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THE SHAPE OF LOW ENERGY CONFIGURATIONS OF A THIN ELASTIC SHEET WITH A SINGLE DISCLINATION

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We consider a geometrically fully nonlinear variational model for thin elastic sheets that contain a single disclination. The free elastic energy contains the thickness h as a small parameter. We give an improvement of a recently proved energy scaling law, removing the next-to-leading-order terms in the lower bound. Then we prove the convergence of (almost-)minimizers of the free elastic energy towards the shape of a radially symmetric cone, up to Euclidean motions, weakly in the spaces $W^{2,2}(B_1 \setminus B_\rho; \mathbb{R}^3)$ for every $0 < \rho < 1$, as the thickness h is sent to 0.

1. Introduction

1.1. ***Setup and previous work.*** The present article continues a program [Müller and Olbermann 2014a; Olbermann 2016; 2017] to explore thin elastic sheets with a single disclination from the variational point of view. The free energy that we consider consists of two parts: (1) the nonconvex membrane energy, which penalizes the difference between the metric that is induced by the deformation and the reference metric, which is the metric of the (singular) cone; (2) the bending energy, which penalizes curvature. The bending energy contains a factor h^2, where the small parameter h is to be thought of as the thickness of the sheet (see (1) below for the definition). Choosing the cone as configuration, one gets infinite energy: While the membrane term vanishes, the bending energy is infinite for this choice. Energetically, there is a competition of the membrane and the bending terms; neither will vanish for configurations of low energy.

Intuitively, it seems quite clear how configurations of low energy should look: they should be identical to the cone far away from the disclination, and near the disclination, there should be some smoothing of the cone, at a scale h (the only length scale in the problem). For such configurations, one gets an energy of $C^*h^2 \log(1/h)$ plus terms of order h^2, where C^* is an explicitly known constant; see Lemma 4 below. It is natural to conjecture that such a scaling behavior should indeed hold true for minimizers. However, a proof of an ansatz-free lower bound with the same scaling is much more difficult than the straightforward construction for the upper bound. In the literature, lower bounds for this setting have been ansatz based [Lidmar et al. 2003; Seung and Nelson 1988; Yavari and Goriely 2013], or have assumed radial symmetry [Müller and Olbermann 2014a].

The idea underlying the recent proofs of ansatz-free lower bounds [Olbermann 2016; 2017] is to control the Gauss curvature (or a linearization thereof) by interpolation between the membrane and the bending term energy. The control over the Gauss curvature allows for a certain control over the Gauss map (or

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the deformation gradient). This information in turn yields lower bounds for the bending energy, using an inequality of Sobolev/isoperimetric type. For the corresponding result from [Olbermann 2017], see (2) below. This lower bound does not quite achieve the conjectured scaling behavior, in that there exist next-to-leading-order terms $O(h^2 \log\log(1/h))$ which are not present in the upper bound.

Here, we are going to improve the results from [Olbermann 2017] in two ways: First, we give an improved lower bound for the elastic energy, which proves the conjecture that the minimum of the energy is given by $C^* h^2 \log(1/h) + O(h^2)$. The observation that allows for this improvement is that it is unnecessary to use interpolation to control the Gauss curvature and the Gauss map (or rather, the linearized Gauss curvature and the deformation gradient). It is enough to use the membrane energy alone to obtain the necessary control, and make more efficient use of the Sobolev/isoperimetric inequality.

Second, we use this improved lower bound to show a statement about the *shape* of configurations that satisfy the energy bounds. We prove that (almost-)minimizers converge to the conical deformation, up to Euclidean motions. It is remarkable that that much information about deformations of small energy can be obtained, considering that we are dealing with a highly nonconvex variational problem. Hitherto, such results had only been achieved for situations in which the energy scales were $O(h^2)$ or less [Friesecke et al. 2002; Pakzad 2004; Hornung 2011a]. The results of these papers will also play an important role in our proof.

1.2. ***Statement of results.*** Let $B_1 := \{x \in \mathbb{R}^2 : |x| < 1\}$ be the sheet in the reference configuration. The singular cone may be described by the mapping $y^\Delta : B_1 \to \mathbb{R}^3$,

$$y^\Delta(x) = \sqrt{1-\Delta^2}\,x + \Delta|x|e_3.$$

Here, $0 < \Delta < 1$ is the height of the singular cone, and is determined by the deficit of the disclination at the origin. The reference metric on B_1 is given by

$$\begin{aligned} g_\Delta(x) &= Dy^\Delta(x)^T Dy^\Delta(x) \\ &= (1-\Delta^2)\mathrm{Id}_{2\times 2} + \Delta^2 \hat{x}\otimes\hat{x} \\ &= \mathrm{Id}_{2\times 2} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp, \end{aligned}$$

where $\hat{x} = x/|x|$ and $\hat{x}^\perp = (-x_2, x_1)/|x|$. The induced metric of a deformation $y \in W^{2,2}(B_1;\mathbb{R}^3)$ is

$$g_y = Dy^T Dy.$$

The free elastic energy $I_{h,\Delta} : W^{2,2}(B_1;\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$I_{h,\Delta}(y) = \int_{B_1} \left(|g_y - g_\Delta|^2 + h^2|D^2 y|^2\right) \mathrm{d}\mathcal{L}^2, \tag{1}$$

where $\mathrm{d}\mathcal{L}^2$ denotes two-dimensional Lebesgue measure. In [Olbermann 2017], we proved the existence of a constant $C = C(\Delta) > 0$ such that

$$2\pi\Delta^2 h^2\Big(\log\frac{1}{h} - 2\log\log\frac{1}{h} - C\Big) \le \min_{y\in W^{2,2}(B_1;\mathbb{R}^3)} I_{h,\Delta}(y) \le 2\pi\Delta^2 h^2\Big(\log\frac{1}{h} + C\Big). \tag{2}$$

Our first aim in the present article is to improve the lower bound for the free elastic energy. The improvement consists in getting rid of the $\log\log(1/h)$ terms on the left-hand side:

Theorem 1. *There exist positive constants C_1, C_2, C_3 that only depend on Δ with the following properties. First,*

$$2\pi\Delta^2h^2\Big(\log\frac{1}{h}-C_1\Big)\leq\min_{y\in W^{2,2}(B_1;\mathbb{R}^3)}I_{h,\Delta}(y)\leq2\pi\Delta^2h^2\Big(\log\frac{1}{h}+C_2\Big) \tag{3}$$

for all $h\in\big(0,\frac{1}{3}\big)$. Furthermore, if y satisfies

$$I_{h,\Delta}(y)\leq2\pi\Delta^2h^2\Big(\log\frac{1}{h}+C_2\Big), \tag{4}$$

then

$$\int_{B_1\setminus B_R}|D^2y|^2\,\mathrm{d}\mathcal{L}^2\leq2\pi\Delta^2\log\frac{1}{R}+C_3\quad\text{for all } R\in(3h,1), \tag{5}$$

$$\int_{B_1}|g_y-g_\Delta|^2\,\mathrm{d}\mathcal{L}^2\leq C_3h^2. \tag{6}$$

As a consequence of Theorem 1, we will be able to prove convergence of (almost)-minimizers of the functional (1) towards the singular cone as $h\to0$:

Theorem 2. *Let $y^h\in W^{2,2}(B_1;\mathbb{R}^3)$ be a sequence with*

$$I_{h,\Delta}(y^h)\leq2\pi\Delta^2h^2\Big(\log\frac{1}{h}+C_2\Big).$$

Then there exists a subsequence $y^{h(k)}$ and a Euclidean motion T such that for every $0<\rho<1$,

$$y^{h(k)}\rightharpoonup Ty^\Delta\quad\text{in } W^{2,2}(B_1\setminus B_\rho;\mathbb{R}^3). \tag{7}$$

1.3. ***Scientific context.*** In the proof of Theorem 1 we show a certain focusing of the elastic energy near the disclination. Phenomena with such elastic energy focusing are also observed in many other settings. In particular, crumpled elastic sheets display networks of vertices and ridges. The investigation of these “sharp” structures in the physics community started in the mid-1990s. For a historical account and an exhaustive list of references, see the very recommendable overview article [Witten 2007]. There has been quite some activity in the analysis of ridge-like structures in particular; see [Lobkovsky 1996; DiDonna and Witten 2001; Lobkovsky and Witten 1997; Kramer and Witten 1997; Lobkovsky et al. 1995; Venkataramani 2004]. Energy focusing in conical shapes has been investigated in [Ben Amar and Pomeau 1997; Cerda et al. 1999; Cerda and Mahadevan 1998; 2005]. Disclinations in thin elastic sheets are particularly interesting as a modeling device for icosahedral elastic structures. This is a popular model for virus capsids [Seung and Nelson 1988; Lidmar et al. 2003] or carbon nanocones [Romanov 2003], the structure one obtains when inserting a single five-valent vertex into a graphene sheet (of otherwise six-valent vertices). The disclinations are located at the vertices of the elastic icosahedra.

In the mathematical literature on thin elastic sheets, there have been two strands of investigation: On the one hand, there are the rigorous derivations of elastic plate models from three-dimensional finite elasticity by means of Γ-convergence; see [Friesecke et al. 2002; 2006; Lewicka et al. 2010]. On the other hand, there has been quite some effort to investigate the qualitative properties of low-energy states in the variational formulation of elasticity, obtained through an analysis of the scaling of the free elastic energy with respect to the relevant parameters in the model; see, e.g., [Bella and Kohn 2014a; 2014b; Ben Belgacem et al. 2002; Kohn and Nguyen 2013]. The present paper belongs of course to the latter group. In more detail, rigorous scaling laws similar to the ones we prove here have been derived for a single fold [Conti and Maggi 2008] and for the so-called *d-cone* [Müller and Olbermann 2014b; Brandman et al. 2013]. The variational problems considered in these references however are of a very special kind: the constraints on the shape of the elastic sheet are quite restrictive, and the lower bounds use these constraints in an essential way; see [Olbermann 2017] for a detailed discussion. This is not the case for our setting, whence our method of proof, which we developed in [Olbermann 2016; 2017] and which we refine here, is completely different.

1.4. *Connection to convex integration and rigidity results.* The Nash–Kuiper theorem [Nash 1954; Kuiper 1955a; 1955b] states that given a two-dimensional Riemannian manifold (M, g), a *short*[1] immersion $y_0 : M \to \mathbb{R}^3$, and $\varepsilon > 0$, there exists an isometric immersion $y_1 \in C^1(M; \mathbb{R}^3)$ such that $\|y_1 - y_0\|_{C^0} < \varepsilon$. This is relevant in our context, since the leading-order term in the energy (1) measures the distance of the deformation y from an isometric immersion with respect to the target metric g_Δ. By the Nash–Kuiper theorem, there exists a vast amount of deformations y that have arbitrarily small membrane energy. A priori, these are all good candidates for energy minimization. One needs a principle that shows that all of these deformations are associated with large bending energy. The energy scaling law from Theorem 1 shows that none of these maps can beat the upper bound construction energetically. Theorem 2 shows the "stronger" statement that maps with low energy cannot look anything like the approximations of C^1 isometric immersions that appear in the proof of the Nash–Kuiper theorem.

The Nash–Kuiper result is an instance of *convex integration*, a concept that has been developed systematically by Gromov [1986]. In particular, the theorem states that solutions to isometric immersion problems are highly nonunique if one requires only C^1-regularity. In stark contrast, there is the uniqueness in the Weyl problem: given a sufficiently smooth metric g on S^2 with positive Gauss curvature, there exists a *unique* isometric immersion $y : S^2 \to \mathbb{R}^3$ of C^2-regularity. Such uniqueness is often called *rigidity*. The dichotomy of convex integration versus rigidity also appears in other contexts, such as the Monge–Ampère equation [Lewicka and Pakzad 2017] and the incompressible Euler equation [Constantin et al. 1994; Isett 2016].

Concerning the uniqueness of solutions in the Weyl problem, the proof is due to Pogorelov [1973]. In fact, he proved that solutions are unique up to Euclidean motions in the class of immersions of *bounded extrinsic curvature*. The latter is the class of immersions for which the pull-back of the volume form on S^2 under the Gauss map is a well-defined signed Radon measure. For smooth maps, this is just the measure $K\,\mathrm{d}A$, where K is the Gauss curvature and $\mathrm{d}A$ is the volume element. We see that control over the Gauss curvature excludes constructions in the style of Nash–Kuiper. This is also the basic concept

[1] An immersion $y : M \to \mathbb{R}^3$ is short with respect to the metric g on M if for every curve $\gamma : [0, 1] \to M$, the length of $y \circ \gamma$ is shorter (measured with the Euclidean metric on $\mathbb{R}^3$) than γ (measured with g).

underlying our proof (with the modification that we consider a linearized version of Gauss curvature). We believe that this hints at a link between questions about rigidity of surfaces and variational problems in the theory of thin elastic sheets.

Notation. For a closed line segment $\{a+t(b-a) : t \in [0,1]\} \subset \mathbb{R}^2$, we write $[a,b]$. For a semiclosed line segment $\{a+t(b-a) : t \in (0,1]\} \subset \mathbb{R}^2$, we write $(a,b]$. Throughout the text, we will assume the deficit of the disclination $0 < \Delta < 1$ to be fixed. A statement such as "$f \leq Cg$" is shorthand for "there exists a constant $C > 0$ that only depends on Δ such that $f \leq Cg$". The value of C may change within the same line.

For $r > 0$, we let $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$. The two-sphere $\{x \in \mathbb{R}^3 : |x| = 1\}$ is denoted by S^2.

The one-dimensional Hausdorff measure is denoted by $\mathcal{H}^1$.

The pairing between a Radon measure μ and a continuous function f will be denoted by $\langle \mu, f \rangle$.

2. Proof of Theorem 1

As in [Olbermann 2017], the proof of the energy scaling law rests on two observations. First, by the weak formulation of the Hessian determinant,

$$\sum_{i=1}^{3} \det D^2 y_i = (y_{,1} \cdot y_{,2})_{,12} - \tfrac{1}{2}(|y_{,1}|^2)_{,22} - \tfrac{1}{2}(|y_{,2}|^2)_{,11} \quad \text{for } y \in C^2(B_1; \mathbb{R}^3), \tag{8}$$

we get that the quantity $\sum_{i=1}^{3} \det D^2 y_i$ is close to $\sum_{i=1}^{3} \det D^2 y_i^{\Delta} = \pi \Delta^2 \delta_0$ (the latter equation holding in the sense of distributions), where δ_0 denotes the Radon measure defined by $\langle \delta_0, f \rangle = f(0)$. The expression $\sum_{i=1}^{3} \det D^2 y_i$ is best thought of as the "linearized Gauss curvature": for a metric of the form $g_y = \mathrm{Id}_{2\times 2} + \varepsilon G$, the Gauss curvature is

$$K = \varepsilon \sum_{i=1}^{3} \det D^2 y_i + O(\varepsilon^2).$$

Second, the following Sobolev/isoperimetric inequality translates estimates for integrals of the Hessian determinant into lower bounds for boundary integrals of the tangential part of the second derivative.

Lemma 3. *For $v \in C^2(\bar{B}_1)$ and $0 \leq r \leq 1$,*

$$\int_{\partial B_r} |D^2 v| \,\mathrm{d}\mathcal{H}^1 \geq \left(4\pi \left| \int_{B_r} \det D^2 v \,\mathrm{d}x \right| \right)^{1/2}. \tag{9}$$

This inequality has been used in the literature in a number of places; see, e.g., [Müller 1990]. The proof of the statement above (including the sharp constant) can be found in [Olbermann 2017].

The main observation that allows for an improvement of the lower bound from [Olbermann 2017] is that we may get a lower bound for the quantity on the left-hand side in (9) from the smallness of the membrane energy directly by integrating a suitable test function against the membrane term $g_y - g_\Delta$. In our previous paper we obtained such an estimate by interpolation instead, which also uses the control over the bending energy. This is unnecessary, and gives slightly worse estimates.

The following calculation indicates how to use the smallness of the membrane term to obtain estimates on integrals of the linearized curvature. Let $\Phi \in L^1(B_1)$ be such that $D^2\Phi$ is a vector-valued Radon measure with support in B_1. In this case, we have $\Phi \in C^0(B_1)$, see [Demengel 1984, Theorem 3.3], and for all $y \in C^2(B_1;\mathbb{R}^3)$ we have

$$\begin{aligned}\int_{B_1}\Big(\sum_{i=1}^3 \det D^2 y_i(x)\Big)\Phi(x)\,\mathrm{d}\mathcal{L}^2 - \pi\Delta^2\langle\delta_0,\Phi\rangle \\ &= \int_{B_1}\big((y_{,1}\cdot y_{,2} - y^\Delta_{,1}\cdot y^\Delta_{,2})\Phi_{,12} - \tfrac12(|y_{,1}|^2 - |y^\Delta_{,1}|^2)\Phi_{,22} - \tfrac12(|y_{,2}|^2 - |y^\Delta_{,2}|^2)\Phi_{,11}\big)\,\mathrm{d}\mathcal{L}^2 \\ &= -\frac12\int_{B_1}(g_y - g_\Delta) : \operatorname{cof} D^2\Phi\,\mathrm{d}\mathcal{L}^2. \end{aligned}\tag{10}$$

Here,

$$\operatorname{cof} D^2\Phi = \begin{pmatrix}\Phi_{,22} & -\Phi_{,12}\\ -\Phi_{,21} & \Phi_{,11}\end{pmatrix}$$

denotes the cofactor matrix of $D^2\Phi$. Note that cof is linear on 2×2 matrices, and hence $\operatorname{cof} D^2\Phi$ is a well-defined Radon measure under our assumptions. After these preliminary remarks, we construct the upper bound in the statement of Theorem 1. It is obtained by a simple mollification of y^Δ on a ball of size h centered at the origin.

Lemma 4. *We have*

$$\inf_{y\in W^{2,2}(B_1;\mathbb{R}^3)} I_{h,\Delta}(y) \le 2\pi\Delta^2 h^2\Big(\log\frac1h + C\Big),$$

where $C = C(\Delta)$ does not depend on h.

Proof. This is the same upper bound construction as in [Olbermann 2017] (see Lemma 2 in that reference), and we will be brief. We choose $\eta \in C^\infty([0,\infty))$ with $\eta = 0$ on $\big[0,\frac12\big]$, $\eta = 1$ on $[1,\infty)$, and $|\eta'| \le C$, $|\eta''| \le C$. We set

$$y_h(x) = \eta\Big(\frac{|x|}{h}\Big)y^\Delta(x).$$

One easily shows

$$\begin{aligned} |g_{y_h} - g_\Delta| &\le C \quad\text{and}\quad & |D^2 y_h| &\le Ch^{-1} \quad \text{on } B_h,\\ g_{y_h} - g_\Delta &= 0 \quad\text{and}\quad & |D^2 y_h(x)| &= \frac{\Delta}{|x|} \quad \text{on } B_1\setminus B_h. \end{aligned}$$

This implies

$$\int_{B_1}|g_{y_h} - g_\Delta|^2\,\mathrm{d}\mathcal{L}^2 \le \int_{B_h} C\,\mathrm{d}\mathcal{L}^2 \le Ch^2,$$

$$\int_{B_1}|D^2 y_h|^2\,\mathrm{d}\mathcal{L}^2 \le \int_{B_1\setminus B_h}\frac{\Delta^2}{|x|^2}\,\mathrm{d}\mathcal{L}^2 + \int_{B_h}\frac{C}{h^2}\,\mathrm{d}\mathcal{L}^2 = 2\pi\Delta^2\int_h^1\frac{\mathrm{d}r}{r} + C = 2\pi\Delta^2\log\frac1h + C.$$

This implies the claim of the lemma. ∎

Proof of Theorem 1. The upper bound is proved by Lemma 4; hence we may choose C_2 to be the constant from that lemma. Now it suffices to show the following: there exist C_1, C_3 such that if $y \in W^{2,2}(B_1;\mathbb{R}^3)$ satisfies (4), then also the lower bound in (3) and (5), (6) hold true.

Let $y \in W^{2,2}(B_1;\mathbb{R}^3)$ satisfy (4). By density of C^2 in $W^{2,2}$, we may assume $y \in C^2(B_1;\mathbb{R}^3)$ for a proof of the remaining statements. Let $0<r<1$. Using Lemma 3, we have for $i=1,2,3$

$$\frac{1}{2\pi}\int_{\partial B_r}|D^2 y_i|\,\mathrm{d}\mathcal{H}^1 \geq \left(\frac{1}{\pi}\left|\int_{B_r}\det D^2 y_i\,\mathrm{d}\mathcal{L}^2\right|\right)^{1/2}.$$

Applying Jensen's inequality, we get

$$\frac{1}{2\pi r}\int_{\partial B_r}|D^2 y_i|^2\,\mathrm{d}\mathcal{H}^1 \geq \left(\frac{1}{2\pi r}\int_{\partial B_r}|D^2 y_i|\,\mathrm{d}\mathcal{H}^1\right)^2.$$

Combining these two estimates, we obtain

$$\int_{\partial B_r}|D^2 y_i|^2\,\mathrm{d}\mathcal{H}^1 \geq \frac{2}{r}\left|\int_{B_r}\det D^2 y_i\,\mathrm{d}\mathcal{L}^2\right|.$$

By the triangle inequality,

$$\int_{\partial B_r}|D^2 y|^2\,\mathrm{d}\mathcal{H}^1 \geq \frac{2}{r}\left|\int_{B_r}\sum_i \det D^2 y_i\,\mathrm{d}\mathcal{L}^2\right|. \tag{11}$$

Now choose $h_0 = h_0(y) \in [h, 2h]$ such that

$$\int_{\partial B_{h_0}}|g_y - g_\Delta|^2\,\mathrm{d}\mathcal{H}^1 \leq h^{-1}\int_{B_1}|g_y - g_\Delta|^2\,\mathrm{d}\mathcal{L}^2. \tag{12}$$

Choosing $R \in (h_0+h, 1)$ and integrating (11) over the range $r \in [h_0, R]$, we get

$$\begin{aligned}\int_{B_R\setminus B_{h_0}}|D^2 y|^2\,\mathrm{d}\mathcal{L}^2 &\geq 2\left|\int_{h_0}^{R}\frac{1}{r}\left(\int_{B_r}\sum_i \det D^2 y_i\,\mathrm{d}\mathcal{L}^2\right)\mathrm{d}r\right| \\ &= 2\left|\int_{h_0}^{R}\left(\int_{B_1}\frac{\chi_{B_r}(x)}{r}\sum_i \det D^2 y_i(x)\,\mathrm{d}\mathcal{L}^2(x)\right)\mathrm{d}r\right| \\ &= 2\left|\int_{B_1}\Phi\left(\sum_i \det D^2 y_i\right)\mathrm{d}\mathcal{L}^2\right|, \end{aligned}\tag{13}$$

where we have used Fubini's theorem to change the order of integration, and have defined the test function

$$\Phi(x) := \int_{h_0}^{R}\frac{1}{r}\chi_{B_r}(x)\,\mathrm{d}r = \begin{cases}\log(R/h_0) & \text{if } |x|\leq h_0,\\ \log(R/|x|) & \text{if } h_0<|x|\leq R,\\ 0 & \text{else.}\end{cases}$$

We add and subtract the term $2\pi\Delta^2\langle\delta_0,\Phi\rangle$, use the triangle inequality and obtain

$$\int_{B_R\setminus B_{h_0}}|D^2 y|^2\,\mathrm{d}\mathcal{L}^2 \geq 2\pi\Delta^2\log\frac{R}{h_0} - 2\left|\pi\Delta^2\langle\delta_0,\Phi\rangle - \int_{B_1}\Phi\left(\sum_i \det D^2 y_i\right)\mathrm{d}\mathcal{L}^2\right|. \tag{14}$$

Now we set

$$A(R) := \int_{B_1} \Phi\Big(\sum_i \det D^2 y_i\Big)\, \mathrm{d}\mathcal{L}^2 - \pi\Delta^2\langle \delta_0, \Phi\rangle = -\frac{1}{2}\int_{B_1} (g_y - g_\Delta) : \operatorname{cof} D^2\Phi\, \mathrm{d}\mathcal{L}^2, \tag{15}$$

where we have used (10) in the second line. An explicit computation yields

$$\begin{aligned} D\Phi(x) &= -\frac{x}{|x|^2}\chi_{B_R\setminus B_{h_0}}(x),\\ D^2\Phi(x) &= \big(-\mathrm{Id}_{2\times 2} + 2\hat{x}\otimes\hat{x}\big)|x|^{-2}\chi_{B_R\setminus B_{h_0}}(x) + |x|^{-1}\hat{x}\otimes\hat{x}\big(\mathcal{H}^1 \llcorner \partial B_R - \mathcal{H}^1 \llcorner \partial B_{h_0}\big). \end{aligned}$$

Inserting these computations in (15), we have

$$|A(R)| \le \int_{B_R\setminus B_{h_0}} \frac{|g_y - g_\Delta|}{|x|^2}\, \mathrm{d}\mathcal{L}^2 + \frac{1}{2R}\int_{\partial B_R} |g_y - g_\Delta|\, \mathrm{d}\mathcal{H}^1 + \frac{1}{2h_0}\int_{\partial B_{h_0}} |g_y - g_\Delta|\, \mathrm{d}\mathcal{H}^1. \tag{16}$$

By Cauchy–Schwarz,

$$\begin{aligned} \int_{B_R\setminus B_{h_0}} \frac{|g_y - g_\Delta|}{|x|^2}\, \mathrm{d}\mathcal{L}^2 &\le \Big(\int_{B_R\setminus B_{h_0}} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{L}^2\Big)^{1/2}\Big(\int_{B_R\setminus B_{h_0}} |x|^{-4}\, \mathrm{d}\mathcal{L}^2\Big)^{1/2}\\ &\le \Big(\int_{B_R\setminus B_{h_0}} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{L}^2\Big)^{1/2}\sqrt{2\pi}\, h_0^{-1},\\ \int_{\partial B_R} |g_y - g_\Delta|\, \mathrm{d}\mathcal{H}^1 &\le C\sqrt{R}\Big(\int_{\partial B_R} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{H}^1\Big)^{1/2},\\ \int_{\partial B_{h_0}} |g_y - g_\Delta|\, \mathrm{d}\mathcal{H}^1 &\le C\sqrt{h_0}\Big(\int_{\partial B_{h_0}} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{H}^1\Big)^{1/2}. \end{aligned} \tag{17}$$

Now choose $R_0 \in [R-h, R]$ such that

$$\int_{\partial B_{R_0}} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{H}^1 \le h^{-1}\int_{B_1} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{L}^2.$$

Together with (12) and (17), inequality (16) becomes

$$|A(R_0)| \le C\frac{E_{\mathrm{m}}(y)^{1/2}}{h_0},$$

where $E_{\mathrm{m}}(y)$ is the membrane energy,

$$E_{\mathrm{m}}(y) := \int_{B_1} |g_y - g_\Delta|^2\, \mathrm{d}\mathcal{L}^2.$$

The lower bound for the bending energy (13) becomes

$$\int_{B_{R_0}\setminus B_{h_0}} |D^2 y|^2\, \mathrm{d}\mathcal{L}^2 \ge 2\pi\Delta^2 \log\frac{R_0}{h_0} - C\frac{E_{\mathrm{m}}(y)^{1/2}}{h_0}. \tag{18}$$

We use (18) with $R \uparrow 1$ to estimate the membrane energy by

$$E_{\mathrm{m}}(y) \le 2\pi\Delta^2 h^2\Big(\log\frac{1}{h}+C_2\Big) - 2\pi\Delta^2 h^2\log\frac{1}{h_0} + Ch^2\frac{E_{\mathrm{m}}(y)^{1/2}}{h_0} \le C(h^2 + hE_{\mathrm{m}}(y)^{1/2}). \tag{19}$$

Using Young's inequality $ab \le \frac{1}{2}((\varepsilon a)^2 + (b/\varepsilon)^2)$, with $\varepsilon = C^{-1}$, we have

$$ChE_{\mathrm{m}}(y)^{1/2} \le \tfrac{1}{2}E_{\mathrm{m}}(y) + Ch^2,$$

and inserting this in (19), we get

$$E_{\mathrm{m}}(y) \le Ch^2,$$

which proves (6). Furthermore, inserting this in (18), we have

$$\int_{B_{R_0}\setminus B_{h_0}} |D^2 y|^2 \, \mathrm{d}\mathcal{L}^2 \ge 2\pi\Delta^2 \log\frac{R_0}{h} - C.$$

Sending $R \to 1$, this proves the lower bound in (3). Furthermore,

$$\begin{aligned}\int_{B_1\setminus B_R} |D^2 y|^2 \, \mathrm{d}\mathcal{L}^2 &\le h^{-2}(I_{h,\Delta}(y) - E_{\mathrm{m}}(y)) - \int_{h_0}^{R_0} |D^2 y|^2 \, \mathrm{d}\mathcal{L}^2 \\ &\le 2\pi\Delta^2\Big(\log\frac{1}{h}+C_2\Big) - 2\pi\Delta^2\log\frac{R_0}{h} \\ &\le 2\pi\Delta^2\log\frac{1}{R} + C,\end{aligned}$$

which proves (5). This completes the proof of the theorem. □

3. Proof of Theorem 2

3.1. ***Isometric immersions of a singular cone.*** The plan of the proof is as follows: The crucial inequality (5) shows that on a fixed annulus $B_1 \setminus B_R$, the $W^{2,2}$ norm of a sequence of deformations y_h satisfying $I_{h,\Delta}(y_h) \le 2\pi\Delta^2 h^2(\log 1/h + C)$ is bounded as $h \to 0$. One gets weak convergence of a subsequence in $W^{2,2}$ to a limit deformation that is an isometric immersion with respect to g_Δ (since the membrane energy of the limit function vanishes by $E_{\mathrm{m}}(y_h) \le Ch^2 \to 0$). We may apply the results on $W^{2,2}$ isometric immersions from [Hornung 2011a; Pakzad 2004] to the limit, which means that the limit deformation is developable. Using our energy estimates, we can show that in fact, it must be identical to the singular cone y^Δ up to a Euclidean motion.

The fact that flat surfaces are locally developable is a classical result from the differential geometry of surfaces. For functions in $W^{2,2}$, this statement has been proved in [Pakzad 2004; Hornung 2011a; 2011b]:

Theorem 5 [Hornung 2011a, Theorem 2]. *Let $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary. Let $y \in W^{2,2}(\Omega;\mathbb{R}^3)$ with $Dy^T Dy = \mathrm{Id}_{2\times 2}$ almost everywhere. Then $y \in C^1(\Omega)$ and there exists a set $\boldsymbol{L}_y$ of mutually disjoint closed line segments in $\overline{\Omega}$ with endpoints on $\partial\Omega$ with the following property. For every $x \in \Omega$, exactly one of the following alternatives hold: either $D^2 y = 0$ in a neighborhood of x, or there exists $L \in \boldsymbol{L}_y$ with $x \in L$ and Dy is constant on L.*

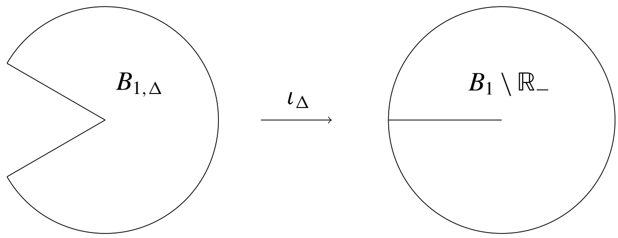

Figure 1. The domain $B_{1,\Delta}$ and the map $\iota_\Delta : B_{1,\Delta} \to B_1 \setminus \mathbb{R}_-$.

Lemma 6. *Let y be as in Theorem 5, and $\widetilde{\Omega} \subseteq \Omega$. Let $\tilde{y}$ be the restriction of y to $\widetilde{\Omega}$. Then for every $\widetilde{L} \in \boldsymbol{L}_{\tilde{y}}$ there exists exactly one $L \in \boldsymbol{L}_y$ such that $\widetilde{L} \subseteq L$. In particular, $\boldsymbol{L}_y$ is unique.*

Proof. From the properties of $\boldsymbol{L}_y$, it is clear that there can be at most one L with the stated property. Suppose there is $\widetilde{L} \in \boldsymbol{L}_{\tilde{y}}$ such that there does not exist $L \in \boldsymbol{L}_y$ with $\widetilde{L} \subseteq L$. Choose $x_0 \in \widetilde{L} \setminus \partial\widetilde{\Omega}$, and choose $r > 0$ such that $B(x_0, 2r) \subset \widetilde{\Omega}$. For every $x \in \widetilde{L} \cap B(x_0, r)$ the following holds true:

There does not exist a neighborhood of x on which $D^2 y$ vanishes. Hence there exists a line segment $L_x \in \boldsymbol{L}_y$ that intersects $\widetilde{L}$ only in x such that Dy is constant on L_x. Hence $Dy(z) = Dy(x) = Dy(x_0)$ for all $z \in L_x$.

Since the line segments $\{L_x : x \in \widetilde{L} \cap B(x_0, r)\}$ are mutually disjoint and their endpoints are outside $B(x_0, r)$, we have that there exists a neighborhood U of x_0 that is covered by the union of these line segments,

$$U \subset \bigcup_{x \in \widetilde{L} \cap B(x_0, r)} L_x.$$

This implies that Dy is constant on U, a contradiction. □

We will need a variant of Theorem 5 for functions whose domain is a singular cone.

To be able to use Theorem 5, we are going to consider the cone in a flat reference configuration. Let $\arccos : [-1, 1] \to [0, \pi]$ denote the inverse of $\cos : [0, \pi] \to [-1, 1]$. Define

$$B_{1,\Delta} := \left\{ x = (x_1, x_2) \in B_1 \setminus \{0\} : 0 \leq \arccos \frac{x_1}{|x|} < \sqrt{1 - \Delta^2}\pi \right\}.$$

Let $\mathbb{R}_- := \{(x_1, 0) : x_1 \leq 0\}$, and let $\varphi : \mathbb{R}^2 \setminus \mathbb{R}_- \to \mathbb{R}$ be the angular coordinate satisfying $x = |x|(\cos\varphi(x), \sin\varphi(x))$ with values in $(-\pi, \pi)$. We define the map $\iota \equiv \iota_\Delta : \mathbb{R}^2 \setminus \mathbb{R}_- \to B_1$ by

$$\iota(x) = \left(|x| \cos \frac{\varphi(x)}{\sqrt{1 - \Delta^2}}, |x| \sin \frac{\varphi(x)}{\sqrt{1 - \Delta^2}} \right).$$

For a sketch of $B_{1,\Delta}$ and ι_Δ, see Figure 1.

On $\iota(B_{1,\Delta}) = B_1 \setminus \mathbb{R}_-$, the map ι has a well-defined inverse, which we denote by

$$j : B_1 \setminus \mathbb{R}_- \to B_{1,\Delta}.$$

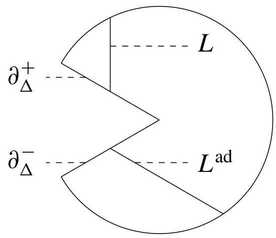

Figure 2. The subsets ∂_Δ^+, ∂_Δ^- of the boundary and adjoint line segments L, L^{ad}.

Furthermore, let $\phi_\Delta := (1-\sqrt{1-\Delta^2})2\pi$ and let the rotation $S_\Delta \in SO(2)$ be defined by

$$S_\Delta = \begin{pmatrix} \cos\phi_\Delta & -\sin\phi_\Delta \\ \sin\phi_\Delta & \cos\phi_\Delta \end{pmatrix}.$$

Finally, let

$$\partial_\Delta := \partial B_{1,\Delta} \setminus (\partial B_1 \cup \{0\}).$$

Note that ∂_Δ has two connected components, one contained in the upper half-plane and one in the lower half-plane. We will denote them by ∂_Δ^+ and ∂_Δ^- respectively; see Figure 2. The rotation matrix S_Δ has been chosen such that $S_\Delta \partial_\Delta^+ = \partial_\Delta^-$.

We define

$$W^{2,2}_{\mathrm{iso}}(B_{1,\Delta}) := \bigl\{ Y \in W^{2,2}_{\mathrm{loc}}(\bar B_{1,\Delta} \setminus \{0\}; \mathbb{R}^3) \\ : g_Y = \mathrm{Id}_{2\times 2},\ Y(S_\Delta x) = Y(x) \text{ and } DY(S_\Delta x) = DY(x) S_\Delta \text{ for every } x \in \partial_\Delta^+ \bigr\} \quad (20)$$

This definition is chosen such that if $y \in W^{2,2}_{\mathrm{loc}}(\bar B_1 \setminus \{0\}; \mathbb{R}^3)$ with $Dy^T Dy = g_\Delta$, then $y \circ \iota \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$.

To $Y \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$, we may apply Theorem 5 with $\Omega = B_{1,\Delta} \setminus B_\rho$ to obtain a set $\boldsymbol{L}_Y^{(\rho)}$ of line segments with the properties stated there. For $\rho < \rho'$ we have by the uniqueness of the line segments stated in Lemma 6 that every line segment in $\boldsymbol{L}_Y^{(\rho')}$ is contained in exactly one line segment of $\boldsymbol{L}_Y^{(\rho)}$.

Hence, by sending $\rho \to 0$, we get a set of (relatively) closed mutually disjoint line segments in $\bar B_{1,\Delta} \setminus \{0\}$, denoted by $\boldsymbol{L}_Y$.

If a line segment in $\boldsymbol{L}_Y$ has only one endpoint in $\bar B_{1,\Delta} \setminus \{0\}$, then we say by slight abuse of terminology that one of its endpoints is the origin.

Remark 7. We note in passing that with obvious modifications of the previous construction, one may extend Theorem 5 to maps with conical singularities, i.e., to maps $y \in W^{2,2}_{\mathrm{loc}}(\bar\Omega \setminus \{x_0\}; \mathbb{R}^3)$ with $x_0 \in \Omega$ and $Dy^T Dy = \mathrm{Id}_{2\times 2}$ almost everywhere.

Next, we are going to define an "adjoint" line segment L^{ad} to any $L \in \boldsymbol{L}_Y$ with an endpoint $x \in \partial_\Delta$. Note that for such L, there exists $v \in \partial B_1$ and $q > 0$ such that

$$L = \{x + tv : t \in [0, q]\}.$$

First let us assume $x \in \partial_\Delta^+$. By the definition of $W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$ in (20), we have that $x' := S_\Delta x \in \partial_\Delta^-$, and $DY(x') = DY(x) S_\Delta$. Moreover, there has to exist $L^{\mathrm{ad}} \in \boldsymbol{L}_y$ with $x' \in L^{\mathrm{ad}}$, and

$$L^{\mathrm{ad}} = \{x' + t S_\Delta v : t \in \mathbb{R}\} \cap \bar B_{1,\Delta}.$$

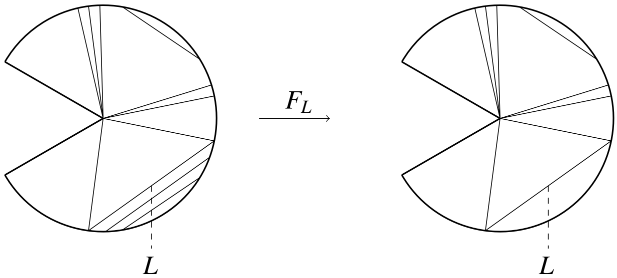

Figure 3. In the left panel, we have the segments that belong to $\boldsymbol{L}_Y$, and $L \in \boldsymbol{L}_Y$ is a bad line segment. We can flatten the deformation Y on the side of L whose closure does not contain the origin, and obtain a deformation $F_L(Y)$, such that $\boldsymbol{L}_{F_L(Y)}$ consists of those line segments in $\boldsymbol{L}_Y$ that are on the same side of L as the origin; see the right panel.

This defines L^{ad} for $x \in \partial_\Delta^+$; for $x \in \partial_\Delta^-$, we define it analogously, replacing S_Δ by S_Δ^{-1}. For a sketch of the construction, see Figure 2.

From now on, the line segments in $\boldsymbol{L}_Y$ for which one of the endpoints is 0 will be called "good", and line segments in the complement of the set of good line segments will be called "bad". The sets of good and bad line segments will be denoted by $\boldsymbol{L}_Y^{(g)}$, $\boldsymbol{L}_Y^{(b)}$ respectively. For any bad line segment, we can lower the elastic energy by "flattening" the deformation Y on one side of the line segment. This is the idea behind the following lemma. For a sketch of this operation, see Figure 3.

Lemma 8. *For every $Y \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$, there exists $Y_\infty \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$ with the following properties*:

(i) $\boldsymbol{L}^{(b)}_{Y_\infty} = \varnothing$ *and* $\boldsymbol{L}^{(g)}_{Y_\infty} = \boldsymbol{L}^{(g)}_Y$.

(ii) *For* $0 < \rho < 1$, *we have*

$$\int_{B_{1,\Delta}\setminus B_\rho} \left|D^2 Y_\infty : ((D\iota)^{-1} \otimes (D\iota)^{-1})\right|^2 \mathrm{d}\mathcal{L}^2 \le \int_{B_{1,\Delta}\setminus B_\rho} \left|D^2 Y : ((D\iota)^{-1} \otimes (D\iota)^{-1})\right|^2 \mathrm{d}\mathcal{L}^2, \tag{21}$$

with equality for all $0 < \rho < 1$ *if and only if* $Y = Y_\infty$.

Proof. For any $L \in \boldsymbol{L}_Y^{(b)}$, we may define a modified map $F_L(Y) \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$ as follows. On L, we have $Y = A_L x + b_L$ for some $A_L \in \mathbb{R}^{3\times 2}$ and $b_L \in \mathbb{R}^3$. We note that $B_{1,\Delta} \setminus L$ has exactly two connected components. Let E_L denote the connected component whose closure does not contain the origin. First let us assume that none of the endpoints of L is in ∂_Δ. Then we define $F_L(Y) \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$ by

$$F_L(Y)(x) = \begin{cases} A_L x + b_L & \text{if } x \in E_L, \\ Y(x) & \text{else.} \end{cases} \tag{22}$$

If one of the endpoints of L is in ∂_Δ, then we set

$$F_L(Y)(x) = \begin{cases} A_L x + b_L & \text{if } x \in E_L, \\ A_{L^{\mathrm{ad}}} x + b_{L^{\mathrm{ad}}} & \text{if } x \in E_{L^{\mathrm{ad}}}, \\ Y(x) & \text{else.} \end{cases}$$

Note that this definition indeed satisfies $F_L(Y) \in W^{2,2}_{\text{iso}}(B_{1,\Delta})$. Obviously, we have $D^2(F_L(Y)) = 0$ on E_L (and on $E_{L^{\text{ad}}}$) and hence, for all $0 < \rho < 1$, we have

$$\int_{B_{1,\Delta}\setminus B_\rho} |D^2 F_L(Y)(D\iota)^{-1}|^2 \, d\mathcal{L}^2 \leq \int_{B_{1,\Delta}\setminus B_\rho} |D^2 Y(D\iota)^{-1}|^2 \, d\mathcal{L}^2. \tag{23}$$

We must distinguish two cases in (23): If $\boldsymbol{L}_{F_L(Y)} \subsetneq \boldsymbol{L}_Y$, then $F_L(Y) \neq Y$ and we must have $|D^2 Y| > 0$ on a subset of positive measure of E_L. Hence, inequality must hold in (23) for some ρ, since we have

$$\sqrt{1-\Delta^2}\,\text{Id}_{2\times 2} \leq (D\iota)^{-1} \leq \text{Id}_{2\times 2} \tag{24}$$

in the sense of positive definite matrices. Equality in (23) only holds in the case $F_L(Y) = Y$.

On $\boldsymbol{L}^{(b)}_Y$, we may define an order relation by $L < L'$ if $E_L \subsetneq E_{L'}$. Since bad line segments are mutually disjoint, we have that either $L < L'$, $L > L'$ or $E_L \cap E_{L'} = \varnothing$. Hence, there exists an at most countable sequence $L_1, L_2, \ldots$ of maximal bad line segments. If for two maximal line segments L, L' we have $L' = L^{\text{ad}}$ then we exclude exactly one of them from that sequence. Now we define a sequence $Y_k \in W^{2,2}_{\text{iso}}(B_{1,\Delta})$ by

$$Y_k = F_{L_k} \circ \cdots \circ F_{L_1}(Y). \tag{25}$$

By (23) and (24), $D^2 Y_k$ is bounded in L^2. Thus the sequence converges weakly in $W^{2,2}(B_{1,\Delta} \setminus B_\rho; \mathbb{R}^3)$ for every $0 < \rho < 1$ to a limit $Y_\infty \in W^{2,2}_{\text{iso}}(B_{1,\Delta})$ such that $\boldsymbol{L}_{Y_\infty}$ does not contain any bad line segments, and $\boldsymbol{L}^{(g)}_{Y_\infty} = \boldsymbol{L}^{(g)}_Y$. The claim (21) follows from (23) and the comment after that equation. □

Remark 9. Letting Y, Y_∞ as in Lemma 8, we have that DY_∞ is constant on every line segment $(0, x)$ for $x \in \partial B_{1,\Delta} \cap \partial B_1$, and

$$Y_\infty \circ j \in W^{2,2}_{\text{loc}}(\bar{B}_1 \setminus \{0\}; \mathbb{R}^3), \qquad g_{Y_\infty \circ j} = g_\Delta.$$

Furthermore,

$$\int_{B_1\setminus B_\rho} |D^2(Y_\infty \circ j)|^2 \, d\mathcal{L}^2 \leq \int_{B_1\setminus B_\rho} |D^2(Y \circ j)|^2 \, d\mathcal{L}^2 \quad \text{for every } 0 < \rho < 1.$$

Proof. The first statement in the remark follows from the fact that for every $x' \in B_{1,\Delta}$, we have that either there exists $x \in \partial B_{1,\Delta} \cap \partial B_1$ such that $x' \in (0, x] \in \boldsymbol{L}^{(g)}_{Y_\infty}$ or there exists a sector containing x' that has empty intersection with every $L \in \boldsymbol{L}_{Y_\infty}$, and hence DY_∞ vanishes in the whole sector.

The second and third statements follow immediately from $Y_\infty \in W^{2,2}_{\text{iso}}(B_{1,\Delta})$. It remains to prove the inequality. Let $\nu = Y_{,1} \wedge Y_{,2}/|Y_{,1} \wedge Y_{,2}|$ be the unit normal. By $DY^T DY = \text{Id}_{2\times 2}$, we have $D^2 Y \perp DY$. Hence

$$\begin{aligned} |D^2(Y \circ j)|^2 &= |D^2 Y : (Dj \otimes Dj) + DY\, D^2 j|^2 \\ &= |D^2 Y : (Dj \otimes Dj)|^2 + |DY\, D^2 j|^2 \\ &= |D^2 Y : (Dj \otimes Dj)|^2 + |D^2 j|^2, \end{aligned} \tag{26}$$

where we used $DY \in O(2,3)$ in the last equality. Now the inequality follows from (21) and a change of variables in the integrals. □

3.2. *Proof of Theorem 2.* Given $0 < R < 1$, we may assume that $h \ll R$. Choose $R_0(h) \in [R-h, R]$ such that

$$\int_{\partial B_{R_0(h)}} |g_{y_h} - g_\Delta|^2 \, \mathrm{d}\mathcal{H}^1 \le h^{-1} \int_{B_1} |g_{y_h} - g_\Delta|^2 \, \mathrm{d}\mathcal{L}^2.$$

By Theorem 1, we have

$$\int_{B_1 \setminus B_R} |D^2 y_h| \, \mathrm{d}\mathcal{L}^2 \le \int_{B_1 \setminus B_{R_0}} |D^2 y_h| \, \mathrm{d}\mathcal{L}^2 \le 2\pi \Delta^2 \log \frac{1}{R} + C, \tag{27}$$

where C depends neither on h nor on R. This proves the boundedness of y_h in $W^{2,2}(B_1 \setminus B_R; \mathbb{R}^3)$ and implies that there exists $\hat{y}_R \in W^{2,2}(B_1 \setminus B_R; \mathbb{R}^3)$ such that (for a subsequence)

$$y_h \rightharpoonup \hat{y}_R \quad \text{in } W^{2,2}(B_1 \setminus B_R; \mathbb{R}^3).$$

After taking a suitable diagonal sequence for $R = 1/j$, $j = 2, 3, \ldots$, we may assume that $\hat{y}_R \in W^{2,2}_{\mathrm{loc}}(B_1 \setminus \{0\}; \mathbb{R}^3)$ is independent of R. We denote this function by y^*. By Theorem 1, we have

$$\int_{B_1} |g_{y^*} - g_\Delta| \, \mathrm{d}\mathcal{L}^2 = 0;$$

i.e., y^* is an isometry with respect to g_Δ.

By (27), we have

$$\int_{B_1 \setminus B_R} |D^2 y^*|^2 \, \mathrm{d}\mathcal{L}^2 \le 2\pi \Delta^2 h^2 \log \frac{1}{R} + C. \tag{28}$$

Let $Y : B_{1,\Delta} \to \mathbb{R}^3$ be defined by

$$Y := y^* \circ \iota.$$

Recalling the definitions from Section 3.1, we have $Y \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$. By an application of Lemma 8 and Remark 9, we obtain $Y_\infty \in W^{2,2}_{\mathrm{iso}}(B_{1,\Delta})$ such that DY_∞ is constant on every line segment $(0, x)$ with $x \in \partial B_{1,\Delta} \cap \partial B_1$. Now we set $y^\infty := Y_\infty \circ j$, and obtain that Dy^∞ is constant on every line segment $(0, x]$ with $x \in \partial B_1$. Hence there exists a curve $\gamma : \partial B_1 \to S^2$ satisfying $|\gamma'| = \sqrt{1 - \Delta^2}$ such that

$$y^\infty(x) = x\gamma\left(\frac{x}{|x|}\right). \tag{29}$$

Using this expression, explicit computation yields

$$\int_{\partial B_\rho} |D^2 y^\infty|^2 \, \mathrm{d}\mathcal{H}^1 = \frac{1}{\rho} \int_{\partial B_1} |D^2 y^\infty|^2 \, \mathrm{d}\mathcal{H}^1. \tag{30}$$

By Remark 9 and (28), we have that for every $0 < \rho < 1$,

$$\int_{B_1 \setminus B_\rho} |D^2 y^\infty|^2 \, \mathrm{d}\mathcal{L}^2 \le 2\pi \Delta^2 \log \frac{1}{\rho} + C. \tag{31}$$

Combining (30) and (31), we see that for every $0 < \rho < 1$, we have

$$\int_{\partial B_\rho} |D^2 y^\infty|^2 \, \mathrm{d}\mathcal{H}^1 \le \frac{2\pi \Delta^2}{\rho},$$

and the constant C in (31) is in fact 0.

By $g_{y^\infty} = g_\Delta$, we have

$$\sum_{i=1}^{3} \det D^2 y_i^\infty = \pi \Delta^2 \delta_0$$

distributionally. We may now estimate using Lemma 3, for any $0 < \rho < 1$,

$$\begin{aligned}
\pi \Delta^2 &= \int_{B_\rho} \sum_{i=1}^{3} \det D^2 y_i^\infty \, \mathrm{d}\mathcal{L}^2 \leq \sum_i \left| \int_{B_\rho} \det D^2 y_i^\infty \, \mathrm{d}\mathcal{L}^2 \right| \\
&\leq \frac{1}{4\pi} \sum_i \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp| \, \mathrm{d}\mathcal{H}^1(x) \right)^2 \\
&\leq \frac{1}{4\pi} \sum_i 2\pi\rho \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp|^2 \, \mathrm{d}\mathcal{H}^1(x) \right) \\
&\leq \frac{\rho}{2} \int_{\partial B_\rho} |D^2 y^\infty(x) \cdot \hat{x}^\perp|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq \pi \Delta^2.
\end{aligned} \tag{32}$$

Here, to obtain the third from the second line, we used Jensen's inequality. By this chain of estimates, all the inequalities must have been equalities, and we have

$$\sum_i \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp| \, \mathrm{d}\mathcal{H}^1(x) \right)^2 = \sum_i 2\pi\rho \left(\int_{\partial B_\rho} |D^2 y_i^\infty(x) \cdot \hat{x}^\perp|^2 \, \mathrm{d}\mathcal{H}^1(x) \right)$$

and thus

$$|D^2 y_i^\infty(x) \cdot \hat{x}^\perp|^2 = \text{constant} \quad \text{for } x \in \partial B_\rho,\ i \in \{1, 2, 3\}. \tag{33}$$

Additionally, (32) implies

$$|D^2 y^\infty(x) \cdot \hat{x}^\perp|^2 = \frac{\Delta^2}{\rho^2} \quad \text{for } x \in \partial B_\rho. \tag{34}$$

By (29), we have $D^2 y^\infty(x) = |x|^{-1}(\gamma + \gamma'') \otimes \hat{x}^\perp \otimes \hat{x}^\perp$. Combining this with (33), we get

$$(\gamma + \gamma'') \cdot e_i = \text{constant on } \partial B_1$$

for $i = 1, 2, 3$. We write $c_i = (\gamma + \gamma'') \cdot e_i$, and have $D^2 y_i^\infty(x) = (c_i/|x|)\hat{x}^\perp \otimes \hat{x}^\perp$, which implies

$$y_i^\infty(x) = c_i |x| + a_i \cdot x + b_i \quad \text{for } i = 1, 2, 3,$$

for some $a_i \in \mathbb{R}^2$, $b_i \in \mathbb{R}$. By (33) we obtain

$$|D^2 y^\infty(x)|^2 = \frac{\sum_i c_i^2}{|x|^2} = \frac{\Delta^2}{|x|^2},$$

and thus $\sum_i c_i^2 = \Delta^2$. By $g_{y^\infty} = g_\Delta$, we have

$$\mathrm{Id}_{2\times 2} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp = (c \otimes \hat{x} + a)^T (c \otimes x + a) = |c|^2 \hat{x} \otimes \hat{x} + (c \cdot a) \otimes \hat{x} + \hat{x} \otimes (c \cdot a) + a^T a.$$

This yields

$$(1 - \Delta^2)\mathrm{Id}_{2\times 2} = (c \cdot a) \otimes \hat{x} + \hat{x} \otimes (c \cdot a) + a^T a,$$

which can only hold true for all $\hat{x} \in \partial B_1$ if $c \cdot a = 0$ and $a^T a = (1-\Delta^2)\mathrm{Id}_{2\times 2}$. This implies

$$R := \left(\frac{a}{\sqrt{1-\Delta^2}}, \frac{c}{\Delta} \right) \in O(3)$$

is an orthogonal matrix, and we have

$$y^\infty(x) = R\big(\sqrt{1-\Delta^2}x + \Delta e_3|x|\big) + b.$$

It remains to show that $y^\infty = y^*$. To see this, note that $y^\infty \circ \iota = Y_\infty$ satisfies

$$\big\{(0, x] : x \in \partial B_{1,\Delta} \cap \partial B_1\big\} = \boldsymbol{L}^{(g)}_{Y_\infty} = \boldsymbol{L}^{(g)}_Y,$$

where the second equality holds by Lemma 8. This implies that for every $x \in B_{1,\Delta}$ there exists an $L \in \boldsymbol{L}^{(g)}_Y$ with $x \in L$. This in turn implies that $\boldsymbol{L}^{(b)}_Y = \varnothing$ (since the line segments in $\boldsymbol{L}_Y$ are pairwise disjoint). By Lemma 8, the latter yields $Y = Y_\infty$. Composing with j on both sides of this last equation, we obtain $y^* = y^\infty$. This completes the proof of the theorem. □

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