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NONAUTONOMOUS MAXIMAL L^p-REGULARITY UNDER FRACTIONAL SOBOLEV REGULARITY IN TIME

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We prove nonautonomous maximal L^p-regularity results on UMD spaces, replacing the common Hölder assumption by a weaker fractional Sobolev regularity in time. This generalizes recent Hilbert space results by Dier and Zacher. In particular, on $L^q(\Omega)$ we obtain maximal L^p-regularity for $p \geq 2$ and elliptic operators in divergence form with uniform VMO-modulus in space and $W^{\alpha,p}$-regularity for $\alpha > \frac{1}{2}$ in time.

1. Introduction

In this work we improve some known results on maximal L^p-regularity of nonautonomous abstract Cauchy problems with time-dependent domains of the form

$$\begin{cases} \dot{u}(t) + A(t)u(t) = f(t), \\ \qquad\qquad u(0) = u_0. \end{cases} \tag{NACP}$$

In particular, we obtain new stronger results if the operators $A(t)$ are elliptic operators in divergence form. Let us right away start with the definition.

Definition 1.1. For a family $(A(t))_{t \in [0,T]}$ of closed linear operators on some Banach space X the problem (NACP) has *maximal L^p-regularity* if for all $f \in L^p([0,T]; X)$ and all initial values u_0 in the real interpolation space $(D(A(0)), X)_{1/p,p}$ there exists a unique solution $u \in L^p([0,T]; X)$ satisfying $u(t) \in D(A(t))$ for almost all $t \in [0,T]$ as well as $\dot{u}, A(\cdot)u(\cdot) \in L^p([0,T]; X)$ and if there exists $C > 0$ such that one has the maximal regularity a priori estimate

$$\|u\|_{W^{1,p}([0,T];X)} + \|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} \leq C(\|f\|_{L^p([0,T];X)} + \|u_0\|_{(D(A(0)),X)_{1/p,p}}).$$

Observe that $W^{1,p}([0,T]; X) \hookrightarrow C([0,T]; X)$ and therefore the initial condition makes sense. Maximal regularity results have profound applications to nonlinear parabolic problems, as we will exemplify in Section 8.

We now give a summary of the previously known results on maximal L^p-regularity. The autonomous case $A(t) = A$ is well understood. Here, maximal L^p-regularity holds for one $p \in (1, \infty)$ if and only if it holds for all $p \in (1, \infty)$. Further, maximal L^p-regularity for $u_0 = 0$ implies maximal L^p-regularity for

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all $u_0 \in D(A, X)_{1/p,p}$. On Hilbert spaces an operator A has maximal L^p-regularity if and only if $-A$ generates an analytic semigroup. In non-Hilbert spaces, not every generator of an analytic semigroup has maximal regularity; see [Kalton and Lancien 2000; Fackler 2014]. Here, an additional $\mathcal{R}$-boundedness assumption is needed. We refer to Section 3, [Denk et al. 2003] and [Kunstmann and Weis 2004] for details.

Let us come to the nonautonomous case. Here the best understood setting is that of nonautonomous forms on Hilbert spaces. For this let V, H be two complex Hilbert spaces with a dense embedding $V \hookrightarrow H$. A mapping $a : [0, T] \times V \times V \to \mathbb{C}$ is called a *coercive, bounded sesquilinear form* if $a(t, \cdot, \cdot)$ is sesquilinear for all $t \in [0, T]$ and if there exist $\alpha, M > 0$ such that for all $u, v \in V$

$$\begin{aligned} \operatorname{Re} a(t, u, u) &\geq \alpha \|u\|_V^2, \\ |a(t, u, v)| &\leq M \|u\|_V \|v\|_V. \end{aligned} \tag{1-1}$$

This induces operators $\mathcal{A}(t) : V \to V'$. We denote their parts in H by $A(t)$. It has been shown in [Haak and Ouhabaz 2015] that the operators $(A(t))_{t\in[0,T]}$ satisfy maximal L^p-regularity for all $p \in (1, \infty)$ if $t \mapsto \mathcal{A}(t)$ is α-Hölder continuous for some $\alpha > \frac{1}{2}$. For $\alpha > \frac{1}{2}$ and maximal L^2-regularity this has been improved to the fractional Sobolev regularity $\mathcal{A} \in \dot{W}^{\alpha,2}([0, T]; \mathcal{B}(V, V'))$ [Dier and Zacher 2017]. If one considers elliptic divergence form operators

$$L(t) = -\operatorname{div}(A(t)\nabla \cdot)$$

for coefficients $A(t) = (a_{ij}(t))$ realized by the form method (see Section 7), this translates into the regularity of the mappings $t \mapsto a_{ij}(t, \cdot) \in L^\infty$, i.e., $a_{ij} \in \dot{W}^{\alpha,2}([0, T]; L^\infty)$ for some $\alpha > \frac{1}{2}$. The less regularity one needs here, the more applicable the results are to nonlinear problems in the form of a priori estimates. In the special case of elliptic operators in divergence form, some more refined results are available; see [Auscher and Egert 2016; Fackler 2017b]. However, all results have in common that one needs some differentiability in time of order at least $\frac{1}{2}$. This is no coincidence. Recent counterexamples to Lions' problem by the author [Fackler 2017a] show that maximal L^p-regularity can fail if $\mathcal{A} \in C^{1/2}([0, T]; \mathcal{B}(V, V'))$. For more details see the recent survey on maximal L^2-regularity of nonautonomous forms [Arendt et al. 2017]. Dealing with nonlinear problems, one needs some form of Sobolev embedding to carry out the usual iteration procedure. In higher dimensional cases maximal regularity on $X = L^2(\Omega)$ is too weak for the embeddings to hold. Therefore one is interested in maximal regularity on $X = L^q(\Omega)$ for q big enough.

Nonautonomous maximal L^p-regularity on Banach spaces is far more involved. The classical works for time-dependent domains are [Hieber and Monniaux 2000a; 2000b]. Although the general method used there is applicable on Banach spaces, maximal L^p-regularity was first only obtained on Hilbert spaces in a nonform setting [Hieber and Monniaux 2000a] and in [Hieber and Monniaux 2000b] extrapolated to $X = L^q(\Omega)$ for smooth bounded domains Ω and elliptic operators assuming $a_{ij} \in C^\alpha([0, T]; C^1(\overline{\Omega}))$ for some $\alpha > \frac{1}{2}$. A true generalization of this approach to Banach spaces was obtained in [Portal and Štrkalj 2006] using the emerging concept of $\mathcal{R}$-boundedness. Already the results in [Hieber and Monniaux 2000b] indicate a fundamental new issue in the non-Hilbert space setting. Whereas on L^2 the coefficients only need to be measurable in space, on L^q all known results require some regularity in space. Recently, the author lowered the needed regularity in space and showed maximal L^p-regularity on $L^q(\Omega)$ for elliptic operators in divergence form if the coefficients have a uniform VMO-modulus [Fackler 2015].

The aim of this work is to generalize the results in both [Dier and Zacher 2017] and [Fackler 2015]. We show maximal L^p-regularity on UMD Banach spaces assuming fractional Sobolev regularity as in [Dier and Zacher 2017]. To give a flavor of the proved results let us formulate a particular consequence of our general result for elliptic operators in divergence form.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain, $T > 0$ and $a_{ij} \in L^\infty([0,T]\times\Omega)$ for $i, j = 1, \ldots, n$. Assume further that there exists $\delta > 0$ such that for almost all $(t,x) \in [0,T]\times\Omega$ and all $\xi \in \mathbb{C}^n$ the ellipticity estimate*

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\bar{\xi}_j \geq \delta|\xi|^2$$

holds and that for $t \in [0,T]$ the functions $x \mapsto a_{ij}(t,x)$ lie in VMO(Ω) *with uniform* VMO*-modulus. Then for all $q \in (1,\infty)$ the nonautonomous problem* (NACP) *associated to the operators $(-\operatorname{div} A\nabla\cdot)_{t\in[0,T]}$ has maximal L^p-regularity*

(a) *for $p \in (1,2]$ if $a_{ij} \in \dot{W}^{1/2+\varepsilon,2}([0,T]; L^\infty(\Omega))$ for some $\varepsilon > 0$,*

(b) *for $p \in [2,\infty)$ if $a_{ij} \in \dot{W}^{1/2+\varepsilon,p}([0,T]; L^\infty(\Omega))$ for some $\varepsilon > 0$.*

Here, the divergence form operators on $L^q(\Omega)$ are compatible with the operator on $L^2(\Omega)$ obtained via the form method (for a precise definition see Section 7). Note that in comparison to [Hieber and Monniaux 2000b], the regularity in space is lowered from $C^1(\overline{\Omega})$ to VMO(Ω) and the time regularity $C^{1/2+\varepsilon}$ is replaced by $\dot{W}^{1/2+\varepsilon,p}$ in the case $p \geq 2$. This is the lower time regularity we aim for and leads to more refined results in nonlinear PDE, as we illustrate in Section 8. The general result makes use of some more technical definitions and we postpone its formulation to Section 3.

The obtained results are even new in the Hilbert space case as [Dier and Zacher 2017] fully relies on Hilbert space methods and therefore only deals with the case $p = 2$. Our result is the first improvement of the time regularity on non-Hilbert spaces since the classical work [Acquistapace and Terreni 1987]. Since we establish maximal L^p-regularity for elliptic operators on $L^q(\Omega)$ for $q > 2$, we obtain existence results for *strong* solutions of quasilinear parabolic equations in divergence form. Such results cannot be obtained with maximal regularity results on Hilbert spaces. We further show that our results are optimal in the sense that in general we cannot relax the regularity to some $\alpha \leq \frac{1}{2}$.

Note that, in contrast, elliptic operators in nondivergence form have time-independent domains and one can therefore obtain maximal L^p-regularity only assuming the time dependence to be measurable; see for example [Gallarati and Veraar 2017b; Dong and Kim 2016] for recent results. However, note that in correspondence with our results, one still needs a variant of VMO-regularity in space.

This work is structured as follows. The first sections introduce the necessary mathematical background. The main result and the strategy of proof is then presented in Section 3. The proof of the main result is given in Section 6. As a consequence, we obtain in Theorem 7.4 the stated result for elliptic operators. Section 8 uses this result to establish strong solutions of quasilinear elliptic equations. We discuss the optimality of our results in Section 9.

2. Extrapolation spaces and the fundamental identity

Using ideas established in [Acquistapace and Terreni 1987] and their previous works, we show that maximal L^p-regularity solutions of (NACP) satisfy a certain integral equation. It turns out that this equation is better approachable with analytic tools. We recall some basic definitions first and introduce the fundamental concept of extrapolation spaces. For $\varphi \in (0, \pi)$ we denote by $\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \varphi\}$ the sector of angle φ. If λ does not lie in the spectrum $\sigma(A) \subset \mathbb{C}$ of A, we write $R(\lambda, A) = (\lambda - A)^{-1}$ for its resolvent.

Definition 2.1. A linear operator $A : D(A) \to X$ on a Banach space X is *sectorial of angle* φ if the spectrum $\sigma(A)$ of A is contained in $\overline{\Sigma}_\varphi$ for some $\varphi \in \left(0, \frac{\pi}{2}\right)$ and if

$$\sup_{\lambda \notin \overline{\Sigma}_\varphi} (|\lambda| + 1) \|R(\lambda, A)\| < \infty.$$

A family of linear operators $A_i : D(A_i) \to X$ for $i \in I$ is *uniformly sectorial* if $\sigma(A_i) \subset \overline{\Sigma}_\varphi$ for some $\varphi \in \left(0, \frac{\pi}{2}\right)$ and all $i \in I$ and if there exists $C > 0$ with

$$\sup_{\lambda \notin \overline{\Sigma}_\varphi} (|\lambda| + 1) \|R(\lambda, A_i)\| \le C \quad \text{for all } i \in I.$$

Recall that a closed operator A is sectorial if and only if $-A$ generates an exponentially stable analytic semigroup [Engel and Nagel 2000, Chapter II, Section 4 and Chapter V, Section I]. In particular, A is invertible.

In the following we need interpolation and extrapolation spaces associated to a sectorial operator A on some Banach space X, a fully developed theory carefully presented in [Amann 1995]. We only discuss spaces associated to the complex interpolation method $[\,\cdot\,,\cdot\,]_\theta$ [Bergh and Löfström 1976, Chapter 4]. The results to be obtained hold for several other, but not all, scales of interpolation and extrapolation spaces. As a unified treatment would lead to a more abstract presentation, we focus on this important setting.

We define $X_{1,A} = D(A)$ endowed with the norm $x \mapsto \|Ax\|$ and $X_{-1,A}$ as the completion of X with respect to the norm $x \mapsto \|A^{-1}x\|$. For $\theta \in (0,1)$ we further let $X_{\theta,A} = [X, X_{1,A}]_\theta$ and $X_{-\theta,A} = [X, X_{-1,A}]_\theta$. The operator $A : X_{1,A} \to X$ and its extension $A_{-1} : X \to X_{-1,A}$ are isometries. By interpolation, for $\theta \in (0,1)$ the part $A_{-\theta}$ of A_{-1} in $X_{-\theta,A}$ is an isometry $A_{-\theta} : X_{1-\theta,A} \to X_{-\theta,A}$. The operator A_{-1} is sectorial on $X_{-1,A}$ with $\rho(A_{-1}) = \rho(A)$ and satisfies the same sectorial estimates as A. By interpolation, the same holds for the operators $A_{-\theta}$ on $X_{-\theta,A}$. Considering duality, if X is reflexive, one has $(X_{\theta,A})' \simeq X'_{-\theta,A}$ and $(A_\theta)' = A'_{-\theta}$ with respect to the pairing induced by $\langle\,\cdot\,,\cdot\,\rangle_{X,X'}$. Extrapolation spaces allow us to define a weaker notion of solution for (NACP).

Proposition 2.2. *Let $(A(t))_{t\in[0,T]}$ for $T > 0$ be uniformly sectorial operators on some Banach space X. If u is a maximal L^p-regularity solution of* (NACP) *for the initial value $u_0 = 0$ in the sense of Definition 1.1, then for every fixed $t \in [0,T]$ one has in $X_{-1,A(t)}$*

$$\begin{aligned} u(t) &= \int_0^t e^{-(t-s)A_{-1}(t)}(A_{-1}(t) - A(s))u(s)\,ds + \int_0^t e^{-(t-s)A(t)} f(s)\,ds \\ &=: \int_0^t K_1(t,s)u(s)\,ds + \int_0^t K_2(t,s) f(s)\,ds =: (S_1 u)(t) + (S_2 f)(t). \end{aligned} \tag{2-1}$$

Proof. Fix $t \in (0,T)$. Consider $v : [0,t] \to X$ given by $v(s) = e^{-(t-s)A(t)}u(s)$. Then v is differentiable in X almost everywhere and for almost every $s \in (0,t)$ we have

$$\begin{aligned}\dot{v}(s) &= A(t)e^{-(t-s)A(t)}u(s) + e^{-(t-s)A(t)}\dot{u}(s)\\ &= e^{-(t-s)A_{-1}(t)}(A_{-1}(t) - A(s))u(s) + e^{-(t-s)A(t)}f(s).\end{aligned}$$

Notice that $(A_{-1}(t) - A(s))u(s) \in X_{-1,A(t)}$ for almost every $s \in (0,T)$. The fundamental theorem of calculus gives

$$v(t) = v(0) + \int_0^t \dot{v}(s)\,ds.$$

Inserting the explicit terms for v and $\dot{v}$ and using $u(0) = 0$ yields (2-1). □

3. Formulation of the main result and strategy of proof

The crucial assumption we make is that on a certain extrapolation space the operators get independent of t. For concrete differential operators endowed with some boundary condition this is often satisfied. For this we refer to [Triebel 1978, Section 4.3] for operators with smooth coefficients and to the results originating from the positive solution of the Kato square root problem in [Auscher et al. 2002] for operators with rough coefficients (see also Section 7).

Definition 3.1. For $\theta \in [0,1]$ a family $(A(t))_{t\in[0,T]}$ of sectorial operators on some Banach space X is called *θ-stable* if there exists a Banach space $X_{\theta,A}$ and $K \geq 0$ such that for all $t \in [0,T]$ the spaces $X_{\theta,A(t)}$ and $X_{\theta,A}$ agree as vector spaces and

$$K^{-1}\|x\|_{\theta,A} \leq \|x\|_{\theta,A(t)} \leq K\|x\|_{\theta,A} \quad \text{for all } x \in X_{\theta,A} \tag{3-1}$$

and if the same also holds for some space $X_{\theta-1,A}$ and all spaces $X_{\theta-1,A(t)}$.

Note that $(A(t))_{t\in[0,T]}$ is 1-stable if and only if the domains $D(A(t))$ agree for all $t \in [0,T]$ and their norms are uniformly equivalent. Further, as already mentioned in the Introduction, even for the autonomous case $A(t) = A$, maximal L^p-regularity may fail on non-Hilbert spaces. This has to do with particular features of harmonic analysis on Banach spaces. In particular, it is by now well-understood that the classical multiplier results only hold in the vector-valued setting if one makes additional assumptions both on the Banach space and the multiplier. We now introduce the necessary background.

Definition 3.2. A Banach space X is called a *UMD space* if for one, or by Hörmander's condition all $p \in (1,\infty)$, the vector-valued Hilbert transform

$$(Hf)(x) = \lim_{\varepsilon\downarrow 0} \int_{|t|\geq\varepsilon} \frac{f(x-t)}{t}\,dt$$

initially defined on $C_c^\infty(\mathbb{R}^n; X)$ extends to a bounded operator $L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)$.

In different words, on UMD spaces one of the most basic Fourier multipliers $m(\xi) = \mathbb{1}_{\mathbb{R}_{>0}}(\xi)$ is bounded. Only on those spaces a reasonable multiplier theory can be developed. For our purposes it is sufficient to know that Hilbert and L^p-spaces for $p \in (1,\infty)$ are UMD spaces and that all UMD spaces are reflexive. For detailed information on UMD spaces we refer to [Rubio de Francia 1986; Burkholder 2001], whereas more on $\mathcal{R}$-boundedness, to be defined now, can be found in [Denk et al. 2003; Kunstmann and Weis 2004].

Definition 3.3. Let X and Y be Banach spaces. A subset $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ is called *$\mathcal{R}$-bounded* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$, $x_1, \dots, x_n \in X$ and all independent identically distributed random variables $\varepsilon_1, \dots, \varepsilon_n$ on some probability space $(\Omega, \Sigma, \mathbb{P})$ with $\mathbb{P}(\varepsilon_k = \pm 1) = \frac{1}{2}$ one has

$$\mathbb{E}\Big\| \sum_{k=1}^{n} \varepsilon_k T_k x_k \Big\|_Y \leq C\, \mathbb{E}\Big\| \sum_{k=1}^{n} \varepsilon_k x_k \Big\|_X .$$

The smallest constant $C \geq 0$ for which this holds is denoted by $\mathcal{R}(\mathcal{T})$. Further, we define $\operatorname{Rad} X$ as the closure in $L^1(\Omega, \Sigma, \mathbb{P}; X)$ of finite sums of the form $\sum_{k=1}^{n} \varepsilon_k x_k$.

Note that the definition of $\mathcal{R}$-boundedness depends only on the distribution of the random variables and is therefore independent of the probability space. The same holds for the definition of $\operatorname{Rad} X$ up to canonical isomorphisms. We write $\mathcal{R}^{X \to Y}$ to indicate between which spaces the mapping is considered if it is not clear from the context. Every $\mathcal{R}$-bounded set is bounded in $\mathcal{B}(X, Y)$. If both $X = Y$ are Hilbert spaces, then the converse holds as well. Further, Kahane's contraction principle sates that $\{z \operatorname{Id} : |z| \leq 1\}$ has $\mathcal{R}$-bound at most 2 on every Banach space. By a celebrated theorem of Weis [2001], on a UMD space the autonomous problem $A(t) = A$ has maximal L^p-regularity for one and then for all $p \in (1, \infty)$ if and only if A is $\mathcal{R}$-sectorial, the $\mathcal{R}$-boundedness analogue of sectorial operators, up to shifts.

Definition 3.4. A linear operator $A : D(A) \to X$ on a Banach space X is called *$\mathcal{R}$-sectorial of angle φ* if $\sigma(A)$ of A is contained in $\overline{\Sigma}_\varphi$ for some $\varphi \in \left(0, \frac{\pi}{2}\right)$ and if

$$\mathcal{R}\{(|\lambda| + 1) R(\lambda, A) : \lambda \notin \overline{\Sigma}_\varphi\} < \infty.$$

A family of linear operators $A_i : D(A_i) \to X$ for $i \in I$ is *uniformly $\mathcal{R}$-sectorial* if $\sigma(A_i) \subset \overline{\Sigma}_\varphi$ for some $\varphi \in \left(0, \frac{\pi}{2}\right)$ and all $i \in I$ and if there exists $C > 0$ with

$$\mathcal{R}\{(|\lambda| + 1) R(\lambda, A_i) : \lambda \notin \overline{\Sigma}_\varphi\} \leq C \quad \text{for all } i \in I.$$

The main point in our maximal L^p-regularity result is that it only assumes the operators to lie in a fractional Sobolev space.

Definition 3.5. Let X be a Banach space, $p \in (1, \infty)$ and $\alpha \in (0, 1)$. A Bochner-measurable function $f : [0, T] \to X$ lies in the *homogeneous fractional Sobolev space* $\dot{W}^{\alpha,p}([0, T]; X)$ provided

$$\|f\|_{\dot{W}^{\alpha,p}([0,T];X)} = \left(\int_0^T \int_0^T \frac{\|f(t) - f(s)\|_X^p}{|t - s|^{1+\alpha p}} \, ds \, dt \right)^{1/p} < \infty.$$

The *inhomogeneous Sobolev space* $W^{\alpha,p}([0, T]; X)$ is the space of all $f \in L^p([0, T]; X)$ such that $\|f\|_{\dot{W}^{\alpha,p}([0,T];X)} < \infty$.

We remark that there exist equivalent definitions of fractional Sobolev spaces based on Littlewood–Paley decompositions [Amann 2000, Section 3, (3.5)]. The usual embedding results for Sobolev spaces into Hölder spaces hold: for $\alpha \in (0,1)$ and $p \in (1,\infty)$ with $\alpha > \frac{1}{p}$ one has $W^{\alpha,p}([0,T];X) \hookrightarrow C^{\alpha-1/p}([0,T];X)$ [Simon 1990, Corollary 26]. We are now ready to present our general maximal L^p-regularity result that in particular implies Theorem 1.2 presented in the Introduction.

Theorem 3.6. *For $T > 0$ and $\theta \in (0,1]$ let $(A(t))_{t\in[0,T]}$ be a θ-stable family of uniformly $\mathcal{R}$-sectorial operators on some UMD space X with fractional regularity $A_{-1} \in \dot{W}^{\alpha,q}([0,T];\mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then the nonautonomous problem (NACP) has maximal L^p-regularity*

(a) *for $p \in \left(1, \frac{1}{1-\theta}\right)$, $q = \frac{1}{1-\theta}$ and $\alpha > 1-\theta$,*

(b) *for $p \in \left[\frac{1}{1-\theta}, \infty\right)$, $q = p$ and $\alpha > 1-\theta$.*

Let us compare the above conditions with the Acquistapace–Terreni condition [1987] used in [Hieber and Monniaux 2000b; Portal and Štrkalj 2006]. Apart from some uniform $\mathcal{R}$-boundedness assumptions they require that there exist constants $0 \le \gamma < \beta \le 1$ such that for all $t, s \in [0,T]$ and all $\lambda \notin \overline{\Sigma}_\varphi$ for some $\varphi \in \left(0, \frac{\pi}{2}\right)$ one has the estimate

$$\left\| A(t)R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1}) \right\|_{\mathcal{B}(X)} \lesssim \frac{|t-s|^\beta}{1+|\lambda|^{1-\gamma}}.$$

In principle, no regularity assumptions on the domain like θ-stability are made. However, in concrete examples some stability is usually necessary and one chooses $\gamma = 1-\theta$ to verify the estimate; see for example [Fackler 2015]. Then one requires $\beta > 1-\theta$ and one arrives at the usual Hölder regularity assumptions. However, for example for elliptic operators with irregular coefficients substantial effort is needed to verify the above inequality from the assumed Hölder regularity on the coefficients. Exactly this is done in [Fackler 2015], where as intermediate steps reformulations of the problem that are close to — but more general than — our setting are used.

The improvement of C^α- to $\dot{W}^{\alpha,p}$-regularity has direct consequences to applications of maximal regularity to nonlinear PDE. As one can see in Theorem 8.1 and Remark 8.2 our result gives existence results under more relaxed regularity assumptions.

Strategy of proof. In Section 4, we first show existence and uniqueness of less regular integrated solutions than is needed for maximal L^p-regularity. This can be done only assuming some continuity on the operators $A(t)$ on the extrapolation spaces. Afterwards in Section 5, we show that we can bootstrap the regularity of these solutions if the operators are α-Hölder continuous for some *arbitrarily small* exponent $\alpha > 0$. With respect to this we note that our assumptions on the fractional Sobolev space are in a such way that the fractional Sobolev space embeds into the space of α-Hölder continuous functions for *some* $\alpha > 0$. After that we show in Section 6 that this higher regularity of the solutions implies maximal L^p-regularity.

4. Existence and uniqueness of integrated solutions

In this section we show that under certain assumptions a unique solution of (2-1) exists. We next show by interpolation that, given an $\mathcal{R}$-sectorial operator, one obtains corresponding $\mathcal{R}$-boundedness estimates on the induced extrapolation spaces. The following result is not new [Haak et al. 2006, Lemma 6.9]; we give a proof for the sake of completeness. For its proof we use the fact that for an interpolation couple (X, Y) of UMD spaces we have by [Kaip and Saal 2012, Proposition 3.14]

$$[\mathrm{Rad}(X), \mathrm{Rad}(Y)]_\theta = \mathrm{Rad}([X, Y]_\theta). \tag{4-1}$$

Here one uses the facts that $[L^1(\Omega, \Sigma, \mathbb{P}; X), L^1(\Omega, \Sigma, \mathbb{P}; Y)]_\theta = L^1(\Omega, \Sigma, \mathbb{P}; [X, Y]_\theta)$ and that the $\mathrm{Rad}(X)$-spaces are complemented in the vector-valued $L^1(\Omega, \Sigma, \mathbb{P}; X)$-spaces if X is UMD.

Lemma 4.1. *Let $A : D(A) \to X$ be an $\mathcal{R}$-sectorial operator on a UMD space X. Then for all $\theta_2, \theta_1 \in [-1, 1]$ with $\theta_2 > \theta_1$ and $\theta_2 - \theta_1 \le 1$ one has with φ as in Definition 3.4 and with constants independent of A*

$$\mathcal{R}^{X_{\theta_1,A} \to X_{\theta_2,A}}\{(1 + |\lambda|)^{1-(\theta_2-\theta_1)} R(\lambda, A) : \lambda \notin \overline{\Sigma}_\varphi\} \lesssim \mathcal{R}^{X \to X}\{(|\lambda| + 1) R(\lambda, A) : \lambda \notin \overline{\Sigma}_\varphi\}.$$

Proof. The assertion holds for $\theta_1 = \theta_2 \in \{-1, 1\}$. By complex interpolation and (4-1) this extends to $\theta_1 = \theta_2 \in [-1, 1]$. Since $AR(\lambda, A) = \lambda R(\lambda, A) - \mathrm{Id}$, one has for all $\theta_1 \in [-1, 0]$

$$\mathcal{R}^{X_{\theta_1,A} \to X_{\theta_1+1,A}}\{R(\lambda, A) : \lambda \notin \overline{\Sigma}_\varphi\} < \infty.$$

For the case of general θ_2 with $\theta_2 - \theta_1 \le 1$ consider for given $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \notin -\overline{\Sigma}_\varphi$ and $x_1, \dots, x_n \in X$ the mapping $\mathcal{S} = \{z \in \mathbb{C} : \mathrm{Re}\, z \in [0, 1]\} \to \mathrm{Rad}(X_{\theta_1,A}) + \mathrm{Rad}(X_{\theta_1+1,A})$ given by

$$\mathcal{T}_z : \sum_{k=1}^n \varepsilon_k x_k \mapsto \sum_{k=1}^n \varepsilon_k (1 + \lambda_k)^z R(\lambda_k, -A) x_k.$$

The mapping $z \mapsto \mathcal{T}_z$ is continuous on $\mathcal{S}$ and analytic in the interior of $\mathcal{S}$ and it follows from Kahane's contraction principle that the norms of $\mathcal{T}_{it}$ and $\mathcal{T}_{1+it}$ as operators in $\mathcal{B}(\mathrm{Rad}(X_{\theta_1,A}), \mathrm{Rad}(X_{\theta_1+1,A}))$ and $\mathcal{B}(\mathrm{Rad}(X_{\theta_1,A}), \mathrm{Rad}(X_{\theta_1,A}))$ are bounded by $e^{|t|\varphi}$ up to a uniform constant. Hence, it follows from the generalized Stein interpolation theorem [Voigt 1992] and (4-1) that for $\alpha \in (0, 1)$

$$\mathcal{T}_\alpha : \mathrm{Rad}(X_{\theta_1,A}) \to \mathrm{Rad}(X_{\theta_1+\alpha,A}),$$

which gives the statement by unwinding the definitions of $\mathcal{R}$-boundedness. □

Remark 4.2. Curiously, the above result fails for the negative Laplacian and the real interpolation method [Haak et al. 2006, Example 6.13]. Hence, this is one step where one cannot work with arbitrary extrapolation spaces.

We establish the existence of a unique solution of (2-1) assuming Hölder regularity of arbitrarily low order.

Definition 4.3. A function $f : [0, T] \to X$ with values in some Banach space X is *α-Hölder continuous* for $\alpha \in (0, 1]$ if $\|f(t) - f(s)\| \le C|t - s|^\alpha$ for some $C \ge 0$ and all $t, s \in [0, T]$. We denote by $C^\alpha([0, T]; X)$ the space of all such functions.

We are now ready to prove the existence of integrated solutions.

Proposition 4.4. *For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t\in[0,T]}$ be a θ-stable family of uniformly $\mathcal{R}$-sectorial operators on some UMD space X. Suppose there exist $\alpha \in (0, 1]$ with $A_{-1} \in C^\alpha([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then for all $p \in (1, \infty)$ and $f \in L^p([0, T]; X)$ there exists a unique solution u of the integral equation (2-1) in $L^p([0, T]; X_{\theta,A})$. Further, one has $u \in W^{1,p}([0, T]; X_{\theta-1,A}) \cap L^p([0, T]; X_{\theta,A})$,*

$$\begin{cases} \dot{u}(t) + A_{\theta-1}(t)u(t) = f(t), \\ \qquad\qquad\qquad u(0) = 0, \end{cases} \tag{WNACP}$$

and $\|u\|_{L^p([0,T];X_{\theta,A})}$ only depends on $\|f\|_{L^p([0,T];X_{\theta-1,A})}$, T, α, θ, K in (3-1) and the constants in the Hölder and $\mathcal{R}$-sectorial estimates.

Proof. First note that by the uniform sectorial estimates and the properties of extrapolation spaces we have the uniform estimate

$$\|e^{-(t-s)A_{-1}(t)}\|_{\mathcal{B}(X_{\theta-1,A},X_{\theta,A})} \lesssim |t-s|^{-1}.$$

Using this together with the assumed Hölder regularity on $A_{-1}(\cdot)$ we get

$$\|K_1(t, s)\|_{\mathcal{B}(X_{\theta,A},X_{\theta,A})} \lesssim |t-s|^{\alpha-1}. \tag{4-2}$$

By Young's inequality for convolutions we then have the norm estimate

$$\|S_1u\|_{L^p([0,T];X_{\theta,A})} \le \int_0^T s^{\alpha-1}\, ds \|u\|_{L^p([0,T];X_{\theta,A})} = \alpha^{-1}T^\alpha \|u\|_{L^p([0,T];X_{\theta,A})}.$$

Let us show the uniqueness of solutions of (2-1) in $L^p([0, T]; X_{\theta,A})$. Since the equation is linear, it suffices to consider a solution with $u = S_1u$. Now, for sufficiently small T_0 we have $\|S_1\| < 1$. Hence, $\mathrm{Id} - S_1$ is invertible and consequently $u_{|[0,T_0]} = 0$. Using this information we see that (2-1) for $t > T_0$ reduces to

$$u(t) = \int_{T_0}^t e^{-(t-s)A_{-1}(t)}(A_{-1}(t) - A_{-1}(s))u(s)\, ds.$$

By the same argument as before we see that the operator defined by the right-hand side is bounded and invertible on $L^p([T_0, 2T_0]; X_{\theta,A})$. Hence, $u_{|[T_0,2T_0]} = 0$. Iterating this argument finitely many times gives $u = 0$.

Since $t \mapsto A_{\theta-1} \in \mathcal{B}(X_{\theta,A}, X_{\theta-1,A})$ is a fortiori continuous, it follows from perturbation arguments and Lemma 4.1 that (WNACP) has nonautonomous maximal L^p-regularity for all $p \in (1, \infty)$; see [Prüss and Schnaubelt 2001, Theorem 2.5; Arendt et al. 2007, Theorem 2.7]. This means there exists a unique $w \in W^{1,p}([0, T]; X_{\theta-1,A}) \cap L^p([0, T]; X_{\theta,A})$ satisfying (WNACP) and the corresponding maximal L^p-regularity estimate. Using the same argument as in Proposition 2.2, we see that w satisfies (2-1). By the uniqueness shown in the first part, we have $w = u$. □

5. Bootstrapping regularity

Again, assuming Hölder regularity of arbitrarily small order, we improve the regularity of the obtained integrated solutions with the help of the following bootstrapping result.

Proposition 5.1. *For $T > 0$ and $\theta \in (0, 1]$ let $(A(t))_{t\in[0,T]}$ be a θ-stable family of uniformly sectorial operators on some Banach space X satisfying $A_{-1} \in C^{\alpha}([0, T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$ for some $\alpha \in (0, 1]$. If either*

(a) $p \in \big(\frac{1}{1-\theta}, \infty\big)$ *and* $q \in (1, \infty]$, *or*

(b) $p = \frac{1}{1-\theta}$ *and* $q \in (1, \infty)$, *or*

(c) $p \in \big(1, \frac{1}{1-\theta}\big)$ *and* $q \in \big(1, \frac{p}{1-p(1-\theta)}\big]$,

then there exists $C_{pq} > 0$ depending only on T, K in (3-1) and the constants of the sectorial and Hölder estimates such that for all solutions $u \in L^p([0, T]; X_{\theta,A})$ of (2-1) for some right-hand side $f \in L^p([0, T]; X)$ one has

$$\|u\|_{L^q([0,T];X_{\theta,A})} \le C_{pq}(\|u\|_{L^p([0,T];X_{\theta,A})} + \|f\|_{L^p([0,T];X)}).$$

Proof. By Young's inequality for convolutions and the kernel estimate (4-2) we have for $q, p, r \in (1, \infty)$ with $\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}$ the estimate

$$\begin{aligned}\bigg(\int_0^T \|(S_1u)(t)\|^q_{X_{\theta,A}}\,dt\bigg)^{1/q} &\le \bigg(\int_0^T \bigg(\int_0^t (t-s)^{\alpha-1}\|u(s)\|_{X_{\theta,A}}\,ds\bigg)^q dt\bigg)^{1/q}\\ &\lesssim \|s \mapsto s^{\alpha-1}\|_{L^{r,\infty}}\bigg(\int_0^T \|u(s)\|^p_{X_{\theta,A}}\,ds\bigg)^{1/p}.\end{aligned}$$

The weak L^r norm is finite for $r \in \big(1, \frac{1}{1-\alpha}\big]$. Hence, S_1 is a bounded operator $L^p([0, T]; X_{\theta,A}) \to L^q([0, T]; X_{\theta,A})$ for all $p \in \big(1, \frac{1}{\alpha}\big)$ and $q \in \big[1, \frac{p}{1-p\alpha}\big]$. If $p > \frac{1}{\alpha}$, then

$$\begin{aligned}\|(S_1u)(t)\|_{X_{\theta,A}} &\le \bigg(\int_0^t \|K_1(t,s)\|^{p'}\,ds\bigg)^{1/p'}\bigg(\int_0^t \|u(s)\|^p_{X_{\theta,A}}\,ds\bigg)^{1/p}\\ &\le \bigg(\int_0^t |t-s|^{p'(\alpha-1)}\,ds\bigg)^{1/p'}\|u\|_{L^p([0,T];X_{\theta,A})}.\end{aligned}$$

Hence, $S_1 : L^p([0, T]; X_{\theta,A}) \to L^\infty([0, T]; X_{\theta,A})$ is bounded for $p > \frac{1}{\alpha}$.

Interpolating the analytic estimate

$$\|e^{-(t-s)A(t)}\|_{\mathcal{B}(X,D(A(t)))} \lesssim |t-s|^{-1}$$

with the boundedness of the semigroups $\|e^{-(t-s)A(t)}\|_{\mathcal{B}(X)} \lesssim 1$, one sees that the kernel of S_2 satisfies

$$\|K_2(t,s)\|_{\mathcal{B}(X,X_{\theta,A(t)})} = \|e^{-(t-s)A(t)}\|_{\mathcal{B}(X,X_{\theta,A(t)})} \lesssim |t-s|^{-\theta}. \tag{5-1}$$

Using Young's inequality together with the kernel estimate (5-1) and θ-stability, we obtain for $p, q, r \in (1, \infty)$ with $\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}$ the estimate

$$\begin{aligned}\bigg(\int_0^T \|(S_2f)(t)\|^q_{X_{\theta,A}}\,dt\bigg)^{1/q} &\lesssim \bigg(\int_0^T \bigg(\int_0^t (t-s)^{-\theta}\|f(s)\|_X\,ds\bigg)^q dt\bigg)^{1/q}\\ &\lesssim \|s \mapsto s^{-\theta}\|_{L^{r,\infty}}\bigg(\int_0^T \|f(s)\|^p_X\,ds\bigg)^{1/p}.\end{aligned}$$

This time the $L^{r,\infty}$-norm is finite for $r \in (1, \theta^{-1}]$. Hence, $S_2 : L^p([0,T]; X) \to L^q([0,T]; X_{\theta,A})$ is bounded for all $p < \frac{1}{1-\theta}$ and $q \in \big[1, \frac{p}{1-p(1-\theta)}\big]$. Further, one has $S_2 : L^p([0,T]; X) \to L^\infty([0,T]; X_{\theta,A})$ for $p > \frac{1}{1-\theta}$. For the stated result, we iterate the obtained regularity improvement finitely often to bootstrap the regularity of u. □

6. Maximal regularity results under fractional Sobolev regularity

In this section we come to the heart of the proof. To the solution obtained in Proposition 4.4 we apply $A_{-1}(t)$ to both sides of (2-1). This gives $A_{-1}(t)u(t) = A_{-1}(t)(S_1 u)(t) + A_{-1}(t)(S_2 f)(t)$. We show that both summands lie in $L^p([0,T]; X)$. The second summand requires some preliminary work. We rely on the following Hölder continuity of the $\mathcal{R}$-boundedness constant.

Lemma 6.1. *For $\theta \in (0,1]$ let $(A(t))_{t\in\mathbb{R}}$ be a θ-stable family of uniformly $\mathcal{R}$-sectorial operators on some UMD space X. Suppose there exists $\alpha \in (0,1]$ with $A_{-1} \in C^\alpha([0,T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then for all $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$ depending only on K in (3-1) and the constants in the Hölder and $\mathcal{R}$-sectorial estimate of Definition 3.4 such that for all $t, h \in \mathbb{R}$*

$$\mathcal{R}^{X\to X}\left\{(1+|\xi|)^k \left(\frac{\partial}{\partial \xi}\right)^k \big[i\xi(R(i\xi, A(t+h)) - R(i\xi, A(t)))\big] : \xi \in \mathbb{R}\right\} \le C_k |h|^\alpha.$$

Proof. We first establish the case $k = 0$. For all $t, h \in \mathbb{R}$ the resolvent identity gives

$$R(i\xi, A(t+h)) - R(i\xi, A(t)) = R(i\xi, A_{-1}(t+h))[A_{-1}(t) - A_{-1}(t+h)]R(i\xi, A(t)).$$

By the assumed Hölder regularity on A_{-1} and Lemma 4.1 we get for all $t, h \in \mathbb{R}$

$$\begin{aligned}
&\mathcal{R}^{X\to X}\{i\xi(R(i\xi, A(t+h)) - R(i\xi, A(t)))\} \\
&\quad \lesssim \mathcal{R}^{X_{\theta-1,A}\to X}\{(1+|\xi|)^\theta R(i\xi, A_{-1}(t+h))\} \|A_{-1}(t+h) - A_{-1}(t)\|_{\mathcal{B}(X_{\theta,A}, X_{\theta-1,A})} \\
&\qquad\qquad \times \mathcal{R}^{X\to X_{\theta,A}}\{(1+|\xi|)^{1-\theta} R(i\xi, A(t))\} \\
&\quad \lesssim |h|^\alpha.
\end{aligned}$$

For the case $k \ge 1$ notice that the map $S : z \mapsto R(z, A(t+h)) - R(z, A(t)) \in \mathcal{B}(X)$ is analytic on the complement of some shifted sector $\Sigma_\varphi + \varepsilon$ and that the above estimate holds there by the same argument. It follows from the Cauchy integral representation of derivatives [Kunstmann and Weis 2004, Example 2.16] that for $S(z) = z(R(z, A(t+h)) - R(z, A(t)))$

$$\mathcal{R}\left\{(1+|z|)^k \left(\frac{d}{dz}\right)^k S(z) : z \notin \Sigma_\varphi\right\} \lesssim \mathcal{R}\left\{S\left(i\xi + \frac{\varepsilon}{2}\right) : \xi \in \mathbb{R}\right\} \lesssim |h|^\alpha. \qquad \square$$

Proposition 6.2. *For $T > 0$ and $\theta \in (0,1]$ let $(A(t))_{t\in\mathbb{R}}$ be a θ-stable family of uniformly $\mathcal{R}$-sectorial operators on some UMD space X. Suppose there exists $\alpha \in (0,1]$ with $A_{-1} \in C^\alpha([0,T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then $A(\cdot)S_2 : L^p([0,T]; X) \to L^p([0,T]; X)$ is bounded for all $p \in (1,\infty)$ and its norm only depends on p, K in (3-1) and the constants in the Hölder and $\mathcal{R}$-sectorial estimates.*

Proof. It is shown in [Hieber and Monniaux 2000b, p. 1053; Fackler 2015, Section 2.4.1] that the boundedness of $A(\cdot)S_2$ follows from the boundedness of the pseudodifferential operator

$$(\widehat{S}f)(t) = \int_{-\infty}^{\infty} a(t,\xi)\hat{f}(\xi)e^{2\pi i t\xi}\,d\xi$$

for the operator-valued symbol $a : \mathbb{R}\times\mathbb{R} \to \mathcal{B}(X)$ given by

$$a(t,\xi) = \begin{cases} i\xi R(i\xi, A(0)), & t < 0, \\ i\xi R(i\xi, A(t)), & t \in [0,T], \\ i\xi R(i\xi, A(T)), & t > T. \end{cases}$$

Such operators are well-studied and understood. Applying [Hytönen and Portal 2008, Theorem 17] and [Hytönen and Portal 2008, Remark 20] (the dependence on the constants is not explicitly stated) in the one-dimensional and one-parameter case, we see that $S : L^p([0,T];X) \to L^p([0,T];X)$ is bounded for all $p \in (1,\infty)$ provided

$$\mathcal{R}\left\{(1+|\xi|)^k\left(\frac{\partial}{\partial\xi}\right)^k\big[a(t+h,\xi)-a(t,\xi)\big] : \xi \in \mathbb{R}\right\} \lesssim |h|^\alpha$$

holds for some $\alpha \in (0,1]$ and all $k = 0,1,2$. This is the $\mathcal{R}$-analogue of the condition considered by Yamazaki [1986] and therefore called an $\mathcal{R}$-Yamazaki symbol. The fact that a is indeed an $\mathcal{R}$-Yamazaki symbol has been verified in Lemma 6.1. □

The next proposition shows that in many cases it is sufficient to show maximal L^p-regularity for initial value zero. This is well known in the autonomous case. The arguments have been used before; see for example [Dier and Zacher 2017, Theorem 6.2].

Proposition 6.3. *Let X be a Banach space, $p \in (1,\infty)$, $T > 0$ and $(A(t))_{t\in[0,T]}$ a family of uniformly sectorial operators*:

(a) *Suppose that the nonautonomous operator $(B(t))_{t\in[0,T+1]}$,*

$$B(t) = \begin{cases} A(0) & \text{for } t \in [0,1], \\ A(t-1) & \text{for } t \in [1,T+1], \end{cases}$$

has maximal L^p-regularity for $u_0 = 0$. Then $(A(t))_{t\in[0,T]}$ has maximal L^p-regularity for all initial values $u_0 \in (D(A(0)),X)_{1/p,p}$. Further, the maximal regularity estimate only additionally depends on a constant controlled by the sectorial estimate for $A(0)$.

(b) *Suppose additionally that for all $t_0 \in (0,T]$ the nonautonomous problem associated to $(C_{t_0}(t))_{t\in[0,t_0+2]}$, where*

$$C_{t_0}(t) = \begin{cases} A(0) & \text{for } t \in [0,1], \\ A(t-1) & \text{for } t \in [1,1+t_0], \\ A(t_0) & \text{for } t \in [1+t_0, 2+t_0], \end{cases}$$

has maximal L^p-regularity for $u_0 = 0$. Then the unique solution of (NACP) for $(A(t))_{t\in[0,T]}$ satisfies $u(t) \in (D(A(t)),X)_{1/p,p}$ for all $t \in [0,T]$ and $u_0 \in (D(A(0)),X)_{1/p,p}$.

Proof. We start with the first part. By the characterization of real interpolation spaces via the trace method [Lunardi 1995, Proposition 1.2.10] and a cut-off argument, there is some $C > 0$ such that for all $u_0 \in (D(A(0)), X)_{1/p,p}$ there exists $v \in W^{1,p}([0,1]; X) \cap L^p([0,1]; D(A(0)))$ with $v(0) = 0$, $v(1) = u_0$ and

$$\|A(0)v\|_{L^p([0,1];X)} + \|\dot{v}\|_{L^p([0,1];X)} \le C\|u_0\|_{(D(A(0)),X)_{1/p,p}}.$$

For given $f \in L^p([0,T]; X)$ we define $g \in L^p([0,T+1]; X)$ as

$$g(t) = \begin{cases} \dot{v}(t) + A(0)v(t) & \text{for } t \in [0,1), \\ f(t-1) & \text{for } t \in [1, T+1]. \end{cases}$$

By assumption $(B(t))_{t\in[0,T+1]}$ has maximal L^p-regularity for $u_0 = 0$. We denote by w the unique solution of (NACP) for $(B(t))_{t\in[0,T+1]}$ with right-hand side g. By the uniqueness of mild solutions in the autonomous case we have $w = v$ on $[0,1]$. In particular, we have $w(1) = v(1) = u_0$. As a consequence we see that $u(t) = w(t+1)$ solves (NACP) for $u(0) = w(1) = u_0$. Further,

$$\begin{aligned} \|u\|_{W^{1,p}([0,T];X)} + \|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} &\lesssim \|g\|_{L^p([0,T+1];X)} \\ &\lesssim \|f\|_{L^p([0,T];X)} + \|u_0\|_{(D(A(0)),X)_{1/p,p}}. \end{aligned}$$

For the uniqueness observe that a second solution $\tilde{u}$ of (NACP) with right-hand side f and $u(0) = u_0$ yields a solution $z = (\dot{v} + A(0)v)\mathbb{1}_{[0,1)} + \tilde{u}(\cdot - 1)\mathbb{1}_{[1,t_0+1)}$ of (NACP) for $(B(t))_{t\in[0,T+1]}$ that agrees with $u(\cdot - 1)$ on $[1, T+1]$ by the uniqueness of solutions.

For the second part and fixed $t_0 \in (0, T]$ let z be the solution of (NACP) for $(C_{t_0}(t))_{t\in[0,t_0+2]}$ and the right-hand side $\tilde{g} = g\,\mathbb{1}_{[0,t_0+1]}$. Then z agrees with the solution w of the first part on $[0, t_0+1]$ and solves the autonomous problem $\dot{z}(s) + A(t_0)z(s) = 0$ on $[t_0+1, t_0+2]$. Since functions in $W^{1,p}([t_0+1, t_0+2]; X) \cap L^p([t_0+1, t_0+2]; D(A(t_0)))$ take values in the corresponding trace spaces [Amann 1995, Theorem III.4.10.2], we have $u(t_0) \in (D(A(t_0)), X)_{1/p,p}$. ☐

We are now ready to prove our general maximal regularity result.

Theorem 6.4. *For $T > 0$ and $\theta \in (0,1]$ let $(A(t))_{t\in[0,T]}$ be a θ-stable family of uniformly $\mathcal{R}$-sectorial operators on some UMD space X with fractional regularity $A_{-1} \in \dot{W}^{\alpha,q}([0,T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. Then the nonautonomous problem (NACP) has maximal L^p-regularity*

(a) *for $p \in \left(1, \frac{1}{1-\theta}\right)$, $q = \frac{1}{1-\theta}$ and $\alpha > 1-\theta$,*

(b) *for $p \in \left[\frac{1}{1-\theta}, \infty\right)$, $q = p$ and $\alpha > 1-\theta$.*

In this case the unique maximal L^p-regularity solution u of (NACP) satisfies $u(t) \in (D(A(t)), X)_{1/p,p}$ for all $t \in [0,T]$ and there exists a constant $C_p > 0$ with

$$\|u\|_{W^{1,p}([0,T];X)} + \|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} \le C(\|f\|_{L^p([0,T];X)} + \|u_0\|_{(D(A(0)),X)_{1/p,p}}),$$

which only depends on T, α, θ, K in (3-1), $\|A_{-1}\|_{\dot{W}^{\alpha,q}([0,T];\mathcal{B}(X_{\theta,A},X_{\theta-1,A}))}$ and the constants in the $\mathcal{R}$-sectorial estimates.

Proof. First note that under the made regularity assumptions, we have $A_{-1} \in C^{\gamma}([0,T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$ for some $\gamma > 0$. Further, let $u \in W^{1,p}([0,T]; X_{\theta-1,A}) \cap L^p([0,T]; X_{\theta,A})$ be the unique solution of (2-1) given by Proposition 4.4. We show that u has the higher regularity $A_{-1}(t)u(t) \in L^p([0,T]; X)$. For this we use the decomposition of $A_{-1}(t)u(t)$ given by (2-1).

Let us start with the integrability of $A_{-1}(t)(S_1 u)(t)$. We will omit subindices in the following estimates. For $g \in L^{p'}([0,T]; X')$ we have, where $A'(t)$ is the adjoint,

$$\int_0^T \int_0^t \langle g(t), A(t)e^{-(t-s)A(t)}(A(t)-A(s))u(s)\rangle_{X',X} \, ds \, dt \\ = \int_0^T \int_0^t \langle A'(t)e^{-(t-s)A'(t)} g(t), (A(t)-A(s))u(s)\rangle_{X'_{1-\theta,A'(t)}, X_{\theta-1,A(t)}} \, ds \, dt. \quad (6\text{-}1)$$

We now distinguish between the cases $p \in \big(\frac{1}{1-\theta}, \infty\big)$, $p = \frac{1}{1-\theta}$ and $p \in \big(1, \frac{1}{1-\theta}\big)$. In the first case we know from Proposition 5.1 that $u \in L^\infty([0,T]; X_{\theta,A})$. Hence, up to constants (6-1) is dominated by

$$\left(\int_0^T \int_0^T \frac{\|(A(t)-A(s))u(s)\|^p_{X_{\theta-1,A}}}{|t-s|^{1+p\alpha}} \, ds \, dt\right)^{1/p} \\ \times \left(\int_0^T \int_0^t \|A'(t)e^{-(t-s)A'(t)} g(t)\|^{p'}_{X_{1-\theta,A'(t)}} |t-s|^{p'(1/p+\alpha)} \, ds \, dt\right)^{1/p'} \\ \lesssim \|A\|_{\dot{W}^{\alpha,p}} \|u\|_{L^\infty([0,T];X_{\theta,A})} \left(\int_0^T \int_0^t (t-s)^{p'(1/p+\alpha+\theta-2)} \, ds \|g(t)\|^{p'}_{X'} \, dt\right)^{1/p'}.$$

The inner integral is finite because of the assumption $\alpha > 1-\theta$. Since $g \in L^{p'}([0,T]; X')$ is arbitrary, we get $A_{-1}(\cdot)S_1 u \in L^p([0,T]; X)$. The case $p = \frac{1}{1-\theta}$ follows similarly, using $u \in L^{q'}([0,T]; X_{\theta,A})$ for some big q' and the fact that the condition $\alpha > 1-\theta$ leaves a little room. Let us come to the case $p \in \big(1, \frac{1}{1-\theta}\big)$. Here Proposition 5.1 shows that $u \in L^{p/(1-p(1-\theta))}([0,T]; X_{\theta,A})$. Hence, using Hölder's inequality, for $\beta > 0$ the expression in (6-1) is dominated by

$$\left(\int_0^T \int_0^T \frac{\|A(t)-A(s)\|^{1/(1-\theta)}_{\mathcal{B}(X_{\theta,A}, X_{\theta-1,A})}}{|t-s|^{1+\alpha(1-\theta)^{-1}}} \, ds \, dt\right)^{1-\theta} \left(\int_0^T \int_0^t (t-s)^{p'(\alpha+\beta-1)} \, ds \|g(t)\|^{p'}_{X'} \, dt\right)^{1/p'} \\ \times \left(\int_0^T \int_s^T (t-s)^{-\beta p/(1-p(1-\theta))} \, dt \|u(s)\|^{p/(1-p(1-\theta))}_{X_{\theta,A}} \, ds\right)^{1/p-(1-\theta)}.$$

The last integral is finite for $\beta < \theta - \frac{1}{p'}$. Since $\alpha > 1-\theta$, we can find $\beta \in \big(0, \theta - \frac{1}{p'}\big)$ for which the second integral is finite as well.

Further, $A_{-1}(\cdot)(S_2 f)(\cdot)$ lies in $L^p([0,T]; X)$ by Proposition 6.2. This shows that the solution satisfies $u(t) \in D(A(t))$ for almost all $t \in [0,T]$ and $A(\cdot)u(\cdot) \in L^p([0,T]; X)$. Since u solves (WNACP), it follows that $\dot{u} \in L^p([0,T]; X)$. This shows maximal L^p-regularity in the case $u_0 = 0$. It remains to verify the maximal regularity estimate. By the estimates obtained in the first part of the proof we have for some case-dependent $q' \in (p, \infty]$

$$\begin{aligned}\|A(\cdot)u(\cdot)\|_{L^p([0,T];X)} &= \|A_{-1}(\cdot)u(\cdot)\|_{L^p([0,T];X)} \\ &\le \|A_{-1}(\cdot)(S_1 u)(\cdot)\|_{L^p([0,T];X)} + \|A_{-1}(\cdot)(S_2 f)(\cdot)\|_{L^p([0,T];X)} \\ &\lesssim \|A\|_{\dot{W}^{\alpha,q}} \|u\|_{L^{q'}([0,T];X_{\theta,A})} + \|f\|_{L^p([0,T];X)} \\ &\lesssim C_{pq'} \|A\|_{\dot{W}^{\alpha,q}} (\|u\|_{L^p([0,T];X_{\theta,A})} + \|f\|_{L^p([0,T];X)}) + \|f\|_{L^p([0,T];X)} \\ &\lesssim C_{pq'} \|A\|_{\dot{W}^{\alpha,q}} \|f\|_{L^p([0,T];X)} + \|f\|_{L^p([0,T];X)}.\end{aligned}$$

Here we have used the estimates obtained in the first part of the proof, Proposition 5.1 and Proposition 4.4 in the third, fourth and fifth lines respectively. Since u solves (NACP) and the operators $(A(t))_{t\in[0,T]}$ are uniformly sectorial, this implies the maximal regularity estimate for $u_0 = 0$.

The case of general initial values $u_0 \in (D(A(0)), X)_{1/p,p}$ follows from Proposition 6.3. Here we use the fact that for $q > \alpha^{-1}$ functions in $\dot{W}^{\alpha,q}$ can be extended with the same regularity by their values at the endpoints [Dier and Zacher 2017, Proposition 7.8]. □

Remark 6.5. Compared to the result in [Portal and Štrkalj 2006] we need a weaker $\mathcal{R}$-boundedness result. Further, the time regularity is lowered to some fractional Sobolev space at the cost of more regularity on the domain spaces. In order to obtain maximal L^p-regularity for all $p \in [(1-\theta)^{-1}, \infty)$ our result requires $A_{-1} \in \bigcap_{p\in[(1-\theta)^{-1},\infty)} \bigcup_{\varepsilon>0} \dot{W}^{1-\theta+\varepsilon,p}([0,T]; \mathcal{B}(X_{\theta,A}, X_{\theta-1,A}))$. This is slightly less restrictive than the α-Hölder continuity for some $\alpha > 1-\theta$ assumed usually.

For nonautonomous problems given by sesquilinear forms on Hilbert spaces one obtains by the same line of thought the following improvement of [Dier and Zacher 2017], where only the case $p = 2$ was treated. Let us shortly recall how the form setting is related to the general setting considered by us. Given, as in (1-1), a coercive, bounded nonautonomous sesquilinear form on some Hilbert space V one gets operators $\mathcal{A}(t) : V \to V'$ with $\mathcal{A}(t)u = a(t, u, \cdot)$. Given a second Hilbert space with dense embedding $V \hookrightarrow H$ and the associated triple $V \hookrightarrow H \hookrightarrow V'$ one considers their restrictions $A(t)$ on H, i.e., $D(A(t)) = \{u \in V : \mathcal{A}(t)u \in H\}$. One then obtains an associated problem (NACP) for $(A(t))_{t\in[0,T]}$ on H. The spaces V and V' can be seen as replacements of $X_{1/2,A}$ and $X_{-1/2,A}$. Hence, $(A(t))$ is $\frac{1}{2}$-stable in some sense.

Corollary 6.6. *Let V, H be Hilbert spaces with dense embedding $V \hookrightarrow H$ and let $a : [0,T] \times V \times V \to \mathbb{C}$ be a coercive, bounded nonautonomous sesquilinear form as in (1-1). Then the associated problem (NACP) on H has maximal L^p-regularity*

(a) *for $p \in (1,2]$ provided $\mathcal{A} \in \dot{W}^{1/2+\varepsilon,2}([0,T]; \mathcal{B}(V, V'))$ for some $\varepsilon > 0$,*

(b) *for $p \in [2,\infty)$ provided $\mathcal{A} \in \dot{W}^{1/2+\varepsilon,p}([0,T]; \mathcal{B}(V, V'))$ for some $\varepsilon > 0$.*

The constants in the maximal L^p-regularity estimate only depend on T, ε, the constants α, M in (1-1) and the fractional Sobolev norm of $\mathcal{A}$.

Proof. Repeat the previous proof for $X = H$ and replace $X_{1/2,A}$ and $X_{-1/2,A}$ with V and V'. □

Note that V and V' only agree with the complex interpolation spaces $X_{1/2,A(t)}$ and $X_{-1/2,A(t)}$ if the operators $A(t)$ satisfy the so-called Kato square root property; see [Auscher 2002] for a short introduction to this topic. However, this is not necessary to carry out the argument. In the UMD setting the case $\theta = \frac{1}{2}$ is also of particular interest. We obtain the following corollary relevant for concrete applications (which holds for other values of θ as well).

Corollary 6.7. *Let $T > 0$ and $(A(t))_{t\in[0,T]}$ be uniformly sectorial on a UMD space X such that for some $\omega \in \left(0, \frac{\pi}{2}\right)$ and $M > 0$ the imaginary powers satisfy*

$$\|A(t)^{is}\| \leq Me^{\omega|s|}$$

uniformly for all $t \in [0, T]$ and $s \in \mathbb{R}$. Further, suppose that there exist Banach spaces $X_{1/2}$ and $X_{-1/2}$ for which for all $t \in [0, T]$ the spaces $D(A(t)^{1/2})$ and $D(A(t)^{-1/2})$ agree with $X_{1/2}$ and $X_{-1/2}$ as vector spaces and the respective norms are uniformly equivalent for some constant $K > 0$. Then the nonautonomous Cauchy problem (NACP) for $(A(t))_{t\in[0,T]}$ has maximal L^p-regularity

(a) *for $p \in (1, 2]$ if $A_{-1} \in \dot{W}^{1/2+\varepsilon,2}([0, T]; \mathcal{B}(X_{1/2}, X_{-1/2}))$ for some $\varepsilon > 0$,*

(b) *for $p \in [2, \infty)$ if $A_{-1} \in \dot{W}^{1/2+\varepsilon,p}([0, T]; \mathcal{B}(X_{1/2}, X_{-1/2}))$ for some $\varepsilon > 0$.*

The constants in the maximal L^p-regularity estimates only depend on p, T, ε, K in (3-1), M, ω, the fractional Sobolev norm of A_{-1} and the constants in the sectorial estimates.

Proof. Since the operators $A(t)$ have uniformly bounded imaginary powers, it follows from [Denk et al. 2003, Theorem 4.5] that for $\varphi \in (\omega, \pi)$

$$\sup_{t\in[0,T]} \mathcal{R}\{\lambda R(\lambda, A(t)) : \lambda \notin \overline{\Sigma}_\varphi\} < \infty.$$

Since uniformly bounded analytic families are uniformly $\mathcal{R}$-bounded on compact subsets of a common domain [Weis 2001, Proposition 2.6], the operators $(A(t))_{t\in[0,T]}$ are uniformly $\mathcal{R}$-sectorial. Further, the fractional domains spaces $D(A(t)^{1/2})$ and $D(A(t)^{-1/2})$ are uniformly equivalent to $X_{1/2,A(t)}$ and $X_{-1/2,A(t)}$ [Fackler 2015, Proposition 2.5]. As a consequence of the assumptions, the family $(A(t))_{t\in[0,T]}$ is $\frac{1}{2}$-stable. This means that we can apply Theorem 6.4. □

Remark 6.8. Corollary 6.7 holds under the slightly weaker assumption that the operators $(A(t))_{t\in[0,T]}$ are uniformly $\mathcal{R}$-sectorial. For this one uses the scale $X_{\theta,A} = D(A^\theta)$ for $|\theta| \in (0, 1)$ and repeats the proof of Theorem 6.4. The main difference is that one has to use [Haak et al. 2006, Lemma 6.9(1)] instead of Lemma 4.1.

7. Nonautonomous maximal regularity for elliptic operators

We now illustrate the consequences of our results for nonautonomous problems governed by elliptic operators in divergence form. We concentrate on pure second-order operators with VMO-coefficients subject to Dirichlet boundary conditions, as the used results are already involved and spread over the literature in this special case. On a bounded domain $\Omega \subset \mathbb{R}^n$ we consider bounded measurable coefficients $A = (a_{ij}) : \Omega \to \mathbb{C}^{n\times n}$ and the bounded sesquilinear form

$$a : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega A\nabla u \overline{\nabla v}.$$

Further, we assume that (a_{ij}) satisfies for some $\delta > 0$ and all $\xi \in \mathbb{C}^n$ the estimate

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \geq \delta|\xi|^2.$$

Then the operator L_2 on $L^2(\Omega)$ associated to a is sectorial. Further, one has for $u \in D(L_2) \subset W_0^{1,2}(\Omega)$ the identity $L_2 u = -\operatorname{div}(A\nabla u)$ in the sense of distributions. One can show that if Ω has C^1-boundary and if the coefficients lie in VMO, then L_2 induces for all $q \in (1, \infty)$ compatible sectorial operators L_q on $L^q(\Omega)$ (see the proof of Theorem 7.2). These operators are realizations of $-\operatorname{div}(A\nabla\,\cdot)$ on $L^q(\Omega)$. For further details on the form method we refer to [Ouhabaz 2005].

Definition 7.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A bounded measurable function $f : \Omega \to \mathbb{C}$ is of *vanishing mean oscillation* if one has $\inf_{r>0} \eta_f(r) = 0$ for the modulus

$$\eta_f(r) := \sup_{B:d(B)\leq r} \left(\frac{1}{|B\cap\Omega|} \int_{B\cap\Omega} |f(x) - f_{B\cap\Omega}|^2\, dx \right)^{1/2},$$

where $f_{\Omega\cap B}$ denotes the mean of f over $B\cap\Omega$ and the supremum is taken over all balls $B \subset \mathbb{R}^n$ centered in Ω whose diameter $d(B)$ does not exceed r.

We need the following variant of the Kato square root property on $L^q(\Omega)$.

Theorem 7.2. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain, $q \in (1, \infty)$ and $A = (a_{ij})_{1\leq i,j\leq n} \in L^\infty(\Omega; \mathbb{C}^{n\times n})$ be complex-valued coefficients with*

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \geq \delta|\xi|^2 \quad \textit{for all } \xi \in \mathbb{C}^n,$$

for some $\delta > 0$ and almost every $x \in \Omega$. Let L_q be the realization of $-\operatorname{div}(A\nabla\,\cdot)$ on $L^q(\Omega)$ subject to Dirichlet boundary conditions. If $a_{ij} \in \mathrm{VMO}(\Omega)$ for all $i, j = 1, \dots, n$, then there exists $\lambda_0 \geq 0$ such that the following holds:

(a) *$L_q + \lambda$ is a sectorial operator on $L^q(\Omega)$ for all $\lambda \geq \lambda_0$ and*

$$\|f\|_q + \|\nabla f\|_q \simeq \|(L_q + \lambda)^{1/2} f\|_q \quad \textit{for all } f \in W_0^{1,q}(\Omega).$$

(b) *The operator L_q extends to an isomorphism $W_0^{1,q}(\Omega) \xrightarrow{\sim} W^{-1,q}(\Omega)$.*

The constant λ_0 only depends on Ω, q, $\eta_{a_{ij}}$, δ and $\|A\|_\infty$. With an additional dependence on λ, the same holds for the constant in the equivalence in (a), the isomorphism in (b) and the sectorial estimates of the operators $L_q + \lambda$.

Proof. Under the made assumptions, the operator L_2 satisfies local Gaussian estimates [Auscher and Tchamitchian 2001a, Theorem 7]. Although not explicitly stated, the coefficients in the estimate only depend on the claimed constants. This has several consequences. First, for λ sufficiently large the operator

$L_2 + \lambda$ satisfies global Gaussian estimates [Auscher and Tchamitchian 1998, Section 1.4.5, Theorem 18] and extends to a sectorial operator $L_q + \lambda$ on $L^q(\Omega)$. Secondly, it essentially follows from [Auscher and Tchamitchian 2001b, Theorem 4] that $\|(L_q + \lambda)^{1/2}\|_q \lesssim \|f\|_q + \|\nabla f\|_q$. Here are two additional points to consider. First, the theorem is only stated in the case $\lambda = 0$. The case $\lambda \neq 0$ can be obtained by including terms of lower order in the argument or by arguing as in [Auscher and Tchamitchian 1998, p. 135]. The second point is the not explicitly stated dependence on the constants. However, taking a close look at the proof in [Auscher and Tchamitchian 2001b] one sees that most auxiliary results give the explicit dependence on the constants (in [Auscher and Tchamitchian 2001b, p. 162] such a dependence is explicitly stated in a special case). One crucial point needed here is the dependence in the case $p = 2$, which is well known. This can be found in [Axelsson et al. 2006, Theorem 1] for a broad class of Lipschitz domains and a combination of [Egert et al. 2014, Theorem 4.2; 2016, Theorems 3.1 and 3.3 and Section 6] yields the dependence for general bounded Lipschitz domains and therefore a fortiori for C^1-domains.

Now, as in [Auscher and Tchamitchian 1998, p. 135], the converse inequality follows if $(L_q + \lambda)^{-1}$ extends to a bounded operator from $W^{-1,q}(\Omega) = (W_0^{1,q'}(\Omega))'$ into $W_0^{1,q}(\Omega)$. Notice that

$$\|u\|_{W^{-1,q}(\Omega)} = \inf\left\{ \|g\|_q + \sum_{k=1}^{n} \|F_k\|_q : g, F_k \in L^q(\Omega) \text{ with } u = g + \operatorname{div} F \right\}.$$

It is shown in [Dong and Kim 2010, Theorem 4] that for $\lambda \geq 0$ there exists $C \geq 0$ such that for all $F_k, g \in L^q(\Omega)$ there is a unique $u \in W_0^{1,q}(\Omega)$ with $-\operatorname{div}(A\nabla u) + \lambda u = g + \operatorname{div} F$ and

$$\|u\|_{W^{1,q}(\Omega)} \leq C\left(\|g\|_q + \sum_{k=1}^{n} \|F_k\|_q \right).$$

Here, our required dependence on the constants can be found in the lemmata in [Dong and Kim 2010, Section 7]. Note that the above estimate is exactly the boundedness of $(\lambda + L_q)^{-1} : W^{-1,q}(\Omega) \to W_0^{1,q}(\Omega)$, which is a uniform isomorphism by the uniqueness of $u \in W_0^{1,q}(\Omega)$. □

Remark 7.3. The estimate $\|L^{1/2} f\|_q \lesssim \|\nabla f\|_q$ is known under more general assumptions on the coefficients and the domain [Auscher and Tchamitchian 2001b, Theorem 4]. The same holds for the boundedness of $(L_q + \lambda)^{-1} : W^{-1,q}(\Omega) \to W_0^{1,q}(\Omega)$ for which originating from [Krylov 2007] many results have been obtained in the last years. For a complete list of references we refer to the introduction of [Dong and Kim 2016] and for a proof of similar results within the framework of maximal regularity to [Gallarati and Veraar 2017a; 2017b].

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain, $T > 0$ and $a_{ij} \in L^\infty([0, T] \times \Omega)$ for $i, j = 1, \ldots, n$. Assume further that the following properties are satisfied*:

(1) *There exists $\delta > 0$ such that for almost all $(t, x) \in [0, T] \times \Omega$ and all $\xi \in \mathbb{C}^n$*

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i \bar{\xi}_j \geq \delta |\xi|^2.$$

(2) *The functions $x \mapsto a_{ij}(t, x)$ lie in $\mathrm{VMO}(\Omega)$ and there is $\eta : [0, 1] \to [0, \infty]$ with $\lim_{r \downarrow 0} \eta(r) = 0$ and $\eta_{a_{ij}(t,\cdot)} \leq \eta$ for all $t \in [0, T]$ and $i, j = 1, \ldots, n$.*

For $q \in (1,\infty)$ let $L_q(t) = -\operatorname{div}(A(t)\nabla\cdot)$ be realizations on $L^q(\Omega)$. Then for all $q \in (1,\infty)$ the nonautonomous problem (NACP) *associated to $(L_q(t))_{t\in[0,T]}$ has maximal L^p-regularity*

(a) *for $p \in (1,2]$ if $a_{ij} \in \dot W^{1/2+\varepsilon,2}([0,T]; L^\infty(\Omega))$ for some $\varepsilon > 0$,*

(b) *for $p \in [2,\infty)$ if $a_{ij} \in \dot W^{1/2+\varepsilon,p}([0,T]; L^\infty(\Omega))$ for some $\varepsilon > 0$.*

The maximal L^p-regularity estimate depends only on $p, q, T, \Omega, \delta, \eta, \varepsilon, \|a_{ij}\|_\infty$ and the homogeneous Sobolev norm in (a) *or* (b).

Proof. Thanks to the Gaussian estimates discussed in the proof of Theorem 7.2, for sufficiently large λ the operators $L_q(t)+\lambda$ have uniformly bounded imaginary powers with $\|(L_q(t)+\lambda)^{is}\| \le Me^{\omega|s|}$ for some $M > 0$ and $\omega \in \big(0, \frac{\pi}{2}\big)$. This follows from the general result [Duong and Robinson 1996, Theorem 4.3] (which even gives a bounded H^∞-calculus), which does not state the dependence on the constants explicitly. Further, it follows from Theorem 7.2 that $D((L_q(t)+\lambda)^{1/2}) \simeq W_0^{1,q}(\Omega)$ holds uniformly in $t \in [0,T]$. Moreover, the operator $L_q(t)+\lambda$ extends to an isomorphism $W_0^{1,q}(\Omega) \xrightarrow{\sim} W^{-1,q}(\Omega)$ which is uniform in $t \in [0,T]$. Consequently, for $u \in L^q(\Omega)$ one has

$$\begin{aligned}\|u\|_{D((L_q(t)+\lambda)^{-1/2})} &= \|(L_q(t)+\lambda)^{-1/2}u\|_{L^q(\Omega)} \\ &= \|(L_q(t)+\lambda)^{1/2}(L_q(t)+\lambda)^{-1}u\|_{L^q(\Omega)} \simeq \|(L_q(t)+\lambda)^{-1}u\|_{W_0^{1,q}(\Omega)} \\ &\simeq \|u\|_{W^{-1,q}(\Omega)}.\end{aligned}$$

Therefore $X_{1/2} = W^{1,q}(\Omega)$ and $X_{-1/2} = W^{-1,q}(\Omega)$ in Corollary 6.7.

It remains to check the time regularity. For $u \in W_0^{1,2}(\Omega) \cap W_0^{1,q}(\Omega)$ and $v \in W_0^{1,2}(\Omega) \cap W_0^{1,q'}(\Omega)$ one has

$$\big|\langle L_q(t)u - L_q(s)u, v\rangle\big| = \left|\int_\Omega (A(t)-A(s))\nabla u \nabla v\right| \le \|A(t)-A(s)\|_\infty \|\nabla u\|_q \|\nabla v\|_{q'}.$$

By density this extends to all $u \in W_0^{1,q}(\Omega)$ and all $v \in W_0^{1,q'}(\Omega)$. Hence, it follows that $L_q(\cdot)+\lambda \in \dot W^{\alpha,r}([0,T]; \mathcal{B}(W_0^{1,q}(\Omega), W^{-1,q}(\Omega)))$ with α and r as in the assumptions. Now, Corollary 6.7 applies and yields maximal L^p-regularity for $(L_q(t)+\lambda)_{t\in[0,T]}$ and λ big enough. By a rescaling argument this is equivalent to the maximal L^p-regularity of $(L_q(t))_{t\in[0,T]}$. ☐

8. Applications to quasilinear parabolic problems

We now use Theorem 7.4 to solve quasilinear parabolic equations. It may be a little bit confusing that in the result below Hölder assumptions on the coefficients are made. The point for concrete applications is not that we can replace Hölder regularity by fractional Sobolev regularity, but that the fractional Sobolev regularity in Theorem 7.4 allows us to loosen the assumed Hölder regularity. We will comment on this point later.

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain and $T > 0$. For coefficients $A = (a_{ij}) : \mathbb{C} \to \mathbb{C}^{n\times n}$, $p \in [2,\infty)$, $q \in (1,\infty)$, an inhomogeneous part $f \in L^p([0,T]; L^q(\Omega))$ and an initial value $u_0 \in L^q(\Omega)$*

satisfying the condition $u_0 \in \big(D(\operatorname{div} A(u_0)\nabla\,\cdot), L^q(\Omega)\big)_{1/p,p}$ *consider the problem*

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \operatorname{div}(A(u(t,x))\nabla u(t,x)) = f(t,x), \\ u(t,x) = 0 \qquad \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = u_0(x) \quad \text{on } \Omega. \end{cases} \tag{QLP}$$

Suppose that the following assumptions are satisfied:

(1) *The coefficients* a_{ij} *are* β*-Hölder continuous for some* $\beta > \frac{1}{2}$.

(2) *For all* $M > 0$ *there exist* $\delta(M) > 0$ *such that for all* $|u| \le M$

$$\operatorname{Re}\sum_{i,j=1}^{n} a_{ij}(u)\xi_i\bar{\xi}_j \ge \delta(M)|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n.$$

If $q > n$ *and* $\beta > \frac{q}{2(q-n)}$, *then there exists* $C \ge 0$ *such that for*

$$\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|_{(D(\operatorname{div} A(u_0)\nabla\cdot),L^q(\Omega))_{1/p,p}} \le C$$

the quasilinear problem (QLP) *has a solution*

$$u \in W^{1,p}([0,T];L^q(\Omega)) \cap \mathrm{BUC}([0,T]\times\bar{\Omega})$$

with $u(t) \in D(\operatorname{div} A(u(t,\cdot))\nabla\,\cdot)$ *for almost every* $t \in [0,T]$ *and* $\operatorname{div} A(u)\nabla u \in L^p([0,T];L^q(\Omega))$. *A fortiori*, $u \in C^{\alpha-1/p}([0,T];C^{1-\alpha-n/q}(\bar{\Omega}))$ *for* $\alpha \in \big(\frac{1}{p}, 1-\frac{n}{q}\big)$.

Proof. Choose $\alpha \in \big(\frac{1}{2\beta}, 1-\frac{n}{q}\big)$, which is possible by our assumptions. Now, choose $\delta > 0$ with $\alpha - \delta > \frac{1}{2\beta}$ and $\alpha + \delta < 1 - \frac{n}{q}$. Further, let

$$\mathcal{M} = \{v \in W^{\alpha-\delta,p}([0,T];W_0^{1-\alpha-\delta,q}(\Omega)) : v(0) = u_0\}$$

and $\mathcal{M}_R$ for $R > 0$ be the ball $B(0,R)$ in $\mathcal{M}_R$. For $v \in \mathcal{M}_{\mathcal{R}}$ consider the problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \operatorname{div}(A(v(t,x))\nabla u(t,x)) = f(t,x), \\ u(t,x) = 0 \qquad \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = u_0(x) \quad \text{on } \Omega. \end{cases} \tag{LP}$$

Since $\alpha+\delta < 1-\frac{n}{q}$ and $\alpha - \delta > \frac{1}{2\beta} \ge \frac{1}{2} \ge \frac{1}{p}$, we have $v \in W^{\alpha-\delta,p}([0,T];\mathrm{BUC}(\bar{\Omega}))$ and $\mathcal{M}$ is compactly embedded in $\mathrm{BUC}([0,T]\times\bar{\Omega})$. By the Arzelà–Ascoli theorem, the functions in $\mathcal{M}_R$ are uniformly equicontinuous on $[0,T]\times\Omega$. As a consequence (2) of Theorem 7.4 is satisfied and one can find uniform ellipticity constants for $A \circ v$ with $v \in \mathcal{M}_R$. For $\varepsilon > 0$ with $r := (\alpha-\delta-\varepsilon)\beta > \frac{1}{2}$ we have

$$\begin{aligned} \|a_{ij}\circ v\|^p_{\dot{W}^{r,p}([0,T];L^\infty(\Omega))} &= \int_0^T\int_0^T \frac{\|a_{ij}(v(t,\cdot)) - a_{ij}(v(s,\cdot))\|^p_\infty}{|t-s|^{1+pr}}\,ds\,dt \\ &\lesssim \int_0^T\int_0^T \frac{\|v(t,\cdot)-v(s,\cdot)\|^{\beta p}_\infty}{|t-s|^{1+pr}}\,ds\,dt = \|v\|^{\beta p}_{\dot{W}^{r\beta^{-1},\beta p}([0,T];L^\infty(\Omega))} \\ &= \|v\|^{\beta p}_{\dot{W}^{\alpha-\delta-\varepsilon,\beta p}([0,T];L^\infty(\Omega))} \lesssim \|v\|^{\beta p}_{\dot{W}^{\alpha-\delta,p}([0,T];L^\infty(\Omega))}. \end{aligned} \tag{8-1}$$

This means that the coefficients $A \circ v$ satisfy the assumptions of Theorem 7.4. Hence, (LP) has a unique solution u and there is $C_R \geq 0$ independent of $v \in \mathcal{M}_R$ with

$$\|u\|_{W^{1,p}([0,T];L^q(\Omega))} + \|\operatorname{div} A(v)\nabla u\|_{L^p([0,T];L^q(\Omega))} \leq C_R(\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|),$$

where the norm of u_0 is taken in $(D(\operatorname{div} A(u_0)\nabla\,\cdot), L^q(\Omega))_{1/p,p}$. Further, by the real interpolation formula for vector-valued Besov spaces [Amann 2000, Corollary 4.3] one has for $\theta \in \left(\frac{1}{2}, 1 - \frac{1}{q}\right)$ and sufficiently small $\varepsilon > 0$, uniformly in $v \in \mathcal{M}_R$, the embeddings

$$\begin{aligned} u \in W^{1,p}([0,T];L^q(\Omega)) \cap L^p([0,T];W_0^{1,q}(\Omega)) &\hookrightarrow \big(L^p([0,T];W_0^{1,q}(\Omega)), W^{1,p}([0,T];L^q(\Omega))\big)_{\theta,p} \\ &\hookrightarrow W^{\theta-\varepsilon,p}\big([0,T];(W_0^{1,q}(\Omega), L^q(\Omega))_{\theta,p}\big) \\ &= W^{\theta-\varepsilon,p}([0,T];B_{0,p}^{1-\theta,q}(\Omega)) \\ &\hookrightarrow W^{\theta-\varepsilon,p}([0,T];W_0^{1-\theta-\varepsilon,q}(\Omega)). \end{aligned} \tag{8-2}$$

All estimates hold uniformly for $v \in \mathcal{M}_R$. The embedding (8-2) implies that for sufficiently small

$$\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|_{(D(\operatorname{div} A(u_0)\nabla\cdot),L^q(\Omega))_{1/p,p}}$$

we obtain a well-defined map

$$S_R : \mathcal{M}_R \to \mathcal{M}_R, \qquad v \mapsto u, \quad \text{where } u \text{ is the solution of (LP).}$$

It follows from (8-2) and the compact embedding results for vector-valued Sobolev spaces [Amann 2000, Theorem 5.1] that $S_R\mathcal{M}_R$ is a precompact subset of $\mathcal{M}_R$. We next show that S_R is continuous. For this let $v_n \to v$ in $\mathcal{M}_R$ and let $u_n = S_R v_n$. After passing to a subsequence we may assume that $v_n \to v$ in $\operatorname{BUC}([0,T] \times \overline{\Omega})$ and that u_n converges weakly to some u in

$$W^{1,p}([0,T];L^q(\Omega)) \cap L^p([0,T];W_0^{1,q}(\Omega)).$$

Now, let $g \in L^{p'}([0,T];W_0^{1,q'}(\Omega))$. Note that $A^T(v_n)\nabla g \to A^T(v)\nabla g$ in $L^{q'}(\Omega)$ by the dominated convergence theorem. Since u_n solves (LP) we have

$$\begin{aligned} \int_0^T \langle f(t), g(t)\rangle\, dt &= \int_0^T \langle \dot{u}_n(t), g(t)\rangle\, dt + \int_0^T \langle A(v_n(t))\nabla u_n(t), \nabla g(t)\rangle\, dt \\ &= \int_0^T \langle \dot{u}_n(t), g(t)\rangle\, dt + \int_0^T \langle \nabla u_n(t), A^T(v_n(t))\nabla g(t)\rangle\, dt. \end{aligned}$$

Taking limits on both sides of the equation, we get

$$\int_0^T \langle f(t), g(t)\rangle\, dt = \int_0^T \langle \dot{u}(t), g(t)\rangle\, dt + \int_0^T \langle A(v(t))\nabla u(t), \nabla g(t)\rangle\, dt.$$

Since g is arbitrary and $u_0 = u_n(0) \to u(0)$, this implies that u solves (LP) on $W^{-1,q}(\Omega)$, i.e., is the unique integrated solution of (LP) given by Proposition 4.4. Hence, $S_R v = u$. Since the same argument works for arbitrary subsequences, we have shown that S_R is continuous. Now, by Schauder's fixed point theorem there is some $u \in \mathcal{M}_R$ with $S_R u = u$. Using Theorem 7.4 for $v = u$ we see that

$$\|u\|_{W^{1,p}([0,T];L^q(\Omega))} + \|\operatorname{div} A(u)\nabla u\|_{L^p([0,T];L^q(\Omega))} \le C(\|f\|_{L^p([0,T];L^q(\Omega))} + \|u_0\|_{(D(\operatorname{div} A(u_0)\nabla\cdot),L^q(\Omega))_{1/p,p}}). \qquad \square$$

Remark 8.2. We illustrate the benefits of Theorem 7.4 for quasilinear equations with the help of Theorem 8.1. First, maximal regularity results for nonautonomous problems governed by elliptic operators before [Fackler 2015] assumed C^1-regularity in space. In particular, such results cannot deal with Hölder continuous coefficients a_{ij} as in Theorem 8.1 because the composition $a_{ij} \circ v$ in (8-1) would fail to have the necessary C^1-smoothness.

Further, in (8-1) one needs from a conceptual point of view that the composition operator $v \mapsto a_{ij} \circ v$ maps into the Sobolev space $\dot{W}^{\alpha,p}([0,T];L^\infty(\Omega))$ for some $\alpha > \frac{1}{2}$ in order to apply Theorem 7.4. Although v lies in some fractional Sobolev space and one only requires the image to lie in a different fractional Sobolev space, the only useful sufficient condition the author is aware of is to assume that the coefficients a_{ij} are Hölder continuous. Nevertheless, the less restrictive fractional Sobolev assumption in Theorem 7.4 is useful as it allows us to relax the assumed regularity. To illustrate this point explicitly, let us calculate the necessary regularity if one needs to check that $a_{ij} \circ v$ is in $C^\alpha([0,T];L^\infty(\Omega))$ for some $\alpha > \frac{1}{2}$. Using the same notation as before one has

$$\|a_{ij}(v(t,\cdot)) - a_{ij}(v(s,\cdot))\|_\infty \lesssim \|v(t,\cdot) - v(s,\cdot)\|_\infty^\beta.$$

Now, ignoring the technical aspect of having an additional $\delta > 0$ of room, for functions $v \in W^{\alpha,p}([0,T];W^{1-\alpha,q}(\Omega))$ we have for $\alpha \in \left(\frac{1}{p}, 1 - \frac{n}{q}\right)$ the embedding

$$W^{\alpha,p}([0,T];W^{1-\alpha,q}(\Omega)) \hookrightarrow C^{\alpha-1/p}([0,T];\mathrm{BUC}(\overline{\Omega})).$$

Consequently, we have

$$\|a_{ij}(v(t,\cdot)) - a_{ij}(v(s,\cdot))\|_\infty \lesssim |t-s|^{\beta(\alpha-1/p)}.$$

Since $\alpha < 1 - \frac{n}{q}$, for maximal regularity with Hölder assumptions one needs

$$\beta \cdot \left(1 - \frac{n}{q} - \frac{1}{p}\right) > \frac{1}{2} \quad \Longleftrightarrow \quad \beta > \frac{q}{2(q-n-q/p)}.$$

In particular, this is a stronger condition than $\beta > \frac{q}{2(q-n)}$, as used in Theorem 8.1. This improvement comes from the fact that the p-integrability is for free in the fractional Sobolev result, whereas in the Hölder case one has to sacrifice some differentiation order for the Sobolev embeddings.

Remark 8.3. We can only deduce the existence of solutions for small data in Theorem 7.4 because the constant in the maximal regularity estimate depends on the VMO-modulus of the coefficients and their fractional Sobolev norm. If one has estimates on solutions of (QLP) independent of these regularity data, the Leray–Schauder principle would yield solutions for arbitrary f and u_0.

Further note that the reasoning of Theorem 8.1 works for a far more general class of problems. For example, the coefficients $A(u)$ may depend in a nonlocal way on u, e.g., on the history of the solution as in [Amann 2005; 2006].

9. Optimality of the results

In this section we show that the maximal regularity results obtained in Theorem 6.4 are optimal or close to optimal. In fact, even in the form setting considered in Corollary 6.6, maximal regularity may fail if one relaxes the assumed regularity, i.e., for maximal L^p-regularity $\mathcal{A} \in \dot{W}^{\alpha,p}([0,T];\mathcal{B}(V,V'))$ for some $\alpha > \frac{1}{2}$. It was shown in [Fackler 2017a, Theorem 5.1] that there is a symmetric nonautonomous form with $\mathcal{A} \in C^{1/2}([0,T];\mathcal{B}(V,V'))$ and $f \in L^\infty([0,T];V)$ for which the unique solution given by Proposition 4.4 satisfies $\dot{u}(t) \notin H$ for almost all $t \in [0,T]$, although $u \in L^\infty([0,T];V)$ holds as one aims for in the bootstrapping result given in Proposition 5.1. As a consequence, maximal L^p-regularity fails for all $p \in [1,\infty]$. Note that $C^{1/2}([0,T];\mathcal{B}(V,V')) \hookrightarrow \dot{W}^{\alpha,q}([0,T];\mathcal{B}(V,V'))$ for all $\alpha \in \left(0,\frac{1}{2}\right)$ and all $q \in [1,\infty]$. Hence, Theorem 6.4 fails for $\alpha < \frac{1}{2}$ in all possible variants.

This leaves open the critical case $\alpha = \frac{1}{2}$. Note that for $q \in (1,2)$ the space $\dot{W}^{1/2,q}([0,T];\mathcal{B}(V,V'))$ contains piecewise constant forms. Hence, as observed by Dier [2014, Section 5.2], the failure of the Kato square root property for general forms implies that maximal L^2-regularity may not hold for $q < 2$. Example 7.2 in [Fackler 2017b] shows that for $p > 2$ maximal L^p-regularity on $L^2(\Omega)$ for $\mathcal{A} \in \dot{W}^{1/2,q}([0,T];\mathcal{L}(V,V'))$ with $q \in (1,2)$ does not even hold for elliptic operators. Note that for $p \in (1,2)$ these arguments based on the incompatibility of trace spaces break down.

Refining the arguments in [Fackler 2017a], we show that for symmetric forms maximal L^p-regularity may fail for all $p \in [1,\infty]$ under the regularity $\mathcal{A} \in \dot{W}^{1/2,q}([0,T];\mathcal{B}(V,V'))$ for some $q > 2$.

Example 9.1. We take $H = L^2\left(\left[0,\frac{1}{2}\right]\right)$ and $V = L^2\left(\left[0,\frac{1}{2}\right],w\right)$ with $w(x) = (x|\log x|)^{-3/2}$. Further, we consider $u(t,x) = c(x)(\sin(t\varphi(x)) + d)$ for $\varphi(x) = w(x)$, $c(x) = x \cdot |\log x|$ and some sufficiently large $d > 0$. Note that for all $t \in [0,T]$

$$\|\dot{u}(t)\|_H^2 \simeq \int_0^{1/2} |c(x)\varphi(x)|^2\,dx = \int_0^{1/2} x^{-1}\frac{1}{|\log x|}\,dx = \infty.$$

Hence, $\dot{u}(t) \notin H$ for all $t \in [0,T]$. Following the ideas and arguments in [Fackler 2017a] we now show that u is indeed an integrated solution of a nonautonomous problem associated to some coercive, bounded symmetric sesquilinear form $a : [0,T] \times V \times V \to \mathbb{C}$ and inhomogeneous part $f(t) = u(t) \in L^\infty([0,T];V)$. For this one defines the form $a(t,\cdot,\cdot)$ on the set $\langle u(t)\rangle \times V$ as

$$a(t, c\cdot u(t), v) = c\left[(f(t)|v)_H - \langle \dot{u}(t), v\rangle_{V',V}\right] \tag{9-1}$$

and then extends the form to $V \times V$ by the same procedure as in [Fackler 2017a, Section 4]. Following Section 5 of that paper, one checks the regularity of the extended forms. By the explicit formula for the extension, one sees that it suffices to control the regularity of (duality) products of the functions $u : [0,T] \to V$, $w^{-1}u : [0,T] \to V$ and $\dot{u} : [0,T] \to V'$. Since $\dot{W}^{\alpha,p} \cap L^\infty$ is an algebra under pointwise multiplication, the regularity question boils down to the regularity of these individual functions. Further, one sees that for our concrete choice of u, the relevant fractional norms are dominated by that of $u : [0, T] \to V$. Hence, one only has to check the regularity of $u : [0, T] \to V$, which we do now.

We show explicitly that $u \in \dot{W}^{1/2,q}([0, T]; V)$ for all $q \in (2, \infty)$. Note that on the one hand

$$|\sin(t\varphi(x)) - \sin(s\varphi(x))|^2 \le |t-s|^2\varphi^2(x) = |t-s|^2 x^{-3}|\log x|^{-3}. \tag{9-2}$$

On the other hand the left-hand side can clearly be estimated by 4. Now, let $\psi(x) = 2x^{3/2}|\log x|^{3/2}$. Then (9-2) gives the sharper estimate if and only if $|t-s| \le \psi(x)$ or equivalently $x \ge \psi^{-1}(|t-s|)$. Splitting the fractional norm, we obtain

$$\begin{aligned}
&\left(\int_0^T\int_0^T \frac{\|u(t)-u(s)\|_V^q}{|t-s|^{1+q/2}}\,dt\,ds\right)^{1/q} \\
&\quad = \left(\int_0^T\int_{-t}^{T-t} \frac{\|u(t)-u(t+r)\|_V^q}{|r|^{1+q/2}}\,dr\,dt\right)^{1/q} \\
&\quad \lesssim \left(\int_0^T\int_{-t}^{T-t}\left(\int_0^{\psi^{-1}(|r|)} x^{1/2}|\log x|^{1/2}\,dx\right)^{q/2}\frac{dr}{|r|^{1+q/2}}\,dt\right)^{1/q} \\
&\qquad\qquad + \left(\int_0^T\int_{-t}^{T-t}\left(\int_{\psi^{-1}(|r|)}^{1/2} x^{-5/2}|\log x|^{-5/2}\,dx\right)^{q/2}\frac{dr}{|r|^{1-q/2}}\,dt\right)^{1/q}.
\end{aligned} \tag{9-3}$$

Now, for the innermost integral of the first term we have for $F(x) = x^{3/2}|\log x|^{1/2}$

$$\begin{aligned}
\int_0^{\psi^{-1}(|r|)} x^{1/2}|\log x|^{1/2}\,dx &\lesssim \int_0^{\psi^{-1}(|r|)} F'(x)\,dx = F(\psi^{-1}(|r|)) \\
&\lesssim \psi(\psi^{-1}(|r|))\big|\log\psi^{-1}(|r|)\big|^{-1} = |r|\,\big|\log\psi^{-1}(|r|)\big|^{-1} \lesssim |r|\,|\log r|^{-1}.
\end{aligned}$$

Analogously, for the second term we have for $F(x) = -x^{-3/2}|\log x|^{-5/2}$

$$\begin{aligned}
\int_{\psi^{-1}(|r|)}^{1/2} x^{-5/2}|\log x|^{-5/2}\,dx &\lesssim \int_{\psi^{-1}(|r|)}^{1/2} F'(x)\,dx \le -F(\psi^{-1}(|r|)) \\
&\lesssim \frac{1}{\psi(\psi^{-1}(|r|))}\big|\log\psi^{-1}(|r|)\big|^{-1} \lesssim |r|^{-1}\,|\log r|^{-1}.
\end{aligned}$$

Hence, (9-3) is dominated up to a constant by the finite expression

$$\left(\int_0^T |\log r|^{-q/2}\frac{dr}{|r|}\right)^{1/q}$$

for $q > 2$.

Hence, for maximal L^2-regularity of forms the only case left open is that of $\dot{W}^{1/2,2}([0, T]; \mathcal{B}(V, V'))$ regularity, which we are not able to answer at the moment. Note that there is also a positive result assuming some half differentiability. Namely, it was shown by Auscher and Egert [2016] that for elliptic operators one has maximal L^2-regularity if the coefficients a_{ij} satisfy $\partial^{1/2}a_{ij} \in \mathrm{BMO}$. This in particular implies $a_{ij} \in \dot{H}^{1/2,q}$ for all $q \in (1, \infty)$, which in turn implies $a_{ij} \in \dot{W}^{1/2,q}$ for all $q \ge 2$, which in general is not sufficient for maximal L^p-regularity by the above example. In the other direction, the inclusion

$\dot{W}^{1/2,q} \hookrightarrow \dot{H}^{1/2,q}$ does only hold for $q \in (1,2]$. Hence, for $q \in (1,2)$ the space $\dot{H}^{1/2,q}$ contains step functions. Note that in the critical case one has $\dot{H}^{1/2,2} = \dot{W}^{1/2,2}$; i.e., the Besov and the Bessel scale give rise to the same problem.

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