ANALYSIS & PDE

Volume 11 No. 2 2018

LAURA CLADEK

RADIAL FOURIER MULTIPLIERS IN $\mathbb{R}^3$ AND $\mathbb{R}^4$

msp

ANALYSIS AND PDE
Vol. 11, No. 2, 2018
dx.doi.org/10.2140/apde.2018.11.467

RADIAL FOURIER MULTIPLIERS IN $\mathbb{R}^3$ AND $\mathbb{R}^4$

LAURA CLADEK

We prove that for radial Fourier multipliers $m : \mathbb{R}^3 \to \mathbb{C}$ supported compactly away from the origin, T_m is restricted strong type (p, p) if $K = \hat{m}$ is in $L^p(\mathbb{R}^3)$, in the range $1 < p < \frac{13}{12}$. We also prove an L^p characterization for radial Fourier multipliers in four dimensions; namely, for radial Fourier multipliers $m : \mathbb{R}^4 \to \mathbb{C}$ supported compactly away from the origin, T_m is bounded on $L^p(\mathbb{R}^4)$ if and only if $K = \hat{m}$ is in $L^p(\mathbb{R}^4)$, in the range $1 < p < \frac{36}{29}$. Our method of proof relies on a geometric argument that exploits bounds on sizes of multiple intersections of 3-dimensional annuli to control numbers of tangencies between pairs of annuli in three and four dimensions.

1. Introduction and statement of results

In this paper we study radial multiplier transformations whose symbol is compactly supported away from the origin. These are operators T_m defined via the Fourier transform by

$$\mathcal{F}[T_m f](\xi) = m(\xi)\hat{f}(\xi),$$

where the function $m : \mathbb{R}^d \to \mathbb{C}$ is bounded, measurable, radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$.

In the cases $p \neq 1, 2$, it is generally believed that it is impossible to give a reasonable characterization of all multiplier operators which are bounded on L^p. However, for *radial* Fourier multipliers, a characterization can be obtained for an appropriate range of p. Heo, Nazarov, and Seeger [Heo et al. 2011] proved a strikingly simple characterization of radial multipliers that are bounded on $L^p(\mathbb{R}^d)$ in dimensions $d \geq 4$ for $1 < p < (2d-2)/(d+1)$.

Theorem A. *Let $d \geq 2$. If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if the kernel $K = \hat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < (2d-2)/(d+1)$.*

The characterization in [Heo et al. 2011] was motivated by the earlier work [Garrigós and Seeger 2008], where the authors obtained a similar characterization of all convolution operators with radial kernels acting on the space L^p_{rad} of radial L^p functions, in the larger range $1 < p < 2d/(d+1)$.

The author would like to thank Andreas Seeger for helpful discussions and comments which improved the presentation of this paper. Part of this work was completed while the author was visiting the Institute for Computational and Experimental Research in Mathematics (ICERM), and the author is grateful to ICERM for its hospitality. Research supported in part by NSF Research and Training grant DMS 1147523.

MSC2010: 42B15.

Keywords: Fourier multipliers, radial functions, incidence geometry, local smoothing.

Theorem B. *Let $d \geq 2$. If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p_{\text{rad}}(\mathbb{R}^d)$ if and only if the kernel $K = \hat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < 2d/(d+1)$.*

This range $1 < p < 2d/(d+1)$ is the optimal range for their result to hold, since for $p \geq 2d/(d+1)$ one may construct radial kernels in L^p that have Fourier transforms which are supported compactly away from the origin but which are also unbounded. By the same reasoning, the range $1 < p < 2d/(d+1)$ is also the largest possible range in which one could hope for the characterization from Theorem A to hold. Thus one might propose the following conjecture, which we will refer to as the "radial Fourier multiplier conjecture".

Conjecture 1.1. *Let $d \geq 2$. If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if the kernel $K = \hat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < 2d/(d+1)$.*

One can appreciate the strength of this conjecture by noting that since $2d/(d+1)$ is the critical value for the Bochner–Riesz conjecture, the Bochner–Riesz conjecture (and hence also the restriction and Kakeya conjectures) would follow as a special case from Conjecture 1.1. However, the statement of Conjecture 1.1 is far more general than the Bochner–Riesz conjecture, since it makes no a priori assumptions whatsoever on the regularity of the multiplier.

The arguments of [Heo et al. 2011] did not yield any results about radial Fourier multipliers in $\mathbb{R}^3$. We will improve a key lemma of that paper in three dimensions to obtain a characterization of restricted strong type (p, p) boundedness of compactly supported radial Fourier multipliers $m : \mathbb{R}^3 \to \mathbb{C}$, in the range $1 < p < \frac{13}{12}$.

Theorem 1.2. *Let m be a radial Fourier multiplier in $\mathbb{R}^3$ supported in $\{\frac{1}{2} < |\xi| < 2\}$ and let $K = \mathcal{F}^{-1}[m]$. Then for $1 < p < \frac{13}{12}$, if $K \in L^p$ then the multiplier operator T_m is restricted strong type (p, p), and moreover,*

$$\|K * f\|_{L^p(\mathbb{R}^3)} \lesssim_p \|K\|_{L^p(\mathbb{R}^3)} \|f\|_{L^{p,1}(\mathbb{R}^3)}.$$

Remark 1.3. Our proof will also show that $\|K * f\|_{L^p} \lesssim_p \|K\|_{L^{p,1}} \|f\|_{L^p}$, and we expect that $\|K\|_{L^{p,1}}$ could be improved to $\|K\|_{L^p}$.

We will also prove a full L^p characterization for compactly supported radial Fourier multipliers in $\mathbb{R}^4$ in the range $1 < p < \frac{36}{29}$, which improves on Heo, Nazarov, and Seeger's result.

Theorem 1.4. *Let m be a radial Fourier multiplier in $\mathbb{R}^4$ supported in $\{\frac{1}{2} < |\xi| < 2\}$ and let $K = \mathcal{F}^{-1}[m]$. Then for $1 < p < \frac{36}{29}$, if $K \in L^p(\mathbb{R}^4)$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^4)$, and moreover,*

$$\|K * f\|_{L^p(\mathbb{R}^4)} \lesssim_p \|K\|_{L^p(\mathbb{R}^4)} \|f\|_{L^p(\mathbb{R}^4)}.$$

Our proofs of Theorems 1.2 and 1.4 refine the arguments of [Heo et al. 2011] while simultaneously incorporating new geometric input. A key divergence from their arguments is the exploitation of the underlying "tensor product structure" inherent in the problem, a notion which will become clearer later. This, combined with a geometric argument involving sizes of multiple intersections of 3-dimensional annuli, allows one to take advantage of improved scalar product estimates which were not used by Heo et al.

However, since we exploit the tensor product structure of the problem, we are currently not able to deduce any local smoothing results for the wave equation as corollaries, as was able to be done in [Heo et al. 2011].

The outline of the paper is as follows. The first portion of the paper will be devoted to the proof of Theorem 1.2, which is less technical than the proof of Theorem 1.4. The second portion will give the proof of Theorem 1.4. At the end, we provide as an Appendix the proof of the geometric lemma used in the proofs of both theorems.

2. Preliminaries and reductions

We will now collect some necessary preliminary results and reductions. Versions of these results can be found in [Heo et al. 2011], but we reproduce them here for completeness. In general, this section will very closely follow that paper, and for convenience we choose to adopt similar notation.

Discretization and density decomposition of sets. The first step will be to discretize our problem, and in preparation for this we will first need to introduce some notation. Let $\mathcal{Y}$ be a 1-separated set of points in $\mathbb{R}^3$ and let $\mathcal{R}$ be a 1-separated set of radii ≥ 1. Let $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ be a finite set that is also a *product*, i.e., $\mathcal{E} = \mathcal{E}_Y \times \mathcal{E}_R$, where $\mathcal{E}_Y \subset \mathcal{Y}$ and $\mathcal{E}_R \subset \mathcal{R}$. (The assumption that $\mathcal{E}$ is a product was not used in [Heo et al. 2011], but will be crucial for our argument.)

Let

$$u \in \mathcal{U} = \{2^\nu : \nu = 0, 1, 2, \ldots\}$$

be a collection of dyadic indices. For each k, let $\mathfrak{B}_k$ denote the collection of all 4-dimensional balls of radius $\leq 2^k$. For a ball B, let rad B denote the radius of B. Following [Heo et al. 2011], define

$$\begin{aligned}
\mathcal{R}_k &:= \mathcal{R} \cap [2^k, 2^{k+1}),\\
\mathcal{E}_k &:= \mathcal{E} \cap (\mathcal{Y} \times \mathcal{R}_k),\\
\widehat{\mathcal{E}}_k(u) &:= \{(y, r) \in \mathcal{E}_k : \exists B \in \mathfrak{B}_k \text{ such that } \#(\mathcal{E}_k \cap B) \geq u \text{ rad } B\},\\
\mathcal{E}_k(u) &= \widehat{\mathcal{E}}_k(u) \setminus \bigcup_{\substack{u' \in \mathcal{U}\\ u' > u}} \widehat{\mathcal{E}}_k(u').
\end{aligned}$$

We will refer to u as the *density* of the set $\mathcal{E}_k(u)$. Note that we have the decomposition

$$\mathcal{E}_k = \bigcup_{u \in \mathcal{U}} \mathcal{E}_k(u).$$

Let σ_r denote the surface measure on rS^2, the 2-sphere of radius r centered at the origin. Now fix a smooth, radial function ψ_0 which is supported in the ball centered at the origin of radius $\frac{1}{10}$ such that $\hat{\psi}_0$ vanishes to order 40 at the origin. Let $\psi = \psi_0 * \psi_0$. For $y \in \mathcal{Y}$ and $r \in \mathcal{R}$, define

$$F_{y,r} = \sigma_r * \psi(\cdot - y).$$

For a given function $c : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$, further define

$$G_{u,k} := \sum_{(y,r) \in \mathcal{E}_k(u)} c(y, r) F_{y,r}, \quad G_u := \sum_{k \geq 0} G_{u,k} \quad \text{and} \quad G_k := \sum_{u \in \mathcal{U}} G_{u,k}.$$

An interpolation lemma. As a preliminary tool, we will need the following dyadic interpolation lemma.

Lemma 2.1. *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j\in\mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that for all j, the inequality*

$$\|F_j\|_{p_v}^{p_v} \le 2^{jp_v} M^{p_v} s_j \tag{2-1}$$

holds for $v = 0$ and $v = 1$. Then for all $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\left\| \sum_j F_j \right\|_p^p \le C^p M^p \sum_j 2^{jp} s_j. \tag{2-2}$$

The discretized L^p inequality. Our goal is to prove the following proposition, which we will see implies our main result for compactly supported multipliers.

Proposition 2.2. *Let $\mathcal{E}$ and $\mathcal{E}_k$ be as above (recall that $\mathcal{E}$ has product structure). Let $c : \mathcal{E} \to \mathbb{C}$ be a function satisfying $|c(y, r)| \le 1$ for all $(y, r) \in \mathcal{E}$. Then for $1 < p < \frac{13}{12}$,*

$$\left\| \sum_{(y,r)\in\mathcal{E}} c(y, r) F_{y,r} \right\|_p^p \lesssim_p \sum_k 2^{2k} \#\mathcal{E}_k.$$

Using the dyadic interpolation lemma (Lemma 2.1), we obtain the following corollary.

Corollary 2.3. *Let E be any measurable set of finite measure, and χ_E its characteristic function. Suppose that f is a measurable function satisfying $|f| \le \chi_E$. Then for $1 < p < \frac{13}{12}$, we have*

$$\left\| \sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} \gamma(r) f(y) F_{y,r} \right\|_p \lesssim \left(\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} |\gamma(r)\chi_E(y)|^p r^2 \right)^{1/p}. \tag{2-3}$$

Also

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty h(r) f(y) F_{y,r} \, dr \, dy \right\|_p \lesssim \left(\int_{\mathbb{R}^d} \int_1^\infty |h(r)\chi_E(y)|^p r^2 \, dr \, dy \right)^{1/p}. \tag{2-4}$$

Proof that Proposition 2.2 implies Corollary 2.3. For $j \in \mathbb{Z}$, define the level sets

$$\mathcal{E}^j := \left\{ (y, r) \in \mathcal{Y} \times \mathcal{R} : 2^{j-1} < |\gamma(r)\chi_E(y)| \le 2^j \right\}.$$

Notice that $\mathcal{E}^j$ has product structure, so Proposition 2.2 implies that for $1 < p < \frac{13}{12}$,

$$\left\| \sum_{(y,r)\in\mathcal{E}^j} \gamma(r) f(y) F_{y,r} \right\|_p^p \lesssim_p 2^{jp} \sum_{(y,r)\in\mathcal{E}^j} r^2.$$

Now apply Lemma 2.1 with $F_j = \sum_{(y,r)\in\mathcal{E}^j} \gamma(r) f(y) F_{y,r}$ and $M = 1$ and $s_j = \sum_{(y,r)\in\mathcal{E}^j} r^2$ to obtain (2-3).

Now we prove (2-4). Let $y = z + w$ for $z \in \mathbb{Z}^3$ and $w \in Q_0 := [0,1)^3$ and $r = n + \tau$ for $n \in \mathbb{N}$ and $0 \le \tau < 1$. By Minkowski's inequality and (2-3),

$$\begin{aligned}\left\| \int_{\mathbb{R}^d} \int_1^\infty h(r) f(y) F_{y,r}\, dr\, dy \right\|_p &\lesssim \iint_{Q_0\times[0,1)} \left\| \sum_{z\in\mathbb{Z}^d} \sum_{n=1}^\infty h(n+\tau) f(z+w) F_{z+w,n+\tau} \right\|_p dw\, d\tau \\ &\lesssim \iint_{Q_0\times[0,1)} \left(\sum_{z\in\mathbb{Z}^d} \sum_{n=1}^\infty \left| h(n+\tau)\chi_E(z+w)\right|^p (n+\tau)^2 \right)^{1/p} dw\, d\tau \\ &\lesssim \left(\int_{\mathbb{R}^d} \int_1^\infty |h(r)\chi_E(y)|^p r^2\, dr\, dy \right)^{1/p},\end{aligned}$$

where in the last step we have used Hölder's inequality. □

Support size estimates vs. L^2 inequalities. As in [Heo et al. 2011], we will show that the functions $G_{u,k}$ either have relatively small support size or satisfy relatively good L^2 bounds. We begin with a support-size bound from that paper that improves as the density u increases.

Lemma C. *For all $u \in \mathcal{U}$, the Lebesgue measure of the support of $G_{u,k}$ is $\lesssim u^{-1} 2^{2k} \#\mathcal{E}_k$.*

We will prove the following L^2 inequality, which in some sense an improved version of Lemma 3.6 from [Heo et al. 2011], although the hypotheses are different since it is crucial that we assume that the underlying set $\mathcal{E}$ has product structure. This inequality improves as the density u decreases. In [Heo et al. 2011], the analogous L^2 inequality proved is

$$\|G_u\|_2^2 \lesssim u^{2/(d-1)} \log(2+u) \sum_k 2^{k(d-1)} \#\mathcal{E}_k, \tag{2-5}$$

and when $d = 3$ the term $u^{2/(d-1)}$ is equal to u. One may check that combining (2-5) with Lemma C as in the proof of Lemma 2.5 below yields no result in three dimensions. We use geometric methods to improve on (2-5) in three dimensions, and our argument will rely on Lemma A.1 proved later in the Appendix.

Lemma 2.4. *Let $\mathcal{E}$, $\mathcal{E}_k$, and G_u be as above (recall that $\mathcal{E}$ has product structure). Assume $|c(y,r)| \le 1$ for $(y,r) \in \mathcal{Y} \times \mathcal{R}$. Then for every $\epsilon > 0$,*

$$\|G_u\|_2^2 \lesssim_\epsilon u^{11/13+\epsilon} \sum_k 2^{2k} \#\mathcal{E}_k.$$

Combining Lemma C and Lemma 2.4, we obtain the following L^p bound.

Lemma 2.5. *For $p \le 2$, for every $\epsilon > 0$,*

$$\|G_u\|_p \lesssim_\epsilon u^{-(1/p-12/13-\epsilon)} \left(\sum_k 2^{2k} \#\mathcal{E}_k \right)^{1/p}.$$

Proof of Lemma 2.5 given Lemma C and Lemma 2.4. By Hölder's inequality,

$$\begin{aligned}\|G_u\|_p &\lesssim (\operatorname{meas}(\operatorname{supp} G_u))^{1/p-1/2} \|G_u\|_2 \\ &\lesssim_\epsilon u^{-1/p+1/2} u^{11/26+\epsilon} \left(\sum_k 2^{2k} \#\mathcal{E}_k \right)^{1/p} \lesssim_\epsilon u^{12/13-1/p+\epsilon} \left(\sum_k 2^{2k} \#\mathcal{E}_k \right)^{1/p}.\end{aligned}$$

□

Summing over $u \in \mathcal{U}$, we obtain Proposition 2.2. Thus to prove Proposition 2.2 it suffices to prove Lemma 2.4.

Compactly supported multipliers. Following [Heo et al. 2011], we now show how one may deduce Theorem 1.2 from Corollary 2.3. Suppose that $m : \mathbb{R}^3 \to \mathbb{C}$ is a bounded, measurable, radial function with compact support inside $\left\{\xi : \frac{1}{2} < |\xi| < 2\right\}$. Then $K = \mathcal{F}^{-1}[m]$ is radial, and so we may write $K(\,\cdot\,) = \kappa(|\cdot|)$ for some $\kappa : \mathbb{R} \to \mathbb{C}$. Fix a radial Schwartz function η_0 such that $\hat{\eta}_0(\xi) = 1$ on $\operatorname{supp} m$ and such that η_0 has Fourier support in $\left\{\frac{1}{4} < |\xi| < 4\right\}$. Set $\eta = \mathcal{F}^{-1}[(\hat{\psi})^{-1}\hat{\eta}_0]$. We have $K * f = \eta * \psi * K * f$. Let $K_0 = K\chi_{\{x:|x|\le 1\}}$ and write $K = K_0 + K_\infty$. Since $\|K_0\|_1 \lesssim \|K\|_p$, it suffices to show that the operator $f \mapsto \eta * \psi * K_\infty * f$ is restricted strong type (p, p) with operator norm $\lesssim_p \|K\|_p$. Let E be a measurable set of finite measure, and suppose that $|f| \le \chi_E$. We may write

$$\psi * K_\infty * f = \int_1^\infty \int \psi * \sigma_r(\,\cdot\, - y)\kappa(r) f(y)\, dy\, dr.$$

By Corollary 2.3, we have

$$\|\eta * \psi * K_\infty * f\|_p \lesssim_p \|\psi * K_\infty * f\|_p \lesssim_p \left(\int |\kappa(r)|^p r^2\, dr\right)^{1/p} \left(\int |\chi_E(y)|^p\, dy\right)^{1/p},$$

which implies the result of Theorem 1.2.

3. Proof of the L^2 inequality

We have shown in Section 2 that to prove our main result Theorem 1.2 it remains to prove Lemma 2.4, and this section is dedicated to the proof of that lemma. The proof will rely on a geometric lemma about sizes of multiple intersections of 3-dimensional annuli, which is stated and proved in the Appendix.

Estimates for scalar products. In order to obtain the desired L^2 estimate, we need to examine pairwise interactions of the form $\langle F_{y,r}, F_{y',r'}\rangle$. By applying Plancherel's theorem and writing $\widehat{F}_{y,r}$ and $\widehat{F}_{y',r'}$ as expressions involving Bessel functions, the authors of [Heo et al. 2011] obtained the following estimates for $|\langle F_{y,r}, F_{y',r'}\rangle|$.

Lemma 3.1. *For any choice of $r, r' > 1$ and $y, y' \in \mathbb{R}^3$,*

$$|\langle F_{y,r}, F_{y',r'}\rangle| \lesssim \frac{rr'}{1 + |y - y'| + |r - r'|}.$$

The proof of this lemma used only the decay and not the oscillation of the Bessel functions. By exploiting the oscillation of the Bessel functions, one may obtain the following improved bounds, which are crucial for our purposes. Since we will use this lemma in three and four dimensions, we state it in terms of dimension d, where the functions $F_{y,r}$ are defined analogously in d dimensions as they are defined previously in three dimensions.

Lemma 3.2. *For any choice of $r, r' > 1$ and $y, y' \in \mathbb{R}^d$ and any $N > 0$,*

$$|\langle F_{y,r}, F_{y',r'}\rangle| \le C_N (rr')^{(d-1)/2} (1 + |y - y'| + |r - r'|)^{-(d-1)/2} \sum_{\pm,\pm} \left(1 + \left|r \pm r' \pm |y - y'|\right|\right)^{-N}.$$

Proof of Lemma 3.2. We may write $\hat{\sigma}_1$ in terms of Bessel functions as $\hat{\sigma}_1(\xi) = B_d(|\xi|)$, where

$$B_d(s) = c_d s^{-(d-2)/2} J_{(d-2)/2}$$

and J denotes the standard Bessel function. This implies

$$\hat{\sigma}_r(\xi) = r^{d-1} B_d(r|\xi|).$$

Since $\hat{\psi}$ is radial, we may write $\hat{\psi}(\xi) = a(|\xi|)$ for some rapidly decaying function a that vanishes to high order (say $10d$) at the origin. By Plancherel, we have

$$\begin{aligned}\langle F_{y,r}, F_{y',r'}\rangle &= \int \hat{\sigma}_r(\xi)\hat{\sigma}_{r'}(\xi)|\hat{\psi}(\xi)|^2 e^{i\langle y'-y,\xi\rangle}\, d\xi \\ &= c_d(rr')^{d-1}\int B_d(r\rho)B_d(r'\rho)B_d(|y-y'|\rho)|a(\rho)|^2\rho^{d-1}\, d\rho.\end{aligned}$$

We will use the following well-known asymptotic expansion, which holds for $|x| \geq 1$ and any M:

$$B_d(x) = \sum_{\nu=0}^{M}(c^+_{\nu,k,d}e^{ix} + c^-_{\nu,k,d}e^{-ix})x^{-\nu-(d-1)/2} + x^{-M}E_{M,k,d}(x)$$

where for any $k_1 \geq 0$,

$$|E^{(k_1)}_{M,k,d}(x)| \leq C(M,k,k_1,d).$$

Using this expansion together with the higher order of vanishing of a at the origin, one sees that there is a fixed Schwartz function η so that we obtain for any $N > 0$,

$$\langle F_{y,r}, F_{y',r'}\rangle \lesssim (rr')^{(d-1)/2}(1+|y-y'|)^{-(d-1)/2}\sum_{\pm,\pm}\eta(r \pm r' \pm |y-y'|) + (1+|r-r'|+|y-y'|)^{-N}.$$

In fact, we may take η to be the Fourier transform of $|a(\cdot)|^2\rho^{\alpha(d)}$ for some appropriate exponent $\alpha(d)$. □

Another preliminary reduction. Recall that our goal is to estimate the L^2 norm of $G_u = \sum_{k\geq 0} G_{u,k}$. Let $N(u)$ be a sufficiently large number to be chosen later (it will be some harmless constant depending on u that is essentially $O(\log(2+u))$). We split the sum in k as $\sum_{k\leq N(u)} G_{u,k} + \sum_{k>N(u)} G_{u,k}$ and apply Cauchy–Schwarz to obtain

$$\Big\|\sum_k G_{u,k}\Big\|_2^2 \lesssim N(u)\Big[\sum_k \|G_{u,k}\|_2^2 + \sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|\Big]. \tag{3-1}$$

We may thus separately estimate $\sum_k \|G_{u,k}\|_2^2$ and $\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|$, which divides the proof of the L^2 estimate into two cases, the first being the case of "comparable radii" and the second being the case of "incomparable radii".

Comparable radii. We will first estimate $\sum_k \|G_{u,k}\|_2^2$. Our goal will be to prove the following lemma.

Lemma 3.3. *For every $\epsilon > 0$,*

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}. \tag{3-2}$$

Fix k and u. We first observe that for $(y, r), (y', r') \in \mathcal{E}_k(u)$, we have $\langle F_{y,r}, F_{y',r'}\rangle = 0$ unless $|(y, r) - (y', r')| \le 2^{k+5}$. To estimate $\|G_{u,k}\|_2^2$ for a fixed k, we would thus like to bound

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

for all $0 \le m \le k + 4$.

Now fix $m \le k + 4$. Let $\mathcal{Q}_{u,k,m}$ be a collection of almost disjoint cubes $Q \subset \mathbb{R}^4$ of side length 2^{m+5} such that $\mathcal{E}_k(u) \subset \bigcup_{Q\in\mathcal{Q}_{u,k,m}} Q$ and so that every Q has nonempty intersection with $\mathcal{E}_k(u)$. Let Q^* denote the 2^5-dilate of Q and $\mathcal{Q}^*_{u,k,m}$ the corresponding collection of dilated cubes. Observe that

$$\begin{aligned}\|G_{u,k}\|_2^2 &\lesssim \sum_{0\le m\le k+4}\Bigg(\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|\Bigg) + \sum_{(y,r)\in\mathcal{E}_k(u)} \|F_{y,r}\|_2^2\\ &\lesssim \sum_{0\le m\le k+4}\Bigg(\sum_{Q\in\mathcal{Q}_{u,k,m}}\Bigg(\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|\Bigg)\Bigg) + \sum_{(y,r)\in\mathcal{E}_k(u)} \|F_{y,r}\|_2^2. \qquad (3\text{-}3)\end{aligned}$$

Now we introduce some terminology that will be useful. For a subset $\mathcal{S} \subset \mathcal{Y} \times \mathcal{R}$, define its $\mathcal{Y}$- and $\mathcal{R}$-projections by

$$\mathcal{S}_Y = \{y \in \mathcal{Y} : \exists\, (y, r) \in \mathcal{S}\},$$
$$\mathcal{S}_R = \{r \in \mathcal{R} : \exists\, (y, r) \in \mathcal{S}\}.$$

Also define the *product-extension* $\mathcal{S}^\times$ of $\mathcal{S} \subset \mathcal{Y} \times \mathcal{R}$ to be the set $\mathcal{S}_Y \times \mathcal{S}_R$. We also define some parameters associated with a fixed $Q \in \mathcal{Q}_{u,k,m}$. Let $N_{R,Q}$ be the cardinality of the $\mathcal{R}$-projection of $\mathcal{E}_k \cap Q^*$, i.e.,

$$N_{R,Q} := \#((\mathcal{E}_k \cap Q^*)_R) = \#\{r : \exists (y, r) \in \mathcal{E}_k \cap Q^*\}.$$

Similarly define

$$N_{Y,Q} := \#((\mathcal{E}_k \cap Q^*)_Y) = \#\{y : \exists (y, r) \in \mathcal{E}_k \cap Q^*\}.$$

We also note the following important observation, which we will use repeatedly. Using the definition of the sets $\mathcal{E}_k(u)$ and the fact that $\mathcal{E}_k$ has product structure, one may see that if $Q \in \mathcal{Q}_{u,k,m}$ is such that $(\mathcal{E}_k(u) \cap Q^*)$ is nonempty, then

$$|N_{Y,Q} \cdot N_{R,Q}| \lesssim |\mathcal{E}_k \cap Q^*| \lesssim u2^m. \qquad (3\text{-}4)$$

We remark that the product structure of the sets $\mathcal{E}_k$ is related to the "tensor product structure" intrinsic to radial Fourier multipliers, mentioned in Section 1. Now with (3-3) in mind, we will prove the following lemma.

Lemma 3.4. *For each $Q \in \mathcal{Q}_{u,k,m}$, we have the estimates*

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim N_{R,Q}(\#(\mathcal{E}_k\cap Q^*))2^{2(k-m/2)}(m\log(u))\max(u^{5/6}2^{5m/6}, u2^{m/2}) \quad (3\text{-}5)$$

and

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k\cap Q^*))u2^m(N_{R,Q})^{-1}. \tag{3-6}$$

We will then choose the better estimate from Lemma 3.4 depending on $N_{R,Q}$ and sum over all $Q\in\mathcal{Q}_{u,k,m}$ and then over all $m\ge u^a$, where a is a number to be chosen later. We will then use other methods to deal with the case $m\le u^a$, from which we will then obtain Lemma 3.3.

Proof of Lemma 3.4. We will first prove (3-5). By incurring a factor of $N_{R,Q}^2$, to estimate

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

it suffices to estimate, for a fixed pair r_1, r_2,

$$\sum_{\substack{(y,r_1),(y',r_2)\in(\mathcal{E}_k(u)^\times\cap Q^*)\\ 2^m\le|(y,r_1)-(y',r_2)|\le 2^{m+1}}} |\langle F_{y,r_1}, F_{y',r_2}\rangle|,$$

i.e., to restrict (y, r) and (y', r') to lie in fixed rows of the product-extension of $\mathcal{E}_k(u)\cap Q^*$. (Our estimates will not depend on the particular choice of r_1 and r_2.)

Now, referring to the estimate in Lemma 3.2, we see that for fixed y, r_1, r_2 we have that $|\langle F_{y,r_1}, F_{y',r_2}\rangle|$ decays rapidly as y' moves away from the set $\{y' : |y-y'|=|r_1-r_2| \text{ or } |y-y'|=r_1+r_2\}$, which is contained in a union of two annuli of thickness 2 and radii $|r_1-r_2|$ and r_1+r_2 centered at y.

Let $s\ge 0$, fix $t\le 2^{m+10}$, and define $K_k(Q,s,t)$ to be the number of points $y\in(\mathcal{E}_k(u)\cap Q^*)_Y$ such that there are at least 2^s many points $y'\in(\mathcal{E}_k\cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$K_k(Q,s,t) := \#\Big\{y\in(\mathcal{E}_k(u)\cap Q^*)_Y : \text{there exist at least } 2^s \text{ many points } y'\in(\mathcal{E}_k\cap Q^*)_Y \text{ such that } \big||y'-y|-\big(t+\tfrac{3}{2}\big)\big|\le\tfrac{3}{2}\Big\}.$$

In view of the observation in the previous paragraph, for a given s and a fixed number $t\le 2^{m+10}$, we would like to prove a bound on $K_k(Q,s,t)$. Our bound will depend on s and m but be independent of the choice of $t\le 2^{m+10}$. For this reason, we define the quantity

$$K_k^*(Q,s) := \max_{0\le t\le 2^{m+10}} K_k(Q,s,t),$$

and we will see that $K_k^*(Q,s)$ satisfies the same bound we prove for $K_k(Q,s,t)$. Our bound for $K_k(Q,s,t)$ will decay as 2^s gets larger and closer to $N_{Y,Q}$; in other words, "most" of the points y in $(\mathcal{E}_k(u)\cap Q^*)_Y$ cannot have a large proportion of other points in $(\mathcal{E}_k\cap Q^*)_Y$ lie in the annulus of inner radius t and thickness 3 centered at y. If we take $t=|r_1-r_2|$ or $t=r_1+r_2$, we see that this implies that "most" of the $F_{y,r}$ with $(y,r)\in(\mathcal{E}_k(u)\cap Q^*)_Y\times\{r_1\}$ do not "interact badly" (where by badly we mean to the worst possible extent allowed by Lemma 3.2, i.e., internal tangencies of annuli) with most of the other $F_{y',r'}$ where $(y',r')\in(\mathcal{E}_k\cap Q^*)_Y\times\{r_2\}$. This will allow us to obtain (3-5), which is a good estimate in the case that $N_{R,Q}$ is small.

More precisely, we will prove

$$K_k^*(Q,s) \lesssim \max\big[u2^m N_{Y,Q}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q}2^{-s}\big]. \tag{3-7}$$

Combining this with the trivial bound $K_k^*(Q,s) \lesssim N_{Y,Q}$ yields

$$K_k^*(Q,s) \lesssim \max\big[\min(u2^m N_{Y,Q}^{5/3}2^{-2s}, N_{Y,Q}), \min(u2^{m/2}N_{Y,Q}2^{-s}, N_{Y,Q})\big]. \tag{3-8}$$

Note that (3-7) gives decay in the number of points $K_k^*(Q,s)$ (i.e., $K_k^*(Q,s) \ll N_{Y,Q}$) if we have that

(1) $N_{Y,Q}^{5/3}2^{-2s}u2^m \ll N_{Y,Q}$, i.e., if $2^s \gg N_{Y,Q}^{1/3}u^{1/2}2^{m/2}$, and also

(2) $N_{Y,Q}2^{-s}u2^{m/2} \ll N_{Y,Q}$, i.e., if $2^s \gg u2^{m/2}$.

Using Lemma 3.2, we may bound

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\quad\lesssim \sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R}\Bigg(\sum_{\substack{y,y'\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ 2^m\le|(y,r_1)-(y',r_2)|\le 2^{m+1}}} |\langle F_{y,r_1}, F_{y',r_2}\rangle|\Bigg) \\
&\quad\lesssim 2^{2(k-m/2)}\sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R}\Bigg(\sum_{0\le a\le m+10}\Bigg(\sum_{y\in(\mathcal{E}_k(u)\cap Q^*)_Y}\ \sum_{\substack{y'\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ \min_{\pm,\pm}\left(1+\left|r_1\pm r_2\pm|y-y'|\right|\right)\approx 2^a}} 2^{-aN}\Bigg)\Bigg) \\
&\quad\lesssim 2^{2(k-m/2)}\sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R}\Bigg(\sum_{0\le a\le m+10}2^{-aN}\Bigg(\sum_{s\ge 0:2^s\le 2N_{Y,Q}} K_k^*(Q,s)2^s\Bigg)\Bigg) \\
&\quad\lesssim 2^{2(k-m/2)}N_{Q,R}^2\sum_{s\ge 0:2^s\le 2N_{Y,Q}} K_k^*(Q,s)2^s.
\end{aligned} \tag{3-9}$$

Assuming (3-8) holds, we have

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\quad\lesssim N_{R,Q}^2 2^{2(k-m/2)}\sum_{s\ge 0:2^s\lesssim N_{Y,Q}} \max\big[\min(u2^m N_{Y,Q}^{5/3}2^{-s},\ N_{Y,Q}2^s), \min(u2^{m/2}N_{Y,Q}, N_{Y,Q}2^s)\big] \\
&\quad\lesssim N_{R,Q}^2 2^{2(k-m/2)}\max\Bigg[\sum_{s\ge 0:2^s\lesssim N_{Y,Q}}\min(u2^m N_{Y,Q}^{5/3}2^{-s}, N_{Y,Q}2^s),\ \sum_{s\ge 0:2^s\lesssim N_{Y,Q}}\min(u2^{m/2}N_{Y,Q}, N_{Y,Q}2^s)\Bigg]
\end{aligned} \tag{3-10}$$

Now, note that $u2^m N_{Y,Q}^{5/3}2^{-s} \ge N_{Y,Q}2^s$ if and only if $2^s \le u^{1/2}2^{m/2}N_{Y,Q}^{1/3}$. Thus choosing the better estimate in the term $\min(u2^m N_{Y,Q}^{5/3}2^{-s}, N_{Y,Q}2^s)$ depending on s yields that

$$\sum_{s\ge 0:2^s\lesssim N_{Y,Q}}\min(u2^m N_{Y,Q}^{5/3}2^{-s}, N_{Y,Q}2^s) \lesssim u^{1/2}2^{m/2}N_{Y,Q}^{4/3}.$$

Note that $u2^{m/2}N_{Y,Q} \geq N_{Y,Q}2^s$ if and only if $2^s \leq u2^{m/2}$. Thus choosing the better estimate in the term $\min(u2^{m/2}N_{Y,Q}, N_{Y,Q}2^s)$ depending on s yields that

$$\sum_{s\geq 0: 2^s \lesssim N_{Y,Q}} \min(u2^{m/2}N_{Y,Q}, N_{Y,Q}2^s) \lesssim \log(N_{Y,Q})N_{Y,Q}\, u2^{m/2}.$$

It follows that the left-hand side of (3-10) is bounded by

$$\begin{aligned} N_{R,Q}^2 2^{2(k-m/2)} N_{Y,Q} \log(N_{Y,Q}) \max(N_{Y,Q}^{1/3} u^{1/2} 2^{m/2},\ u2^{m/2}) \\ \lesssim N_{R,Q}^2 2^{2(k-m/2)} N_{Y,Q} (m\log(u)) \max(u^{5/6}2^{5m/6},\ u2^{m/2}) \\ \lesssim N_{R,Q}(\#(\mathcal{E}_k \cap Q^*)) 2^{2(k-m/2)} (m\log(u)) \max(u^{5/6}2^{5m/6},\ u2^{m/2}), \quad (3\text{-}11) \end{aligned}$$

which proves (3-5). This will be a good estimate when $N_{R,Q}$ is small.

Thus to prove (3-5) it remains to prove (3-7). We will in fact prove (3-7) with $K_k^*(Q,s)$ replaced by $K_k(Q,s,t)$, uniformly in $t \leq 2^{m+10}$. Fix $t \leq 2^{m+10}$ and let $j = \lceil \log_2(t) \rceil$ and cover $(\mathcal{E}_k(u) \cap Q^*)_Y$ by $\lesssim 2^{3(m-j)}$ many 3-dimensional almost disjoint balls of radius 2^{j+5}; denote this collection of balls as $\mathfrak{B} = \{B_i\}$. For each i, we define a collection of "special" points $A_i(Q,s,t)$ to be the set of all points $y \in (\mathcal{E}_k(u) \cap Q^*)_Y \cap B_i$ such that there are $\geq 2^s$ many points $y' \in (\mathcal{E}_k \cap Q^*)_Y$ such that y' lies in the annulus of radius t and thickness 3 centered at y. That is, we define

$$\begin{aligned} A_{k,i}(Q,s,t) := \big\{ y \in (\mathcal{E}_k(u) \cap Q^*)_Y \cap B_i :\ &\text{there exist at least } 2^s \text{ many points} \\ &y' \in (\mathcal{E}_k \cap Q^*)_Y \text{ such that } \big||y'-y| - \big(t+\tfrac{3}{2}\big)\big| \leq \tfrac{3}{2}\big\}. \end{aligned}$$

Let $K_{k,i}(Q,s,t)$ denote the cardinality of $A_{k,i}(Q,s,t)$. Now cover each B_i with $\lesssim 2^{3(j-l)}$ many almost disjoint 3-dimensional balls $\{B_{i,\alpha}\}_\alpha$ of radius 2^l for some $l \leq j$. Each such ball contains at most $u2^l$ many points of $A_{k,i}(Q,s,t)$, so for a fixed i there must be $\gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}$ many balls $B_{i,\alpha}$ that contain at least one point in $A_{k,i}(Q,s,t)$. Thus there must be at least $\gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}$ many such points in $B_i \cap A_{k,i}(Q,s,t)$ spaced apart by $\gtrsim 2^l$; call this set $D_{k,i}(Q,s,t)$. But by Lemma A.1, which we prove later in the Appendix, the size of 3-fold intersections of annuli of radius $t \approx 2^j$ and thickness 3 spaced apart by $\approx 2^l$ with centers lying in a ball of radius 2^{j-5} is bounded above by $2^{3(j-l)}$ provided that $l \geq j/2 + 20$.

It follows that if $l \geq j/2+20$, then for each of these $\approx K_{k,i}(Q,s,t)(u2^l)^{-1}$ many points $p \in D_{k,i}(Q,s,t)$, there can be

$$\lesssim K_{k,i}(Q,s,t)^2 (u2^l)^{-2} 2^{3(j-l)}$$

points lying inside the t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}(Q,s,t)$. This implies that if $N_{Y,Q,i}$ denotes the cardinality of $(\mathcal{E}_k \cap Q^*)_Y \cap B_i^*$, where $B_i^* = 10B_i$, then we have

$$N_{Y,Q,i} \gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}2^s, \quad (3\text{-}12)$$

which is essentially 2^s times the number of points in $D_{k,i}(Q,s,t)$, provided that 2^s is much bigger than the total number of points lying inside a t-annulus centered at p that are simultaneously contained in at

least two other different t-annuli centered at points in $D_{k,i}(Q,s,t)$, i.e., provided that

$$K_{k,i}(Q,s,t)^2(u2^l)^{-2}2^{3(j-l)} \ll 2^s \tag{3-13}$$

and

$$l \geq j/2 + 20.$$

Solving for 2^l in (3-13) yields

$$2^l \gg K_{k,i}(Q,s,t)^{2/5}2^{3j/5}u^{-2/5}2^{-s/5}. \tag{3-14}$$

Thus choosing a minimal l such that

$$2^l \gg \max\big[K_{k,i}(Q,s,t)^{2/5}2^{3j/5}u^{-2/5}2^{-s/5},\ 2^{j/2}\big]$$

for a sufficiently large implied constant and substituting into (3-12) yields

$$K_{k,i}(Q,s,t) \lesssim \max\big[u2^m N_{Y,Q,i}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,i}2^{-s}\big], \tag{3-15}$$

and summing over all i and using the almost-disjointness of the B_i^* gives

$$K_k(Q,s,t) \lesssim \max\big[u2^m N_{Y,Q}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q}2^{-s}\big]. \tag{3-16}$$

Taking the maximum over all $0 \leq t \leq 2^{m+10}$ proves (3-7) and hence also (3-5).

It remains to prove (3-6), which will be a good estimate in the case that $N_{R,Q}$ is large. For a fixed $(y,r) \in Q^*$ and a fixed $y' \in (\mathcal{E}_k(u) \cap Q^*)_Y$, there are at most two values of r' away from which $\langle F_{y,r}, F_{y',r'}\rangle$ decays rapidly. Thus using Lemma 3.2 we may estimate

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\quad\lesssim \sum_{0\leq a\leq m+10}\Bigg(\sum_{(y,r)\in(\mathcal{E}_k(u)\cap Q^*)}\Bigg(\sum_{y'\in(\mathcal{E}_k(u)\cap Q^*)_Y}\Bigg(\sum_{\substack{r'\in(\mathcal{E}_k(u)\cap Q^*)_R\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}\\ \min_{\pm,\pm}(1+|r\pm r'\pm|y-y'||)\approx 2^a}} 2^{-Na}2^{2(k-m/2)}\Bigg)\Bigg)\Bigg) \\
&\quad\lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k(u)\cap Q^*))N_{Y,Q} \lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k(u)\cap Q^*))u2^m(N_{Q,R})^{-1},
\end{aligned} \tag{3-17}$$

and the proof of (3-6) is complete. □

We will now use Lemma 3.4 to prove Lemma 3.3.

Proof of Lemma 3.3. Fix an $a > 0$ to be determined later. As in [Heo et al. 2011], we let $G_k = \sum_\mu G_{k,\mu}$, where for each positive integer μ we set

$$\begin{gathered}
I_{k,\mu} = \big[2^k + (\mu-1)u^a, 2^k + \mu u^a\big), \\
\mathcal{E}_{k,\mu} = \mathcal{E}_k \cap (\mathcal{Y} \times I_{k,\mu}), \\
G_{k,\mu} = \sum_{(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)F_{y,r} \quad\text{and}\quad G_{k,\mu,r} = \sum_{y:(y,r)\in\mathcal{E}_k} c(y,r)F_{y,r}.
\end{gathered}$$

We have

$$\|G_k\|_2^2 \lesssim \Big\| \sum_\mu G_{k,\mu} \Big\|_2^2 \lesssim \sum_\mu \|G_{k,\mu}\|_2^2 + \sum_{\mu' > \mu+10} |\langle G_{k,\mu'}, G_{k,\mu}\rangle|. \tag{3-18}$$

By Cauchy–Schwarz,

$$\|G_{k,\mu}\|_2^2 \lesssim u^a \sum_{r \in \mathcal{I}_{k,\mu} \cap \mathcal{R}} \|G_{k,\mu,r}\|_2^2.$$

Write

$$G_{k,\mu,r} = \Big(\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)\psi_0(\cdot - y) \Big) * (\sigma_r * \psi_0).$$

By the Fourier decay of σ_r and the order of vanishing of ψ_0 at the origin, we have

$$\|\hat{\sigma}_r \hat{\psi}_0\|_\infty \lesssim r.$$

Since the square of the L^2 norm of $\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)\psi_0(\cdot - y)$ is $\lesssim \#\{y \in \mathcal{Y} : (y,r) \in \mathcal{E}_{k,\mu}\}$, we have

$$\sum_\mu \|G_{k,\mu}\|_2^2 \lesssim u^a \sum_\mu \sum_{r \in \mathcal{I}_{k,\mu}\cap\mathcal{R}} \|G_{k,\mu,r}\|_2^2 \lesssim u^a 2^{2k} \#\mathcal{E}_k. \tag{3-19}$$

By (3-18), it remains to estimate $\sum_{\mu' > \mu+10} |\langle G_{k,\mu'}, G_{k,\mu}\rangle|$.

Fix $\epsilon > 0$. We will use (3-5) when $N_{R,Q} \le 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$ and (3-6) when $N_{R,Q} \ge 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$. We write

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\ge u^a}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

$$\lesssim \sum_{m:2^m \ge u^a} \Bigg(\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\approx 2^m}} \Bigg(\sum_{\substack{Q\in\mathcal{Q}_{u,k,m}\\ N_{R,Q}\le 2^{m\epsilon}\min(u^{1/12+a/12},u^{a/4})}} |\langle F_{y,r}, F_{y',r'}\rangle| + \sum_{\substack{Q\in\mathcal{Q}_{u,k,m}\\ N_{R,Q}\ge 2^{m\epsilon}\min(u^{1/12+a/12},u^{a/4})}} |\langle F_{y,r}, F_{y',r'}\rangle| \Bigg) \Bigg).$$

One sees that

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\ge u^a}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I + II, \tag{3-20}$$

where using (3-5) when $N_{R,Q} \le 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$ and summing over all $Q \in \mathcal{Q}_{u,k,m}$ and over all m such that $2^m \ge u^a$ we have

$$I := 2^{2k}(\#\mathcal{E}_k)\log(u) \sum_{m:2^m\ge u^a} u^\epsilon \max\big[2^{-m/6+\epsilon}\min(u^{11/12+a/12}, u^{5/6+a/4}), 2^{-m/2+\epsilon}\min(u^{13/12+a/12}, u^{1+a/4})\big]$$

$$\lesssim 2^{2k}(\#\mathcal{E}_k)u^\epsilon \max\big[u^{-a/6}\min(u^{11/12+a/12}, u^{5/6+a/4}), u^{-a/2}\min(u^{13/12+a/12}, u^{1+a/4})\big], \tag{3-21}$$

and using (3-6) when $N_{R,Q} \ge 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$ and summing over all Q and over all m such that $2^m \ge u^a$ we have

$$II := 2^{2k}(\#\mathcal{E}_k)u^\epsilon \sum_{m:2^m\ge u^a} 2^{-m\epsilon}\max(u^{11/12-a/12}, u^{1-a/4}) \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^\epsilon \max(u^{11/12-a/12}, u^{1-a/4}). \tag{3-22}$$

Combining (3-18), (3-19) and (3-20), we thus have the estimate

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)\big[u^a + u^\epsilon \max\big\{u^{-a/6}\min(u^{11/12+a/12}, u^{5/6+a/4}), u^{-a/2}\min(u^{13/12+a/12}, u^{1+a/4})\big\} + u^\epsilon \max(u^{11/12-a/12},\ u^{1-a/4})\big].$$

Choose $a = \frac{11}{13}$ to obtain

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}$$

for every $\epsilon > 0$, which is (3-2). □

Incomparable radii. We now want to estimate $\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|$. Our estimate will be much better than in the comparable radii case. In view of (3-1), we will in fact prove the following.

Lemma 3.5. *Let $\epsilon > 0$. For the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have*

$$\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle| \lesssim_\epsilon \sum_k 2^{2k}\#\mathcal{E}_k. \tag{3-23}$$

Fix u and k. Similar to the case of comparable radii, the first step is to cover $\mathcal{E}_k(u)$ by a collection $\mathcal{Q}_{u,k}$ of almost-disjoint cubes Q of side length 2^{k+5}. By the almost-disjointness of the cubes, it is enough to estimate $|\langle G_{u,k'}, G_{u,k}\rangle|$ when we restrict our points in $\mathcal{E}_k(u)$ and $\mathcal{E}_{k'}(u)$ to points in a fixed Q^* and get an estimate in terms of $\#(\mathcal{E}_k \cap Q^*)$, after which we may sum in $Q \in \mathcal{Q}_{u,k}$. So fix such a cube Q, and let $N_{R,Q,k}$ denote the cardinality of $(\mathcal{E}_k \cap Q^*)_R$ and for a fixed k', let $N_{R,Q,k'}$ denote the cardinality of $(\mathcal{E}_{k'} \cap Q^*)_R$. Similarly, let $N_{Y,Q,k}$ denote the cardinality of $(\mathcal{E}_k \cap Q^*)_Y$ and for a fixed k', let $N_{Y,Q,k'}$ denote the cardinality of $(\mathcal{E}_{k'} \cap Q^*)_Y$. Next, we prove a lemma that plays a role similar to Lemma 3.4 in the comparable radii case.

Lemma 3.6. *For each $Q \in \mathcal{Q}_{u,k}$, we have the estimates*

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \lesssim R^2\#(\mathcal{E}_k \cap Q^*)u(N_{R,Q,k'})^{-1} \tag{3-24}$$

and

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \lesssim N_{R,Q,k'}(\#(\mathcal{E}_k \cap Q^*))2^k(k\log(u))\max(u^{5/6}2^{5k/6},\ u2^{k/2}). \tag{3-25}$$

Proof of Lemma 3.6. We will first prove (3-24), which will be a good estimate in the case that $N_{R,Q,k'}$ is large. For each $(Y, R) \in (\mathcal{E}_k(u) \cap Q^*)$ we need only consider $y \in (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ lying in an annulus of width $2^{k'+5}$ built upon the sphere of radius R centered at Y in $\mathbb{R}^3$. Cover the intersection of this annulus with $(\mathcal{E}_{k'}(u) \cap Q^*)_Y$ by a collection $\mathcal{C}$ of $\lesssim R^2 2^{-2k'}$ 3-dimensional cubes C of side length $2^{k'+3}$ in $\mathbb{R}^3$ such that each $C \cap (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ is nonempty. For each $C \in \mathcal{C}$, let $\widetilde{C}$ denote the 4-dimensional cube $\widetilde{C} = C \times [2^{k'} - 2^{k'+2}, 2^{k'} + 2^{k'+2}]$, and let $\widetilde{\mathbb{C}}$ denote the corresponding collection of cubes $\widetilde{C}$. Now note that $C \cap (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ nonempty implies that $(\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_R = (\mathcal{E}_{k'} \cap Q^*)_R$, and also that $\#(\widetilde{C} \cap \mathcal{E}_{k'}) \lesssim u2^{k'}$, and hence by the product structure of $\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*$,

$$\#\big((\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_Y\big) \lesssim \#(\widetilde{C} \cap \mathcal{E}_{k'})\big(\#(\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_R\big)^{-1} \lesssim u2^{k'}(N_{R,Q,k'})^{-1}. \tag{3-26}$$

Next, note that for a fixed $Y \in (\mathcal{E}_k \cap Q^*)_Y$, a fixed $R \in (\mathcal{E}_k \cap Q^*)_R$, and a fixed $y \in (\mathcal{E}_{k'} \cap Q^*)_Y$, Lemma 3.2 gives rapid decay for $|\langle F_{Y,R}, F_{y,r}\rangle|$ as r moves away from two possible values of r', that is, when r moves far away from $r' = R - |Y-y|$ and $r' = |Y-y| - R$. For these values of r' we have $|\langle F_{Y,R}, F_{y,r'}\rangle| \lesssim 2^{k'}$. Using (3-26) and our bound on the size of the collection $\mathcal{C}$, we thus have

$$\begin{aligned}
&\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \\
&\quad\lesssim \sum_{(Y,R)\in\mathcal{E}_k\cap Q^*} \Bigg(\sum_{\widetilde{C}\in\widetilde{\mathbb{C}}} \Bigg(\sum_{(y,r)\in\mathcal{E}_{k'}\cap Q^*\cap\widetilde{C}} |\langle F_{Y,R}, F_{y,r}\rangle| \Bigg)\Bigg) \\
&\quad\lesssim \sum_{(Y,R)\in\mathcal{E}_k\cap Q^*} \Bigg(\sum_{\widetilde{C}\in\widetilde{\mathbb{C}}} \Bigg(\sum_{y\in(\mathcal{E}_{k'}\cap Q^*\cap\widetilde{C})_Y} \Bigg(\sum_{a\in\mathbb{Z},a\geq 0} \Bigg(\sum_{\substack{r\in(\mathcal{E}_{k'}\cap Q^*)_R \\ \max\left(\left|r'-r+|Y-y'|\right|,\left|r'+r-|Y-y|\right|\right)\approx 2^a}} 2^{-aN}2^{k'} \Bigg)\Bigg)\Bigg)\Bigg) \\
&\quad\lesssim R^2 \#(\mathcal{E}_k\cap Q^*)(N_{R,Q,k'})^{-1}u,
\end{aligned}$$

which is (3-24).

Now we prove (3-25), which is the estimate that we will use in the case that $N_{R,Q,k'}$ is small. This estimate is similar to (3-5), and the proof is very similar with only minor modifications, but we give all the details anyways.

By incurring a factor of $N_{R,Q,k}\cdot N_{R,Q,k'}$, to estimate

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle|,$$

it suffices to estimate for a fixed pair $r_1 \in (\mathcal{E}_k\cap Q^*)_R$ and $r_2 \in (\mathcal{E}_{k'}\cap Q^*)_R$

$$\sum_{(Y,r_1)\in\mathcal{E}_k\cap Q^*} \sum_{(y,r_2)\in\mathcal{E}_{k'}\cap Q^*} |\langle F_{Y,r_1}, F_{y,r_2}\rangle|.$$

Similar to the proof of (3-5), for $s \geq 0$, let $N'_{Y,Q,k} = 2^s \leq N_{Y,Q,k}$ be a given dyadic number. Fix $t \leq 2^{k+10}$, and define $K_{k,k'}(Q,s,t)$ to be the number of points $y \in (\mathcal{E}_k(u)\cap Q^*)_Y$ such that there are $\geq N'_{Y,Q,k} = 2^s$ many points $y' \in (\mathcal{E}_{k'}\cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$\begin{aligned}
K_{k,k'}(Q,s,t) := \#\Big\{ y \in (\mathcal{E}_k(u)\cap Q^*)_Y : &\text{ there exist at least } 2^s \text{ many points} \\
&y' \in (\mathcal{E}_{k'}\cap Q^*)_Y \text{ such that } \Big||y'-y| - \big(t+\tfrac{3}{2}\big)\Big| \leq \tfrac{3}{2}\Big\}.
\end{aligned}$$

Also define

$$K^*_{k,k'}(Q,s) := \max_{0\leq t\leq 2^{k+10}} K_{k,k'}(Q,s,t).$$

Note that the product structure of $\mathcal{E}$ implies that if both $\mathcal{E}_k\cap Q^*$ and $\mathcal{E}_{k'}\cap Q^*$ are nonempty, then their $\mathcal{Y}$-projections are equal, and so (3-8) implies the bound

$$K_{k,k'}(Q,s,t) \lesssim \max\big[\min(u2^k N_{Y,Q,k}^{5/3}2^{-2s}, N_{Y,Q,k}), \min(u2^{k/2}N_{Y,Q,k}2^{-s}, N_{Y,Q,k})\big]. \tag{3-27}$$

Using Lemma 3.2, we may bound

$$\begin{aligned}\sum_{\substack{(Y,R)\in(\mathcal{E}_k(u)\cap Q^*)\\(y,r)\in(\mathcal{E}_{k'}(u)\cap Q^*)}} |\langle F_{Y,R}, F_{y,r}\rangle| &\lesssim \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \Bigg(\sum_{\substack{Y\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ y\in(\mathcal{E}_{k'}\cap Q^*)_Y}} |\langle F_{Y,r_1}, F_{y,r_2}\rangle|\Bigg)\\ &\lesssim 2^k \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \Bigg(\sum_{0\le a\le m+10}\Bigg(\sum_{Y\in(\mathcal{E}_k(u)\cap Q^*)_Y} \sum_{\substack{y\in(\mathcal{E}_{k'}\cap Q^*)_Y\\ \min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}} 2^{-aN}\Bigg)\Bigg)\\ &\lesssim 2^k \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \Bigg(\sum_{0\le a\le m+10} 2^{-aN}\Bigg(\sum_{s\ge 0: 2^s\le 2N_{Y,Q,k}} K^*_{k,k'}(Q,s)2^s\Bigg)\Bigg)\\ &\lesssim 2^k N_{R,Q,k}\, N_{R,Q,k'} \sum_{s\ge 0:2^s\le 2N_{Y,Q,k}} K^*_{k,k'}(Q,s)2^s. \end{aligned} \tag{3-28}$$

Applying (3-27), we have

$$\begin{aligned}&\sum_{\substack{(Y,R)\in(\mathcal{E}_k(u)\cap Q^*)\\(y,r)\in(\mathcal{E}_{k'}(u)\cap Q^*)}} |\langle F_{Y,R}, F_{y,r}\rangle|\\ &\lesssim N_{R,Q,k}\, N_{R,Q,k'}\, 2^k \sum_{s\ge 0:2^s\lesssim N_{Y,Q,k}} \max\big[\min(u2^k N_{Y,Q,k}^{5/3}2^{-s},\, N_{Y,Q,k}2^s), \min(u2^{k/2}N_{Y,Q,k}, N_{Y,Q,k}2^s)\big]. \end{aligned} \tag{3-29}$$

Now, note that $u2^k N_{Y,Q}^{5/3}2^{-s} \ge N_{Y,Q}2^s$ if and only if $2^s \le u^{1/2}2^{k/2}N_{Y,Q}^{1/3}$. Also note that $u2^{k/2}N_{Y,Q,k} \ge N_{Y,Q,k}2^s$ if and only if $2^s \le u2^{k/2}$. Thus choosing the better estimate in the term $\min(u2^m N_{Y,Q}^{5/3}2^{-s}, N_{Y,Q}2^s)$ depending on s and the better estimate in the term $\min(u2^{k/2}N_{Y,Q,k}, N_{Y,Q,k}2^s)$ yields that the left-hand side of (3-29) is bounded by

$$N_{R,Q,k}\, N_{R,Q,k'}\, 2^k\, N_{Y,Q,k} \log(N_{Y,Q,k}) \max(N_{Y,Q,k}^{1/3}u^{1/2}2^{k/2},\, u2^{k/2}). \tag{3-30}$$

Using $N_{Y,Q,k} \lesssim u2^k$, (3-30) is bounded by

$$\begin{aligned}&N_{R,Q,k}\, N_{R,Q,k'}\, 2^k\, N_{Y,Q,k}(k\log(u)) \max(u^{5/6}2^{5k/6},\, u2^{k/2})\\ &\qquad\lesssim N_{R,Q,k'}(\#(\mathcal{E}_k\cap Q^*))2^k(k\log(u)) \max(u^{5/6}2^{5k/6},\, u2^{k/2}),\end{aligned}$$

which completes the proof of (3-25). □

Proof of Lemma 3.5. Fix $\epsilon > 0$, and set $N(u) = 100\epsilon^{-1}\log_2(2+u)$. We apply (3-24) when $N_{R,Q,k'} \ge 2^{k'\epsilon}$ and (3-25) when $N_{R,Q,k'} \le 2^{k'\epsilon}$, and then we sum over $N(u) < k' < k$ for k fixed to obtain

$$\begin{aligned}&\sum_{\substack{N(u)<k'<k\\ k \text{ fixed}}} \sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y',r'}\rangle|\\ &\qquad\lesssim_\epsilon R^2\#(\mathcal{E}_k\cap Q^*)\max(1, \log(u)u^{5/6}2^{-k/6+\epsilon}, \log(u)u2^{-k/2+\epsilon}). \end{aligned} \tag{3-31}$$

Next we sum over $Q \in \mathcal{Q}_{u,k}$ and $k > N(u)$ to obtain

$$\sum_k \sum_{Q\in\mathcal{Q}_{u,k}} \sum_{\substack{N(u)<k'<k \\ k \text{ fixed}}} \sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y',r'}\rangle| \lesssim_\epsilon \sum_k 2^{2k} \#\mathcal{E}_k. \tag{3-32}$$

We have thus shown that for the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have

$$\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle| \lesssim_\epsilon \sum_k 2^{2k} \#\mathcal{E}_k. \qquad \square$$

Putting it together. Combining (3-1), (3-2) and (3-23), we have that for every $\epsilon > 0$,

$$\|G_u\|_2^2 = \Big\|\sum_k G_{u,k}\Big\|_2^2 \lesssim_\epsilon \log_2(2+u) \sum_k 2^{2k}(\#\mathcal{E}_k) u^{11/13+\epsilon}. \tag{3-33}$$

This completes the proof of Lemma 2.4 and hence the proof of Proposition 2.2. Thus we have finished the proof of Theorem 1.2. The rest of the paper will be devoted to the (more technical) proof of Theorem 1.4.

4. Preliminaries and reductions: part II

Similarly to Section 2, we will collect necessary preliminary results and reductions to prove Theorem 1.4. Much of the proof of Theorem 1.4 will be similar to the proof of Theorem 1.2, but there are nontrivial additional technical difficulties to the proof of Theorem 1.4 that will make the proof more involved. The main reason for this is the fact that Theorem 1.4 is a full L^p characterization rather than a restricted strong type (p, p) result, and therefore we cannot simply assume that our discrete sets $\mathcal{E}$ have product structure as we were able to do in the proof of Theorem 1.2. The obstacle in applying these techniques to the 3-dimensional case is in fact the case of "incomparable radii". While this case is very easy to deal with in dimensions $d \geq 4$, we currently do not know how to handle it in three dimensions without the product structure assumption we are allowed to make when proving restricted strong type inequalities.

Discretization and density decomposition of sets. Again, the first step will be to discretize our problem, and as before we will first need to introduce some notation. Let $\mathcal{Y}$ be a 1-separated set of points in $\mathbb{R}^4$ and let $\mathcal{R}$ be a 1-separated set of radii ≥ 1. Let $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ be a finite set, and let

$$u \in \mathcal{U} = \{2^\nu : \nu = 0, 1, 2, \ldots\}$$

be a collection of dyadic indices. For each k, let $\mathfrak{B}_k$ denote the collection of all 5-dimensional balls of radius $\leq 2^k$. For a ball B, let rad B denote the radius of B. Following [Heo et al. 2011], define

$$\mathcal{R}_k := \mathcal{R} \cap [2^k, 2^{k+1}),$$
$$\mathcal{E}_k := \mathcal{E} \cap (\mathcal{Y} \times \mathcal{R}_k),$$
$$\widehat{\mathcal{E}}_k(u) := \{(y, r) \in \mathcal{E}_k : \exists B \in \mathfrak{B}_k \text{ such that } \#(\mathcal{E}_k \cap B) \geq u \text{ rad } B\},$$
$$\mathcal{E}_k(u) = \widehat{\mathcal{E}}_k(u) \setminus \bigcup_{\substack{u'\in\mathcal{U} \\ u'>u}} \widehat{\mathcal{E}}_k(u').$$

We will refer to u as the *density* of the set $\mathcal{E}_k(u)$. Note that we have the decomposition

$$\mathcal{E}_k = \bigcup_{u \in \mathcal{U}} \mathcal{E}_k(u).$$

Let σ_r denote the surface measure on rS^3, the 3-sphere centered at the origin of radius r. Now fix a smooth, radial function ψ_0 which is supported in the ball centered at the origin of radius $\frac{1}{10}$ such that $\hat{\psi}_0$ vanishes to order 40 at the origin. Let $\psi = \psi_0 * \psi_0$. For $y \in \mathcal{Y}$ and $r \in \mathcal{R}$, define

$$F_{y,r} = \sigma_r * \psi(\cdot - y).$$

For a given function $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ and finite set $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$, further define

$$G_{u,k}^{\gamma,\mathcal{E}} := \sum_{(y,r) \in \mathcal{E}_k(u)} \gamma(y,r) F_{y,r}, \quad G_u^{\gamma,\mathcal{E}} := \sum_{k \geq 0} G_{u,k}^{\gamma,\mathcal{E}}, \quad G_k^{\gamma,\mathcal{E}} := \sum_{u \in \mathcal{U}} G_{u,k}^{\gamma,\mathcal{E}}.$$

The discretized L^p inequality. We will prove the following proposition, which implies our main result for compactly supported multipliers.

Proposition 4.1. *Let $1 < p < \frac{36}{29}$. Let $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ be a function that is a tensor product; i.e., $\gamma(y,r) = \gamma_1(y)\gamma_2(r)$. For each $j \in \mathbb{Z}$, define*

$$\mathcal{E}^{\gamma,j} := \{(y,r) \in \mathcal{Y} \times \mathcal{R} : 2^j \leq |\gamma(y,r)| < 2^{j+1}\},$$
$$\mathcal{E}_k^{\gamma,j} := \{(y,r) \in \mathcal{Y} \times \mathcal{R} : r \in \mathcal{R}_k,\ 2^j \leq |\gamma(y,r)| < 2^{j+1}\}.$$

Then

$$\left\| \sum_{(y,r) \in \mathcal{E}^{\gamma,j}} \gamma(y,r) F_{y,r} \right\|_p^p \lesssim_p 2^{jp} \sum_{l \geq j} 2^{(l-j)/5} \sum_k 2^{3k} \#\mathcal{E}_k^{\gamma,l}. \tag{4-1}$$

Using the dyadic interpolation lemma (Lemma 2.1), we obtain the following corollary.

Corollary 4.2. *Let $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ be a function that is a tensor product; i.e., $\gamma(y,r) = \gamma_1(y)\gamma_2(r)$. Let $h : \mathbb{R}^5 \to \mathbb{C}$ be a function that is a tensor product; i.e., $h(y,r) = h_1(y)h_2(r)$. Then for $1 < p < \frac{36}{29}$, we have*

$$\left\| \sum_{(y,r) \in \mathcal{Y} \times \mathcal{R}} \gamma(y,r) F_{y,r} \right\|_p \lesssim_p \left(\sum_{(y,r) \in \mathcal{Y} \times \mathcal{R}} |\gamma(y,r)|^p r^3 \right)^{1/p}. \tag{4-2}$$

Also

$$\left\| \int_{\mathbb{R}^3} \int_1^\infty h(y,r) F_{y,r} \, dr \, dy \right\|_p \lesssim_p \left(\int_{\mathbb{R}^3} \int_1^\infty |h(y,r)|^p r^3 \, dr \, dy \right)^{1/p}. \tag{4-3}$$

Proof that Proposition 4.1 implies Corollary 4.2. Apply Lemma 2.1 with $F_j = \sum_{(y,r) \in \mathcal{E}^{\gamma,j}} \gamma(y,r) F_{y,r}$, M^p the implied constant from (4-1), and

$$s_j = \sum_{l \geq j} 2^{(l-j)/5} \sum_k 2^{3k} \#\mathcal{E}_k^{\gamma,l}$$

to obtain (4-2).

Now we prove (4-3). Let $y = z + w$ for $z \in \mathbb{Z}^4$ and $w \in Q_0 := [0,1)^4$ and $r = n + \tau$ for $n \in \mathbb{N}$ and $0 \le \tau < 1$. By Minkowski's inequality and (4-2),

$$\begin{aligned}\left\| \int_{\mathbb{R}^4} \int_1^\infty h(y,r) F_{y,r}\, dr\, dy \right\|_p &\lesssim_p \iint_{Q_0\times[0,1)} \left\| \sum_{z\in\mathbb{Z}^4} \sum_{n=1}^\infty h_2(n+\tau) h_1(z+w) F_{z+w,n+\tau} \right\|_p dw\, d\tau \\ &\lesssim_p \iint_{Q_0\times[0,1)} \left(\sum_{z\in\mathbb{Z}^4} \sum_{n=1}^\infty |h_2(n+\tau)h_1(z+w)|^p (n+\tau)^3 \right)^{1/p} dw\, d\tau \\ &\lesssim_p \left(\int_{\mathbb{R}^3} \int_1^\infty |h(y,r)|^p r^3\, dr\, dy \right)^{1/p},\end{aligned}$$

where in the last step we have used Hölder's inequality. □

Support-size estimates vs. L^2 inequalities. Fix a function $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ that is a tensor product, i.e., $\gamma(y,r) = \gamma_1(y)\gamma_2(r)$, and fix $j \in \mathbb{Z}$. Let

$$\widetilde{\mathcal{E}}^{\gamma,j} := \{(y,r) \in \mathcal{Y}\times\mathcal{R} : 2^{j-5} \le |\gamma(y,r)| \le 2^{j+5}\},$$

and recall the density decomposition

$$\widetilde{\mathcal{E}}_k^{\gamma,j} = \bigcup_{u\in\mathcal{U}} \widetilde{\mathcal{E}}_k^{\gamma,j}(u)$$

defined previously. Define a function $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} : \mathcal{Y}\times\mathcal{R} \to \mathbb{C}$ to be the restriction of the function $G_{u,k}^{\gamma,\widetilde{\mathcal{E}}^{\gamma,j}}$ to the set $\mathcal{E}_k^{\gamma,j}$, i.e.,

$$\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}(y,r) = \begin{cases} G_{u,k}^{\gamma,\widetilde{\mathcal{E}}^{\gamma,j}}(y,r) & \text{if } (y,r) \in \mathcal{E}_k^{\gamma,j}, \\ 0 & \text{if } (y,r) \notin \mathcal{E}_k^{\gamma,j}. \end{cases}$$

Similarly define

$$\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}} = \sum_{k\ge 0} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} \quad \text{and} \quad \widetilde{G}_k^{\gamma,\mathcal{E}^{\gamma,j}} = \sum_{u\in\mathcal{U}} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}.$$

Note that $\widetilde{G}_k^{\gamma,\mathcal{E}^{\gamma,j}} = G_k^{\gamma,\mathcal{E}^{\gamma,j}}$, and $\sum_k G_k^{\gamma,\mathcal{E}^{\gamma,j}}$ appears on the left-hand side of the inequality in Proposition 4.1. Similarly to [Heo et al. 2011], we will show that the functions $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{j,\gamma}}$ either have relatively small support size or satisfy relatively good L^2 bounds. As in the previous part of the paper, we begin with a support-size bound which follows immediately from the similar bound in [Heo et al. 2011] that improves as the density u increases.

Lemma C′. *For all $u \in \mathcal{U}$, the Lebesgue measure of the support of $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}$ is $\lesssim u^{-1}2^{3k}\#\big(\bigcup_{l:|l-j|\le 10} \mathcal{E}_k^{\gamma,l}\big)$.*

We will prove the following L^2 inequality, which in some sense an improved version of Lemma 3.6 from [Heo et al. 2011], although the hypotheses are different since it is crucial that we assume that the underlying set is of the form $\mathcal{E}^{\gamma,j}$, i.e., the $\approx 2^j$ level set of some function $\gamma(y,r) = \gamma_1(y)\gamma_2(r)$. This inequality improves as the density u decreases. In [Heo et al. 2011], the analogous L^2 inequality proved is

$$\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim u^{2/(d-1)} \log(2+u) 2^{2j} \sum_k 2^{k(d-1)} \#\Big(\bigcup_{l:|l-j|\le 10} \mathcal{E}_k^{\gamma,j} \Big). \tag{4-4}$$

We use geometric methods to improve on (4-4) in four dimensions, and our argument will rely on Lemma A.1 proved later in the Appendix.

Lemma 4.3. *Let $\mathcal{E}^{\gamma,j}$, $\mathcal{E}_k^{\gamma,j}$, and $\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}$ be as above. Then for every $\epsilon > 0$,*

$$\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim_\epsilon u^{11/18+\epsilon} 2^{2j} \sum_{l\geq j} 2^{(l-j)/10} \sum_k 2^{3k} \#\mathcal{E}_k^{\gamma,l}.$$

Combining Lemma C′ and Lemma 4.3, we obtain the following L^p bound.

Lemma 4.4. *For $p \leq 2$, for every $\epsilon > 0$,*

$$\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_p \lesssim_{\epsilon,p} u^{-(1/p-29/36-\epsilon)} 2^j \left(\sum_{l\geq j} 2^{(l-j)/10} \sum_k 2^{3k} \#\mathcal{E}_k^{\gamma,l}\right)^{1/p}.$$

Proof of Lemma 4.4 given Lemma C′ and Lemma 4.3. By Hölder's inequality,

$$\begin{aligned}\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_p &\lesssim_p (\mathrm{meas}(\mathrm{supp}(\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}})))^{1/p-1/2} \|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_2 \\ &\lesssim_{\epsilon,p} u^{29/36-1/p+\epsilon} 2^j \left(\sum_{l\geq j} 2^{(l-j)/10} \sum_k 2^{3k} \#\mathcal{E}_k^{\gamma,l}\right)^{1/p}.\end{aligned}$$

□

Summing over $u \in \mathcal{U}$, we obtain Proposition 4.1. Thus to prove Proposition 4.1 it suffices to prove Lemma 4.3. One may deduce Theorem 1.4 from Corollary 4.2 in the same way as one deduces Theorem 1.2 from Corollary 2.3.

5. Proof of the L^2 inequality: part II

We have shown in Section 4 that to prove our main result Theorem 1.4 it remains to prove Lemma 4.3, and the goal of this section is to prove Lemma 4.3. The intuition and reasoning behind our arguments will be loosely as follows. Unlike the previous case where we worked with characteristics functions, the level sets $\mathcal{E}^{\gamma,j}$ in Lemma 4.3 are no longer product sets in $\mathbb{R}^d \times \mathbb{R}$ since we no longer have the assumption that we are working with characteristic functions. However, they are still very well structured, since they are level sets of tensor products $g(y)h(r)$ of functions, where $y \in \mathbb{R}^d$ and $r \in \mathbb{R}$. The dyadic level sets $\mathcal{E}^{\gamma,j}$ may be written as a sum of product sets, and if there are not too many of them (e.g., logarithmic in the relevant parameters) then we may simply crudely sum over the total number of product sets and proceed with the same argument as in the characteristic function case. On the other hand, if there are a large number of such product sets, then this forces the underlying function to take on values much larger than 2^j, and we may then control sums over the sets $\mathcal{E}^{\gamma,j}$ by cardinalities of sets $\mathcal{E}_k^{\gamma,l}$ with $l \geq j$.

Another preliminary reduction. Recall that our goal is to estimate the L^2 norm of $\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}} = \sum_{k\geq 0} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}$. Let $N(u)$ be a sufficiently large number to be chosen later (it will be some harmless constant depending on u that is essentially $O(\log(2+u))$). We split the sum in k as $\sum_{k\leq N(u)} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} + \sum_{k>N(u)} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}$ and

apply Cauchy–Schwarz to obtain

$$\Big\|\sum_k \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\Big\|_2^2 \lesssim N(u)\Big(\sum_k \|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2 + \sum_{k>k'>N(u)} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k'}, \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\rangle|\Big). \tag{5-1}$$

We may thus separately estimate $\sum_k \|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2$ and

$$\sum_{k>k'>N(u)} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k'}, \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\rangle|,$$

which divides the proof of the L^2 estimate into two cases, the first being the case of "comparable radii" and the second being the case of "incomparable radii".

Comparable radii. We will first estimate $\sum_k \|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2$. Our goal will be to prove the following lemma.

Lemma 5.1. *For every $\epsilon > 0$,*

$$\|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2 \lesssim_\epsilon u^{11/18+\epsilon} 2^{2j} \sum_{l\geq j} 2^{(l-j)/10} 2^{3k} (\#\mathcal{E}^{\gamma,l}_k). \tag{5-2}$$

Fix k and u. As in [Heo et al. 2011], we first observe that for (y, r), $(y', r') \in \widetilde{\mathcal{E}}^{\gamma,j}_k(u) \cap \mathcal{E}^{\gamma,j}_k$, we have $\langle F_{y,r}, F_{y',r'}\rangle = 0$ unless $|(y, r) - (y', r')| \leq 2^{k+5}$. To estimate $\|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2$ for a fixed k, we would thus like to bound

$$2^{2j} \sum_{\substack{(y,r),(y',r')\in \widetilde{\mathcal{E}}^{\gamma,j}_k(u)\cap\mathcal{E}^{\gamma,j}_k \\ 2^m\leq |(y,r)-(y',r')|\leq 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

for all $0 \leq m \leq k + 4$.

Now fix $m \leq k + 4$. Let $\mathcal{Q}_{u,j,k,m}$ be a collection of almost disjoint cubes $Q \subset \mathbb{R}^5$ of side length 2^{m+5} such that

$$\widetilde{\mathcal{E}}^{\gamma,j}_k(u) \cap \mathcal{E}^{\gamma,j}_k \subset \bigcup_{Q\in\mathcal{Q}_{u,k,j,m}} Q$$

and so that every Q has nonempty intersection with $\widetilde{\mathcal{E}}^{\gamma,j}_k(u) \cap \mathcal{E}^{\gamma,j}_k$. Let Q^* denote the 2^5-dilate of Q and $\mathcal{Q}^*_{u,k,j,m}$ the corresponding collection of dilated cubes. Observe that

$$\begin{aligned}
\|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2 &\lesssim 2^{2j} \sum_{0\leq m\leq k+4} \Bigg(\sum_{\substack{(y,r),(y',r')\in \widetilde{\mathcal{E}}^{\gamma,j}_k(u)\cap\mathcal{E}^{\gamma,j}_k \\ 2^m\leq |(y,r)-(y',r')|\leq 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| + \sum_{(y,r)\in \widetilde{\mathcal{E}}^{\gamma,j}_k(u)\cap\mathcal{E}^{\gamma,j}_k} \|F_{y,r}\|_2^2 \Bigg) \\
&\lesssim 2^{2j} \sum_{0\leq m\leq k+4} \Bigg(\sum_{Q\in\mathcal{Q}_{u,k,j,m}} \Bigg(\sum_{\substack{(y,r),(y',r')\in (\widetilde{\mathcal{E}}^{\gamma,j}_k(u)\cap\mathcal{E}^{\gamma,j}_k\cap Q^*) \\ 2^m\leq |(y,r)-(y',r')|\leq 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \Bigg) + \sum_{(y,r)\in \widetilde{\mathcal{E}}^{\gamma,j}_k(u)\cap\mathcal{E}^{\gamma,j}_k} \|F_{y,r}\|_2^2 \Bigg).
\end{aligned} \tag{5-3}$$

For each integer $b \in \mathbb{Z}$ define

$$\mathcal{E}^{\gamma,j,b}_k := \{(y, r) \in \mathcal{Y} \times \mathcal{R}_k : 2^{b-3} \leq \gamma_1(y) \leq 2^{b+3},\ 2^{j-b-3} \leq \gamma_2(r) \leq 2^{j-b+3}\}.$$

Note that

$$\mathcal{E}_k^{\gamma,j} \subset \bigcup_{b\in\mathbb{Z}} \mathcal{E}_k^{\gamma,j,b} \subset \widetilde{\mathcal{E}}_k^{\gamma,j}.$$

Note also that each set $\mathcal{E}_k^{\gamma,j,b}$ is a product, that is, a set of the form $Y \times R$, where $Y \subset \mathcal{Y}$ and $R \subset \mathcal{R}$. It follows that $\mathcal{E}_k^{\gamma,j,b} \cap Q$ is a product for any cube $Q \subset \mathbb{R}^{d+1}$.

We also define some parameters associated with a fixed $Q \in \mathcal{Q}_{u,k,j,m}$ and $b \in \mathbb{Z}$. Let $N_{R,Q,b}$ be the cardinality of the $\mathcal{R}$-projection of $\mathcal{E}_k^{\gamma,j,b} \cap Q^*$; i.e.,

$$N_{R,Q,b} := \#((\mathcal{E}_k^{\gamma,j,b} \cap Q^*)_R) = \#\{r : \exists (y,r) \in \mathcal{E}_k^{\gamma,j,b} \cap Q^*\}.$$

Similarly define

$$N_{Y,Q,b} := \#((\mathcal{E}_k^{\gamma,j,b} \cap Q^*)_Y) = \#\{y : \exists (y,r) \in \mathcal{E}_k^{\gamma,j,b} \cap Q^*\}.$$

We also note the following important observation, which we will use repeatedly. Using the definition of the sets $\widetilde{\mathcal{E}}_k^{\gamma,j}(u)$ and the fact that for each $b \in \mathbb{Z}$, the set $\mathcal{E}_k^{\gamma,j,b}$ has product structure, one may see that if $Q \in \mathcal{Q}_{u,k,j,m}$ is such that $\widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap \mathcal{E}_k^{\gamma,j,b} \cap Q^*$ is nonempty, then

$$|N_{Y,Q,b} \cdot N_{R,Q,b}| \lesssim \#(\mathcal{E}_k^{\gamma,j,b} \cap Q^*) \lesssim \#(\widetilde{\mathcal{E}}_k^{\gamma,j} \cap Q^*) \lesssim u2^m. \tag{5-4}$$

Now, we will organize our sets $\mathcal{E}_k^{\gamma,j,b}$ as follows. For a fixed m, given $Q \in \mathcal{Q}_{u,k,j,m}$, we would like to group together those $b \in \mathbb{Z}$ for which $\#(\mathcal{E}_k^{\gamma,j,b} \cap Q)$ has essentially equal cardinality and for which the ratio $N_{Y,Q,b}/N_{R,Q,b}$ is essentially equal. For each pair of integers $(c,d) \in \mathbb{Z}^2$, we define

$$\mathcal{B}_{Q,c,d} := \{b \in \mathbb{Z} : 2^{c-1} \le \#(\mathcal{E}_k^{\gamma,j,b} \cap Q^*) < 2^c,\ 2^{d-1} \le N_{Y,Q,b}/N_{R,Q,b} < 2^d\}.$$

Now with (5-3) in mind, we will prove the following lemma.

Lemma 5.2. *For each $Q \in \mathcal{Q}_{u,k,j,m}$ and each quadruple $(c,d,c',d') \in \mathbb{Z}^4$, we have the estimates*

$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{\max((c-d)/2,(c'-d')/2)}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 \times 2^{\max(c,c')}2^{3(k-m/2)}(m\log(u))\max(u^{5/6}2^m, u2^{m/2}) \tag{5-5}$$

and

$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{3(k-m/2)}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 \times u2^m(2^{\max((c-d)/2,(c'-d')/2)})^{-1}. \tag{5-6}$$

Notice that (5-5) is the better estimate when $2^{\max((c-d)/2,(c'-d')/2)}$ is small and (5-6) is the better estimate when $2^{\max((c-d)/2,(c'-d')/2)}$ is large. We will use (5-5) when $2^{\max((c-d)/2,(c'-d')/2)} \le u^{1/12}$ and (5-6) when $2^{\max((c-d)/2,(c'-d')/2)} > u^{1/12}$. This yields the following corollary.

Corollary 5.3.
$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim_\epsilon I + II, \tag{5-7}$$

where

$$I := 2^{3k}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 u^\epsilon 2^{m\epsilon}\max(u^{11/12}2^{-m/2}, u^{13/12}2^{-m}), \tag{5-8}$$

$$II := 2^{3k}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 u^\epsilon 2^{m\epsilon}u^{11/12}2^{-m/2}. \tag{5-9}$$

Now note that if $(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'})) > 10000m\log(u)$, then for some l such that $l > j + \frac{1}{10}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))$, we have $\#(Q^*\cap\mathcal{E}_k^{\gamma,l}) \ge 1$, because this implies there must be (y, r) such that $\gamma(y, r) \ge 2^l$, where $l > j + \frac{1}{10}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'})$. Since $2^{\max(c,c')} \lesssim \#(Q^*\cap\mathcal{E}_k^{\gamma,j}) \lesssim u2^m \lesssim 2^{(l-j)/20}$, this implies

$$2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 \lesssim 2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*).$$

Thus Corollary 5.3 implies the following.

Corollary 5.4.
$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I + II, \tag{5-10}$$

where

$$I := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}\max(u^{11/12}2^{-m/2}, u^{13/12}2^{-m}), \tag{5-11}$$

$$II := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}u^{11/12}2^{-m/2}. \tag{5-12}$$

By (5-4) there are $\lesssim m^4\log(u)^4$ quadruples (c, d, c', d') for which both $\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)$ and $\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)$ are nonempty, so Corollary 5.4 implies the following.

Corollary 5.5.
$$\sum_{\substack{(y,r),(y',r')\in\bigcup_{b\in\mathbb{Z}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I + II, \tag{5-13}$$

where

$$I := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}\max(u^{11/12}2^{-m/2}, u^{13/12}2^{-m}) \tag{5-14}$$

$$II := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}u^{11/12}2^{-m/2}. \tag{5-15}$$

Proof of Lemma 5.2. We will first prove (5-5). Fix $b\in\mathcal{B}_{Q,c,d}$ and $b'\in\mathcal{B}_{Q,c',d'}$. Set

$$N_{Y,Q,b,b'} = \max(N_{Y,Q,b}, N_{Y,Q,b'}) \approx 2^{\max((c+d)/2,(c'+d')/2)}.$$

It suffices to prove

$$\sum_{\substack{(y,r)\in\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*\\ (y',r')\in\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r},\, F_{y',r'}\rangle| \lesssim N_{R,Q,b}N_{R,Q,b'}N_{Y,Q,b,b'}\times 2^{3(k-m/2)}(m\log(u))\max(u^{5/6}2^m,\ u2^{m/2}). \tag{5-16}$$

After incurring a factor of $N_{R,Q,b}N_{R,Q,b'}$, to estimate the left-hand side of (5-16) it suffices to estimate for a fixed pair r_1, r_2,

$$\sum_{\substack{(y,r_1)\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times\\ (y',r_2)\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r},\, F_{y',r'}\rangle|, \tag{5-17}$$

i.e., to restrict (y, r) and (y', r') to lie in fixed rows of the product-extensions of $(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times$ and $(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times$, respectively. (Our estimates will not depend on the particular choice of r_1 and r_2.)

Now, referring to the estimate in Lemma 3.2, we see that for a fixed y, r_1, r_2 we have that $|\langle F_{y,r_1}, F_{y',r_2}\rangle|$ decays rapidly as y' moves away from the set $\{y' : |y-y'| = |r_1-r_2| \text{ or } |y-y'| = r_1+r_2\}$, which is contained in a union of two annuli of thickness 2 and radii $|r_1-r_2|$ and r_1+r_2 centered at y.

Let $s\ge 0$, fix $t\le 2^{m+10}$, and define $K_k^{\gamma,j,b,b'}(Q,s,t)$ to be the number of points $y\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y$ such that there are $\ge 2^s$ many points $y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$K_k^{\gamma,j,b,b'}(Q,s,t) := \#\Big\{y\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y : \text{there exist at least } 2^s \text{ many points } y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap Q^*)_Y \text{ such that } \big||y'-y|-\big(t+\tfrac{3}{2}\big)\big|\le\tfrac{3}{2}\Big\}.$$

In view of the observation in the previous paragraph, for a given s sufficiently large but smaller than $u2^m$, say $s > m+100$, and a fixed number $t\le 2^{m+10}$, we would like to prove a bound on $K_k^{\gamma,j,b,b'}(Q,s,t)$. Our bound will depend on s and m but be independent of the choice of $t\le 2^{m+10}$. For this reason, we define the quantity

$$K_k^{\gamma,j,b,b',*}(Q,s) := \max_{0\le t\le 2^{m+10}} K_k^{\gamma,j,b,b'}(Q,s,t).$$

We will prove

$$K_k^{\gamma,j,b,b',*}(Q,s)\lesssim \max\big[u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,b,b'}2^{-s}\big],\quad s>m+100. \tag{5-18}$$

Combining this with the trivial bound $K_k^{\gamma,j,b,b',*}(Q,s)\lesssim N_{Y,Q,b,b'}$ yields

$$K_k^{\gamma,j,b,b',*}(Q,s)$$
$$\lesssim \max\big[\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-2s}, N_{Y,Q,b,b'}), \min(u2^{m/2}N_{Y,Q,b,b'}2^{-s}, N_{Y,Q,b,b'})\big],\quad s>m+100. \tag{5-19}$$

Note that (5-18) gives decay in the number of points $K_k^{\gamma,j,b,b',*}(Q,s)$ (i.e., $K_k^{\gamma,j,b,b'*}(Q,s)\ll N_{Y,Q,b,b'}$) if we have that both

(1) $N_{Y,Q,b,b'}^{5/3}2^{-2s}u2^{4m/3}\ll N_{Y,Q,b,b'}$, that is, if $2^s\gg N_{Y,Q,b,b'}^{1/3}u^{1/2}2^{2m/3}$, and

(2) $N_{Y,Q,b,b'}2^{-s}u2^{m/2}\ll N_{Y,Q,b,b'}$, that is, if $2^s\gg u2^{m/2}$.

Using Lemma 3.2, we may bound

$$\begin{aligned}
&\sum_{\substack{(y,r)\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\(y',r')\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r},F_{y',r'}\rangle|\\
&\lesssim \sum_{\substack{r_1\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R}}\Bigg(\sum_{\substack{y,y'\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y\\2^m\le|(y,r_1)-(y',r_2)|\le 2^{m+1}}}|\langle F_{y,r_1},F_{y',r_2}\rangle|\Bigg)\\
&\lesssim 2^{3(k-m/2)}\sum_{\substack{r_1\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R}}\Bigg(\sum_{0\le a\le m+10}\Bigg(\sum_{y\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y}\sum_{\substack{y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y\\ \min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}}2^{-aN}\Bigg)\Bigg)\\
&\lesssim 2^{3(k-m/2)}\sum_{\substack{r_1\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R}}\Bigg(\sum_{0\le a\le m+10}2^{-aN}\Bigg(\sum_{s\ge 0:2^s\le 2N_{Y,Q,b}}K_k^{\gamma,j,b,b',*}(Q,s)2^s\Bigg)\Bigg)\\
&\lesssim 2^{3(k-m/2)}N_{R,Q,b}\,N_{R,Q,b'}\sum_{s\ge 0:2^s\le 2N_{Y,Q,b}}K_k^{\gamma,j,b,b',*}(Q,s)2^s. \qquad (5\text{-}20)
\end{aligned}$$

Assuming (5-19) holds, we have

$$\begin{aligned}
&\sum_{\substack{(y,r)\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\(y',r')\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r},F_{y',r'}\rangle|\\
&\lesssim N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)}\sum_{\substack{s\ge 0\\2^s\lesssim N_{Y,Q,b,b'}}}\max\big[\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s},N_{Y,Q,b,b'}2^s),\\
&\qquad\qquad \min(u2^{m/2}N_{Y,Q,b,b'},N_{Y,Q,b,b'}2^s)\big]\\
&\lesssim N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)}\Bigg[m2^mN_{Y,Q,b,b'}+\max\Bigg\{\sum_{\substack{s\ge 0\\2^s\lesssim N_{Y,Q,b,b'}}}\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s},N_{Y,Q,b,b'}2^s),\\
&\qquad\qquad\sum_{\substack{s\ge 0\\2^s\lesssim N_{Y,Q,b,b'}}}\min(u2^{m/2}N_{Y,Q,b,b'},N_{Y,Q,b,b'}2^s)\Bigg\}\Bigg]. \qquad (5\text{-}21)
\end{aligned}$$

Now, note that $u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s}\ge N_{Y,Q,b,b'}2^s$ if and only if $2^s\le u^{1/2}2^{2m/3}N_{Y,Q,b,b'}^{1/3}$. Thus choosing the better estimate in the term $\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s},N_{Y,Q,b,b'}2^s)$ depending on s yields that

$$\sum_{s\ge 0:2^s\lesssim N_{Y,Q,b,b'}}\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s},N_{Y,Q,b,b'}2^s)\lesssim u^{1/2}2^{2m/3}N_{Y,Q,b,b'}^{4/3}.$$

Note that $u2^{m/2}N_{Y,Q,b,b'} \geq N_{Y,Q,b,b'}2^s$ if and only if $2^s \leq u2^{m/2}$. Thus choosing the better estimate in the term $\min(u2^{m/2}N_{Y,Q,b,b'}, N_{Y,Q,b,b'}2^s)$ depending on s yields that

$$\sum_{s\geq 0: 2^s \lesssim N_{Y,Q,b,b'}} \min(u2^{m/2}N_{Y,Q,b,b'}, N_{Y,Q,b,b'}2^s) \lesssim \log(N_{Y,Q,b,b'})N_{Y,Q,b,b'}\, u2^{m/2}.$$

Using that $N_{Y,Q,b,b'}^{1/3} \lesssim u^{1/3}2^{m/3}$, it follows that the left-hand side of (5-21) is bounded by

$$\begin{aligned} N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)}N_{Y,Q,b,b'}\big[m2^m + \log(N_{Y,Q,b,b'})\max(N_{Y,Q,b,b'}^{1/3}u^{1/2}2^{2m/3},\ u2^{m/2})\big] \\ \lesssim N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)}N_{Y,Q,b,b'}(m\log(u))\max(u^{5/6}2^m,\ u2^{m/2}) \\ \lesssim \max(N_{R,Q,b}, N_{R,Q,b'})2^{\max(c,c')}2^{3(k-m/2)}(m\log(u))\max(u^{5/6}2^m,\ u2^{m/2}), \quad (5\text{-}22) \end{aligned}$$

which proves (5-5). This will be a good estimate when $\max(N_{R,Q,b}, N_{R,Q,b'})$ is small.

Thus to prove (5-5) it remains to prove (5-18). We will in fact prove (5-18) with $K_k^{\gamma,j,b,b',*}(Q,s)$ replaced by $K_k^{\gamma,j,b,b'}(Q,s,t)$, uniformly in $t \leq 2^{m+10}$. Fix $t \leq 2^{m+10}$ and let $\alpha = \lceil \log_2(t)\rceil$ and cover $(\mathcal{E}_k^{\gamma,j,b} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y$ by $\lesssim 2^{4(m-\alpha)}$ many 4-dimensional almost-disjoint balls of radius $2^{\alpha+5}$; denote this collection of balls as $\mathfrak{B} = \{B_i\}$. For each i, we define a collection of "special" points $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$ to be the set of all points

$$y \in (\mathcal{E}_k^{\gamma,j,b} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y \cap B_i$$

such that there are $\geq 2^s$ many points $y' \in (\mathcal{E}_k^{\gamma,j,b'} \cap Q^*)_Y$ such that y' lies in the annulus of radius t and thickness 3 centered at y. That is, we define

$$\begin{aligned} A_{k,i}^{\gamma,j,b,b'}(Q,s,t) := \big\{ y \in (\mathcal{E}_k^{\gamma,j,b} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y \cap B_i : \text{ there exist at least } 2^s \text{ many points} \\ y' \in (\mathcal{E}_k^{\gamma,j,b'} \cap Q^*)_Y \text{ such that } \big||y'-y| - \big(t+\tfrac{3}{2}\big)\big| \leq \tfrac{3}{2}\big\}. \end{aligned}$$

Let $K_{k,i}^{\gamma,j,b,b'}(Q,s,t)$ denote the cardinality of $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. Now cover each B_i with $\lesssim 2^{4(\alpha-l)}$ many almost disjoint 4-dimensional balls $B_{i,\alpha}$ of radius 2^l for some $l \leq \alpha$. Each such ball contains at most $u2^l$ many points of $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, so for a fixed i there must be at least $\gtrsim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}$ many balls $B_{i,\alpha}$ that contain at least one point in $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. Thus there must be at least $\gtrsim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}$ many such points in $B_i \cap A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$ spaced apart by $\gtrsim 2^l$; call this set $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. But by Corollary A.2, which we prove later in the Appendix, the size of 3-fold intersections of 4-dimensional annuli of radius $t \approx 2^\alpha$ and thickness 3 spaced apart by $\gtrsim 2^l$ with centers lying in a ball of radius $2^{\alpha-5}$ is bounded above by $\lesssim 2^{3(\alpha-l)}2^\alpha$ provided that $l \geq \alpha/2 + 20$. This is true because for each point p in $D_{k,i}(Q,s,t)$, there will be at least $\gtrsim 2^s$ many points contained in the t-annulus centered at p that are contained in no more than two other different t annuli centered at different points p' and p'' in $D_{k,i}(Q,s,t)$, so the total number of points is $\gtrsim 2^s \times \operatorname{card}(D_{k,i}(Q,s,t))$.

It follows that if $l \geq \alpha/2 + 20$, then for each of these $\approx K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}$ many points $p \in D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, there can be

$$\lesssim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^2(u2^l)^{-2}2^{3(\alpha-l)}2^\alpha$$

points lying inside the t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. This implies that if $N_{Y,Q,b',i}$ denotes the cardinality of $(\mathcal{E}_k^{\gamma,j,b'} \cap Q^*)_Y \cap B_i^*$, where $B_i^* = 10B_i$, then we have

$$N_{Y,Q,b',i} \gtrsim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}2^s, \tag{5-23}$$

which is essentially 2^s times the number of points in $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, provided that 2^s is much bigger than the total number of points lying inside a t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, i.e., provided that

$$K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^2(u2^l)^{-2}2^{3(\alpha-l)}2^\alpha \ll 2^s \tag{5-24}$$

and

$$l \geq \alpha/2 + 20.$$

Solving for 2^l in (5-24) yields

$$2^l \gg K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^{2/5}2^{4\alpha/5}u^{-2/5}2^{-s/5}. \tag{5-25}$$

Since $s \gg m$, we may choose a minimal l such that

$$2^l \gg \max\big[K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^{2/5}2^{4\alpha/5}u^{-2/5}2^{-s/5},\ 2^{\alpha/2}\big]$$

for a sufficiently large implied constant. Substituting into (5-23) yields

$$K_{k,i}^{\gamma,j,b,b'}(Q,s,t) \lesssim \max\big[u2^{4m/3}N_{Y,Q,b',i}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,b',i}2^{-s}\big], \tag{5-26}$$

and summing over all i and using the almost-disjointness of the B_i^* gives

$$K_k^{\gamma,j,b,b'}(Q,s,t) \lesssim \max\big[u2^{4m/3}N_{Y,Q,b'}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,b'}2^{-s}\big]. \tag{5-27}$$

Taking the maximum over all $0 \leq t \leq 2^{m+10}$ proves (5-18) and hence also (5-5).

It remains to prove (5-6), which we reproduce again below for convenience:

$$\begin{aligned}\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}}} &|\langle F_{y,r}, F_{y',r'}\rangle| \\ &\lesssim 2^{3(k-m/2)}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2u2^m(2^{\max((c-d)/2,(c'-d')/2)})^{-1}.\end{aligned} \tag{5-28}$$

This will be a good estimate in the case that $2^{\max((c-d)/2,(c'-d')/2)}$ is large. Without loss of generality, assume that $c'-d' \geq c-d$. For a fixed $(y,r) \in Q^*$ and a fixed

$$y' \in (\mathcal{E}_k^{\gamma,j,b'} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y,$$

there are at most two values of r' away from which $\langle F_{y,r}, F_{y',r'}\rangle$ decays rapidly. Thus using Lemma 3.2 we may estimate

$$\begin{aligned}
&\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\lesssim \sum_{0\le a\le m+10}\Bigg(\sum_{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)}\Bigg(\sum_{y'\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y}\\
&\qquad\Bigg(\sum_{\substack{r'\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}\\ \min_{\pm,\pm}(1+|r\pm r'\pm|y-y'||)\approx 2^a}} 2^{-Na}2^{3(k-m/2)}\Bigg)\Bigg)\Bigg)\\
&\lesssim 2^{3(k-m/2)}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2\max_{b\in\mathcal{B}_{Q,c,d}}(\#(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*))\\
&\qquad\times\max_{b'\in\mathcal{B}_{Q,c,d}}(\#(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y)\\
&\lesssim 2^{3(k-m/2)}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2 2^{\max(c,c')}u2^m(2^{(c'-d')/2})^{-1},
\end{aligned}\tag{5-29}$$

and the proof of (5-6) is complete. □

We will now use Lemma 5.2 to prove Lemma 5.1.

Proof of Lemma 5.1. Fix an $a>0$ to be determined later. Similar to [Heo et al. 2011], we let $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}=\sum_\mu \widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}$, where for each positive integer μ we set

$$I_{k,\mu}=[2^k+(\mu-1)u^a,\,2^k+\mu u^a),$$

$$\mathcal{E}_{k,\mu}=\mathcal{Y}\times I_{k,\mu},$$

$$\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}=\sum_{(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)}\gamma(y,r)F_{y,r}\quad\text{and}\quad\widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}=\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)}\gamma(y,r)F_{y,r}.$$

We have

$$\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2\lesssim\Big\|\sum_\mu\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\Big\|_2^2\lesssim\sum_\mu\|\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2+\sum_{\mu'>\mu+10}|\langle\widetilde{G}_{u,k,\mu'}^{\gamma,\mathcal{E}^{\gamma,j}},G_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle|.\tag{5-30}$$

By Cauchy–Schwarz,

$$\|\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2\lesssim u^a\sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}}\|\widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2.$$

Write

$$\widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}=\Big(\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)}\gamma(y,r)\psi_0(\cdot-y)\Big)*(\sigma_r*\psi_0).$$

By the Fourier decay of σ_r and the order of vanishing of ψ_0 at the origin, we have

$$\|\hat{\sigma}_r \hat{\psi}_0\|_\infty \lesssim r^{3/2}.$$

Since

$$\Big\| \sum_{y:(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)} \gamma(y,r)\psi_0(\cdot - y)\Big\|_2^2 \lesssim 2^{2j}\#\{y\in\mathcal{Y} : (y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}^{\gamma,j}(u)\},$$

we have

$$\sum_\mu \|\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim u^a \sum_\mu \sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}} \|\widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim 2^{2j}u^a 2^{3k}\#\mathcal{E}_k^{\gamma,j}. \tag{5-31}$$

By (5-30), it remains to estimate $\sum_{\mu'>\mu+10}|\langle \widetilde{G}_{u,k,\mu'}^{\gamma,\mathcal{E}^{\gamma,j}}, \widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle|$.

Note that we have the bound

$$\begin{aligned}\sum_{\mu'>\mu+10}|\langle \widetilde{G}_{u,k,\mu'}^{\gamma,\mathcal{E}^{\gamma,j}}, \widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle| &\lesssim 2^{2j}\sum_{\substack{m:2^m\geq u^a\\ (y,r),(y',r')\in\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}}}|\langle F_{y,r}, F_{y',r'}\rangle| \\ &\lesssim 2^{2j}\sum_{m:2^m\geq u^a}\Bigg(\sum_{Q\in\mathcal{Q}_{u,k,j,m}}\Bigg(\sum_{\substack{(y,r),(y',r')\in Q\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}}}|\langle F_{y,r}, F_{y',r'}\rangle|\Bigg)\Bigg).\end{aligned} \tag{5-32}$$

To estimate the inner sum above, we will use Corollary 5.5. Summing over all $Q\in\mathcal{Q}_{u,k,j,m}$ and over all m such that $2^m\geq u^a$ we have

$$\sum_{\mu'>\mu+10}|\langle \widetilde{G}_{u,k,\mu'}^{\gamma,\mathcal{E}^{\gamma,j}}, \widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle| \lesssim_\epsilon 2^{2j}(I+II), \tag{5-33}$$

where

$$I := 2^{3k}\sum_{l\geq j}2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})u^\epsilon \max(u^{11/12-a/2},\ u^{13/12-a}), \tag{5-34}$$

$$II := 2^{3k}\sum_{l\geq j}2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})u^\epsilon u^{11/12-a/2}. \tag{5-35}$$

Combining (5-30), (5-31) and (5-33), we thus have the estimate

$$\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim_\epsilon 2^{2j}2^{3k}\sum_{l\geq j}2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})(u^a + u^{11/12-a/2+\epsilon} + u^{13/12-a+\epsilon}).$$

Choose $a=\frac{11}{18}$ to obtain

$$\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim_\epsilon 2^{2j}2^{3k}\sum_{l\geq j}2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})u^{11/18+\epsilon}$$

for every $\epsilon>0$, which is (5-2). □

Incomparable radii. We now want to estimate

$$\sum_{k>k'>N(u)} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k'}, \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k} \rangle|.$$

Our estimate will be much better than in the comparable radii case. In fact, since $d = 4$, we may simply use the estimate proved for incomparable radii in [Heo et al. 2011], which is more than sufficient for our purposes. We restate this estimate using our notation as follows.

Lemma 5.6. *Let $\epsilon > 0$. For the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have*

$$\sum_{k>k'>N(u)} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k'}, G^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k} \rangle| \lesssim_\epsilon 2^{2j} \sum_k 2^{3k} \sum_{l:|l-j|\le 10} \#\mathcal{E}^{\gamma,l}_k. \tag{5-36}$$

For a proof of Lemma 5.6, see [Heo et al. 2011].

Putting it together. Combining (5-1), (5-2) and (5-36), we have that for every $\epsilon > 0$,

$$\Big\| \sum_k \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k} \Big\|_2^2 \lesssim_\epsilon u^{11/18+\epsilon} \sum_k 2^{3k} 2^{2j} \sum_{l\ge j} 2^{(l-j)/10} (\#\mathcal{E}^{\gamma,l}_k). \tag{5-37}$$

This completes the proof of Lemma 4.3 and hence the proof of Proposition 4.1.

Appendix: A geometric lemma

In this section we prove the geometric lemma used in the previous section.

Lemma A.1. *Fix integers j, l with $l \le j$. Let $2^{j-1} \le t \le 2^{j+1}$. Then the size of the intersection of three annuli in $\mathbb{R}^3$ of thickness 4 and inner radius t such that the distance between the centers of any pair is at least 2^l and no greater than $2^j/10$ is $\lesssim 2^{3(j-l)}$, provided that $l \ge j/2 + 10$.*

We will use the following basic lemma which gives an estimate on the size of intersections of 2-dimensional annuli. This is an immediate corollary of Lemma 3.1 in [Wolff 1999].

Lemma D. *Let A_1 and A_2 be two annuli in $\mathbb{R}^2$ of thickness 1 built upon circles C_1 and C_2 of radius R, and let d denote the distance between the centers of C_1 and C_2. If $d \le R/5$, then $A_1 \cap A_2$ is contained in the 10-neighborhood of an arc of C_1 of length $\lesssim R/d$.*

Proof of Lemma A.1. Let A_1, A_2, A_3 denote the three annuli. Let $\ell_{1,2}$ denote the line through the centers of A_1 and A_2, and let $\ell_{1,3}$ denote the line through the centers of A_1 and A_3. Let P be any plane containing both $\ell_{1,2}$ and $\ell_{1,3}$. Then $A_1 \cap A_2$ is the 3-dimensional solid formed by rotating the intersection of the two (circular) annuli $A_1 \cap P$ and $A_2 \cap P$ about the line $\ell_{1,2}$. Similarly, $A_1 \cap A_3$ is the 3-dimensional solid formed by rotating the intersection of the two (circular) annuli $A_1 \cap P$ and $A_3 \cap P$ about the line $\ell_{1,3}$.

Now, by Lemma D, $A_1 \cap A_2 \cap P$ is contained in the 10-neighborhood of two arcs of length $\lesssim 2^{j-l}$ of the circle that $A_1 \cap P$ is built upon. Rotating $A_1 \cap A_2 \cap P$ about the line $\ell_{1,2}$ to get $A_1 \cap A_2$, this implies that $A_1 \cap A_2$ is the union of $\lesssim 2^{j-l}$ many 10-neighborhoods of circular annuli of radius $\lesssim 2^j$ lying in a plane normal to the line $\ell_{1,2}$. The same holds for $A_1 \cap A_3$ with $\ell_{1,2}$ replaced by $\ell_{1,3}$. Suppose first

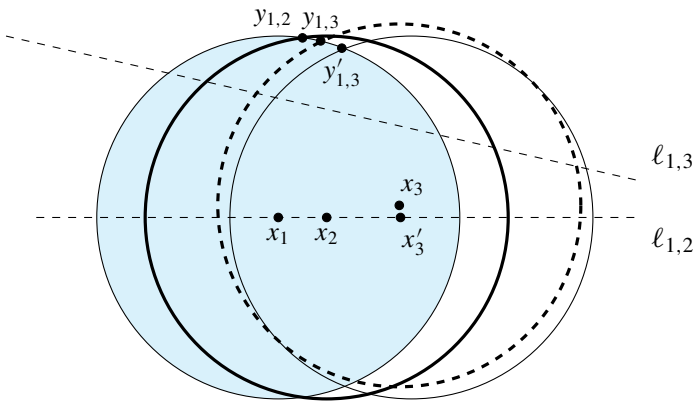

Figure 1. The circles C_1, C_2, C_3, and C'_3 in the plane P, from the proof of Lemma A.1. The shaded-in circle is C_1, the thick circle is C_2, the dashed circle is C_3, and the remaining circle is C'_3.

that the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $\geq 2^{l-j-3}$, in radians. Then $|A_1 \cap A_2 \cap A_3|$ is bounded by $\lesssim 2^{2(j-l)}$ times the largest possible size of the intersection of two 10-neighborhoods of circular annuli, where the first lies in a plane normal to $\ell_{1,2}$ and the second lies in a plane normal to $\ell_{1,3}$. One computes that the largest possible size of such an intersection is $\lesssim 2^{j-l}$.

It remains to consider the case when the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $< 2^{l-j-3}$, in radians. We now define the following coordinates associated to the lines $\ell_{1,2}$ and $\ell_{1,3}$. Let x_1, x_2, x_3 denote the centers of A_1, A_2, A_3 respectively. For $x \in \mathbb{R}^3$, we define the $\ell_{1,2}$-coordinate

$$(x)_{1,2} = \frac{\langle x - x_1, x_2 - x_1 \rangle}{|x_2 - x_1|}.$$

Similarly define the $\ell_{1,3}$-coordinate

$$(x)_{1,3} = \frac{\langle x - x_1, x_3 - x_1 \rangle}{|x_3 - x_1|}.$$

By interchanging the order of A_1, A_2, A_3, we may assume without loss of generality that $(x_3)_{1,2} \geq (x_2)_{1,2} = 1$. We will show that $l \geq j/2 + 10$ implies that $A_1 \cap A_2$ and $A_2 \cap A_3$ are actually disjoint. Observe that since the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $< 2^{l-j-3}$, we have that $(x_3 - x_2)_{1,2} \geq 2^{l-1}$. Now, let x'_3 be the closest point on the line $\ell_{1,2}$ whose distance from x_1 is the same as the distance from x_1 to x_3. Clearly, we also have $(x'_3 - x_2)_{1,2} \geq 2^{l-1}$. Let C_3 be the circle in P with center at x_3 and radius t and let C'_3 be the circle in P with center at x'_3 and radius t. Then if $y'_{1,3}$ denotes either of the two points in $C_1 \cap C'_3$ and $y_{1,2}$ either of the two points in $C_1 \cap C_2$, then $(x'_3 - x_2)_{1,2} \geq 2^{l-1}$ implies that $(y'_{1,3} - y_{1,2})_{1,2} \geq 2^{l-2}$. This is because with respect to the $\ell_{1,2}$-coordinate, $y'_{1,3}$ lies at the midpoint of x_1 and x_3 and $y_{1,2}$ lies at the midpoint of x_1 and x_2. Note that $C_1 \cap C_3$ is the rotation within P of $C_1 \cap C'_3$ by an angle of $< 2^{l-j-3}$, where the rotation is based at x_1. This implies that if $y_{1,3}$ is either of the two points in $C_1 \cap C_3$, then $|y'_{1,3} - y_{1,3}| \leq 2^{l-3}$. It follows that $(y_{1,3} - y_{1,2})_{1,2} \geq (y'_{1,3} - y_{1,2})_{1,2} - |y'_{1,3} - y_{1,3}| \geq 2^{l-2} - 2^{l-3} = 2^{l-3}$.

But by Lemma D, $A_1 \cap A_2$ is the rotation in $\mathbb{R}^3$ of a 10-neighborhood of an arc of C_1 of length $\lesssim 2^{j-l}$ that contains $y_{1,2}$ about $\ell_{1,2}$, and so $A_1 \cap A_2$ lives in the slab $\{z \in \mathbb{R}^3 : |(z-y_{1,2})_{1,2}| \le 2^{j-l+4}\}$. Similarly, $A_1 \cap A_3$ is the rotation in $\mathbb{R}^3$ of a 10-neighborhood of an arc of C_1 of length $\lesssim 2^{j-l}$ that contains $y_{1,3}$ about $\ell_{1,3}$, and so $A_1 \cap A_3$ lives in the half-infinite slab $\{z \in \mathbb{R}^3 : (z-y_{1,3})_{1,2} \ge -2^{j-l+4}\}$, and since $l \ge j/2+10$, we have $j-l+4 \le l-10$. Since $(y_{1,3}-y_{1,2})_{1,2} \ge 2^{l-3}$, it follows that $A_1 \cap A_2$ and $A_2 \cap A_3$ are disjoint. □

Corollary A.2. *Fix integers j, l with $l \le j$. Let $2^{j-1} \le t \le 2^{j+1}$. Then the size of the intersection of three annuli in $\mathbb{R}^4$ of thickness 4 and inner radius t such that the distance between the centers of any pair is at least 2^l and no greater than $2^j/10$ is $\lesssim 2^{3(j-l)}2^j$, provided that $l \ge j/2+10$.*

Proof. Let P be a hyperplane in $\mathbb{R}^4$ containing the centers of the three annuli, and for each $t \in \mathbb{R}$, let P_t be the one-parameter family of hyperplanes with normals parallel to the normal to P. For each t, the intersection of each annulus with P_t is a 3-dimensional annulus of a fixed radius depending on t that is $\lesssim 2^j$ and a fixed width depending on t that is $\lesssim 1$, and with centers spaced apart by $\gtrsim 2^l$. By Lemma A.1, $A_1 \cap A_2 \cap A_3 \cap P_t$ has size $\lesssim 2^{3(j-l)}$. It follows that $A_1 \cap A_2 \cap A_3$ has size $\lesssim 2^{3(j-l)}2^j$. □

References

[Garrigós and Seeger 2008] G. Garrigós and A. Seeger, "Characterizations of Hankel multipliers", *Math. Ann.* **342**:1 (2008), 31–68. MR Zbl

[Heo et al. 2011] Y. Heo, F. Nazarov, and A. Seeger, "Radial Fourier multipliers in high dimensions", *Acta Math.* **206**:1 (2011), 55–92. MR Zbl

[Wolff 1999] T. Wolff, "Recent work connected with the Kakeya problem", pp. 129–162 in *Prospects in mathematics* (Princeton, NJ, 1996), edited by H. Rossi, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

Received 12 Feb 2017. Revised 9 Jul 2017. Accepted 10 Aug 2017.

LAURA CLADEK: cladek@math.wisc.edu
Department of Mathematics, University of Wisconsin, Madison, WI, United States

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2018 is US $275/year for the electronic version, and $480/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 11 No. 2 2018