ANALYSIS & PDE

Volume 11 No. 3 2018

JEFFREY GALKOWSKI AND JOHN A. TOTH

EIGENFUNCTION SCARRING AND IMPROVEMENTS IN L^∞ BOUNDS

msp

ANALYSIS AND PDE
Vol. 11, No. 3, 2018
dx.doi.org/10.2140/apde.2018.11.801

EIGENFUNCTION SCARRING AND IMPROVEMENTS IN L^∞ BOUNDS

JEFFREY GALKOWSKI AND JOHN A. TOTH

We study the relationship between L^∞ growth of eigenfunctions and their L^2 concentration as measured by defect measures. In particular, we show that scarring in the sense of concentration of defect measure on certain submanifolds is incompatible with maximal L^∞ growth. In addition, we show that a defect measure which is too diffuse, such as the Liouville measure, is also incompatible with maximal eigenfunction growth.

1. Introduction

Let (M, g) be a C^∞ compact manifold of dimension n without boundary. Consider the eigenfunctions

$$(-\Delta_g - \lambda_j^2)u_{\lambda_j} = 0, \quad \|u_{\lambda_j}\|_{L^2} = 1 \tag{1-1}$$

as $\lambda_j \to \infty$. It is well known [Avakumović 1956; Levitan 1952; Hörmander 1968], see also [Zworski 2012, Chapter 7], that solutions to (1-1) satisfy

$$\|u_{\lambda_j}\|_{L^\infty(M)} \le C\lambda_j^{(n-1)/2} \tag{1-2}$$

and that this bound is saturated, e.g., on the sphere. It is natural to consider the situations which produce sharp examples for (1-2). In many cases, one expects polynomial improvements to (1-2), but rigorous results along these lines are few and far between [Iwaniec and Sarnak 1995]. In the case of negatively curved manifolds, log improvements can be obtained [Bérard 1977]. However, at present, under general dynamical assumptions, known results involve o-improvements to (1-2) [Toth and Zelditch 2002; Sogge et al. 2011; Sogge and Zelditch 2002; 2003; 2016a; 2016b]. These papers all study the connections between the growth of L^∞ norms of eigenfunctions and the global geometry of the manifold (M, g). In this note, we examine the relationship between L^∞ growth and L^2 concentration of eigenfunctions. We measure L^2 concentration using the concept of a *defect measure* — a sequence $\{u_{\lambda_j}\}$ has defect measure μ if for any $a \in S^0_{\mathrm{hom}}(T^*M \setminus \{0\})$,

$$\langle a(x, D)u_{\lambda_j}, u_{\lambda_j}\rangle \to \int_{S^*M} a(x, \xi)\, d\mu. \tag{1-3}$$

By an elementary compactness/diagonalization argument, it follows that any sequence of eigenfunctions u_{λ_j} solving (1-1) possesses a further subsequence that has a defect measure in the sense of (1-3) [Zworski 2012, Chapter 5; Gérard 1991]. Moreover, a standard commutator argument shows that if $\{u_{\lambda_j}\}$ is any

MSC2010: 35P20, 58J50.
Keywords: eigenfunction, sup norms, defect measure.

sequence of L^2-normalized Laplace eigenfunctions, the associated defect measure μ is invariant under the geodesic flow; that is, if $G_t : S^*M \to S^*M$ is the geodesic flow, i.e., the Hamiltonian flow of $p = \frac{1}{2}|\xi|_g^2$, $(G_t)_*\mu = \mu$ for all $t \in \mathbb{R}$.

Definition 1.1. We say that an eigenfunction subsequence is *strongly scarring* provided that for any defect measure μ associated to the sequence, $\operatorname{supp}\mu$ is a finite union of periodic geodesics.

Theorem 1. *Let $\{u_{\lambda_j}\}$ be a strongly scarring sequence of solutions to (1-1). Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{(n-1)/2}).$$

We also have improved L^∞ bounds when eigenfunctions are *quantum ergodic*, that is, their defect measure is the Liouville measure on S^*M, μ_L; see, e.g., [Shnirelman 1974; Colin de Verdière 1985; Zelditch 1987] for the standard quantum ergodicity theorem.

Theorem 2. *Let $\{u_{\lambda_j}\}$ be a quantum ergodic sequence of solutions to (1-1). Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{(n-1)/2}).$$

Theorems 1 and 2 are corollaries of our next theorem, where we relax the assumptions on μ and make the following definitions. Define the *time-T flow-out* by

$$\Lambda_{x,T} := \bigcup_{t=-T}^{T} G_t(S_x^*M).$$

Definition 1.2. Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on S^*M induced by the Sasaki metric on T^*M; see for example [Blair 2010, Chapter 9] for a treatment of the Sasaki metric. We say that the subsequence u_{λ_j}, $j = 1, 2, \ldots$, is *admissible at x* if for any defect measure μ associated to the sequence there exists $T > 0$ such that

$$\mathcal{H}^n(\operatorname{supp}\mu|_{\Lambda_{x,T}}) = 0. \tag{1-4}$$

We say that the subsequence is *admissible* if it is admissible at x for every $x \in M$.

We note that in (1-4), $\mu|_{\Lambda_{x,T}}$ denotes the defect measure restricted to the flow-out $\Lambda_{x,T}$; for any A that is μ-measurable,

$$\mu|_{\Lambda_{x,T}}(A) := \mu(A \cap \Lambda_{x,T}).$$

Theorem 3. *Let $\{u_{\lambda_j}\}$ be a sequence of L^2-normalized Laplace eigenfunctions that is admissible in the sense of (1-4). Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{(n-1)/2}).$$

Remark 1.3. We choose to use the Sasaki metric to define $\mathcal{H}^n$ for concreteness, but this is not important and we could replace the Sasaki metric by any other metric on S^*M.

Theorem 3 can be interpreted as saying that eigenfunctions which strongly scar are too concentrated to have maximal L^∞ growth, while diffuse eigenfunctions are too spread out to have maximal growth. However, the reason the admissibility assumption is satisfied differs in these cases. In the diffuse case (see Theorem 2), one has $\mu|_{\Lambda_{x,T}} = 0$, so that the admissibility assumption is trivially verified. In the case where the eigenfunctions strongly scar (see Theorem 1), $\mu|_{\Lambda_{x,T}} \neq 0$ but the Hausdorff dimension of $\operatorname{supp} \mu|_{\Lambda_{x,T}}$ is $< n$; so again, (1-4) is satisfied. The zonal harmonics on the sphere S^2, which saturate the L^∞ bound (1-2), lie precisely between being diffuse and strongly scarring (see Section 4).

Observe that the condition μ is diffuse is much more general than $\mu = \mu_L$. Jakobson and Zelditch [1999] show that any invariant measure on S^*S^n where S^n is the round sphere can be obtained as a defect measure for a sequence of eigenfunctions and in particular many non-Liouville but diffuse measures occur.

Remark 1.4. We note that the results here hold for any $o(\lambda)$ quasimode of $(-\Delta_g - \lambda^2)$ that is compactly microlocalized in frequency; see [Galkowski 2017].

Relation with previous results. Theorem 2 is related to [Sogge et al. 2011, Theorem 3], where the $o(h^{(1-n)/2})$ sup bound is proved for all Laplace eigenfunctions on a C^ω surface with ergodic geodesic flow. However, in Theorem 2, we make no analyticity or dynamical assumptions on (M, g) whatsoever, only an assumption on the particular defect measure associated with the eigenfunction sequence. Recently, Hezari [2016] and Sogge [2016] gave independent proofs of Theorem 2.

One consequence of the work of Sogge is the relation between L^p norms for eigenfunctions and the push forward of defect measures to the base manifold M. In particular, he showed [Sogge 2016, (3.3)] that

$$\|u_\lambda\|_{L^\infty(M)} \leq C\lambda^{(n-1)/2} \sup_{x\in M} \delta^{-1/2} \|u_\lambda\|_{L^2(B_\delta(x))} \tag{1-5}$$

when $\lambda^{-1} \leq \delta \leq \operatorname{inj}(M, g)$ and $\lambda \geq 1$. We note that when u_λ are quantum ergodic, $\|u_\lambda\|_{L^2(B_\delta(x))} \approx \delta^{n/2}$ and so the $o(\lambda^{(n-1)/2})$-bound in Theorem 2 follows from (1-5) as well; see also Corollary 1.2 in [Sogge 2016].

However, neither the scarring result in Theorem 1 nor the more general bound in Theorem 3 follow from (1-5). To compare and contrast with (1-5), we observe that (1-5) implies for any $\delta > 0$ independent of λ,

$$\limsup_{\lambda\to\infty} \lambda^{(1-n)/2} \|u_\lambda\|_{L^\infty(M)} \leq C \sup_{x\in M} \delta^{-1/2} \left(\mu(S^* B_\delta(x))\right)^{1/2}.$$

Our main estimate in (3-12) says that for any $x(\lambda)$ with $d(x(\lambda), x) = o(1)$,

$$\limsup_{\lambda\to\infty} \lambda^{(1-n)/2} |u_\lambda(x(\lambda))| \leq C'_\delta \big(\mathcal{H}^n(\operatorname{supp} \mu|_{A_x(\delta/2, 3\delta)})\big)^{1/2}, \tag{1-6}$$

where for $\delta_2 > \delta_1$ we have $A_x(\delta_1, \delta_2) = \Lambda_{x,\delta_2} \setminus \Lambda_{x,\delta_1}$. This microlocalized bound allows us to deal with the more general scarring-type cases as well. In particular, the key differences are that we have replaced $S^* B_\delta(x)$ by $A_x(\delta/2, 2\delta) \subset \Lambda_x$ and the defect measure by Hausdorff n-measure. We note however that unlike (1-5), $\delta > 0$ can be arbitrarily small but is fixed independent of λ in (1-6).

Sogge and Zelditch [2002] proved that any manifold on which (1-2) is sharp must have a self-focal point. That is, a point x such that $|\mathcal{L}_x| > 0$, where

$$\mathcal{L}_x := \{\xi \in S^*_x M : \text{ there exists } T \text{ such that } \exp_x T\xi = x\}$$

and $|\cdot|$ denotes the normalized surface measure on the sphere. Subsequently, in [Sogge et al. 2011] the authors showed that one can replace $\mathcal{L}_x$ by the set of recurrent directions $\mathcal{R}_x \subset \mathcal{L}_x$ and the assumption $|\mathcal{R}_x| > 0$ for some $x \in M$ is necessary to saturate the maximal bound in (1-2). Here,

$$\mathcal{R}_x := \left\{ \xi \in S_x^*M : \xi \in \left(\bigcap_{T>0} \overline{\bigcup_{t \geq T} G_t(x,\xi) \cap S_x^*M} \right) \bigcap \left(\bigcap_{T>0} \overline{\bigcup_{t \leq -T} G_t(x,\xi) \cap S_x^*M} \right) \right\}.$$

The example of the triaxial ellipsoid with x equal to an umbilic point shows that latter assumption is weaker than the former. Indeed, in such a case $|\mathcal{L}_x| = 1$, whereas $|\mathcal{R}_x| = 0$. Most recently, in [Sogge and Zelditch 2016a; 2016b], it was proved that for real-analytic surfaces, the maximal L^∞ bound can only be achieved if there exists a periodic point $x \in M$ for the geodesic flow, i.e., a point (x,ξ) such that all geodesics starting at $(x,\xi) \in S^*M$ close up smoothly after some finite time $T > 0$.

Together with our analysis, the results of [Sogge et al. 2011] imply that any sequence of eigenfunctions, $\{u_\lambda\}$ having maximal L^∞ growth near x and defect measure μ must have $\mu(\Lambda_{x,T}) > 0$ for all $T > 0$ and $|\mathcal{R}_x| > 0$. In particular, the results of that paper show that u_λ can only have maximal L^∞ growth near a point with a positive measure set of recurrent points and Theorem 3 shows that a point with maximal L^∞ growth must have $\mu(\Lambda_{x,T}) > 0$. As far as the authors are aware, the results in [Sogge et al. 2011; Sogge and Zelditch 2016a; 2016b] do not give additional information about μ.

On the other hand, under an additional regularity assumption on the measure μ, Theorem 3 can be used to show that when u_λ has maximal growth near x, the measure $\mu|_{\Lambda_{x,T}}$ is not mutually singular with respect to $\mathcal{H}^n$. Since the measure for a zonal harmonic is a smooth multiple of $\mathcal{H}^n$ (see Section 4), this implies that the measure μ resembles the defect measure of a zonal harmonic. In [Galkowski 2017], the first author removed the necessity for any additional regularity assumption and gave a full characterization of defect measures for eigenfunctions with maximal L^∞ growth, in particular proving that if u_λ has maximal growth near x and defect measure μ, then $\mu|_{\Lambda_{x,T}}$ is not mutually singular with respect to $\mathcal{H}^n$. Finally, we note that unlike [Sogge et al. 2011; Sogge and Zelditch 2002; 2016a; 2016b], the analysis here is entirely local.

2. A local version of Theorem 3

In the following, we will freely use semiclassical pseudodifferential calculus where the semiclassical parameter is h with $h^{-1} = \lambda \in \operatorname{Spec}\sqrt{-\Delta_g}$. We write $r(x,y) : M \times M \to \mathbb{R}$ for the Riemannian distance from x to y and write $B(x,\delta)$ for the geodesic ball of radius δ around x. We start with a local result:

Theorem 4. *Let $\{u_h\}$ be sequence of Laplace eigenfunctions that is admissible at x. Then for any $\delta(h) = o(1)$,*

$$\|u_h\|_{L^\infty(B(x,\delta(h)))} = o(h^{(1-n)/2}).$$

Theorem 3 is an easy consequence of Theorem 4.

Proof that Theorem 4 implies Theorem 3. Suppose that u is admissible and

$$\limsup_{h\to 0} h^{(n-1)/2} \|u_h\|_{L^\infty} \neq 0.$$

Then, there exist $c > 0,\ h_k \to 0,\ x_{h_k}$ so that

$$|u_{h_k}(x_{h_k})| \geq c h_k^{-(n-1)/2}.$$

Since M is compact, by taking a subsequence, we may assume $x_{h_k} \to x$. But then $r(x, x_{h_k}) = o(1)$ and since u is admissible at x, Theorem 4 implies

$$\limsup_{k\to\infty} h_k^{(n-1)/2} |u_{h_k}(x_{h_k})| = 0. \qquad \square$$

3. Proof of Theorem 4

In view of the above, it suffices to prove the local result: Theorem 4.

Proof. Fix $T > 3\delta > 0$ and let $\rho \in \mathcal{S}(\mathbb{R})$ with $\rho(0) = 1$ and $\operatorname{supp} \hat{\rho} \subset (\delta, 2\delta)$. Let

$$S^*M(\varepsilon) := \big\{(x,\xi) : \big||\xi|_x - 1\big| \leq \varepsilon\big\}$$

and $\chi(x,\xi) \in C_0^\infty(T^*M)$ be a cutoff near the cosphere S^*M with $\chi(x,\xi) = 1$ for $(x,\xi) \in S^*M(\varepsilon)$ and $\chi(x,\xi) = 0$ when $(x,\xi) \in T^*M \setminus S^*M(2\varepsilon)$. Let $\chi(x, hD) \in \mathrm{Op}_h(C_0^\infty(T^*M))$ be the corresponding h-pseudodifferential cutoff. Also, in the following, we will use the notation

$$\Gamma_x := \operatorname{supp} \mu|_{\Lambda_{x,T}}$$

to denote the support of the restricted defect measure corresponding to the eigenfunction sequence $\{u_{h_j}\}$ in Theorem 3.

Then, we have

$$u_h = \rho\Big(\frac{1}{2h}[-h^2\Delta - 1]\Big) u_h = \int_{\mathbb{R}} \hat{\rho}(t) e^{i(t/2)[-h^2\Delta-1]/h} \chi(y, hD_y) u_h \, dt + O_\varepsilon(h^\infty). \tag{3-1}$$

Microlocalization to the flow-out $\mathbf{\Lambda_x}$. Set

$$V(t,x,y,h) := \big(\hat{\rho}(t) e^{i(t/2)[-h^2\Delta-1]/h} \chi(y, hD_y)\big)(t,x,y).$$

Then, by Egorov's theorem [Zworski 2012, Theorem 11.1]

$$\mathrm{WF}_h'(V(t,\cdot,\cdot,h)) \subset \big\{(x,\xi,y,\eta) : (x,\xi) = G_t(y,\eta),\ \big||\xi|_x - 1\big| \leq 2\varepsilon\big\}; \tag{3-2}$$

see, e.g., [Dyatlov and Zworski 2017, Definition E.37] for a definition of WF_h'.

Let $b_{x,\varepsilon}(x, hD) \in \mathrm{Op}_h(C_0^\infty(T^*M))$ be a family of h-pseudodifferential cutoffs with principal symbols

$$b_{x,\varepsilon} \in C_0^\infty\big(\big\{(y,\eta) : (y,\eta) = G_t(x_0,\xi) \text{ for some } (x_0,\xi) \in S^*_{x_0}M(3\varepsilon) \text{ with } r(x,x_0) < 2\varepsilon,\ \delta/2 < t < 3\delta\big\}\big),$$

with

$$b_{x,\varepsilon} \equiv 1 \quad \text{on } \big\{(y,\eta) : (y,\eta) = G_t(x_0,\xi) \text{ for some } (x_0,\xi) \in S^*_{x_0}M(2\varepsilon) \text{ with } r(x,x_0) < \varepsilon,\ \delta < t < 2\delta\big\}.$$

By the definition of WF_h' together with (3-1) and (3-2), it follows that for $r(x(h),x) = o(1)$,

$$u_h(x(h)) = \int_M \overline{V}(x(h),y,h)\, b_{x,\varepsilon}(y, hD_y)\, u_h(y)\, dy + O_\varepsilon(h^\infty), \tag{3-3}$$

where,

$$\bar{V}(x(h), y, h) := \int_{\mathbb{R}} \hat{\rho}(t)\big(e^{i(t/2)[-h^2\Delta-1]/h}\chi(y, hD_y)\big)(t, x(h), y)\, dt.$$

By a standard stationary phase argument,

$$\bar{V}(x, y, h) = h^{(1-n)/2} e^{-ir(x,y)/h} a(x, y, h)\, \hat{\rho}(r(x, y)) + O_\varepsilon(h^\infty), \tag{3-4}$$

where $a(x, y, h) \in S^0(1)$.

To see this, observe that by [Zworski 2012, Theorem 10.4]

$$\bar{V}(x, y, h) = (2\pi h)^{-n} \int e^{i\varphi(t,x,y,\eta)/h}\, \alpha(t, x, y, \eta, h)\, \hat{\rho}(t)\, d\eta\, dt + O(h^\infty),$$

where $b \in C_c^\infty$ and φ solves

$$\partial_t\varphi = \tfrac{1}{2}(|\partial_x\varphi|^2_{g(x)} - 1), \quad \varphi(0, x, y, \eta) = \langle x - y, \eta\rangle. \tag{3-5}$$

In particular, for all (t, x, y, η), we have $\exp(tH_{|\xi|_g^2/2})(\partial_\eta\varphi + y, \eta) = (x, \partial_x\varphi)$. The phase function

$$\varphi(t, x, y, \eta) = \langle \exp_y^{-1}(x), \eta\rangle + \tfrac{1}{2}t(|\eta|_y^2 - 1)$$

satisfies (3-5).

We next perform stationary phase in (t, η). First, observe that the phase is stationary at

$$\exp(tH_{|\xi|_g^2/2})(y, \eta) = (x, \partial_x\varphi), \quad |\partial_x\varphi|_{g(x)} = 1.$$

In particular, $t = r(x, y)$ and the geodesic through (y, η) passes through x. Since $\operatorname{supp}\hat{\rho} \subset (\delta, 2\delta)$, by performing nonstationary phase, we may assume $t \in (\delta, 2\delta)$ and hence $\delta < r(x, y) < 2\delta$. Then, we observe that $\partial^2_{(t,\eta)}\varphi$ is nondegenerate for $t \in (\delta, 2\delta)$. The solutions (t_c, η_c) of the critical point equations $\partial_t\varphi = 0$ and $\partial_\eta\varphi = 0$ are given by

$$t_c = |\exp_y^{-1}(x)| = r(x, y), \quad \eta_c = -\frac{\exp_y^{-1}(x)}{r(x, y)}.$$

Consequently, (3-4) follows from an application of stationary phase; see also [Sogge 1993, Lemma 5.1.3; Burq et al. 2007, Theorem 4].

Then, in view of (3-4) and (3-3),

$$\begin{gathered} u_h(x(h)) = v_h(x(h)) + O_\varepsilon(h^\infty), \\ v_h(x(h)) = h^{(1-n)/2} \int_{\delta/2 < r(x,y) < 2\delta} e^{-ir(x(h),y)/h} a(x(h), y, h)\, \hat{\rho}(r(x(h), y))\, b_{x,\varepsilon}(y, hD_y)\, u_h(y)\, dy. \end{gathered} \tag{3-6}$$

Now, note that for any $\psi \in C_0^\infty(M)$,

$$v_h(x(h)) = I_1(x(h), h) + I_2(x(h), h), \tag{3-7}$$

where

$$I_1 := (2\pi h)^{(1-n)/2} \int_{\delta/2 < r(x,y) < 2\delta} e^{-ir(x(h),y)/h} a(x(h), y, h)\, \hat{\rho}(r(x(h), y))\, \psi(y)\, (b_{x,\varepsilon}(y, hD_y)u_h)\, dy,$$

$$I_2 := (2\pi h)^{(1-n)/2} \int_{\delta/2 < r(x,y) < 2\delta} e^{-ir(x(h),y)/h} a(x(h), y, h)\, \hat{\rho}(r(x(h), y))\, (1-\psi(y))\, (b_{x,\varepsilon}(y, hD_y)u_h)\, dy.$$

Therefore, by Cauchy–Schwarz applied to I_1 and I_2,

$$|h^{(n-1)/2} v_h(x(h))| \le C_\delta \big(\|\psi\|_{L^2} \|b_{x,\varepsilon}(y, hD_y) u_h(y)\|_{L^2} + \big\| (1-\psi(y))[b_{x,\varepsilon}(y, hD_y) u_h] \big\|_{L^2} \big).$$

Hence letting $h \to 0$ then $\varepsilon \to 0$, and using that

$$\|b_{x,\varepsilon}(y, hD_y) u_h(y)\|_{L^2} \le (\sup |b_{x,\varepsilon}| + o_\varepsilon(1)) \|u_h\|_{L^2},$$

see for example [Zworski 2012, Theorem 5.1], we have

$$\limsup_{h\to 0} h^{(n-1)/2} |u_h(x(h))| \le C_\delta \bigg(\|\psi\|_{L^2} + \bigg(\int_{\Lambda_{x,3\delta} \setminus \Lambda_{x,\delta/2}} (1-\psi(y))^2 \, d\mu \bigg)^{\frac{1}{2}} \bigg). \tag{3-8}$$

Further microlocalization along $\mathbf{supp}\, \boldsymbol{\mu}|_{\boldsymbol{\Lambda_x}}$. Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on the flow-out Λ_x. By assumption, $\mathcal{H}^n(\operatorname{supp} \mu|_{\Lambda_x}) = 0$. In view of the microlocalization above, we are only interested in the annular subset

$$A_x(\delta/2, 3\delta) := \Lambda_{x,3\delta} \setminus \Lambda_{x,\delta/2}.$$

Since $\mathcal{H}^n$ is Radon, for any $\varepsilon_1 > 0$, there exist n-dimensional balls $B(r_j) \subset A_x(\delta/4, 4\delta)$, $j = 1, 2, \dots,$ with radii $r_j > 0$, $j = 1, 2, \dots,$ such that

$$\operatorname{supp} \mu|_{A_x(\delta/2, 3\delta)} \subset \bigcup_{j=1}^\infty B(r_j), \quad \mathcal{H}^n\Big(\bigcup_{j=1}^\infty B(r_j) \Big) < \mathcal{H}^n(\operatorname{supp} \mu|_{A_x(\delta/2,3\delta)}) + \varepsilon_1.$$

Note that for $\delta > 0$ small enough, the canonical projection $\pi : T^*M \to M$ restricts to a diffeomorphism

$$\pi : A_x\big(\tfrac{1}{4}\delta, 4\delta\big) \to \{ y \in M : \delta/4 < r(x,y) < 4\delta \}.$$

Consider the closed set

$$K = \pi(\operatorname{supp} \mu|_{A_x(\delta/2,3\delta)}) \subset M$$

with open covering

$$G := \pi\Big(\bigcup_{j=1}^\infty B(r_j) \Big) \quad \text{satisfying } \mathcal{H}^n(G) = \mathcal{H}^n(K) + O(\varepsilon_1). \tag{3-9}$$

By the C^∞ Urysohn lemma, there exists $\chi_{\Gamma_x} \in C_0^\infty(M; [0,1])$ with

$$\chi_{\Gamma_x}|_K = 1, \quad \operatorname{supp} \chi_{\Gamma_x} \subset G. \tag{3-10}$$

(Note that χ_{Γ_x} depends on ε_1, but we suppress this dependence to simplify notation.) We now apply (3-8) with $\psi = \chi_{\Gamma_x}$. First, observe that by (3-9) and (3-10)

$$\|\chi_{\Gamma_x}\|_{L^2} \le (\mathcal{H}^n(G))^{1/2} \le (\mathcal{H}^n(K))^{1/2} + O(\varepsilon_1^{1/2}). \tag{3-11}$$

Next, by construction, for all $\varepsilon_1 > 0$,

$$(1 - \chi_{\Gamma_x})(y) = 0 \quad \text{for all } y \in \pi(\operatorname{supp} \mu|_{\Lambda_{x,4\delta} \setminus \Lambda_{x,\delta/4}})$$

and hence

$$\int_{\Lambda_{x,3\delta}\setminus\Lambda_{x,\delta/2}} (1-\chi_{\Gamma_x})^2\, d\mu = 0.$$

Using this together with (3-11) in (3-8) and sending $\varepsilon_1 \to 0$ gives

$$\limsup_{h\to 0} h^{(n-1)/2}|u_h(x(h))| \le C_\delta\big(\mathcal{H}^n(\pi(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)}))\big)^{1/2} \le C'_\delta\big(\mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)})\big)^{1/2}, \quad (3\text{-}12)$$

where the last inequality follows from the fact that $\pi|_{A(\delta/2,3\delta)}$ is a diffeomorphism. Finally, since u_h is admissible at x,

$$\mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)}) = 0. \qquad \square$$

Remark 3.1. For $r(x(h), x) = o(1)$, the estimate

$$\limsup_{h\to 0} h^{(n-1)/2}|u_h(x(h))| \le C'_\delta\big(\mathcal{H}^n(\operatorname{supp}\mu|_{A_x(\delta/2,3\delta)})\big)^{1/2}$$

in (3-12) holds for any sequence of eigenfunctions with defect measure μ. It gives a quantitative estimate relating the behaviour of the defect measure to L^∞ norms of eigenfunctions. This estimate can also be obtained as a consequence of [Galkowski 2017, Theorem 2] by replacing the absolutely continuous part of μ with $1_{\operatorname{supp}\mu|_{\Lambda_x}}\, d\mathcal{H}^n$.

4. The example of zonal harmonics

Let (S^2, g_{can}) be the round sphere and (r, θ) be polar variables centred at the north pole $p = (0, 0, 1) \in \mathbb{R}^3$. The geodesic flow is a completely integrable system with Hamiltonian

$$H = |\xi|_g^2 = \xi_r^2 + (\sin r)^{-2}\xi_\theta^2, \qquad r \in (0, \pi), \quad (4\text{-}1)$$

and Claurault integral $p = \xi_\theta$ satisfying $\{H, p\} = 0$. The associated moment mapping is $\mathcal{P} = (H, p) : T^*S^2 \to \mathbb{R}^2$ and the connected components of the level sets are, by the Liouville–Arnold theorem, Lagrangian tori Λ_c indexed by the values of the moment map $(1, c) \in \mathcal{P}(T^*S^2)$.

The associated quantum integrable system is given by the Laplacian Δ_g and the rotation operator hD_θ. The corresponding L^2-normalized joint eigenfunctions are the standard spherical harmonics Y_m^k with

$$-\Delta_g Y_m^k = k(k+1)Y_m^k, \qquad hD_\theta Y_m^k = mY_m^k.$$

These eigenfunctions can be separated into various sequences (i.e., *ladders*) associated with different values ($\in \mathcal{P}(T^*S^2)$; specifically, the correspondence is given by $c = \lim_{m\to\infty} m/k$). The eigenfunctions with maximal L^∞ blow-up are the sequence of *zonal* harmonics given by

$$u_h(r,\theta) = Y_0^k(r,\theta) = \frac{\sqrt{2k+1}}{2\pi}\int_0^{2\pi} (\cos r + i\sin r\cos\tau)^k\, d\tau, \qquad h = k^{-1},\ k = 1, 2, 3\ldots. \quad (4\text{-}2)$$

It is obvious from (4-2) that

$$|Y_0^k(p)| \approx k^{1/2}$$

and thus attains the maximal sup growth at p (similarly, at the south pole). At the classical level, the zonals $u_h = Y_0^k$ concentrate microlocally on the Lagrangian tori $\Lambda_0 = \mathcal{P}^{-1}(1, 0)$. From the formula (4-1) it is clear that away from the poles (where (r, θ) are honest coordinates),

$$\Lambda_0 \setminus \{\pm p\} = \{(r, \theta, \xi_r = \pm 1, \xi_\theta = 0) : r \in (0, \pi)\} \cong S^2 \setminus \{\pm p\}. \tag{4-3}$$

The choice of $\xi_r = \pm 1$ determines the Lagrangian torus (there are two of them) and also, either torus clearly covers the entire sphere. At the poles themselves, the projection $\pi_{\Lambda_0} : \Lambda_0 \to S^2$ has a blowdown singularity with

$$\pi_{\Lambda_0}^{-1}(\pm p) = S^*_{\pm}(S^2) \cong S^1. \tag{4-4}$$

To see this, consider the behaviour at p (with a similar computation at $-p$). Rewriting the integral in involution in Euclidean coordinates $(x, y, z) \in \mathbb{R}^3$, one has $H = (x\xi_y - y\xi_x)^2 + (x\xi_z - z\xi_x)^2 + (y\xi_z - z\xi_y)^2$ and $\xi_\theta = x\xi_y - y\xi_x$. Setting $H = 1$, $x\xi_y - y\xi_x = 0$ and $(x, y, z) = (0, 0, 1)$ gives

$$\pi_{\Lambda_0}^{-1}(p) \cong \{(\xi_x, \xi_y) \in \mathbb{R}^2 : \xi_x^2 + \xi_y^2 = 1\}.$$

It is then clear from (4-3) and (4-4) that $\pi_{\Lambda_0} : \Lambda_0 \to S^2$ is surjective and a diffeomorphism away from the poles (modulo choice of Lagrangian cover) and the fibres above the poles are $S^*_{\pm}(S^2) \cong S^1$. We also note that the Lagrangian $\Lambda_0 = \Lambda_{p,2\pi}$ is the 2π-flow-out Lagrangian of $S^*_p(S^2)$ and the cylinder $A_p(\delta/2, 3\delta)$ is just a local slice of this Lagrangian.

The defect measure μ associated with the zonals is

$$d\mu = |d\theta_1 \, d\theta_2|,$$

where $(\theta_1, \theta_2; I_1, I_2) \in \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2$ are symplectic action-angle variables defined in a neighbourhood of the Lagrangian torus Λ_0 [Toth and Zelditch 2003]. One can choose one of the angle variables $\theta_1 \in S^*_p(S^2)$ to parametrize the circle fibre above p (a homology generator of the torus). Then, by the Liouville–Arnold theorem, the geodesic flow on the torus $\Lambda_0 = \{I_1 = c_1, I_2 = c_2\}$ is affine with

$$\theta_j(t) = \theta_j(0) + \alpha_j t, \qquad \alpha_j = \frac{\partial H}{\partial I_j} \neq 0.$$

It is then clear that

$$\mu(\Lambda_{p,\delta}) = \int_0^{2\pi} d\theta_1 \cdot \int_{|t|<\delta} \alpha_2 \, dt \approx \delta \neq 0$$

and $\operatorname{supp} \mu|_{\Lambda_p} = \Lambda_p$. Therefore, this case violates the assumption in Theorem 3 and that is of course consistent with the maximal L^∞ growth of zonal harmonics.

The analysis above extends in a straightforward fashion to the case of a more general sphere of rotation [Toth and Zelditch 2003].

5. Eigenfunctions of Schrödinger operators

Consider a Schrödinger operator $P(h) = -h^2\Delta_g + V$ with $V \in C^\infty(M; \mathbb{R})$ on a compact, closed Riemannian manifold (M, g) and let u_h be an L^2-normalized eigenfunction with

$$P(h)u_h = E(h)u_h, \qquad E(h) = E + o(1), \quad E > \min V, \qquad \|u_h\|_{L^2} = 1. \tag{5-1}$$

Any sequence u_h of solutions to (5-1) has a subsequence u_{h_k} with a defect measure μ in the sense that for $a \in C_0^\infty(T^*M)$

$$\langle a(x, hD)u_h, u_h\rangle \to \int_{T^*M} a\, d\mu.$$

Such a measure μ is supported on $\{p=0\}$ and is invariant under the bicharacteristic flow $G_t := \exp(tH_p)$.

In analogy with the homogeneous case, we define for $x \in M$ the *time-T flow-out* by

$$\Lambda_{x,T,V} := \bigcup_{t=-T}^{T} G_t(\Sigma_x),$$

where

$$\Sigma_x = \{\xi \in T_x^*M : |\xi|_g^2 + V(x) = E\}.$$

Definition 5.1. Let $\mathcal{H}^n$ be n-dimensional Hausdorff measure on $\{|\xi|_g^2 + V(x) = E\}$ induced by the Sasaki metric on T^*M. We say that the sequence u_h of solutions to (5-1) is *admissible at x* if for any defect measure μ associated to the sequence, there exists $T > 0$ so that

$$\mathcal{H}^n(\operatorname{supp} \mu|_{\Lambda_{x,T,V}}) = 0. \tag{5-2}$$

With these definitions we have the analog of Theorem 3:

Theorem 5. *Let $B \subset V^{-1}(E)$ be a closed ball in the classically allowable region and μ be a defect measure associated with the eigenfunction sequence u_h. Then, if the eigenfunction sequence is admissible for all $x \in B$ in the sense of (5-2),*

$$\sup_{x\in B} |u_h(x)| = o(h^{(1-n)/2}).$$

Proof. In analogy with the homogeneous case [Christianson et al. 2015, Lemma 5.1], we have

$$\rho(h^{-1}[P(h) - E]))(x, y) = h^{(1-n)/2} a(x, y, h)e^{-iA(x,y)/h} + R(x, y, h),$$

where $A(x, y) \in [(2C_0)^{-1}\varepsilon, 2C_0\varepsilon]$ for some $C_0 > 1$ and is the action function defined to be the integral of the Lagrangian $L(x, \xi) = |\xi|_g^2 - V(x)$ along the bicharacteristic in $\{p = E\}$ starting at (y, η) and ending at (x, ξ). For (x, y) in a small neighbourhood of the diagonal, there is a unique such η satisfying this condition. The remainder $R(x, y, h)$ is equal to $O(h^\infty)$ pointwise and with all derivatives. The proof then follows using the same argument as in the homogeneous case. ☐

Acknowledgments

The authors would like to thank the anonymous referees for their detailed reading and many helpful comments. Galkowski is grateful to the National Science Foundation for support under the Mathematical Sciences Postdoctoral Research Fellowship DMS-1502661. The research of Toth was partially supported by NSERC Discovery Grant #OGP0170280 and an FRQNT Team Grant. Toth was also supported by the French National Research Agency project Gerasic-ANR-13-BS01-0007-0.

References

[Avakumović 1956] V. G. Avakumović, "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten", *Math. Z.* **65** (1956), 327–344. MR Zbl

[Bérard 1977] P. H. Bérard, "On the wave equation on a compact Riemannian manifold without conjugate points", *Math. Z.* **155**:3 (1977), 249–276. MR Zbl

[Blair 2010] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, 2nd ed., Progress in Mathematics **203**, Birkhäuser, Boston, 2010. MR Zbl

[Burq et al. 2007] N. Burq, P. Gérard, and N. Tzvetkov, "Restrictions of the Laplace–Beltrami eigenfunctions to submanifolds", *Duke Math. J.* **138**:3 (2007), 445–486. MR Zbl

[Christianson et al. 2015] H. Christianson, A. Hassell, and J. A. Toth, "Exterior mass estimates and L^2-restriction bounds for Neumann data along hypersurfaces", *Int. Math. Res. Not.* **2015**:6 (2015), 1638–1665. MR Zbl

[Dyatlov and Zworski 2017] S. Dyatlov and M. Zworski, "Mathematical theory of scattering resonances", book in progress, 2017, available at http://math.mit.edu/~dyatlov/res/res_20171019.pdf.

[Galkowski 2017] J. Galkowski, "Defect measures of eigenfunctions with maximal L^∞ growth", preprint, 2017. arXiv

[Gérard 1991] P. Gérard, "Microlocal defect measures", *Comm. Partial Differential Equations* **16**:11 (1991), 1761–1794. MR Zbl

[Hezari 2016] H. Hezari, "Quantum ergodicity and L^p norms of restrictions of eigenfunctions", preprint, 2016. arXiv

[Hörmander 1968] L. Hörmander, "The spectral function of an elliptic operator", *Acta Math.* **121** (1968), 193–218. MR Zbl

[Iwaniec and Sarnak 1995] H. Iwaniec and P. Sarnak, "L^∞ norms of eigenfunctions of arithmetic surfaces", *Ann. of Math.* (2) **141**:2 (1995), 301–320. MR Zbl

[Jakobson and Zelditch 1999] D. Jakobson and S. Zelditch, "Classical limits of eigenfunctions for some completely integrable systems", pp. 329–354 in *Emerging applications of number theory* (Minneapolis, MN, 1996), edited by D. A. Hejhal et al., IMA Vol. Math. Appl. **109**, Springer, 1999. MR Zbl

[Levitan 1952] B. M. Levitan, "On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order", *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **16** (1952), 325–352. In Russian. MR

[Shnirelman 1974] A. I. Shnirelman, "Ergodic properties of eigenfunctions", *Uspehi Mat. Nauk* **29**:6(180) (1974), 181–182. In Russian. MR Zbl

[Sogge 1993] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics **105**, Cambridge University Press, 1993. MR Zbl

[Sogge 2016] C. D. Sogge, "Localized L^p-estimates of eigenfunctions: a note on an article of Hezari and Rivière", *Adv. Math.* **289** (2016), 384–396. MR Zbl

[Sogge and Zelditch 2002] C. D. Sogge and S. Zelditch, "Riemannian manifolds with maximal eigenfunction growth", *Duke Math. J.* **114**:3 (2002), 387–437. MR Zbl

[Sogge and Zelditch 2016a] C. D. Sogge and S. Zelditch, "Focal points and sup-norms of eigenfunctions", *Rev. Mat. Iberoam.* **32**:3 (2016), 971–994. MR Zbl

[Sogge and Zelditch 2016b] C. D. Sogge and S. Zelditch, "Focal points and sup-norms of eigenfunctions, II: the two-dimensional case", *Rev. Mat. Iberoam.* **32**:3 (2016), 995–999. MR Zbl

[Sogge et al. 2011] C. D. Sogge, J. A. Toth, and S. Zelditch, "About the blowup of quasimodes on Riemannian manifolds", *J. Geom. Anal.* **21**:1 (2011), 150–173. MR Zbl

[Toth and Zelditch 2002] J. A. Toth and S. Zelditch, "Riemannian manifolds with uniformly bounded eigenfunctions", *Duke Math. J.* **111**:1 (2002), 97–132. MR Zbl

[Toth and Zelditch 2003] J. A. Toth and S. Zelditch, "Norms of modes and quasi-modes revisited", pp. 435–458 in *Harmonic analysis at Mount Holyoke* (South Hadley, MA, 2001), edited by W. Beckner et al., Contemp. Math. **320**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Colin de Verdière 1985] Y. Colin de Verdière, "Ergodicité et fonctions propres du laplacien", *Comm. Math. Phys.* **102**:3 (1985), 497–502. MR Zbl

[Zelditch 1987] S. Zelditch, "Uniform distribution of eigenfunctions on compact hyperbolic surfaces", *Duke Math. J.* **55**:4 (1987), 919–941. MR Zbl

[Zworski 2012] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics **138**, American Mathematical Society, Providence, RI, 2012. MR Zbl

Received 4 Apr 2017. Revised 18 Aug 2017. Accepted 16 Oct 2017.

JEFFREY GALKOWSKI: jeffrey.galkowski@mcgill.ca
Department of Mathematics and Statistics, McGill University, Montréal, QC, Canada

JOHN A. TOTH: jtoth@math.mcgill.ca
Department of Mathematics and Statistics, McGill University, Montréal, QC, Canada

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2018 is US $275/year for the electronic version, and $480/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 11 No. 3 2018