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THE SEMIGROUP GENERATED BY THE DIRICHLET LAPLACIAN OF FRACTIONAL ORDER

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In the whole space $\mathbb{R}^d$, linear estimates for heat semigroup in Besov spaces are well established, which are estimates of L^p–L^q type, with maximal regularity, etc. This paper is concerned with such estimates for the semigroup generated by the Dirichlet Laplacian of fractional order in terms of the Besov spaces on an arbitrary open set of $\mathbb{R}^d$.

1. Introduction

Let Ω be an arbitrary open set of $\mathbb{R}^d$ with $d \geq 1$. We consider the Dirichlet Laplacian A on $L^2(\Omega)$,

$$A = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

with the domain

$$\mathcal{D}(A) := \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}.$$

We consider the fractional Laplacian and the semigroup

$$A^{\frac{\alpha}{2}} = \int_{-\infty}^{\infty} \lambda^{\frac{\alpha}{2}}\, dE_A(\lambda), \quad e^{-tA^{\alpha/2}} = \int_{-\infty}^{\infty} e^{-t\lambda^{\alpha/2}}\, dE_A(\lambda), \quad t \geq 0.$$

Here, $\alpha > 0$ and $\{E_A(\lambda)\}_{\lambda \in \mathbb{R}}$ denotes the spectral resolution of identity, which is determined uniquely for the self-adjoint operator A by the spectral theorem. The motivation of the study of the fractional Laplacian comes from the study of fluid mechanics, stochastic processes, finance etc.; see for instance [Applebaum 2009; Bertoin 1996; Chen et al. 2010; Vlahos et al. 2008]. We also refer to [Di Nezza et al. 2012; Vázquez 2012; 2014], where one can find some results on fundamental properties of fractional Sobolev spaces and applications to partial differential equations.

In the paper [Iwabuchi et al. 2016a], based on spectral theory for the Dirichlet Laplacian A on $L^2(\Omega)$, a kind of L^p theory was established and the Besov spaces on an open set Ω were introduced, where regularity of functions is measured by A. The purpose of this paper is to develop linear estimates for the semigroup generated by the Dirichlet Laplacian of fractional order in the homogeneous Besov spaces $\dot{B}^s_{p,q}(A)$, namely, the estimate of L^p–L^q type, smoothing effects, continuity in time of the semigroup, equivalent norms with the semigroup and maximal regularity estimates. Such estimates with the heat semigroup in the case when $\Omega = \mathbb{R}^d$ are well established; see [Bahouri et al. 2011; Chemin 2004; Danchin 2005; 2007; Danchin and Mucha 2009; Hieber and Prüss 1997; Kozono et al. 2003; Lemarié-Rieusset

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2002; Ogawa and Shimizu 2010; 2016; Peetre 1976]. In this paper we consider open sets of $\mathbb{R}^d$ and the semigroup generated by the fractional Laplacian with the Dirichlet boundary condition.

Let us recall the definitions of spaces of test functions and tempered distributions and the Besov spaces associated with the Dirichlet Laplacian; see [Iwabuchi et al. 2016a]. We take $\phi_0(\,\cdot\,) \in C_0^\infty(\mathbb{R})$ to be a nonnegative function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi_0 \subset \{\lambda \in \mathbb{R} : 2^{-1} \le \lambda \le 2\}, \quad \sum_{j\in\mathbb{Z}} \phi_0(2^{-j}\lambda) = 1 \quad \text{for } \lambda > 0, \tag{1-1}$$

and $\{\phi_j\}_{j\in\mathbb{Z}}$ is defined by letting

$$\phi_j(\lambda) := \phi_0(2^{-j}\lambda) \quad \text{for } \lambda \in \mathbb{R}. \tag{1-2}$$

Definition. (i) (linear topological spaces $\mathcal{X}_0(\Omega)$ and $\mathcal{X}_0'(\Omega)$) $\mathcal{X}_0(\Omega)$ is defined by letting

$$\mathcal{X}_0(\Omega) := \{ f \in L^1(\Omega) \cap \mathcal{D}(A) : A^M f \in L^1(\Omega) \cap \mathcal{D}(A) \text{ for all } M \in \mathbb{N} \},$$

equipped with the family of seminorms $\{p_{0,M}(\,\cdot\,)\}_{M=1}^\infty$ given by

$$p_{0,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{N}} 2^{Mj} \|\phi_j(\sqrt{A}) f\|_{L^1(\Omega)}.$$

(ii) (linear topological spaces $\mathcal{Z}_0(\Omega)$ and $\mathcal{Z}_0'(\Omega)$) $\mathcal{Z}_0(\Omega)$ is defined by letting

$$\mathcal{Z}_0(\Omega) := \Big\{ f \in \mathcal{X}_0(\Omega) : \sup_{j\le 0} 2^{M|j|} \|\phi_j(\sqrt{A}) f\|_{L^1(\Omega)} < \infty \text{ for all } M \in \mathbb{N} \Big\},$$

equipped with the family of seminorms $\{q_{0,M}(\,\cdot\,)\}_{M=1}^\infty$ given by

$$q_{0,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{Z}} 2^{M|j|} \|\phi_j(\sqrt{A}) f\|_{L^1(\Omega)}.$$

Definition. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, $\dot{B}^s_{p,q}(A)$ is defined by letting

$$\dot{B}^s_{p,q}(A) := \{ f \in \mathcal{Z}_0'(\Omega) : \|f\|_{\dot{B}^s_{p,q}(A)} < \infty \},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(A)} := \big\| \{ 2^{sj} \|\phi_j(\sqrt{A}) f\|_{L^p(\Omega)} \}_{j\in\mathbb{Z}} \big\|_{\ell^q(\mathbb{Z})}.$$

Let us mention the basic properties of $\mathcal{X}_0(\Omega)$, $\mathcal{Z}_0(\Omega)$, their duals, and $\dot{B}^s_{p,q}(A)$ and explain the operators $\phi_j(\sqrt{A})$ and the Laplacian of fractional order.

Proposition [Iwabuchi et al. 2016a]. *Let $s, \alpha \in \mathbb{R}$ and $1 \le p, q, r \le \infty$. Then the following hold*:

(i) *$\mathcal{X}_0(\Omega)$ and $\mathcal{Z}_0(\Omega)$ are Fréchet spaces and enjoy $\mathcal{X}_0(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{X}_0'(\Omega)$ and $\mathcal{Z}_0(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{Z}_0'(\Omega)$.*

(ii) *$\dot{B}^s_{p,q}(A)$ is a Banach space and enjoys $\mathcal{Z}_0(\Omega) \hookrightarrow \dot{B}^s_{p,q}(A) \hookrightarrow \mathcal{Z}_0'(\Omega)$.*

(iii) *If $p, q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, the dual space of $\dot{B}^s_{p,q}(A)$ is $\dot{B}^{-s}_{p',q'}(A)$.*

(iv) *If $r \le p$, then $\dot{B}^{s+d(\frac{1}{r}-\frac{1}{p})}_{r,q}(A)$ is embedded to $\dot{B}^s_{p,q}(A)$.*

(v) *For any $f \in \dot{B}^{s+\alpha}_{p,q}(A)$, we have $A^{\frac{\alpha}{2}} f \in \dot{B}^s_{p,q}(A)$.*

It should be noted that $\phi_j(\sqrt{A})$ and A are defined in $L^2(\Omega)$ initially and by the argument in [Iwabuchi et al. 2016a] they can be realized as operators in $\mathcal{Z}_0'(\Omega)$ and Besov spaces. In the proof, the uniform boundedness in $L^p(\Omega)$ of $\phi_j(\sqrt{A})$ with respect to $j \in \mathbb{Z}$ is essential. Uniformity in $L^2(\Omega)$ is proved easily by the spectral theorem, while that in $L^1(\Omega)$ is not trivial. For any open set $\Omega \subset \mathbb{R}^d$, $L^1(\Omega)$ boundedness is known in some papers; see Proposition 6.1 in [Thinh Duong et al. 2002] and also Theorem 1.1 in [Iwabuchi et al. 2017]. Let us explain the strategy of its proof as in [Iwabuchi et al. 2017] (see also the comment below Lemma 2.2). The uniform boundedness in $L^1(\Omega)$ is proved via estimates in amalgam spaces $\ell^1(L^2)_\theta$, where the side length of each cube is scaled by $\theta^{\frac{1}{2}}$, $\theta = 2^{-2j}$ (see Section 2), together with the Gaussian upper bounds of the kernel of e^{-tA}. That scaling fits for the scaled operator $\phi_j(\sqrt{A}) = \phi_0(2^{-j}\sqrt{A})$, and we can handle the norm in $\ell^1(L^2)_\theta$ through the estimates in $L^2(\Omega)$, since its norm is defined locally with $L^2(\Omega)$. The Gaussian upper bounds of the kernel of e^{-tA} are necessary in order to estimate the $L^1(\Omega)$ norm via $\ell^1(L^2)_\theta$. Once the $L^1(\Omega)$ estimate is proved, the $L^p(\Omega)$ case is assured by interpolation and a duality argument.

As for the Laplacian of fractional order, it was shown in the proof of Proposition 3.2 in [Iwabuchi et al. 2016a] that $A^{\frac{\alpha}{2}}$ is a continuous operator from $\mathcal{Z}_0'(\Omega)$ to itself, which is proved as follows: Show the continuity of $A^{\frac{\alpha}{2}}$ in $\mathcal{Z}_0(\Omega)$ first with the boundedness of spectral multipliers

$$\|A^{\frac{\alpha}{2}}\phi_j(\sqrt{A})\|_{L^1(\Omega)\to L^1(\Omega)} \le C2^{\alpha j}$$

for all $j \in \mathbb{Z}$ and consider their dual operator together with the approximation of the identity

$$f = \sum_{j\in\mathbb{Z}} \phi_j(\sqrt{A})f \quad \text{in } \mathcal{Z}_0'(\Omega) \text{ for any } f \in \mathcal{Z}_0'(\Omega).$$

Hence, we define $A^{\frac{\alpha}{2}}$ by

$$A^{\frac{\alpha}{2}} f = \sum_{j\in\mathbb{Z}} (A^{\frac{\alpha}{2}}\phi_j(\sqrt{A}))f \quad \text{in } \mathcal{Z}_0'(\Omega) \text{ for any } f \in \mathcal{Z}_0'(\Omega).$$

Noting that $e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})$ with $t \ge 0$ is also bounded in $L^1(\Omega)$ (see Lemma 2.1 and (3-1) below), we also define $e^{-tA^{\alpha/2}}$ by

$$e^{-tA^{\alpha/2}} f = \sum_{j\in\mathbb{Z}} (e^{-tA^{\alpha/2}}\phi_j(\sqrt{A}))f \quad \text{in } \mathcal{Z}_0'(\Omega) \text{ for any } f \in \mathcal{Z}_0'(\Omega).$$

We state four theorems on the semigroup generated by $A^{\frac{\alpha}{2}}$: the estimates of L^p–L^q type and smoothing effects, continuity in time, equivalent norms with semigroup and maximal regularity estimates, referring to the results in the case when $\Omega = \mathbb{R}^d$ and $\alpha = 2$.

We start by considering estimates of L^p–L^q type and smoothing effects. When $\Omega = \mathbb{R}^n$, it is well known that

$$\|e^{t\Delta} f\|_{L^q(\mathbb{R}^d)} \le Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}, \quad \|\nabla e^{t\Delta} f\|_{L^p(\mathbb{R}^d)} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\mathbb{R}^d)},$$

where $1 \le p, q \le \infty$ and $f \in L^p(\mathbb{R}^d)$. Hence one can show that

$$\|e^{t\Delta} f\|_{\dot{B}^{s_2}_{p_2,q}(A)} \le Ct^{-\frac{d}{2}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2-s_1}{2}}\|f\|_{\dot{B}^{s_1}_{p_1,q}(A)},$$

where $s_2 \ge s_1,\ 1 \le p_1 \le p_2 \le \infty$ and $1 \le q \le \infty$. The following gives the linear estimates for the semigroup generated by $A^{\frac{\alpha}{2}}$ on an open set.

Theorem 1.1. *Let $\alpha > 0,\ t \ge 0,\ s, s_1, s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$:*

(i) *$e^{-tA^{\alpha/2}}$ is a bounded linear operator in $\dot{B}^s_{p,q}(A)$; i.e., there exists a constant $C > 0$ such that for any $f \in \dot{B}^s_{p,q}(A)$*

$$e^{-tA^{\alpha/2}} f \in \dot{B}^s_{p,q}(A) \quad \text{and} \quad \|e^{-tA^{\alpha/2}} f\|_{\dot{B}^s_{p,q}(A)} \le C \|f\|_{\dot{B}^s_{p,q}(A)}. \tag{1-3}$$

(ii) *If $s_2 \ge s_1,\ p_1 \le p_2$ and*

$$d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) + s_2 - s_1 > 0,$$

then there exists a constant $C > 0$ such that

$$\|e^{-tA^{\alpha/2}} f\|_{\dot{B}^{s_2}_{p_2,q_2}(A)} \le C t^{-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2-s_1}{\alpha}} \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(A)} \tag{1-4}$$

for any $f \in \dot{B}^{s_1}_{p_1,q_1}(A)$.

Remark. In the estimate (1-4), the regularity on indices q_1 and q_2 is gained without loss of the singularity at $t = 0$. This estimate is known in the case when $\Omega = \mathbb{R}^n$ and $\alpha = 2$; see [Kozono et al. 2003].

As for the continuity in time of the heat semigroup $e^{t\Delta}$ when $\Omega = \mathbb{R}^d$, it is well known that for $1 \le p < \infty$

$$\lim_{t \to 0} \|e^{t\Delta} f - f\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{for any } f \in L^p(\mathbb{R}^d).$$

In the case when $p = \infty$, the above strong convergence does not hold in general, while it holds in the dual weak sense. The following theorem is concerned with such continuity in the Besov spaces on an open set.

Theorem 1.2. *Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$:*

(i) *Assume that $q < \infty$ and $f \in \dot{B}^s_{p,q}(A)$. Then*

$$\lim_{t \to 0} \|e^{-tA^{\alpha/2}} f - f\|_{\dot{B}^s_{p,q}(A)} = 0.$$

(ii) *Assume that $1 < p \le \infty$, $q = \infty$ and $f \in \dot{B}^s_{p,\infty}(A)$. Then $e^{-tA^{\alpha/2}} f$ converges to f in the dual weak sense as $t \to 0$; namely,*

$$\lim_{t \to 0} \sum_{j \in \mathbb{Z}} \int_\Omega \{\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} f - f)\} \overline{\Phi_j(\sqrt{A}) g}\, dx = 0$$

for any $g \in \dot{B}^{-s}_{p',1}(A)$.

Remark. Related to Theorem 1.2(ii), it should be noted that the predual of $\dot{B}^s_{p,q}(A)$ is $\dot{B}^{-s}_{p',q'}(A)$ for $1 < p, q \le \infty$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. In fact, we can regard $f \in \dot{B}^s_{p,q}(A)$ as an element of the dual of $\dot{B}^{-s}_{p',q'}(A)$ by

$$\langle f, g \rangle = \sum_{j \in \mathbb{Z}} \int_\Omega \{\phi_j(\sqrt{A}) f\} \overline{\Phi_j(\sqrt{A}) g}\, dx$$

for any $g \in \dot{B}^{-s}_{p',q'}(A)$, see [Iwabuchi et al. 2016a], where $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$.

As for the characterization of the norm by using the semigroup when $\Omega = \mathbb{R}^d$, it is known that

$$\|f\|_{\dot{B}^s_{p,q}(A)} \simeq \left\{ \int_0^\infty \left(t^{-\frac{s}{2}} \|e^{t\Delta} f\|_{L^p(\mathbb{R}^d)}\right)^q \frac{dt}{t} \right\}^{\frac{1}{q}},$$

where $s < 0$; see, e.g., [Lemarié-Rieusset 2002]. We consider the equivalent norm of Besov spaces on an open set by using the semigroup generated by $A^{\frac{\alpha}{2}}$.

Theorem 1.3. *Let $\alpha > 0$, $s, s_0 \in \mathbb{R}$, $s_0 > \frac{s}{\alpha}$ and $1 \le p, q \le \infty$. Then there exists a constant $C > 0$ such that*

$$C^{-1} \|f\|_{\dot{B}^s_{p,q}(A)} \le \left\{ \int_0^\infty \left(t^{-\frac{s}{\alpha}} \|(tA^{\frac{\alpha}{2}})^{s_0} e^{-tA^{\alpha/2}} f\|_X\right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \le C \|f\|_{\dot{B}^s_{p,q}(A)} \tag{1-5}$$

for any $f \in \dot{B}^s_{p,q}(A)$, where $X = L^p(\Omega)$, $\dot{B}^0_{p,r}(A)$ with $1 \le r \le \infty$.

Since the equivalence (1-5) is closely related to the real interpolation in the Besov spaces, we mention that the interpolation is also available; see, e.g., [Bergh and Löfström 1976; Triebel 1983] and also Proposition A.1 in the Appendix.

The last result is concerned with the maximal regularity estimates. When $\Omega = \mathbb{R}^d$, the Cauchy problem which we should consider is

$$\begin{cases} \partial_t u - \Delta u = f, & t > 0,\ x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

For $1 < p, q < \infty$, the solution u of the above problem satisfies

$$\|\partial_t u\|_{L^q(0,\infty;L^p(\mathbb{R}^d))} + \|\Delta u\|_{L^q(0,\infty;L^p(\mathbb{R}^d))} \le C \|u_0\|_{\dot{B}^{2-2/q}_{p,q}(A)} + C \|f\|_{L^q(0,\infty;L^p(\mathbb{R}^d))},$$

provided that $u_0 \in \dot{B}^{2-\frac{2}{q}}_{p,q}(A)$ and $f \in L^q(0, \infty; L^p(\mathbb{R}^d))$; see [Hieber and Prüss 1997; Lemarié-Rieusset 2002]. We note that maximal regularity such as the above is well-studied in the general framework on Banach spaces with unconditional martingale differences (UMD); see [Amann 1995; Da Prato and Grisvard 1975; Denk et al. 2003; Dore and Venni 1987; Ladyzhenskaya and Ural'tseva 1968; Weis 2001]. We also note that the cases when $p, q = 1, \infty$ require a different treatment from UMD since the spaces are not reflexive. In terms of Besov spaces, one can consider $\dot{B}^0_{p,q}(A)$ for all indices p, q with $1 \le p, q \le \infty$; see [Danchin 2005; 2007; Danchin and Mucha 2009; Hieber and Prüss 1997; Ogawa and Shimizu 2010; 2016]. Our result on the maximal regularity estimates on open sets is formulated in the following way.

Theorem 1.4. *Let $s \in \mathbb{R}$, $\alpha > 0$ and $1 \le p, q \le \infty$. Assume that $u_0 \in \dot{B}^{s+\alpha-\frac{\alpha}{q}}_{p,q}(A)$, $f \in L^q(0, \infty; \dot{B}^s_{p,q}(A))$. Let u be given by*

$$u(t) = e^{-tA^{\alpha/2}} u_0 + \int_0^t e^{-(t-\tau)A^{\alpha/2}} f(\tau)\, d\tau.$$

Then there exists a constant $C > 0$ independent of u_0 and f such that

$$\|\partial_t u\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))} + \|A^{\frac{\alpha}{2}} u\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))} \le C \|u_0\|_{\dot{B}^{s+\alpha-\alpha/q}_{p,q}(A)} + C \|f\|_{L^q(0,\infty;\dot{B}^s_{p,q}(A))}. \tag{1-6}$$

The proofs of our theorems are based on the boundedness of the spectral multiplier of the operator $e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})$:

$$\left\|e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})\right\|_{L^p(\Omega)\to L^p(\Omega)} \le C\left\|e^{-t2^{\alpha j}(\cdot)^\alpha}\phi_0(\sqrt{\cdot})\right\|_{H^s(\mathbb{R})} \quad \text{for all } j\in\mathbb{Z},$$

where $s > \frac{d}{2}+\frac{1}{2}$ (see Lemma 2.1 below). The above inequality implies

$$\left\|e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})\right\|_{L^p(\Omega)\to L^p(\Omega)} \le Ce^{-C^{-1}t2^{\alpha j}},$$

and this estimate allows us to show our theorems in a method analogous to those in the case when $\Omega=\mathbb{R}^d$. In this paper, we give proofs of all theorems by estimating directly so that the paper is self-contained. Here, we note that our proofs can be applicable to the estimates for e^{-tA} in the inhomogeneous Besov spaces and hence similar theorems are able to be obtained. On the other hand, for the semigroup generated by the fractional Laplacian, since there appear to be problems around low frequencies, we show only the result for the heat semigroup in Section 7 (see Theorem 7.2 below). It should be also noted that our argument can be applied not only to the Dirichlet Laplacian but also to more general operators A such that the Gaussian upper bounds for e^{-tA} hold.

This paper is organized as follows. In Section 2, we prepare a lemma to prove our results. Sections 3–6 are devoted to proofs of theorems. In Section 7, we state the result for the inhomogeneous Besov spaces. In the Appendix, we show the characterization of Besov spaces by real interpolation.

Before closing this section, we introduce some notation. We denote by $\|\cdot\|_{L^p}$ the $L^p(\Omega)$ norm and by $\|\cdot\|_{\dot{B}^s_{p,q}}$ the $\dot{B}^s_{p,q}(A)$ norm. We use the notation $\|\cdot\|_{H^s(\mathbb{R})}$ to represent the $H^s(\mathbb{R})$ norm for functions, e.g., $\phi_j(\lambda)$, $e^{-t\lambda^{\alpha/2}}$, whose variables are spectral parameters. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class.

2. Preliminaries

In this section we introduce the following lemma on the boundedness of the scaled spectral multiplier.

Lemma 2.1. *Let $N>\frac{d}{2}$, $1\le p\le\infty$, $\delta>0$ and $a,b>0$. Then there exists a constant $C>0$ such that for any $\phi\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\phi\subset[a,b]$, $G\in C^\infty((0,\infty))\cap C(\mathbb{R})$ and $f\in L^p(\Omega)$ we have*

$$\left\|G(\sqrt{A})\phi(2^{-j}\sqrt{A})f\right\|_{L^p} \le C\left\|G(2^j\sqrt{\cdot})\phi(\sqrt{\cdot})\right\|_{H^{N+1/2+\delta}(\mathbb{R})}\|f\|_{L^p} \tag{2-1}$$

for all $j\in\mathbb{Z}$.

Remark. As is seen from the proof below, the constant C on the right-hand side of (2-1) depends on the interval $[a,b]$ containing the support of ϕ.

To prove Lemma 2.1, we introduce a set $\mathscr{A}_N$ of some bounded operators on $L^2(\Omega)$ and scaled amalgam spaces $\ell^1(L^2)_\theta$ for $\theta>0$ to prepare a lemma. Hereafter, for $k\in\mathbb{Z}^d$, $C_\theta(k)$ denotes a cube with the center $\theta^{\frac{1}{2}}k$ and side length $\theta^{\frac{1}{2}}$, namely,

$$C_\theta(k) := \left\{x\in\Omega : |x_j-\theta^{\frac{1}{2}}k_j| \le 2^{-1}\theta^{\frac{1}{2}} \text{ for } j=1,2,\ldots,d\right\},$$

and $\chi_{C_\theta(k)}$ is a characteristic function whose support is $C_\theta(k)$.

Definition. For $N \in \mathbb{N}$, we denote by $\mathscr{A}_N$ the set of all bounded operators T on $L^2(\Omega)$ such that

$$\|T\|_{\mathscr{A}_N} := \sup_{k\in\mathbb{Z}^d} \big\| |\cdot - \theta^{\frac{1}{2}} k|^N T \chi_{C_\theta(k)} \big\|_{L^2\to L^2} < \infty.$$

Definition. The space $\ell^1(L^2)_\theta$ is defined by letting

$$\ell^1(L^2)_\theta := \{ f \in L^2_{\mathrm{loc}}(\overline{\Omega}) : \|f\|_{\ell^1(L^2)_\theta} < \infty \},$$

where

$$\|f\|_{\ell^1(L^2)_\theta} := \sum_{k\in\mathbb{Z}^d} \|f\|_{L^2(C_\theta(k))}.$$

Lemma 2.2 [Iwabuchi et al. 2017; Iwabuchi et al. 2016b]. (i) *Let $N \in \mathbb{N}$ and $N > \frac{d}{2}$. Then there exists a constant $C > 0$ such that*

$$\|T\|_{\ell^1(L^2)_\theta \to \ell^1(L^2)_\theta} \le C\big(\|T\|_{L^2\to L^2} + \theta^{-\frac{d}{4}} \|T\|_{\mathscr{A}_N}^{\frac{d}{2N}} \|T\|_{L^2\to L^2}^{1-\frac{d}{2N}} \big) \tag{2-2}$$

for any $T \in \mathscr{A}_N$ and $\theta > 0$.

(ii) *Let $N \in \mathbb{N}$. Then there exists a constant $C > 0$ such that*

$$\|\psi((M+\theta A)^{-1})\|_{\mathscr{A}_N} \le C\theta^{\frac{N}{2}} \int_{-\infty}^{\infty} (1+|\xi|^2)^{\frac{N}{2}} |\hat{\psi}(\xi)|\, d\xi \tag{2-3}$$

for any $\psi \in \mathcal{S}(\mathbb{R})$ and $\theta > 0$.

(iii) *Let $M > 0$ and $\beta > \frac{d}{4}$. Then there exists a constant $C > 0$ such that*

$$\|(M+\theta A)^{-\beta}\|_{L^1\to\ell^1(L^2)_\theta} \le C\theta^{-\frac{d}{2}} \tag{2-4}$$

for any $\theta > 0$.

Remark. Lemma 2.2 is useful to prove the L^1 boundedness of spectral multipliers and let us briefly remind how to prove Lemma 2.2 as in [Iwabuchi et al. 2017; Iwabuchi et al. 2016b]. The original idea is by Jensen and Nakamura [1994; 1995], who studied the Schrödinger operators on $\mathbb{R}^d$. In the first inequality (2-2), we start with the decomposition $T = \sum_{m,k\in\mathbb{Z}^d} \chi_{C_\theta(m)} T \chi_{C_\theta(k)}$, and it suffices to show that for each $k \in \mathbb{Z}$ a sum of operator norms $\sum_{m\in\mathbb{Z}} \|\chi_{C_\theta(m)} T \chi_{C_\theta(k)}\|_{L^2\to L^2}$ is bounded by the right-hand side of (2-2). The first term $\|T\|_{L^2\to L^2}$ is obtained just by applying $L^2(\Omega)$ boundedness to the $L^2(C_\theta(m))$ norm with $m = k$. The second term is obtained by decomposing the sum into two cases when $0 < |m-k| \le \omega$ and $|m-k| > \omega$ for $\omega > 0$, applying the $L^2(\Omega)$ boundedness to the case $|m-k| \le \omega$ and the Schwarz inequality to the case $|m-k| > \omega$ for sequences $|m-k|^{-N}$, $|m-k|^N \|\chi_{C_\theta(m)} T \chi_{C_\theta(k)}\|_{L^2}$, and minimizing by taking suitable ω. As for the second one (2-3), we utilize the formula

$$\psi((M+\theta A)^{-1}) = (2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-it(M+\theta A)^{-1}} \hat{\psi}(t)\, dt.$$

To estimate $\|e^{-it(M+\theta A)^{-1}}\|_{A_N}$, we consider the commutator of $(x - \theta^{1/2} k)$ and $e^{-it(M+\theta A)^{-1}}$, which is rewritten with θ, $(M+\theta A)^{-1}$, $\nabla(M+\theta A)^{-1}$ and is able to be handled by the use of $L^2(\Omega)$

boundedness, which proves (2-3). As for the last one (2-4), thanks to the formula

$$(M+\theta A)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-Mt} e^{-t\theta A}\, dt$$

and the Young inequality, we get

$$\|(M+\theta A)^{-\beta} f\|_{\ell^1(L^2)_\theta} \le \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-Mt} \left(\int_\Omega \|e^{-t\theta A}(\,\cdot\,, y)\|_{\ell^1(L^2)_\theta} |f(y)|\, dy \right) dt,$$

where $\Gamma(\beta)$ is the Gamma function. By the Gaussian upper bounds of $e^{-t\theta A}$, we have

$$\|e^{-t\theta A}(\,\cdot\,, y)\|_{\ell^1(L^2)_\theta} \le C\theta^{-\frac{d}{4}}(1+t^{-\frac{d}{4}}).$$

These estimates yield the inequality (2-4), since the integrability with respect to $t \in (0,\infty)$ is assured by $\beta > \frac{4}{d}$.

Proof of Lemma 2.1. Instead of the inequality (2-1), by replacing $2^{-j}\sqrt{A}$ and $\sqrt{A}$ with $2^{-2j}A$ and A, respectively, it is sufficient to show that

$$\|G(A)\phi(2^{-2j}A) f\|_{L^p} \le C\|G(2^{2j}\,\cdot)\phi(\,\cdot\,)\|_{H^{N+1/2+\delta}(\mathbb{R})} \|f\|_{L^p}, \tag{2-5}$$

where $\operatorname{supp}\phi \subset [a^2, b^2]$.

First we consider the case when $p=1$. By decomposing Ω into cubes $C_\theta(k)$ and the Hölder inequality, we get

$$\|G(A) f\phi(2^{-2j}A)\|_{L^1} \le C\theta^{\frac{d}{2}} \|G(A)\phi(2^{-2j}A) f\|_{\ell^1(L^2)_\theta}. \tag{2-6}$$

For fixed real numbers $M>0$ and $\beta > \frac{d}{2}$, let ψ be such that

$$\psi(\mu) := G(2^{2j}(\mu^{-1}-M))\phi(\mu^{-1}-M)\mu^{-\beta}. \tag{2-7}$$

It is easy to check that

$$\psi \in C_0^\infty((0,\infty)) \quad \text{and} \quad \operatorname{supp}\psi \subset \left[\frac{1}{M+b}, \frac{1}{M+a}\right],$$

and

$$G(\lambda)\phi(2^{-2j}\lambda) = G(2^{2j}\cdot 2^{-2j}\lambda)\phi(2^{-2j}\lambda)\mu^{-\beta}\cdot\mu^{\beta} = \psi(\mu)\mu^\beta,$$

where λ and μ are real numbers with

$$2^{-2j}\lambda = \mu^{-1} - M.$$

The above equality yields that

$$G(A)\phi(2^{-2j}A) = \psi((M+2^{-2j}A)^{-1})(M+2^{-2j}A)^{-\beta}. \tag{2-8}$$

Then it follows from (2-6), (2-8) and the estimate (2-4) in Lemma 2.2 that

$$\begin{aligned}
&\|G(A)\phi(2^{-2j}A) f\|_{L^1} \\
&\quad \le C\theta^{\frac{d}{2}} \big\|\psi((M+2^{-2j}A)^{-1})(M+2^{-2j}A)^{-\beta} f\big\|_{\ell^1(L^2)_\theta} \\
&\quad \le C\theta^{\frac{d}{2}} \|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^1)_\theta\to\ell^1(L^2)_\theta} \|(M+2^{-2j}A)^{-\beta}\|_{L^1\to\ell^1(L^2)_\theta} \|f\|_{L^1} \\
&\quad \le C\|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^1)_\theta\to\ell^1(L^2)_\theta} \|f\|_{L^1}.
\end{aligned} \tag{2-9}$$

By comparing the estimates (2-5) and (2-9), all we have to do is to show that

$$\|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^2)_\theta\to\ell^1(L^2)_\theta} \le C\|G(2^j\,\cdot)\phi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}. \tag{2-10}$$

To apply the estimate (2-2), we consider the operator norms $\|\cdot\|_{L^2\to L^2}$ and $\|\cdot\|_{\mathscr{A}_N}$ of $\psi((M+2^{-2j}A)^{-1})$. On the operator norm $\|\cdot\|_{L^2\to L^2}$, we have from $N>\frac{d}{2}$ and the embedding $H^{N+\frac{1}{2}+\delta}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ that

$$\|\psi((M+2^{-2j}A)^{-1})\|_{L^2\to L^2} \le \|\psi\|_{L^\infty(\mathbb{R})} \le \|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}$$

for any $\delta>0$. As for $\|\psi((M+2^{-2j}A)^{-1})\|_{\mathscr{A}_N}$, by applying the estimate (2-3) and the Hölder inequality, for any $\delta>0$ there exists $C>0$ such that

$$\begin{aligned}\|\psi((M+\theta A)^{-1})\|_{\mathscr{A}_N} &\le C\theta^{\frac{N}{2}}\int_{-\infty}^{\infty}(1+|\xi|^2)^{\frac{N}{2}}|\hat{\psi}(\xi)|\,d\xi\\ &\le C\theta^{\frac{N}{2}}\|(1+|\xi|^2)^{-\frac{1}{2}-\delta}\|_{L^2(\mathbb{R})}\|(1+|\xi|^2)^{\frac{N}{2}+\frac{1}{2}+\delta}\hat{\psi}\|_{L^2(\mathbb{R})}\\ &\le C\theta^{\frac{N}{2}}\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}.\end{aligned}$$

Then we deduce from the above two estimates and (2-2) that

$$\begin{aligned}&\|\psi((M+2^{-2j}A)^{-1})\|_{\ell^1(L^2)_\theta\to\ell^1(L^2)_\theta}\\ &\qquad\le C\bigl\{\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}+\theta^{-\frac{d}{4}}(\theta^{\frac{N}{2}}\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})})^{\frac{d}{2N}}(\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})})^{1-\frac{d}{2N}}\bigr\}\\ &\qquad\le C\|\psi\|_{H^{N+1/2+\delta}(\mathbb{R})}.\end{aligned}$$

Since ψ is defined by (2-7) and the support is bounded and away from the origin, we see from the change of variables by $\mu=(\lambda+M)^{-1}$ that

$$\|\psi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})} \le C\|G(2^{2j}\,\cdot)\phi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}.$$

Hence the estimate (2-10) is obtained by the above two estimates, and the estimate (2-5) in the case when $p=1$ is proved.

We next consider the case when $p=\infty$. Since the dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$, the following holds:

$$\|G(A)\phi(2^{-j}A)f\|_{L^\infty} = \sup_{g\in C_0^\infty,\,\|g\|_{L^1}=1}\left|\int_\Omega\bigl(G(A)\phi(2^{-j}A)f\bigr)\bar{g}\,dx\right|.$$

On the right-hand side of the above equality, we have from the duality argument for the operator $G(A)\phi(2^{-j}A)$, the Hölder inequality and the estimate (2-5) with $p=1$ that

$$\begin{aligned}\left|\int_\Omega\bigl(G(A)\phi(2^{-j}A)f\bigr)\,\bar{g}\,dx\right| &= \bigl|{}_{\mathcal{X}_0'}\langle G(A)\phi(2^{-j}A)f,g\rangle_{\mathcal{X}_0}\bigr| = \bigl|{}_{\mathcal{X}_0'}\langle f,G(A)\phi(2^{-j}A)g\rangle_{\mathcal{X}_0}\bigr|\\ &= \left|\int_\Omega f\,\overline{G(A)\phi(2^{-j}A)g}\,dx\right| \le \|f\|_{L^\infty}\|G(A)\phi(2^{-j}A)g\|_{L^1}\\ &\le \|f\|_{L^\infty}\|G(2^{2j}\,\cdot)\phi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|g\|_{L^1},\end{aligned}$$

where $g\in C_0^\infty$. This proves (2-5) in the case when $p=\infty$.

As for the case when $1<p<\infty$, the Riesz–Thorin theorem allows us to obtain the estimate (2-5). □

3. Proof of Theorem 1.1

Proof of (1-3). Put $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. By applying the estimate (2-1) in Lemma 2.1 with

$$G = G_t(\lambda) = e^{-t\lambda^\alpha},$$

we have

$$\begin{aligned}\big\|\phi_j(\sqrt{A})e^{-tA^{\alpha/2}}f\big\|_{L^p} &= \big\|\big(G_t(\sqrt{A})\Phi_j(\sqrt{A})\big)(\phi_j(\sqrt{A})f)\big\|_{L^p}\\ &\le C\big\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\big\|_{H^{N+1/2+\delta}(\mathbb{R})}\big\|\phi_j(\sqrt{A})f\big\|_{L^p},\end{aligned}$$

where $N > \frac{d}{2}$ and $\delta > 0$. Here it is easy to check that there exists $C > 0$ such that

$$\big\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\big\|_{H^{N+1/2+\delta}(\mathbb{R})} \le Ce^{-C^{-1}t2^{\alpha j}} \quad \text{for any } j \in \mathbb{Z},$$

and hence,

$$\big\|\phi_j(\sqrt{A})e^{-tA^{\alpha/2}}f\big\|_{L^p} \le Ce^{-C^{-1}t2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p} \quad \text{for any } j \in \mathbb{Z}. \tag{3-1}$$

By multiplying 2^{sj} and taking the $\ell^q(\mathbb{Z})$ norm in the above inequality, we obtain the assertion (1-3). □

Proof of (1-4). By the inequalities

$$\|e^{-tA^{\alpha/2}}f\|_{\dot{B}^{s_2}_{p_2,q_2}} \le \|e^{-tA^{\alpha/2}}f\|_{\dot{B}^{s_2}_{p_2,1}}, \quad \|f\|_{\dot{B}^{s_1}_{p_1,\infty}} \le \|f\|_{\dot{B}^{s_1}_{p_1,q_1}},$$

which are assured from the embedding relations in the Besov spaces, and taking $s_1 = 0$ for the sake of simplicity, it is sufficient to show

$$\|e^{-tA^{\alpha/2}}f\|_{\dot{B}^{s_2}_{p_2,1}} \le Ct^{-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})-\frac{s_2}{\alpha}}\|f\|_{\dot{B}^0_{p_1,\infty}}, \tag{3-2}$$

where

$$s_2 \ge 0, \quad p_1 \le p_2 \quad \text{and} \quad d\Big(\frac{1}{p_1}-\frac{1}{p_2}\Big)+s_2 > 0.$$

It follows from the embedding $\dot{B}^{s_2+d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_1,1} \hookrightarrow \dot{B}^{s_2}_{p_2,1}$ and the estimate (3-1) that

$$\|e^{-tA^{\alpha/2}}f\|_{\dot{B}^{s_2}_{p_2,1}} \le C\|e^{-tA^{\alpha/2}}f\|_{\dot{B}^{s_2+d(1/p_1-1/p_2)}_{p_1,1}} \le C\sum_{j\in\mathbb{Z}}2^{s_2j+d(\frac{1}{p_1}-\frac{1}{p_2})j}e^{-ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^{p_1}}.$$

Since $s_2 + d\big(\frac{1}{p_1}-\frac{1}{p_2}\big) > 0$, we get

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}2^{s_2j+d(\frac{1}{p_1}-\frac{1}{p_2})j}e^{-ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^{p_1}}\\ &\qquad = t^{-\frac{s_2}{\alpha}-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})}\sum_{j\in\mathbb{Z}}\big\{(t2^{\alpha j})^{\frac{s_2}{\alpha}+\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})}e^{-ct2^{\alpha j}}\big\}\big\|\phi_j(\sqrt{A})f\big\|_{L^{p_1}}\\ &\qquad \le Ct^{-\frac{s_2}{\alpha}-\frac{d}{\alpha}(\frac{1}{p_1}-\frac{1}{p_2})}\|f\|_{\dot{B}^0_{p_1,\infty}},\end{aligned}$$

which proves (3-2). □

4. Proof of Theorem 1.2

Proof of (i). Let $f \in \dot{B}^s_{p,q}(A)$. We take f_N such that

$$f_N := \sum_{|j|\le N} \phi_j(\sqrt{A}) f \quad \text{for } N \in \mathbb{N}.$$

Since $q < \infty$, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$\|f_N - f\|_{\dot{B}^s_{p,q}} < \varepsilon \quad \text{for any } N \ge N_0.$$

The above inequality and boundedness (1-3) in Theorem 1.1 imply

$$\begin{aligned}\|e^{-tA^{\alpha/2}} f - f\|_{\dot{B}^s_{p,q}} &\le \|e^{-tA^{\alpha/2}} f_N - f_N\|_{\dot{B}^s_{p,q}} + \|e^{-tA^{\alpha/2}}(f_N - f)\|_{\dot{B}^s_{p,q}} + \|f_N - f\|_{\dot{B}^s_{p,q}} \\ &\le \|e^{-tA^{\alpha/2}} f_N - f_N\|_{\dot{B}^s_{p,q}} + C\|f_N - f\|_{\dot{B}^s_{p,q}} \\ &\le \|e^{-tA^{\alpha/2}} f_N - f_N\|_{\dot{B}^s_{p,q}} + C\varepsilon\end{aligned}$$

for any $t > 0$ provided that $N \ge N_0$. Then all we have to do is to show that

$$\lim_{t\to 0} \|e^{-tA^{\alpha/2}} f_N - f_N\|_{\dot{B}^s_{p,q}} = 0. \tag{4-1}$$

We prove (4-1). Noting that the spectrum of f_N is restricted and

$$\|e^{-tA^{\alpha/2}} f_N - f_N\|_{\dot{B}^s_{p,q}} = \left\{ \sum_{j=-N-1}^{N+1} \left(2^{sj} \left\|\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f_N\right\|_{L^p}\right)^q \right\}^{\frac{1}{q}},$$

we may consider the convergence of $\left\|\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f_N\right\|_{L^p}$ for each j. For each $j = 0, \pm 1, \pm 2, \dots, \pm(N+1)$, it follows from (2-1) in Lemma 2.1 with

$$G = G_t(\lambda) = e^{-t\lambda^\alpha} - 1$$

that

$$\begin{aligned}\left\|\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f_N\right\|_{L^p} &= \left\|(G_t(\sqrt{A})\Phi_j(\sqrt{A}))(\phi_j(\sqrt{A}) f_N)\right\|_{L^p} \\ &\le C\left\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\right\|_{H^{N+d/2+\delta}} \left\|\phi_j(\sqrt{A}) f_N\right\|_{L^p},\end{aligned}$$

where $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. Here it is readily checked that

$$\lim_{t\to 0}\left\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\right\|_{H^{N+d/2+\delta}} = 0 \quad \text{for each } j,$$

and hence, (4-1) is obtained. □

Proof of (ii). Put $\Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. By considering the dual operator of $e^{-tA^{\alpha/2}} - 1$, we have

$$\sum_{j\in\mathbb{Z}} \int_\Omega \{\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) f\} \overline{\Phi_j(\sqrt{A}) g}\, dx = \sum_{j\in\mathbb{Z}} \int_\Omega \{\phi_j(\sqrt{A}) f\} \overline{\Phi_j(\sqrt{A})(e^{-tA^{\alpha/2}} - 1) g}\, dx. \tag{4-2}$$

It follows from the Hölder inequality that

$$\sum_{j\in\mathbb{Z}}\int_\Omega \big|\{\phi_j(\sqrt{A})f\}\overline{\Phi_j(\sqrt{A})(e^{-tA^{\alpha/2}}-1)g}\big|\,dx$$
$$\le \sum_{j\in\mathbb{Z}} 2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p}\cdot 2^{-sj}\|\Phi_j(\sqrt{A})(e^{-tA^{\alpha/2}}-1)g\|_{L^{p'}}$$
$$\le C\|f\|_{\dot B^s_{p,\infty}}\|(e^{-tA^{\alpha/2}}-1)g\|_{\dot B^{-s}_{p',1}}, \tag{4-3}$$

which assures the absolute convergence of the series in (4-2) by the boundedness of $e^{-tA^{\alpha/2}}$ in $\dot B^{-s}_{p',1}$ from (1-3) in Theorem 1.1. The above estimate and the assertion (i) of Theorem 1.2 imply

$$\left|\sum_{j\in\mathbb{Z}}\int_\Omega \{\phi_j(\sqrt{A})(e^{-tA^{\alpha/2}}-1)f\}\overline{\Phi_j(\sqrt{A})g}\,dx\right| \le C\|f\|_{\dot B^s_{p,\infty}}\|(e^{-tA^{\alpha/2}}-1)g\|_{\dot B^{-s}_{p',1}}\to 0 \quad \text{as } t\to 0. \quad \square$$

5. Proof of Theorem 1.3

To prove Theorem 1.3 we will need the following lemma.

Lemma 5.1. *Let $\alpha>0$, $s_0\in\mathbb{R}$ and $1\le p\le\infty$. Then there exists $C>0$ such that*

$$\begin{aligned}C^{-1}(t2^{\alpha j})^{s_0}e^{-Ct2^{\alpha j}}\|\phi_j(\sqrt{A})f\|_{L^p} &\le \|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\|_{L^p}\\ &\le C(t2^{\alpha j})^{s_0}e^{-C^{-1}t2^{\alpha j}}\|\phi_j(\sqrt{A})f\|_{L^p}\end{aligned} \tag{5-1}$$

for any $t>0$, $j\in\mathbb{Z}$ and $f\in L^p(\Omega)$.

Proof. Put $\Phi_j:=\phi_{j-1}+\phi_j+\phi_{j+1}$. We start by proving the second inequality of the estimate (5-1). By applying the estimate (2-1) in Lemma 2.1 with

$$G=G_t(\lambda)=(t\lambda^\alpha)^{s_0}e^{-t\lambda^\alpha},$$

we have

$$\begin{aligned}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\|_{L^p} &= \big\|\big(G_t(\sqrt{A})\Phi_j(\sqrt{A})\big)(\phi_j(\sqrt{A})f)\big\|_{L^p}\\ &\le C\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|\phi_j(\sqrt{A})f\|_{L^p},\end{aligned} \tag{5-2}$$

where $N>\frac{d}{2}$ and $\delta>0$. Here it is easy to check that there exists $C>0$ such that

$$\|G_t(2^j\sqrt{\cdot}\,)\Phi_0(\sqrt{\cdot}\,)\|_{H^{N+1/2+\delta}(\mathbb{R})}\le C(t2^{\alpha j})^{s_0}e^{-C^{-1}t2^{\alpha j}} \quad \text{for any } j\in\mathbb{Z}, \tag{5-3}$$

and hence,

$$\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\|_{L^p}\le C(t2^{\alpha j})^{s_0}e^{-C^{-1}t2^{\alpha j}}\|\phi_j(\sqrt{A})f\|_{L^p} \quad \text{for any } j\in\mathbb{Z}.$$

This proves the second inequality of (5-1).

We turn to the first inequality of (5-1). Since $\phi_j(\sqrt{A})f$ is written as

$$\begin{aligned}\phi_j(\sqrt{A})f &= \big((tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})\big)\big((tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}\phi_j(\sqrt{A})f\big)\\ &=: \big((tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})\big)F,\end{aligned}$$

all we have to do is to show that

$$\big\|(tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})F\big\|_{L^p} \le C(t2^{\alpha j})^{-s_0}e^{Ct\lambda^\alpha}\|F\|_{L^p}. \tag{5-4}$$

Applying (2-1) in Lemma 2.1 with

$$G = \widetilde{G}_t(\lambda) = (t\lambda^\alpha)^{-s_0}e^{t\lambda^\alpha}$$

to the left-hand side of (5-4), we have from an similar argument to (5-2) and (5-3) that

$$\begin{aligned}\big\|(tA^{\frac{\alpha}{2}})^{-s_0}e^{tA^{\alpha/2}}\Phi_j(\sqrt{A})F\big\|_{L^p} &\le C\big\|\widetilde{G}_t(2^j\sqrt{\cdot})\Phi_0(\sqrt{\cdot})\big\|_{H^{N+1/2+\varepsilon}(\mathbb{R})}\|F\|_{L^p}\\ &\le C(t2^{\alpha j})^{-s_0}e^{Ct\lambda^\alpha}\|F\|_{L^p}.\end{aligned}$$

This proves (5-4) and the first inequality of (5-1) is obtained. □

In what follows, we show the inequality (1-5) for $f \in \dot{B}^s_{p,q}(A)$ to prove Theorem 1.3. We note that the proof below concerns the case when $q < \infty$ only, since the case when $q = \infty$ is also shown analogously with some modification.

Proof of the first inequality of (1-5). By the embedding $L^p(\Omega), \dot{B}^0_{p,r}(A) \hookrightarrow \dot{B}^0_{p,\infty}(A)$, it is sufficient to show that

$$C^{-1}\|f\|_{\dot{B}^s_{p,q}} \le \bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\big\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\big\|_{\dot{B}^0_{p,\infty}}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}. \tag{5-5}$$

We have from the definition of norm $\|\cdot\|_{\dot{B}^0_{p,\infty}}$ and the first inequality of estimate (5-1) in Lemma 5.1 that

$$\begin{aligned}&\bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\big\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\big\|_{\dot{B}^0_{p,\infty}}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}\\ &\qquad\ge C^{-1}\bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\sup_{j\in\mathbb{Z}}(t2^{\alpha j})^{s_0}e^{-Ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}.\end{aligned}$$

Decomposing $(0,\infty)$ in the last line by

$$(0,\infty) = \bigcup_{k\in\mathbb{Z}}[2^{-\alpha(k+1)}, 2^{-\alpha k}], \tag{5-6}$$

we get

$$\begin{aligned}&\bigg\{\int_0^\infty \big(t^{-\frac{s}{\alpha}}\big\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\big\|_{\dot{B}^0_{p,\infty}}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}\\ &\qquad\ge C^{-1}\bigg\{\sum_{k\in\mathbb{Z}}\int_{2^{-\alpha(k+1)}}^{2^{-\alpha k}} \big(t^{-\frac{s}{\alpha}}\sup_{j\in\mathbb{Z}}(t2^{\alpha j})^{s_0}e^{-Ct2^{\alpha j}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p}\big)^q\,\frac{dt}{t}\bigg\}^{\frac{1}{q}}\\ &\qquad\ge C^{-1}\bigg\{\sum_{k\in\mathbb{Z}}\big(2^{sk}\sup_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\big\|\phi_j(\sqrt{A})f\big\|_{L^p}\big)^q\bigg\}^{\frac{1}{q}}.\end{aligned} \tag{5-7}$$

Here it follows from the Hölder inequality that

$$\sup_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p} \\ \geq C^{-1}\Big\{\sum_{j\in\mathbb{Z}}\Big(\frac{1}{1+\alpha^2|j-k|^2}\cdot(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}.$$

Then we deduce from (5-7) and the above inequality that

$$\begin{aligned}\Big\{\int_0^\infty\big(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,\infty}}\big)^q\frac{dt}{t}\Big\}^{\frac{1}{q}} &\geq C^{-1}\Big\{\sum_{k\in\mathbb{Z}}(2^{sk})^q\sum_{j\in\mathbb{Z}}\Big(\frac{1}{1+\alpha^2|j-k|^2}\cdot(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}\\ &= C^{-1}\Big\{\sum_{j\in\mathbb{Z}}(2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p})^q\sum_{k\in\mathbb{Z}}\Big(\frac{2^{-s(j-k)}}{1+\alpha^2|j-k|^2}\cdot(2^{\alpha(j-k)})^{s_0}e^{-C2^{\alpha(j-k)}}\Big)^q\Big\}^{\frac{1}{q}}\\ &= C^{-1}\|f\|_{\dot B^s_{p,q}}\Big\{\sum_{k\in\mathbb{Z}}\Big(\frac{2^{(s_0\alpha-s)k}}{1+\alpha^2|k|^2}\cdot e^{-C2^{\alpha k}}\Big)^q\Big\}^{\frac{1}{q}}.\end{aligned}$$

Since $s_0 > \frac{s}{\alpha}$ and the summation appearing in the last line converges, we obtain (5-5). □

Proof of the second inequality of (1-5). By the embedding $\dot B^0_{p,1}(A) \hookrightarrow L^p(\Omega), \dot B^0_{p,q}(A)$, it is sufficient to show that

$$\Big\{\int_0^\infty\big(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,1}}\big)^q\frac{dt}{t}\Big\}^{\frac{1}{q}} \leq C\|f\|_{\dot B^s_{p,q}(A)}. \tag{5-8}$$

Analogously to the proof of (5-5), we apply the second inequality of (5-1) in Lemma 5.1 instead of the first one and the decomposition (5-6) to get

$$\Big\{\int_0^\infty\big(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,\infty}}\big)^q\frac{dt}{t}\Big\}^{\frac{1}{q}} \\ \leq\Big\{\sum_{k\in\mathbb{Z}}\Big(2^{sk}\sum_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}}.$$

Here the Hölder inequality yields that

$$\sum_{j\in\mathbb{Z}}(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p} \\ \leq C\Big\{\sum_{j\in\mathbb{Z}}\big((1+\alpha^2|j-k|^2)(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\big)^q\Big\}^{\frac{1}{q}}.$$

Then we have from the above two estimates that

$$\begin{aligned}
&\left\{\int_0^\infty \left(t^{-\frac{s}{\alpha}}\|(tA^{\frac{\alpha}{2}})^{s_0}e^{-tA^{\alpha/2}}f\|_{\dot B^0_{p,\infty}}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\
&\qquad\le C\left\{\sum_{k\in\mathbb{Z}}(2^{sk})^q\sum_{j\in\mathbb{Z}}\left((1+\alpha^2|j-k|^2)(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\|\phi_j(\sqrt{A})f\|_{L^p}\right)^q\right\}^{\frac{1}{q}}\\
&\qquad= C\left\{\sum_{j\in\mathbb{Z}}\left(2^{sj}\|\phi_j(\sqrt{A})f\|_{L^p}\right)^q\sum_{k\in\mathbb{Z}}\left(2^{-s(j-k)}(1+\alpha^2|j-k|^2)(2^{\alpha(j-k)})^{s_0}e^{-C^{-1}2^{\alpha(j-k)}}\right)^q\right\}^{\frac{1}{q}}\\
&\qquad= C\|f\|_{\dot B^s_{p,q}}\left\{\sum_{k\in\mathbb{Z}}\left((1+\alpha^2|k|^2)2^{(s_0\alpha-s)k}e^{-C^{-1}2^{\alpha k}}\right)^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Since $s_0 > \frac{s}{\alpha}$ and the summation appearing in the last line converges, we obtain (5-8). □

6. Proof of Theorem 1.4

Proof of (1-6). It is sufficient to prove the case when $s = 0$ thanks to the lifting property in the proposition on page 684. We also consider the case when $q < \infty$ only, since the case when $q = \infty$ is also shown analogously. First we prove that

$$\|A^{\frac{\alpha}{2}}u\|_{L^q(0,\infty;\dot B^0_{p,q})} \le C\|u_0\|_{\dot B^{\alpha-\alpha/q}_{p,q}} + C\|f\|_{L^q(0,\infty;\dot B^0_{p,q})}. \tag{6-1}$$

By the definition of u and the triangle inequality, we get

$$\|A^{\frac{\alpha}{2}}u\|_{L^q(0,\infty;\dot B^0_{p,q})} \le \|A^{\frac{\alpha}{2}}e^{-tA^{\alpha/2}}u_0\|_{L^q(0,\infty;\dot B^0_{p,q})} + \left\|A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^q(0,\infty;\dot B^0_{p,q})}. \tag{6-2}$$

On the first term of the right-hand side in the above inequality, it follows from the estimate (1-5) for $s_0 = 1,\ s = \alpha - \frac{\alpha}{q}$ that

$$\|A^{\frac{\alpha}{2}}e^{-tA^{\alpha/2}}u_0\|_{L^q(0,\infty;\dot B^0_{p,q})} \le C\|u_0\|_{L^q(0,\infty;\dot B^{\alpha-\alpha/q}_{p,q})}. \tag{6-3}$$

As for the second one, we start by proving that

$$\left\|\phi_j(\sqrt{A})A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^p} \le C2^{\frac{\alpha}{q}j}\left\{\int_0^t \left(e^{-C^{-1}(t-\tau)2^{\alpha j}}\|\phi_j(\sqrt{A})f\|_{L^p}\right)^q d\tau\right\}^{\frac{1}{q}}. \tag{6-4}$$

The above estimate (6-4) is verified by applying the estimate (5-1) in Lemma 5.1 and the Hölder inequality; in fact, we get

$$\left\|\phi_j(\sqrt{A})A^{\frac{\alpha}{2}}\int_0^t e^{-(t-\tau)A^{\alpha/2}}f(\tau)\,d\tau\right\|_{L^p} \le C2^{\alpha j}\int_0^t e^{-C^{-1}(t-\tau)2^{\alpha j}}\|\phi_j(\sqrt{A})f(\tau)\|_{L^p}\,d\tau$$

$$\le C 2^{\alpha j} \| e^{-(2C)^{-1}(t-\tau)2^{\alpha j}} \|_{L^{q/(q-1)}(\{0\le\tau\le t\})} \left\{ \int_0^t \left(e^{-(2C)^{-1}(t-\tau)2^{\alpha j}} \left\| \phi_j(\sqrt{A}) f(\tau) \right\|_{L^p}\right)^q d\tau \right\}^{\frac{1}{q}}$$

$$\le C 2^{\frac{\alpha}{q} j} \left\{ \int_0^t \left(e^{-(2C)^{-1}(t-\tau)2^{\alpha j}} \left\| \phi_j(\sqrt{A}) f(\tau) \right\|_{L^p}\right)^q d\tau \right\}^{\frac{1}{q}}.$$

By the estimate (6-4), we have

$$\begin{aligned} &\left\| A^{\frac{\alpha}{2}} \int_0^t e^{-(t-\tau)A^{\alpha/2}} f(\tau)\, d\tau \right\|_{L^q(0,\infty;\dot{B}^0_{p,q})} \\ &\qquad \le C \left[\int_0^\infty \sum_{j\in\mathbb{Z}} \left\{ 2^{\frac{\alpha}{q} j} \left(\int_0^t \left(e^{-(2C)^{-1}(t-\tau)2^{\alpha j}} \left\| \phi_j(\sqrt{A}) f(\tau) \right\|_{L^p}\right)^q d\tau \right)^{\frac{1}{q}} \right\}^q dt \right]^{\frac{1}{q}} \\ &\qquad = C \left[\int_0^\infty \sum_{j\in\mathbb{Z}} \left\| \phi_j(\sqrt{A}) f(\tau) \right\|_{L^p}^q \left(2^{\alpha j} \int_\tau^\infty e^{-q(2C)^{-1}(t-\tau)2^{\alpha j}}\, dt \right) d\tau \right]^{\frac{1}{q}} \\ &\qquad = C \| f \|_{L^q(0,\infty;\dot{B}^0_{p,q}(A))}. \end{aligned} \tag{6-5}$$

Then the estimates (6-2), (6-3) and (6-5) imply the inequality (6-1). The estimate for $\partial_t u$, i.e., the inequality

$$\|\partial_t u\|_{L^q(0,\infty;\dot{B}^0_{p,q})} \le C \|u_0\|_{\dot{B}^{\alpha-\alpha/q}_{p,q}} + C \|f\|_{L^q(0,\infty;\dot{B}^0_{p,q})},$$

is verified by the estimate (6-1) and the equality

$$\partial_t u = -A^{\frac{\alpha}{2}} u + f.$$

Hence we obtain the estimate (1-6) and the proof is complete. □

7. Results for the inhomogeneous Besov spaces

We should mention that similar theorems also hold for the heat semigroup in the inhomogeneous Besov spaces $B^s_{p,q}(A)$. We also note that the semigroup generated by the fractional Laplacian cannot be treated analogously by the direct application of boundedness of the scaled spectral multiplier in Lemma 2.1 (see the comment below Theorem 7.2).

First we recall the definition of $B^s_{p,q}(A)$. Let ψ be as in $C_0^\infty((-\infty,\infty))$ such that

$$\psi(\lambda^2) + \sum_{j\in\mathbb{N}} \phi_j(\lambda) = 1 \quad \text{for any } \lambda \ge 0.$$

The inhomogeneous Besov space $B^s_{p,q}(A)$ is defined as follows; see [Iwabuchi et al. 2016a].

Definition. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, $B^s_{p,q}(A)$ is defined by letting

$$B^s_{p,q}(A) := \{ f \in \mathcal{X}_0'(\Omega) : \ \|f\|_{B^s_{p,q}(A)} < \infty \},$$

where

$$\|f\|_{B^s_{p,q}(A)} := \|\psi(A) f\|_{L^p} + \left\| \left\{ 2^{sj} \left\| \phi_j(\sqrt{A}) f \right\|_{L^p} \right\}_{j\in\mathbb{N}} \right\|_{\ell^q(\mathbb{N})}.$$

The high-frequency part is able to be treated in the same way as the proof for the homogeneous case by using Lemma 2.1. As for the low-frequency part, we employ the pointwise estimate of the kernel of e^{-tA}

$$0 \le e^{-tA}(x,y) \le (4\pi t)^{-\frac{d}{2}} \exp\biggl(\frac{|x-y|^2}{4t}\biggr),$$

which assures the boundedness of e^{-tA} in $L^p(\Omega)$ and also $B^s_{p,q}(A)$ as well as the case when $\Omega = \mathbb{R}^d$. In order to treat continuity in time of e^{-tA}, we need the following obtained by a proof similar to that of Lemma 2.1.

Lemma 7.1. *Let $N > \frac{d}{2}$, $1 \le p \le \infty$, $\delta > 0$, $\psi \in C_0^\infty((-\infty,\infty))$ and $G \in H^{N+\frac{1}{2}+\delta}(\mathbb{R})$. Then there exists a positive constant C such that for any $f \in L^p(\Omega)$*

$$\|G(A)\psi(A)f\|_{L^p} \le C\|G(\cdot)\psi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})}\|f\|_{L^p}. \tag{7-1}$$

We take G such that

$$G(\lambda) := e^{-t\lambda} - 1 \quad \text{for any } \lambda \in \mathbb{R}$$

to apply the above lemma. For the above G it is easy to check that

$$\|G(\cdot)\psi(\cdot)\|_{H^{N+1/2+\delta}(\mathbb{R})} \to 0 \quad \text{as } t \to 0.$$

Hence for any $f \in B^s_{p,q}(A)$, it follows from (7-1) that

$$\lim_{t\to 0} \|\psi(A)(e^{-tA}f - f)\|_{L^p} = 0.$$

According to the boundedness and the continuity of e^{-tA}, we obtain the following result for the inhomogeneous Besov spaces.

Theorem 7.2. *Let $s \in \mathbb{R}$, $1 \le p, p_1, p_2, q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let Ψ and Ψ_j with $j \in \mathbb{N}$ be such that*

$$\begin{aligned}\Psi(A) &:= \psi(A) + \phi_1(\sqrt{A}),\\ \Phi_1(\sqrt{A}) &:= \psi(A) + \phi_1(\sqrt{A}) + \phi_2(\sqrt{A}),\\ \Phi_j(\sqrt{A}) &:= \phi_{j-1}(\sqrt{A}) + \phi_j(\sqrt{A}) + \phi_{j+1}(\sqrt{A}) \quad \textit{for } j \ge 2:\end{aligned}$$

(i) *There exists a constant $C > 0$ such that*

$$\|e^{-tA}f\|_{B^s_{p,q}(A)} \le C\|f\|_{B^s_{p,q}(A)}$$

for any $f \in B^s_{p,q}(A)$. If $p_1 \le p_2$, then there exists a constant $C > 0$ such that

$$\|e^{-tA}f\|_{B^s_{p_2,q}(A)} \le Ct^{-\frac{d}{2}(\frac{1}{p_1}-\frac{1}{p_2})}\|f\|_{B^s_{p_1,q}(A)}$$

for any $f \in B^s_{p_1,q}(A)$.

(ii) *If $q < \infty$ and $f \in B^s_{p,q}(A)$, then*

$$\lim_{t\to 0}\|e^{-tA}f - f\|_{B^s_{p,q}(A)} = 0.$$

If $q=\infty$, $1<p\le\infty$ *and* $f\in B^s_{p,\infty}(A)$, *then* $e^{-tA}f$ *converges to* f *in the dual weak sense as* $t\to 0$; *namely,*

$$\lim_{t\to 0}\biggl[\int_\Omega \{\psi(A)(e^{-tA}f-f)\}\,\overline{\Psi(A)g}\,dx+\sum_{j\in\mathbb{N}}\int_\Omega\{\phi_j(\sqrt{A})(e^{-tA}f-f)\}\,\overline{\Phi_j(\sqrt{A})g}\,dx\biggr]=0$$

for any $g\in \dot B^{-s}_{p',1}(A)$.

(iii) *Let* $T>0$, $s,s_0\in\mathbb{R}$ *and* $s_0>\frac{s}{2}$. *Then*

$$\|f\|_{B^s_{p,q}(A)}\simeq\|\psi(TA)f\|_{L^p}+\biggl\{\int_0^T\bigl(t^{-\frac{s}{2}}\|(tA)^{s_0}e^{-tA}f\|_X\bigr)^q\,\frac{dt}{t}\biggr\}^{\frac1q}$$

for any $f\in B^s_{p,q}(A)$, *where* $X=L^p(\Omega)$, $B^0_{p,r}(A)$ *with* $1\le r\le\infty$.

(iv) *Let* $T>0$, $u_0\in B^{s+2-2/q}_{p,q}(A)$ *and* $f\in L^q(0,T;B^s_{p,q}(A))$. *Assume that u satisfies*

$$u(t)=e^{-tA}u_0+\int_0^t e^{-(t-\tau)A}f(\tau)\,d\tau.$$

Then there exists a constant $C=C(T)>0$ *independent of* u_0 *and* f *such that*

$$\|\partial_t u\|_{L^q(0,T;B^s_{p,q}(A))}+\|A^{\frac{\alpha}{2}}u\|_{L^q(0,T;B^s_{p,q}(A))}\le C\|u_0\|_{B^{s+2-2/q}_{p,q}(A)}+C\|f\|_{L^q(0,T;B^s_{p,q}(A))}.$$

Remark. Let us mention what is obtained by the abstract theory for sectorial operators by Da Prato and Grisvard [1975]; see also [Haase 2006; Lunardi 1995]. Let $X=B^0_{p,q}(A)$. We can consider A as a sectorial operator with the domain $D(A^\alpha)=B^2_{p,q}(A)$. Let $0<T<\infty$, $1<q<\infty$, $1\le p,r\le\infty$, $\theta\in(0,1)$ and $\alpha>0$. Then for any $f\in L^q(0,T;(X,D(A^\alpha))_{\theta,r})$ the equation

$$\begin{cases}\dfrac{du}{dt}+Au=f, & 0<t<T,\\ u(0)=0\end{cases}$$

admits a unique solution u satisfying

$$\Bigl\|\frac{du}{dt}\Bigr\|_{L^q(0,T;(X,D(A^\alpha))_{\theta,r})}+\|Au\|_{L^q(0,T;(X,D(A^\alpha))_{\theta,r})}\le C\|f\|_{L^q(0,T;(X,D(A^\alpha))_{\theta,r})},$$

where C depends on T. Here we note that $(X,D(A^\alpha))_{\theta,r}=B^{2\alpha\theta}_{p,r}(A)$ and $2\alpha\theta$ is possibly an arbitrary positive number since $\alpha>0$ and $\theta\in(0,1)$.

Let us give a few remarks on the semigroup generated by $A^{\frac{\alpha}{2}}$. If we consider applying Lemma 7.1 directly, it is impossible to obtain the boundedness of $e^{-tA^{\alpha/2}}$ for general α. In fact, taking

$$G=G_t(\lambda)=e^{-t|\lambda|^{\alpha/2}},$$

and applying (7-1), we see that the $H^{N+\frac12+\delta}(\mathbb{R})$ norm of the above $G=G_t(\lambda)$ is not finite for small $\lambda>0$ because of less regularity around $\lambda=0$. On the other hand, if α is even or sufficiently large, the $H^{N+\frac12+\delta}(\mathbb{R})$ norm of $e^{-t|\lambda|^{\alpha/2}}$ is finite and we can get some results. However this argument does not reach the optimal estimate, and hence, we do not treat it in this paper and will treat it in a future work.

Appendix: Real interpolation

We now give a remark that real interpolation can be considered in the Besov spaces $\dot{B}^s_{p,q}(A)$ and $B^s_{p,q}(A)$ on open sets as well as the whole space case. We recall the definition of real interpolation spaces $(X_0, X_1)_{\theta,q}$ for Banach spaces X_0 and X_1; see, e.g., [Bergh and Löfström 1976; Peetre 1968; Triebel 1983].

Definition. Let $0 < \theta < 1$ and $1 \le q \le \infty$. $(X_0, X_1)_{\theta,q}$ is defined by letting

$$(X_0, X_1)_{\theta,q} := \left\{ a \in X_0 + X_1 : \|a\|_{(X_0,X_1)_{\theta,q}} := \left\{ \int_0^\infty (t^{-\theta} K(t,a))^q \, \frac{dt}{t} \right\}^{\frac{1}{q}} < \infty \right\},$$

where $K(t,a)$ is Peetre's K-function

$$K(t,a) := \inf\bigl\{ \|a_0\|_{X_0} + t\|a_1\|_{X_1} : a = a_0 + a_1,\ a_0 \in X_0,\ a_1 \in X_1 \bigr\}.$$

As well as in the case when $\Omega = \mathbb{R}^d$, we obtain the following.

Proposition A.1. *Let $0 < \theta < 1$, $s, s_0, s_1 \in \mathbb{R}$ and $1 \le p, q, q_0, q_1 \le \infty$. Assume that $s_0 \ne s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(\dot{B}^{s_0}_{p,q_0}(A),\ \dot{B}^{s_1}_{p,q_1}(A))_{\theta,q} = \dot{B}^s_{p,q}(A),$$
$$(B^{s_0}_{p,q_0}(A),\ B^{s_1}_{p,q_1}(A))_{\theta,q} = B^s_{p,q}(A).$$

We omit the proof of the above proposition since one can show it analogously to the whole space case; see, e.g., [Triebel 1983].

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