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LARGE SETS AVOIDING PATTERNS

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We construct subsets of Euclidean space of large Hausdorff dimension and full Minkowski dimension that do not contain nontrivial patterns described by the zero sets of functions. The results are of two types. Given a countable collection of v-variate vector-valued functions $f_q : (\mathbb{R}^n)^v \to \mathbb{R}^m$ satisfying a mild regularity condition, we obtain a subset of $\mathbb{R}^n$ of Hausdorff dimension $m/(v-1)$ that avoids the zeros of f_q for every q. We also find a set that simultaneously avoids the zero sets of a family of uncountably many functions sharing the same linearization. In contrast with previous work, our construction allows for nonpolynomial functions, as well as uncountably many patterns. In addition, it highlights the dimensional dependence of the avoiding set on v, the number of input variables.

1. Introduction

Identification of geometric and algebraic patterns in large sets has been a focal point of interest in modern analysis, geometric measure theory and additive combinatorics. A fundamental and representative result in the discrete setting that has been foundational in the development of a rich theory is Szemerédi's theorem [1975], which states that every subset of the integers with positive asymptotic density contains an arbitrarily long arithmetic progression. There is now an abundance of similar results in the continuum setting, all of which guarantee existence of configurations under appropriate assumptions on size, often stated in terms of Lebesgue measure, Hausdorff dimension or Banach density. While this body of work has contributed significantly to our understanding of such phenomena, a complete picture concerning existence or avoidance of patterns in sets is yet to emerge. In this paper, we will be concerned with the "avoidance" aspect of the problem. Namely, given a function $f : \mathbb{R}^{nv} \to \mathbb{R}^m$ satisfying certain conditions, how large a set $E \subset \mathbb{R}^n$ can one construct that carries no nontrivial solution of the equation $f(x_1, \dots, x_v) = 0$? In other words, we aim to find as large a set E as possible such that $f(x_1, \dots, x_v)$ is nonzero for any choice of distinct points $x_1, \dots, x_v \in E$.

In the discrete regime, results of this type can be traced back to Salem and Spencer [1942] and Behrend [1946], who identified large subsets of the integers avoiding progressions. The Euclidean formulation of this problem appears to be of relatively recent vintage. Keleti [1999] constructed a subset E of the real numbers of full Hausdorff dimension avoiding all nontrivial "one-dimensional rectangles". More precisely, this means that there exist no solutions of the equation $x_2 - x_1 - x_4 + x_3 = 0$ with $x_1 < x_2 \le x_3 < x_4$, $x_i \in E$, $1 \le i \le 4$. In particular, such a set contains no nontrivial arithmetic progression, as can be seen by setting $x_2 = x_3$. A counterpoint to [Keleti 1999] is a result of Łaba and the second author [Łaba and

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Pramanik 2009], who established existence of three-term progressions in special "random-like" subsets of $\mathbb{R}$ that support measures satisfying an appropriate ball condition and a Fourier decay estimate. Higher-dimensional variants of this theme may be found in [Chan et al. 2016; Henriot et al. 2016]. On the other hand, large Hausdorff dimensionality, while failing to ensure specific patterns, is sometimes sufficient to ensure existence or even abundance of certain configuration classes; see for instance [Greenleaf and Iosevich 2012; Greenleaf et al. 2015; 2017; Bennett et al. 2016]. Harangi, Keleti, Kiss, Maga, Máthé, Mattila, and Strenner [Harangi et al. 2013] showed that sets of sufficiently large Hausdorff dimension contain points that generate specific angles.

Nonexistence of patterns such as the one proved by Keleti [1999] is the primary focus of this article. A main contribution of [Keleti 1999] is best described as a Cantor-type construction with memory, where selection of basic intervals at each stage is contingent on certain selections made at a much earlier step of the construction, so as to prevent certain algebraic relations from taking place. This idea has been instrumental in a large body of subsequent work involving nonexistence of configurations. For example, Keleti [2008] used this to show that for any countable set A, it is possible to construct a full-dimensional subset E of $\mathbb{R}$ such that

$$x_2 - x_1 + a(x_3 - x_2) = 0$$

has no solutions for any $a \in A$, where x_1, x_2 and x_3 are distinct points in E. Maga [2010] exploited this idea to demonstrate a full-dimensional subset $E \subset \mathbb{R}^n$ not containing the vertices of any parallelogram. He also constructed a full-dimensional planar set that misses all similar copies of a given triangle. Other results in this direction of considerable generality, extending their predecessors in [Keleti 1999; 2008; Maga 2010], are due to Máthé [2012]. Given any countable collection of polynomials $p_j : \mathbb{R}^{nm_j} \to \mathbb{R}$ of degree at most d with rational coefficients, the main result of [Máthé 2012] ensures the existence of a subset $E \subseteq \mathbb{R}^n$ of Hausdorff dimension n/d such that $p_j(x_1, \dots, x_{m_j})$ is nonzero for any choice of distinct points $x_1, \dots, x_{m_j} \in E$. The same conclusion continues to hold if the polynomials p_j are replaced by $p_j(\Phi_{j,1}(x_1), \dots, \Phi_{j,m_j}(x_{m_j}))$, where $\Phi_{j,k}$ are C^1-diffeomorphisms of $\mathbb{R}^n$. Interestingly, the Hausdorff dimension bound in [Máthé 2012], while depending on the ambient dimension n and the maximum degree d of the polynomials, is independent of the number of input vectors m_j in p_j, which may continue to grow without bound.

This paper uses similar ideas to present two results in a somewhat different direction. The first complements Máthé's result mentioned above. It applies to a countable family of functions $f : \mathbb{R}^{nv} \to \mathbb{R}^m$ with a fixed v that are not necessarily polynomials with rational coefficients. Further, in contrast with [Máthé 2012], the Hausdorff dimension of the obtained set depends on the number of vector variables v. The second result is of a perturbative flavour, and gives a set of positive Hausdorff dimension that simultaneously avoids zeros of *all* functions with a common linearization and bounded higher-order terms. To the best of our knowledge, such uniform avoidance results are new. Some points of tenuous similarity may be found in [Harangi et al. 2013], where the authors construct sets that avoid angles within a specific range, but the ideas, methods and goals are very different.

1A. ***Main results.*** Our first result is most general in dimension one, where we need very mild restrictions on the functions whose zeros we want to avoid. The higher-dimensional, vector-valued version of this result applies with some additional restrictions. We state these two separately.

Theorem 1.1. *For any $\eta > 0$ and integer $v \geq 3$, let $f_q : \mathbb{R}^v \to \mathbb{R}$ be a countable family of functions in v variables with the following properties*:

(a) *There exists $r_q < \infty$ such that $f_q \in C^{r_q}([0, \eta]^v)$.*

(b) *For each q, some partial derivative of f_q of order $r_q \geq 1$ does not vanish at any point of $[0, \eta]^v$.*

Then there exists a set $E \subseteq [0, \eta]$ of Hausdorff dimension at least $1/(v-1)$ and Minkowski dimension 1 such that $f_q(x_1, \dots, x_v)$ is not equal to zero for any v-tuple of distinct points $x_1, \dots, x_v \in E$ and any function f_q.

Theorem 1.2. *Fix $\eta > 0$ and positive integers m, n, v such that $v \geq 3$, and $m \leq n(v-1)$. Let $f_q : \mathbb{R}^{nv} \to \mathbb{R}^m$ be a countable family of C^2 functions with the following property*: *for every q on $[0, \eta]^{nv}$, the derivative $Df_q(x_1, \dots, x_v)$ has full rank at every point $(x_1, \dots, x_v)$ in the zero set of f_q such that $x_r \neq x_s$ for all $r \neq s$.*

Then there exists a set $E \subseteq [0, \eta]^n$ of Hausdorff dimension at least $m/(v-1)$ and Minkowski dimension n such that $f_q(x_1, \dots, x_v)$ is not equal to zero for any v-tuple of distinct points $x_1, \dots, x_v \in E^n$ and any function f_q.

Remarks. (a) If one seeks to avoid zeros of a single function f, then Theorem 1.1 is nontrivial only when the components of $\nabla f(x)$ sum to zero at every point x in the zero set of f. If this is not the case, then there is necessarily some interval I such that $f(x_1, \dots, x_v)$ is nonzero for points x_i in the interval I.

(b) The points $x_1, \dots, x_v \in E$ that ensure $f(x_1, \dots, x_v) \neq 0$ in Theorems 1.1 and 1.2 are taken to be distinct. This assumption, while needed for the proof, is often nonrestrictive for the purpose of applications. In fact, one can typically augment the family $\{f_q\}$ by $\{g_q\}$, where the function g_q equals f_q with certain input variables coincident, and apply the theorems above to the augmented family provided the nonvanishing derivative assumptions are met. For instance, Keleti's function $f(x_1, x_2, x_3, x_4) = (x_2 - x_1) - (x_4 - x_3) = -x_1 + x_2 + x_3 - x_4$ identifies "one-dimensional rectangles" in general, and three-term arithmetic progressions only if $x_2 = x_3$. In order to obtain a set that avoids both using our setup, we would need to apply our Theorem 1.1 to the collection $\{f, g\}$, where $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_2, x_4) = -x_1 + 2x_2 - x_4$.

(c) Theorem 1.2 is sharp in certain instances, for example when $m = n(v-1)$. On the other hand, Theorem 1.1 need not be sharp for specific choices of f_q, as Keleti's example shows. Our result would only ensure a set of Hausdorff dimension $\frac{1}{3}$ for this example. Given the similarity in our respective methods of proof, the contrast in the results requires a word of explanation. In [Keleti 1999], one had explicit knowledge of the function f (which was linear), and hence of the structure of its zero set. This arithmetic structure was exploited heavily in the construction. Our assumptions on $\{f_q\}$ are too weak to offer explicit information concerning algebraic dependencies in the zero set, and hence our proof is based on a "worst-case analysis", which is true generically, but results in worse bounds. However, our method of proof is robust enough to accommodate special structures in zero sets, and yields better dimensional bounds in those settings. We substantiate this comment with more precise details at the appropriate juncture of the proof; see the remark on page 1092.

(d) Our conjecture is that the dimensional lower bound of $1/(v-1)$ in Theorem 1.1 is sharp for certain generic functions, but we are currently unaware of any result in the literature that addresses the optimality of this bound in the setup that we describe. Partial evidence in support of this conjecture in provided in [Körner 2009], where the author constructs a set of Fourier dimension $1/(v-1)$ avoiding all v-variate rational linear relations. We hope to return to this issue in the future.

(e) Even though our results do not recover those of [Keleti 2008; Maga 2010; Máthé 2012] in all instances where these results are applicable, the Hausdorff dimension provided in Theorems 1.1 and 1.2 offers new bounds in settings where previously none were available, for instance where the functions are nonpolynomials with mild regularity. It also improves the bound given in [Máthé 2012] for polynomials with rational coefficients in the regime where the degree d is much larger than the number of variables v. On the other hand, for polynomials of low degree the result in [Máthé 2012] improves ours, obtaining the best bound when $d = 1$.

(f) Finding the optimal dimension of a zero-avoiding set for a *specific and explicitly stated function* remains an interesting open question. For the quadratic polynomial $f(x_1, x_2, x_3) = (x_3 - x_1) - (x_2 - x_1)^2$, the zero-avoiding set is guaranteed to be of Hausdorff dimension at least $\frac{1}{2}$, both according to [Máthé 2012] and Theorem 1.1. It is not known whether this bound is optimal.

Our second result is about a set on which no function f with a given linearization and controlled higher-order term is zero.

Theorem 1.3. *Given any constant $K > 0$ and a vector $\alpha \in \mathbb{R}^v$ such that*

$$\alpha \cdot u \neq 0 \quad \text{for every } u \in \{0, 1\}^v \text{ with } u \neq 0,\ u \neq (1, 1, \dots, 1), \tag{1-1}$$

and such that

$$\sum_{j=1}^{v} \alpha_j = 0, \tag{1-2}$$

there exists a positive constant $c(\alpha)$ and a set $E = E(K, \alpha) \subseteq [0, 1]$ of Hausdorff dimension $c(\alpha) > 0$ with the following property.

The set E does not contain any nontrivial solution of the equation

$$f(x_1, \dots, x_v) = 0, \quad (x_1, \dots, x_v) \text{ not all identical},$$

for any C^2 function f of the form

$$f(x_1, \dots, x_v) = \sum_{j=1}^{v} \alpha_j x_j + G(x_1, \dots, x_v), \tag{1-3}$$

where

$$|G(x)| \leq K \sum_{j=2}^{v} (x_j - x_1)^2. \tag{1-4}$$

Remarks. (a) The condition (1-1) implies that α does not lie in any coordinate hyperplane.

(b) The proof of Theorem 1.3 can be used to obtain a corresponding result with *finitely* many linearizations. There is a loss in the Hausdorff dimension as more linear functions are added to the family, so the proof fails for families of functions with countably many linearizations.

(c) It is interesting to note that the dimensional constant $c(\alpha)$ does not depend on K. Of course the set E does, and is uniform for all functions f obeying (1-3) and (1-4) with a fixed value of K.

1B. ***Layout.*** Section 2 is devoted to geometric applications of Theorems 1.1, 1.2 and 1.3. Optimality of these results (or lack thereof) in various settings is discussed, and comparison with earlier work is presented. Section 3 is a collection of geometric algorithms needed for the proofs of Theorems 1.1 and 1.2. The proofs themselves are executed in Sections 4 and 5.

2. Examples

2A. ***Subsets of curves avoiding isosceles triangles.*** This subsection is given over to the following question: suppose we are given a small segment of a simple C^2 curve $\Gamma \subset \mathbb{R}^n$ with nonvanishing curvature bounded above by K, parameterized by a C^2 function $\gamma : [0, \eta] \to \mathbb{R}^n$ with nonvanishing derivative. How large can the Hausdorff dimension of a subset $E \subseteq [0, \eta]$ be if there do not exist three points $x_1, x_2, x_3 \in E$ such that $\{\gamma(x_1), \gamma(x_2), \gamma(x_3)\} \subseteq \Gamma$ are the vertices of an isosceles triangle?

The existence of an isosceles triangle with vertices on Γ will be determined using one of the functions

$$f_1(t_1, t_2, t_3) = |\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2, \tag{2-1}$$

$$f_2(t_1, t_2, t_3) = d(\gamma(t_1), \gamma(t_2)) - d(\gamma(t_2), \gamma(t_3)). \tag{2-2}$$

Here d is the "signed distance" along the curve Γ defined by

$$d(\gamma(t_1), \gamma(t_2)) = \begin{cases} |\gamma(t_1) - \gamma(t_2)| & \text{if } t_1 \geq t_2, \\ -|\gamma(t_1) - \gamma(t_2)| & \text{if } t_1 < t_2. \end{cases} \tag{2-3}$$

For reasons to be explained shortly, we will want to avoid the zero set of f_1 or f_2. In order to apply Theorem 1.1, we need to verify that these functions are differentiable. This is evident for f_1. In Lemma A.1 of the Appendix, we have shown that the signed distance d is differentiable, which provides the same conclusion for f_2.

Let f be either the function f_1 or f_2 given in (2-1) or (2-2). In either case, we have that if $f(t_1, t_2, t_3) = 0$, then $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ form the vertices of an isosceles triangle or points in an arithmetic progression. Conversely, let x, y, z be distinct points of Γ that form an isosceles triangle, with $|x - y| = |y - z|$. Then there exist $t_1 < t_2 < t_3$ such that some permutation of $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ will be the points x, y, z. It is not difficult to see that if η is sufficiently small depending on $|\gamma'(0)|$ and the curvature K, then y can be neither $\gamma(t_1)$ nor $\gamma(t_3)$. We include a proof of this in Lemma A.2 in the Appendix. Therefore $y = \gamma(t_2)$, in which case $f(t_1, t_2, t_3) = 0$.

2A1. *A set avoiding isosceles triangles along a single curve.* We will first discuss the problem of avoiding isosceles triangles along a single curve Γ. For this variant of the problem, γ may be any parameterization of Γ satisfying the conditions laid out above.

Let us first consider the case where Γ is parameterized by a polynomial function γ of degree d with rational coefficients; i.e., $\gamma(t) = (p_1(t), p_2(t), \dots, p_n(t))$. Let us observe that the result in [Máthé 2012] does not apply to the nonpolynomial function $f_2(t_1, t_2, t_3)$, but does apply to

$$f_1(t_1, t_2, t_3) = \big[(p_1(t_1)-p_1(t_2))^2+\cdots+(p_n(t_1)-p_n(t_2))^2\big]-\big[(p_1(t_2)-p_1(t_3))^2+\cdots+(p_n(t_2)-p_n(t_3))^2\big],$$

which is a polynomial of degree at most $2d$. Applying [Máthé 2012] then gives a subset of Γ of Hausdorff dimension $1/(2d)$ that does not contain the vertices of any isosceles triangle.

If Γ is a general (not necessarily polynomial) C^2 curve with parameterization $\gamma(t)$, and $f(t_1, t_2, t_3)$ is either f_1 or f_2 described above, then Theorem 1.1 demonstrates the existence of a subset E of [0, 1] of Hausdorff dimension $\frac{1}{2}$ such that $f(t_1, t_2, t_3) \neq 0$ for any choice of $t_1, t_2, t_3 \in E$. Under γ, this lifts to a subset of Γ of Hausdorff dimension $\frac{1}{2}$ that does not contain the vertices of an isosceles triangle. Even for the case of functions with a rational polynomial parameterization, this set has a larger Hausdorff dimension than the one provided by [Máthé 2012].

Incidentally, it is instructive to compare the above with the case where the curve γ is a line, even though the curvature for the latter is zero. Here we will view three-term arithmetic progressions as degenerate isosceles triangles. Set $\gamma(t) = at + b$ for some $a, b \in \mathbb{R}^n$, $a \neq 0$. Then the function $f(t_1, t_2, t_3) = t_1 + t_3 - 2t_2$ is equal to zero precisely when $\gamma(t_1)$, $\gamma(t_2)$ and $\gamma(t_3)$ lie in arithmetic progression. Keleti's result [1999], as well as [Máthé 2012], applied to this f show that there is a subset of Γ of Hausdorff dimension 1 that does not contain any arithmetic progressions. Theorem 1.1 on the other hand provides a set with Hausdorff dimension $\frac{1}{2}$, which is suboptimal.

2A2. *A set avoiding isosceles triangles along all curves with bounded curvature.* We will also ask a question related to the one above, this time considering only C^2 curves given by arclength parameterization. How large a set $E \subset [0, 1]$ can we construct such that $\gamma(E)$ does not contain any isosceles triangle for any $\gamma : [0, 1] \to \mathbb{R}^n$ with $|\gamma'(t)| \equiv 1$ and with curvature at most K?

For any such curve γ, the function f_2 defined in (2-2) will be differentiable everywhere, with $\partial f_2/\partial t_1 = \partial f_2/\partial t_3 \equiv 1$ and $\partial f_2/\partial t_2 \equiv -2$, as we have verified in Lemma A.1(b). Thus the function f_2 will satisfy the conditions of Theorem 1.3. One therefore obtains a subset $E \subset [0, 1]$ of positive Hausdorff dimension such that $f_2(t_1, t_2, t_3) \neq 0$ whenever $t_1, t_2, t_3 \in E$ are distinct, no matter which γ we choose in this class. Thus the points parameterized by E manage to avoid isosceles triangles on all curves Γ with a fixed bounded curvature.

How large a Hausdorff dimension can we get? A careful scrutiny of Lemma 5.1, Proposition 5.2 and Theorem 1.3 shows that one can ensure sets of Hausdorff dimension at least $\log 2/\log 3$. For more details, we refer the reader to the proofs of these results in Section 5A and the remarks following them.

2A3. *Discussion on optimality.* Clearly Theorem 1.2 is optimal when $m = n(v-1)$. On the other hand, we can use Theorem 1.3 together with the example above to give a polynomial with rational coefficients for which neither [Máthé 2012] nor Theorem 1.1 gives the optimal bounds. Consider a polynomial of the form

$$p(t_1, t_2, t_3) = t_1 - 2t_2 + t_3 + q(t_1, t_2, t_3),$$

where $q(t_1, t_2, t_3)$ is a nontrivial homogeneous quadratic polynomial in $(t_2 - t_1)$ and $(t_3 - t_1)$ with rational coefficients. We are of course interested in finding a set E (as large as possible) such that $p(t_1, t_2, t_3) \neq 0$

for any choice of distinct points $t_1, t_2, t_3 \in E$. Both [Máthé 2012] and Theorem 1.1 provide such a set E, with dimension at least $\frac{1}{2}$ in both cases. Theorem 1.3 provides such a set E as well. Note that p has the same linearization as the functions f described in the previous section above. Hence, as described at the end of Section 2A2, the set E obtained via Theorem 1.3 is a set of dimension at least $\log 2/\log 3 > \frac{1}{2}$, proving the claimed suboptimality statement.

In fact, we can use this framework to construct other examples. Notice that it is possible to ask for sets E that avoid triangles that are not necessarily isosceles, for instance triangles where the side-length ratio is a prescribed constant κ. The results in [Keleti 1999; Máthé 2012] and Theorem 1.1 all apply to give a set with the same Hausdorff dimension $\frac{1}{2}$ as above not containing t_1, t_2, t_3 such that $|\gamma(t_2) - \gamma(t_1)| = \kappa|\gamma(t_3) - \gamma(t_1)|$. However, the Hausdorff dimension bound in Theorem 1.3 becomes worse as κ moves farther away from 1. Still, for κ close to 1, Theorem 1.3 outperforms Theorem 1.1, giving rise to a family of polynomials whose zeros can be avoided by a set of unusually large Hausdorff dimension.

2B. ***A subset of a curve not containing certain kinds of trapezoids.*** The following is a geometric example of Theorem 1.2. Call a trapezoid $ABCD$ with AD parallel to BC "special" if the side lengths obey the restriction $|BC|^2 = |AB||CD|$. Given a curve $\Gamma \subset \mathbb{R}^2$ parameterized by a smooth function $\gamma : [0, \eta] \to \mathbb{R}^2$, we aim to find a subset E of $[0, \eta]$ with the following property: for any choice of $t_1 < t_2 < t_3 < t_4$ in E, the trapezoid $ABCD$ with

$$A = \gamma(t_1), \quad B = \gamma(t_2), \quad C = \gamma(t_3), \quad D = \gamma(t_4)$$

is not special. For simplicity and ease of exposition, we may assume that the components of γ' are strictly positive on $[0, \eta]$ and that the curvature is also of constant sign, say Γ is strictly convex.

Notice that the special trapezoid assumption places two essentially independent conditions on $\gamma(t_1)$, $\gamma(t_2)$, $\gamma(t_3)$, and $\gamma(t_4)$. One is that two sides need to be parallel, and the other is the condition on the side lengths. Accordingly, we define two functions f_1 and f_2 as follows:

$$f_1(t_1, t_2, t_3, t_4) = \det\big[(\gamma(t_4) - \gamma(t_1))^t, (\gamma(t_3) - \gamma(t_2))^t\big], \tag{2-4}$$

$$f_2(t_1, t_2, t_3, t_4) = d(\gamma(t_4), \gamma(t_3))\, d(\gamma(t_2), \gamma(t_1)) - d(\gamma(t_3), \gamma(t_2))^2. \tag{2-5}$$

Here in (2-4), a^t and b^t represent the transpose of the planar row vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ respectively, while

$$\det[a^t, b^t] = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

denotes the (signed) length of the cross-product $a \times b$. Alternatively, $\det[a^t, b^t]$ may be interpreted as the signed area of the parallelogram whose sides are the vectors a and b. The determinant vanishes if either a or b is zero, or if the two vectors are parallel.

Returning to (2-4) and (2-5), f_1 is zero if and only if AD is parallel to BC, while f_2 is zero if and only if $|BC|^2 = |AB||CD|$. We therefore seek to avoid the zeros of the smooth vector-valued function $f = (f_1, f_2)$. We verify in Lemma A.3 of the Appendix that the derivative Df is of full rank on the zero set of f. Applying Theorem 1.2 with $n = 1,\ m = 2,\ v = 4$, we obtain a set E of Hausdorff dimension $\frac{2}{3}$ such that the points on Γ indexed by E avoids special trapezoids as explained above. Thus, there is a subset of Γ of Hausdorff dimension $\frac{2}{3}$ that does not contain any special trapezoids.

3. Avoidance of zeros on a single scale

The proofs of Theorems 1.1 and 1.2 are based on an iterative construction whose primary building block relies on an algorithm: given a set $T \subseteq \mathbb{R}^{nv}$ contained in the domain of a suitably nonsingular function $f : \mathbb{R}^{nv} \to \mathbb{R}^m$, one identifies a subset $S \subseteq T$ that stays away from the zero set of f. This zero-avoiding subset S, which is a union of cubes in $\mathbb{R}^{nv}$ (and as such of positive Lebesgue measure and full Hausdorff dimension), does not immediately yield the set we seek because it is typically not the v-fold Cartesian product of a set in $\mathbb{R}^n$ with itself, and hence does not meet the specifications of the theorems. However, the algorithm can be used iteratively on many different scales and for many functions in the construction of the set E whose existence has been asserted in the theorems. Our objective in this section is to describe this algorithm. The versions that we need for Theorems 1.1 and 1.2 are very similar in principle, although the exact statements differ somewhat. These appear in Propositions 3.1 and 3.4 below respectively.

3A. ***Building block in dimension one.*** Let f be a real-valued C^1 function of v variables and nonvanishing gradient defined in a neighbourhood of the origin containing $[0, 1]^v$. Suppose that we are given an index $i_0 \in \{1, 2, \ldots, v\}$, an integer $M \geq 1$, a small constant $c_0 > 0$ and compact subsets $T_1, \ldots, T_v \subseteq [0, 1]$ with the following properties:

$$\text{Each } T_i \text{ is a union of closed intervals of length } M^{-1} \text{ with disjoint interiors. Let us denote by } \mathcal{J}_M(T_i) \text{ this collection of intervals.} \tag{3-1}$$

$$\operatorname{int}(T_i) \cap \operatorname{int}(T_{i'}) = \varnothing \text{ if } i \neq i'. \tag{3-2}$$

$$\left|\frac{\partial f}{\partial x_{i_0}}(x)\right| \geq c_0 \text{ and } |\nabla f(x)| \leq c_0^{-1} \text{ for all } x \in T_1 \times \cdots \times T_v. \tag{3-3}$$

Proposition 3.1. *Given f, M, i_0, c_0 and $\mathbb{T} = (T_1, \ldots, T_v)$ obeying* (3-1) *and* (3-3) *above, there exist a small rational constant $c_1 > 0$ and an integer N_0 (depending on all these quantities), for which the following conclusions hold.*

There is a sequence of arbitrarily large integers $N \geq N_0$ with $N/M, c_1 N \in \mathbb{N}$ such that for each N in this sequence, one can find compact subsets $S_i \subseteq T_i$ for all $1 \leq i \leq v$ such that:

(a) *There are no solutions of $f(x) = 0$ with $x \in S_1 \times \cdots \times S_v$.*

(b) *For each $J \in \mathcal{J}_M(T_i)$, let us decompose J into closed intervals of length N^{-1} with disjoint interiors and call the resulting collection of intervals $\mathcal{I}_N(J, i)$. Then for each $i \neq i_0$ and each $I \in \mathcal{I}_N(J, i)$, the set $S_i \cap I$ is an interval of length $c_1 N^{1-v}$.*

(c) *For every $J \in \mathcal{J}_M(T_{i_0})$, there exists $\mathcal{I}'_N(J, i_0) \subseteq \mathcal{I}_N(J, i_0)$ with*

$$\#(\mathcal{I}'_N(J, i_0)) \geq \Big(1 - \frac{1}{M}\Big)\#(\mathcal{I}_N(J, i_0)) \tag{3-4}$$

such that for each $I \in \mathcal{I}'_N(J, i_0)$,

$$|S_{i_0} \cap I| \geq \frac{c_1}{N}. \tag{3-5}$$

Unlike part (b), $S_{i_0} \cap I$ *need not be an interval*; *however*, *it can be written as a union of intervals of length* $c_1 N^{1-v}$ *with disjoint interiors.*

Proof. Without loss of generality, we may set $i_0 = v$. For $i \neq v$, we define

$$S_i = \bigcup \bigl\{ [a_i, a_i + c_1 N^{1-v}] \,:\, [a_i, b_i] = I \in \mathcal{I}_N(J, i) \text{ for some } J \in \mathcal{J}_M(T_i) \bigr\},$$

where the small positive constant c_1 and the integer N will be specified shortly. In other words, S_i consists of the leftmost $c_1 N^{1-v}$-subintervals of all the $1/N$-intervals that constitute T_i. It is clear that the conclusion (b) holds for this choice of S_i.

We now proceed to define the subcollection $\mathcal{I}'_N(J, v)$ and the set S_v that obey the requirements in (c). Consider the collection

$$\mathbb{A}_N := \prod_{i=1}^{v-1} \bigl\{ a_i : [a_i, b_i] = I \in \mathcal{I}_N(J, i) \text{ for some } J \in \mathcal{J}_M(T_i) \bigr\}$$

consisting of $(v-1)$-tuples of the form $a' = (a_1, \dots, a_{v-1})$, where each a_i is a left endpoint of an interval in $\mathcal{I}_N(J, i)$ for some $J \in \mathcal{J}_M(T_i)$. For each i, the number of possible choices for $1/N$-intervals $I \subseteq [0, 1]$ and hence for a_i is at most N. Thus

$$\#(\mathbb{A}_N) \leq N^{v-1}. \tag{3-6}$$

We will prove in Lemma 3.2 below that for every fixed $a' \in \mathbb{A}_N$,

$$\#\{x_v : f(a', x_v) = 0\} \leq M. \tag{3-7}$$

Assuming this for the moment, define

$$\mathbb{B} := \{x_v : \exists\, a' \in \mathbb{A}_N \text{ such that } f(a', x_v) = 0\}.$$

In light of (3-6) and (3-7), we find that

$$\#(\mathbb{B}) \leq M N^{v-1}. \tag{3-8}$$

The subcollection $\mathcal{I}'_N(J, v) \subseteq \mathcal{I}_N(J, v)$ specified in part (c) is chosen as follows: we declare

$$I \in \mathcal{I}'_N(J, v) \quad \text{if } \#(\mathbb{B} \cap I) \leq M^3 N^{v-2}.$$

In view of (3-8) and the pigeonhole principle, it follows that

$$\#\bigl(\mathcal{I}_N(J, v) \setminus \mathcal{I}'_N(J, v)\bigr) \leq \frac{M N^{v-1}}{M^3 N^{v-2}} = \frac{N}{M^2}. \tag{3-9}$$

The fact that $\#(\mathcal{I}_N(J, v)) = N/M$ then implies (3-4).

We now decompose each $I \in \mathcal{I}'_N(J, v)$ into consecutive subintervals of length $C_0 c_1 / N^{v-1}$ with disjoint interiors, and denote the successive intervals by $\tilde{I}_\ell(I)$:

$$I = \bigcup \{\tilde{I}_\ell(I) : 1 \leq \ell \leq N^{v-2}/(C_0 c_1)\}.$$

Here C_0 is a constant integer depending on f, M and $T_1, \dots T_v$, as has been specified in Lemma 3.3 below. The integer N is chosen large enough so that $N^{v-2}/(C_0c_1)$ is an integer. All intervals $\tilde{I}_\ell(I)$ that intersect $\mathbb{B}$, together with their adjacent neighbours, are then discarded. This still leaves open the possibility that the subintervals $\tilde{I}_\ell(I)$ at the edges of I, namely $\ell = 1$ and $\ell = N^{v-2}/(C_0c_1)$, are proximate to a part of $\mathbb{B}$ lying in an adjacent interval I', so we remove these edge subintervals as well. The remaining subset of T_v is defined to be S_v. More specifically,

$$S_v = \bigcup \left\{ \tilde{I}_\ell(I) : \tilde{I}_k(I) \cap \mathbb{B} = \varnothing \ \text{ for } \begin{array}{c} I \in \mathcal{I}'_N(J, v),\ J \in \mathcal{J}_M(T_v), \\ |k-\ell| \le 1,\ 1 < \ell < N^{v-2}/(C_0c_1) \end{array} \right\}.$$

Clearly S_v can be viewed a union of intervals of length c_1/N^{v-1}. The definition of $\mathcal{I}'_N(J, v)$ implies that the total length of the discarded subintervals in each $I \in \mathcal{I}'_N(J, v)$ is at most $3C_0c_1M^3N^{v-2}/N^{v-1} = 3M^3C_0c_1/N$. The claim (3-5) now follows by choosing $c_1 > 0$ small enough so as to satisfy $3M^3C_0c_1 < (1-c_1)$.

Finally, Lemma 3.3 below shows that given $x' = (x_1, \dots, x_{v-1}) \in S_1 \times S_2 \times \dots \times S_{v-1}$, any x_v obeying $f(x', x_v) = 0$ should necessarily lie within a C_0c_1/N^{v-1} neighbourhood of $\mathbb{B}$. Since the set $S_v \subseteq T_v$ was created so as to avoid these neighbourhoods, conclusion (b) follows. □

Remark. We take this opportunity to point out the distinction of our selection algorithm as compared to, say, [Keleti 1999; Máthé 2012]. The length c_1N^{1-v} of the intervals $S_i \cap I$ (for $i \ne i_0$) is the main contributing factor to the dimensional lower bound of Theorem 1.1. These intervals can be chosen slightly differently and also possibly longer if additional information is available about the zero set of f, as indicated in part (c) of the remark on page 1085.

For example, suppose that $f : \mathbb{R}^v \to \mathbb{R}$ is a linear function, say

$$f(x_1, \dots, x_v) = \sum_{i=1}^{v} \alpha_i x_i \tag{3-10}$$

with nonzero integer coefficients as in [Keleti 1999], and that $i_0 = v$. Without loss of generality suppose also that in the notation of Proposition 3.1 each T_i is a finite union of intervals J of the form $\mathbb{Z}/M + [0, 1/M]$. Then for $i < v$, a possible choice of S_i could be as follows: for each $I = [k/N, (k+1)/N] \in \mathcal{I}_N(J, i)$ with k any integer, we set

$$S_i \cap I := \left[\frac{k}{N}, \frac{k+c_1}{N} \right]$$

for some small positive constant c_1 to be chosen shortly. If $x_i \in S_i$ for $i < v$, then any x_v with $f(x_1, x_2, \dots, x_v) = 0$ has to be of the form

$$x_v = -\frac{1}{\alpha_v} \sum_{i=1}^{v-1} \alpha_i x_i, \quad \text{so that } \operatorname{dist}\Big(x_v, \frac{\mathbb{Z}}{|\alpha_v|N}\Big) < \frac{1}{4|\alpha_v|N}, \quad \text{where } c_1 \sum_{i=1}^{v-1} |\alpha_i| < \frac{1}{4}.$$

Let us then choose S_v as follows: for any $I = |\alpha_v|^{-1}[k/N, (k+1)/N] \subseteq T_v$,

$$S_v \cap I := \frac{1}{|\alpha_v|} \left[\frac{k}{N} + \frac{1-c_1}{2N}, \frac{k}{N} + \frac{1+c_1}{2N} \right].$$

This implies

$$\operatorname{dist}\left(S_v \cap I, \frac{\mathbb{Z}}{|\alpha_v|N}\right) > \frac{1}{4|\alpha_v|N},$$

provided $c_1 > 0$ is chosen small enough. Thus the construction above ensures that $S_1 \times S_2 \cdots \times S_v$ contains no zeros of f. Further, the size of this Cartesian product is significantly larger than the one obtained in Proposition 3.1. Tracking these new choices of S_i through the rest of the proof yields a set E of full Hausdorff dimension that avoids all zeros of (3-10), which is the result of [Keleti 1999].

Lemma 3.2. *For f and $\mathbb{A}_N$ as in Proposition 3.1, the inequality* (3-7) *holds for every fixed $a' \in \mathbb{A}_N$.*

Proof. Given $a' \in \mathbb{A}_N$, we claim that for every $J \in \mathcal{J}_M(T_v)$, there exists at most one $x_v \in J$ such that $f(a', x_v) = 0$. Since the number of $J \in \mathcal{J}_M(T_v)$ is at most M, the desired conclusion would follow once the claim is established.

To prove the claim, let us assume if possible that there exist $x_v, y_v \in J$, $x_v \neq y_v$, such that $f(a', x_v) = f(a', y_v) = 0$. By Rolle's theorem, this ensures the existence of some point $z_v \in J$ where $\partial f/\partial x_v(a', x_v) = 0$. But this contradicts the hypothesis (3-3) that the partial derivative $\partial f/\partial x_v$ is nonzero on $T_1 \times \cdots \times T_v$. □

Lemma 3.3. *Let f, M and $T_1, \ldots, T_v$ be as in Proposition 3.1. Then there exists a constant C_0 depending on these quantities, and in particular on c_0, such that for the choice of $S_1, S_2, \ldots, S_{v-1}$ as specified in the proof of the proposition,*

$$\operatorname{dist}(x_v, \mathbb{B}) \leq \frac{C_0 c_1}{N^{v-1}}$$

for any x_v obeying $f(x) = 0$, with $x' = (x_1, \ldots, x_{v-1}) \in S_1 \times \cdots \times S_{v-1}$.

Proof. Let $\mathbb{J} = J_1 \times \cdots \times J_v = \mathbb{J}' \times J_v \in \prod_{i=1}^{v} \mathcal{J}_M(T_i)$ be a v-dimensional cube of side length $1/M$ such that the zero set of f intersects $\mathbb{J}$. The nonvanishing derivative condition (3-3) then implies, in view of the implicit function theorem, that there exists a $(v-1)$-variate C^1 function $g_{\mathbb{J}}$ defined on $\mathbb{J}'$ and a constant $C_0 > 0$ depending on c_0, M, $T_1, \ldots, T_v$ such that

$$f(x) = 0,\ x \in \mathbb{J}, \quad \text{implies} \quad x_v = g_{\mathbb{J}}(x'),\ x' \in \mathbb{J}', \tag{3-11}$$

$$|\nabla g_{\mathbb{J}}| \leq \frac{C_0}{\sqrt{v}} \quad \text{on } \mathbb{J}'. \tag{3-12}$$

Given $x = (x', x_v) \in S_1 \times \cdots \times S_v$ such that $f(x) = 0$, let $\mathbb{J}_x$ denote the v-dimensional $1/M$-cube $\mathbb{J}$ in which x lies, and let $\mathbb{I}'_x = I_1 \times \cdots \times I_{v-1} = \prod_{i=1}^{v-1}[a_i, b_i] \in \prod_{i=1}^{v-1} \mathcal{I}_N(J_i, i)$ be the $(v-1)$-dimensional subcube of $\mathbb{J}'_x$ of side length $1/N$ containing x'. Then

$$x_v = g_{\mathbb{J}}(x'), \quad a' = (a_1, \ldots, a_{v-1}) \in \mathbb{A}_N, \quad g_{\mathbb{J}}(a') \in \mathbb{B}, \quad \text{and} \quad |x' - a'| \leq \frac{c_1\sqrt{v}}{N^{v-1}}.$$

Further, (3-12) implies

$$\operatorname{dist}(x_v, \mathbb{B}) \leq |g_{\mathbb{J}}(a') - g_{\mathbb{J}}(x')| \leq \|\nabla g_{\mathbb{J}}\|_\infty\, |x' - a'| \leq \frac{C_0}{\sqrt{v}} \times \frac{c_1\sqrt{v}}{N^{v-1}} = \frac{C_0 c_1}{N^{v-1}},$$

which is the conclusion of the lemma. □

3B. ***Building block in higher dimensions.*** Given positive integers $m, n \geq 1$ and $v \geq 3$ with $m \leq n(v-1)$, let $f : \mathbb{R}^{nv} \to \mathbb{R}^m$ be a C^2 function whose zero set has nontrivial intersection with $[0, 1]^{nv}$. Suppose that $M \geq M_0$ is a large integer, $c_0 > 0$ is a small constant and $T_1, \dots, T_v \subseteq [0, 1]^n$ are sets with the following properties:

$$\begin{gathered}\text{Each } T_i \text{ is expressible as a union of closed axis-parallel cubes of side length } M^{-1} \\ \text{with disjoint interiors, the collection of which will be called } \mathcal{J}_M(T_i). \\ \text{As before, } \operatorname{int}(T_i) \cap \operatorname{int}(T_{i'}) = \varnothing \text{ if } i \neq i'.\end{gathered} \tag{3-13}$$

$$\begin{gathered}\text{On } \{x \in T_1 \times \cdots \times T_v : f(x) = 0\} \text{ the matrix } Df \text{ is of full rank, with} \\ \text{the singular values of } Df \text{ bounded above and below by } c_0^{-1} \text{ and } c_0 \text{ respectively.}\end{gathered} \tag{3-14}$$

$$\text{On } [0, 1]^{nv}, \text{ the matrix norm of the Hessian } D^2 f \text{ is bounded above by } c_0^{-1}. \tag{3-15}$$

Proposition 3.4. *Given f, M and c_0 as above, there exists a rational constant $c_1 > 0$ and an integer N_0 depending on these quantities for which the following conclusions hold. For $N \geq N_0$, set $\ell = c_1 N^{n(1-v)/m}$. If N is such that $N/M, 1/(\ell N) \in \mathbb{Z}$, then one can find compact subsets $S_i \subseteq T_i$ for all $1 \leq i \leq v$ such that:*

(a) *There are no solutions to $f(x) = 0$ with $x \in S_1 \times \cdots \times S_v$.*

(b) *For each $1 \leq i \leq v$ and $J \in \mathcal{J}_M(T_i)$, let us decompose J into closed axis-parallel cubes of length N^{-1} with disjoint interiors and call the resulting collection of cubes $\mathcal{I}_N(J, i)$. There exists $\mathcal{I}'_N(J, i) \subseteq \mathcal{I}_N(J, i)$ such that*

$$S_i \subseteq \bigcup \{I : J \in \mathcal{J}_M(T_i),\ I \in \mathcal{I}'_N(J, i)\}.$$

More precisely, for each $I \in \mathcal{I}'_N(J, i)$, the set $S_i \cap I$ is a single axis-parallel cube of side length $\ell = c_1 N^{n(1-v)/m}$, provided $i \neq v$. For $i = v$ and $I \in \mathcal{I}'_N(J, v)$, the set $S_v \cap I$ is not necessarily a single cube of side length ℓ, but a union of such cubes, with the property that

$$|S_v \cap I| \geq \left(1 - \frac{1}{M}\right)\frac{1}{N^n}. \tag{3-16}$$

(c) *The subcollections $\mathcal{I}'_N(J, i)$ of cubes are large subsets of the ambient collection $\mathcal{I}_N(J, i)$, in the sense that for all $1 \leq i \leq v,\ J \in \mathcal{J}_M(T_i)$,*

$$\#(\mathcal{I}'_N(J, i)) \geq \left(1 - \frac{1}{M}\right)\#(\mathcal{I}_N(J, i)). \tag{3-17}$$

Remarks. (a) The proof will show that the constant c_1 in Proposition 3.4 may be chosen as a small constant multiple of M^{-R}, where $R = [(n+1)v + 1]/m$. For the purposes of application, M is negligible compared to N, and hence the specific power of M that appears in the expression for ℓ is not critical to the proof. The power of N, which is $-(n/m)(v-1)$, is of utmost importance and the principal reason that the Hausdorff dimension of the set $E \subseteq \mathbb{R}^n$ in Theorem 1.2 is equal to $m/(v-1)$.

(b) The restriction $m \leq n(v-1)$ justifies on one hand the dimensional constraint on the set E which lies in $\mathbb{R}^n$. On a technical note, it is also necessary for the assumption $\ell \ll N^{-1}$ that permeates the proof. If $m < n(v-1)$, the chosen value of $\ell = \epsilon_0 M^{-R} N^{-n(v-1)/m}$ will be less than $1/N$ if N is sufficiently large. If $m = n(v-1)$, the chosen value of ℓ will be less than $1/N$ provided that M is sufficiently large.

(c) The special treatment of the variable x_v in the proposition is for convenience only. The result holds for x_v replaced by x_{i_0} for any $1 \le i_0 \le v$.

Proof. Let $Z_f = \{x = (x_1, \dots, x_v) \in ([0, 1]^n)^v : f(x) = 0\}$ be the zero set of the function f, which we wish to avoid. The assumptions (3-14) and (3-15) ensure that $Z_f \cap (T_1 \times \cdots \times T_v)$ is an $(nv-m)$-dimensional submanifold of $[0, 1]^{nv}$; see for example [Sharpe 1997, Theorem 2.13]. Further, the coarea formula gives that Z_f is coverable by at most $C\epsilon^{m-nv}$ many cubes of side length ϵ, for all sufficiently small ϵ. Here C is a large constant depending only on c_0 and independent of ϵ. The proof consists of projecting Z_f successively onto the coordinates $x_1, x_2, \dots$ and selecting the sets S_i so as to avoid the projected zero sets. The main ingredient of this argument is described in Lemma 3.5. We ask the reader to view the statement of the lemma first. Assuming the lemma, the remainder of the proof proceeds as follows.

Fix a parameter $\ell \ll 1/N$ soon to be specified. Recalling that $\mathcal{I}_{\alpha^{-1}}(J, i)$ denotes the collection of axis-parallel subcubes of side length α that constitute a partition of $J \in \mathcal{J}_M(T_i)$, let us define the collection of "bad boxes" $\mathbb{B}_1$ as

$$\mathbb{B}_1 = \left\{ Q \in \prod_{i=1}^{v} \mathcal{I}_{\ell^{-1}}(J_i, i) : Q \cap Z_f \neq \varnothing \text{ for some } J_i \in J_M(T_i) \right\}. \tag{3-18}$$

In other words, a box of side length ℓ in $T_1 \times \cdots \times T_v$ is considered bad if it contains a point in the zero set of the function f. The discussion in the preceding paragraph shows that

$$\#(\mathbb{B}_1) \le C\ell^{m-nv}, \tag{3-19}$$

where C is a constant that depends only on the function f and the value c_0.

The construction of $S_1, \dots, S_v$ now proceeds as follows. At the first step, we project the boxes in $\mathbb{B}_1$ onto their $(x_2, \dots, x_v)$-coordinates (each n-dimensional), and use Lemma 3.5 below with $r = v$, $T = T_1$, $T' = T_2 \times \cdots \times T_v$ and $\mathbb{B} = \mathbb{B}_1$ to arrive at a set $S_1 \subseteq T_1$ and a family of $n(v-1)$-dimensional boxes $\mathbb{B}' = \mathbb{B}_2$ obeying the conclusions of that lemma. Clearly the set S_1 obeys the requirements of part (b) of the proposition. Lemma 3.5 also ensures that

$$\#(\mathbb{B}_2) \le M^{n+1} N^n \ell^n \#(\mathbb{B}_1) \le C M^{n+1} N^n \ell^{m-n(v-1)},$$

and that $f(x) \neq 0$ for any $x = (x_1, x')$ such that $x_1 \in S_1$ and any $x' \in T_2 \cdots \times T_v$ that is not contained in the cubes constituting $\mathbb{B}_2$.

We now inductively follow a procedure similar to the above. At the end of step j, we will have selected sets $S_1 \subseteq T_1, \dots, S_j \subseteq T_j$ and will be left with a family $\mathbb{B}_{j+1}$ of $n(v-j)$-dimensional cubes of side length ℓ, such that

$$\#(\mathbb{B}_{j+1}) \le C M^{(n+1)j} N^{jn} \ell^{m-n(v-j)} \tag{3-20}$$

and

$$f(x'', x') \neq 0 \quad \text{for} \quad x'' = (x_1, \dots, x_j) \in \prod_{i=1}^{j} S_i,\ x' \in \prod_{i=j+1}^{v} T_i,\ x' \text{ not contained in any of the cubes in } \mathbb{B}_{j+1}. \tag{3-21}$$

We can then apply Lemma 3.5 with

$$T = T_{j+1}, \quad T' = T_{j+2} \times \cdots \times T_v, \quad \mathbb{B} = \mathbb{B}_{j+1},$$

to arrive at a set $S_{j+1} \subseteq T_{j+1}$ meeting the requirement of part (b) of the proposition. The lemma also gives a family $\mathbb{B}' = \mathbb{B}_{j+2}$ of $n(v-j-1)$-dimensional cubes of side length ℓ, whose cardinality obeys the inequality (3-20) with j replaced by $j+1$, allowing us to carry the induction forward.

We continue this construction for $v-1$ steps, obtaining sets $S_1, \dots, S_{v-1}$ and a collection $\mathbb{B}_v$ consisting of at most $CM^{(n+1)(v-1)}N^{n(v-1)}\ell^{m-n}$ cubes of side length ℓ and dimension n contained in T_v. The set S_v is then defined according to the prescription of Lemma 3.6, the conclusion of which verifies part (a) of the proposition for $S_1, \dots, S_v$. □

3B1. *Projections of bad boxes.* It remains to justify the projection mechanism used repeatedly in Proposition 3.4. We set this up below.

Fix $2 \le r \le v$, and consider sets $T \subseteq [0,1]^n$ and $T' \subseteq [0,1]^{n(r-1)}$ expressible as unions of closed axis-parallel cubes of side length M^{-1} and disjoint interiors. As before, we denote by $\mathcal{J}_M(T)$ and $\mathcal{J}_M(T')$ the respective collections of these cubes. Given any $J \in \mathcal{J}_M(T)$, we decompose J into axis-parallel subcubes of side length N^{-1}; the corresponding collection is termed $\mathcal{I}_N(J)$. We will also need to fix a subset $B \subseteq T \times T'$, which we view as a union of a collection $\mathbb{B}$ of cubes of side length ℓ. Here M, N and ℓ are as specified in Proposition 3.4. Since ℓ/N is taken to be an integer, we may assume that each cube in $\mathbb{B}$ is contained in exactly one cube in $\mathcal{I}_N(J)$.

Lemma 3.5. *Given T, T', B as above, there exist sets $S \subseteq T$, $B' \subseteq T'$ and a collection of boxes $\mathbb{B}' \subseteq \mathbb{T}'$ with the following properties*:

(a) *The set S is a union of closed axis-parallel cubes with side length ℓ and disjoint interiors. More precisely, for every $J \in \mathcal{J}_M(T)$, there exists $\mathcal{I}'_N(J) \subseteq \mathcal{I}_N(J)$ such that*
$$\#(\mathcal{I}'_N(J)) \ge (1 - M^{-1})\#(\mathcal{I}_N(J)),$$
and $S \cap I$ is a single ℓ-cube for each $I \in \mathcal{I}'_N(J)$. For $I \in \mathcal{I}_N(J) \setminus \mathcal{I}'_N(J)$, the interior of the set $S \cap I$ is empty.

(b) *The set B' is the union of the ℓ-cubes in $\mathbb{B}'$.*

(c) $\#(\mathbb{B}') \le M^{n+1}N^n\ell^n\#(\mathbb{B})$.

(d) $(S \times T') \cap B \subseteq S \times B'$.

Proof. Fix $J \in \mathcal{J}_M(T)$. For $I \in \mathcal{I}_N(J)$, define a "slab"
$$W_N[I] := \bigcup \{Q = I \times I' \subseteq T \times T' : Q \text{ is a cube of side length } N^{-1}\}.$$

Thus a slab is the union of all of the axis-parallel boxes in $T \times T'$ of side length $1/N$ whose projection onto the x_1-coordinate is the cube I. Similarly, given an n-dimensional cube I of side length ℓ, we define a "wafer" $W_{\ell^{-1}}[I]$ to be the union of all cubes of side length ℓ that project onto I in the x_1-space. Let us observe that a slab is an essentially disjoint union of exactly $N^{-n}\ell^{-n}$ wafers, and that the total number of wafers supported by J is $M^{-n}\ell^{-n}$. A wafer in turn is a union of ℓ-cubes.

Let us agree to call a wafer $W_{\ell^{-1}}[I]$ "good" if it contains at most $M^{n+1}\ell^n\#(\mathbb{B})$ boxes of $\mathbb{B}$. The pigeonhole principle dictates that the proportion of bad wafers is $\le 1/M$. We will call a slab $W_N[I]$

"good" if it contains at least one good wafer. Again pigeonholing implies that no more than a $1/M$-fraction of the slabs can be bad. Let us define $\mathcal{I}'_N(J)$ as the collection all cubes $I \in \mathcal{I}_N(J)$ such that $W_N[I]$ is good. For each cube $I \in \mathcal{I}'_N(J)$, we select one cube $I_0 = I_0(I) \subset I$ of side length ℓ such that $W_{\ell^{-1}}[I_0]$ is a good wafer. The set S is now defined to be the union of all selected ℓ-cubes $I_0(I)$, with $I \in \mathcal{I}'_N(J)$ and $J \in \mathcal{J}_M(T)$. Clearly, S satisfies part (a) of the lemma.

Let B' be the union of the collection $\mathbb{B}'$ of all ℓ-cubes $Q' \subseteq T'$ such that $Q \times Q' \in \mathbb{B}$ for some ℓ-cube $Q \subseteq S$. Then (b) and (d) hold by definition. The selection algorithm for S gives that for a given cube $Q \subseteq S$, the number of Q' such that $Q \times Q' \in \mathbb{B}$ is $\leq M^{n+1}\ell^n\#(\mathbb{B})$. On the other hand, each $Q \subseteq S$ comes from a distinct slab. Hence the total number of possible choices for $Q \subseteq S$ is no more than the total number of slabs, which is bounded above by N^n. Combining all of this we get (c) as desired. □

A version of the lemma above is needed for the extreme case $r = 1$. We needed this in the final step of the iterative process described in Proposition 3.4, specifically in the construction of S_v.

Lemma 3.6. *Fix parameters $\ell \ll N^{-1} \ll M^{-1}$. Let $T \subseteq [0, 1]^n$ be a union of closed axis-parallel cubes with side length M^{-1} and disjoint interiors. Let $B \subseteq T$ be a union of similar cubes with side length ℓ. Decompose T into similar axis-parallel cubes of side length N^{-1}, denoting the corresponding collection by $\mathbb{T}$. The collection of ℓ-cubes composing B is termed $\mathbb{B}$. Suppose that*

$$\#(\mathbb{B}) \leq CM^{(n+1)(v-1)}N^{n(v-1)}\ell^{m-n},$$

with

$$\ell \leq C^{-1/m}M^{-(1/m)((n+1)v+1)}N^{-n(v-1)/m}.$$

Then there exist $S \subseteq T$ and $\mathbb{T}^* \subseteq \mathbb{T}$ such that

(a) $S \cap B = \varnothing$,

(b) $\#(\mathbb{T}^*) \geq (1 - 1/M)\#(\mathbb{T})$,

(c) *S is a union of a large number of ℓ-cubes coming from $\mathbb{T}^*$. More precisely, $|S \cap I| \geq (1 - M^{-1})N^{-n}$ for each $I \in \mathbb{T}^*$.*

Proof. Decomposing each cube $I \in \mathbb{T}$ into subcubes of side length ℓ, we declare I to be good if it contains $\leq M^{n+1}N^{-n}\#(\mathbb{B})$ subcubes that are in $\mathbb{B}$. As in the proof of Lemma 3.5, the pigeonhole principle ensures that the fraction of bad cubes in $\mathbb{T}$ is at most M^{-1}. Define $\mathbb{T}^*$ to be the collection of good cubes in $\mathbb{T}$, and S to be the union of all subcubes of side length ℓ that are contained in the cubes of $\mathbb{T}^*$ but which are disjoint from B. The relation between ℓ, M and N implies that for every $I \in \mathbb{T}^*$,

$$|I \cap B| \leq M^{n+1}N^{-n}\#(\mathbb{B})\ell^n \leq CM^{(n+1)v}N^{n(v-2)}\ell^m \leq M^{-1}N^{-n},$$

which justifies the size conclusion for S. □

4. Proof of Theorems 1.1 and 1.2

We present the construction of the set E in Theorem 1.1 in complete detail. The construction for Theorem 1.2 is similar. The small variations needed for this have been discussed in Section 4C.

4A. ***A sequence of differential operators.*** We will need to define a sequence of privileged derivatives in order to prove Theorem 1.1. For η and r_q as in the statement of Theorem 1.1, let α_q be a v-dimensional multi-index with $|\alpha_q| = r_q$ such that $\partial^{\alpha_q} f_q / \partial x^{\alpha_q}$ is nonvanishing everywhere on $[0, \eta]$. Here $\partial^\beta / \partial x^\beta$ denotes, following standard convention, the differential operator $\partial^{\beta_1 + \cdots + \beta_v} / \partial x_1^{\beta_1} \cdots \partial x_v^{\beta_v}$ of order $|\beta| = \beta_1 + \cdots = \beta_v$, if $\beta = (\beta_1, \ldots, \beta_v)$. We now define for each q a finite sequence of privileged differential operators of diminishing order

$$\mathcal{D}_q^k = \frac{\partial^{\alpha_{qk}}}{\partial x^{\alpha_{qk}}}, \quad 0 \le k \le r_q. \tag{4-1}$$

Here $\alpha_{q,r_q} = \alpha_q$, and $\alpha_{q,k-1}$ is obtained by reducing the largest entry of α_{qk} by 1 and leaving the others unchanged. If there are multiple entries of α_{qk} with the largest value, we pick any one. Clearly $|\alpha_{qk}| = k$.

4B. ***Construction of E.*** The construction is of Cantor type with a certain memory-retaining feature inspired by the constructions of Keleti [1999; 2008]. This distinctive feature is the existence of an accompanying queue that is, on one hand, generated by the construction and on the other, contributes to it. More precisely, the j-th iteration of the construction is predicated on the j-th member of the queue; at the same time the j-th step also adds a large number of new members to the queue, which become significant at a later stage.

Step 0: At the initializing step, we set for $k = 1, \ldots, v$,

$$I_k[0] = \left[(k-1)\frac{\eta}{v}, \frac{k\eta}{v}\right], \quad \mathcal{E}_0 = \{I_1[0], \ldots, I_v[0]\}, \quad M_0 = \frac{v}{\eta}.$$

Letting Σ_0 denote the collection of injective mappings from $\{1, \ldots, v-1\}$ into $\{1, \ldots, v\}$, we define an ordered queue

$$\mathcal{Q}_0 = \{(1, m, \mathbb{I}_\sigma[0]) : 0 \le m \le r_1 - 1, \sigma \in \Sigma_0\},$$

where

$$\mathbb{I}_\sigma[0] = (I_{\sigma(1)}[0], \ldots, I_{\sigma(v-1)}[0]).$$

The ordering in $\mathcal{Q}_0$ is as follows: Viewing Σ_0 as a collection of $(v-1)$-tuples with values from $\{1, \ldots, v\}$, we first endow Σ_0 with the lexicographic ordering, writing $\Sigma_0 = \{\sigma_1 < \sigma_2 < \cdots\}$. Then $(1, m, \mathbb{I}_{\sigma_r}[0])$ precedes $(1, m', \mathbb{I}_{\sigma_{r'}}[0])$ in the list $\mathcal{Q}_0$ if one of the following scenarios holds: (a) $r < r'$, no matter what m, m' might be, or (b) $r = r'$ and $m > m'$.

Step 1: Consider the first member of $\mathcal{Q}_0$, which is $(1, r_1 - 1, \mathbb{I}_{\sigma_1}[0])$. Recalling the definition (4-1), we proceed to verify the hypotheses of Proposition 3.1, with

$$f = \mathcal{D}_1^{r_1 - 1} f_1, \quad (T_i : i \ne v) = \mathbb{I}_{\sigma_1}[0], \quad M = M_0.$$

Here $i_0 = i_0(1)$ is the unique index in $\{1, 2, \ldots, v\}$ such that $\partial f / \partial x_{i_0} = \mathcal{D}_1^{r_1} f_1$, which is nonzero on $[0, \eta]$. The set T_v will be the complement in $[0, \eta]$ of $\bigcup_i \{T_i : i \ne v\}$. The conclusion of Proposition 3.1 therefore holds for some small constant $d_0 = c_1(M_0, \mathbb{T}) > 0$ and for arbitrarily large integers N_1 obeying the divisibility criteria of the proposition. We choose such an integer N_1 large enough so that $N_1 > e^{M_0/d_0}$.

Proposition 3.1 then ensures the existence of subsets $S_j \subset T_j$ for $1 \le j \le v$, each of which is a union of intervals of length $\ell_1 = d_0/N_1^{v-1}$ with

$$\mathcal{D}_1^{r_1-1} f_1(x) \neq 0 \quad \text{for } x = (x_1, \ldots, x_v) \in S_1 \times \cdots \times S_v.$$

These constitute the basic intervals for the first stage.

Let $\mathcal{E}_1 = \{I_1[1], I_2[1], \ldots, I_{L_1}[1]\}$ be an enumeration of the first-stage basic intervals, and Σ_1 the collection of injective mappings from $\{1, \ldots, v-1\}$ to $\{1, \ldots, L_1\}$. We view an element of Σ_1 as an ordered $(v-1)$-tuple of distinct indices from $\{1, \ldots, L_1\}$. As before, Σ_1 is arranged lexicographically. Set

$$\mathcal{Q}_1' = \{(q, k, \mathbb{I}_\sigma[1]) : 1 \le q \le 2,\ 0 \le k \le r_q - 1,\ \sigma \in \Sigma_1\},$$

with $\mathbb{I}_\sigma[1] = (I_{\sigma(1)}[1], \ldots, I_{\sigma(v-1)}[1])$. The list $\mathcal{Q}_1'$ is assigned the following ordering. An element of the form $(q, k, \mathbb{I}_\sigma[1])$ will precede $(q', k', \mathbb{I}_{\sigma'}[1])$ if one of the following conditions holds:

(a) $\sigma < \sigma'$ (irrespective of the relative values of q, q', k, k'), or

(b) $\sigma = \sigma',\ q < q'$ (irrespective of the relative values of k, k'), or

(c) $\sigma = \sigma',\ q = q'$ and $k > k'$.

The list $\mathcal{Q}_1'$ is appended to $\mathcal{Q}_0$ to arrive at the updated queue $\mathcal{Q}_1$ at the end of the first step.

The general step: In general, at the end of step j, we have the following quantities:

- The j-th iterate of the construction E_j, which is the union of the j-th level basic intervals of length $\ell_j = d_{j-1}/N_j^{v-1}$. Here d_j is a sequence of small constants obtained as c_1 from repeated applications of Proposition 3.1 and depending on the collection of functions $\{f_q : q \le j\}$. In particular, d_j only depends on parameters involved in the first j steps of the construction. The sequence N_j is chosen to be rapidly increasing. For instance, choosing

$$N_{j+1} > \exp\left[\prod_{k=1}^{j} \left(\frac{N_k}{d_k}\right)^R\right] \quad \text{for all } j \ge 1 \tag{4-2}$$

and some fixed large constant R would suffice.

- The collection of the j-th level basic intervals that constitute E_j, which we denote by

$$\mathcal{E}_j = \{I_1[j], I_2[j], \ldots, I_{L_j}[j]\}.$$

- The updated queue $\mathcal{Q}_j = \mathcal{Q}_{j-1} \cup \mathcal{Q}_j'$, with

$$\mathcal{Q}_j' = \{(q, k, \mathbb{I}_\sigma[j]) :\ 1 \le q \le j+1,\ 0 \le k \le r_q - 1,\ \sigma \in \Sigma_j\}.$$

Here Σ_j is the collection of all injective maps from $\{1, \ldots, v-1\}$ to $\{1, \ldots, L_j\}$, which is viewed as the collection of all $(v-1)$-dimensional vectors with distinct entries taking values in $\{1, \ldots, L_j\}$ and endowed with the lexicographical order. The new list $\mathcal{Q}_j'$ is ordered in the same way as described in Step 1 and appended to $\mathcal{Q}_{j-1}$. Notice that the number of members in the list $\mathcal{Q}_j$ is much larger than j.

We also know that $\mathcal{D}_q^k f_q(x)$ is nonzero for certain choices of k, q, and x with $k \le r_q - 1$. Specifically, given any tuple of the form $(q, k, \mathbb{I})$ that appears among the first j members of the list $\mathcal{Q}_j$, the construction yields that

$$|\mathcal{D}_q^k f_q(x)| > 0 \quad \text{if } x_i \in E_j \cap I_i \ \text{ for } \ i \ne v,\ x_v \in E_j \setminus (I_1 \cup \cdots \cup I_{v-1}). \tag{4-3}$$

Here the $(v-1)$-tuple of intervals $\mathbb{I}$ has been labelled as $\mathbb{I} = (I_i : i \ne v)$. We will continue to use this notation for the remainder of this subsection.

At step $j+1$, we refer to the $(j+1)$-th entry of the queue $\mathcal{Q}_j$, which we denote by $(q_0, k_0, \mathbb{I})$. Set i_0 to be the distinguished index such that

$$\frac{\partial}{\partial x_{i_0}}[\mathcal{D}_{q_0}^{k_0} f_{q_0}] = \mathcal{D}_{q_0}^{k_0+1} f_{q_0}.$$

Two cases can occur, depending on whether k_0 is maximal for the given q_0 or not. If it is, that means $k_0 = r_{q_0} - 1$ for some $1 \le q_0 \le j+1$. We want to apply Proposition 3.1 with $M^{-1} = \ell_j$, the index i_0 as described in the paragraph above, and

$$f = \mathcal{D}_{q_0}^{r_{q_0}-1} f_{q_0}, \quad T_i = \begin{cases} E_j \cap I_i & \text{if } i \ne v, \\ E_j \setminus \bigcup_{i \ne v} T_i & \text{if } i = v. \end{cases} \tag{4-4}$$

In this case, the nonvanishing derivative condition required for the application of Proposition 3.1 is ensured by the hypothesis of Theorem 1.1.

The other possibility is when $k_0 < r_{q_0} - 1$. Given the specified ordering on $\mathcal{Q}_j$, we conclude that $(q_0, k_0+1, \mathbb{I})$ must be the j-th member of $\mathcal{Q}_j$, and hence, by the induction hypothesis, (4-3) holds with $q = q_0$ and $k = k_0 + 1$. We can now apply Proposition 3.1 with $f = \mathcal{D}_{q_0}^{k_0} f_{q_0}$, $M^{-1} = \ell_j$, and the same choices of i_0 and $T_1, \dots, T_v$ as in (4-4) above.

In either case, we obtain a collection $\mathcal{E}_{j+1}$ of $(j+1)$-th level basic cubes of length $\ell_{j+1} = d_j / N_{j+1}^{v-1}$, the union of which is E_{j+1}, and for which (4-3) holds with $q = q_0$, $k = k_0$ and j replaced by $j+1$. This completes the induction.

The set E is now defined as $E = \bigcap_{j=1}^{\infty} E_j$. We will establish shortly that E meets the requirements of Theorem 1.1.

4C. ***Modifications to the construction of E for Theorem 1.2.*** The main distinction for Theorem 1.2 is that we only need to consider the first derivative Df_q of f_q, so there is no need for the higher-order differential operators $\mathcal{D}_q^k$, and hence no need for distinguished indices i_0. What this means is that the elements of the queue $\mathcal{Q}_j'$ are of the form $(q, \mathbb{I}_\sigma[j])$, where q ranges from 1 to j and $\mathbb{I}_\sigma$ is a tuple of cubes instead of intervals, and one needs to appeal to Proposition 3.4 instead of Proposition 3.1. The number of subcubes of $[0, \eta]^{nv}$ at the initializing step needs to be chosen large enough, so that their side lengths do not exceed M_0^{-1}, as specified in the hypotheses of Proposition 3.4. This is simply to ensure that Proposition 3.4 is applicable. The small parameters d_j and large parameters N_j are still assumed to obey a relation of the form (4-2), with the constant R possibly depending on v, n, m. The side length ℓ_j of a j-th level basic cube is now

$$\ell_j = d_{j-1} N_j^{-n(v-1)/m}. \tag{4-5}$$

From this point onward, no distinction will be made between Theorem 1.1 and the $m = 1,\ n = 1$ case of Theorem 1.2. The computation of the Hausdorff and Minkowski dimensions of the set E in these two cases proceeds in exactly the same manner.

4D. ***Nonexistence of solutions.*** Fix any $q \geq 1$, and a tuple $x = (x_1, \dots, x_v)$ of distinct points in E. Since $\ell_j \to 0$, the minimum separation between the points $x_1, \dots, x_v$ exceeds ℓ_j for some j. In other words, there exists a step $j \geq q$ in the construction of E where these points lie in distinct basic intervals (in the case of Theorem 1.1) or cubes (in the case of Theorem 1.2) of that step. Suppose that $\mathbb{I}^* = (I_1^*, \dots, I_{v-1}^*)$ is the tuple of j-th stage basic intervals such that $x_i \in I_i^*$ for $1 \leq i \leq v-1$ and $x_v \in E_j \setminus (I_1^* \cup \dots \cup I_{v-1}^*)$. Then the tuple $(q, 0, \mathbb{I}^*)$ (or $(q, \mathbb{I}^*)$ in the case of Theorem 1.2) belongs to the list $\mathcal{Q}_j$. Suppose that it is the j_0-th member of $\mathcal{Q}_j$, $j_0 \gg j$. This tuple then plays a decisive role at the j_0-th step of the construction, at the end of which we obtain (either from Proposition 3.1 or 3.4) that f_q does not vanish on $\prod_{i=1}^{v} E_{j_0} \cap I_i^*$. Since x lies in this set, we are done.

4E. ***Hausdorff dimension of E.*** Frostman's lemma dictates that the Hausdorff dimension of a Borel set E is the supremum value of $\alpha > 0$ for which one can find a probability measure supported on E with $\sup_{x,r} \mu(B(x;r))/r^\alpha < \infty$, where $B(x;r)$ denotes a ball centred at x of radius r. Keeping in mind that any ball is coverable by a fixed number of cubes, we aim to construct a probability measure μ on E with the property that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\mu(I) \leq C_\epsilon l(I)^{m/(v-1)-\epsilon} \quad \text{for all cubes } I. \tag{4-6}$$

Here $l(I)$ denotes the side length of I.

Let us recall that $\mathcal{E}_j$ denotes the collection of all basic cubes with side length ℓ_j at step j of the construction. Decomposing each cube in $\mathcal{E}_j$ into equal subcubes of length $1/N_{j+1}$, we denote by $\mathcal{F}_{j+1}$ the resulting collection of subcubes that contain a cube from $\mathcal{E}_{j+1}$. Let F_{j+1} be the union of the cubes in $\mathcal{F}_{j+1}$. We define a sequence of measures ν_{j+1} and μ_j supported respectively on F_{j+1} and E_j as follows. The measure μ_0 is the uniform measure on $[0, \eta]^n$. Given μ_j, the measure ν_{j+1} will be supported on F_{j+1} and will be defined by evenly splitting the measure μ_j of each cube in $\mathcal{E}_j$ among its children in $\mathcal{F}_{j+1}$. Given ν_j, the measure μ_j will be supported on E_j and will be defined by evenly splitting the measure ν_j of each cube in $\mathcal{F}_j$ among its children in $\mathcal{E}_j$. It follows from the mass distribution principle that the measures μ_j have a weak limit μ. We claim that μ obeys the desired requirement (4-6).

The proof of the claim rests on the following proposition, which describes the mass distribution on the basic cubes of the construction.

Proposition 4.1. *Let $K \in \mathcal{E}_j$, $J \in \mathcal{F}_{j+1}$ with $J \subset K$. Then*:

(a) $\mu(K)/|K| \leq \mu(J)/|J| \leq 2\mu(K)/|K|$.

(b) $\mu(J) \leq M_j|J|$, *where* $M_j = \prod_{k=1}^{j} 2(\ell_k N_k)^{-n}$.

Proof. We first prove part (a). Each $K \in \mathcal{E}_j$ decomposes into $(\ell_j N_{j+1})^n$ subcubes of side length $1/N_{j+1}$. Propositions 3.1 and 3.4 assert that at least a $(1-1/M)$-fraction of these subcubes contain a cube from $\mathcal{E}_{j+1}$ and hence lie in $\mathcal{F}_{j+1}$. The number of descendants $J \in \mathcal{F}_{j+1}$ of a given cube $K \in \mathcal{E}_j$ is therefore at most $(\ell_j N_{j+1})^n = |K|/|J|$ and at least $\frac{1}{2}(\ell_j N_{j+1})^n = |K|/(2|J|)$. Since $\mu(K)$ is evenly distributed among such J, part (a) follows.

We prove part (b) by applying part (a) iteratively. Suppose that $\bar{J}$ is the cube in $\mathcal{F}_j$ that contains K. Then

$$\frac{\mu(J)}{|J|} \le 2\frac{\mu(K)}{|K|} \le 2\frac{\mu(\bar{J})}{|K|} = \frac{2|\bar{J}|}{|K|}\frac{\mu(\bar{J})}{|\bar{J}|} = \frac{2}{(\ell_j N_j)^n}\frac{\mu(\bar{J})}{|\bar{J}|}. \qquad \square$$

We are now ready to apply Proposition 4.1 to prove (4-6). Suppose that I is a cube with side length between ℓ_{j+1} and ℓ_j. There are two possibilities: either $1/N_{j+1} \le l(I) \le \ell_j$ or $\ell_{j+1} \le l(I) < 1/N_{j+1}$.

In the first case I can be covered by at most $C|I|N_{j+1}^n$ cubes of side length $1/N_{j+1}$, all of which could be in $\mathcal{F}_{j+1}$. If J is a generic member of $\mathcal{F}_{j+1}$, we obtain from Proposition 4.1 that

$$\begin{aligned}\mu(I) &\le C|I|N_{j+1}^n \mu(J) \le C|I|N_{j+1}^n M_j|J| \le CM_j|I| \\ &\le C\frac{2M_{j-1}}{(\ell_j N_j)^n}|I| \le CM_{j-1}d_{j-1}^{-m/(v-1)}\ell_j^{m/(v-1)-n}|I| \le C_\epsilon \ell_j^{m/(v-1)-n-\epsilon}|I| \le C_\epsilon l(I)^{m/(v-1)-\epsilon}.\end{aligned}$$

Here the penultimate inequality follows from the relation (4-5) and the rapid growth condition (4-2) .

Let us turn to the complementary case, when $\ell_{j+1} \le l(I) \le N_{j+1}^{-1}$. If $\mu(I) > 0$, the cube I intersects at least one cube J in $\mathcal{F}_{j+1}$ in which case it is contained in the union of at most $3^n - 1$ cubes of the same dimension adjacent to it. Proposition 4.1 then yields

$$\mu(I) \le C_n\mu(J) \le C_n M_j|J| = C_n M_j N_{j+1}^{-n} = C_n M_j d_j^{-m/(v-1)}\ell_{j+1}^{m/(v-1)} \le C_\epsilon \ell_{j+1}^{m/(v-1)-\epsilon} \le C_\epsilon l(I)^{m/(v-1)-\epsilon},$$

applying (4-2) as before at the penultimate stage. This establishes the claim (4-6).

4F. ***Minkowski dimension of E.*** In order to establish the full Minkowski dimension of E, we show that for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$\mathcal{N}_\ell(E) \ge c_\epsilon \ell^{-n+\epsilon} \quad \text{for any } 0 < \ell \ll 1. \tag{4-7}$$

Here $\mathcal{N}_\ell(A)$ denotes the ℓ-covering number of a Borel set A, i.e., the smallest number of closed cubes of side length ℓ required to cover A. Given $0 < \ell \ll 1$, we first fix the index j such that $\ell_{j+1} \le \ell < \ell_j$. As before, we study two cases.

4F1. *Case 1.* If $\ell \in [\ell_{j+1}, 1/N_{j+1})$, we select $I \in \mathcal{E}_j$ of side length ℓ_j to be one of the "special cubes" for step $j+1$; i.e.,

$$\begin{aligned} &I \subseteq T_{i_0(j+1)} && \text{for Theorem 1.1,} \\ &I \subseteq T_v[j+1] && \text{for Theorem 1.2.}\end{aligned} \tag{4-8}$$

Here $i_0(j+1) \in \{1, \dots, v\}$ denotes the preferred index at step $j+1$ of the construction, based on which Proposition 3.1 is applied. On the other hand, $T_v[j+1]$ denotes the choice of T_v at the $(j+1)$-th step for the purpose of applying Proposition 3.4. In either case, $I \in \mathcal{E}_j$ can be partitioned into $(\ell_j N_{j+1})^n$ subcubes of side length $1/N_{j+1}$. It follows from (3-4) and (3-17) in Propositions 3.1 and 3.4 that at least half of these subcubes lie in $\mathcal{F}_{j+1}$. Further, the conclusions (3-5) and (3-16) of the propositions say that for each $J \in \mathcal{F}_{j+1},\ J \subset I$,

$$|J \cap E_{j+1}| \geq \begin{cases} d_j N_{j+1}^{-1} & \text{for Theorem 1.1,} \\ \frac{1}{2} N_{j+1}^{-n} & \text{for Theorem 1.2,} \end{cases}$$

so that combining the two

$$|J \cap E_{j+1}| \geq d_j N_{j+1}^{-n} \quad \text{for any } n. \tag{4-9}$$

Let $\mathcal{Q}_\ell$ be a collection of cubes of side length ℓ that cover $I \cap E$, with $\#(\mathcal{Q}_\ell) = \mathcal{N}_\ell(I \cap E)$. Given any $Q \in \mathcal{Q}_\ell$, let Q^* denote the axis-parallel cube with the same centre as Q, but side length $4\ell\sqrt{n}$. Our main claim is that

$$I \cap E_{j+1} = \bigcup \{J \cap E_{j+1} : J \subseteq I,\, J \in \mathcal{F}_{j+1}\} \tag{4-10}$$

$$\subseteq \bigcup \{Q^* : Q \in \mathcal{Q}_\ell\}, \tag{4-11}$$

so that

$$|I \cap E_{j+1}| \leq \#(\mathcal{Q}_\ell)(4\sqrt{n}\ell)^n. \tag{4-12}$$

Assuming the claim for now, the proof of (4-7) proceeds as follows,

$$\mathcal{N}_\ell(E) \geq \mathcal{N}_\ell(I \cap E) = \#(\mathcal{Q}_\ell) \geq c_n \frac{|I \cap E_{j+1}|}{\ell^n} \geq c_n \sum_J \left\{ \frac{|J \cap E_{j+1}|}{\ell^n} : J \subseteq I,\ J \in \mathcal{F}_{j+1} \right\}$$

$$\geq \frac{c_n}{\ell^n} \times \frac{(\ell_j N_{j+1})^n}{2} \times d_j N_{j+1}^{-n} = c_n \frac{d_j \ell_j^n}{2\ell^n} \geq c_{\epsilon,n} \ell^{-n+\epsilon}.$$

Let us pause for a moment to explain the steps above. The second inequality in the sequence follows from (4-12) with $c_n = (4\sqrt{n})^{-n}$. The third inequality uses (4-10) and the disjointness of the cubes J; the fourth follows from (4-9) and the counting argument for $\#(\mathcal{F}_{j+1})$ preceding it. The final inequality is a consequence of the rapid growth condition (4-2) and the assumption $\ell \leq 1/N_{j+1}$. Together they imply that for any $\epsilon > 0$, there is a constant $c_\epsilon > 0$ such that $d_j \ell_j^n \geq c_\epsilon N_{j+1}^{-\epsilon} \geq c_\epsilon \ell^\epsilon$.

Proof of the claim. It remains to verify (4-10)–(4-12). The equality in (4-10) is part of the definitions of $\mathcal{E}_j$ and $\mathcal{F}_{j+1}$. The estimate (4-12) is an easy consequence of (4-11). To establish (4-11), pick any $x \in I \cap E_{j+1}$. Since E_{j+1} is by definition a union of the cubes in $\mathcal{E}_{j+1}$, there must exist a basic interval $I' \in \mathcal{E}_{j+1}$ containing x. The set $I' \cap E$ is nonempty by construction, so we pick an element y in this set. Then $|x - y| \leq \operatorname{diam}(I') = \sqrt{n}\ell_{j+1}$. Since $y \in I \cap E$, there must be a cube $Q_y \in \mathcal{Q}_\ell$ containing y; let $c(Q_y)$ denote the centre of Q_y. The assumption $\ell \geq \ell_{j+1}$ gives that

$$|x - c(Q_y)| \leq |x - y| + |y - c(Q_y)| \leq \sqrt{n}\ell_{j+1} + \tfrac{1}{2}\ell\sqrt{n} \leq 2\ell\sqrt{n}.$$

This means that $x \in Q_y^*$, as desired. □

4F2. *Case 2.* In the second case, where $\ell \in [1/N_{j+1}, \ell_j)$, the analysis is similar, with minor variations in numerology. Since ℓ is larger, we need to start from a coarser scale. Pick a cube $I \in \mathcal{E}_{j-1}$ that is "special" for the j-th step, in the sense that (4-8) holds with j replaced by $j-1$. As before, we decompose I into cubes $J \in \mathcal{F}_j$; the number of such cubes J is at least $\frac{1}{2}(\ell_{j-1}N_j)^n$. Let $\mathcal{Q}_\ell$ again denote a covering of

$I \cap E$ by ℓ-cubes, with $\#(\mathcal{Q}_\ell) = \mathcal{N}_\ell(I \cap E)$. Set F_j to be the union of the intervals in $\mathcal{F}_j$. This time, we will need the following analogues of (4-10)–(4-12), to be proven shortly:

$$I \cap F_{j+1} = \bigcup \{J \cap F_{j+1} : J \subseteq I,\ J \in \mathcal{F}_j\} \subseteq \bigcup \{Q^* : Q \in \mathcal{Q}_\ell\}, \tag{4-13}$$

so that

$$|I \cap F_{j+1}| \le \#(\mathcal{Q}_\ell)(4\sqrt{n}\ell)^n. \tag{4-14}$$

Further,

$$|J \cap F_{j+1}| \ge \tfrac{1}{2} d_{j-1} N_j^{-n} \quad \text{for each } J \in \mathcal{F}_j,\, J \subseteq I. \tag{4-15}$$

Assuming these, an argument analogous to the previous case leads to

$$\begin{aligned} \mathcal{N}_\ell(E) \ge \mathcal{N}_\ell(E \cap I) = \#(\mathcal{Q}_\ell) &\ge c_n \ell^{-n} |I \cap F_{j+1}| \\ &\ge c_n \ell^{-n} \sum \{|J \cap F_{j+1}| : J \subseteq I,\ J \in \mathcal{F}_j\} \\ &\ge c_n \ell^{-n} \tfrac{1}{2}(\ell_{j-1} N_j)^n \times \tfrac{1}{2} d_{j-1} N_j^{-n} = c_n \frac{d_{j-1}\ell_{j-1}^n}{4\ell^n} \ge c_{\epsilon,n} \ell^{-n+\epsilon}. \end{aligned}$$

The second inequality in the sequence above uses (4-14), and the fourth uses (4-15). The last step uses the assumption $\ell < \ell_j$, which implies in view of (4-2) that

$$\ell^\epsilon < \ell_j^\epsilon < c_\epsilon^{-1} d_{j-1} \ell_{j-1}^n \quad \text{for every } \epsilon > 0.$$

Proof of the claim. Returning to the claims surrounding $I \cap F_{j+1}$, we briefly comment on (4-13) and (4-15), whose proofs constitute the only points of departure from the previous case. Let us start with (4-13). For any $x \in I \cap F_{j+1}$, we focus on a cube $J' \in \mathcal{F}_{j+1}$ such that $x \in J'$. Choosing $y \in J' \cap E$ and $Q_y \in \mathcal{Q}_\ell$ containing y, we see that $|x - y| \le \operatorname{diam}(J') = \sqrt{n}/N_{j+1}$. Keeping in mind that $\ell \ge 1/N_{j+1}$, one obtains

$$|x - c(Q_y)| \le |x - y| + |y - c(Q_y)| \le \frac{\sqrt{n}}{N_{j+1}} + \ell\sqrt{n} \le 2\ell\sqrt{n},$$

where $c(Q_y)$ denotes the centre of Q_y, as before. This in turn implies (4-13).

To prove (4-15), let us fix $J \in \mathcal{F}_j,\ J \subseteq I$, and observe that $J \cap E_j$ is a union of basic ℓ_j-cubes. The special choice of $I \in \mathcal{E}_{j-1}$ dictates that (4-9) holds with j replaced by $j-1$; i.e., $|J \cap E_j| \ge d_{j-1} N_j^{-n}$. Thus, the number of basic ℓ_j-cubes in $J \cap E_j$ at the j-th level is at least $d_{j-1} N_j^{-n}/\ell_j^n$. At step $j+1$, each j-th level basic cube contributes at least $\frac{1}{2}(\ell_j N_{j+1})^n$ subcubes of side length $1/N_{j+1}$ to $\mathcal{F}_{j+1}$, according to Propositions 3.1 and 3.4. Combining all of this yields,

$$|J \cap F_{j+1}| \ge \frac{d_{j-1} N_j^{-n}}{\ell_j^n} \times \tfrac{1}{2}(\ell_j N_{j+1})^n \times N_{j+1}^{-n} = \tfrac{1}{2} d_{j-1} N_j^{-n},$$

as claimed. □

5. Zero sets of functions with a common linearization

We now turn our attention to the proof of Theorem 1.3. Not surprisingly in view of the other results in this paper, it is also predicated on an iterative algorithm, which has been encapsulated in Proposition 5.2 below. The following lemma provides a preparatory step.

Let $\alpha \in \mathbb{R}^v$ be as in the statement of Theorem 1.3, and let $\mathfrak{C}$ be a nonempty proper subset of the index set $\{1, 2, \ldots, v\}$. Let $\delta > 0$. Consider disjoint intervals $[a_1, b_1]$ and $[a_2, b_2]$ of length λ, with $a_1 < b_1 \le a_2 < b_2$. We define two quantities ϵ_{left} and ϵ_{right} depending on $\mathfrak{C}, a_1, b_1, a_2, b_2$ and δ as follows:

$$\epsilon_{\text{left}} := \sup\left\{\epsilon : \left|\sum_{j=1}^{v} \alpha_j z_j\right| \ge \delta\lambda \;\text{ for }\; \begin{matrix} z_j \in [a_1, a_1 + \epsilon\lambda] \;\text{ for all } j \notin \mathfrak{C}, \\ z_j \in [a_2, a_2 + \epsilon\lambda] \;\text{ for all } j \in \mathfrak{C} \end{matrix}\right\}, \tag{5-1}$$

$$\epsilon_{\text{right}} := \sup\left\{\epsilon : \left|\sum_{j=1}^{v} \alpha_j z_j\right| \ge \delta\lambda \;\text{ for }\; \begin{matrix} z_j \in [a_1, a_1 + \epsilon\lambda] \;\text{ for all } j \notin \mathfrak{C}, \\ z_j \in [b_2 - \epsilon\lambda, b_2] \;\text{ for all } j \in \mathfrak{C} \end{matrix}\right\}. \tag{5-2}$$

Lemma 5.1. *Given any $\alpha \in \mathbb{R}^v$ as in Theorem 1.3, there exists $\delta_0 > 0$ depending only on α such that for any $\lambda > 0$ and any choice of intervals $\mathfrak{I}_1 = [a_1, b_1]$ and $\mathfrak{I}_2 = [a_2, b_2]$ of equal length λ with $a_1 < b_1 \le a_2 < b_2$, the following property holds. For any $\delta < \delta_0$, there exists $\epsilon_0 = \epsilon_0(\mathfrak{C}, \delta)$ (not depending on a_1, a_2, b_1, b_2, or λ) such that* $\max(\epsilon_{\text{left}}, \epsilon_{\text{right}}) \ge \epsilon_0$.

In particular, there exist subintervals $\widehat{\mathfrak{I}}_1 \subseteq \mathfrak{I}_1$ and $\widehat{\mathfrak{I}}_2 \subseteq \mathfrak{I}_2$ with $|\widehat{\mathfrak{I}}_1| = |\widehat{\mathfrak{I}}_2| = \epsilon_0\lambda$ and $\operatorname{dist}(\widehat{\mathfrak{I}}_1, \widehat{\mathfrak{I}}_2) \ge (1 - \epsilon_0)\lambda$ *such that*

$$|\alpha \cdot x| \ge \delta\lambda \quad \textit{for all } x \in \mathbb{R}^v \textit{ such that } \begin{cases} x_j \in \widehat{\mathfrak{I}}_1 & \textit{for } j \notin \mathfrak{C}, \\ x_j \in \widehat{\mathfrak{I}}_2 & \textit{for } j \in \mathfrak{C}. \end{cases}$$

Proof. Set $g(y) = \sum_j \alpha_j y_j$, and consider $g(z^*)$, where $z^* = (z_1^*, \ldots, z_v^*)$ is defined to be the v-dimensional vector with $z_j^* = a_1$ if $j \notin \mathfrak{C}$ and $z_j^* = a_2$ if $j \in \mathfrak{C}$. Setting $C^* = \sum_j |\alpha_j|$, we note that

$$|g(z) - g(z^*)| \le C^*\epsilon\lambda \quad \text{whenever } |z_j - z_j^*| \le \epsilon\lambda,\; 1 \le j \le v. \tag{5-3}$$

If $|g(z^*)| > (\delta + \epsilon_0 C^*)\lambda$, then (5-3) implies that $|g(z)| \ge \delta\lambda$ for any z as in (5-1). Therefore $\epsilon_{\text{left}} \ge \epsilon_0$, and the conclusion of the lemma holds with $\widehat{\mathfrak{I}}_1 = [a_1, a_1 + \epsilon_0\lambda]$, $\widehat{\mathfrak{I}}_2 = [a_2, a_2 + \epsilon_0\lambda]$. Otherwise, let $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_v)$ be the v-dimensional vector with $\hat{z}_j = a_1$ if $j \notin \mathfrak{C}$ and $\hat{z}_j = b_2$ if $j \in \mathfrak{C}$. Then $g(\hat{z}) = g(z^*) + \alpha \cdot (\hat{z} - z^*) = g(z^*) \pm (b_2 - a_2)C_0 = g(z^*) \pm \lambda C_0$, where $C_0 = \left|\sum_{j\in\mathfrak{C}} \alpha_j\right| > 0$. Thus, for z as in (5-2), we obtain the estimate

$$\begin{aligned} |g(z)| &\ge |g(\hat{z})| - |\alpha \cdot (z - \hat{z})| \ge |C_0\lambda \pm g(z^*)| - C^*\epsilon_0\lambda \\ &\ge C_0\lambda - (\delta + C^*\epsilon_0)\lambda - C^*\epsilon_0\lambda \ge C_0\lambda - (\delta + 2\epsilon_0 C^*)\lambda, \end{aligned}$$

which is greater than or equal to $\delta\lambda$ provided that $\delta < \frac{1}{2}C_0 =: \delta_0$ and $\epsilon_0 \le (C_0 - 2\delta)/(2C^*)$. One has $\epsilon_{\text{right}} \ge \epsilon_0$ for this choice of ϵ_0, with the conclusion of the lemma verified for $\widehat{\mathfrak{I}}_1 = [a_1, a_1 + \epsilon_0\lambda]$, $\widehat{\mathfrak{I}}_2 = [b_2 - \epsilon_0\lambda, b_2]$. □

Remarks. (a) Let us consider the example $\alpha = (1, -2, 1)$, which corresponds to a linear function g that picks out three-term arithmetic progressions. Choose $\mathfrak{C}$ to be $\{3\}$. For $x_1, x_2 \in [a_1, a_1 + \epsilon\lambda]$ and $x_3 \in [a_2, a_2 + \epsilon\lambda]$, it is easy to see that

$$x_1 - 2x_2 + x_3 \ge a_1 + a_2 - 2(a_1 + \epsilon\lambda) = a_2 - a_1 - 2\epsilon\lambda \ge (1 - 2\epsilon)\lambda.$$

We can thus take $\epsilon_{\text{left}} = \frac{1}{2}(1-\delta)$. On the other hand, if $x_1, x_2 \in [a_1, a_1+\epsilon\lambda]$ and $x_3 \in [b_2-\epsilon\lambda, b_2]$, then

$$x_1 - 2x_2 + x_3 \geq a_1 + b_2 - \epsilon\lambda - 2(a_1+\epsilon\lambda) = b_2 - a_1 - 3\epsilon\lambda \geq (2-3\epsilon)\lambda.$$

Thus $\epsilon_{\text{right}} = \frac{1}{3}(2-\delta)$. The point is that, in the above lemma, it is possible in certain instances for both ϵ_{left} and ϵ_{right} to be bounded from below. The lemma guarantees that at least one of them will be.

(b) It is important to be aware that the above proof does not necessarily give the best possible ϵ_0 for a given δ because the signs of the components of α are not taken into account. When dealing with a specific α, it is often possible to improve the bound on ϵ_0 given above.

Proposition 5.2. *Given any $\alpha \in \mathbb{R}^v$ obeying the hypotheses of Theorem 1.3, there exist fixed small constants $0 < \epsilon < \frac{3}{4}$ and $\delta(\epsilon) > 0$ depending on α with the following property.*

Let I be any interval say of length ℓ, and let I_1 and I_2 denote the two halves of I. Then one can find subintervals I_1' and I_2' of I_1 and I_2 of length $\epsilon\ell$ such that

$$|\alpha \cdot x| \geq \delta\ell \quad \textit{for every sufficiently small } \delta \leq \delta(\epsilon),$$

and for any choice of $x_1, x_2, \dots, x_v \in I_1' \cup I_2'$, not all of which are in I_i' for a single $i = 1, 2$. The subintervals I_1' and I_2' are separated by at least $\frac{1}{4}\ell$.

Proof. Let $\{\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_R\}$ be an enumeration of all nonempty, proper subsets of $\{1, 2, \dots, v\}$. Given any $x = (x_1, \dots, x_v)$ such that $x_j \in I$ for all j but not all the x_j lie in a single I_1 or I_2, there exists $1 \leq m \leq R$ such that $j \in \mathfrak{C}_m$ if and only if $x_j \in I_2$. Set

$$C_m := \Bigl|\sum_{j \in \mathfrak{C}_m} \alpha_j\Bigr| \quad \text{and} \quad \delta_0 = \tfrac{1}{2}\min(C_1, \dots, C_R),$$

so that Lemma 5.1 can be applied for any $\delta < \delta_0$ and any $\mathfrak{C} = \mathfrak{C}_m,\ 1 \leq m \leq R$.

Starting with I_1 and I_2, we apply Lemma 5.1 with $\mathfrak{C} = \mathfrak{C}_1,\ \mathfrak{I}_1 = I_1,\ \mathfrak{I}_2 = I_2$ and $\lambda = \frac{1}{2}\ell$. For a small but fixed $\delta_1 > 0$ with $2\delta_1 \leq \delta_0$, this gives a constant $\epsilon_1 = \epsilon_0(\mathfrak{C}_1, 2\delta_1) > 0$ and two subintervals $I_1^{(1)} \subseteq I_1$ and $I_2^{(1)} \subseteq I_2$ of length $\frac{1}{2}\epsilon_1\ell$ obeying the conclusions of the lemma. Without loss of generality, we can assume that $\epsilon_1 \leq \frac{1}{2}$, so that

$$\operatorname{dist}(I_1^{(1)}, I_2^{(1)}) \geq (1-\epsilon_1)\tfrac{1}{2}\ell \geq \tfrac{1}{4}\ell. \tag{5-4}$$

For $2 \leq k \leq R$, we continue to apply Lemma 5.1 recursively, with the same value δ_1, and

$$\mathfrak{C} = \mathfrak{C}_k, \quad \mathfrak{I}_1 = I_1^{(k-1)}, \quad \mathfrak{I}_2 = I_2^{(k-1)}, \quad \lambda = \tfrac{1}{2}\epsilon_1 \cdots \epsilon_{k-1}\ell.$$

At the end of the k-th step, this yields a constant $\epsilon_k = \epsilon_0(\mathfrak{C}_k, 2\delta)$ and subintervals $I_1^{(k)} \subseteq I_1^{(k-1)} \subseteq I_1$, $I_2^{(k)} \subseteq I_2^{(k-1)} \subseteq I_2$ each of length $\frac{1}{2}\epsilon_1 \cdots \epsilon_k\ell$ such that for any $m \leq k$,

$$|\alpha \cdot x| \geq \delta_1\epsilon_1 \cdots \epsilon_{k-1}\ell \quad \text{for all } x \text{ such that } \begin{cases} x_j \in I_1^{(k)} & \text{for } j \notin \mathfrak{C}_m, \\ x_j \in I_2^{(k)} & \text{for } j \in \mathfrak{C}_m. \end{cases}$$

The conclusion of the proposition then holds for

$$I_1' = I_1^{(R)}, \quad I_2' = I_2^{(R)}, \quad \epsilon = \frac{1}{2}\prod_{k=1}^{R}\epsilon_k \quad \text{and} \quad \delta(\epsilon) = \delta_1\epsilon_1 \cdots \epsilon_{R-1}.$$

The separation condition is an easy consequence of the one in Lemma 5.1. Specifically, since $I_i' \subseteq I_i^{(1)}$ for $i = 1, 2$, the relation (5-4) yields

$$\operatorname{dist}(I_1', I_2') \geq \operatorname{dist}(I_1^{(1)}, I_2^{(1)}) \geq \tfrac{1}{4}\ell. \qquad \square$$

Remarks. (a) Tracking the parameters from Lemma 5.1, we find that the constant ϵ claimed in Proposition 5.2 obeys the estimate

$$\epsilon \geq \frac{1}{2} \prod_{m=1}^{R} \frac{(C_m - 2\delta_1)}{(2C^*)}, \tag{5-5}$$

where recall

$$C^* = \sum_{j=1}^{v} |\alpha_j|$$

and C_m and δ_1 are as in the proof of the proposition.

(b) In view of the remarks made at the end of Lemma 5.1, it is not surprising that the bound on ϵ in the preceding inequality is not always optimal. Returning to the example $\alpha = (1, -2, 1)$, we leave the reader to verify that given any small $\delta > 0$ and $I = [a, a + \ell]$, the choice $I_1' = \big[a, a + \frac{1}{3}(1 - \delta)\ell\big]$ and $I_2' = \big[a + \frac{1}{3}(2 + \delta)\ell, a + \ell\big]$ meets the requirements of the proposition. Thus for this α, the best choice of ϵ is at least $\frac{1}{3}(1 - \delta)$, which is much better than the one provided by the proof.

5A. *Proof of Theorem 1.3.* Take ϵ and $\delta = \delta(\epsilon)$ to be the positive α-dependent constants given by Proposition 5.2. Recall that $g(x_1, \ldots, x_v) = \sum_{j=1}^{v} \alpha_j x_j$.

Start with $E_0 = [0, \eta]$ where $0 < \eta \ll 1$ is chosen sufficiently small so as to ensure $2Kv\eta < \delta$. Applying Proposition 5.2 with $I = E_0$, we arrive at subintervals $I_1' = J_1 \subseteq \big[0, \frac{1}{2}\eta\big]$ and $I_2' = J_2 \subseteq \big[\frac{1}{2}\eta, \eta\big]$ of length $\ell_1 = \epsilon\eta$ that obey its conclusions. Let $E_1 = J_1 \cup J_2$ with $|J_1| = |J_2| = \ell_1$. In general, if E_j is a disjoint union of 2^j basic intervals of length $\ell_j = \epsilon^j \eta$, then at step $j + 1$, we apply Proposition 5.2 to each such interval to find two subintervals of length $\ell_{j+1} = \epsilon\ell_j = \epsilon^{j+1}\eta$ and separated by a length of at least $\frac{1}{4}\ell_j$, which form the basic intervals of E_{j+1}.

Defining $E = \bigcap_{j=1}^{\infty} E_j$, we now show that $f(x_1, \ldots, x_v) \neq 0$ if $x_1, \ldots, x_v$ are not all identical and f is of the form (1-3). For any such choice of $x_1, \ldots, x_v$, there exists a largest index j such that $x_1, x_2, \ldots, x_v$ all lie in a basic interval I at step j. This means that if I_1' and I_2' are the two subintervals of I generated by Proposition 5.2, then $x_1, \ldots, x_v$ lie in $I_1' \cup I_2'$, but not all of them lie in a single I_i'. If I is of length ℓ_j, it follows from Proposition 5.2 that $|g(x)| \geq \delta\ell_j$. But $|f(x) - g(x)| \leq Kv\ell_j^2$ according to (1-4), so this implies $|f(x)| \geq \frac{1}{2}\delta\ell_j$ for $\ell_j < \eta$.

We recall that the $(j+1)$-th step of the construction generates exactly two children from each parent, and these are separated by at least $\frac{1}{4}\ell_j$. It now follows from standard results, see for instance [Falconer 2003, Example 4.6, page 64], that the Hausdorff dimension of E is bounded from below by

$$\lim_{j\to\infty} \frac{\log(2^j)}{-\log(2\ell_j/4)} = \lim_{j\to\infty} \frac{\log(2^j)}{-\log(\epsilon^j\eta/2)} = \frac{\log 2}{-\log \epsilon}.$$

This establishes the existence of the set claimed by the theorem, with $c(\alpha) = \log 2/\log(1/\epsilon)$, where ϵ is at least as large as the bound given in (5-5). $\square$

Remark. We return to the example $\alpha = (1, -2, 1)$ that we have been following across this section to show that the avoiding set in this instance can be chosen to have Hausdorff dimension $\log 2/\log 3$. We have referred to this fact in certain examples occurring in Sections 2A2 and 2A3.

Choose a slowly decreasing sequence $\delta_j = 1/(j + C)$ for some fixed large constant C. We have seen, in part (b) of the remark on page 1107, that $\epsilon(\delta_j) = \epsilon_j$ can be chosen as $\frac{1}{3}(1 - \delta_j)$. Let us now use the same Cantor construction as in the proof given above, but using the parameter δ_j at step j instead of a fixed δ. The following consequences are immediate:

$$\ell_j = \epsilon_1 \cdots \epsilon_j \eta \quad \text{so that } \ell_j \le \frac{C\eta 3^{-j}}{j+C},$$

$$|g(x)| \ge \delta_j \ell_j \quad \text{and} \quad |f(x) - g(x)| \le K v \ell_j^2 \quad \text{so that } |f(x)| \ge (\delta_j - K v \ell_j)\ell_j \ge \left(\frac{1}{j+C} - \frac{K v \eta C}{j+C} 3^{-j}\right)\ell_j > 0,$$

where $x = (x_1, \dots, x_v)$ is as in the second paragraph of Section 5A. This proves the nonexistence of nontrivial zeros of f. Further, the Hausdorff dimension is bounded from below by

$$\lim_{j\to\infty} \frac{\log(2^j)}{-\log(2\ell_j/4)} = \lim_{j\to\infty} \frac{\log(2^j)}{-\log\bigl(3^{-j}\eta \prod_{k=1}^{j}(1-\delta_k)/2\bigr)} = \frac{\log 2}{\log 3},$$

establishing the claim.

Appendix

We collect here the proofs of a few technical facts mentioned in Section 2.

Lemma A.1. *Given a C^2 parameterization $\gamma : [0, \eta] \to \mathbb{R}^n$ of a curve Γ, let us recall the definition of the signed distance function d from (2-3). Set $F(t_1, t_2) = d(\gamma(t_1), \gamma(t_2))$. Then:*

(a) *F is differentiable on $[0, \eta]^2$.*

(b) *If γ is the arclength parameterization, i.e., $|\gamma'(t)| \equiv 1$, then*

$$\frac{\partial F}{\partial t_1}(t, t) = 1, \quad \frac{\partial F}{\partial t_2}(t, t) = -1.$$

Proof. Since differentiability is obvious for $t_1 \ne t_2$, it suffices to verify it when $t_1 = t_2 = t$. We consider two cases. If $h \ge k$, then

$$\begin{aligned} F(t+h, t+k) = d(\gamma(t+h), \gamma(t+k)) &= |\gamma(t+h) - \gamma(t+k)| \\ &= |\gamma'(t)|\,|h-k| + O(h^2+k^2) \\ &= |\gamma'(t)|(h-k) + O(h^2+k^2). \end{aligned}$$

On the other hand if $h < k$, we have

$$\begin{aligned} d(\gamma(t+h), \gamma(t+k)) &= -|\gamma(t+h) - \gamma(t+k)| \\ &= -|\gamma'(t)|\,|h-k| + O(h^2+k^2) \\ &= |\gamma'(t)|(h-k) + O(h^2+k^2). \end{aligned}$$

This establishes the first part of the lemma, with

$$\frac{\partial F}{\partial t_1}(t,t) = |\gamma'(t)|, \quad \frac{\partial F}{\partial t_2}(t,t) = -|\gamma'(t)|.$$

The second part is now obvious. □

Lemma A.2. *Let $\gamma : [0,\eta] \to \mathbb{R}^n$ be an injective parameterization of a C^2 curve with*

$$\gamma'(0) \neq 0 \quad \textit{and} \quad \sup\{\|\gamma''(t)\| : t \in [0,\eta]\} \le K.$$

If η is sufficiently small depending on $|\gamma'(0)|$ and K, then there are no isosceles triangles $\gamma(t_1)$, $\gamma(t_2)$, $\gamma(t_3)$ with $0 \le t_1 < t_2 < t_3 \le \eta$ whose sides of equal length meet at $\gamma(t_1)$ or at $\gamma(t_3)$.

Proof. Since d has already been shown to be differentiable in the previous lemma, we compute

$$d(\gamma(t_3),\gamma(t_1)) - d(\gamma(t_2),\gamma(t_1)) = \int_{t_2}^{t_3} \frac{\partial}{\partial t} d(\gamma(t),\gamma(t_1)) = \int_{t_2}^{t_3} \gamma'(t) \cdot \frac{\gamma(t)-\gamma(t_1)}{|\gamma(t)-\gamma(t_1)|}. \tag{A-1}$$

For $t, t_1 \in [0,\eta]$ with $t > t_1$, we obtain

$$\begin{aligned}\frac{\gamma(t)-\gamma(t_1)}{|\gamma(t)-\gamma(t_1)|} &= \frac{[\gamma'(t_1)(t-t_1) + O(K(t-t_1)^2)]}{|[\gamma'(t_1)(t-t_1) + O(K(t-t_1)^2)]|} \\ &= \frac{\gamma'(t_1) + O(K\eta)}{|\gamma'(t_1) + O(K\eta)|} = \frac{\gamma'(0) + O(K\eta)}{|\gamma'(0) + O(K\eta)|} \\ &= \frac{\gamma'(0)}{|\gamma'(0)|}\left[1 + O\left(\frac{K\eta}{|\gamma'(0)|}\right)\right].\end{aligned}$$

Using this, the integrand in (A-1) may be estimated as follows:

$$\gamma'(t) \cdot \frac{\gamma(t)-\gamma(t_1)}{|\gamma(t)-\gamma(t_1)|} = [\gamma'(0) + O(K\eta)] \cdot \frac{\gamma'(0)}{|\gamma'(0)|}\left[1 + O\left(\frac{K\eta}{|\gamma'(0)|}\right)\right] \ge \tfrac{1}{2}|\gamma'(0)| \neq 0,$$

provided $K\eta$ is small relative to $|\gamma'(0)|$. This shows that

$$d(\gamma(t_3),\gamma(t_1)) - d(\gamma(t_2),\gamma(t_1)) \ge \tfrac{1}{2}|\gamma'(0)|(t_3 - t_2) \neq 0,$$

proving that $\gamma(t_1)$ cannot be the vertex at the intersection of two equal sides in an isosceles triangle. A similar argument works for $\gamma(t_3)$. □

Lemma A.3. *Given a curve Γ as described in Section 2B, let us recall the function $f = (f_1, f_2)$ given by* (2-4) *and* (2-5). *Then $Df(t)$ is of full rank at every point $t = (t_1,t_2,t_3,t_4)$ with distinct entries and $f(t) = 0$.*

Proof. To prove that Df has rank 2 on the zero set of f, it suffices to show that the 2×2 submatrix with entries $\partial f_i/\partial t_j$ with $i = 1, 2$ and $j = 1, 4$ is nonsingular. We will do this by proving that $\partial f_1/\partial t_j$ are nonzero and of the same sign for $j = 1, 4$, whereas for $\partial f_2/\partial t_j$ the signs are reversed.

We begin by computing $\partial f_1/\partial t_j$ on the zero set of f_1, where

$$\frac{\gamma_2(t_3)-\gamma_2(t_2)}{\gamma_1(t_3)-\gamma_1(t_2)} = \frac{\gamma_2(t_4)-\gamma_2(t_1)}{\gamma_1(t_4)-\gamma_1(t_1)}. \tag{A-2}$$

Feeding this into the formula for the derivatives, we find that

$$\frac{\partial f_1}{\partial t_1} = -\gamma_1'(t_1)(\gamma_2(t_3)-\gamma_2(t_2)) + \gamma_2'(t_1)(\gamma_1(t_3)-\gamma_1(t_2)) = \gamma_1'(t_1)(\gamma_1(t_3)-\gamma_1(t_2))F_1,$$
$$\frac{\partial f_1}{\partial t_4} = \gamma_1'(t_4)(\gamma_2(t_3)-\gamma_2(t_2)) - \gamma_2'(t_4)(\gamma_1(t_3)-\gamma_1(t_2)) = \gamma_1'(t_4)(\gamma_1(t_3)-\gamma_1(t_2))F_4,$$

where

$$F_1 = -\frac{\gamma_2(t_4)-\gamma_2(t_1)}{\gamma_1(t_4)-\gamma_1(t_1)} + \frac{\gamma_2'(t_1)}{\gamma_1'(t_1)} \quad \text{and} \quad F_4 = \frac{\gamma_2(t_4)-\gamma_2(t_1)}{\gamma_1(t_4)-\gamma_1(t_1)} - \frac{\gamma_2'(t_4)}{\gamma_1'(t_4)}.$$

Since γ_1' is assumed to be of fixed positive sign on $[0, \eta]$, we have

$$\operatorname{sign}\left(\frac{\partial f_1}{\partial t_1}\cdot\frac{\partial f_1}{\partial t_4}\right) = \operatorname{sign}(F_1F_4).$$

But $\gamma_2'(t_j)/\gamma_1'(t_j)$ is the slope of the tangent to the curve Γ at the point t_j, whereas $(\gamma_2(t_4)-\gamma_2(t_1))/(\gamma_1(t_4)-\gamma_1(t_1))$ is the slope of the chord joining t_1 and t_4. Since we have assumed that Γ is strictly convex, this yields that F_1 and F_4 are of the same sign, which is the desired conclusion.

We turn to $\partial f_2/\partial t_j$ for $j = 1, 4$. Let us observe that f_2 is nonzero if $t_4 - t_3$ and $t_2 - t_1$ have opposite signs. In what follows, we will therefore restrict to the case where $(t_4 - t_3)(t_2 - t_1) > 0$. We find that

$$\frac{\partial}{\partial t_4} d(\gamma(t_4), \gamma(t_3)) = \gamma'(t_4)\cdot\frac{\gamma(t_4)-\gamma(t_3)}{|\gamma(t_4)-\gamma(t_3)|},$$

so

$$\frac{\partial f_2}{\partial t_4} = \gamma'(t_4)\cdot\frac{\gamma(t_4)-\gamma(t_3)}{|\gamma(t_4)-\gamma(t_3)|} d(\gamma(t_2), \gamma(t_1)).$$

Similarly

$$\frac{\partial f_2}{\partial t_1} = -\gamma'(t_1)\cdot\frac{\gamma(t_2)-\gamma(t_1)}{|\gamma(t_2)-\gamma(t_1)|} d(\gamma(t_4), \gamma(t_3)).$$

In the regime where $(t_4 - t_3)(t_2 - t_1) > 0$, these two quantities are of opposite signs, completing the proof. □

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