_{1y}|^2 - \theta D\tfrac{1}{2}(1+\mu)\int_\Omega \phi_{2x}\phi_{1y}\,dx\,dy - \theta k|\phi_1|^2 - \theta k\int_\Omega \psi_x\phi_1\,dx\,dy\\ &-\theta\int_\Omega \phi_{1t}\phi_1\,dx\,dy + \theta\tfrac{1}{12}\rho h^3|\phi_{1t}|^2 - \theta D|\phi_{2y}|^2 - \theta D\tfrac{1}{2}(1-\mu)|\phi_{2x}|^2 - \theta D\tfrac{1}{2}(1+\mu)\int_\Omega \phi_{1y}\phi_{2x}\,dx\,dy\\ &-\theta k|\phi_2|^2 - \theta k\int_\Omega \psi_y\phi_2\,dx\,dy + \theta\tfrac{1}{12}\rho h^3|\phi_{2t}|^2 - \theta\int_\Omega \phi_{2t}\phi_2\,dx\,dy - \theta k|\psi_x|^2\\ &-\theta k\int_\Omega \phi_1\psi_x\,dx\,dy - \theta k|\psi_y|^2 - \theta k\int_\Omega \phi_2\psi_y\,dx\,dy - \theta\int_\Omega [N_1\psi_x + N_{12}\psi_y]\psi_x\,dx\,dy\\ &-\theta\int_\Omega [N_2\psi_y + N_{12}\psi_x]\psi_y\,dx\,dy + \theta\rho h|\psi_t|^2 - \theta\int_\Omega \psi_t\psi\,dx\,dy - 2\theta\int_\Omega N_1\eta_{1x}\,dx\,dy\\ &-2\theta\int_\Omega N_{12}\eta_{1y}\,dx\,dy + 2\theta\rho h|\eta_{1t}|^2 - 2\theta\int_\Omega N_2\eta_{2y}\,dx\,dy - 2\theta\int_\Omega N_{12}\eta_{2x}\,dx\,dy\\ &+2\theta\rho h|\eta_{2t}|^2 - 2\theta\int_\Omega \eta_{1t}\eta_1\,dx\,dy - 2\theta\int_\Omega \eta_{2t}\eta_2\,dx\,dy. \end{aligned} \tag{3-12}$$

We bound each term on the right-hand side of identity (3-12) separately.

- Analysis of $-\theta(\phi_{1t},\phi_1) - \theta(\phi_{2t},\phi_2)$:

$$\begin{aligned}-\theta(\phi_{1t},\phi_1) - \theta(\phi_{2t},\phi_2) &\le \frac{\theta^2}{2\xi}|\phi_{1t}|^2 + \frac{\xi}{2}|\phi_1|^2 + \frac{\theta^2}{2\xi}|\phi_{2t}|^2 + \frac{\xi}{2}|\phi_2|^2\\ &\le \frac{\theta^2}{2\xi}[|\phi_{1t}|^2 + |\phi_{2t}|^2] + \frac{\xi C}{2}(|\phi_{1x}|^2 + |\phi_{1y}|^2 + |\phi_{2x}|^2 + |\phi_{2y}|^2)\\ &= \frac{\theta^2}{2\xi}[|\phi_{1t}|^2 + |\phi_{2t}|^2] + \frac{\xi C}{2}\left(|\phi_{1x}|^2 + |\phi_{2y}|^2 + |\phi_{1y} + \phi_{2x}|^2 - 2\int_\Omega \phi_{1y}\phi_{2x}\,dx\,dy\right)\\ &\le \frac{\theta^2}{2\xi}[|\phi_{1t}|^2 + |\phi_{2t}|^2] + \xi C E_k(t), \end{aligned} \tag{3-13}$$

where $\xi > 0$ is a real number to be appropriately chosen.

- Analysis of $-\theta(\psi_t(t),\psi(t))$:

$$\begin{aligned}-\theta(\psi_t,\psi) &\le \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi}{2}|\psi|^2\\ &\le \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi C}{2}(|\psi_x|^2 + |\psi_y|^2)\\ &\le \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi C}{2}(|\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2 + |\phi_1|^2 + |\phi_2|^2)\\ &\le \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi C}{2}(|\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2 + |\phi_{1x}|^2 + |\phi_{1y}|^2 + |\phi_{2x}|^2 + |\phi_{2y}|^2)\\ &\le \frac{\theta^2}{2\xi}|\psi_t|^2 + \xi C E_k(t). \end{aligned} \tag{3-14}$$

- Analysis of $-2\theta(\eta_{1t},\eta_1)-2\theta(\eta_{1t},\eta_1)$:

$$
\begin{aligned}
-2\theta(\eta_{1t},\eta_1)&-2\theta(\eta_{1t},\eta_1)\\
&\le \frac{\theta^2}{2\xi}|\eta_{1t}|^2+\frac{\xi}{2}|\eta_1|^2+\frac{\theta^2}{2\xi}|\eta_{2t}|^2+\frac{\xi}{2}|\eta_2|^2\\
&\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\frac{\xi C}{2}[|\eta_{1x}|^2+|\eta_{1y}|^2+|\eta_{2x}|^2+|\eta_{2y}|^2]\\
&\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\frac{\xi C}{2}\Big[\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}\,dx\,dy+\tfrac12\psi_x^2+\tfrac12\psi_y^2\Big]\\
&\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\frac{\xi C}{2}\Big[\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}\,dx\,dy+|\nabla\psi|^2\Big]\\
&\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\xi C E_k(t).
\end{aligned}\tag{3-15}
$$

Using bounds (3-13)–(3-15), one obtains, from (3-12),

$$
\begin{aligned}
\frac{d}{dt}F(t)\le{}&-\theta D|\phi_{1x}|^2-\theta D|\phi_{2y}|^2-\theta k|\phi_1+\psi_x|^2-\theta k|\phi_2+\psi_y|^2-\theta D\frac{1-\mu}{2}|\phi_{1y}+\phi_{2x}|^2\\
&-2\theta D\mu\int_\Omega\phi_{1y}\phi_{2x}\,dx\,dy-2\theta\frac{Eh}{1-\mu^2}\frac{1-\mu}{2}|\eta_{1y}+\eta_{2x}+\psi_x\psi_y|^2-2\theta\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2\\
&-2\theta\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2-2\mu\theta\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2-2\mu\theta\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+3\xi C E_k(t)+\theta\frac{\rho h^3}{12}\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]\\
&+\theta\rho h\big[|\psi_t|^2+2|\eta_{1t}|^2+2|\eta_{2t}|^2\big]+\frac{\theta^2}{2\xi}\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]+\frac{\theta^2}{2\xi}|\psi_t|^2+\frac{\theta^2}{2\xi}\big[|\eta_{1t}|^2+|\eta_{2t}|^2\big]\\
\le{}&-(\theta-3\xi C)E_k(t)+\Big(\theta\frac{\rho h^3}{12}+\frac{\theta^2}{2\xi}\Big)\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]+\Big(\theta\rho h+\frac{\theta^2}{2\xi}\Big)|\psi_t|^2\\
&+\Big(2\theta\rho h+\frac{\theta^2}{2\xi}\Big)\big[|\eta_{1t}|^2+|\eta_{2t}|^2\big].
\end{aligned}\tag{3-16}
$$

Therefore,

$$\frac{d}{dt}F(t)\le-(\theta-3\xi C)E_k(t)+C\big[|\phi_{1t}|^2+|\phi_{2t}|^2+|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\big].\tag{3-17}$$

Considering the derivative of the expression (3-5), and observing (3-2) and (3-17), one has

$$\frac{d}{dt}G_\lambda(t)\le-\lambda(\theta-3\xi C)E_k(t)-(1-\lambda C)\big[|\phi_{1t}|^2+|\phi_{2t}|^2+|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\big].$$

Choosing $\lambda\le 1/(2C)$ and $\xi<\theta/3$, one obtains, according to (3-11),

$$\frac{d}{dt}G_\lambda(t)\le-\lambda(\theta-3\xi C)E_k(t)\le-\tfrac12 wG_\lambda(t)\quad\forall t\ge0,$$

where $\omega=\lambda(\theta-3\xi C)$. Therefore,

$$G_\lambda(t)\le G_\lambda(0)e^{-wt/2}.\tag{3-18}$$

Combining (3-11) and (3-18), one gets (3-4). □

4. Stability: boundary feedback

In this section we analyze the plate model in the case where the energy of the Mindlin–Timoshenko system is dissipated through boundary feedback mechanisms. Let us assume that $\Gamma_i \neq \varnothing$ $(i = 0, 1)$, and we consider the system (1-1) with boundary conditions

$$\begin{gathered}\phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0,\\ \{\mathcal{B}_1(\phi_1, \phi_2),\ \mathcal{B}_2(\phi_1, \phi_2),\ \mathcal{B}_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2),\ \mathcal{B}_4(\eta_1, \eta_2),\ \mathcal{B}_5(\eta_1, \eta_2)\} \\ = -\{\phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}\} \quad \text{on } \Sigma_1,\end{gathered} \tag{4-1}$$

and initial data (1-3). The energy of this system obeys the dissipation law

$$\frac{d}{dt}E_k(t) = -\int_{\Gamma_1}\big[(\phi_{1t}^k)^2 + (\phi_{2t}^k)^2 + (\psi_t^k)^2 + (\eta_{1t}^k)^2 + (\eta_{2t}^k)^2\big]\,d\Gamma.$$

Consequently,

$$E_k(t) \le E_k(0) \quad \forall\, t \ge 0.$$

We are interested in studying the asymptotic behavior of $E_k(t)$ as $t \to \infty$.

The variational formulation of (1-1), (4-1), (1-3) is given by

$$\begin{aligned}&\tfrac{1}{12}\rho h^3\frac{d}{dt}(\phi_{1t}^k, a) + \tfrac{1}{12}\rho h^3\frac{d}{dt}(\phi_{2t}^k, b) + \rho h\frac{d}{dt}(\psi_t^k, c) + \rho h\frac{d}{dt}(\eta_{1t}^k, d) + \rho h\frac{d}{dt}(\eta_{2t}^k, e)\\ &\quad + k\big[(\phi_1^k + \psi_x^k, a + c_x) + (\phi_2^k + \psi_y^k, b + c_y)\big]\\ &\quad + D\big[(\phi_{1x}^k, a_x) + \tfrac{1}{2}(1-\mu)(\phi_{1y}^k, a_y) + \tfrac{1}{2}(1+\mu)(\phi_{2x}^k, a_y) + (\phi_{2y}^k, b_y) + \tfrac{1}{2}(1-\mu)(\phi_{2x}^k, b_x) + \tfrac{1}{2}(1+\mu)(\phi_{1y}^k, b_x)\big]\\ &\quad + (N_1^k\psi_x^k + N_{12}^k\psi_y^k, c_x) + (N_1^k, d_x) + (N_{12}^k, d_y) + (N_2^k\psi_y^k + N_{12}^k\psi_x^k, c_y)\\ &\quad + (N_2^k, e_y) + (N_{12}^k, e_x) + \int_{\Gamma_1}\big[\phi_{1t}^k a + \phi_{2t}^k b + \psi_t^k c + \eta_{1t}^k d + \eta_{2t}^k e\big]\,d\Gamma = 0\end{aligned} \tag{4-2}$$

for all $\{a, b, c, d, e\} \in [H^1_{\Gamma_0}(\Omega)]^5$.

Remark 4.1. Using arguments similar to those in Section 2, considering initial data in a suitable class and satisfying (2-4), we can prove that the system (1-1), (4-1), (1-3) converges (as $k \to \infty$) toward the dissipative von Kármán system (1-5) with boundary conditions

$$\begin{gathered}\psi = \frac{\partial\psi}{\partial\nu} = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0,\\ D\big[\Delta\psi + (1-\mu)(2\nu_1\nu_2\psi_{xy} - \nu_1^2\psi_{yy} - \nu_2^2\psi_{xx})\big] = -(\nu_1\psi_{xt} + \nu_2\psi_{yt}) \quad \text{on } \Sigma_1,\\ D\bigg[\frac{\partial(\Delta\psi)}{\partial\nu} + (1-\mu)\frac{\partial}{\partial\tau}\big[(\nu_1^2 - \nu_2^2)\psi_{xy} + \nu_1\nu_2(\psi_{yy} - \psi_{xx})\big]\bigg]\\ -\tfrac{1}{12}\rho h^3\frac{\partial\psi_{tt}}{\partial\nu} - (\nu_1 N_1 + \nu_2 N_{12})\psi_x - (\nu_2 N_2 + \nu_1 N_{12})\psi_y = \frac{\partial}{\partial\tau}(-\nu_1\psi_{yt} + \nu_2\psi_{xt}) - \psi_t \quad \text{on } \Sigma_1,\\ \nu_1 N_1 + \nu_2 N_{12} = -\eta_{1t} \quad \text{on } \Sigma_1,\\ \nu_2 N_2 + \nu_1 N_{12} = -\eta_{2t} \quad \text{on } \Sigma_1,\end{gathered} \tag{4-3}$$

and initial data (1-7).

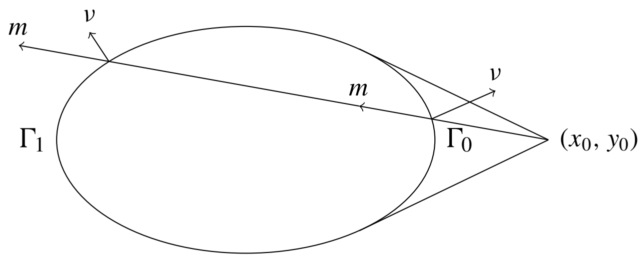

Figure 1. Example for which condition (4-4) is satisfied.

In order to establish the uniform asymptotic stability of system (1-1), (4-1), (1-3), some restrictions are needed on the geometry of Ω, Γ_0 and Γ_1. Let us introduce a vector field $m = m(x, y)$ in $\mathbb{R}^2$ defined by

$$m(x, y) = (x, y) - (x_0, y_0),$$

where (x_0, y_0) is a fixed point of $\mathbb{R}^2$. We assume that Γ_0 and Γ_1 are such that

$$m \cdot \nu \leq 0 \quad \text{on } \Gamma_0, \qquad m \cdot \nu \geq 0 \quad \text{on } \Gamma_1. \tag{4-4}$$

Let us consider $G = [g_{ij}]$ the 5×5 matrix such that

$$g_{ij} = 0, \quad i \neq j, \qquad \text{and} \qquad (m \cdot \nu) g_{ii} = 1, \quad i = 1, \dots, 5.$$

Note that $g_{ij} \in C^1(\bar{\Gamma}_1)$. Moreover, there are positive constants g_0 and G_0 such that

$$g_0 |\varsigma|^2 \leq G\varsigma \cdot \varsigma \leq G_0 |\varsigma|^2 \quad \forall\, \varsigma \in \mathbb{R}^5, \text{ on } \Gamma_1. \tag{4-5}$$

Before establishing the main result of this section, we will state and prove the following two lemmas.

Lemma 4.2. *Let* $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ *and* $\{\phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}\}$ *be regular enough. Then*

$$\begin{aligned}\int_\Omega \big[&\phi_{1t} L_1(\phi_1, \phi_2, \psi) + \phi_{2t} L_2(\phi_1, \phi_2, \psi) \\ &+ \psi_t L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} L_4(\psi, \eta_1, \eta_2) + \eta_{2t} L_5(\psi, \eta_1, \eta_2)\big]\, dx\, dy \\ &\quad + a(\phi_1, \phi_2, \psi, \eta_1, \eta_2, \phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}) \\ = \int_\Gamma &\big[\phi_{1t} \mathcal{B}_1(\phi_1, \phi_2) + \phi_{2t} \mathcal{B}_2(\phi_1, \phi_2) + \psi_t \mathcal{B}_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} \mathcal{B}_4(\psi, \eta_1, \eta_2) + \eta_{2t} \mathcal{B}_5(\psi, \eta_1, \eta_2)\big]\, d\Gamma,\end{aligned} \tag{4-6}$$

with

$$a(\phi_1, \phi_2, \psi, \eta_1, \eta_2, \phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}) := a_0(\phi_1, \phi_2, \phi_{1t}, \phi_{2t}) + k a_1(\phi_1, \phi_2, \psi, \phi_{1t}, \phi_{2t}, \psi_t) + a_2(\psi, \eta_1, \eta_2, \psi_t, \eta_{1t}, \eta_{2t}),$$

where

$$a_0(\phi_1, \phi_2, \phi_{1t}, \phi_{2t}) = D \int_\Omega \big[\phi_{1x}\phi_{1tx} + \phi_{2y}\phi_{2ty} + \mu \phi_{1x}\phi_{2ty} + \mu \phi_{1tx}\phi_{2y} + \tfrac{1}{2}(1-\mu)(\phi_{1y} + \phi_{2x})(\phi_{1ty} + \phi_{2tx})\big]\, dx\, dy,$$

$$a_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t)=\int_\Omega\big[(\phi_1+\psi_x)(\phi_{1t}+\psi_{tx})+(\phi_2+\psi_y)(\phi_{2t}+\psi_{ty})\big]\,dx\,dy,$$

and

$$\begin{aligned}a_2(\psi,&\eta_1,\eta_2,\psi_t,\eta_{1t},\eta_{2t})\\&=\frac{Eh}{1-\mu^2}\int_\Omega\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)(\eta_{1tx}+\psi_x\psi_{tx})\\&\qquad+(1-\mu)\big(\eta_{2y}+\tfrac{1}{2}\psi_y^2\big)(\eta_{2ty}+\psi_y\psi_{ty})\\&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)(\eta_{1tx}+\eta_{2ty}+\nabla\psi\cdot\nabla\psi_t)\\&\qquad+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)(\eta_{1ty}+\eta_{2tx}+\psi_x\psi_{ty}+\psi_y\psi_{tx})\Big]\,dx\,dy.\end{aligned}$$

Remark 4.3. Here and elsewhere in this section we use the term “regular enough” to ensure that all integrals are well defined (see Section 5 for additional comments on this point).

Proof of Lemma 4.2. By definition of the operators $L_i(\phi_1,\phi_2,\psi,\eta_1,\eta_2)$ $(i=1,\dots,5)$, one has

$$\begin{aligned}&\int_\Omega\big[\phi_{1t}L_1(\phi_1,\phi_2,\psi)+\phi_{2t}L_2(\phi_1,\phi_2,\psi)+\psi_tL_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\&\qquad\qquad+\eta_{1t}L_4(\psi,\eta_1,\eta_2)+\eta_{2t}L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy\\&=\int_\Omega\Big\{\phi_{1t}\big[D\big(\phi_{1xx}+\tfrac{1}{2}(1-\mu)\phi_{1yy}+\tfrac{1}{2}(1+\mu)\phi_{2xy}\big)-k(\phi_1+\psi_x)\big]\\&\qquad+\phi_{2t}\big[D\big(\phi_{2yy}+\tfrac{1}{2}(1-\mu)\phi_{2xx}+\tfrac{1}{2}(1+\mu)\phi_{1xy}\big)-k(\phi_2+\psi_y)\big]\\&\qquad+\psi_t\big\{k\big[(\psi_x+\phi_1)_x+(\psi_y+\phi_2)_y\big]+(N_1\psi_x+N_{12}\psi_y)_x+(N_2\psi_y+N_{12}\psi_x)_y\big\}\\&\qquad+\eta_{1t}[N_{1x}+N_{12y}]+\eta_{2t}[N_{2y}+N_{12x}]\Big\}\,dx\,dy.\end{aligned}$$

Through integration by parts one obtains

$$\begin{aligned}&\int_\Omega\big[\phi_{1t}L_1(\phi_1,\phi_2,\psi)+\phi_{2t}L_2(\phi_1,\phi_2,\psi)+\psi_tL_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\&\qquad\qquad+\eta_{1t}L_4(\psi,\eta_1,\eta_2)+\eta_{2t}L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy\\&=-a_0(\phi_1,\phi_2,\phi_{1t},\phi_{2t})-ka_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t)\\&\quad-\int_\Omega\big[(N_1\psi_x+N_{12}\psi_y)\psi_{tx}+(N_{2\psi_y+N_{12}\psi_x)})\psi_{ty}+\eta_{1tx}N_1+\eta_{1tyN_{12}}+\eta_{2ty}N_2+\eta_{2tx}N_{12}\big]\,dx\,dy\\&\quad+\int_\Gamma\Big\{\phi_{1t}D\big[\phi_{1x}\nu_1+\tfrac{1}{2}(1-\mu)\phi_{1y}\nu_2+\tfrac{1}{2}(1-\mu)\phi_{2x}\nu_2+\mu\phi_{2y\nu_1}\big]\\&\qquad+\phi_{2t}D\big[\phi_{2y}\nu_2+\tfrac{1}{2}(1-\mu)\phi_{2x}\nu_1+\tfrac{1}{2}(1-\mu)\phi_{1y}\nu_1+\mu\phi_{1x\nu_2}\big]\\&\qquad+\psi_tk\big[(\phi_1+\psi_x)\nu_1+(\phi_2+\psi_y)\nu_2\big]+(N_1\psi_x+N_{12}\psi_y)\nu_1\\&\qquad+(N_2\psi_y+N_{12}\psi_x)\nu_2+\eta_{1t}(N_1\nu_1+N_{12}\nu_2)+\eta_{2t}(N_2\nu_2+N_{12}\nu_1)\Big\}\,d\Gamma.\end{aligned}$$

Finally, using the definition of N_1, N_2 and N_{12}, one has

$$\begin{aligned}&\int_\Omega\big[\phi_{1t}L_1(\phi_1,\phi_2,\psi)+\phi_{2t}L_2(\phi_1,\phi_2,\psi)+\psi_tL_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\&\qquad\qquad+\eta_{1t}L_4(\psi,\eta_1,\eta_2)+\eta_{2t}L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy\end{aligned}$$

$$= -a_0(\phi_1,\phi_2,\phi_{1t},\phi_{2t}) - ka_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t) - a_2(\psi,\eta_1,\eta_2,\psi_t,\eta_{1t},\eta_{2t}) + \int_\Gamma \{\phi_{1t}\mathcal{B}_1(\phi_1,\phi_2) + \phi_{2t}\mathcal{B}_2(\phi_1,\phi_2) + \psi_t\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \eta_{1t}\mathcal{B}_4(\psi,\eta_1,\eta_2) + \eta_{2t}\mathcal{B}_5(\psi,\eta_1,\eta_2)\}\,d\Gamma. \quad \square$$

Lemma 4.4. *Consider* $\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}$ *to be regular enough. Then*

$$\int_\Omega \big[(m\cdot\nabla\phi_1)L_1(\phi_1,\phi_2,\psi)+(m\cdot\nabla\phi_2)L_2(\phi_1,\phi_2,\psi)+(m\cdot\nabla\psi)L_3(\phi_1,\phi_2,\psi) +(m\cdot\nabla\eta_1)L_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy$$
$$= k\int_\Omega \big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy$$
$$-\tfrac12\int_\Gamma m\cdot\nu\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big] + \frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12(1)\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2 +\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma$$
$$+\int_\Gamma \big[(m\cdot\nabla\phi_1)\mathcal{B}_1(\phi_1,\phi_2)+(m\cdot\nabla\phi_2)\mathcal{B}_2(\phi_1,\phi_2)+(m\cdot\nabla\psi)\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) +(m\cdot\nabla\eta_1)\mathcal{B}_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)\mathcal{B}_5(\psi,\eta_1,\eta_2)\big]\,d\Gamma. \quad (4\text{-}7)$$

Proof. Analogously to the proof of Lemma 4.2,

$$\int_\Omega \big[(m\cdot\nabla\phi_1)L_1(\phi_1,\phi_2,\psi)+(m\cdot\nabla\phi_2)L_2(\phi_1,\phi_2,\psi)+(m\cdot\nabla\psi)L_3(\phi_1,\phi_2,\psi) + (m\cdot\nabla\eta_1)L_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy$$
$$= -a\big(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2\big)$$
$$+\int_\Gamma \big[(m\cdot\nabla\phi_1)\mathcal{B}_1(\phi_1,\phi_2)+(m\cdot\nabla\phi_2)\mathcal{B}_2(\phi_1,\phi_2)+(m\cdot\nabla\psi)\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + (m\cdot\nabla\eta_1)\mathcal{B}_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)\mathcal{B}_5(\psi,\eta_1,\eta_2)\big]\,d\Gamma. \quad (4\text{-}8)$$

In this way, to prove (4-7) we have only to study the term

$$a\big(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2\big). \quad (4\text{-}9)$$

Note that

$$a_0(\phi_1,\phi_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2)$$
$$= D\int_\Omega \big[\phi_{1x}(m\cdot\nabla\phi_1)_x+\phi_{2y}(m\cdot\nabla\phi_2)_y+\mu\phi_{1x}(m\cdot\nabla\phi_2)_y+\mu\phi_{2y}(m\cdot\nabla\phi_1)_x + \tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})\big((m\cdot\nabla\phi_1)_y+(m\cdot\nabla\phi_2)_x\big)\big]\,dx\,dy$$
$$= \frac{D}{2}\int_\Omega \operatorname{div}\big\{m\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\big\}\,dx\,dy$$
$$= \frac{D}{2}\int_\Gamma m\cdot\nu\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\,d\Gamma, \quad (4\text{-}10)$$

$$
\begin{aligned}
&a_1(\phi_1,\phi_2,\psi,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi)\\
&\quad=\int_\Omega\big[(\phi_1+\psi_x)((m\cdot\nabla\phi_1)+(m\cdot\nabla\psi)_x)+(\phi_2+\psi_y)((m\cdot\nabla\phi_2)+(m\cdot\nabla\psi)_y)\big]\,dx\,dy\\
&\quad=\frac{1}{2}\int_\Omega \operatorname{div}\{m[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2]\}\,dx\,dy-\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\\
&\quad=\frac{1}{2}\int_\Gamma\{m\cdot\nu[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2]\}\,d\Gamma-\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy,
\end{aligned}
\tag{4-11}
$$

and

$$
\begin{aligned}
&a_2(\psi,\eta_1,\eta_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2)\\
&=\frac{Eh}{1-\mu^2}\int_\Omega\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)\big((m\cdot\nabla\eta_1)_x+\psi_x(m\cdot\nabla\psi)_x\big)+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)\big((m\cdot\nabla\eta_2)_y+\psi_y(m\cdot\nabla\psi)_y\big)\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)\big((m\cdot\nabla\eta_1)_x+(m\cdot\nabla\eta_2)_y+\nabla\psi\cdot\nabla(m\cdot\nabla\psi)\big)\\
&\qquad+\tfrac12(1-\mu)u(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)\big((m\cdot\nabla\eta_1)_y+(m\cdot\nabla\eta_2)_x+\psi_x(m\cdot\nabla\psi)_y+\psi_y(m\cdot\nabla\psi)_x\big)\Big]\,dx\,dy\\
&=\frac{Eh}{2(1-\mu^2)}\int_\Omega\operatorname{div}\Big\{m\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\phi_{2y}+\tfrac12\psi_y^2\big)^2\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)u(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,dx\,dy\\
&=\frac{Eh}{2(1-\mu^2)}\int_\Gamma m\cdot\nu\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)u(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\,d\Gamma.
\end{aligned}
\tag{4-12}
$$

Plugging (4-10)–(4-12) in (4-9) we get

$$
\begin{aligned}
&a(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2)\\
&=\tfrac12\int_\Gamma m\cdot\nu\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\
&\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12(1)\psi_x^2\big)^2+(1-\mu)\big(\phi_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2\\
&\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma\\
&\qquad-k\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy.
\end{aligned}
\tag{4-13}
$$

Equation (4-7) follows directly from (4-8) and (4-13). □

The main result in this section is the following.

Theorem 4.5. *Assume the geometric condition (4-4) holds. Let $\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}$ be a regular enough solution of system (1-1), (4-1), (1-3). Then, there exist positive constants C and ω such that*

$$E_k(t)\le CE_k(0)e^{-\omega t}\quad\forall t\ge 0. \tag{4-14}$$

Remark 4.6. For regular enough initial data satisfying (2-4), one obtains, as a consequence of inequality (4-14), exponential decay for the energy $E(t)$ associated to system (1-5), (4-3), (1-7) as $k\to\infty$. This decay rate for the limit system is in agreement with the results from [Perla Menzala et al. 2002].

Remark 4.7. The case $\Gamma_0=\varnothing$ is not considered in this paper. In this case, one cannot ensure that the energy decays to zero for every finite energy solution of (1-1), (4-1), (1-3) regardless of how the feedbacks are chosen. Indeed, defining

$$\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}=\bigl\{\alpha,\,\beta,\,-\alpha x-\beta y+\gamma,\,-\tfrac12\alpha^2x-\tfrac12\alpha\beta y+c_1,\,-\tfrac12\beta^2y-\tfrac12\alpha\beta x+c_2\bigr\},$$

where α, β, γ, c_1 and c_2 are nonzero constants, and $\{\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}\}$ such that

$$L_i(\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20})=0,\qquad i=1,\dots,5,$$

$$\bigl\{\mathcal{B}_1(\phi_{10},\phi_{20}),\mathcal{B}_2(\phi_{10},\phi_{20}),\mathcal{B}_3(\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}),\mathcal{B}_4(\psi_0,\eta_{10},\eta_{20}),\mathcal{B}_5(\psi_0,\eta_{10},\eta_{20})\bigr\}=-\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\},$$

it is not difficult to check that

$$\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}=t\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}+\{\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}\}$$

is a solution of (1-1), (4-1), (1-3). However, for this solution,

$$E(t)=\tfrac12\bigl[\tfrac1{12}\rho h^3(|\phi_{11}|^2+|\phi_{21}|^2)+\rho h(|\psi_1|^2+|\eta_1|^2+|\eta_2|^2)\bigr]=\text{const.}>0.$$

Proof of Theorem 4.5. We divide the proof into three steps:

Step 1: We apply Lemma 4.4 to the solution of (1-1), (4-1), (1-3) and integrate the resulting identity with respect to t from 0 to T to obtain

$$\begin{aligned}
&\rho h\int_0^T\bigl[\tfrac1{12}h^2(\phi_{1tt},m\cdot\nabla\phi_1)+\tfrac1{12}h^2(\phi_{2tt},m\cdot\nabla\phi_2)+(\psi_{tt},m\cdot\nabla\psi)+(\eta_{1tt},m\cdot\nabla\eta_1)+(\eta_{2tt},m\cdot\nabla\eta_2)\bigr]dt\\
&\qquad\qquad-k\int_0^T\int_\Omega\bigl[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\bigr]dx\,dy\\
&=-\tfrac12\int_0^T\int_\Gamma m\cdot\nu\Bigl\{D\bigl[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\bigr]+k\bigl[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\bigr]\\
&\qquad+\frac{Eh}{1-\mu^2}\Bigl[(1-\mu)\bigl(\eta_{1x}+\tfrac12\psi_x^2\bigr)^2+(1-\mu)(\eta_{2y}+\tfrac12\psi_y^2)^2\\
&\qquad\qquad+\mu(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2)^2+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Bigr]\Bigr\}d\Gamma\\
&\quad+\int_0^T\int_{\Gamma_0}\bigl[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\bigr]d\Gamma\\
&\qquad-\int_0^T\int_{\Gamma_1}\bigl[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\bigr]d\Gamma.
\end{aligned}\tag{4-15}$$

Both of the integrals on the left-hand side of (4-15) may be interpreted in the $L^2(Q)$ scalar product since $\{\phi_{1tt},\phi_{2tt},\psi_{tt},\eta_{1tt},\eta_{2tt}\}\in C([0,\infty),[L^2(\Omega)]^5)$. The first integral on the left-hand side may be written as

$$\begin{aligned}
&\rho h\int_0^T\int_\Omega\bigl\{\tfrac1{12}h^2\bigl[\phi_{1tt}(m\cdot\nabla\phi_1)+\phi_{2tt}(m\cdot\nabla\phi_2)\bigr]+\psi_{tt}(m\cdot\nabla\psi)+\eta_{1tt}(m\cdot\nabla\eta_1)+\eta_{2tt}(m\cdot\nabla\eta_2)\bigr\}dx\,dy\,dt\\
&\quad=\Upsilon_1-\rho h\int_0^T\int_\Omega\bigl[\tfrac1{12}h^2\bigl(\phi_{1t}(m\cdot\nabla\phi_{1t})+\phi_{2t}(m\cdot\nabla\phi_{2t})\bigr)\\
&\qquad\qquad+\psi_t(m\cdot\nabla\psi_t)+\eta_{1t}(m\cdot\nabla\eta_{1t})+\eta_{2t}(m\cdot\nabla\eta_{2t})\bigr]dx\,dy\,dt,
\end{aligned}\tag{4-16}$$

where

$$\mathrm{Y}_1=\rho h\int_\Omega\left\{\tfrac{1}{12}h^2\left[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)\right]+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\right\}dx\,dy\Big|_0^T. \quad (4\text{-}17)$$

A typical term of the last integral in (4-16) is (except for a constant factor)

$$\begin{aligned}\int_0^T(\phi_{1t},m\cdot\nabla\phi_{1t})\,dt&=\tfrac{1}{2}\int_0^T\!\!\int_\Omega \operatorname{div}(m\phi_{1t}^2)\,dx\,dy\,dt-\int_0^T\!\!\int_\Omega\phi_{1t}^2\,dx\,dy\,dt\\&=\tfrac{1}{2}\int_0^T\!\!\int_{\Gamma_1}(m\cdot\nu)\phi_{1t}^2\,d\Gamma\,dt-\int_0^T\!\!\int_\Omega\phi_{1t}^2\,dx\,dy\,dt.\end{aligned}$$

The other terms of that integral are treated similarly. Thus, it follows that

$$\begin{aligned}&\rho h\int_0^T\!\!\int_\Omega\left\{\tfrac{1}{12}h^2\left[\phi_{1t}(m\cdot\nabla\phi_{1t})+\phi_{2t}(m\cdot\nabla\phi_{2t})\right]+\psi_t(m\cdot\nabla\psi_t)+\eta_{1t}(m\cdot\nabla\eta_{1t})+\eta_{2t}(m\cdot\nabla\eta_{2t})\right\}dx\,dy\,dt\\&=\tfrac{1}{2}\rho h\int_0^T\!\!\int_{\Gamma_1}m\cdot\nu\left[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]d\Gamma\,dt\\&\qquad-\int_0^T\!\!\int_\Omega\rho h\left[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]dx\,dy\,dt. \quad (4\text{-}18)\end{aligned}$$

Combining (4-15), (4-16) and (4-18), one has

$$\begin{aligned}&\mathrm{Y}_1+\int_0^T\!\!\int_\Omega\rho h\left[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]dx\,dy\,dt-k\int_0^T\!\!\int_\Omega\left[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\right]dx\,dy\,dt\\&=\mathrm{J}_1-\mathrm{J}_2+\int_0^T\!\!\int_{\Gamma_0}\left[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\right]d\Gamma\,dt\\&\quad-\int_0^T\!\!\int_{\Gamma_1}\left[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\right]d\Gamma\,dt, \quad (4\text{-}19)\end{aligned}$$

where

$$\mathrm{J}_1=\tfrac{1}{2}\rho h\int_0^T\!\!\int_{\Gamma_1}m\cdot\nu\left[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]d\Gamma\,dt, \quad (4\text{-}20)$$

and

$$\begin{aligned}\mathrm{J}_2=\tfrac{1}{2}\int_0^T\!\!\int_{\Gamma_1}m\cdot\nu\Big\{&D\left[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\right]\\&+k\left[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\right]\\&+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)(\eta_{2y}+\tfrac{1}{2}\psi_y^2)^2\\&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}d\Gamma\,dt. \quad (4\text{-}21)\end{aligned}$$

Let us examine the integrals on Γ_0 in the right-hand side of (4-19). Since $\phi_1=\phi_2=\psi=\eta_1=\eta_2=0$ on Γ_0, we have $\nabla\phi_1=\nu((\partial\phi_1)/(\partial\nu))$ on Γ_0 and similarly for the other functions. Therefore,

$$
\begin{aligned}
&\int_{\Gamma_0} m\cdot\nu\Big\{D\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)\big]\Big\}\,d\Gamma\\
&\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2\\
&\qquad\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\\
&=\int_{\Gamma_0} m\cdot\nu\Bigg\{D\bigg[\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2-(1-\mu)\nu_1\nu_2\frac{\partial\phi_1}{\partial\nu}\frac{\partial\phi_2}{\partial\nu}\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2\\
&\qquad+\frac{Eh}{1-\mu^2}\bigg[(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac12\Big(\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2\\
&\qquad\qquad-2(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac12\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\Big)\Big(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac12\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\Big)\\
&\qquad\qquad+\mu\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac12\Big|\Big(\frac{\partial\psi}{\partial\nu}\Big)\Big|^2\Big)^2\\
&\qquad\qquad+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\Big)\bigg]\Bigg\}\,d\Gamma. \qquad (4\text{-}22)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\int_{\Gamma_0}\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\,d\Gamma\\
&=\int_{\Gamma_0} m\cdot\nu\Bigg\{\frac{D}{2}\bigg[(1-\mu)\Big(\Big(\frac{\partial\phi_1}{\partial\nu}\Big)^2+\Big(\frac{\partial\phi_2}{\partial\nu}\Big)^2\Big)+(1+\mu)\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2+N_1\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\\
&\qquad+2N_{12}\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}+N_2\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2+N_1\nu_1\frac{\partial\eta_1}{\partial\nu}+N_{12}\nu_2\frac{\partial\eta_1}{\partial\nu}+N_2\nu_2\frac{\partial\eta_2}{\partial\nu}+N_{12}\nu_1\frac{\partial\eta_2}{\partial\nu}\Bigg\}\,d\Gamma. \qquad (4\text{-}23)
\end{aligned}
$$

Since

$$
\begin{aligned}
&-\tfrac12\Big\{D\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)\big]\\
&\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2\\
&\qquad\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\\
&\qquad+\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\\
&=\frac12\Bigg\{D\bigg[\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\phi_1}{\partial\nu}-\nu_1\frac{\partial\psi_2}{\partial_\nu}\Big)^2\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2\\
&\qquad+\frac{Eh}{1-\mu^2}\bigg[(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac12\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2+(1-\mu)\Big(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac12\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2\\
&\qquad\qquad+\mu\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac12|\nabla\psi|^2\Big)^2+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\bigg]\Bigg\}, \qquad (4\text{-}24)
\end{aligned}
$$

we conclude from (4-19) and (4-24) that

$$\mathrm{Y}_1+\int_0^T\int_\Omega \rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt-k\int_0^T\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\,dt$$
$$=\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2+\int_0^T\int_{\Gamma_0}\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\,d\Gamma\,dt$$
$$-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\big]\,d\Gamma\,dt,\tag{4-25}$$

where

$$\mathrm{J}_0=\frac{1}{2}\int_0^T\int_{\Gamma_0}\Bigg\{D\bigg[\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\phi_1}{\partial\nu}-\nu_1\frac{\partial\psi_2}{\partial_\nu}\Big)^2\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2$$
$$+\frac{Eh}{1-\mu^2}\bigg[(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac{1}{2}\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2+(1-\mu)\Big(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2$$
$$+\mu\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}|\nabla\psi|^2\Big)^2+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\bigg]\Bigg\}\,d\Gamma.$$

Now, use (4-6) with $\{\phi_1,\phi_2,0,\eta_1,\eta_2\}$ in the third term on the left-hand side of (4-25) to obtain

$$\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1tt}\phi_1+\phi_{2tt}\phi_2)+\eta_{1tt}\eta_1+\eta_{2tt}\eta_2\big]\,dx\,dy$$
$$-k\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)$$
$$=-\int_{\Gamma_1}\big[\phi_{1t}\phi_1+\phi_{2t}\phi_2+\eta_{1t}\eta_1+\eta_{2t}\eta_2\big]\,d\Gamma.\tag{4-26}$$

Integrate identity (4-26) with respect to t from 0 to T. After an integration by parts in the first term, one obtains

$$\mathrm{Y}_2-\rho h\int_0^T\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt$$
$$+k\int_0^T\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\,dt+\int_0^T\big[a_0(\phi_1,\phi_2)+a_2(\eta_1,\eta_2)\big]\,dt$$
$$=-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}\phi_1+\phi_{2t}\phi_2+\eta_{1t}\eta_1+\eta_{2t}\big]\,d\Gamma\,dt\tag{4-27}$$

where

$$\mathrm{Y}_2=\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}\phi_1+\phi_{2t}\phi_2)+\eta_{1t}\eta_1+\eta_{2t}\eta_2\big]\,dx\,dy\bigg|_0^T.\tag{4-28}$$

Multiply (4-27) by $1-\varepsilon$, with $\varepsilon \in (0,1)$, and add the product to (4-25) to get

$$(1-\varepsilon)\rho h\int_0^T\int_\Omega \psi_t^2\,dx\,dy\,dt+\varepsilon\int_0^T\int_\Omega \rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt$$
$$+(1-\varepsilon)\int_0^T a_0(\phi_1,\phi_2)\,dt+(1-\varepsilon)\int_0^T a_2(\psi,\eta_1,\eta_2)\,dt-\varepsilon k\int_0^T a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\,dt+\mathrm{Y}_1+(1-\varepsilon)\mathrm{Y}_2$$
$$=\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi)$$
$$+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\,dt. \quad (4\text{-}29)$$

Now, use (4-6) with $\{0, 0, \psi, 0, 0\}$. After an integration by parts in t one obtains

$$\mathrm{Y}_3-\rho h\int_0^T\int_\Omega \psi_t^2\,dx\,dy\,dt+k\int_0^T a_1(\phi_1,\phi_2,\psi,0,0,\psi)\,dt+\int_0^T a_2(\psi,\eta_1,\eta_2,\psi,0,0)\,dt$$
$$=-\int_0^T\int_{\Gamma_1}\psi\psi_t\,d\Gamma\,dt, \quad (4\text{-}30)$$

where

$$\mathrm{Y}_3=\rho h\int_\Omega \psi_t\psi\,dx\,dy\Big|_0^T. \quad (4\text{-}31)$$

Multiply identity (4-30) by ε and add the product to (4-29) to obtain

$$(1-2\varepsilon)\rho h\int_0^T\int_\Omega \psi_t^2\,dx\,dy\,dt+\varepsilon\int_0^T\int_\Omega \rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt$$
$$+(1-\varepsilon)\int_0^T\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2,0,\eta_1\eta_2)\big]\,dt$$
$$+\varepsilon k\int_0^T a_1(\phi_1,\phi_2,\psi)\,dt-2\varepsilon k\int_0^T a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\,dt$$
$$+\varepsilon\int_0^T a_2(\psi,\eta_1,\eta_2,\psi,0,0)\,dt+\mathrm{Y}_1+(1-\varepsilon)\mathrm{Y}_2+\varepsilon\mathrm{Y}_3$$
$$=\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)$$
$$+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\,dt. \quad (4\text{-}32)$$

Step 2: Define the functional

$$\rho_\varepsilon(t)=\rho h\Big[\tfrac{1}{12}h^2(\phi_{1t}(t),m\cdot\nabla\phi_1(t))+\tfrac{1}{12}h^2(\phi_{2t}(t),m\cdot\nabla\phi_2(t))+(\psi_t(t),m\cdot\nabla\psi(t))$$
$$+(\eta_{1t}(t),m\cdot\nabla\eta_1(t))+(\eta_{2t}(t),m\cdot\nabla\eta_2(t))\Big]$$
$$+(1-\varepsilon)\rho h\big\{\tfrac{1}{12}h^2\big[(\phi_{1t}(t),\phi_1(t))+(\phi_{2t}(t),\phi_2(t))\big]+(\eta_{1t}(t),\eta_1(t))+(\eta_{2t}(t),\eta_2(t))\big\}$$
$$+\varepsilon\rho h(\psi_t(t),\psi(t)). \quad (4\text{-}33)$$

From identities (4-17), (4-28), and (4-31), one sees that

$$\mathrm{Y}_1+(1-\varepsilon)\mathrm{Y}_2+\varepsilon \mathrm{Y}_3=\rho_\varepsilon(T)-\rho_\varepsilon(0). \tag{4-34}$$

Since (4-32) is valid for all $T>0$, we differentiate in T and obtain, writing t in place of T,

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t)=\frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\\ -(1-\varepsilon)\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)\big]-\varepsilon k a_1(\phi_1,\phi_2,\psi)\,dt\\ +2\varepsilon k a_1(\phi_1\phi_2,\psi,\phi_1,\phi_2,0)-\varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0)\\ -\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)\\ +\psi_t(m\cdot\nabla\psi+\varepsilon\psi)+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma,\end{aligned} \tag{4-35}$$

where the right-hand side is evaluated at t. Now, let $\delta>0$ and consider the perturbed energy $F_{\varepsilon,\delta}(t)$ given by

$$F_{\varepsilon,\delta}(t)=E_k(t)+\delta\rho_\varepsilon(t). \tag{4-36}$$

We are going to prove that for all ε, δ sufficiently small, one has

$$\frac{d}{dt}F_{\varepsilon,\delta}(t)\le -\tfrac12\varepsilon\delta E_k(t)-\tfrac12\delta E_\Gamma(t), \tag{4-37}$$

where

$$\begin{aligned}E_\Gamma(t)=\tfrac12\rho h\int_{\Gamma_1} m\cdot\nu\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,d\Gamma\\ +\tfrac12\int_\Gamma |m\cdot\nu|\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\ +\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2\\ +\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma.\end{aligned} \tag{4-38}$$

We begin the proof of inequality (4-37) estimating $(d/dt)\rho_\varepsilon(t)$. First of all, we bound the term $a_1(\phi_1,\phi_2,\psi,\phi_1\phi_2,0)$ in (4-35). For any $\xi>0$, we have

$$\big|a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\big|\le\frac{\xi}{2}a_1(\phi_1,\phi_2,\psi)+\frac{1}{2\xi}a_1(\phi_1,\phi_2,0).$$

Since $\Gamma_0\ne\varnothing$, according to [Lagnese 1989, Lemma 2.1] there is a constant γ_0 (depending on the geometry of Ω and on the parameters μ and D) such that

$$a_1(\phi_1,\phi_2,0)=\|\phi_1\|^2+\|\phi_2\|^2\le\gamma_0 a_0(\phi_1,\phi_2).$$

Therefore,

$$\big|a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\big|\le\frac{\xi}{2}a_1(\phi_1,\phi_2,\psi)+\frac{\gamma_0}{2\xi}a_0(\phi_1,\phi_2). \tag{4-39}$$

Use inequality (4-39) in identity (4-35) to get

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\\ -\Big(1-\varepsilon-\frac{\varepsilon\gamma_0 k}{\xi}\Big)a_0(\phi_1,\phi_2)-(1-\varepsilon)a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)\\ -\varepsilon k(1-\xi)a_1(\phi_1,\phi_2,\psi)\,dt-\varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0)\\ -\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)\\ +\psi_t(m\cdot\nabla\psi+\varepsilon\psi)+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma.\end{aligned}$$

Fix $\xi=\frac{1}{2}$, and then choose $\varepsilon>0$ so that $1-\varepsilon-2\varepsilon\gamma_0 k\ge\varepsilon$; that is,

$$0<\varepsilon\le\frac{1}{2(1+\gamma_0 k)}. \tag{4-40}$$

For such ε, one has

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) &\le \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\\ &\quad -\varepsilon a_0(\phi_1,\phi_2)-(1-\varepsilon)a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)-\tfrac{1}{2}k\varepsilon a_1(\phi_1,\phi_2,\psi)-\varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0)\\ &\quad -\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ &\qquad +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\,dt\\ &\le \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon E_k(t)\\ &\quad -\tfrac{1}{2}\varepsilon\Big\{\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\Big\}\\ &\quad -\int_{\Gamma_1}[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ &\qquad +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)]\,d\Gamma\,dt.\end{aligned} \tag{4-41}$$

We estimate the last term on the right-hand side of (4-41) as follows:

$$\begin{aligned}&\Big|\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ &\qquad +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\Big|\\ &\le\frac{1}{2\xi}\int_{\Gamma_1}[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2]\,d\Gamma\\ &\qquad +\frac{\xi}{2}\int_{\Gamma_1}\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2+(m\cdot\nabla\psi+\varepsilon\psi)^2\\ &\qquad +(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma\\ &=-\frac{1}{2\xi}\frac{d}{dt}E_k(t)+\frac{\xi}{2}\int_{\Gamma_1}\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2+(m\cdot\nabla\psi+\varepsilon\psi)^2\\ &\qquad +(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma.\end{aligned} \tag{4-42}$$

Looking for the last integral in (4-42), it follows by (4-5) that

$$\begin{aligned}\int_{\Gamma_1}\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2+(m\cdot\nabla\psi+\varepsilon\psi)^2\\ +(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma\\ \le G_0\int_{\Gamma_1} m\cdot\nu\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2\\ +(m\cdot\nabla\psi+\varepsilon\psi)^2+(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma.\end{aligned}\tag{4-43}$$

We now bound the right-hand side of inequality (4-43). Its first term is bounded by

$$\begin{aligned}\int_{\Gamma_1} m\cdot\nu(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2\,d\Gamma &\le 2\int_{\Gamma_1} m\cdot\nu\big[(m\cdot\nabla\phi_1)^2+(1-\varepsilon)^2\phi_1^2\big]\,d\Gamma\\ &\le 2R^2\int_{\Gamma_1} m\cdot\nu|\nabla\phi_1|^2\,d\Gamma+2(1-\varepsilon)^2R\int_{\Gamma_1}\phi_1^2\,d\Gamma,\end{aligned}$$

where $R=\sup_{\Gamma_1} m(x,y)$. The other terms can be bounded analogously. Therefore, one gets

$$\begin{aligned}\int_{\Gamma_1}\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2\\ +(m\cdot\nabla\psi+\varepsilon\psi)^2+(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma\\ \le 2G_0R^2\int_{\Gamma_1} m\cdot\nu\big[|\nabla\phi_1|^2+|\nabla\phi_2^2+|\nabla\psi|^2+|\nabla\eta_1|^2+|\nabla\eta_2|^2\big]\,d\Gamma\\ +2G_0(1-\varepsilon)^2R\int_{\Gamma_1}\big[\phi_1^2+\phi_2^2+\psi^2+\eta_1^2+\eta_2^2\big]\,d\Gamma.\end{aligned}\tag{4-44}$$

For $k\ge k_0>0$ we have, according to [Lagnese 1989, Lemma 2.1] and to trace theory,

$$\int_{\Gamma_1}\big[\phi_1^2+\phi_2^2+\psi^2+\eta_1^2+\eta_2^2\big]\,d\Gamma\le\gamma_1\big[a_0(\phi_1,\phi_2)+ka_1(\phi_1,\phi_2,\psi)+a_2(\psi,\eta_1,\eta_2)\big].\tag{4-45}$$

In addition,

$$\begin{aligned}\int_{\Gamma_1} m\cdot\nu\big[|\nabla\phi_1|^2+|\nabla\phi_2|^2+|\nabla\psi|^2+|\nabla\eta_1|^2+|\nabla\eta_2|^2\big]\,d\Gamma\\ \le\tilde\gamma_2\Big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+\int_{\Gamma_1}(\phi_1^2+\phi_2^2+\eta_1^2+\eta_2^2)\Big]\\ \le\gamma_2\big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\eta_1,\eta_2)\big],\end{aligned}\tag{4-46}$$

where the constants γ_1, γ_2 depend only on Ω, D, μ, and k_0, and

$$\begin{aligned}&a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\ &=2\frac{d}{dt}\mathrm{J}_2=\int_\Gamma m\cdot\nu\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\\ &\quad+k[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2]\\ &\quad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)(\eta_{2y}+\tfrac12\psi_y^2)^2\\ &\quad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)(\eta_{1y}\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma.\end{aligned}\tag{4-47}$$

From (4-42)–(4-47), we obtain the estimate

$$\begin{aligned}\Big|\int_{\Gamma_1}\big[&\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\&+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\Big|\\&\le-\frac{1}{2\xi}\frac{d}{dt}E_k(t)+\xi G_0R^2\gamma_2\big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big]\\&\qquad+\xi G_0\gamma_1(1-\varepsilon)^2R\big[a_0(\phi_1,\phi_2)+ka_1(\phi_1,\phi_2,\psi)+a_2(\psi,\eta_1,\eta_2)\big]\\&\le-\frac{1}{2\xi}\frac{d}{dt}E_k(t)+\xi G_0R^2\gamma_2\big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big]\\&\qquad+\xi G_0\gamma_1(1-\varepsilon)^2RE_k(t).\end{aligned}\tag{4-48}$$

Using (4-48) in (4-41), it follows that

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t)\le\frac{d}{dt}\mathrm{J}_0+\tfrac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{1}{2\xi}\frac{d}{dt}E_k(t)-\big(\tfrac{1}{2}-\xi\gamma_2G_0R^2\big)a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\-\big[\varepsilon-2\xi\gamma_1G_0(1-\varepsilon)^2R\big]E_k(t)-\big[\tfrac{1}{2}\varepsilon-\xi\gamma_2G_0R^2\big]\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big],\end{aligned}\tag{4-49}$$

where

$$c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)=2\frac{d}{dt}J_1=\rho h\int_{\Gamma_1}m\cdot\nu\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,d\Gamma.$$

From the definition of J_0 and the first of the geometric assumptions in (4-4), we have

$$\begin{aligned}\frac{d}{dt}\mathrm{J}_0=\tfrac{1}{2}\int_0^T\int_{\Gamma_0}m\cdot\nu\Big\{&D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\\&+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\&+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac{1}{2}\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2\\&\qquad+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma\,dt\\&+\tfrac{1}{4}(1-\mu)\int_{\Gamma_0}m\cdot\nu\left(\nu_2\frac{\partial\phi_1}{\partial\nu}+\nu_1\frac{\partial\phi_2}{\partial\nu}\right)^2d\Gamma\\&\le-\tfrac{1}{2}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}\tag{4-50}$$

where

$$\begin{aligned}&a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\&=\tfrac{1}{2}\int_0^T\int_{\Gamma_0}|m\cdot\nu|\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\&\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac{1}{2}\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2\\&\qquad\qquad+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma.\end{aligned}$$

Substituting (4-50) in the right-hand side of (4-49), one gets the estimate

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le \tfrac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \frac{1}{2\xi}\frac{d}{dt}E_k(t)\\ - \big(\tfrac{1}{2}-\xi\gamma_2G_0R^2\big)a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - [\varepsilon-2\xi\gamma_1G_0(1-\varepsilon)^2R]E_k(t)\\ - \big[\tfrac{1}{2}\varepsilon-\xi\gamma_2G_0R^2\big]\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big] - \tfrac{1}{2}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).\end{aligned}\tag{4-51}$$

Now, taking $\varepsilon < \frac{1}{2}$ and choosing $\xi > 0$ small enough such that

$$2\xi\gamma_1G_0(1-\varepsilon)^2R \le \tfrac{1}{2}\varepsilon, \quad \xi\gamma_2G_0R^2 \le \tfrac{1}{4}\varepsilon < \tfrac{1}{4},$$

we can guarantee from inequality (4-51) that

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le -\frac{1}{2\xi}\frac{d}{dt}E_k(t) - \tfrac{1}{2}\varepsilon E_k(t) + \tfrac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\ - \tfrac{1}{4}a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \tfrac{1}{4}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).\end{aligned}$$

Let us consider

$$F_{\varepsilon,\delta}(t) = E_k(t) + \delta\rho_\varepsilon(t)$$

with $\delta > 0$. Therefore,

$$\begin{aligned}\frac{d}{dt}F_{\varepsilon,\delta}(t) &= \frac{d}{dt}E_k(t) + \delta\frac{d}{dt}\rho_\varepsilon(t)\\ &= \Big(1-\frac{\delta}{2\xi}\Big)\frac{d}{dt}E_k(t) - \frac{\delta\varepsilon}{2}E_k(t) + \frac{\delta}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \frac{\delta}{4}a_\Gamma(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}$$

where

$$a_\Gamma(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).$$

From (4-5), we get

$$\begin{aligned}\frac{d}{dt}E_k(t) &= -\int_{\Gamma_1}[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2]\,d\Gamma\\ &\le -g_0\int_{\Gamma_1} m\cdot\nu[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2]\,d\Gamma\\ &\le -\frac{g_0}{\rho h}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}\tag{4-52}$$

provided $\frac{1}{12}h^2 \le 1$ (as we may assume). Therefore

$$\begin{aligned}\frac{d}{dt}F_{\varepsilon,\delta}(t) &= -\bigg[\frac{g_0}{\rho h}\Big(1-\frac{\delta}{2\xi}\Big)-\frac{\delta}{2}\bigg]c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \frac{\varepsilon\delta}{2}E_k(t) - \frac{\delta}{4}a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\ &\le -\frac{\varepsilon\delta}{2}E_k(t) - \frac{\delta}{4}\big[c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\big]\\ &= -\frac{\varepsilon\delta}{2}E_k(t) - \frac{\delta}{2}E_\Gamma(t),\end{aligned}\tag{4-53}$$

with $\delta > 0$ being chosen such that

$$\frac{g_0}{\rho h}\Big(1-\frac{\delta}{2\xi}\Big) - \frac{\delta}{2} \ge \frac{\delta}{4}.$$

Step 3: To get the exponential decay of $E_k(t)$ using inequality (4-53), we need to compare $E_k(t)$ and $F_{\varepsilon,\delta}(t)$. To carry this out, we use the definition (4-33) of $\rho_\varepsilon(t)$ and [Lagnese 1989, Lemma 2.1] to obtain

$$|\rho_\varepsilon(t)| \le C E_k(t),$$

where C depends on Ω, D, μ, and K_0 ($K \ge K_0 > 0$) but not on ε. Consequently

$$|F_{\varepsilon,\delta} - E(t)| = \delta\rho_\varepsilon(t) \le \delta C E_k(t).$$

Therefore,

$$(1-\delta C)E_k(t) \le F_{\varepsilon,\delta}(t) \le (1+\delta C)E_k(t).$$

Moreover, since

$$E_k(t) + \frac{1}{\varepsilon}E_\Gamma(t) \ge E_k(t),$$

one gets

$$\frac{d}{dt}F_{\varepsilon,\delta} \le -\omega F_{\varepsilon,\delta},$$

where $\omega = \delta\varepsilon/(2(1+\delta C))$. As a consequence of (4-33), (4-36) and of the choice of ε (see (4-40)), we conclude that there exist positive constants $C > 0$ and $\omega > 0$ such that

$$E_k(t) \le C E_k(0)\mathrm{e}^{-\omega t}$$

for every $t > 0$ and every solution of (1-1), (4-1), (1-3). □

5. Further comments and open problems

(1) Although we know the physical deduction for the nonlinear Mindlin–Timoshenko system (1-1)–(1-3), see for example [Lagnese and Lions 1988; Rahmani 2014], we are not aware of results concerning well-posedness and regularity for all $k > 0$. However, since our main goal was to give a positive response to the Lagnese–Lions conjecture, what we can say is that, for k large enough and for initial data in the space $\mathcal{X}$, the system (1-1)–(1-3) is very close to the known von Kármán system (1-5)–(1-7) (see Theorem 2.1). On the other hand, there is extensive literature dealing with well-posedness, regularity, stability, etc., for system (1-5)–(1-7); see [Favini et al. 1996; Lagnese 1989; Lagnese and Leugering 1991; Lasiecka 1998; Perla Menzala et al. 2002]. In Section 4 we analyzed the asymptotic behavior (as $t \to \infty$) for the solution of the nonlinear Mindlin–Timoshenko system with boundary feedback. To this end, we had to request an additional regularity for their solutions. For this reason, in all results of that section, we have used the expression "regular enough" to the solutions, in order to ensure that, under certain restrictions, the results hold. In our case, for instance, if we consider the solution $\{\phi_1(t), \phi_2(t), \psi(t), \eta_1(t), \eta_2(t)\} \in [H^2 \cap H^1_{\Gamma_0}]^2 \times [H^3 \cap H^1_{\Gamma_0}] \times [H^2 \cap H^1_{\Gamma_0}]^2$, the stability results hold. For the linear Mindlin–Timoshenko system, this issue was treated in [Lagnese 1989, Remark 3.1].

(2) In the proofs of Theorems 2.1, 3.1, and 4.5, we have considered the case where the initial data are fixed. The same results hold if we consider the case where they do depend on k, provided we assume the initial data $\{\phi^k_{10}, \phi^k_{11}, \phi^k_{20}, \phi^k_{21}, \psi^k_0, \psi^k_1, \eta^k_{10}, \eta^k_{11}, \eta^k_{20}, \eta^k_{21}\}$ to be such that the initial energy $E_k(0)$ remains bounded and such that they converge weakly to $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ in the corresponding spaces.

(3) It would be interesting to analyze whether the same stabilization results (Theorems 3.1, 4.5) hold considering the systems (3-1), (1-2), (1-3) and (1-1), (4-1), (1-3) with less damping terms. To eliminate some of these dissipative terms is a difficult task due to the complex nonlinearities involved. In this context, we can mention the works [Alabau-Boussouira 2007; Alabau-Boussouira and Léautaud 2012; Alabau-Boussouira et al. 2011; Ammar-Khodja et al. 2007; Soufyane 1999], which have obtained stability for some hyperbolic systems without damping terms in some of their equations.

(4) Another interesting and difficult problem is to obtain the same result in Theorem 3.1 when the damping mechanisms act in an arbitrary small region of the plate. The difficulty for this case, of course, consists in getting a unique continuation result for the Mindlin–Timoshenko system. On this subject, we mention [Cavalcanti et al. 2014; Charles et al. 2013; Geredeli and Lasiecka 2013; Komornik and Zuazua 1990; Zuazua 1990], which have obtained decay rates for the energy of various hyperbolic systems considering both linear and nonlinear localized damping terms.

References

[Alabau-Boussouira 2007] F. Alabau-Boussouira, "Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control", *NoDEA Nonlinear Differential Equations Appl.* **14**:5-6 (2007), 643–669. MR Zbl

[Alabau-Boussouira and Léautaud 2012] F. Alabau-Boussouira and M. Léautaud, "Indirect stabilization of locally coupled wave-type systems", *ESAIM Control Optim. Calc. Var.* **18**:2 (2012), 548–582. MR Zbl

[Alabau-Boussouira et al. 2011] F. Alabau-Boussouira, J. E. Muñoz Rivera, and D. da S. Almeida Júnior, "Stability to weak dissipative Bresse system", *J. Math. Anal. Appl.* **374**:2 (2011), 481–498. MR Zbl

[Ammar-Khodja et al. 2003] F. Ammar-Khodja, A. Benabdallah, J. E. Muñoz Rivera, and R. Racke, "Energy decay for Timoshenko systems of memory type", *J. Differential Equations* **194**:1 (2003), 82–115. MR Zbl

[Ammar-Khodja et al. 2007] F. Ammar-Khodja, S. Kerbal, and A. Soufyane, "Stabilization of the nonuniform Timoshenko beam", *J. Math. Anal. Appl.* **327**:1 (2007), 525–538. MR Zbl

[Araruna and Zuazua 2008] F. D. Araruna and E. Zuazua, "Controllability of the Kirchhoff system for beams as a limit of the Mindlin–Timoshenko system", *SIAM J. Control Optim.* **47**:4 (2008), 1909–1938. MR Zbl

[Araruna et al. 2010] F. D. Araruna, P. Braz e Silva, and E. Zuazua, "Asymptotic limits and stabilization for the 1D nonlinear Mindlin–Timoshenko system", *J. Syst. Sci. Complex.* **23**:3 (2010), 414–430. MR Zbl

[Bradley and Lasiecka 1992] M. Bradley and I. Lasiecka, "Local exponential stabilization for a non-linearly perturbed von Kármán plate", *Nonlinear Anal.* **18**:4 (1992), 333–343. MR Zbl

[Cavalcanti et al. 2014] M. M. Cavalcanti, V. N. Domingos Cavalcanti, F. A. Falcão Nascimento, I. Lasiecka, and J. H. Rodrigues, "Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping", *Z. Angew. Math. Phys.* **65**:6 (2014), 1189–1206. MR Zbl

[Charles et al. 2013] W. Charles, J. A. Soriano, F. A. Falcão Nascimento, and J. H. Rodrigues, "Decay rates for Bresse system with arbitrary nonlinear localized damping", *J. Differential Equations* **255**:8 (2013), 2267–2290. MR Zbl

[Chueshov and Lasiecka 2006] I. Chueshov and I. Lasiecka, "Global attractors for Mindlin–Timoshenko plates and for their Kirchhoff limits", *Milan J. Math.* **74** (2006), 117–138. MR Zbl

[Doyle 1997] J. F. Doyle, *Wave propagation in structures: spectral analysis using fast discrete fourier transforms*, Springer, 1997. Zbl

[Favini et al. 1996] A. Favini, M. A. Horn, I. Lasiecka, and D. Tataru, "Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation", *Differential Integral Equations* **9**:2 (1996), 267–294. MR Zbl

[Fernández Sare 2009] H. D. Fernández Sare, "On the stability of Mindlin–Timoshenko plates", *Quart. Appl. Math.* **67**:2 (2009), 249–263. MR Zbl

[Geredeli and Lasiecka 2013] P. G. Geredeli and I. Lasiecka, "Asymptotic analysis and upper semicontinuity with respect to rotational inertia of attractors to von Karman plates with geometrically localized dissipation and critical nonlinearity", *Nonlinear Anal.* **91** (2013), 72–92. MR Zbl

[Grobbelaar-Van Dalsen 2015] M. Grobbelaar-Van Dalsen, "Polynomial decay rate of a thermoelastic Mindlin–Timoshenko plate model with Dirichlet boundary conditions", *Z. Angew. Math. Phys.* **66**:1 (2015), 113–128. MR Zbl

[Kang 2013] J.-R. Kang, "Exponential decay for nonlinear von Kármán equations with memory", *Abstr. Appl. Anal.* **2013** (2013), art. id. 484596. MR Zbl

[Kim and Renardy 1987] J. U. Kim and Y. Renardy, "Boundary control of the Timoshenko beam", *SIAM J. Control Optim.* **25**:6 (1987), 1417–1429. MR Zbl

[Komornik and Zuazua 1990] V. Komornik and E. Zuazua, "A direct method for the boundary stabilization of the wave equation", *J. Math. Pures Appl.* (9) **69**:1 (1990), 33–54. MR Zbl

[Lagnese 1989] J. E. Lagnese, *Boundary stabilization of thin plates*, SIAM Studies in Applied Mathematics **10**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989. MR Zbl

[Lagnese and Leugering 1991] J. E. Lagnese and G. Leugering, "Uniform stabilization of a nonlinear beam by nonlinear boundary feedback", *J. Differential Equations* **91**:2 (1991), 355–388. MR Zbl

[Lagnese and Lions 1988] J. Lagnese and J.-L. Lions, *Modelling analysis and control of thin plates*, Recherches en Mathématiques Appliquées **6**, Masson, Paris, 1988. MR Zbl

[Lasiecka 1998] I. Lasiecka, "Weak, classical and intermediate solutions to full von Karman system of dynamic nonlinear elasticity", *Appl. Anal.* **68**:1-2 (1998), 121–145. MR Zbl

[Nicaise 2011] S. Nicaise, "Internal stabilization of a Mindlin–Timoshenko model by interior feedbacks", *Math. Control Relat. Fields* **1**:3 (2011), 331–352. MR Zbl

[Perla Menzala and Zuazua 1997] G. Perla Menzala and E. Zuazua, "Explicit exponential decay rates for solutions of von Kármán's system of thermoelastic plates", *C. R. Acad. Sci. Paris Sér. I Math.* **324**:1 (1997), 49–54. MR Zbl

[Perla Menzala and Zuazua 1999] G. Perla Menzala and E. Zuazua, "The beam equation as a limit of a 1-D nonlinear von Kármán model", *Appl. Math. Lett.* **12**:1 (1999), 47–52. MR Zbl

[Perla Menzala and Zuazua 2000a] G. Perla Menzala and E. Zuazua, "Timoshenko's beam equation as limit of a nonlinear one-dimensional von Kármán system", *Proc. Roy. Soc. Edinburgh Sect. A* **130**:4 (2000), 855–875. MR Zbl

[Perla Menzala and Zuazua 2000b] G. Perla Menzala and E. Zuazua, "Timoshenko's plate equation as a singular limit of the dynamical von Kármán system", *J. Math. Pures Appl.* (9) **79**:1 (2000), 73–94. MR Zbl

[Perla Menzala et al. 2002] G. Perla Menzala, A. F. Pazoto, and E. Zuazua, "Stabilization of Berger–Timoshenko's equation as limit of the uniform stabilization of the von Kármán system of beams and plates", *M2AN Math. Model. Numer. Anal.* **36**:4 (2002), 657–691. MR Zbl

[Rahmani 2014] L. Rahmani, "Modeling of the nonlinear vibrations of a stiffened moderately thick plate", *C. R. Math. Acad. Sci. Paris* **352**:3 (2014), 223–227. MR Zbl

[Rahmani 2015] L. Rahmani, "Reinforcement of a Mindlin–Timoshenko plate by a thin layer", *Z. Angew. Math. Phys.* **66**:6 (2015), 3499–3517. MR Zbl

[Simon 1987] J. Simon, "Compact sets in the space $L^p(0, T; B)$", *Ann. Mat. Pura Appl.* (4) **146** (1987), 65–96. MR Zbl

[Soufyane 1999] A. Soufyane, "Stabilisation de la poutre de Timoshenko", *C. R. Acad. Sci. Paris Sér. I Math.* **328**:8 (1999), 731–734. MR Zbl

[Zuazua 1990] E. Zuazua, "Exponential decay for the semilinear wave equation with locally distributed damping", *Comm. Partial Differential Equations* **15**:2 (1990), 205–235. MR Zbl

Received 21 Jul 2016. Revised 5 May 2017. Accepted 5 Sep 2017.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 11 No. 2 2018