ANALYSIS & PDE

Volume 11 No. 3 2018

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ANALYSIS AND PDE
Vol. 11, No. 3, 2018
dx.doi.org/10.2140/apde.2018.11.583

SQUARE FUNCTION ESTIMATES FOR DISCRETE RADON TRANSFORMS

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We show $\ell^p(\mathbb{Z}^d)$-boundedness, for $p \in (1, \infty)$, of discrete singular integrals of Radon type with the aid of appropriate square function estimates, which can be thought of as a discrete counterpart of Littlewood–Paley theory. It is a very robust approach which allows us to proceed as in the continuous case.

1. Introduction

Assume that $K \in \mathcal{C}^1(\mathbb{R}^k \setminus \{0\})$ is a Calderón–Zygmund kernel satisfying the differential inequality

$$|y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \le 1 \tag{1-1}$$

for all $y \in \mathbb{R}^k$ with $|y| \ge 1$ and the cancellation condition

$$\sup_{\lambda \ge 1} \left| \int_{1 \le |y| \le \lambda} K(y)\, dy \right| \le 1. \tag{1-2}$$

Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \to \mathbb{Z}^{d_0}$ be a polynomial mapping, where each component $\mathcal{P}_j : \mathbb{Z}^k \to \mathbb{Z}$ is a polynomial of k variables with integer coefficients and $\mathcal{P}_j(0) = 0$. In the present article, as in [Ionescu and Wainger 2006], we are interested in the discrete singular Radon transform $T^{\mathcal{P}}$ defined by

$$T^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y) \tag{1-3}$$

for a finitely supported function $f : \mathbb{Z}^{d_0} \to \mathbb{R}$. We prove the following theorem.

Theorem A. *For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$ we have*

$$\|T^{\mathcal{P}} f\|_{\ell^p} \le C_p \|f\|_{\ell^p}. \tag{1-4}$$

Moreover, the constant C_p is independent of the coefficients of the polynomial mapping $\mathcal{P}$.

Theorem A was proven by Ionescu and Wainger [2006]. The operator $T^{\mathcal{P}}$ is a discrete analogue of the continuous Radon transform $R^{\mathcal{P}}$ defined by

$$R^{\mathcal{P}} f(x) = \text{p.v.} \int_{\mathbb{R}^k} f(x - \mathcal{P}(y)) K(y)\, dy. \tag{1-5}$$

The author was partially supported by the Schmidt Fellowship and the IAS School of Mathematics and by the National Science Centre in Poland, NCN grant DEC-2015/19/B/ST1/01149.
MSC2010: 11L07, 42B20, 42B25.
Keywords: discrete harmonic analysis, Radon operators, Hardy–Littlewood circle method, exponential sums.

Nowadays the operators $R^{\mathcal{P}}$ and their $L^p(\mathbb{R}^{d_0})$-boundedness properties for $p \in (1, \infty)$ are very well understood. We refer to [Stein 1993] for a detailed exposition, and see also [Christ et al. 1999] for more general cases. The key ingredient in proving $L^p(\mathbb{R}^{d_0})$ bounds for $R^{\mathcal{P}}$ is Littlewood–Paley theory. More precisely, we begin with $L^2(\mathbb{R}^{d_0})$ theory which, based on some oscillatory integral estimates for dyadic pieces of the multiplier corresponding to $R^{\mathcal{P}}$, provides bounds with acceptable decays. Then appealing to Littlewood–Paley theory and interpolation it is possible to obtain general $L^p(\mathbb{R}^{d_0})$ bounds for all $p \in (1, \infty)$. Now, one would like to follow the same scheme in the discrete case. However, the situation for $T^{\mathcal{P}}$ is much more complicated due to arithmetic nature of this operator. Although $\ell^2(\mathbb{Z}^{d_0})$ theory is based on estimates for some oscillatory integrals, or rather exponential sums associated with dyadic pieces of the multiplier corresponding to $T^{\mathcal{P}}$ as was shown in [Ionescu and Wainger 2006], $\ell^p(\mathbb{Z}^{d_0})$ theory does not fall under the Littlewood–Paley paradigm as it does in the continuous case.

The main aim of this paper is to give a new proof of Theorem A using square function techniques. We construct a suitable square function which allows us to proceed as in the continuous case to obtain $\ell^p(\mathbb{Z}^{d_0})$ theory for the operator (1-3). Our square function gives a new insight for these sort of problems, see especially [Mirek et al. 2015; 2017], and can be thought as a discrete counterpart of Littlewood–Paley theory.

There is also an interesting open question concerning the estimates of $T^{\mathcal{P}}$ at the endpoint for $p = 1$. This is unknown even in the continuous case. For instance, if we consider a Radon transform $R^{\mathcal{P}}$ along the parabola $\mathcal{P}(y) = (y, y^2)$ in $\mathbb{R}^2$, i.e.,

$$R^{\mathcal{P}} f(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}} f(x_1 - y, x_2 - y^2) \,\frac{dy}{y},$$

then the question about weak-type (1, 1)-estimates for $R^{\mathcal{P}}$ is one of the major unsolved problems in harmonic analysis. The best known result to date belongs to Seeger, Tao and Wright [Seeger et al. 2004].

In view of the recent negative results of [Buczolich and Mauldin 2010] and [LaVictoire 2011], at the endpoint for $p = 1$, for Bourgain's maximal functions corresponding to the discrete averaging operators along n^2 or n^k with $k \ge 2$, we expect that similar phenomena may occur for discrete singular Radon transforms.

Outline of the strategy of our proof. Recall from [Stein 1993, Chapter 6, §4.5, Chapter 13, §5.3] that given a kernel K satisfying (1-1) and (1-2) there are functions $(K_n : n \in \mathbb{Z})$ and a constant $C > 0$ such that for $x \neq 0$,

$$K(x) = \sum_{n \in \mathbb{Z}}^{\infty} K_n(x), \tag{1-6}$$

where for each $n \in \mathbb{Z}$, the kernel K_n is supported inside $2^{n-2} \le |x| \le 2^n$, satisfies

$$|x|^k |K_n(x)| + |x|^{k+1} |\nabla K_n(x)| \le C \tag{1-7}$$

for all $x \in \mathbb{R}^k$ such that $|x| \ge 1$, and has integral 0. Thus in view of (1-7), instead of (1-4), it suffices to show that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\Big\| \sum_{n \ge 0} T_n^{\mathcal{P}} f \Big\|_{\ell^p} \le C_p \|f\|_{\ell^p} \tag{1-8}$$

for all $f \in \ell^p(\mathbb{Z}^d)$, where

$$T_n^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{P}(y)) K_n(y) \tag{1-9}$$

for each $n \in \mathbb{Z}$. The summation in (1-8) can be taken over nonnegative integers, since $\sum_{n<0} T_n^{\mathcal{P}} f \equiv 0$.

As we mentioned before, the proof of inequality (1-8) will strongly follow the scheme of the proof of the corresponding inequality from the continuous setup. Now we describe the key points of our approach. To avoid some technicalities assume that $\mathcal{P}(x) = (x^d, \dots, x)$ is a moment curve for some $d = d_0 \geq 2$ and $k = 1$. Let m_n be the Fourier multiplier associated with the operator $T_n^{\mathcal{P}}$; i.e., $T_n^{\mathcal{P}} f = \mathcal{F}^{-1}(m_n \hat{f})$. As in [Mirek et al. 2015; 2017], we introduce a family of appropriate projections $(\Xi_n(\xi) : n \geq 0)$ which will localize the asymptotic behaviour of $m_n(\xi)$. Namely, let η be a smooth bump function with a small support, fix $l \in \mathbb{N}$ and define for each integer $n \geq 0$ the projection

$$\Xi_n(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(\mathcal{E}_n^{-1}(\xi - a/q)), \tag{1-10}$$

where $\mathcal{E}_n$ is a diagonal $d \times d$ matrix with positive entries $(\varepsilon_j : 1 \leq j \leq d)$ such that $\varepsilon_j \leq e^{-n^{1/5}}$ and

$$\mathscr{U}_{n^l} = \left\{ a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a = (a_1, \dots, a_d) \in \mathbb{N}_q^d \text{ and } \gcd(a_1, \dots, a_d, q) = 1 \text{ and } q \in P_{n^l} \right\}$$

for some family P_{n^l} such that $\mathbb{N}_{n^l} \subseteq P_{n^l} \subseteq \mathbb{N}_{e^{n^{1/10}}}$. All details are described in Section 2. Exploiting the ideas of [Ionescu and Wainger 2006], we prove that for every $p \in (1, \infty)$ there is a constant $C_{l,p} > 0$ such that

$$\|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \leq C_{l,p} \log(n+2) \|f\|_{\ell^p}. \tag{1-11}$$

Inequality (1-11) will be essential in our proof. Observe that (1-10) allows us to dominate (1-8) as

$$\left\| \sum_{n \geq 0} T_n^{\mathcal{P}} f \right\|_{\ell^p} \leq \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \right\|_{\ell^p} + \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n (1 - \Xi_n) \hat{f}) \right\|_{\ell^p}, \tag{1-12}$$

and we can employ the ideas from the circle method of Hardy and Littlewood, which are implicit in the behaviour of the projections Ξ_n and $1 - \Xi_n$. Namely, the second norm on the right-hand side of (1-12) is bounded, since the multiplier $m_n(1 - \Xi_n)$ is highly oscillatory. Thus appealing to (1-11) and a variant of Weyl's inequality with logarithmic decay, which has been proven in [Mirek et al. 2015], see Theorem 3.1, we can conclude that there is a constant $C_p > 0$ such that for each $n \geq 0$ we have

$$\left\| \mathcal{F}^{-1}(m_n (1 - \Xi_n) \hat{f}) \right\|_{\ell^p} \leq C_p (n+1)^{-2} \|f\|_{\ell^p}.$$

Now the whole difficulty lies in proving

$$\left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}. \tag{1-13}$$

For this purpose we construct new multipliers of the form

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \left(\eta(\mathcal{E}_{n+j}^{-1}(\xi - a/q)) - \eta(\mathcal{E}_{n+j+1}^{-1}(\xi - a/q)) \right) \eta(\mathcal{E}_s^{-1}(\xi - a/q)) \tag{1-14}$$

such that

$$\Xi_n(\xi) \simeq \sum_{j\in\mathbb{Z}} \sum_{s\geq 0} \Delta^j_{n,s}(\xi).$$

Moreover, we will be able to show in Theorem 3.3, using Theorem 2.2, that for each $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\bigg\| \bigg(\sum_{n\in\mathbb{Z}} |\mathcal{F}^{-1}(\Delta^j_{n,s}\hat{f})|^2 \bigg)^{1/2} \bigg\|_{\ell^p} \leq C_p \log(s+2) \|f\|_{\ell^p} \tag{1-15}$$

for any $s \geq 0$, uniformly in $j \in \mathbb{Z}$. Estimate (1-15) can be thought of as a discrete counterpart of the Littlewood–Paley inequality; see Theorem 3.3. This is the key ingredient in our proof, which combined with the robust $\ell^2(\mathbb{Z}^d)$ estimate

$$\bigg\| \bigg(\sum_{n\in\mathbb{Z}} |\mathcal{F}^{-1}(m_n \Delta^j_{n,s}\hat{f})|^2 \bigg)^{1/2} \bigg\|_{\ell^2} \leq C 2^{-\varepsilon|j|}(s+1)^{-\delta l} \|f\|_{\ell^2}, \tag{1-16}$$

allows us to deduce (1-13). The last bound follows, since for each $a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}$ we have

$$m_n(\xi) = G(a/q)\Phi_n(\xi - a/q) + \mathcal{O}(2^{-n/2}),$$

where $G(a/q)$ is the Gaussian sum and Φ_n is a continuous counterpart of m_n; precise definitions can be found at the beginning of Section 3. This observation leads to (1-16), because $|G(a/q)| \leq Cq^{-\delta}$ and $q \geq s^l$ if $a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}$. The decay in $|j|$ in (1-16) follows from the assumption on the support of $\Delta^j_{n,s}$ and the behaviour of $\Phi_n(\xi - a/q)$; see Section 3 for more details.

The ideas of exploiting projection (1-10) were initiated in [Mirek et al. 2015] in the context of $\ell^p(\mathbb{Z}^{d_0})$-boundedness of maximal functions corresponding to the averaging Radon operators

$$M_N^{\mathcal{P}} f(x) = N^{-k} \sum_{y\in\mathbb{N}_N^k} f(x - \mathcal{P}(y)), \tag{1-17}$$

where $\mathbb{N}_N^k = \{1, 2, \dots, N\}^k$, and the truncated singular Radon transforms

$$T_N^{\mathcal{P}} f(x) = \sum_{y\in\mathbb{B}_N\setminus\{0\}} f(x - \mathcal{P}(y))K(y), \tag{1-18}$$

where $\mathbb{B}_N = \{x \in \mathbb{Z}^k : |x| \leq N\}$. These ideas, on the one hand, resulted in a new proof for Bourgain's maximal operators [Bourgain 1988a; 1988b; 1989]. On the other hand, they turned out to be flexible enough to attack $\ell^p(\mathbb{Z}^{d_0})$-boundedness of maximal functions for operators with signs like in (1-18). In fact, in [Mirek et al. 2015] we provided some vector-valued estimates for the maximal functions associated with (1-17) and (1-18). These estimates found applications in variational estimates for (1-17) and (1-18), which were the subject of [Mirek et al. 2017]. Our approach falls within the scope of a general scheme which was recently developed in [Mirek et al. 2015; 2017] and resulted in some unification in the theory of discrete analogues in harmonic analysis. The novelty of this paper is that it provides a counterpart of the Littlewood–Paley square function, which is useful in the problems with arithmetic flavour. Furthermore, this square function theory is also an invaluable ingredient in the estimates of variational seminorm in [Mirek et al. 2017].

The paper is organized as follows. In Section 2 we prove Theorem 2.2, which is essential in our approach and guarantees (1-11). Ionescu and Wainger [2006] proved this result with $(\log N)^D$ loss in norm, where $D > 0$ is a large power. In [Mirek et al. 2015] we provided a slightly different proof and showed that $\log N$ is possible. Moreover $\log N$ loss is sharp for the method which we used. Since Theorem 2.2 is a deep theorem, which uses the most sophisticated tools developed to date in the field of discrete analogues, we have decided, for the sake of completeness, to provide necessary details. In Section 3 we prove Theorem A. To understand more quickly the proof of Theorem A, the reader may begin by looking at Section 3 first. These sections can be read independently, assuming the results from Section 2.

Basic reductions. We set

$$\deg \mathcal{P} = \max\{\deg \mathcal{P}_j : 1 \le j \le d_0\}$$

and define the set

$$\Gamma = \{\gamma \in \mathbb{Z}^k \setminus \{0\} : 0 < |\gamma| \le \deg \mathcal{P}\}$$

with the lexicographic order. Let d be the cardinality of Γ. Then we can identify $\mathbb{R}^d$ with the space of all vectors whose coordinates are labelled by multi-indices $\gamma \in \Gamma$. Next we introduce the canonical polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \to \mathbb{Z}^d,$$

where $\mathcal{Q}_\gamma(x) = x^\gamma$ and $x^\gamma = x_1^{\gamma_1} \cdot \dots \cdot x_k^{\gamma_k}$. The canonical polynomial mapping $\mathcal{Q}$ determines anisotropic dilations. Namely, let A be a diagonal $d \times d$ matrix such that

$$(Av)_\gamma = |\gamma| v_\gamma$$

for any $v \in \mathbb{R}^d$ and $\gamma \in \Gamma$, where $|\gamma| = \gamma_1 + \dots + \gamma_k$. Then for every $t > 0$ we set

$$t^A = \exp(A \log t);$$

i.e., $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$ for $x \in \mathbb{R}^d$, and we see that $\mathcal{Q}(tx) = t^A \mathcal{Q}(x)$.

Observe also that each $\mathcal{P}_j$ can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some $c_j^\gamma \in \mathbb{R}$. Moreover, the coefficients $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$ define a linear transformation $L : \mathbb{R}^d \to \mathbb{R}^{d_0}$ such that $L\mathcal{Q} = \mathcal{P}$. Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each $j \in \{1, \dots, d_0\}$ and $v \in \mathbb{R}^d$. Now instead of proving Theorem A we show the following.

Theorem B. *For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\|T^{\mathcal{Q}} f\|_{\ell^p} \le C_p \|f\|_{\ell^p}. \tag{1-19}$$

In view of [Stein 1993, Section 11] we can perform a lifting procedure, which allows us to replace the underlying polynomial mapping $\mathcal{P}$ from (1-4) by the canonical polynomial mapping $\mathcal{Q}$. Moreover, it shows that (1-19) implies (1-4) with the same constant C_p; see also [Ionescu and Wainger 2006] for more details. Therefore, the matters are reduced to proving (1-19) for the canonical polynomial mapping. The advantage of working with the canonical polynomial mapping $\mathcal{Q}$ is that it has all coefficients equal to 1, and the uniform bound in this case is immediate. From now on for simplicity of notation we will write $T = T^{\mathcal{Q}}$.

Notation. Throughout the whole article $C > 0$ will stand for a positive constant (possibly large constant) whose value may change from occurrence to occurrence. If there is an absolute constant $C > 0$ such that $A \le CB$ ($A \ge CB$) then we will write $A \lesssim B$ ($A \gtrsim B$). Moreover, we will write $A \simeq B$ if $A \lesssim B$ and $A \gtrsim B$ hold simultaneously, and we will write $A \lesssim_\delta B$ ($A \gtrsim_\delta B$) to indicate that the constant $C > 0$ depends on some $\delta > 0$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for $N \in \mathbb{N}$ we set

$$\mathbb{N}_N = \{1, 2, \dots, N\}.$$

For a vector $x \in \mathbb{R}^d$ we will use the norms

$$|x|_\infty = \max\{|x_j| : 1 \le j \le d\} \quad \text{and} \quad |x| = \Big(\sum_{j=1}^{d} |x_j|^2\Big)^{1/2}.$$

If γ is a multi-index from $\mathbb{N}_0^k$ then $|\gamma| = \gamma_1 + \cdots + \gamma_k$. Although, we use $|\cdot|$ for the length of a multi-index $\gamma \in \mathbb{N}_0^k$ and the Euclidean norm of $x \in \mathbb{R}^d$, their meaning will be always clear from the context and it will cause no confusions in the sequel.

2. Ionescu–Wainger-type multipliers

For a function $f \in L^1(\mathbb{R}^d)$ let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$ defined as

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x}\, dx.$$

If $f \in \ell^1(\mathbb{Z}^d)$ we set

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x}.$$

To simplify the notation, we denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^d$ and the inverse Fourier transform on the torus $\mathbb{T}^d \equiv [0, 1)^d$ (Fourier coefficients). The meaning of $\mathcal{F}^{-1}$ will be always clear from the context. Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $0 \le \eta(x) \le 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \le 1/(16d), \\ 0 & \text{for } |x| \ge 1/(8d). \end{cases}$$

Remark 2.1. We will additionally assume that η is a convolution of two nonnegative smooth functions ϕ and ψ with compact supports contained in $(-1/(8d), 1/(8d))^d$.

This section is intended to prove Theorem 2.2, which is inspired by the ideas of [Ionescu and Wainger 2006]. Let $\rho > 0$ and for every $N \in \mathbb{N}$ define

$$N_0 = \lfloor N^{\rho/2} \rfloor + 1 \quad \text{and} \quad Q_0 = (N_0!)^D,$$

where $D = D_\rho = \lfloor 2/\rho \rfloor + 1$. Let $\mathbb{P}_N = \mathbb{P} \cap (N_0, N]$, where $\mathbb{P}$ is the set of all prime numbers. For any $V \subseteq \mathbb{P}_N$ we define

$$\Pi(V) = \bigcup_{k \in \mathbb{N}_D} \Pi_k(V),$$

where for any $k \in \mathbb{N}_D$

$$\Pi_k(V) = \{p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} : \ \gamma_l \in \mathbb{N}_D \text{ and } p_l \in V \text{ are distinct for all } 1 \le l \le k\}.$$

In other words $\Pi(V)$ is the set of all products of prime factors from V of length at most D, at powers between 1 and D. Now we introduce the sets

$$P_N = \big\{q = Q \cdot w : Q \mid Q_0 \text{ and } w \in \Pi(\mathbb{P}_N) \cup \{1\}\big\}.$$

It is not difficult to see that every integer $q \in \mathbb{N}_N$ can be uniquely written as $q = Q \cdot w$, where $Q \mid Q_0$ and $w \in \Pi(\mathbb{P}_N) \cup \{1\}$. Moreover, for sufficiently large $N \in \mathbb{N}$ we have

$$q = Q \cdot w \le Q_0 \cdot w \le (N_0!)^D N^{D^2} \le e^{N^\rho};$$

thus we have $\mathbb{N}_N \subseteq P_N \subseteq \mathbb{N}_{e^{N^\rho}}$. Furthermore, if $N_1 \le N_2$ then $P_{N_1} \subseteq P_{N_2}$.

For a subset $S \subseteq \mathbb{N}$ we define

$$\mathcal{R}(S) = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a \in A_q \text{ and } q \in S\},$$

where for each $q \in \mathbb{N}$

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma)) = 1\}.$$

Finally, for each $N \in \mathbb{N}$ we will consider the sets

$$\mathscr{U}_N = \mathcal{R}(P_N). \tag{2-1}$$

It is easy to see, if $N_1 \le N_2$ then $\mathscr{U}_{N_1} \subseteq \mathscr{U}_{N_2}$.

We will assume that Θ is a multiplier on $\mathbb{R}^d$ and for every $p \in (1, \infty)$ there is a constant $\boldsymbol{A}_p > 0$ such that for every $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ we have

$$\|\mathcal{F}^{-1}(\Theta \mathcal{F} f)\|_{L^p} \le \boldsymbol{A}_p \|f\|_{L^p}. \tag{2-2}$$

For each $N \in \mathbb{N}$ we define the new periodic multiplier

$$\Delta_N(\xi) = \sum_{a/q \in \mathscr{U}_N} \Theta(\xi - a/q)\, \eta_N(\xi - a/q), \tag{2-3}$$

where $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$ and $\mathcal{E}_N$ is a diagonal $d \times d$ matrix with positive entries $(\varepsilon_\gamma : \gamma \in \Gamma)$ such that $\varepsilon_\gamma \le e^{-N^{2\rho}}$. The main result is the following.

Theorem 2.2. *Let Θ be a multiplier on $\mathbb{R}^d$ obeying (2-2). Then for every $\rho > 0$ and $p \in (1, \infty)$ there is a constant $C_{\rho,p} > 0$ such that for any $N \in \mathbb{N}$ and $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\|\mathcal{F}^{-1}(\Delta_N \hat{f})\|_{\ell^p} \le C_{\rho,p}(\boldsymbol{A}_p + 1)\log N \|f\|_{\ell^p}. \tag{2-4}$$

The main constructing blocks have been gathered in the next three subsections. Theorem 2.2 is a consequence of Theorem 2.6 and Proposition 2.7 proved below. To prove Theorem 2.2 we find some $C_\rho > 0$ and disjoint sets $\mathscr{U}_N^i \subseteq \mathscr{U}_N$ such that

$$\mathscr{U}_N = \bigcup_{1 \le i \le C_\rho \log N} \mathscr{U}_N^i$$

and we show that Δ_N with the summation restricted to $\mathscr{U}_N^i$ is bounded on $\ell^p(\mathbb{Z}^d)$ for every $p \in (1, \infty)$. In order to construct $\mathscr{U}_N^i$, we need a suitable partition of integers from the set $\Pi(\mathbb{P}_N) \cup \{1\}$; see also [Ionescu and Wainger 2006].

Fundamental combinatorial lemma. We begin with the following definition.

Definition 2.3. A subset $\Lambda \subseteq \Pi(V)$ has property $\mathcal{O}$ if there is $k \in \mathbb{N}_D$ and there are sets $S_1, S_2, \dots, S_k$ with the following properties:

(i) For each $1 \le j \le k$ there is $\beta_j \in \mathbb{N}$ such that $S_j = \{q_{j,1}, \dots, q_{j,\beta_j}\}$.

(ii) For every $q_{j,s} \in S_j$ there are $p_{j,s} \in V$ and $\gamma_j \in \mathbb{N}_D$ such that $q_{j,s} = p_{j,s}^{\gamma_j}$.

(iii) For every $w \in \Lambda$ there are unique numbers $q_{1,s_1} \in S_1, \dots, q_{k,s_k} \in S_k$ such that $w = q_{1,s_1} \cdot \dots \cdot q_{k,s_k}$.

(iv) If $(j, s) \ne (j', s')$ then $(q_{j,s}, q_{j',s'}) = 1$.

Now three comments are in order.

- The set $\Lambda = \{1\}$ has property $\mathcal{O}$ corresponding to $k = 0$.
- If Λ has property $\mathcal{O}$, then each subset $\Lambda' \subseteq \Lambda$ has property $\mathcal{O}$ as well.
- If a set Λ has property $\mathcal{O}$ then each element of Λ has the same number of prime factors $k \le D$.

The main result is the following.

Lemma 2.4. *For every $\rho > 0$ there exists a constant $C_\rho > 0$ such that for every $N \in \mathbb{N}$ the set $\mathscr{U}_N$ can be written as a disjoint union of at most $C_\rho \log N$ sets $\mathscr{U}_N^i = \mathcal{R}(P_N^i)$, where*

$$P_N^i = \{q = Q \cdot w : Q \mid Q_0 \text{ and } w \in \Lambda_i(\mathbb{P}_N)\} \tag{2-5}$$

and $\Lambda_i(\mathbb{P}_N) \subseteq \Pi(\mathbb{P}_N) \cup \{1\}$ has property $\mathcal{O}$ for each integer $1 \le i \le C_\rho \log N$.

Proof. We have to prove that for every $V \subseteq \mathbb{P}_N$ the set $\Pi(V)$ can be written as a disjoint union of at most $C_k \log N$ sets with property $\mathcal{O}$. Fix $k \in \mathbb{N}_D$, let $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{N}_D^k$ be a multi-index and observe that

$$\Pi_k(V) = \bigcup_{\gamma \in \mathbb{N}_D^k} \Pi_k^\gamma(V),$$

where

$$\Pi_k^\gamma(V) = \{p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} : p_l \in V \text{ are distinct for all } 1 \le l \le k\}.$$

Since there are D^k possible choices of exponents $\gamma_1, \dots, \gamma_k \in \mathbb{N}_D$ when $k \in \mathbb{N}_D$, it only suffices to prove that every $\Pi_k^{\gamma}(V)$ can be partitioned into a union (not necessarily disjoint) of at most $C_k \log N$ sets with property $\mathcal{O}$.

We claim that for each $k \in \mathbb{N}$ there is a constant $C_k > 0$ and a family

$$\pi = \{\pi_i(V) : 1 \le i \le C_k \log |V|\} \tag{2-6}$$

of partitions of V such that

(i) for every $1 \le i \le C_k \log |V|$, each $\pi_i(V) = \{V_1^i, \dots, V_k^i\}$ consists of pairwise disjoint subsets of V and $V = V_1^i \cup \dots \cup V_k^i$;

(ii) for every $E \subseteq V$ with at least k elements, there exists $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$ such that $E \cap V_j^i \neq \varnothing$ for every $1 \le j \le k$.

Assume for a moment we have constructed a family π as in (2-6). Then one sees that for a fixed $\gamma \in \mathbb{N}_D^k$ we have

$$\Pi_k^{\gamma}(V) = \bigcup_{1 \le i \le C_k \log |V|} \Pi_{k,i}^{\gamma}(V), \tag{2-7}$$

where

$$\Pi_{k,i}^{\gamma}(V) = \{p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} : p_j \in V_j^i \text{ and } V_j^i \in \pi_i(V) \text{ for each } 1 \le j \le k\}.$$

Indeed, the sum on the right-hand side of (2-7) is contained in $\Pi_k^{\gamma}(V)$ since each $\Pi_{k,i}^{\gamma}(V)$ is. For the opposite inclusion take $p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} \in \Pi_k^{\gamma}(V)$ and let $E = \{p_1, \dots, p_k\}$; then property (ii) for the family (2-6) ensures that there is $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$ such that $E \cap V_j^i \neq \varnothing$ for every $1 \le j \le k$. Therefore, $p_1^{\gamma_1} \cdot \dots \cdot p_k^{\gamma_k} \in \Pi_{k,i}^{\gamma}(V)$. Furthermore, we see that for each $1 \le i \le C_k \log N$, the sets $\Pi_{k,i}^{\gamma}(V)$ have property $\mathcal{O}$.

The proof will be completed if we construct the family π as in (2-6) for the set V. We assume, for simplicity, that $V = \mathbb{N}_N$ but the result is true for all $V \subseteq \mathbb{N}_N$ containing at least k elements. Now it will be more comfortable to work with surjective mappings $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ rather than with partitions of $\mathbb{N}_N$ into k nonempty subsets. It will cause no changes to us, since every surjection $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ determines a partition $\{f^{-1}[\{m\}] : 1 \le m \le k\}$ of $\mathbb{N}_N$ into k nonempty subsets.

For the proof we employ a probabilistic argument. Indeed, let $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ be a random surjective mapping. Assume that for every $n \in \mathbb{N}_N$ and $m \in \mathbb{N}_k$ we have $\mathbb{P}(\{f(n) = m\}) = 1/k$ independently of all other $n \in \mathbb{N}_N$. For every $E \subseteq \mathbb{N}_N$ with k elements we have $\mathbb{P}(\{|f[E]| = k\}) = k!/k^k$. It suffices to show that for some $r \simeq_k \log N$ and $f_1, \dots, f_r$ random surjections we have

$$\mathbb{P}\big(\{\forall_{E \subseteq \mathbb{N}_N} \ |E| = k \ \exists_{1 \le l \le r} \ |f_l[E]| = k\}\big) > 0.$$

In other words, for each $E \subseteq \mathbb{N}_N$ with cardinality k it is always possible to find, with a positive probability, among at most $C_k \log N$ random surjections at least one $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ such that $|f[E]| = k$. Then the set $\{f^{-1}[\{m\}] : 1 \le m \le k\}$ is a partition of $\mathbb{N}_N$ and $E \cap f^{-1}[\{m\}] \neq \varnothing$ for every $1 \le m \le k$.

The task now is the determine the exact value of $r \simeq_k \log N$. Take now $1 \le r \le N$ independent random surjections $f_1, \dots, f_r$ and observe that

$$\mathbb{P}\big(\big\{\exists_{E\subseteq\mathbb{N}_N}\ |E|=k\ \forall_{1\le l\le r}\ |f_l[E]|<k\big\}\big) \le \sum_{E\subseteq\mathbb{N}_N:|E|=k} \mathbb{P}\big(\{\forall_{1\le l\le r}\ |f_l[E]|<k\}\big) = \sum_{E\subseteq\mathbb{N}_N:|E|=k} \Big(1-\frac{k!}{k^k}\Big)^r$$
$$= \binom{N}{k}\Big(1-\frac{k!}{k^k}\Big)^r \le \Big(\frac{eN}{k}\Big)^k e^{-rk!/k^k} = e^{k\log(eN/k)-rk!/k^k}.$$

Therefore

$$\mathbb{P}\big(\{\exists_{E\subseteq\mathbb{N}_N}\ |E|=k\ \forall_{1\le l\le r}\ |f_l[E]|<k\}\big) < 1$$

if and only if

$$r > \frac{k^{k+1}}{k!}\log\Big(\frac{eN}{k}\Big).$$

Thus taking

$$r = \Big\lceil \frac{k^{k+1}}{k!}\log\Big(\frac{eN}{k}\Big)\Big\rceil + 1 \simeq C_k \log N,$$

we see that it does the job. This completes the proof of Lemma 2.4. □

Further reductions and square function estimates. Now we can write

$$\Delta_N = \sum_{1\le i\le C_\rho \log N} \Delta_N^i,$$

where for each $1 \le i \le C_\rho \log N$

$$\Delta_N^i(\xi) = \sum_{a/q\in\mathscr{U}_N^i} \Theta(\xi - a/q)\,\eta_N(\xi - a/q) \tag{2-8}$$

with $\mathscr{U}_N^i$ as in Lemma 2.4. The proof of Theorem 2.2 will be completed if we show that for every $p \in (1,\infty)$ and $\rho > 0$, there is a constant $C > 0$ such that for any $N \in \mathbb{N}$ and $1 \le i \le C_\rho \log N$ we have

$$\|\mathcal{F}^{-1}(\Delta_N^i \hat{f})\|_{\ell^p} \le C(\boldsymbol{A}_p + 1)\|f\|_{\ell^p} \tag{2-9}$$

for every $f \in \ell^p(\mathbb{Z}^d)$.

Let

$$\Lambda \subseteq \Pi(\mathbb{P}_N) \cup \{1\} \tag{2-10}$$

be a set with property $\mathcal{O}$; see Definition 2.3. Define

$$\mathscr{U}_N^\Lambda = \mathcal{R}\big(\{q = Q\cdot w : Q \mid Q_0 \text{ and } w \in \Lambda\}\big)$$

and $\mathscr{W}_N = \mathcal{R}(\Lambda)$, and we introduce

$$\Delta_N^\Lambda(\xi) = \sum_{a/q\in\mathscr{U}_N^\Lambda} \Theta(\xi - a/q)\,\eta_N(\xi - a/q). \tag{2-11}$$

We show that for every $p \in (1, \infty)$ and $\rho > 0$, there is a constant $C > 0$ such that for any $N \geq 8^{\max\{p,p'\}/\rho}$ and for any set Λ as in (2-10) and for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\|\mathcal{F}^{-1}(\Delta_N^\Lambda \hat{f})\|_{\ell^p} \leq C(\boldsymbol{A}_p + 1)\|f\|_{\ell^p}. \tag{2-12}$$

For $N \leq 8^{\max\{p,p'\}/\rho}$ the bound in (2-12) is obvious, since we allow the constant $C > 0$ to depend on p and ρ. Moreover, by the duality and interpolation, it suffices to prove (2-12) for $p = 2r$, where $r \in \mathbb{N}$. If $\Lambda = \Lambda_i(\mathbb{P}_N)$, as in Lemma 2.4, for some $1 \leq i \leq C_\rho \log N$, then we see that $\mathscr{U}_N^\Lambda = \mathscr{U}_N^i$ and $\Delta_N^\Lambda = \Delta_N^i$, and consequently (2-12) implies (2-9) as desired.

The function $\Theta(\xi)\eta_N(\xi)$ is regarded as a periodic function on $\mathbb{T}^d$; thus

$$\Delta_N^\Lambda(\xi) = \sum_{a/q \in \mathscr{U}_N^\Lambda} \Theta(\xi - a/q)\eta_N(\xi - a/q) = \sum_{b \in \mathbb{N}_{Q_0}^d} \sum_{a/w \in \mathscr{W}_N} \Theta(\xi - b/Q_0 - a/w)\eta_N(\xi - b/Q_0 - a/w),$$

where we have used the fact that if $(q_1, q_2) = 1$ then for every $a \in \mathbb{Z}^d$, there are unique $a_1, a_2 \in \mathbb{Z}^d$, such that $a_1/q_1, a_2/q_2 \in [0, 1)^d$ and

$$\frac{a}{q_1 q_2} = \frac{a_1}{q_1} + \frac{a_2}{q_2} \pmod{\mathbb{Z}^d}. \tag{2-13}$$

Since Λ has property $\mathcal{O}$, according to Definition 2.3 there is an integer $1 \leq k \leq 2/\rho + 1$ and there are sets $S_1, \ldots, S_k$ such that for any $j \in \mathbb{N}_k$ we have $S_j = \{q_{j,1}, \ldots, q_{j,\beta_j}\}$ for some $\beta_j \in \mathbb{N}$.

Now for each $j \in \mathbb{N}_k$ we introduce

$$\mathcal{U}_{\{j\}} = \big\{a_{j,s}/q_{j,s} \in \mathbb{T}^d \cap \mathbb{Q}^d : s \in \mathbb{N}_{\beta_j} \text{ and } a_{j,s} \in A_{q_{j,s}}\big\}$$

and for any $M = \{j_1, \ldots, j_m\} \subseteq \mathbb{N}_k$ let

$$\mathcal{U}_M = \big\{u_{j_1} + \cdots + u_{j_m} \in \mathbb{T}^d \cap \mathbb{Q}^d : u_{j_l} \in \mathcal{U}_{\{j_l\}} \text{ for any } l \in \mathbb{N}_m\big\}.$$

For any sequence $\sigma = (s_{j_1}, \ldots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_m}}$ determined by the set M, let us define

$$\mathcal{V}_M^\sigma = \big\{a_{j_1,s_{j_1}}/q_{j_1,s_{j_1}} + \cdots + a_{j_m,s_{j_m}}/q_{j_m,s_{j_m}} \in \mathbb{T}^d \cap \mathbb{Q}^d : a_{j_l,s_{j_l}} \in A_{q_{j_l,s_{j_l}}} \text{ for any } l \in \mathbb{N}_m\big\}.$$

Note that $\mathcal{V}_M^\sigma$ is a subset of $\mathcal{U}_M$ with fixed denominators $q_{j_1,s_{j_1}}, \ldots, q_{j_m,s_{j_m}}$. If $M = \varnothing$ then we have $\mathcal{U}_M = \mathcal{V}_M = \{0\}$. Let

$$\chi(\xi) = \mathbb{1}_\Lambda(\xi) \quad \text{and} \quad \Omega_N(\xi) = \Theta(\xi)\eta_N(\xi).$$

Then again by (2-13) we obtain

$$\begin{aligned}\Delta_N^\Lambda(\xi) &= \sum_{a/w \in \mathscr{W}_N} \sum_{b \in \mathbb{N}_{Q_0}^d} \Theta(\xi - b/Q_0 - a/w)\eta_N(\xi - b/Q_0 - a/w) \\ &= \sum_{s_1 \in \mathbb{N}_{\beta_1}} \sum_{a_{1,s_1} \in A_{q_{1,s_1}}} \cdots \sum_{s_k \in \mathbb{N}_{\beta_k}} \sum_{a_{k,s_k} \in A_{q_{k,s_k}}} m_{a_{1,s_1}/q_{1,s_1} + \cdots + a_{k,s_k}/q_{k,s_k}}(\xi) = \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} m_u(\xi),\end{aligned} \tag{2-14}$$

where

$$m_u(\xi) = m_{a_{1,s_1}/q_{1,s_1}+\cdots+a_{k,s_k}/q_{k,s_k}}(\xi) = \chi(q_{1,s_1}\cdot\cdots\cdot q_{k,s_k}) \sum_{b\in\mathbb{N}^d_{Q_0}} \Omega_N\Big(\xi - b/Q_0 - \sum_{j=1}^{k} a_{j,s_j}/q_{j,s_j}\Big) \tag{2-15}$$

for $u = a_{1,s_1}/q_{1,s_1} + \cdots + a_{k,s_k}/q_{k,s_k}$.

From now on we will write, for every $u \in \mathcal{U}_{\mathbb{N}_k}$,

$$f_u(x) = \mathcal{F}^{-1}(m_u \hat{f})(x) \tag{2-16}$$

with $f \in \ell^{2r}(\mathbb{Z}^d)$ and $r \in \mathbb{N}$. Therefore,

$$\mathcal{F}^{-1}(\Delta_N^{\Lambda}\hat{f})(x) = \sum_{u\in\mathcal{U}_{\mathbb{N}_k}} f_u(x) \tag{2-17}$$

and the proof of inequality (2-12) will follow from the theorem below.

Theorem 2.5. *Suppose that $\rho > 0$ and $r \in \mathbb{N}$ are given. Then there is a constant $C_{\rho,r} > 0$ such that for any $N > 8^{2r/\rho}$ and for any set Λ as in (2-10) and for every $f \in \ell^{2r}(\mathbb{Z}^d)$ we have*

$$\Big\| \sum_{u\in\mathcal{U}_{\mathbb{N}_k}} f_u \Big\|_{\ell^{2r}} \le C_{\rho,r}\|f\|_{\ell^{2r}}. \tag{2-18}$$

Moreover, the integer $k \in \mathbb{N}_D$, the set $\mathcal{U}_{\mathbb{N}_k}$ and consequently the sets $S_1, \dots, S_k$ are determined by the set Λ as it was described above.

The estimate (2-18) will follow from Theorem 2.6 and Proposition 2.7 formulated below. Let us introduce a suitable square function which will be useful in bounding (2-18). For any $M \subseteq \mathbb{N}_k$ and $L = \{j_1, \dots, j_l\} \subseteq M$ and any sequence $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_l}}$ determined by the set L, let us define the square function $\mathcal{S}^{\sigma}_{L,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$ associated with the sequence $(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$ of complex-valued functions as in (2-16) by setting

$$\mathcal{S}^{\sigma}_{L,M}(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) = \Big(\sum_{w\in\mathcal{U}_{M^c}} \Big| \sum_{u\in\mathcal{U}_{M\setminus L}} \sum_{v\in\mathcal{V}^{\sigma}_L} f_{w+u+v}(x) \Big|^2 \Big)^{1/2}, \tag{2-19}$$

where $M^c = \mathbb{N}_k \setminus M$. For some $s_{j_i} \in \{s_{j_1}, \dots, s_{j_l}\}$ we will write

$$\|\mathcal{S}^{\sigma}_{L,M}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^2_{s_{j_i}}} = \Big(\sum_{s_{j_i}\in\mathbb{N}_{\beta_{j_i}}} \big|\mathcal{S}^{(s_{j_1},\dots,s_{j_l})}_{L,M}(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k})\big|^2 \Big)^{1/2}, \tag{2-20}$$

which defines some function which depends on $x \in \mathbb{Z}^d$ and on each $s_{j_n} \in \{s_{j_1}, \dots, s_{j_l}\} \setminus \{s_{j_i}\}$.

For the proof of (2-18) we have to exploit the fact that the Fourier transform of f_u is defined as a sum of disjointly supported smooth cut-off functions. Then appropriate subsums of $\sum_{u\in\mathcal{U}_{\mathbb{N}_k}} f_u$ should be strongly orthogonal to each other.

Theorem 2.5 will be proved as a consequence of Theorem 2.6 and Proposition 2.7 below.

Theorem 2.6. *Suppose that $\rho > 0$ and $r \in \mathbb{N}$ are given. Then there is a constant $C_{\rho,r} > 0$ such that for any $N > 8^{2r/\rho}$ and for any set Λ as in (2-10) and for every $f \in \ell^{2r}(\mathbb{Z}^d)$ we have*

$$\Big\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \Big\|_{\ell^{2r}}^{2r} \le C_{\rho,r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M=\{j_1,\dots,j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_m}}} \|\mathcal{S}_{M,M}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r}. \tag{2-21}$$

Moreover, the integer $k \in \mathbb{N}_D$, the set $\mathcal{U}_{\mathbb{N}_k}$ and consequently the sets $S_1, \dots, S_k$ are determined by the set Λ as it was described above the formulation of Theorem 2.5.

Proof. Under the assumptions of Theorem 2.5, there is a constant $C_r > 0$ such that for any $M \subseteq \mathbb{N}_k$ and $L = \{j_1, \dots, j_l\} \subseteq M$ and $j_n \in M \setminus L$ and for any $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_l}}$ determined by the set L we have

$$\|\mathcal{S}_{L,M}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}} \le C_r \Big\| \|\mathcal{S}_{L\cup\{j_n\},M}^{\sigma\oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^2_{s_{j_n}}} \Big\|_{\ell^{2r}}, \tag{2-22}$$

where $\sigma \oplus s_{j_n} = (s_{j_1}, \dots, s_{j_l}, s_{j_n}) \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_l}} \times \mathbb{N}_{\beta_{j_n}}$ is the sequence determined by the set $L \cup \{s_{j_n}\}$. Moreover, the right-hand side in (2-22) can be controlled in the following way:

$$\begin{aligned} &\Big\| \|\mathcal{S}_{L\cup\{j_n\},M}^{\sigma\oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^2_{s_{j_n}}} \Big\|_{\ell^{2r}}^{2r} \\ &\qquad \le C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \|\mathcal{S}_{L\cup\{j_n\},M}^{\sigma\oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + C_r \|\mathcal{S}_{L,M\setminus\{j_n\}}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r}. \end{aligned} \tag{2-23}$$

The proofs of (2-22) and (2-23) can be found in [Mirek et al. 2015]. Therefore, (2-22) combined with (2-23) yields

$$\|\mathcal{S}_{L,M}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \le C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \|\mathcal{S}_{L\cup\{j_n\},M}^{\sigma\oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + C_r \|\mathcal{S}_{L,M\setminus\{j_n\}}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r}. \tag{2-24}$$

Applying (2-24) recursively we obtain

$$\begin{aligned} \Big\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \Big\|_{\ell^{2r}}^{2r} &= \|\mathcal{S}_{\varnothing,\mathbb{N}_k}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\ &\lesssim_r \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \|\mathcal{S}_{\{k\},\mathbb{N}_k}^{(s_k)}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + \|\mathcal{S}_{\varnothing,\mathbb{N}_{k-1}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\ &\lesssim_r \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \|\mathcal{S}_{\{k-1,k\},\mathbb{N}_k}^{(s_{k-1},s_k)}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \|\mathcal{S}_{\{k\},\mathbb{N}_k\setminus\{k-1\}}^{(s_k)}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\ &\quad + \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \|\mathcal{S}_{\{k-1\},\mathbb{N}_{k-1}}^{(s_{k-1})}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + \|\mathcal{S}_{\varnothing,\mathbb{N}_{k-2}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\ &\lesssim_r \cdots \lesssim_{\rho,r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M=\{j_1,\dots,j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \cdots \times \mathbb{N}_{\beta_{j_m}}} \sum_{x \in \mathbb{Z}^d} \bigg(\sum_{w \in \mathcal{U}_{M^c}} \Big| \sum_{v \in \mathcal{V}_M^{\sigma}} f_{w+v}(x) \Big|^2 \bigg)^r. \end{aligned} \tag{2-25}$$

The proof of (2-21) is completed. ☐

Concluding remarks and the proof of Theorem 2.5. Now Theorem 2.6 reduces the proof of inequality (2-18) to showing the estimate

$$\sum_{\substack{M\subseteq\mathbb{N}_k\\ M=\{j_1,\dots,j_m\}}}\ \sum_{\sigma\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}} \|\mathcal{S}^{\sigma}_{M,M}(f_u:u\in\mathcal{U}_{\mathbb{N}_k})\|^{2r}_{\ell^{2r}} \lesssim_r \|f\|^{2r}_{\ell^{2r}} \tag{2-26}$$

for any $f\in\ell^{2r}(\mathbb{Z}^d)$ which is a characteristic function of a finite set in $\mathbb{Z}^d$. Firstly, we prove the following.

Proposition 2.7. *Under the assumptions of Theorem 2.5, there exists a constant $C_{\rho,r}>0$ such that for any $M=\{j_1,\dots,j_m\}\subseteq\mathbb{N}_k$, any $\sigma=(s_{j_1},\dots,s_{j_m})\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}$ determined by the set M and $f\in\ell^{2r}(\mathbb{Z}^d)$ we have*

$$\|\mathcal{S}^{\sigma}_{M,M}(f_u:u\in\mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}\le C_{\rho,r}\,\boldsymbol{A}_r\bigg\|\mathcal{S}^{\sigma}_{M,M}\bigg(\mathcal{F}^{-1}\bigg(\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-u)\hat f(\xi)\bigg):u\in\mathcal{U}_{\mathbb{N}_k}\bigg)\bigg\|_{\ell^{2r}}. \tag{2-27}$$

Proof. We assume, without of loss of generality, that $N\in\mathbb{N}$ is large. Let $B_h=q_{j_1,s_{j_1}}\cdot\dots\cdot q_{j_m,s_{j_m}}\cdot Q_0\le e^{N^\rho}$ and observe that according to the notation from (2-16) and (2-14), we have

$$\begin{aligned}
&\|\mathcal{S}^{\sigma}_{M,M}(f_u:u\in\mathcal{U}_{\mathbb{N}_k})\|^{2r}_{\ell^{2r}}\\
&\qquad=\sum_{x\in\mathbb{Z}^d}\bigg(\sum_{w\in\mathcal{U}_{M^c}}\bigg|\sum_{v\in\mathcal{V}^{\sigma}_M}f_{w+v}(x)\bigg|^2\bigg)^r\\
&\qquad\le\sum_{x\in\mathbb{Z}^d}\bigg(\sum_{w\in\mathcal{U}_{M^c}}\bigg|\mathcal{F}^{-1}\bigg(\sum_{v\in\mathcal{V}^{\sigma}_M}\sum_{b\in\mathbb{N}_{Q_0}}\Theta(\xi-b/Q_0-v-w)\eta_N(\xi-b/Q_0-v-w)\hat f(\xi)\bigg)(x)\bigg|^2\bigg)^r\\
&\qquad=\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\bigg(\sum_{w\in\mathcal{U}_{M^c}}|\mathcal{F}^{-1}(\Theta\eta_N G(\xi;n,w))(B_hx+n)|^2\bigg)^r,
\end{aligned}\tag{2-28}$$

where

$$G(\xi;n,w)=\sum_{v\in\mathcal{V}^{\sigma}_M}\sum_{b\in\mathbb{N}^d_{Q_0}}\hat f(\xi+b/Q_0+v+w)e^{-2\pi i(b/Q_0+v)\cdot n}. \tag{2-29}$$

We know that for each $0<p<\infty$ there is a constant $C_p>0$ such that for any $d\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_d\in\mathbb{C}^d$ we have

$$\bigg(\int_{\mathbb{C}^d}|\lambda_1z_1+\dots+\lambda_dz_d|^pe^{-\pi|z|^2}\,\mathrm{d}z\bigg)^{1/p}=C_p(|\lambda_1|^2+\dots+|\lambda_d|^2)^{1/2}. \tag{2-30}$$

By Proposition 4.5 from [Mirek et al. 2015], with the sequence of multipliers $\Theta_N=\Theta$ for all $N\in\mathbb{N}$ and Θ as in (2-2), we have

$$\big\|\mathcal{F}^{-1}(\Theta\eta_N G(\xi;n,w))(B_hx+n)\big\|_{\ell^{2r}(x)}\le C_{\rho,r}\,\boldsymbol{A}_{2r}\big\|\mathcal{F}^{-1}(\eta_N G(\xi;n,w))(B_hx+n)\big\|_{\ell^{2r}(x)} \tag{2-31}$$

since $\inf_{\gamma\in\Gamma}\varepsilon_\gamma^{-1}\ge e^{N^{2\rho}}\ge 2e^{(d+1)N^\rho}\ge B_h$ for sufficiently large $N\in\mathbb{N}$.

Therefore, combining (2-31) with (2-30), we obtain

$$\begin{aligned}\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big(\sum_{w\in\mathcal{U}_{M^c}}\big|\mathcal{F}^{-1}(\Theta\eta_N G(\xi;n,w))(B_hx+n)\big|^2\Big)^r\\ &= C_{2r}^{2r}\int_{\mathbb{C}^d}\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big|\mathcal{F}^{-1}\Big(\Theta\eta_N\Big(\sum_{w\in\mathcal{U}_{M^c}} z_w G(\xi;n,w)\Big)\Big)(B_hx+n)\Big|^{2r}e^{-\pi|z|^2}\,\mathrm{d}z\\ &\lesssim_r \int_{\mathbb{C}^d}\sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big|\mathcal{F}^{-1}\Big(\sum_{w\in\mathcal{U}_{M^c}} z_w\eta_N G(\xi;n,w)\Big)(B_hx+n)\Big|^{2r}e^{-\pi|z|^2}\,\mathrm{d}z\\ &\lesssim_r \sum_{n\in\mathbb{N}^d_{B_h}}\sum_{x\in\mathbb{Z}^d}\Big(\sum_{w\in\mathcal{U}_{M^c}}\big|\mathcal{F}^{-1}(\eta_N G(\xi;n,w))(B_hx+n)\big|^2\Big)^r\\ &\lesssim_r \sum_{x\in\mathbb{Z}^d}\Big(\sum_{w\in\mathcal{U}_{M^c}}\Big|\mathcal{F}^{-1}\Big(\sum_{v\in\mathcal{V}^\sigma_M}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\hat{f}(\xi)\Big)(x)\Big|^2\Big)^r. \qquad (2\text{-}32)\end{aligned}$$

This completes the proof of Proposition 2.7. □

Now we are able to finish the proof of Theorem 2.5.

Proof of Theorem 2.5. It remains to show that there exists a constant $C_{\rho,r}>0$ such that for any $M=\{j_1,\dots,j_m\}\subseteq\mathbb{N}_k$ any $\sigma=(s_{j_1},\dots,s_{j_m})\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}$ determined by the set M and $f\in\ell^{2r}(\mathbb{Z}^d)$ we have

$$\sum_{\sigma\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}}\Big\|\mathcal{S}^\sigma_{M,M}\Big(\mathcal{F}^{-1}\Big(\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-u)\hat{f}(\xi)\Big):u\in\mathcal{U}_{\mathbb{N}_k}\Big)\Big\|_{\ell^{2r}}^{2r}\le C_{\rho,r}^{2r}\|f\|_{\ell^{2r}}. \qquad (2\text{-}33)$$

Since there are 2^k possible choices of sets $M\subseteq\mathbb{N}_k$ and $k\in\mathbb{N}_D$, (2-26) will follow and the proof of Theorem 2.5 will be completed. If $r=1$ then Plancherel's theorem does the job since the functions $\eta_N(\xi-b/Q_0-v-w)$ are disjointly supported for all $b/Q_0\in\mathbb{N}_{Q_0}$, $w\in\mathcal{U}_{M^c}$, $v\in\mathcal{V}^\sigma_M$ and $\sigma=(s_{j_1},\dots,s_{j_m})\in\mathbb{N}_{\beta_{j_1}}\times\cdots\times\mathbb{N}_{\beta_{j_m}}$. For general $r\ge 2$, since $\|f\|_{\ell^{2r}}^{2r}=\|f\|_{\ell^2}^2$ because we have assumed that f is a characteristic function of a finite set in $\mathbb{Z}^d$, it suffices to prove for any $x\in\mathbb{Z}^d$ that

$$\sum_{w\in\mathcal{U}_{M^c}}\Big|\mathcal{F}^{-1}\Big(\sum_{v\in\mathcal{V}^\sigma_M}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\hat{f}(\xi)\Big)(x)\Big|^2\le C_{\rho,r}. \qquad (2\text{-}34)$$

In fact, since $\|f\|_{\ell^\infty}=1$, it is enough to show

$$\Big\|\mathcal{F}^{-1}\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)\sum_{v\in\mathcal{V}^\sigma_M}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\Big)\Big\|_{\ell^1}\le C_{\rho,r} \qquad (2\text{-}35)$$

for any sequence of complex numbers $(\alpha(w):w\in\mathcal{U}_{M^c})$ such that

$$\sum_{w\in\mathcal{U}_{M^c}}|\alpha(w)|^2=1. \qquad (2\text{-}36)$$

Computing the Fourier transform we obtain

$$\mathcal{F}^{-1}\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)\sum_{v\in\mathcal{V}_M^\sigma}\sum_{b\in\mathbb{N}_{Q_0}}\eta_N(\xi-b/Q_0-v-w)\Big)(x)$$
$$=\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i x\cdot w}\Big)\cdot\det(\mathcal{E}_N)\mathcal{F}^{-1}\eta(\mathcal{E}_N x)\cdot\Big(\sum_{v\in\mathcal{V}_M^\sigma}\sum_{b\in\mathbb{N}_{Q_0}}e^{-2\pi i x\cdot(b/Q_0+v)}\Big). \tag{2-37}$$

The function

$$\sum_{v\in\mathcal{V}_M^\sigma}\sum_{b\in\mathbb{N}_{Q_0}}e^{-2\pi i x\cdot(b/Q_0+v)} \tag{2-38}$$

can be written as a sum of 2^m functions

$$\sum_{b\in\mathbb{N}_Q}e^{-2\pi i x\cdot(b/Q)}=\begin{cases}Q^d & \text{if } x\equiv 0 \pmod{Q},\\ 0 & \text{otherwise,}\end{cases} \tag{2-39}$$

where possible values of Q are products of Q_0 and $p_{j_i,s_{j_i}}^{\gamma_i}$ or $p_{j_i,s_{j_i}}^{\gamma_i-1}$ for $i\in\mathbb{N}_m$. Therefore, the proof of (2-35) will be completed if we show that

$$\Big\|\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i Qx\cdot w}\Big)\cdot Q^d\det(\mathcal{E}_N)\mathcal{F}^{-1}\eta(Q\mathcal{E}_N x)\Big\|_{\ell^1(x)}\le C_{\rho,r} \tag{2-40}$$

for any integer $Q\le e^{N^\rho}$ such that $(Q,q_{j,s})=1$, for all $j\in M^c$ and $s\in\mathbb{N}_{\beta_j}$.

Recall that, according to Remark 2.1, in our case $\eta=\phi*\psi$ for some smooth functions ϕ, ψ supported in $(-1/(8d),1/(8d))^d$. Therefore, by the Cauchy–Schwarz inequality we only need to prove that

$$Q^{d/2}\det(\mathcal{E}_N)^{1/2}\|\mathcal{F}^{-1}\phi(Q\mathcal{E}_N x)\|_{\ell^2(x)}\le C_{\rho,r} \tag{2-41}$$

and

$$Q^{d/2}\det(\mathcal{E}_N)^{1/2}\Big\|\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i Qx\cdot w}\Big)\cdot\mathcal{F}^{-1}\psi(Q\mathcal{E}_N x)\Big\|_{\ell^2(x)}\le C_{\rho,r}. \tag{2-42}$$

Since $(Q,q_{j,s})=1$, for all $j\in M^c$ and $s\in\mathbb{N}_{\beta_j}$, we know $Qw\notin\mathbb{Z}^d$ for any $w\in\mathcal{U}_{M^c}$ and its denominator is bounded by N^D. We can assume, without of loss of generality, that $Qw\in[0,1)^d$ by the periodicity of $x\mapsto e^{-2\pi i x\cdot Qw}$. Inequality (2-41) easily follows from Plancherel's theorem. In order to prove (2-42), observe that by the change of variables one has

$$\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i x\cdot Qw}\Big)\cdot\mathcal{F}^{-1}\psi(Q\mathcal{E}_N x)=Q^{-d}\det(\mathcal{E}_N)^{-1}\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)\mathcal{F}^{-1}\big(\psi(Q^{-1}\mathcal{E}_N^{-1}(\cdot-Qw))\big)(x).$$

Therefore, Plancherel's theorem and the last identity yield

$$Q^d\det(\mathcal{E}_N)\Big\|\Big(\sum_{w\in\mathcal{U}_{M^c}}\alpha(w)e^{-2\pi i Qx\cdot w}\Big)\cdot\mathcal{F}^{-1}\psi(Q\mathcal{E}_N x)\Big\|_{\ell^2(x)}^2$$
$$=\sum_{w\in\mathcal{U}_{M^c}}|\alpha(w)|^2\int_{\mathbb{R}^d}\big|\psi(\xi-\mathcal{E}_N^{-1}w)\big|^2\,d\xi+\sum_{\substack{w_1,w_2\in\mathcal{U}_{M^c}\\ w_1\neq w_2}}\alpha(w_1)\overline{\alpha(w_2)}\int_{\mathbb{R}^d}\psi(\xi)\psi(\xi-\mathcal{E}_N^{-1}(w_1-w_2))\,d\xi. \tag{2-43}$$

The first sum on the right-hand side of (2-43) is bounded in view of (2-36). The second one vanishes since the function ψ is supported in $(-1/(8d), 1/(8d))^d$ and $|\mathcal{E}_N^{-1}(w_1 - w_2)|_\infty \geq e^{N^{2\rho}} N^{-2D} > 1$ for sufficiently large N. The proof of Theorem 2.5 is completed. □

3. Proof of Theorem B

To prove inequality (1-19) in Theorem B, in view of the decomposition of the kernel K into dyadic pieces as in (1-6), it suffices to show that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\Big\| \sum_{n \geq 0} T_n f \Big\|_{\ell^p} \leq C_p \|f\|_{\ell^p}, \tag{3-1}$$

where

$$T_n f(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{Q}(y)) K_n(y) \tag{3-2}$$

with the kernel K_n as in (1-6) for each $n \in \mathbb{Z}$.

Exponential sums and $\ell^2(\mathbb{Z}^d)$ approximations. Recall that for $q \in \mathbb{N}$

$$A_q = \big\{ a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma)) = 1 \big\}.$$

Now for $q \in \mathbb{N}$ and $a \in A_q$ we define the Gaussian sums

$$G(a/q) = q^{-k} \sum_{y \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(y)}.$$

Let us observe that there exists $\delta > 0$ such that

$$|G(a/q)| \lesssim q^{-\delta}. \tag{3-3}$$

This follows from the multidimensional variant of Weyl's inequality; see [Stein and Wainger 1999, Proposition 3].

Let P be a polynomial in $\mathbb{R}^k$ of degree $d \in \mathbb{N}$ such that

$$P(x) = \sum_{\gamma \in \Gamma} \xi_\gamma x^\gamma.$$

Given $N \geq 1$, let Ω_N be a convex set in $\mathbb{R}^k$ such that

$$\Omega_N \subseteq \{x \in \mathbb{R}^k : |x - x_0| \leq cN\}$$

for some $x_0 \in \mathbb{R}^k$ and $c > 0$. We define the Weyl sums

$$S_N = \sum_{n \in \Omega_N \cap \mathbb{Z}^k} e^{2\pi i P(n)} \varphi(n), \tag{3-4}$$

where $\varphi : \mathbb{R}^k \mapsto \mathbb{C}$ is a continuously differentiable function which for some $C > 0$ satisfies

$$|\varphi(x)| \leq C \quad \text{and} \quad |\nabla \varphi(x)| \leq C(1 + |x|)^{-1}. \tag{3-5}$$

In [Mirek et al. 2015] we proved Theorem 3.1, which is a refinement of the estimates for the multidimensional Weyl sums S_N, where the limitations $N^{\varepsilon} \le q \le N^{k-\varepsilon}$ from [Stein and Wainger 1999, Proposition 3] are replaced by the weaker restrictions $(\log N)^{\beta} \le q \le N^k (\log N)^{-\beta}$ for appropriate β. Namely:

Theorem 3.1. *Assume that there is a multi-index γ_0 such that $0 < |\gamma_0| \le d$ and*

$$\left|\xi_{\gamma_0} - \frac{a}{q}\right| \le \frac{1}{q^2}$$

for some integers a, q such that $0 \le a \le q$ and $(a, q) = 1$. Then for any $\alpha > 0$ there is $\beta_\alpha > 0$ so that, for any $\beta \ge \beta_\alpha$, if

$$(\log N)^{\beta} \le q \le N^{|\gamma_0|} (\log N)^{-\beta} \tag{3-6}$$

then there is a constant $C > 0$ such that

$$|S_N| \le C N^k (\log N)^{-\alpha}. \tag{3-7}$$

The implied constant C is independent of N.

Let $(m_n : n \ge 0)$ be a sequence of multipliers on $\mathbb{T}^d$, corresponding to the operators (3-2). Then for any finitely supported function $f : \mathbb{Z}^d \mapsto \mathbb{C}$ we see that

$$T_n f(x) = \mathcal{F}^{-1}(m_n \hat{f})(x),$$

where

$$m_n(\xi) = \sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y).$$

For $n \ge 0$ we set

$$\Phi_n(\xi) = \int_{\mathbb{R}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y)\, dy.$$

Using multidimensional version of van der Corput's lemma, see [Stein and Wainger 2001, Proposition 2.1], we obtain

$$|\Phi_n(\xi)| \lesssim \min\{1, |2^{nA}\xi|_\infty^{-1/d}\}. \tag{3-8}$$

Moreover, if $n \ge 1$ we have

$$|\Phi_n(\xi)| = \left|\Phi_n(\xi) - \int_{\mathbb{R}^k} K_n(y)\, dy\right| \lesssim \min\{1, |2^{nA}\xi|_\infty\}. \tag{3-9}$$

The next proposition shows relations between m_n and Φ_n.

Proposition 3.2. *There is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for every $\xi \in \left[\frac{1}{2}, \frac{1}{2}\right)^d$ satisfying*

$$\left|\xi_\gamma - \frac{a_\gamma}{q}\right| \le L_1^{-|\gamma|} L_2$$

for all $\gamma \in \Gamma$, where $1 \le q \le L_3 \le 2^{n/2}$, $a \in A_q$, $L_1 \ge 2^n$ and $L_2 \ge 1$ we have

$$\left|m_n(\xi) - G(a/q)\Phi_n(\xi - a/q)\right| \le C\left(L_3 2^{-n} + L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\right) \le C L_2 L_3 2^{-n}. \tag{3-10}$$

Proof. Let $\theta = \xi - a/q$. For any $r \in \mathbb{N}_q^k$, if $y \equiv r \pmod q$ then for each $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod 1;$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod 1.$$

Therefore,

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) = \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} K_n(qy+r).$$

If $2^{n-2} \le |qy+r|, |qy| \le 2^n$ then by the mean value theorem we obtain

$$\bigl|\theta \cdot \mathcal{Q}(qy+r) - \theta \cdot \mathcal{Q}(qy)\bigr| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot 2^{n(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 2^{n(|\gamma|-1)} \lesssim L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}$$

and

$$\bigl|K_n(qy+r) - K_n(qy)\bigr| \lesssim 2^{-n(k+1)} L_3.$$

Thus

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) = G(a/q) \cdot q^k \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) + \mathcal{O}\Bigl(L_3 2^{-n} + L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\Bigr).$$

Now one can replace the sum on the right-hand side by the integral. Indeed, again by the mean value theorem we obtain

$$\Bigl|\sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) - \int_{\mathbb{R}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} K_n(qt)\,\mathrm{d}t\Bigr|$$

$$= \Bigl|\sum_{y \in \mathbb{Z}^k} \int_{[0,1)^k} \bigl(e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) - e^{2\pi i \theta \cdot \mathcal{Q}(q(y+t))} K_n(q(y+t))\,\mathrm{d}t\bigr)\Bigr|$$

$$= \mathcal{O}\Bigl(q^{-k} L_3 2^{-n} + q^{-k} L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\Bigr). \qquad \square$$

Discrete Littlewood–Paley theory. Fix $j, n \in \mathbb{Z}$ and $N \in \mathbb{N}$ and let $\mathcal{E}_N$ be a diagonal $d \times d$ matrix with positive entries $(\varepsilon_\gamma : \gamma \in \Gamma)$ such that $\varepsilon_\gamma \le e^{-N^{2\rho}}$ with $\rho > 0$ as in Section 2. Let us consider the multipliers

$$\Omega_N^{j,n}(\xi) = \sum_{a/q \in \mathscr{U}_N} \Phi_{j,n}(\xi - a/q)\,\eta_N(\xi - a/q) \tag{3-11}$$

with $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$ and $\Phi_{j,n}(\xi) = \Phi(2^{nA+jI}\xi)$, where Φ is a Schwartz function such that $\Phi(0) = 0$.

If $\mathscr{U}_N = \{0\}$ then $\Omega_N^{j,n}(\xi)$ can be treated as a standard Littlewood–Paley projector. Now we formulate an abstract theorem which can be thought of as a discrete variant of Littlewood–Paley theory. Its proof will be based on Theorem 2.2. Here we obtain a square function estimate which will be used in the proof of inequality (3-1).

Theorem 3.3. *For every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for all $-\infty \le M_1 \le M_2 \le \infty$, $j \in \mathbb{Z}$ and $N \in \mathbb{N}$ and every $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\Big\| \Big(\sum_{M_1 \le n \le M_2} |\mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f})|^2 \Big)^{1/2} \Big\|_{\ell^p} \le C_p \log N \|f\|_{\ell^p}. \tag{3-12}$$

Proof. By Khintchine's inequality, (3-12) is equivalent to

$$\Big(\int_0^1 \Big\| \sum_{M_1 \le n \le M_2} \varepsilon_n(t) \mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f}) \Big\|_{\ell^p}^p \, \mathrm{d}t \Big)^{1/p} \lesssim \log N \|f\|_{\ell^p}. \tag{3-13}$$

Observe that the multiplier from (3-13) can be rewritten as

$$\sum_{M_1 \le n \le M_2} \varepsilon_n(t) \Omega_N^{j,n}(\xi) = \sum_{a/q \in \mathscr{U}_N} \sum_{M_1 \le n \le M_2} \mathfrak{m}_n(\xi - a/q) \eta_N(\xi - a/q)$$

with the functions

$$\mathfrak{m}_n(\xi) = \varepsilon_n(t) \Phi(2^{nA + jI} \xi).$$

We observe that

$$|\mathfrak{m}_n(\xi)| \lesssim \min\{ |2^{nA+jI}\xi|_\infty, |2^{nA+jI}\xi|_\infty^{-1} \}.$$

The first bound follows from the mean value theorem, since

$$|\Phi(2^{nA+jI}\xi)| = \big|\Phi(2^{nA+jI}\xi) - \Phi(0)\big| \lesssim |2^{nA+jI}\xi| \sup_{\xi \in \mathbb{R}^d} |\nabla \Phi(\xi)| \lesssim |2^{nA+jI}\xi|_\infty.$$

The second bound follows since Φ is a Schwartz function. Moreover, for every $p \in (1, \infty)$ there is $C_p > 0$ such that

$$\Big\| \sup_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\mathfrak{m}_n \mathcal{F} f)| \Big\|_{L^p} \le C_p \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^d)$. Therefore, by [Stein 1993], the multiplier

$$\sum_{M_1 \le n \le M_2} \mathfrak{m}_n(\xi)$$

corresponds to a continuous singular integral; thus it defines a bounded operator on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$ with the bound independent of $j \in \mathbb{Z}$ and $-\infty \le M_1 \le M_2 \le \infty$. Hence, Theorem 2.2 applies and the multiplier

$$\sum_{M_1 \le n \le M_2} \varepsilon_n(t) \Omega_N^{j,n}(\xi)$$

defines a bounded operator on $\ell^p(\mathbb{Z}^d)$ with the $\log N$ loss, and (3-13) is established. □

Remark 3.4. If the function Φ is a real-valued function then we have

$$\Big\| \sum_{M_1 \le n \le M_2} \mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f}_n) \Big\|_{\ell^p} \le C_p \log N \Big\| \Big(\sum_{M_1 \le n \le M_2} |f_n|^2 \Big)^{1/2} \Big\|_{\ell^p}. \tag{3-14}$$

This is the dual version of inequality (3-12) for any sequence of functions $(f_n : M_1 \le n \le M_2)$ such that

$$\Big\| \Big(\sum_{M_1 \le n \le M_2} |f_n|^2 \Big)^{1/2} \Big\|_{\ell^p} < \infty.$$

We have gathered all necessary ingredients to prove inequality (3-1).

Proof of inequality (3-1). Let $\chi > 0$ and $l \in \mathbb{N}$ be the numbers whose precise values will be adjusted later. As in [Mirek et al. 2015], we will consider for every $n \in \mathbb{N}_0$ the multipliers

$$\Xi_n(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(2^{n(A-\chi I)}(\xi - a/q))^2 \tag{3-15}$$

with $\mathscr{U}_N$ as defined in Section 2. Theorem 2.2 yields, for every $p \in (1, \infty)$, that

$$\|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}. \tag{3-16}$$

The implicit constant in (3-16) depends on $\rho > 0$ from Theorem 2.2. From now on we will assume that $l \in \mathbb{N}$ and $\rho > 0$ are related by the equation

$$10\rho l = 1. \tag{3-17}$$

Assume that $f : \mathbb{Z}^d \mapsto \mathbb{C}$ has finite support and $f \ge 0$. Observe that

$$\Big\| \sum_{n \ge 0} T_n f \Big\|_{\ell^p} \le \Big\| \sum_{n \ge 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \Big\|_{\ell^p} + \Big\| \sum_{n \ge 0} \mathcal{F}^{-1}(m_n(1 - \Xi_n) \hat{f}) \Big\|_{\ell^p}. \tag{3-18}$$

Without of loss of generality we may assume that $p \ge 2$; the case $1 < p \le 2$ follows by the duality then.

The estimate of the second norm in (3-18). It suffices to show that

$$\big\| \mathcal{F}^{-1}(m_n(1 - \Xi_n) \hat{f}) \big\|_{\ell^p} \lesssim (n+1)^{-2} \|f\|_{\ell^p}. \tag{3-19}$$

For this purpose we define for every $x \in \mathbb{Z}^d$ the Radon averages

$$M_N f(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} f(x - \mathcal{Q}(y)).$$

From [Mirek et al. 2015] follows that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\big\| \sup_{N \in \mathbb{N}} |M_N f| \big\|_{\ell^p} \le C_p \|f\|_{\ell^p}. \tag{3-20}$$

Then for every $1 < p < \infty$, by (3-16) and (3-20) we obtain

$$\big\| \mathcal{F}^{-1}(m_n(1 - \Xi_n) \hat{f}) \big\|_{\ell^p} \le \| \sup_{N \in \mathbb{N}} M_N f \|_{\ell^p} + \big\| \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_n \hat{f})|) \big\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p} \tag{3-21}$$

since we have a pointwise bound

$$|\mathcal{F}^{-1}(m_n \hat{f})(x)| = |T_n f(x)| \lesssim M_{2^n} f(x). \tag{3-22}$$

We show that it is possible to improve estimate (3-21) for $p=2$. Indeed, by Theorem 3.1 we will show that for big enough $\alpha>0$, which will be specified later, and for all $n\in\mathbb{N}_0$ we have

$$\big|m_n(\xi)(1-\Xi_n(\xi))\big|\lesssim(n+1)^{-\alpha}. \tag{3-23}$$

By Dirichlet's principle, we have for every $\gamma\in\Gamma$

$$|\xi_\gamma-a_\gamma/q_\gamma|\le q_\gamma^{-1}n^\beta 2^{-n|\gamma|},$$

where $1\le q_\gamma\le n^{-\beta}2^{n|\gamma|}$. In order to apply Theorem 3.1 we must show that there exists some $\gamma\in\Gamma$ such that $n^\beta\le q_\gamma\le n^{-\beta}2^{n|\gamma|}$. Suppose for a contradiction that for every $\gamma\in\Gamma$ we have $1\le q_\gamma<n^\beta$; then for some $q\le\operatorname{lcm}(q_\gamma:\gamma\in\Gamma)\le n^{\beta d}$ we have

$$|\xi_\gamma-a'_\gamma/q|\le n^\beta 2^{-n|\gamma|},$$

where $\gcd(q,\gcd(a'_\gamma:\gamma\in\Gamma))=1$. Hence, taking $a'=(a'_\gamma:\gamma\in\Gamma)$ we have $a'/q\in\mathscr{U}_{n^l}$ provided that $\beta d<l$. On the other hand, if $1-\Xi_n(\xi)\neq0$ then for every $a'/q\in\mathscr{U}_{n^l}$ there exists $\gamma\in\Gamma$ such that

$$|\xi_\gamma-a'_\gamma/q|>(16d)^{-1}2^{-n(|\gamma|-\chi)}.$$

Therefore

$$2^{\chi n}<16dn^\beta$$

but this is impossible when $n\in\mathbb{N}$ is large. Hence, there is $\gamma\in\Gamma$ such that $n^\beta\le q_\gamma\le n^{-\beta}2^{n|\gamma|}$. Thus by Theorem 3.1,

$$|m_n(\xi)|\lesssim(n+1)^{-\alpha}$$

provided that $1-\Xi_n(\xi)\neq0$. This yields (3-23) and we obtain

$$\big\|\mathcal{F}^{-1}(m_n(1-\Xi_n)\hat f)\big\|_{\ell^2}\lesssim(1+n)^{-\alpha}\log(n+2)\|f\|_{\ell^2}. \tag{3-24}$$

Interpolating (3-24) with (3-21) we obtain

$$\big\|\mathcal{F}^{-1}(m_n(1-\Xi_n)\hat f)\big\|_{\ell^p}\lesssim(1+n)^{-c_p\alpha}\log(n+2)\|f\|_{\ell^p}. \tag{3-25}$$

for some $c_p>0$. Choosing $\alpha>0$ and $l\in\mathbb{N}$ appropriately large, one obtains (3-19).

***The estimate of the first norm in* (3-18).** Note that for any $\xi\in\mathbb{T}^d$ such that

$$|\xi_\gamma-a_\gamma/q|\le2^{-n(|\gamma|-\chi)}$$

for every $\gamma\in\Gamma$ with $1\le q\le e^{n^{1/10}}$, we have

$$m_n(\xi)=G(a/q)\Phi_n(\xi-a/q)+q^{-\delta}E_{2^n}(\xi), \tag{3-26}$$

where

$$|E_{2^n}(\xi)|\lesssim2^{-n/2}. \tag{3-27}$$

Proposition 3.2, with $L_1=2^n$, $L_2=2^{\chi n}$ and $L_3=e^{n^{1/10}}$, establishes (3-26) and (3-27), since for sufficiently large $n\in\mathbb{N}$ we have

$$q^\delta|E_{2^n}(\xi)|\lesssim q^\delta L_2L_32^{-n}\lesssim(e^{-n((1-\chi)\log2-2n^{-9/10})})\lesssim2^{-n/2}$$

provided $\chi > 0$ is sufficiently small. Now for every $j, n \in \mathbb{N}_0$ we introduce the multiplier

$$\Xi_n^j(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(2^{nA+jI}(\xi - a/q))^2 \tag{3-28}$$

and we note that

$$\begin{aligned}\Big\| \sum_{n\ge 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \Big\|_{\ell^p} &\\ \le \Big\| \sum_{n\ge 0} \mathcal{F}^{-1}\Big(\sum_{-\lfloor \chi n \rfloor \le j < n} m_n(\Xi_n^j - \Xi_n^{j+1})\hat{f}\Big) \Big\|_{\ell^p} &+ \Big\| \sum_{n\ge 0} \mathcal{F}^{-1}\big(m_n(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor})\hat{f} + m_n \Xi_n^n \hat{f}\big)\Big\|_{\ell^p} \\ = I_p^1 + I_p^2. &\end{aligned} \tag{3-29}$$

We will estimate I_p^1 and I_p^2 separately. For this purpose observe that by (3-26) and (3-27), for every $a/q \in \mathscr{U}_{n^l}$ we have

$$\begin{aligned}|m_n(\xi)| &\lesssim q^{-\delta}|\Phi_n(\xi - a/q)| + q^{-\delta}|E_{2^n}(\xi)| \\ &\lesssim q^{-\delta}\big(\min\{1, |2^{nA}(\xi - a/q)|_\infty, |2^{nA}(\xi - a/q)|_\infty^{-1/d}\} + 2^{-n/2}\big),\end{aligned} \tag{3-30}$$

where the last inequality follows from (3-8) and (3-9). Therefore by (3-30) we get

$$\big|m_n(\xi)\big(\eta(2^{nA-\chi nI}(\xi - a/q))^2 - \eta(2^{nA-\lfloor \chi n \rfloor I}(\xi - a/q))\big)^2\big| \lesssim q^{-\delta}(2^{-\chi n/d} + 2^{-n/2}) \tag{3-31}$$

since $\eta(2^{nA-\chi nI}(\xi - a/q)) \ge \eta(2^{nA-\lfloor \chi n \rfloor I}(\xi - a/q)$. Moreover, for any integer $-\chi n \le j < n$ we get

$$\big|m_n(\xi)\big(\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2\big)\big| \lesssim q^{-\delta}(2^{-|j|/d} + 2^{-n/2}). \tag{3-32}$$

Bounding I_p^2. It will suffice to show, for some $\varepsilon = \varepsilon_p > 0$, that

$$\big\|\mathcal{F}^{-1}\big(m_n(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor})\hat{f} + m_n \Xi_n^n \hat{f}\big)\big\|_{\ell^p} \lesssim 2^{-\varepsilon n}\|f\|_{\ell^p}. \tag{3-33}$$

Observe that for any $1 < p < \infty$, by (3-22), (3-20) and (3-16) we have

$$\big\|\mathcal{F}^{-1}(m_n \Xi_n^n \hat{f})\big\|_{\ell^p} \le \big\|\sup_{N\in\mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_n^n \hat{f})|)\big\|_{\ell^p} \lesssim \big\|\mathcal{F}^{-1}(\Xi_n^n \hat{f})\big\|_{\ell^p} \lesssim \log(n+2)\|f\|_{\ell^p} \tag{3-34}$$

and in a similar way we obtain

$$\big\|\mathcal{F}^{-1}\big(m_n(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor})\hat{f}\big)\big\|_{\ell^p} \lesssim \log(n+2)\|f\|_{\ell^p}. \tag{3-35}$$

For $p = 2$, by Plancherel's theorem and (3-30) we obtain

$$\big\|\mathcal{F}^{-1}(m_n \Xi_n^n \hat{f})\big\|_{\ell^2} = \Big(\int_{\mathbb{T}^d} \sum_{a/q \in \mathscr{U}_{n^l}} |m_n(\xi)|^2 \eta(2^{nA+nI}(\xi - a/q))^4 |\hat{f}(\xi)|^2 \, d\xi\Big)^{1/2} \lesssim 2^{-n/(2d)}\|f\|_{\ell^2}. \tag{3-36}$$

By (3-31) we obtain

$$\begin{aligned}&\big\|\mathcal{F}^{-1}\big(m_n(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor})\hat{f}\big)\big\|_{\ell^2} \\ &\quad = \Big(\int_{\mathbb{T}^d} \sum_{a/q \in \mathscr{U}_{n^l}} |m_n(\xi)|^2 \big(\eta(2^{nA-\chi nI}(\xi - a/q))^2 - \eta(2^{nA-\lfloor \chi n \rfloor I}(\xi - a/q))^2\big)^2 |\hat{f}(\xi)|^2 \, d\xi\Big)^{1/2} \\ &\quad \lesssim 2^{-\chi n/(2d)}\|f\|_{\ell^2}.\end{aligned} \tag{3-37}$$

Therefore, by interpolation (3-34) with (3-36) and (3-35) with (3-37) we obtain for every $p \in (1, \infty)$ that

$$\big\|\mathcal{F}^{-1}\big(m_n(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor})\hat{f} + m_n \Xi_n^n \hat{f}\big)\big\|_{\ell^p} \lesssim 2^{-\varepsilon n}\|f\|_{\ell^p},$$

which in turn implies (3-33) and $I_p^2 \lesssim \|f\|_{\ell^p}$.

Bounding I_p^1. Define for any $0 \le s < n$ the new multiplier

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2\big)\eta(2^{s(A-\chi I)}(\xi - a/q))^2$$

and we observe that by the definition (3-28) we have

$$\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi) = \sum_{0 \le s < n} \Delta_{n,s}^j(\xi).$$

Moreover,

$$\eta(2^{nA+jI}\xi)^2 - \eta(2^{nA+(j+1)I}\xi)^2 = \big(\eta(2^{nA+jI}\xi)^2 - \eta(2^{nA+(j+1)I}\xi)^2\big)\cdot\big(\eta(2^{nA+(j-1)I}\xi) - \eta(2^{nA+(j+2)I}\xi)\big).$$

Thus we see

$$\Delta_{n,s}^j(\xi) = \Delta_{n,s}^{j,1}(\xi)\cdot\Delta_{n,s}^{j,2}(\xi),$$

where

$$\Delta_{n,s}^{j,1}(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(2^{nA+(j-1)I}(\xi - a/q)) - \eta(2^{nA+(j+2)I}(\xi - a/q))\big)\eta(2^{s(A-\chi I)}(\xi - a/q))$$

and

$$\Delta_{n,s}^{j,2}(\xi) = \sum_{a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}} \big(\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2\big)\eta(2^{s(A-\chi I)}(\xi - a/q)).$$

Moreover, $\Delta_{n,s}^{j,1}$ and $\Delta_{n,s}^{j,2}$ are the multipliers which satisfy the assumptions of Theorem 3.3. Therefore,

$$\begin{aligned} I_p^1 &= \Big\|\sum_{n \ge 0}\mathcal{F}^{-1}\Big(\sum_{-\chi n \le j < n}\ \sum_{0 \le s < n} \Delta_{n,s}^{j,1} m_n \Delta_{n,s}^{j,2}\hat{f}\Big)\Big\|_{\ell^p} \\ &\le \sum_{s \ge 0}\sum_{j \in \mathbb{Z}}\Big\|\sum_{n \ge \max\{j, -j/\chi, s\}} \mathcal{F}^{-1}(\Delta_{n,s}^{j,1} m_n \Delta_{n,s}^{j,2}\hat{f})\Big\|_{\ell^p} \\ &\lesssim \sum_{s \ge 0}\sum_{j \in \mathbb{Z}} \log s \Big\|\Big(\sum_{n \ge \max\{j, -j/\chi, s\}} \big|\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p}. \end{aligned} \tag{3-38}$$

In the last step we used (3-14). The task now is to show that for some $\varepsilon = \varepsilon_p > 0$

$$\Big\|\Big(\sum_{n \ge \max\{j, -j/\chi, s\}} \big|\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2}\hat{f})\big|^2\Big)^{1/2}\Big\|_{\ell^p} \lesssim s^{-2}2^{-\varepsilon j}\|f\|_{\ell^p}. \tag{3-39}$$

This in turn will imply $I_p^1 \lesssim \|f\|_{\ell^p}$ and the proof will be completed. We have assumed that $p \ge 2$; then for every $g \in \ell^r(\mathbb{Z}^d)$ such that $g \ge 0$ with $r = (p/2)' > 1$ we have by (3-22), the Cauchy–Schwarz inequality and (3-20) that

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(m_n\Delta^{j,2}_{n,s}\hat{f})(x)\big|^2 g(x) &\lesssim \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}} M_{2^n}\big(\big|\mathcal{F}^{-1}(\Delta^{j,2}_{n,s}\hat{f})\big|\big)(x)^2 g(x)\\ &\le \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}} M_{2^n}\big(\big|\mathcal{F}^{-1}(\Delta^{j,2}_{n,s}\hat{f})\big|^2\big)(x) g(x)\\ &= \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta^{j,2}_{n,s}\hat{f})(x)\big|^2 M^*_{2^n} g(x)\\ &\lesssim \bigg\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta^{j,2}_{n,s}\hat{f})\big|^2\Big)^{1/2}\bigg\|^2_{\ell^p}\Big\|\sup_{N\in\mathbb{N}} M^*_N g\Big\|_{\ell^r}\\ &\lesssim \bigg\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta^{j,2}_{n,s}\hat{f})\big|^2\Big)^{1/2}\bigg\|^2_{\ell^p}\|g\|_{\ell^r}. \end{aligned}\tag{3-40}$$

Therefore, by Theorem 3.3 we have

$$\bigg\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(m_n\Delta^{j,2}_{n,s}\hat{f})\big|^2\Big)^{1/2}\bigg\|_{\ell^p}\lesssim\bigg\|\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{F}^{-1}(\Delta^{j,2}_{n,s}\hat{f})\big|^2\Big)^{1/2}\bigg\|_{\ell^p}\lesssim \log s\|f\|_{\ell^p}. \tag{3-41}$$

We refine the estimate in (3-41) for $p=2$. Indeed, define

$$\varrho_{n,j}(\xi)=\big(\eta(2^{nA+jI}\xi)^2-\eta(2^{nA+(j+1)I}\xi)^2\big)\eta(2^{s(A-\chi I)}\xi),$$
$$\Psi_n(\xi)=\min\big\{|2^{nA}\xi|_\infty,\,|2^{nA}\xi|_\infty^{-1/d},\,1\big\}.$$

By Plancherel's theorem we have

$$\begin{aligned}&\bigg\|\Big(\sum_{n\ge\max\{j,-j/\chi,s\}}\big|\mathcal{F}^{-1}(m_n\Delta^{j,2}_{n,s}\hat{f})\big|^2\Big)^{1/2}\bigg\|_{\ell^2}\\ &\qquad=\bigg(\int_{\mathbb{T}^d}\sum_{n\ge\max\{j,-j/\chi,s\}}\ \sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}}|m_n(\xi)|^2\varrho_{n,j}(\xi-a/q)^2|\hat{f}(\xi)|^2\,d\xi\bigg)^{1/2}\\ &\qquad\lesssim (s+1)^{-\delta l}2^{-|j|/(2d)}\|f\|_{\ell^2}.\end{aligned}\tag{3-42}$$

The last estimate is implied by (3-30). Namely, by (3-30) we may write

$$\begin{aligned}&\sum_{n\ge\max\{j,-j/\chi,s\}}\ \sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}}|m_n(\xi)|^2\varrho_{n,j}(\xi-a/q)^2\\ &\quad\lesssim\sum_{n\ge\max\{j,-j/\chi,s\}}\ \sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}} q^{-2\delta}(\Psi_n(\xi-a/q)+2^{-n/2})(2^{-|j|/d}+2^{-n/2})\eta(2^{s(A-\chi I)}(\xi-a/q))^2\\ &\quad\lesssim (s+1)^{-2\delta l}2^{-|j|/(2d)}.\end{aligned}\tag{3-43}$$

The last line follows, since we have used the lower bound for $q\ge s^l$ if $a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}$. Moreover,

$$\sum_{n\ge 0}(\Psi_n(\xi-a/q)+2^{-n/2})\lesssim 1\quad\text{and}\quad\sum_{a/q\in\mathscr{U}_{(s+1)^l}\setminus\mathscr{U}_{s^l}}\eta(2^{s(A-\chi I)}(\xi-a/q))\lesssim 1$$

by the disjointness of the supports of the $\eta(2^{s(A-\chi l)}(\xi - a/q))$ whenever $a/q \in \mathscr{U}_{(s+1)^l} \setminus \mathscr{U}_{s^l}$. Since $l \in \mathbb{N}$ can be as large as we wish, interpolating (3-42) with (3-41) we obtain (3-39) and the proof of (3-1) and consequently Theorem A is completed. □

Acknowledgments

I am grateful to Wojtek Samotij for explaining a very nice probabilistic argument which simplified the proof of Lemma 2.4.

I am also grateful to the referee for careful reading of the manuscript and useful remarks that led to the improvement of the presentation.

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Received 18 Jun 2016. Revised 22 Jul 2017. Accepted 20 Sep 2017.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANALYSIS & PDE

Volume 11 No. 3 2018