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DINI AND SCHAUDER ESTIMATES FOR NONLOCAL FULLY NONLINEAR PARABOLIC EQUATIONS WITH DRIFTS

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We obtain Dini- and Schauder-type estimates for concave fully nonlinear nonlocal parabolic equations of order $\sigma \in (0, 2)$ with rough and nonsymmetric kernels and drift terms. We also study such linear equations with only measurable coefficients in the time variable, and obtain Dini-type estimates in the spacial variable. This is a continuation of work by the authors Dong and Zhang.

1. Introduction and main results

The paper is a continuation of [Dong and Zhang 2016a; 2016b] by the first and last authors, where they obtained Schauder-type estimates for concave fully nonlinear nonlocal parabolic equations and Dini-type estimates for concave fully nonlinear nonlocal elliptic equations. Here, we consider concave fully nonlinear nonlocal parabolic equations with Dini continuous coefficients, drifts and nonhomogeneous terms, and establish a C^σ estimate under these assumptions.

The study of second-order equations with Dini continuous coefficients and data dates back to at least 1970s, when Burch [1978] first considered divergence-type linear elliptic equations with Dini continuous coefficients and data, and estimated the modulus of continuity of the derivatives of solutions. Later work for second-order linear or concave fully nonlinear elliptic and parabolic equations with Dini data includes, for example, [Sperner 1981; Lieberman 1987; Safonov 1988; Kovats 1997; Bao 2002; Duzaar and Gastel 2002; Wang 2006; Maz'ya and McOwen 2011; Li 2017], and many others.

The regularity theory for nonlocal elliptic and parabolic equations has been developed extensively in recent years. For example, C^α estimates, $C^{1,\alpha}$ estimates, an Evans–Krylov-type theorem, and Schauder estimates were established in the past decade. See, for instance, [Caffarelli and Silvestre 2009; 2011; Dong and Kim 2012; 2013; Kim and Lee 2013; Lara and Dávila 2014; Mikulevičius and Pragarauskas 2014; Chang-Lara and Kriventsov 2017; Jin and Xiong 2015; 2016; Serra 2015; Mou 2016; Imbert et al. 2016]. In particular, Mou [2016] investigated a class of concave fully nonlinear nonlocal elliptic equations with smooth symmetric kernels, and obtained the C^σ estimate under a slightly stronger assumption than the usual Dini continuity on the coefficients and data. He implemented a recursive Evans–Krylov theorem, which was first studied by Jin and Xiong [2016], as well as a perturbation-type argument. By using a novel perturbation-type argument, the first and last authors proved the C^σ estimate for

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concave fully nonlinear elliptic equations in [Dong and Zhang 2016a], which relaxed the regularity assumption to simply Dini continuity and also removed the symmetry and smoothness assumptions on the kernels.

In this paper, we extend the results in [Dong and Zhang 2016a] from elliptic equations to parabolic equations with drifts; that is, we study fully nonlinear nonlocal parabolic equations in the form

$$\partial_t u = \inf_{\beta \in \mathcal{A}} (L_\beta u + b_\beta Du + f_\beta), \tag{1-1}$$

where $\mathcal{A}$ is an index set and for each $\beta \in \mathcal{A}$,

$$L_\beta u = \int_{\mathbb{R}^d} \delta u(t,x,y) K_\beta(t,x,y)\, dy,$$

$$\delta u(t,x,y) = \begin{cases} u(t,x+y) - u(t,x) & \text{for } \sigma \in (0,1), \\ u(t,x+y) - u(t,x) - y \cdot Du(t,x)\chi_{B_1} & \text{for } \sigma = 1, \\ u(t,x+y) - u(t,x) - y \cdot Du(t,x) & \text{for } \sigma \in (1,2), \end{cases}$$

and

$$K_\beta(t,x,y) = a_\beta(t,x,y)|y|^{-d-\sigma}.$$

This type of nonlocal operator was first investigated by Komatsu [1984], Mikulevičius and Pragarauskas [1992; 2014], and later by Dong and Kim [2012; 2013], and Schwab and Silvestre [2016].

We assume

$$(2-\sigma)\lambda \le a_\beta(\cdot,\cdot,\cdot) \le (2-\sigma)\Lambda \quad \text{for all } \beta \in \mathcal{A},$$

for some ellipticity constants $0 < \lambda \le \Lambda$, and is merely measurable with respect to the y-variable. When $\sigma = 1$, we additionally assume

$$\int_{S_r} y K_\beta(t,x,y)\, ds_y = 0 \tag{1-2}$$

for any $r > 0$, where S_r is the sphere of radius r centered at the origin.

We also assume $b_\beta \equiv 0$ when $\sigma < 1$ and $b_\beta = b(t,x)$ is independent of β when $\sigma = 1$.

We say that a function f is Dini continuous if its modulus of continuity ω_f is a Dini function, i.e.,

$$\int_0^1 \frac{\omega_f(r)}{r}\, dr < \infty.$$

We need the Dini continuity assumptions on the coefficients of (1-1):

$$\begin{gathered} \sup_{\beta \in \mathcal{A}} \int_{B_{2r} \setminus B_r} |a_\beta(t,x,y) - a_\beta(t',x',y)|\, dy \le \Lambda r^d \omega_a(\max\{|x-x'|, |t-t'|^{\frac{1}{\sigma}}\}) \quad \text{for all } r > 0, \\ \sup_{\beta \in \mathcal{A}} \|f_\beta\|_{L_\infty(Q_1)} < \infty, \quad \sup_{\beta \in \mathcal{A}} |f_\beta(t,x) - f_\beta(t',x')| \le \omega_f(\max\{|x-x'|, |t-t'|^{\frac{1}{\sigma}}\}), \\ \sup_{\beta \in \mathcal{A}} \|b_\beta\|_{L_\infty(Q_1)} \le N_0, \quad \sup_{\beta \in \mathcal{A}} |b_\beta(t,x) - b_\beta(t',x')| \le \omega_b(\max\{|x-x'|, |t-t'|^{\frac{1}{\sigma}}\}), \\ \text{where } N_0 > 0, \text{ and } \omega_a, \omega_b, \omega_f \text{ are all Dini functions.} \end{gathered} \tag{1-3}$$

In Theorem 1.1 below, ω_u denotes the modulus of continuity of u in $(-1,0)\times\mathbb{R}^d$; that is,

$$|u(t,x)-u(t',x')|\le\omega_u(\max\{|x-x'|,|t-t'|^{\frac{1}{\sigma}}\})\quad\text{for all }(t,x),(t,x')\in(-1,0)\times\mathbb{R}^d.$$

We also use the notation $C^{1,\sigma^+}(Q_1)$ to denote $C^{1,\sigma+\varepsilon}_{t,x}(Q_1)$ for some arbitrarily small $\varepsilon>0$. This condition is only needed for $L_\beta u$ to be well defined, and may be replaced by other weaker conditions.

Theorem 1.1. *Let $\sigma\in(0,2)$, $0<\lambda\le\Lambda<\infty$, and $\mathcal{A}$ be an index set. Assume for each $\beta\in\mathcal{A}$, K_β satisfies (1-2) when $\sigma=1$, and the Dini continuity assumption (1-3) holds for all $(t,x),(t',x')\in Q_1$. Suppose $u\in C^{1,\sigma^+}(Q_1)$ is a solution of (1-1) in Q_1 and is Dini continuous in $(-1,0)\times\mathbb{R}^d$. Then we have $\partial_t u$ is uniformly continuous and the a priori estimate*

$$\|\partial_t u\|_{L_\infty(Q_{1/2})}+[u]^x_{\sigma;Q_{1/2}}\le C\sum_{j=0}^{\infty}(2^{-j\sigma}\omega_u(2^j)+\omega_u(2^{-j})+\omega_f(2^{-j})),\tag{1-4}$$

where $C>0$ is a constant depending only on d, σ, λ, Λ, N_0, ω_b, and ω_a. Moreover, when $\sigma\neq1$, we have

$$\sup_{(t_0,x_0)\in Q_{1/2}}[u]^x_{\sigma;Q_r(t_0,x_0)}\to0\quad\text{as }r\to0$$

with a decay rate depending only on d, σ, λ, Λ, ω_a, ω_f, ω_u, N_0, and ω_b. When $\sigma=1$, Du is uniformly continuous in $Q_{1/2}$ with a modulus of continuity controlled by the quantities before.

This theorem improves Theorem 1.1 in [Dong and Zhang 2016a] in the following two ways. First, (1-1) is parabolic and has drift terms. Second, the right-hand side of the estimate (1-4) depends only on the seminorms of u and f, in particular, not on $\sup_{\beta\in\mathcal{A}}\|f_\beta\|_{L_\infty(Q_1)}$.

Remark 1.2. When $\sigma\in(1,2)$ in Theorem 1.1, by interpolation inequalities we have

$$[Du]^t_{\frac{\sigma-1}{\sigma};Q_{1/2}}\le C(\|\partial_t u\|_{L_\infty(Q_{1/2})}+[u]^x_{\sigma;Q_{1/2}})\le C\sum_{j=0}^{\infty}(2^{-j\sigma}\omega_u(2^j)+\omega_u(2^{-j})+\omega_f(2^{-j})).$$

The same proof of Theorem 1.1 can be used to prove Schauder estimates for concave fully nonlinear nonlocal parabolic equations with drifts. To this end, we need the Hölder continuity assumptions on the coefficients of (1-1):

$$\begin{gathered}\sup_{\beta\in\mathcal{A}}\int_{B_{2r}\setminus B_r}|a_\beta(t,x,y)-a_\beta(t',x',y)|\,dy\le\Lambda r^d\max\{|x-x'|^\gamma,|t-t'|^{\frac{\gamma}{\sigma}}\}\quad\text{for all }r>0,\\ \sup_{\beta\in\mathcal{A}}\|f_\beta\|_{L_\infty(Q_1)}<\infty,\quad \sup_{\beta\in\mathcal{A}}|f_\beta(t,x)-f_\beta(t',x')|\le C_f\max\{|x-x'|^\gamma,|t-t'|^{\frac{\gamma}{\sigma}}\},\\ \sup_{\beta\in\mathcal{A}}\|b_\beta\|_{L_\infty(Q_1)}\le N_0,\quad \sup_{\beta\in\mathcal{A}}|b_\beta(t,x)-b_\beta(t',x')|\le C_b\max\{|x-x'|^\gamma,|t-t'|^{\frac{\gamma}{\sigma}}\},\\ \text{where }N_0,C_f,C_b>0,\text{ and }\gamma\in(0,1).\end{gathered}\tag{1-5}$$

Recall that we assume $b_\beta\equiv0$ when $\sigma<1$, and $b_\beta=b(t,x)$ is independent of β when $\sigma=1$.

Theorem 1.3. *Let $\sigma \in (0,2)$, $0 < \lambda \le \Lambda < \infty$, and $\mathcal{A}$ be an index set. There exists $\hat{\alpha}$ depending only on d, λ, Λ and σ (uniform as $\sigma \to 2^-$) such that the following holds. Let $\gamma \in (0, \hat{\alpha})$ such that $\sigma + \gamma < 2$ is not an integer. Assume for each $\beta \in \mathcal{A}$, K_β satisfies (1-2) when $\sigma = 1$, and the Hölder continuity assumptions (1-5) hold for all $(t,x), (t',x') \in Q_1$. Suppose $u \in C^{1+\frac{\gamma}{\sigma}, \sigma+\gamma}(Q_1) \cap C^{\frac{\gamma}{\sigma},\gamma}((-1,0)\times\mathbb{R}^d)$ is a solution of (1-1) in Q_1; then we have the a priori estimate*

$$[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma;(-1,0)\times\mathbb{R}^d} + CC_f, \tag{1-6}$$

where $C > 0$ is a constant depending only on d, σ, γ, λ, Λ, N_0, and C_b.

The essential new part of Theorem 1.3 is for the case $\sigma = 1$. For $\sigma < 1$, Theorem 1.3 is just Theorem 1.1 in [Dong and Zhang 2016b]. Even though the Hölder continuity assumption appeared slightly differently, the proof in [Dong and Zhang 2016b] can be carried out with minimal modifications. For $\sigma > 1$, the drift is a lower-order perturbation and the conclusion can be proved without assuming $\sigma + \gamma < 2$ by using Theorem 1.1 in [Dong and Zhang 2016b] and interpolation inequalities.

In the case of the linear equation

$$\partial_t u = Lu + bDu + f, \tag{1-7}$$

the estimate (1-6) holds for all $\gamma \in (0,\sigma)$. Again, we assume $b \equiv 0$ when $\sigma < 1$.

Theorem 1.4. *Let $\sigma \in (0,2)$, $0 < \lambda \le \Lambda < \infty$, and $\gamma \in (0,\sigma)$ such that $\sigma+\gamma$ is not an integer. Assume K satisfies (1-2) when $\sigma = 1$, and the Hölder continuity assumptions (1-5) hold for all $(t,x), (t',x') \in Q_1$. Suppose $u \in C^{1+\frac{\gamma}{\sigma},\sigma+\gamma}(Q_1) \cap C^{\frac{\gamma}{\sigma},\gamma}((-1,0)\times\mathbb{R}^d)$ is a solution of (1-7) in Q_1; then we have the a priori estimate*

$$[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma;(-1,0)\times\mathbb{R}^d)} + CC_f, \tag{1-8}$$

where $C > 0$ is a constant depending only on d, σ, γ, λ, Λ, N_0, and C_b.

It is natural to assume $\gamma < \sigma$ in Theorem 1.4, since (1-5) will imply that f is independent of t if $0 < \sigma < \gamma$. In many applications, a will be independent of t as well under the assumptions of (1-5) and $\sigma < \gamma$. Then, we can always differentiate (1-7) in t, and obtain higher-order regularity in t by applying the result of Theorem 1.4 above.

We are also interested in the linear equation (1-7) when K, b, and f are Dini continuous in x but only measurable in the time variable t, that is, they satisfy

$$\begin{aligned} &\int_{B_{2r}\setminus B_r} |a(t,x,y) - a(t,x',y)|\,dy \le \Lambda r^d \omega_a(|x-x'|) \quad \text{for all } r > 0,\\ &\|f\|_{L_\infty(Q_1)} < \infty, \quad |f(t,x) - f(t,x')| \le \omega_f(|x-x'|),\\ &\|b\|_{L_\infty(Q_1)} \le N_0, \quad |b(t,x) - b(t,x')| \le \omega_b(|x-x'|),\\ &\text{where } N_0 > 0, \text{ and } \omega_a, \omega_b, \omega_f \text{ are all Dini functions.} \end{aligned} \tag{1-9}$$

In Theorem 1.5 below, ω_u denotes the modulus of continuity of u in x uniform for all t; that is,

$$|u(t,x) - u(t,x')| \le \omega_u(|x-x'|) \quad \text{for all } (t,x), (t,x') \in (-1,0)\times\mathbb{R}^d.$$

Theorem 1.5. *Let $\sigma \in (0,2)$, $0 < \lambda \le \Lambda < \infty$. Assume K satisfies (1-2) when $\sigma = 1$, and the Dini continuity assumption (1-9) holds for all $(t,x), (t,x') \in Q_1$. Suppose $u \in C^{1,\sigma^+}(Q_1)$ is a solution of (1-7) in Q_1 and is Dini continuous in x in $(-1,0)\times\mathbb{R}^d$. Then we have the a priori estimate: for $\sigma \in (0,2)$,*

$$\|\partial_t u\|_{L_\infty(Q_{1/2})} + [u]^x_{\sigma;Q_{1/2}} \le C \sum_{j=0}^{\infty} (2^{-j\sigma}\omega_u(2^j) + \omega_u(2^{-j}) + \omega_f(2^{-j})), \tag{1-10}$$

where $C > 0$ is a constant depending only on d, σ, λ, Λ, N_0, ω_b, and ω_a. Moreover, when $\sigma \ne 1$, we have

$$\sup_{(t_0,x_0)\in Q_{1/2}} [u]^x_{\sigma;Q_r(t_0,x_0)} \to 0 \quad \text{as } r \to 0$$

with a decay rate depending only on d, σ, λ, Λ, ω_a, ω_f, ω_u, N_0, and ω_b. When $\sigma = 1$, Du is uniformly continuous in x in $Q_{1/2}$ with a modulus of continuity controlled by the quantities before. Also, $\partial_t u$ is uniformly continuous in x in $Q_{1/2}$ with a modulus of continuity controlled by d, σ, λ, Λ, ω_a, ω_f, ω_u, N_0, ω_b, and $\|u\|_{L_\infty}$.

If K, b, and f in (1-7) are Hölder continuous in x locally but only measurable in the time variable t, that is, they satisfy

$$\begin{aligned}
&\int_{B_{2r}\setminus B_r} |a(t,x,y) - a(t,x',y)|\,dy \le \Lambda r^d |x-x'|^\gamma \quad \text{for all } r > 0,\\
&\|f\|_{L_\infty(Q_1)} < \infty, \quad |f(t,x) - f(t,x')| \le C_f |x-x'|^\gamma,\\
&\|b\|_{L_\infty(Q_1)} \le N_0, \quad |b(t,x) - b(t,x')| \le C_b |x-x'|^\gamma,\\
&\text{where } N_0, C_a, C_b > 0, \text{ and } \gamma \in (0,1),
\end{aligned} \tag{1-11}$$

then we have:

Theorem 1.6. *Let $\sigma \in (0,2)$, $0 < \lambda \le \Lambda < \infty$, and $\gamma \in (0,1)$ such that $\sigma + \gamma$ is not an integer. Assume K satisfies (1-2) when $\sigma = 1$, and the Hölder continuity assumptions (1-11) hold for all $(t,x), (t,x') \in Q_1$. Suppose $u \in C^{1,\sigma+\gamma}(Q_1) \cap C^\gamma_x((-1,0)\times\mathbb{R}^d)$ is a solution of (1-7) in Q_1; then we have the a priori estimate*

$$[\partial_t u]^x_{\gamma;Q_{1/2}} + [u]^x_{\sigma+\gamma,;Q_{1/2}} \le C\|u\|^x_{\gamma;(-1,0)\times\mathbb{R}^d} + CC_f, \tag{1-12}$$

where $C > 0$ is a constant depending only on d, σ, γ, λ, Λ, N_0, and C_b.

Note that here we assume $\gamma \in (0,1)$ for all $\sigma \in (0,2)$, since all the estimates only involve x. This theorem improves Theorem 1.1 in [Jin and Xiong 2015], which does not include drifts and requires the Hölder continuity of a and f in the time variable t as well. In the second-order case, similar results were obtained a long time ago by Knerr [1980/81] and Lieberman [1992].

A few remarks are in order.

Remark 1.7. It is evident that Theorems 1.1, 1.3, 1.4, 1.5, and 1.6 hold for corresponding elliptic equations as well.

Remark 1.8. Our proof does not tell whether the a priori estimates in Theorems 1.1 and 1.5 can be made uniformly bounded as $\sigma \to 2^-$, even if we replace Λ by $(2-\sigma)\Lambda$ in both (1-5) and (1-9).

The ideas of our proofs are in the spirit of the approach first developed in [Campanato 1966], which has been used in [Dong and Zhang 2016a] for nonlocal fully nonlinear elliptic equations. A similar idea was also used in the literature to derive Cordes–Nirenberg-type estimates; see, e.g., [Nirenberg 1954]. Here, we adapt the methods in [Dong and Zhang 2016a] from elliptic settings to parabolic settings, with extra efforts to deal with the drift term especially when $\sigma = 1$ and some simplification of the proofs.

The key idea is that instead of estimating the C^σ seminorm of the solution, we construct and bound certain seminorms of the solution; see Lemma 2.1. When $\sigma < 1$, we define such a seminorm as a series of lower-order Hölder seminorms of u. In order for the nonlocal operator to be well defined, the solution needs to be smoother than C^σ. This motivates us to divide the integral domain into annuli, and use a lower-order seminorm to estimate the integral in each annulus. The proof of the case when $\sigma \geq 1$ is more involved mainly due to the fact that the series of lower-order Hölder seminorms of the solution itself is no longer sufficient to estimate the C^σ norm. Therefore, we need to subtract a polynomial from the solution in the construction of the seminorm. In some sense, the polynomial should be chosen to minimize the series. It turns out that when $\sigma \geq 1$, we can make use of the first-order Taylor's expansion of the mollification of the solution.

The organization of this paper is as follows. In the next section, we introduce some notation and preliminary results that are necessary in the proofs of our main theorems. In Section 3, we show the Dini estimates for nonlocal nonlinear parabolic equations in Theorem 1.1. In Section 4, we prove the Schauder estimates for equations with a drift in Theorems 1.3 and 1.4. The last section is devoted to linear parabolic equations with measurable coefficients in the time variable t, where Theorems 1.5 and 1.6 are proved.

2. Preliminaries

We will use the following notation:

- For $r > 0$, we set $Q_r(t_0, x_0) = (t_0 - r^\sigma, t_0] \times B_r(x_0)$ and $\hat{Q}_r(t_0, x_0) = (t_0 - r^\sigma, t_0 + r^\sigma) \times B_r(x_0)$, where $B_r(x_0) \subset \mathbb{R}^d$ is the ball of radius r centered at x_0. We write $Q_r = Q_r(0, 0)$ for brevity.
- $\mathcal{P}_t$ (or $\mathcal{P}_x$) is the set of first-order polynomials in t (or x), respectively.
- $\mathcal{P}_1$ is the set of first-order polynomials in both t and x.
- For $\alpha, \beta > 0$,

$$
\begin{aligned}
[u]_{\alpha,\beta;Q_r(t_0,x_0)} &= [u]_{C^{\alpha,\beta}_{t,x}(Q_r(t_0,x_0))},\\
[u]^x_{\beta;Q_r(t_0,x_0)} &= \sup_{t\in(t_0-r^\sigma,t_0)} [u(t,\cdot)]_{C^\beta(B_r(x_0))},\\
[u]^t_{\alpha;Q_r(t_0,x_0)} &= \sup_{x\in B_r(x_0)} [u(\cdot,x)]_{C^\alpha((t_0-r^\sigma,t_0))}.
\end{aligned}
$$

If β (or α) is an integer, the above seminorms mean the Lipschitz norm of $D^{|\beta|-1}$ (or $\partial_t^{|\alpha|-1}$). If there is no subscript about the region where the norm is taken, then it means the whole domain where the function is defined (e.g., $\mathbb{R}^d$ or $(-t_0, 0] \times \mathbb{R}^d$ for some $t_0 > 0$).

- We say $u \in C^{1,\sigma^+}(Q_1)$ if $u \in C^{1,\sigma+\varepsilon}(Q_1)$ for some small $\varepsilon > 0$.

- We will also use the following Lipschitz–Zygmund seminorms. Let $\Omega \subset \mathbb{R}^d$ be a domain, $r > 0$, and $Q = (t_0 - r, t_0] \times \Omega$. For $\alpha, \beta \in (0,2)$, we define

$$[u]^x_{\mathbf{\Lambda}^\beta(Q)} = \sup_{t\in(t_0-r^\sigma,t_0]} [u(t,\cdot)]_{\mathbf{\Lambda}^\beta(\Omega)} = \sup_{t\in(t_0-r^\sigma,t_0]} \sup_{\substack{x_1,x_2,x_3\in\Omega\\ x_1\neq x_3,\ x_1+x_3=2x_2}} \frac{|u(t,x_1)+u(t,x_3)-2u(t,x_2)|}{|x_1-x_2|^\alpha},$$

$$[u]^t_{\mathbf{\Lambda}^\alpha(Q)} = \sup_{x\in\Omega}[u(\cdot,x)]_{\mathbf{\Lambda}^\alpha((t_0-r^\sigma,t_0))} = \sup_{x\in\Omega} \sup_{\substack{t_1,t_2,t_3\in(t_0-r^\sigma,t_0]\\ t_1\neq t_3,\ t_1+t_3=2t_2}} \frac{|u(t_1,x)+u(t_3,x)-2u(t_2,x)|}{|t_1-t_2|^\alpha},$$

$$[u]_{\mathbf{\Lambda}^{\alpha,\beta}(Q)} = \sup_{\substack{(t_1,x_1),(t_2,x_2),(t_3,x_3)\in Q\\ (t_1,x_1)\neq(t_3,x_3),\ (t_1,x_1)+(t_3,x_3)=2(t_2,x_2)}} \frac{|u(t_1,x_1)+u(t_3,x_3)-2u(t_2,x_2)|}{|t_1-t_2|^\alpha+|x_1-x_2|^\beta}.$$

We will frequently use the identities

$$2^j(u(t,x+2^{-j}l)-u(t,x))-(u(t,x+l)-u(t,x)) = \sum_{k=1}^{j} 2^{k-1}\big(2u(t,x+2^{-k}l)-u(t,x+2^{-k+1}l)-u(t,x)\big), \tag{2-1}$$

$$2^j(u(t-2^{-j},x)-u(t,x))-(u(t-1,x)-u(t,x)) = \sum_{k=1}^{j} 2^{k-1}\big(2u(t-2^{-k},x)-u(t-2^{-k+1},x)-u(t,x)\big), \tag{2-2}$$

which hold for any unit vector $l \in \mathbb{R}^d$ and $j \in \mathbb{N}$.

Lemma 2.1. *Let $\alpha \in (0,\sigma)$ be a constant. Let Q be a convex cylinder such that $Q_{\frac{1}{2}} \subset Q \subset Q_1$.*

(i) *When $\sigma \in (0,1)$, we have*

$$[u]^x_{\sigma;Q} + \|\partial_t u\|_{L_\infty(Q)} \le C\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} + C\|u\|_{L_\infty(Q_{2^{1/\sigma}})}, \tag{2-3}$$

where C is a constant depending only on d, σ, and α. Moreover, the modulus of continuity of $\partial_t u$ is bounded by the tail of the summation on the right-hand side of (2-3).

(ii) *When $\sigma \in (1,2)$, we have*

$$[u]^x_{\sigma;Q} + \|\partial_t u\|_{L_\infty(Q)} + [Du]^t_{\frac{\sigma-1}{\sigma};Q} \le C\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} + C\|u\|_{L_\infty(Q_2)}, \tag{2-4}$$

where C is a constant depending on d, α, and σ. The modulus of continuity of $\partial_t u$ is bounded by the tail of the summation above.

(iii) *When $\sigma = 1$, we have*

$$\|Du\|_{L_\infty(Q)} + \|\partial_t u\|_{L_\infty(Q)} \le C \sum_{k=0}^{\infty} 2^{k(1-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)} + C \sup_{\substack{(t,x),(t',x')\in Q_2 \\ \max\{|t-t'|,|x-x'|\}=1}} |u(t,x)-u(t',x')|, \quad (2\text{-}5)$$

where C is a constant depending on d, α, and σ. The modulus of continuity of $\partial_t u$ and Du are bounded by the tail of the summation above.

Proof. We first prove the estimate of $\partial_t u$ for $\sigma \in (0,2)$ by showing that

$$\|\partial_t u\|_{L_\infty(Q)} \le C \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_t} [u-p]^t_{\frac{\alpha}{\sigma};Q_{2^{-k}}(t_0,x_0)} + 2 \sup_{(t_0,x_0)\in Q} |u(t_0-1,x_0)-u(t_0,x_0)|. \quad (2\text{-}6)$$

Indeed, from (2-2),

$$\begin{aligned} 2^j |u(t-2^{-j},x)-u(t,x)| &\le |u(t-1,x)-u(t,x)| + \sum_{k=1}^{\infty} 2^{k-1} |2u(t-2^{-k},x)-u(t-2^{-k+1},x)-u(t,x)| \\ &\le |u(t-1,x)-u(t,x)| + C \sum_{k=1}^{\infty} 2^{k(1-\frac{\alpha}{\sigma})} [u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t,x))}, \end{aligned} \quad (2\text{-}7)$$

where C only depends on σ and $k^* = [(k-1)/\sigma]$, i.e., the largest integer which is smaller than $(k-1)/\sigma$. The right-hand side of the above inequality is less than

$$\begin{aligned} &|u(t-1,x)-u(t,x)| + C \sum_{k=1}^{\infty} 2^{(k^*\sigma+\sigma)(1-\frac{\alpha}{\sigma})} [u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t,x))} \\ &\qquad \le |u(t-1,x)-u(t,x)| + C \sum_{k=1}^{\infty} 2^{k^*(\sigma-\alpha)} \inf_{p\in\mathcal{P}_t} [u-p]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t,x))}. \end{aligned}$$

By using the definition of k^*, it is easy to see the second term on the right-hand side of the above inequality is bounded by

$$C \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_t} [u-p]^t_{\frac{\alpha}{\sigma};Q_{2^{-k}}(t_0,x_0)}.$$

Therefore, by sending $j \to \infty$ in (2-7), we prove that $\|\partial_t u\|_{L_\infty(Q)}$ is bounded by the right-hand side of (2-6). Since

$$\inf_{p\in\mathcal{P}_t} [u-p]^t_{\frac{\alpha}{\sigma};Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)},$$
$$\inf_{p\in\mathcal{P}_t} [u-p]^t_{\frac{\alpha}{\sigma};Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)},$$

the right-hand side of (2-6) is bounded by that of (2-3)–(2-5). We obtain the bound of $\|\partial_t u\|_{L_\infty(Q)}$.

Next, we bound the modulus of continuity of $\partial_t u$ in Q. Assume

$$|t-t'|+|x-x'|^\sigma \in [2^{-(i+1)}, 2^{-i}) \quad \text{for some } i \geq 1.$$

From (2-2), for any $j \geq i+1$,

$$2^j(u(t-2^{-j},x)-u(t,x))-2^i(u(t-2^{-i},x)-u(t,x)) = \sum_{k=i+1}^{j} 2^{k-1}(2u(t-2^{-k},x)-u(t-2^{-k+1},x)-u(t,x)),$$

and the same identity holds with (t',x') in place of (t,x). Then we have

$$\begin{aligned}|\partial_t u(t,x)-\partial_t u(t',x')| &= \lim_{j\to\infty} |2^j(u(t-2^{-j},x)-u(t,x))-2^j(u(t'-2^{-j},x')-u(t',x'))| \\ &\leq |2^i(u(t-2^{-i},x)-u(t,x))-2^i(u(t'-2^{-i},x')-u(t',x'))| \\ &\quad + C\sum_{k=i+1}^{\infty} \sup_{(t_0,x_0)\in Q} 2^{k(1-\frac{\alpha}{\sigma})}[u]^t_{\Lambda^{\alpha/\sigma}(Q_{2^{-k^*}}(t_0,x_0))},\end{aligned}$$

where k^* is defined above. By the triangle inequality, the first term on the right-hand side is bounded by

$$2^i|u(t-2^{-i},x)+u(t',x')-2u(\bar{t},\bar{x})|+2^i|u(t'-2^{-i},x')-2u(\bar{t},\bar{x})+u(t,x)|,$$

where $\bar{t}=(t+t'-2^{-i})/2$ and $\bar{x}=(x+x')/2$. This is further bounded by

$$2^{i(1-\frac{\alpha}{\sigma})} \sup_{(t_0,x_0)\in Q} [u]_{\Lambda^{\alpha/\sigma,\alpha}(Q_{2^{-i^*}}(t_0,x_0))},$$

where $i^*=[(i-1)/\sigma]$. Therefore,

$$\begin{aligned}|\partial_t u(t,x)-\partial_t u(t',x')| &\leq C\sum_{k=i}^{\infty} \sup_{(t_0,x_0)\in Q} 2^{k(1-\frac{\alpha}{\sigma})}[u]_{\Lambda^{\alpha/\sigma,\alpha}(Q_{2^{-i^*}}(t_0,x_0))} \\ &\leq C\sum_{k=i}^{\infty} \sup_{(t_0,x_0)\in Q} 2^{k(1-\frac{\alpha}{\sigma})} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-i^*}}(t_0,x_0)},\end{aligned}$$

which, from the definition of i^*, converges to 0 as $i\to\infty$.

In the rest of the proof, we consider the three cases separately.

Case 1: $\sigma\in(0,1)$. The estimates of $[u]^x_\sigma$ are the same as [Dong and Zhang 2016a, Lemma 2.1] and we only provide a sketch here. Let $(t,x),(t,x')\in Q$ be two different points. Suppose $h:=|x-x'|\in(0,1)$. Since

$$h^{-\sigma}|u(t,x')-u(t,x)| \leq \sup_{x\in Q} h^{\alpha-\sigma}[u(t,\cdot)]_{\alpha;B_h(x)},$$

by taking the supremum with respect to t, x, and x' for $h<1$ on both sides, we get

$$[u]^x_{\sigma;Q} \leq \sup_{(t_0,x_0)\in Q} \sup_{0<h<1} h^{\alpha-\sigma}[u]^x_{\alpha;Q_h(t_0,x_0)} \leq C\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} [u]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

Notice that

$$[u]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} = \inf_{p\in\mathcal{P}_t}[u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_t}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

The proof of Case 1 is completed.

Case 2: $\sigma \in (1,2)$. Similar to the previous case, we only provide the sketch of the proof following that of [Dong and Zhang 2016a, Lemma 2.1]. Let $\ell \in \mathbb{R}^d$ be a unit vector and $\varepsilon \in \left(0, \frac{1}{16}\right)$ be a small constant to be specified later. For any two distinct points $(t,x),(t,x') \in Q$ such that $h = |x-x'| < \frac{1}{2}$, there exist $\bar{x}, \bar{x}' \in Q$ such that $|x-\bar{x}| < \varepsilon h$, $\bar{x}+\varepsilon h l \in Q$, and $|x'-\bar{x}'| < \varepsilon h$, $\bar{x}'+\varepsilon h l \in Q$. By the triangle inequality,

$$h^{1-\sigma}|D_\ell u(t,x) - D_\ell u(t,x')| \le I_1 + I_2 + I_3, \tag{2-8}$$

where

$$\begin{aligned} I_1 &:= h^{1-\sigma}|D_\ell u(t,x) - (\varepsilon h)^{-1}(u(t,\bar{x}+\varepsilon h\ell) - u(t,\bar{x}))|,\\ I_2 &:= h^{1-\sigma}|D_\ell u(t,x') - (\varepsilon h)^{-1}(u(t,\bar{x}'+\varepsilon h\ell) - u(t,\bar{x}'))|,\\ I_3 &:= h^{1-\sigma}(\varepsilon h)^{-1}|(u(t,\bar{x}+\varepsilon h\ell) - u(t,\bar{x})) - (u(t,\bar{x}'+\varepsilon h\ell) - u(t,\bar{x}'))|. \end{aligned}$$

By the mean value theorem,

$$I_1 + I_2 \le 2^\sigma \varepsilon^{\sigma-1}[u]^x_{\sigma;Q}. \tag{2-9}$$

Now we choose and fix an ε sufficiently small depending only on σ such that $2^\sigma\varepsilon^{\sigma-1} \le \frac{1}{2}$. Using the triangle inequality, we have

$$I_3 \le Ch^{-\sigma}\big(|u(t,\bar{x}+\varepsilon h\ell) + u(t,\bar{x}') - 2u(t,\tilde{x})| + |u(t,\bar{x}'+\varepsilon h\ell) + u(t,\bar{x}) - 2u(t,\tilde{x})|\big),$$

where $\tilde{x} = (\bar{x}+\varepsilon h\ell+\bar{x}')/2$. Thus,

$$I_3 \le Ch^{\alpha-\sigma}[u(t,\cdot)]^x_{\Lambda^\alpha(Q_h(t,\tilde{x}))}. \tag{2-10}$$

Combining (2-8), (2-9), and (2-10), we get

$$[u]^x_{\sigma;Q} \le C\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q}\ \inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

Because

$$\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)},$$

we bound $[u]^x_{\sigma;Q}$ by the right-hand side of (2-4).

It follows from [Krylov 1996, Section 3.3] that $[Du]^t_{\frac{\sigma-1}{\sigma};Q}$ is bounded by $\|\partial_t u\|_{L_\infty(Q)} + [u]^x_{\sigma;Q}$. Therefore, (2-4) is proved.

Case 3: $\sigma = 1$. We give the estimate of $\|Du\|_{L_\infty}$. It follows from (2-1) that

$$2^j|u(t,x+2^{-j}\ell) - u(t,x)| \le |u(t,x+\ell) - u(t,x)| + \sum_{k=1}^{j} 2^{k(1-\alpha)}[u(t,\cdot)]^x_{\Lambda^\alpha(B_{2^{-k}}(x+2^{-k}\ell))}.$$

Taking $j \to \infty$, we obtain that

$$\|Du\|_{L_\infty} \le C \sum_{k=1}^{\infty} 2^{k(1-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} + \sup_{\substack{(t,x),(t,x')\in Q_2 \\ |x-x'|=1}} |u(t,x)-u(t,x')|.$$

The estimate of the continuity of Du is the same as $\partial_t u$, and thus omitted. □

Let η be a smooth nonnegative function in $\mathbb{R}$ with unit integral and vanishing outside $(0,1)$. For $R>0$ and $\sigma\in(0,1)$, we define the mollification of u with respect to t as

$$u^{(R)}(t,x) = \int_{\mathbb{R}} u(t-R^\sigma s, x)\eta(s)\,ds.$$

For the case $\sigma\in[1,2)$, we define $u^{(R)}$ differently by mollifying the x-variable as well. Let $\zeta\in C_0^\infty(B_1)$ be a radial nonnegative function with unit integral. For $R>0$, we define

$$u^{(R)}(t,x) = \int_{\mathbb{R}^{d+1}} u(t-R^\sigma s, x-Ry)\eta(s)\zeta(y)\,dy\,ds.$$

The following lemma is for the case $\sigma\in(0,1)$.

Lemma 2.2. *Let $\sigma\in(0,1)$, $\alpha\in(0,\sigma)$, and $R>0$ be constants. Let $p_0=p_0(t)$ be the first-order Taylor expansion of $u^{(R)}$ at the origin in t and $\tilde u = u-p_0$. Then for any integer $j\ge 0$, we have*

$$[\tilde u]_{\frac{\alpha}{\sigma},\alpha;(-R^\sigma,0)\times B_{2^jR}} \le C \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;(-R^\sigma,0)\times B_{2^jR}}, \tag{2-11}$$

where C is a constant only depending on d and α.

Proof. It is easily seen that $\tilde u$ is invariant up to a constant if we replace u by $u-p$ for any $p\in\mathcal{P}_t$. Thus to prove the lemma, we only need to bound the left-hand side of (2-11) by

$$C[u]_{\frac{\alpha}{\sigma},\alpha;(-R^\sigma,0)\times B_{2^jR}}.$$

Since $\tilde u = u-p(t)$, it suffices to observe that

$$[p]^t_{\frac{\alpha}{\sigma};(-R^\sigma,0)} = R^{\sigma-\alpha}|\partial_t u^{(R)}(0,0)| \le C[u]^t_{\frac{\alpha}{\sigma};Q_R}.$$ □

The following lemma is useful in dealing with the case $\sigma\in(1,2)$.

Lemma 2.3. *Let $\alpha\in(0,1)$ and $\sigma\in(1,2)$ be constant. Then for any $u\in C^1$ and any cylinder Q, we have*

$$\sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} [u-p_0]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le C \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}, \tag{2-12}$$

where p_0 is the first-order Taylor's expansion of u in the x-variable at (t_0,x_0), and $C>0$ is a constant depending only on d, α, and σ.

Proof. Define

$$b_k := 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

Then for any $(t_0, x_0) \in Q$ and each $k = 0, 1, \dots,$ there exists $\mathfrak{p}_k \in \mathcal{P}_x$ such that

$$[u - \mathfrak{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le 2b_k 2^{-k(\sigma-\alpha)}.$$

By the triangle inequality, for $k \ge 1$ we have

$$[\mathfrak{p}_{k-1} - \mathfrak{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \le 2b_k 2^{-k(\sigma-\alpha)} + 2b_{k-1} 2^{-(k-1)(\sigma-\alpha)}. \tag{2-13}$$

It is easily seen that

$$[\mathfrak{p}_{k-1} - \mathfrak{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} = |\nabla \mathfrak{p}_{k-1} - \nabla \mathfrak{p}_k| 2^{-(k-1)(1-\alpha)},$$

which together with (2-13) implies

$$|\nabla \mathfrak{p}_{k-1} - \nabla \mathfrak{p}_k| \le C 2^{-k(\sigma-1)} (b_{k-1} + b_k). \tag{2-14}$$

Since $\sum_k b_k < \infty$, from (2-14) we see that $\{\nabla \mathfrak{p}_k\}$ is a Cauchy sequence in $\mathbb{R}^d$. Let $q = q(t_0, x_0) \in \mathbb{R}^d$ be the limit, which clearly satisfies for each $k \ge 0$,

$$|q - \nabla \mathfrak{p}_k| \le C \sum_{j=k}^{\infty} 2^{-j(\sigma-1)} b_j.$$

By the triangle inequality, we get

$$\begin{aligned} [u - q \cdot x]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} &\le [u - \mathfrak{p}_k]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} + [\mathfrak{p}_k - q \cdot x]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \\ &\le C 2^{-k(1-\alpha)} \sum_{j=k}^{\infty} 2^{-j(\sigma-1)} b_j \le C 2^{-k(\sigma-\alpha)}, \end{aligned} \tag{2-15}$$

which implies

$$\|u(t_0, \cdot) - u(t_0, x_0) - q \cdot (x - x_0)\|_{L_\infty(B_{2^{-k}}(x_0))} \le C 2^{-k\sigma},$$

and thus $q = \nabla u(t_0, x_0)$. It then follows from (2-15) that

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q} [u - p_0]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} &\le C \sum_{k=0}^{\infty} 2^{k(\sigma-1)} \sum_{j=k}^{\infty} 2^{-j(\sigma-1)} b_j \\ &= C \sum_{j=0}^{\infty} 2^{-j(\sigma-1)} b_j \sum_{k=0}^{j} 2^{k(\sigma-1)} \le C \sum_{j=0}^{\infty} b_j. \end{aligned}$$

This completes the proof of (2-12). $\square$

The last lemma in this section is for the case when $\sigma \in [1, 2)$.

Lemma 2.4. *Let $\alpha \in (0, 1)$, $\sigma \in [1, 2)$, and $R > 0$ be constants. Let $p_0 = p_0(t, x)$ be the first-order Taylor's expansion of $u^{(R)}$ at the origin and $\tilde{u} = u - p_0$. Then for any integer $j \ge 0$, we have*

$$\sup_{\substack{(t,x),(t',x')\in(-R^\sigma,0)\times B_{2^j R} \\ (t,x)\ne(t',x'),\, 0\le|x-x'|<2R}} \frac{|\tilde{u}(t,x) - \tilde{u}(t',x')|}{|x - x'|^\alpha + |t - t'|^{\frac{\alpha}{\sigma}}} \le C \inf_{p\in\mathcal{P}_1} [u - p]_{\frac{\alpha}{\sigma},\alpha;(-R^\sigma,0)\times B_{2^j R}}, \tag{2-16}$$

where $C > 0$ is a constant depending only on d, α, and σ.

Proof. It is easily seen that $\tilde{u}$ is invariant up to a constant if we replace u by $u-p$ for any $p \in \mathcal{P}_1$. Thus to show (2-16), we only bound the left-hand side of (2-16) by

$$C[u]_{\frac{\alpha}{\sigma},\alpha;(-R^\sigma,0)\times B_{2^j R}}.$$

Since $\tilde{u} = u - p_0$, it suffices to observe that for any two distinct $(t,x), (t',x') \in (-R^\sigma, 0) \times B_{2^j R}$ such that $0 \le |x-x'| < 2R$,

$$\begin{aligned}|p_0(t,x) - p_0(t',x')| &\le |x-x'|\,|Du^{(R)}(0,0)| + |t-t'|\,|\partial_t u^{(R)}(0,0)| \\ &\le C|x-x'|R^{\alpha-1}[u]^x_{\alpha;Q_R} + C|t-t'|R^{\sigma(\frac{\alpha}{\sigma}-1)}[u]^t_{\frac{\alpha}{\sigma};Q_R} \\ &\le C(|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{\sigma}})[u]_{\frac{\alpha}{\sigma},\alpha;Q_R}.\end{aligned}$$ □

3. Dini estimates for nonlocal nonlinear parabolic equations

The following proposition is a further refinement of [Dong and Zhang 2016b, Corollary 4.6].

Proposition 3.1. *Let $\sigma \in (0,2)$ and $0 < \lambda \le \Lambda$. Assume for any $\beta \in \mathcal{A}$ that K_β only depends on y. There is a constant $\hat{\alpha}$ depending on d, σ, λ, and Λ (uniformly as $\sigma \to 2^-$) so that the following holds. Let $\alpha \in (0,\hat{\alpha})$ such that $\sigma+\alpha$ is not an integer. Suppose $u \in C^{1+\frac{\alpha}{\sigma},\sigma+\alpha}(Q_1) \cap C^{\frac{\alpha}{\sigma},\alpha}((-1,0)\times\mathbb{R}^d)$ is a solution of*

$$\partial_t u = \inf_{\beta\in\mathcal{A}} (L_\beta u + f_\beta) \quad \text{in } Q_1.$$

Then,

$$[u]_{1+\frac{\alpha}{\sigma},\alpha+\sigma;Q_{1/2}} \le C\sum_{j=1}^\infty 2^{-j\sigma} M_j + C\sup_\beta [f_\beta]_{\frac{\alpha}{\sigma},\alpha;Q_1},$$

where

$$M_j = \sup_{\substack{(t,x),(t',x')\in(-1,0)\times B_{2^j}\\ (t,x)\neq(t',x'),\, 0\le|x-x'|<2}} \frac{|u(t,x)-u(t',x')|}{|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{\sigma}}},$$

and $C > 0$ depends only on $d, \lambda, \Lambda, \alpha$ and σ, and is uniformly bounded as $\sigma \to 2^-$.

Proof. This follows from the proof of [Dong and Zhang 2016b, Corollary 4.6] by observing that in the estimate of $[h_\beta]_{\frac{\alpha}{\sigma},\alpha;Q_1}$, the term $[u]_{\frac{\alpha}{\sigma},\alpha;(-1,0)\times B_{2^j}}$ can be replaced by M_j. Moreover, by replacing u by $u - u(0,0)$, we see that

$$\|u\|_{\frac{\alpha}{\sigma},\alpha;(-1,0)\times B_2} \le C[u]_{\frac{\alpha}{\sigma},\alpha;(-1,0)\times B_2}.$$ □

In the rest of this section, we consider three cases separately.

The case $\sigma \in (0,1)$.

Proposition 3.2. *Suppose (1-1) is satisfied in $Q_{2^{1/\sigma}}$. Then under the conditions of Theorem 1.1, we have*

$$[u]^x_{\sigma;Q_{1/2}} + \|\partial_t u\|_{L_\infty;Q_{1/2}} \le C\|u\|_{\frac{\alpha}{\sigma},\alpha} + C\sum_{k=1}^\infty \omega_f(2^{-k}), \tag{3-1}$$

where $C > 0$ is a constant depending only on d, λ, Λ, ω_a, and σ.

Proof. For $k \in \mathbb{N}$, let v be the solution of

$$\begin{cases} \partial_t v = \inf_{\beta\in\mathcal{A}}(L_\beta(0,0)v + f_\beta(0,0) - \partial_t p_0) & \text{in } Q_{2^{-k}}, \\ v = u - p_0(t) & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t=-2^{-k\sigma}\}\times B_{2^{-k}}), \end{cases}$$

where $L_\beta(0,0)$ is the operator with kernel $K_\beta(0,0,y)$, and $p_0(t)$ is the Taylor's expansion of $u^{(2^{-k})}$ in t at the origin. Then by Proposition 3.1 with scaling, we have

$$[v]_{1+\frac{\alpha}{\sigma},\alpha+\sigma;Q_{2^{-k-1}}} \le C\sum_{j=1}^{\infty} 2^{(k-j)\sigma} M_j + C2^{k\sigma}[v]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}, \tag{3-2}$$

where $\alpha \in (0,\hat{\alpha})$ satisfying $\sigma+\alpha<1$,

$$M_j = \sup_{\substack{(t,x),(t',x')\in(-2^{-k\sigma},0)\times B_{2^{j-k}} \\ (t,x)\neq(t',x'),\, 0\le|x-x'|<2^{-k+1}}} \frac{|\tilde{u}(t,x)-\tilde{u}(t',x')|}{|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{\sigma}}},$$

and $\tilde{u} = u - p_0$.

Let $k_0 \ge 1$ be an integer to be specified and $p_1 = p_1(t)$ be the Taylor's expansion of v in t at the origin. By the mean value formula,

$$\|v-p_1\|_{L_\infty(Q_{2^{-k-k_0}})} \le 2^{-(k+k_0)(\sigma+\alpha)}[v]_{1+\frac{\alpha}{\sigma},\sigma+\alpha;Q_{2^{-k-k_0}}},$$

and the interpolation inequality

$$[v-p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}} \le C\big(2^{(k+k_0)\alpha}\|v-p_1\|_{L_\infty(Q_{2^{-k-k_0}})} + 2^{-(k+k_0)\sigma}[v-p_1]_{1+\frac{\alpha}{\sigma},\alpha+\sigma;Q_{2^{-k-k_0}}}\big),$$

we obtain

$$[v-p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}} \le C2^{-(k+k_0)\sigma}[v]_{1+\frac{\alpha}{\sigma},\alpha+\sigma;Q_{2^{-k-k_0}}}.$$

From Lemma 2.2, we have

$$M_j \le C\inf_{p\in\mathcal{P}_t}[u-p]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}} \le C[u]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d}. \tag{3-3}$$

These and (3-2) give

$$\begin{aligned} &[v-p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}} \\ &\quad\le C2^{-(k+k_0)\sigma}\sum_{j=1}^{k}2^{(k-j)\sigma}M_j + C2^{-(k+k_0)\sigma}[u]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{-k_0\sigma}[v]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} \\ &\quad\le C2^{-(k+k_0)\sigma}\sum_{j=1}^{k}2^{(k-j)\sigma}M_j + C2^{-(k+k_0)\sigma}[u]_{\frac{\alpha}{\sigma},\alpha} + C2^{-k_0\sigma}[v]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}. \end{aligned} \tag{3-4}$$

Next, $w := u - p_0 - v$ satisfies

$$\begin{cases} w_t - \mathcal{M}^+ w \le C_k & \text{in } Q_{2^{-k}}, \\ w_t - \mathcal{M}^- w \ge -C_k & \text{in } Q_{2^{-k}}, \\ w = 0 & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t=-2^{-k\sigma}\}\times B_{2^{-k}}), \end{cases} \tag{3-5}$$

where $\mathcal{M}^+$ and $\mathcal{M}^-$ are the Pucci extremal operators, see, e.g., [Dong and Zhang 2016b], and

$$C_k = \sup_{\beta\in\mathcal{A}} \| f_\beta - f_\beta(0,0) + (L_\beta - L_\beta(0,0))u\|_{L_\infty(Q_{2^{-k}})}.$$

It is easily seen that

$$C_k \le \omega_f(2^{-k}) + C\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big).$$

Then by the Hölder estimate [Dong and Zhang 2016b, Lemma 2.5], we have

$$\begin{aligned}[w]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} &\le C2^{-k(\sigma-\alpha)}C_k \\ &\le C2^{-k(\sigma-\alpha)}\Big[\omega_f(2^{-k})+\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big)\Big]\end{aligned} \tag{3-6}$$

for some $\alpha > 0$. This α can be the same as the one in (3-2) since α is always small. By the triangle inequality and Lemma 2.2 with $j = 0$

$$\begin{aligned}[v]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} &\le [w]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} + [u-p_0]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} \\ &\le [w]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} + C\inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}.\end{aligned} \tag{3-7}$$

Combining (3-4), (3-6), (3-3), and (3-7) yields

$$\begin{aligned}&2^{(k+k_0)(\sigma-\alpha)}[u-p_0-p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}} \\ &\quad = 2^{(k+k_0)(\sigma-\alpha)}[w+v-p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}} \\ &\quad \le C2^{-(k+k_0)\alpha}\sum_{j=1}^{k} 2^{(k-j)\sigma}\inf_{p\in\mathcal{P}_t}[u-p]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}} \\ &\qquad + C2^{-(k+k_0)\alpha}[u]_{\frac{\alpha}{\sigma},\alpha} + C2^{-k_0\alpha+k(\sigma-\alpha)}\inf_{p\in\mathcal{P}_t}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} + C2^{k_0(\sigma-\alpha)}\omega_f(2^{-k}) \\ &\qquad + C2^{k_0(\sigma-\alpha)}\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}(t_0,x_0)}} + \|u\|_{L_\infty}\Big).\end{aligned} \tag{3-8}$$

Let $\ell_0 \ge 1$ be an integer such that

$$\frac{1}{2^\sigma} + \sum_{l=\ell_0+1}^{\infty}\frac{1}{2^{l\sigma}} \le 1.$$

Set $Q^{\ell_0} = Q_{\frac{1}{2}}$ and for $l = \ell_0+1, \ell_0+2, \dots,$ we define

$$Q^l := \Big(-\frac{1}{2^\sigma} - \sum_{j=\ell_0+1}^{l}\frac{1}{2^{j\sigma}}, 0\Big] \times \Big\{x : |x| < \frac{1}{2} + \sum_{j=\ell_0+1}^{l}\frac{1}{2^j}\Big\}.$$

The choice of ℓ_0 will ensure that $Q^l \subset Q_1$ for all $l \geq \ell_0$, and the definition of Q^l will ensure that for $l \geq \ell_0,\ k \geq l+1$, there holds

$$Q^l + Q_{2^{-k}}(t_0, x_0) \subset Q^{l+1} \quad \text{for all } (t_0, x_0) \in Q^l.$$

By translation of the coordinates, from (3-8) we have for any $l \geq \ell_0$ and $k \geq l+1$,

$$\begin{aligned} &2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u-p_0-p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\ &\quad \leq C2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} + C2^{-(k+k_0)\alpha}[u]_{\frac{\alpha}{\sigma},\alpha} \\ &\qquad + C2^{k_0(\sigma-\alpha)}\Big[\omega_f(2^{-k}) \\ &\qquad\qquad + \omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q^{l+1}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big)\Big]. \end{aligned} \tag{3-9}$$

Then we take the sum (3-9) in $k = l+1, l+2, \dots$ to obtain

$$\begin{aligned} &\sum_{k=l+1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\ &\quad \leq C \sum_{k=l+1}^{\infty} 2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\ &\qquad + C2^{-(l+k_0)\alpha}[u]_{\frac{\alpha}{\sigma},\alpha} + C2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_f(2^{-k}) \\ &\qquad + C2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_a(2^{-k}) \Big(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Big). \end{aligned}$$

By switching the order of summations and then replacing k by $k+j$, the first term on the right-hand side is bounded by

$$\begin{aligned} &C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{-j\sigma} \sum_{k=j}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\ &\qquad \leq C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{-k}}(x_0)} \\ &\qquad \leq C2^{-k_0\alpha} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)}. \end{aligned}$$

With the above inequality, we have

$$\begin{aligned}\sum_{k=l+1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} &\sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\ &\le C2^{-k_0\alpha}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\ &\quad + C2^{-(l+k_0)\alpha}[u]_{\frac{\alpha}{\sigma},\alpha} + C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}\omega_f(2^{-k}) \\ &\quad + C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}\omega_a(2^{-k})\Bigl(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Bigr).\end{aligned}$$

The bound above, together with the obvious inequality

$$\sum_{j=0}^{l+k_0} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \le C2^{(l+k_0)(\sigma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha},$$

implies

$$\begin{aligned}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} &\sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\ &\le C2^{-k_0\alpha}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\ &\quad + C2^{(l+k_0)(\sigma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha} + C2^{k_0(\sigma-\alpha)}\sum_{k=l}^{\infty}\omega_f(2^{-k}) \\ &\quad + C2^{k_0(\sigma-\alpha)}\sum_{k=l}^{\infty}\omega_a(2^{-k})\Bigl(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Bigr).\end{aligned}$$

By first choosing k_0 sufficiently large, and then ℓ_0 sufficiently large (recalling that $l \ge \ell_0$), we get

$$\begin{aligned}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} &\sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} \\ &\le \frac{1}{4}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} + C2^{(l+k_0)(\sigma-\alpha)}\|u\|_{\frac{\alpha}{\sigma},\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}).\end{aligned}$$

Multiplying both sides by 4^{-l}, taking the sum in l, we have

$$4^{-l}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} \le C\|u\|_{\frac{\alpha}{\sigma},\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}). \tag{3-10}$$

This, together with Lemma 2.1(i) and the fact that $Q^{\ell_0} = Q_{\frac{1}{2}}$, gives (3-1) and the continuity of $\partial_t u$. $\square$

The case when $\sigma \in (1,2)$.

Proposition 3.3. *Suppose (1-1) is satisfied in Q_2. Then under the conditions of Theorem 1.1, we have for $\sigma \in (1,2)$*

$$[u]^x_{\sigma;Q_{1/2}} + [Du]^t_{\frac{\sigma-1}{\sigma};Q_{1/2}} + \|\partial_t u\|_{L_\infty(Q_{1/2})} \le C\|u\|_{\frac{\alpha}{\sigma},\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{3-11}$$

where $C>0$ is a constant depending only on d, λ, Λ, ω_a, ω_b, N_0, and σ.

Proof. For $k \in \mathbb{N}$, let v_M be the solution of

$$\begin{cases} \partial_t v_M = \inf_{\beta\in\mathcal{A}}\big(L_\beta(0,0)v_M + f_\beta(0,0) + b_\beta(0,0)Du(0,0) - \partial_t p_0\big) & \text{in } Q_{2^{-k}},\\ v_M = g_M & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t=-2^{-k\sigma}\}\times B_{2^{-k}}),\end{cases}$$

where $M \ge 2\|u-p_0\|_{L_\infty(Q_{2^{-k}})}$ is a constant to be specified later,

$$g_M = \max(\min(u-p_0, M), -M),$$

and $p_0 = p_0(t,x)$ is the first-order Taylor's expansion of $u^{(2^{-k})}$ at the origin. By Proposition 3.1, we have

$$[v_M]_{1+\frac{\alpha}{\sigma},\alpha+\sigma;Q_{2^{-k-1}}} \le C\sum_{j=1}^{\infty} 2^{(k-j)\sigma}M_j + C2^{k\sigma}[v_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}},$$

where $\alpha \in (0, \min\{\hat\alpha, (\sigma-1)/2, 2-\sigma\})$ and

$$M_j = \sup_{\substack{(t,x),(t',x')\in(-2^{-k\sigma},0)\times B_{2^{j-k}}\\ (t,x)\neq(t',x'),\,0\le|x-x'|<2^{-k+1}}} \frac{|u(t,x)-p_0(t,x)-u(t',x')+p_0(t',x')|}{|t-t'|^{\frac{\alpha}{\sigma}}+|x-x'|^\alpha}.$$

From Lemma 2.4 with $\sigma \in (1,2)$, it follows

$$M_j \le C\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}}. \tag{3-12}$$

In particular, for $j>k$, we have

$$M_j \le C[u]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d},$$

and thus,

$$\begin{aligned}[v_M]_{1+\frac{\alpha}{\sigma},\alpha+\sigma;Q_{2^{-k-1}}} &\le C\sum_{j=1}^{\infty}2^{(k-j)\sigma}M_j + C2^{k\sigma}[v_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}\\ &\le C\sum_{j=1}^{k}2^{(k-j)\sigma}M_j + C[u]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{k\sigma}[v_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}. \end{aligned} \tag{3-13}$$

From (3-13), and the mean value formula (recalling that $\alpha < 2-\sigma$),

$$\|v_M - p_1\|_{L_\infty(Q_{2^{-k-k_0}})} \le C2^{-(k+k_0)(\sigma+\alpha)} \sum_{j=1}^{k} 2^{(k-j)\sigma} M_j + C2^{-(k+k_0)(\sigma+\alpha)} [u]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{-k\alpha-k_0(\sigma+\alpha)}[v_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}},$$

where p_1 is the first-order Taylor's expansion of v_M at the origin. The above inequality, (3-13), and the interpolation inequality imply

$$[v_M - p_1]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}} \le C2^{-(k+k_0)\sigma} \sum_{j=1}^{k} 2^{(k-j)\sigma} M_j + C2^{-(k+k_0)\sigma}[u]_{\frac{\alpha}{\sigma},\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{-k_0\sigma}[v_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}. \tag{3-14}$$

Next $w_M := g_M - v_M$ satisfies

$$\begin{cases} \partial_t w_M \le \mathcal{M}^+ w_M + h_M + C_k & \text{in } Q_{2^{-k}}, \\ \partial_t w_M \ge \mathcal{M}^- w_M + \hat{h}_M - C_k & \text{in } Q_{2^{-k}}, \\ \quad w_M = 0 & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\}\times B_{2^{-k}}), \end{cases}$$

where

$$h_M := \mathcal{M}^+(u - p_0 - g_M), \quad \hat{h}_M := \mathcal{M}^-(u - p_0 - g_M).$$

Here

$$C_k = \sup_{\beta\in\mathcal{A}} \big\| f_\beta - f_\beta(0,0) + b_\beta Du - b_\beta(0,0)Du(0,0) + (L_\beta - L_\beta(0,0))u \big\|_{L_\infty(Q_{2^{-k}})}.$$

It follows easily that

$$C_k \le \omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} + \sup_\beta \|b_\beta\|_{L_\infty} 2^{-k\alpha}[Du]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} + C\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} [u - p_{t_0,x_0}]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|Du\|_{L_\infty(Q_{2^{-k}})} + \|u\|_{L_\infty} \Big),$$

where $p_{t_0,x_0} = p_{t_0,x_0}(x)$ is the first-order Taylor's expansion of u with respect to x at (t_0,x_0). From Lemma 2.3, we obtain

$$C_k \le \omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} + \sup_\beta \|b_\beta\|_{L_\infty} 2^{-k\alpha}[Du]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} + C\omega_a(2^{-k})\Big(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q_{2^{-k}}} \inf_{p\in\mathcal{P}_x} [u - p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|Du\|_{L_\infty(Q_{2^{-k}})} + \|u\|_{L_\infty} \Big).$$

By the dominated convergence theorem, it is easy to see that

$$\|h_M\|_{L_\infty(Q_{2^{-k}})}, \quad \|\hat{h}_M\|_{L_\infty(Q_{2^{-k}})} \to 0 \quad \text{as } M \to \infty.$$

Thus similar to (3-6), choosing M sufficiently large so that

$$\|h_M\|_{L_\infty(Q_{2^{-k}})}, \quad \|\hat{h}_M\|_{L_\infty(Q_{2^{-k}})} \le \tfrac{1}{2} C_k,$$

we have

$$
\begin{aligned}
&[w_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} \\
&\quad\le C2^{-k(\sigma-\alpha)}\Big[\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q_{2^{-k}})}+2^{-k\alpha}[Du]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} \\
&\qquad\qquad+\omega_a(2^{-k})\Big(\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q_{2^{-k}}}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big)\Big]. \qquad (3\text{-}15)
\end{aligned}
$$

Clearly,

$$
\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)}\le\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)}. \qquad (3\text{-}16)
$$

From the triangle inequality and Lemma 2.4 with $j=0$,

$$
[v_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}\le[w_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}+[u-p_0]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}\le[w_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}+C\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}}.
$$

For all $l=1,2,\dots$, we define $Q^l=Q_{1-2^{-l}}$. Combining (3-14), (3-15) with (3-16), and (3-12), similar to (3-9), we get that for all $l\ge1$ and $k\ge l+1$,

$$
\begin{aligned}
&2^{(k+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-(k_0+k)}}(t_0,x_0)} \\
&\le C2^{-(k+k_0)\alpha}\sup_{(t_0,x_0)\in Q^l}\sum_{j=0}^{k}2^{(k-j)\sigma}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\quad+C2^{-(k+k_0)\alpha}[u]_{\frac{\alpha}{\sigma},\alpha}+C2^{-k\alpha+k_0(\sigma-\alpha)}[Du]_{\frac{\alpha}{\sigma},\alpha;Q^{l+1}} \\
&\quad+C2^{k_0(\sigma-\alpha)}\Big[\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q^{l+1})} \\
&\qquad\qquad+\omega_a(2^{-k})\Big(\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big)\Big]. \qquad (3\text{-}17)
\end{aligned}
$$

Summing the above inequality in $k=l+1,l+2,\dots$ as before, we obtain

$$
\begin{aligned}
&\sum_{k=l+1}^{\infty}2^{(k+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\le C2^{-k_0\alpha}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\quad+C2^{-(k_0+l)\alpha}[u]_{\frac{\alpha}{\sigma},\alpha}+C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}2^{-k\alpha}[Du]_{\frac{\alpha}{\sigma},\alpha;Q^{l+1}} \\
&\quad+C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}\big(\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q^{l+1})}\big) \\
&\quad+C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}\omega_a(2^{-k})\Big(\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\inf_{p\in\mathcal{P}_1}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big), \qquad (3\text{-}18)
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\quad \le C2^{(k_0+l)(\sigma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha} + C2^{-k_0\alpha}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\qquad + C2^{k_0(\sigma-\alpha)-l\alpha}[Du]_{\frac{\alpha}{\sigma},\alpha;Q^{l+1}} \\
&\qquad + C2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \bigl(\omega_f(2^{-k}) + (\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q^{l+1})}\bigr) \\
&\qquad + C2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_a(2^{-k}) \Biggl(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\Biggr).
\end{aligned}$$

By choosing k_0 and l sufficiently large, and using (2-4) and interpolation inequalities (recalling that $\alpha < (\sigma-1)/2$), we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\quad \le \frac{1}{4}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + C2^{(k_0+l)(\sigma-\alpha)}\|u\|_{\frac{\alpha}{\sigma},\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}).
\end{aligned}$$

Therefore,

$$\frac{1}{4^l}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \le C\|u\|_{\frac{\alpha}{\sigma},\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{3-19}$$

which together with Lemma 2.1(ii) gives (3-11) and the continuity of $\partial_t u$. □

The case when $\sigma = 1$.

Proposition 3.4. *Suppose (1-1) is satisfied in Q_2. Then under the conditions of Theorem 1.1,*

$$\|Du\|_{L_\infty(Q_{1/2})} + \|\partial_t u\|_{L_\infty(Q_{1/2})} \le C\|u\|_{\alpha,\alpha} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{3-20}$$

where $C > 0$ is a constant depending only on d, λ, Λ, N_0, ω_a, and ω_b.

Proof. Set $b_0 = b(0,0)$ and we define

$$\hat{u}(t,x) = u(t,x-b_0t), \quad \hat{f}_\beta(t,x) = f_\beta(t,x-b_0t), \quad \text{and} \quad \hat{b}(t,x) = b(t,x-b_0t).$$

It is easy to see that in Q_δ for some $\delta > 0$,

$$\partial_t\hat{u}(t,x) = \partial_t u(t,x-b_0t) - b_0\nabla u(t,x-b_0t),$$

and for $(t,x)\in Q_{2^{-k}}$,

$$|\hat{f}_\beta(t,x)-\hat{f}_\beta(0,0)|\le \omega_f((1+N_0)2^{-k}),$$
$$|\hat{b}-b_0|\le \omega_b((1+N_0)2^{-k}).$$

It follows immediately that

$$\hat{u}_t=\inf_\beta(\hat{L}_\beta\hat{u}+\hat{f}_\beta+(\hat{b}-b_0)\nabla\hat{u}), \tag{3-21}$$

where $\hat{L}$ is the operator with kernel $a(t,x-b_0t,y)|y|^{-d-\sigma}$. Furthermore,

$$\|Du\|_{L_\infty}+\|\partial_t u\|_{L_\infty}\le(1+N_0)(\|D\hat{u}\|_{L_\infty}+\|\partial_t\hat{u}\|_{L_\infty}).$$

Therefore, it is sufficient to bound $\hat{u}$. In the rest of the proof, we estimate the solution to (3-21) and abuse the notation to use u instead of $\hat{u}$ for simplicity. By scaling, translation and covering arguments, we also assume u satisfies the equation in Q_2.

The proof is similar to the case $\sigma\in(1,2)$ and we indeed proceed as in the previous case. Take p_0 to be the first-order Taylor's expansion of $u^{(2^{-k})}$ at the origin. We also assume that the solution v to the equations

$$\begin{cases}\partial_t v=\inf_{\beta\in\mathcal{A}}(L_\beta(0,0)v+f_\beta(0,0)-\partial_t p_0) & \text{in } Q_{2^{-k}},\\ \quad v=u-p_0 & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}})\cup(\{t=-2^{-k\sigma}\}\times B_{2^{-k}})\end{cases}$$

exists without carrying out another approximation argument. By Proposition 3.1 and Lemma 2.4 with $\sigma=1$,

$$\begin{aligned}[v]_{1+\alpha,1+\alpha;Q_{2^{-k-1}}}&\le C\sum_{j=1}^{\infty}2^{k-j}M_j+C2^k[v]_{\alpha,\alpha;Q_{2^{-k}}}\\&\le C\sum_{j=1}^{\infty}2^{k-j}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}}+C2^k[v]_{\alpha,\alpha;Q_{2^{-k}}}\\&\le C\sum_{j=1}^{k}2^{k-j}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}}+C[u]_{\alpha,\alpha}+C2^k[v]_{\alpha,\alpha;Q_{2^{-k}}}.\end{aligned}\tag{3-22}$$

From (3-22) and the interpolation inequality, we obtain

$$\begin{aligned}&[v-p_1]_{\alpha,\alpha;Q_{2^{-k-k_0}}}\\&\le C2^{-(k+k_0)}\sum_{j=1}^{k}2^{k-j}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(-2^{-k},0)\times B_{2^{j-k}}}+C2^{-k_0}[v]_{\alpha,\alpha;Q_{2^{-k}}}+C2^{-(k+k_0)}[u]_{\alpha,\alpha},\end{aligned}\tag{3-23}$$

where p_1 is the first-order Taylor's expansion of v at the origin. Next $w:=u-p_0-v$ satisfies (3-5), where by the cancellation property,

$$\begin{aligned}C_k\le{}&\omega_f((1+N_0)2^{-k})+\omega_b((1+N_0)2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})}\\&+C\omega_a((1+N_0)2^{-k})\biggl(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^{\infty}2^{j(1-\alpha)}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\biggr).\end{aligned}$$

Clearly, for any $r \geq 0$,

$$\omega_\bullet((1+N_0)r) \leq (2+N_0)\omega_\bullet(r).$$

Therefore, similar to (3-6), we have

$$\begin{aligned}
&[w]_{\alpha,\alpha;Q_{2^{-k}}} \\
&\quad \leq C2^{-k(1-\alpha)}\bigg[\omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} \\
&\qquad\qquad + \omega_a(2^{-k})\bigg(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q_{2^{-k}}} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg)\bigg]. \qquad (3\text{-}24)
\end{aligned}$$

From (2-16) and the triangle inequality,

$$[v]_{\alpha,\alpha;Q_{2^{-k}}} \leq [w]_{\alpha,\alpha;Q_{2^{-k}}} + [u-p_0]_{\alpha,\alpha;Q_{2^{-k}}} \leq [w]_{\alpha,\alpha;Q_{2^{-k}}} + C\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}}.$$

For all $l = 1, 2, \ldots$, we define $Q^l = Q_{1-2^{-l}}$. Similar to (3-9), by combining (3-23) and (3-24), shifting the coordinates, and using the above inequality, we obtain for all $l \geq 1$ and $k \geq l+1$,

$$\begin{aligned}
&2^{(k+k_0)(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\quad \leq C2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{k-j} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;(t_0-2^{-k},t_0)\times B_{2^{j-k}}(x_0)} \\
&\qquad + C2^{k_0(1-\alpha)}\bigg[\omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q^{l+1})} \\
&\qquad\qquad + \omega_a(2^{-k})\bigg(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha,Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg)\bigg] \\
&\qquad + C2^{-(k+k_0)\alpha}[u]_{\alpha,\alpha}, \qquad (3\text{-}25)
\end{aligned}$$

which by summing in $k = l+1, l+2, \ldots$ implies

$$\begin{aligned}
&\sum_{k=l+1}^{\infty} 2^{(k+k_0)(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\quad \leq C2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\qquad + C2^{-(k_0+l)\alpha}[u]_{\alpha,\alpha} + C2^{k_0(1-\alpha)} \sum_{k=l+1}^{\infty} \omega_f(2^{-k}) \\
&\qquad + C2^{k_0(1-\alpha)} \sum_{k=l+1}^{\infty} \bigg[\omega_b(2^{-k})\|Du\|_{L_\infty(Q^{l+1})} + \omega_a(2^{-k}) \\
&\qquad\qquad \cdot \bigg(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_x} [u-p]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg)\bigg],
\end{aligned}$$

where for the first term on the right-hand side, we replaced j by $k-j$, switched the order of the summation, and bounded it by

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-(k+k_0)\alpha} \sum_{j=0}^{k} 2^j \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \\ = 2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^j \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \sum_{k=j}^{\infty} 2^{-k\alpha} \\ \le C 2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \\ &\le C 2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \\ &\quad + C 2^{(l+k_0)(1-\alpha)} [u]_{\alpha,\alpha} + C 2^{k_0(1-\alpha)} \sum_{k=l+1}^{\infty} \omega_f(2^{-k}) \\ &\quad + C 2^{k_0(1-\alpha)} \sum_{k=l+1}^{\infty} \omega_b(2^{-k}) \|Du\|_{L_\infty(Q^{l+1})} \\ &\quad + C 2^{k_0(1-\alpha)} \sum_{k=l+1}^{\infty} \omega_a(2^{-k}) \Big(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \Big). \qquad (3\text{-}26)\end{aligned}$$

Then we choose k_0 and l sufficiently large, and apply Lemma 2.1(iii) to obtain

$$\begin{aligned}&\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \\ &\quad \le \frac{1}{4} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} + C 2^{(l+k_0)(1-\alpha)} \|u\|_{\frac{\alpha}{\sigma},\alpha} + C \sum_{k=1}^{\infty} \omega_f(2^{-k}),\end{aligned}$$

and thus,

$$\frac{1}{4^l} \sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l}} \inf_{p\in\mathcal{P}_1} [u-p]_{\alpha,\alpha;Q_{2^{-j}}(t_0,x_0)} \le C \|u\|_{\frac{\alpha}{\sigma},\alpha} + C \sum_{k=1}^{\infty} \omega_f(2^{-k}), \qquad (3\text{-}27)$$

from which (3-20) follows. The proposition is proved. □

Proof of Theorem 1.1. We use the localization argument to prove Theorem 1.1.

Without loss of generality, we assume the equation holds in Q_3. We divide the proof into three steps.

Step 1: For $k = 1, 2, \ldots$, define $Q^k := Q_{1-2^{-k}}$. Let $\eta_k \in C_0^\infty(\widehat{Q}^{k+3})$ be a sequence of nonnegative smooth cutoff functions satisfying $\eta_k \equiv 1$ in Q^{k+2}, $|\eta_k| \le 1$ in Q^{k+3}, $\|\partial_t^j D^i \eta_k\|_{L_\infty} \le C 2^{k(i+j)}$ for each $i, j \ge 0$. Set $v_k := u\eta_k \in C^{1,\sigma+}$ and notice that in Q^{k+1},

$$\begin{aligned}\partial_t v_k = \eta_k \partial_t u + \partial_t \eta_k u &= \inf_{\beta\in\mathcal{A}} (\eta_k L_\beta u + \eta_k b_\beta Du + \eta_k f_\beta + \partial_t \eta_k u)\\ &= \inf_{\beta\in\mathcal{A}} (L_\beta v_k + b_\beta D v_k - b_\beta u D\eta_k + h_{k\beta} + \eta_k f_\beta + \partial_t \eta_k u),\end{aligned}$$

where

$$h_{k\beta} = \eta_k L_\beta u - L_\beta v_k = \int_{\mathbb{R}^d} \frac{\xi_k(t,x,y) a_\beta(t,x,y)}{|y|^{d+\sigma}}\, dy,$$

and

$$\begin{aligned}\xi_k(t,x,y) &= u(t,x+y)(\eta_k(t,x+y) - \eta_k(t,x)) - y \cdot D\eta_k(t,x) u(t,x)(\chi_{\sigma=1}\chi_{B_1} + \chi_{\sigma>1})\\ &= u(t,x+y)(\eta_k(t,x+y) - \eta_k(t,x)) \quad \text{since } D\eta_k \equiv 0 \text{ in } Q^{k+1}.\end{aligned}$$

We will apply Proposition 3.3 to the equation of v_k in Q^{k+1} and obtain corresponding estimates for v_k in Q^k.

Obviously, in Q^{k+1} we have $\eta_k f_\beta \equiv f_\beta$, $b_\beta u D\eta_k \equiv 0$, and $\partial_t \eta_k u \equiv 0$. Thus, we only need to estimate the modulus of continuity of $h_{k\beta}$ in Q^{k+1}.

Step 2: For $(t,x) \in Q^{k+1}$ and $|y| \le 2^{-k-3}$, we have

$$\xi_k(t,x,y) = 0.$$

Also,

$$\begin{aligned}|\xi_k(t,x,y)| &= |u(t,x+y)(\eta_k(t,x+y) - \eta_k(t,x))|\\ &\le \begin{cases} 2\omega_u(|y|) + 2|u(t,x)| & \text{when } |y| \ge 1,\\ C2^k |u(t,x+y)|\,|y| & \text{when } 2^{-k-3} < |y| < 1.\end{cases}\end{aligned}$$

For $(t,x), (t',x') \in Q^{k+1}$, by the triangle inequality,

$$\begin{aligned}&|h_{k\beta}(t,x) - h_{k\beta}(t',x')|\\ &\quad\le \int_{\mathbb{R}^d} \frac{|(\xi_k(t,x,y) - \xi_k(t',x',y)) a_\beta(t,x,y)|}{|y|^{d+\sigma}} + \frac{|\xi_k(t',x',y)(a_\beta(t,x,y) - a_\beta(t',x',y))|}{|y|^{d+\sigma}}\, dy\\ &\quad := \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3-28}$$

By the estimates of $|\xi_k(t,x,y)|$ above, we have

$$\mathrm{II} \le C\left(2^{k(\sigma+1)} \|u\|_{L_\infty(Q_2)} + \sum_{j=0}^{\infty} 2^{-j\sigma} \omega_u(2^j)\right) \omega_a(\max\{|x-x'|, |t-t'|^{\frac{1}{\sigma}}\}), \tag{3-29}$$

where C depends on d, σ, and Λ. For I, by the fundamental theorem of calculus,

$$\xi_k(t,x,y) - \xi_k(t',x',y) = y \cdot \int_0^1 \big(u(t,x+y) D\eta_k(t,x+sy) - u(t',x'+y) D\eta_k(t',x'+sy)\big)\, ds.$$

When $2^{-k-3} \le |y| < 2$, similar to the estimate of $\xi_k(t,x,y)$, it follows that

$$|\xi_k(t,x,y)-\xi_k(t',x',y)| \le C|y|\big(2^k\omega_u(\max\{|x-x'|,|t-t'|^{\frac{1}{\sigma}}\})+2^{2k}\|u\|_{L_\infty(Q_3)}(|x-x'|+|t-t'|)\big). \tag{3-30}$$

When $|y|\ge 2$, we have

$$|\xi_k(t,x,y)-\xi_k(t',x',y)| = |u(t,x+y)-u(t',x'+y)| \le \omega_u(\max\{|x-x'|,|t-t'|^{\frac{1}{\sigma}}\}),$$

which implies

$$\mathrm{I} \le C2^{k(\sigma+1)}\omega_u(\max\{|x-x'|,|t-t'|^{\frac{1}{\sigma}}\})+C2^{k(\sigma+2)}\|u\|_{L_\infty(Q_3)}(|x-x'|+|t-t'|).$$

Therefore,

$$|h_{k\beta}(t,x)-h_{k\beta}(t',x')| \le \omega_h(\max\{|x-x'|,|t-t'|^{\frac{1}{\sigma}}\}),$$

where

$$\omega_h(r) := C\Big(2^{k(\sigma+1)}\|u\|_{L_\infty(Q_3)}+\sum_{j=0}^{\infty}2^{-j\sigma}\omega_u(2^j)\Big)\omega_a(r) + C2^{k(\sigma+1)}\omega_u(r)+C2^{k(\sigma+2)}\|u\|_{L_\infty(Q_3)}(r+r^\sigma) \tag{3-31}$$

is a Dini function.

Step 3: In this last step, we only present the detailed proof for $\sigma\in(1,2)$. We omit the details for the proof of the case $\sigma\in(0,1]$, since it is almost the same as and actually even simpler than the case $\sigma\in(1,2)$. We apply Proposition 3.3, together with a scaling and covering argument, to v_k to obtain

$$\begin{aligned}\|\partial_t v_k\|_{L_\infty(Q^k)}+[v_k]^x_{\sigma;Q^k}+[Dv_k]^t_{\frac{\sigma-1}{\sigma};Q^k} &\le C2^{k\sigma}\|v_k\|_{L_\infty}+C2^{k(\sigma-\alpha)}[v_k]_{\frac{\alpha}{\sigma},\alpha}+C\sum_{j=1}^{\infty}(\omega_h(2^{-j})+\omega_f(2^{-j}))\\ &\le C2^{k(\sigma+2)}\|u\|_{L_\infty(Q_3)}+C_02^{k(\sigma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha;Q^{k+3}}\\ &\quad+C\sum_{j=0}^{\infty}2^{-j\sigma}\omega_u(2^j)+C\sum_{j=0}^{\infty}(2^{k(\sigma+1)}\omega_u(2^{-j})+\omega_f(2^{-j})),\end{aligned}$$

where C and C_0 depend on d, λ, Λ, σ, N_0, ω_b, and ω_a, but are independent of k. Since $\eta_k\equiv 1$ in Q^k, it follows that

$$\begin{aligned}\|\partial_t u\|_{L_\infty(Q^k)}+[u]^x_{\sigma;Q^k}+[Du]^t_{\frac{\sigma-1}{\sigma};Q^k} &\le C2^{k(\sigma+2)}\|u\|_{L_\infty(Q_3)}+C_02^{k(\sigma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha;Q^{k+3}}\\ &\quad+C\sum_{j=0}^{\infty}2^{-j\sigma}\omega_u(2^j)+C\sum_{j=0}^{\infty}(2^{k(\sigma+1)}\omega_u(2^{-j})+\omega_f(2^{-j})).\end{aligned} \tag{3-32}$$

By the interpolation inequality, for any $\varepsilon \in (0,1)$,

$$[u]_{\frac{\alpha}{\sigma},\alpha;Q^{k+3}} \le \varepsilon(\|\partial_t u\|_{L_\infty(Q^{k+3})} + [u]^x_{\sigma;Q^{k+3}}) + C\varepsilon^{-\frac{\alpha}{\sigma-\alpha}}\|u\|_{L_\infty(Q_3)}. \tag{3-33}$$

Recall that $\alpha \le (\sigma-1)/2$ and thus,

$$\frac{\alpha}{\sigma-\alpha} \le \frac{\sigma-1}{\sigma+1} < \frac{1}{2}.$$

Combining (3-32) and (3-33) with $\varepsilon = C_0^{-1}2^{-3k-16}$, we obtain

$$\begin{aligned}\|\partial_t u\|_{L_\infty(Q^k)} &+ [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k}\\ &\le 2^{-16}([u]^x_{\sigma;Q^{k+3}} + \|\partial_t u\|_{L_\infty(Q^{k+3})} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^{k+3}}) + C2^{4k}\|u\|_{L_\infty(Q_3)}\\ &\qquad + C\sum_{j=0}^\infty 2^{-j\sigma}\omega_u(2^j) + C\sum_{j=0}^\infty (2^{k(\sigma+1)}\omega_u(2^{-j}) + \omega_f(2^{-j})).\end{aligned}$$

Then we multiply 2^{-5k} to both sides of the above inequality and get

$$\begin{aligned}2^{-5k}(\|\partial_t u\|_{L_\infty(Q^k)} &+ [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k})\\ &\le 2^{-5(k+3)-1}(\|\partial_t u\|_{L_\infty(Q^{k+3})} + [u]^x_{\sigma;Q^{k+3}} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^{k+3}})\\ &\qquad + C2^{-k}\|u\|_{L_\infty(Q_3)} + C2^{-2k}\sum_{j=0}^\infty (2^{-j\sigma}\omega_u(2^j) + \omega_u(2^{-j}) + \omega_f(2^{-j})).\end{aligned}$$

We sum up the both sides of the above inequality and obtain

$$\begin{aligned}\sum_{k=1}^\infty 2^{-5k}(\|\partial_t u\|_{L_\infty(Q^k)} &+ [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k})\\ &\le \frac{1}{2}\sum_{k=4}^\infty 2^{-5k}(\|\partial_t u\|_{L_\infty(Q^k)} + [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k})\\ &\qquad + C\|u\|_{L_\infty(Q_3)} + C\sum_{j=0}^\infty (2^{-j\sigma}\omega_u(2^j) + \omega_u(2^{-j}) + \omega_f(2^{-j})),\end{aligned}$$

which further implies

$$\begin{aligned}\sum_{k=1}^\infty 2^{-5k}(\|\partial_t u\|_{L_\infty(Q^k)} &+ [u]^x_{\sigma;Q^k} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^k})\\ &\le C\|u\|_{L_\infty(Q_3)} + C\sum_{j=0}^\infty (2^{-j\sigma}\omega_u(2^j) + \omega_u(2^{-j}) + \omega_f(2^{-j})),\end{aligned}$$

where C depends on d, λ, Λ, σ, ω_b, N_0, and ω_a. By applying this estimate to $u - u(0,0)$, we obtain

$$\|\partial_t u\|_{L_\infty(Q^4)} + [u]^x_{\sigma;Q^4} + [Du]^t_{\frac{\sigma-1}{\sigma};Q^4} \le C\sum_{j=0}^\infty (2^{-j\sigma}\omega_u(2^j) + \omega_u(2^{-j}) + \omega_f(2^{-j})). \tag{3-34}$$

This proves (1-4).

Finally, since $\|v_1\|_{\frac{\alpha}{\sigma},\alpha}$ is bounded by the right-hand side of (3-34), from (3-19), we see that

$$\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1} [v_1-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} \le C$$

for some large l. This and (3-18) with u replaced by v_1 and f_β replaced by $h_{1\beta}+\eta_1 f_\beta+\partial_t\eta_1 u-b_\beta u D\eta_1$ give

$$\sum_{j=k_1+1}^{\infty} 2^{(j+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{k_1}} \inf_{p\in\mathcal{P}_1} [v_1-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j-k_0}}(t_0,x_0)}$$
$$\le C2^{-k_0\alpha} + C2^{k_0(\sigma-\alpha)} \sum_{j=k_1}^{\infty} (\omega_f(2^{-j})+\omega_a(2^{-j})+\omega_u(2^{-j})+\omega_b(2^{-j})+2^{-j\alpha}).$$

Here we also used (3-31) with $k=1$. Therefore, for any small $\varepsilon>0$, we can find k_0 sufficiently large, then k_1 sufficiently large, depending only on C, σ, N_0, α, ω_f, ω_a, ω_f, ω_b, and ω_u, such that

$$\sum_{j=k_1+1}^{\infty} 2^{(j+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{k_1}} \inf_{p\in\mathcal{P}_1} [v_1-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j-k_0}}(t_0,x_0)} < \varepsilon,$$

which, together with the fact that $v_1=u$ in $Q_{\frac{1}{2}}$ and the proof of Lemma 2.1(ii), indicates that

$$\sup_{(t_0,x_0)\in Q_{1/2}} ([u]^x_{\sigma;Q_r(t_0,x_0)} + [Du]^t_{\frac{\sigma-1}{\sigma};Q_r(t_0,x_0)}) \to 0 \quad \text{as } r\to 0$$

with a decay rate depending only on d, λ, N_0, Λ, ω_a, ω_f, ω_b, ω_u, and σ. Hence, the proof of the case when $\sigma\in(1,2)$ is completed. □

4. Schauder estimates for equations with drifts

We now are going to prove Theorems 1.3 and 1.4. Here, the main difference from the theorems in [Dong and Zhang 2016b] is that our equation may have a drift, especially for $\sigma=1$.

We first prove a weaker version of Theorem 1.3.

Proposition 4.1. *Suppose (1-1) is satisfied in Q_2. Then under the conditions of Theorem 1.3, for any $\gamma\in(0,\min\{\hat\alpha,2-\sigma\})$ with $\hat\alpha$ being the one in Proposition 3.1, and any $\alpha\in(\gamma,\min\{\hat\alpha,2-\sigma\})$, we have*

$$[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \le C(\|u\|_{\frac{\alpha}{\sigma},\alpha}+C_f),$$

where $C>0$ is a constant depending only on $d,\gamma,\alpha,\sigma,\lambda,\Lambda,N_0$, and C_b.

Proof. The proof is very similar to that of Propositions 3.2, 3.3, and 3.4. We fix an $\alpha\in(\gamma,\hat\alpha)$.

Case 1: $\sigma\in(0,1)$. We start from (3-9). Let Q^l and ℓ_0 be as in the proof of Proposition 3.2. Multiplying $2^{(k+k_0)\gamma}$ to both sides of (3-9) and making use of the Hölder continuity of a and f, we have for all $l\ge\ell_0$

and $k \geq l+1$,

$$
\begin{aligned}
&2^{(k+k_0)(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\quad \leq C 2^{(k+k_0)(\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\qquad + C 2^{(k+k_0)(\gamma-\alpha)} [u]_{\frac{\alpha}{\sigma},\alpha} \\
&\qquad + C 2^{k_0(\sigma+\gamma-\alpha)} \Bigg[C_f + \sup_{(t_0,x_0)\in Q^{l+1}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \Bigg].
\end{aligned}
$$

Taking the supremum in $k \geq \ell_0+1$ and using the fact that $\gamma<\alpha$, we have

$$
\begin{aligned}
&\sup_{k\geq \ell_0+k_0+1} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&\quad \leq C 2^{k_0(\gamma-\alpha)} \sup_{k\geq 0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(x_0)} \\
&\qquad + C 2^{(\ell_0+k_0+1)(\gamma-\alpha)} [u]_{\frac{\alpha}{\sigma},\alpha} \\
&\qquad + C 2^{k_0(\sigma+\gamma-\alpha)} \Bigg[C_f + \sup_{(t_0,x_0)\in Q^{l+1}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \Bigg].
\end{aligned}
$$

By taking k_0 large, $l=\ell_0$, using (3-10), and noticing that

$$
\sup_{0\leq k\leq \ell_0+k_0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}(t_0,x_0)}} \leq C 2^{(\ell_0+k_0)(\sigma+\gamma-\alpha)} [u]_{\frac{\alpha}{\sigma},\alpha},
$$

we have

$$
\sup_{k\geq 0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q_{1/2}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} \leq C[C_f + \|u\|_{\frac{\alpha}{\sigma},\alpha}].
$$

Since

$$
[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \leq C \sup_{k\geq 0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q_{1/2}} \inf_{p\in\mathcal{P}_t} [u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} + C[u]_{\frac{\alpha}{\sigma},\alpha},
$$

we obtain

$$
[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \leq C(\|u\|_{\frac{\alpha}{\sigma},\alpha} + C_f).
$$

Case 2: $\sigma\in(1,2)$. We start from (3-17). Let Q^l be as in the proof of Proposition 3.3. Multiplying $2^{(k+k_0)\gamma}$ to both sides of (3-17) and making use of the Hölder continuity of a, b, and f, we have for all $l\geq 1$ and $k\geq l+1$,

$$
\begin{aligned}
&2^{(k+k_0)(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k-k_0}}(t_0,x_0)}\\
&\le C2^{(k+k_0)(\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{(k-j)\sigma} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)}\\
&\quad +C2^{(k+k_0)(\gamma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha}+C2^{(k+k_0)(\gamma-\alpha)+k_0\sigma}[Du]_{\frac{\alpha}{\sigma},\alpha;Q^{l+1}}\\
&\quad +C2^{k_0(\sigma+\gamma-\alpha)}\Big[C_f+\|Du\|_{L_\infty(Q^{l+1})}+\sup_{(t_0,x_0)\in Q^{l+1}}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big].
\end{aligned}
$$

Note that this $\hat{\alpha}$ can be chosen very small, at least strictly smaller than $\sigma-1$. Taking the supremum in $k\ge 2$ and using the fact that $\gamma<\alpha$, we have

$$
\begin{aligned}
&\sup_{k\ge k_0+2} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)}\\
&\le C2^{k_0(\gamma-\alpha)} \sup_{k\ge 0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(x_0)}\\
&\quad +C2^{(k_0+2)(\gamma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha}+C2^{(2+k_0)(\gamma-\alpha)+k_0\sigma}[Du]_{\frac{\alpha}{\sigma},\alpha;Q^{l+1}}\\
&\quad +C2^{k_0(\sigma+\gamma-\alpha)}\Big[C_f+\|Du\|_{L_\infty(Q^{l+1})}+\sup_{(t_0,x_0)\in Q^{l+1}}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big].
\end{aligned}
$$

By taking k_0 large, $l=1$, using (3-19) and (2-4), and noticing that

$$
\sup_{0\le k\le 1+k_0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}(t_0,x_0)}} \le C2^{(1+k_0)(\sigma+\gamma-\alpha)}[u]_{\frac{\alpha}{\sigma},\alpha},
$$

we have

$$
\sup_{k\ge 0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q_{1/2}} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)} \le C[C_f+\|u\|_{\frac{\alpha}{\sigma},\alpha}].
$$

Since

$$
[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \le C\sup_{k\ge 0} 2^{k(\sigma+\gamma-\alpha)} \sup_{(t_0,x_0)\in Q_{1/2}} \inf_{p\in\mathcal{P}_1}[u-p]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}(t_0,x_0)}+C[u]_{\frac{\alpha}{\sigma},\alpha},
$$

we obtain

$$
[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \le C(\|u\|_{\frac{\alpha}{\sigma},\alpha}+C_f).
$$

<u>Case 3</u>: $\sigma=1$. We start from (3-25). Multiplying $2^{(k+k_0)\gamma}$ to both sides of (3-25) and making use of the Hölder continuity of a,b,f, we have for all $l\ge 1$ and $k\ge l+1$,

$$
\begin{aligned}
&2^{(k+k_0)(1+\gamma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k-k_0}}(t_0,x_0)}\\
&\le C2^{(k+k_0)(\gamma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\sum_{j=0}^{k}2^{k-j}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;(t_0-2^{-k},t_0)\times B_{2^{j-k}}(x_0)}\\
&\quad+C2^{k_0(1+\gamma-\alpha)}\Big[C_f+\|Du\|_{L_\infty(Q^{l+1})}+\sum_{j=0}^{\infty}2^{j(1-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha,Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big]\\
&\quad+C2^{(k+k_0)(\gamma-\alpha)}[u]_{\alpha,\alpha}.
\end{aligned}
$$

Taking the supremum in $k\ge 2$ and using the fact that $\gamma<\alpha$, we have

$$
\begin{aligned}
&\sup_{k\ge k_0+2}2^{k(1+\gamma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)}\\
&\le C2^{(k_0-1)(\gamma-\alpha)}\sup_{k\ge 0}2^{k(1+\gamma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)}\\
&\quad+C2^{k_0(1+\gamma-\alpha)}\Big[C_f+\|Du\|_{L_\infty(Q^{l+1})}+\sum_{j=0}^{\infty}2^{j(1-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\inf_{p\in\mathcal{P}_x}[u-p]^x_{\alpha,Q_{2^{-j}}(t_0,x_0)}+\|u\|_{L_\infty}\Big]\\
&\quad+C2^{k_0(\gamma-\alpha)}[u]_{\alpha,\alpha}.
\end{aligned}
$$

By taking k_0 large, $l=1$, using (3-27), and noticing that

$$
\sup_{0\le k\le k_0+1}2^{k(1+\gamma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)}\le C2^{k_0(1+\gamma-\alpha)}[u]_{\alpha,\alpha},
$$

we have

$$
\sup_{k\ge 0}2^{k(1+\gamma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)}\le C[C_f+\|u\|_{\alpha,\alpha}].
$$

Since

$$
[Du]_{\gamma,\gamma;Q_{1/2}}+[\partial_t u]_{\gamma,\gamma;Q_{1/2}}\le C\sup_{k\ge 0}2^{k(1+\gamma-\alpha)}\sup_{(t_0,x_0)\in Q_{1/2}}\inf_{p\in\mathcal{P}_1}[u-p]_{\alpha,\alpha;Q_{2^{-k}}(t_0,x_0)}+C[u]_{\alpha,\alpha},
$$

we obtain

$$
[Du]_{\gamma,\gamma;Q_{1/2}}+[\partial_t u]_{\gamma,\gamma;Q_{1/2}}\le C[C_f+\|u\|_{\alpha,\alpha}]. \qquad \square
$$

Proof of Theorem 1.3. The proof is the same as that of Theorem 1.1 using localizations. We sketch the proof here. We use the same notation as in the proof of Theorem 1.1. Without loss of generality, we assume (1-1) holds in Q_3.

Let $\eta_k\in C_0^\infty(\hat{Q}^{k+3})$ be a sequence of nonnegative smooth cutoff functions satisfying $\eta\equiv 1$ in Q^{k+2}, $|\eta|\le 1$ in Q^{k+3}, $\|\partial_t^j D^i\eta_k\|_{L_\infty}\le C2^{k(i+j)}$ for each $i,j\ge 0$. Set $v_k:=u\eta_k\in C^{1+\frac{\gamma}{\sigma},\sigma+\gamma}$ and notice that in Q_1,

$$
\partial_t v_k=\inf_{\beta\in\mathcal{A}}(L_\beta v_k+b_\beta Dv_k-b_\beta uD\eta_k+h_{k\beta}+\eta_k f_\beta+\partial_t\eta_k u),
$$

where

$$h_{k\beta}(t,x)=\int_{\mathbb{R}^d}\frac{\xi_k(t,x,y)a_\beta(t,x,y)}{|y|^{d+1}}\,dy,$$

and

$$\begin{aligned}\xi_k(t,x,y)&:=u(t,x+y)(\eta_k(t,x+y)-\eta_k(t,x))-u(t,x)y\cdot D\eta_k(t,x)(\chi_{\sigma=1}\chi_{B_1}+\chi_{\sigma>1})\\&=u(t,x+y)(\eta_k(t,x+y)-\eta_k(t,x))\quad\text{since } D\eta_k\equiv 0 \text{ in } Q^{k+1}.\end{aligned}$$

We will apply Proposition 4.1 to the equation of v_k in Q^{k+1} and obtain corresponding estimates for v_k in Q^k.

Obviously, in Q^{k+1} we have $\eta_k f_\beta\equiv f_\beta$, $buD\eta_k\equiv 0$, and $\partial_t\eta_k u\equiv 0$. Thus, we only need to estimate the modulus of continuity of $h_{k\beta}$ in Q^{k+1}. Since

$$\xi_k(t,x,y):=u(t,x+y)(\eta_k(t,x+y)-\eta_k(t,x)),$$

which is the same as in Theorem 1.1, we also have (3-31) here. Therefore,

$$[h_{k\beta}]_{\frac{\gamma}{\sigma},\gamma;Q^{k+1}}\le C\Bigl(2^{k(\sigma+1)k}\|u\|_{L_\infty(Q_3)}+\sum_{j=0}^{\infty}2^{-j\sigma}\omega_u(2^j)\Bigr)+C2^{k(\sigma+1)}[u]_{\frac{\gamma}{\sigma},\gamma}+C2^{k(\sigma+2)}\|u\|_{L_\infty(Q_3)}.$$

The rest is almost the same as (actually much simpler than) the proof of Theorem 1.1, by using Proposition 4.1 (recalling $\gamma<\alpha$), and we omit the details. □

In the following, we prove Theorem 1.4 using Theorem 1.3 and difference quotients.

Proof of Theorem 1.4. We only provide the proof for $\sigma+\gamma>2$. We know from Theorem 1.3 that there exists γ_0 such that $\sigma+\gamma_0<2$ is not an integer, and the theorem holds for $0<\gamma\le\gamma_0$. Below we will prove the theorem for all $\gamma\in(\gamma_0,\sigma)$ using difference quotients.

We suppose (1-7) holds in Q_4. We will consider the difference quotients in x first. For $h\in\bigl(0,\frac14\bigr)$, $e\in\mathbb{S}^{d-1}$, let

$$u^h(t,x)=\frac{u(t,x+he)-u(t,x)}{h^{\gamma-\gamma_0}},\qquad f^h(t,x)=\frac{f(t,x+he)-f(t,x)}{h^{\gamma-\gamma_0}},$$

and

$$a^h(t,x,y)=\frac{a(t,x+he,y)-a(t,x,y)}{h^{\gamma-\gamma_0}},\qquad b^h(t,x)=\frac{b(t,x+he)-b(t,x)}{h^{\gamma-\gamma_0}}.$$

Then u^h satisfies

$$\partial_t u^h(t,x)=L_h u^h+b(t,x+he)Du^h+f^h+b^hDu+g\quad\text{in } Q_1,$$

where

$$L_hu=\int_{\mathbb{R}^d}\frac{\delta u(t,x,y)a(t,x+he,y)}{|y|^{d+\sigma}}\,dy,\qquad g=\int_{\mathbb{R}^d}\frac{\delta u(t,x,y)a^h(t,x,y)}{|y|^{d+\sigma}}\,dy.$$

Applying the result for $\gamma=\gamma_0$ gives

$$[u^h]_{1+\frac{\gamma_0}{\sigma},\sigma+\gamma_0;Q_{3/4}}\le C\|u^h\|_{\frac{\gamma_0}{\sigma},\gamma_0}+C[f^h+b^hDu+g]_{\frac{\gamma_0}{\sigma},\gamma_0;Q_1}.$$

It follows from direct calculations that

$$[g]_{\frac{\gamma_0}{\sigma},\gamma_0;Q_1}\le C[u]_{1+\frac{\gamma_0}{\sigma},\sigma+\gamma_0;Q_{5/4}}+C\|u\|_{\frac{\gamma_0}{\sigma},\gamma_0}.$$

Applying the $C^{1+\frac{\gamma_0}{\sigma},\sigma+\gamma_0}$ estimate as mentioned at the beginning of this proof, we have

$$[g]_{\frac{\gamma_0}{\sigma},\gamma_0;Q_1} \le C\|u\|_{\frac{\gamma_0}{\sigma},\gamma_0} + C[f]_{\frac{\gamma_0}{\sigma},\gamma_0;Q_2}.$$

Similarly, we have

$$[b^h Du]_{\frac{\gamma_0}{\sigma},\gamma_0;Q_1} \le C\|Du\|_{\frac{\gamma_0}{\sigma},\gamma_0;Q_1} \le C\|u\|_{\frac{\gamma_0}{\sigma},\gamma_0} + C[f]_{\frac{\gamma_0}{\sigma},\gamma_0;Q_2}.$$

Therefore,

$$[u^h]_{1+\frac{\gamma_0}{\sigma},\sigma+\gamma_0;Q_{3/4}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2}.$$

Note that we assumed that $\sigma+\gamma>2$ and thus, $\sigma>1$. Also $1<\sigma+\gamma_0<2$. Then we have

$$[(Du)^h]^x_{\sigma+\gamma_0-1;Q_{3/4}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2},$$

that is,

$$\frac{|Du(t,x+2he)-2Du(t,x+he)+Du(t,x)|}{h^{\sigma+\gamma-1}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2}$$

for all $(t,x)\in Q_{\frac{1}{2}}$ and $h\le\frac{1}{20}$. Making use of (2-1) and sending $j\to\infty$ there, we have

$$\begin{aligned}|Du(t,x+he)-Du(t,x)-hD^2u(t,x)\cdot e| &\le Ch^{\sigma+\gamma-1}\sum_{k=1}^{\infty}2^{-k(\sigma+\gamma-2)}(\|u\|_{\frac{\gamma}{\sigma},\gamma}+[f]_{\frac{\gamma}{\sigma},\gamma;Q_2})\\ &\le C(\|u\|_{\frac{\gamma}{\sigma},\gamma}+[f]_{\frac{\gamma}{\sigma},\gamma;Q_2})h^{\sigma+\gamma-1},\end{aligned}$$

from which we have

$$[u]^x_{\sigma+\gamma;Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2}. \tag{4-1}$$

Similarly, we can use the difference quotients in t. For $s\in\left(0,\frac{1}{10}\right)$, let

$$u^s(t,x)=\frac{u(t,x)-u(t-s,x)}{s^{\frac{\gamma-\gamma_0}{\sigma}}}. \tag{4-2}$$

By similar arguments, we have

$$[u^s]^t_{1+\frac{\gamma_0}{\sigma};Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2},$$

that is

$$[(u_t)^s]^t_{\frac{\gamma_0}{\sigma};Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2}.$$

The same arguments as the above (noticing $\sigma>\gamma$) will lead to

$$[u]^t_{1+\frac{\gamma}{\sigma};Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2}.$$

This estimate, together with (4-1), implies

$$[u]_{1+\frac{\gamma}{\sigma},\sigma+\gamma;Q_{1/2}} \le C\|u\|_{\frac{\gamma}{\sigma},\gamma} + C[f]_{\frac{\gamma}{\sigma},\gamma;Q_2}.$$

We remark that actually the proof of the other situation $\sigma+\gamma\in(0,1)\cup(1,2)$ is exactly the same as above. ☐

5. Linear parabolic equations with measurable coefficient in t

We now consider the linear equation (1-7), where K, b, and f are Dini continuous in x but only measurable in the time variable t. We first need a proposition for the case that K does not depend on x, and $b \equiv 0$.

Proposition 5.1. *Let $\sigma \in (0,2)$ and $0 < \lambda \le \Lambda$. Assume K does not depend on x, and $b \equiv 0$. Let $\alpha \in (0,1)$ such that $\sigma+\alpha$ is not an integer. Suppose $u \in C_x^{\sigma+\alpha}(Q_1) \cap C^{\frac{\alpha}{\sigma},\alpha}((-2^\sigma,0)\times\mathbb{R}^d)$ is a solution of (1-7) in Q_1. Then,*

$$[u]^x_{\alpha+\sigma;Q_{1/2}} \le C\sum_{j=1}^{\infty} 2^{-j\sigma} M_j + C[f]^x_{\alpha;Q_1},$$

where

$$M_j = \sup_{\substack{(t,x),(t,x')\in(-1,0)\times B_{2^j}\\ 0<|x-x'|<2}} \frac{|u(t,x)-u(t,x')|}{|x-x'|^\alpha}$$

and $C>0$ is a constant depending only on d, σ, λ, Λ, and N_0, and is uniformly bounded as $\sigma \to 2^-$.

Proof. We only prove the case that $\sigma+\alpha>2$ as before. Let η be a cut-off function such that $\eta \in C_c^\infty(\hat{Q}_{\frac{3}{4}})$ and $\eta \equiv 1$ in $Q_{\frac{1}{2}}$. Let $w(t,x) = u(t,x)-u(t,0)$, $\tilde{f}(t,x) = f(t,x)-f(t,0)$ and $v=\eta w$. Then v satisfies

$$v_t = Lv + h + \eta_t w + \eta\tilde{f} - \eta g(t) \quad \text{in } (-2^\sigma,0)\times\mathbb{R}^d,$$

where

$$h = \eta L w - L(\eta w) = \int_{\mathbb{R}} \big((\eta(t,x)-\eta(t,x+y))w(t,x+y) + y^T D\eta(t,x)w(t,x)\big)K(t,y)\,dy$$

and

$$g(t) = (Lu)(t,0).$$

By Theorem 4 in [Mikulevičius and Pragarauskas 2014], we have

$$\|v\|^x_{\sigma+\alpha} \le C\|h+\eta_t w + \eta\tilde{f} - \eta g(t)\|^x_\alpha.$$

From (3.18) and (3.23) in [Dong and Zhang 2016a], we have

$$\|h\|^x_\alpha \le C(\|w\|^x_{\alpha;(-1,0)\times\mathbb{R}^d} + [\nabla w]^x_{\alpha;Q_{15/16}}) \le C(\|u\|^x_{\alpha;(-1,0)\times\mathbb{R}^d} + [\nabla u]^x_{\alpha;Q_{15/16}}).$$

It is clear that

$$\|\eta g(t)\|^x_\alpha \le C(\|D^2u\|_{L_\infty(Q_{7/8})} + \|Du\|_{L_\infty(Q_{7/8})} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}),$$
$$\|\eta\tilde{f}\|^x_\alpha \le C[f]^x_{\alpha,Q_{3/4}}.$$

Therefore, we have

$$[u]^x_{\sigma+\alpha;Q_{1/2}} \le C(\|D^2u\|_{L_\infty(Q_{7/8})} + \|Du\|_{L_\infty(Q_{7/8})} + [\nabla u]^x_{\alpha;Q_{15/16}} + \|u\|^x_{\alpha;(-1,0)\times\mathbb{R}^d} + [f]^x_{\alpha;Q_{3/4}}).$$

The same interpolation arguments of the proof of Theorem 1.1 in [Dong and Zhang 2016b] lead to

$$[u]^x_{\sigma+\alpha;Q_{1/2}} \le C(\|u\|^x_{\alpha;(-1,0)\times\mathbb{R}^d} + [f]^x_{\alpha;Q_{3/4}}).$$

Then as in the proof of Proposition 3.1, see also [Dong and Zhang 2016b, Corollary 4.6], applying this estimate to the equation of $\tilde{v} := \tilde{\eta}(u(t,x) - u(t,0))$, where $\tilde{\eta} \in C_0^\infty(\hat{Q}_{\frac{15}{16}})$ satisfying $\tilde{\eta} = 1$ in $Q_{\frac{3}{4}}$, we have the desired estimates for $[u]^x_{\alpha+\sigma;Q_{1/2}}$. ☐

Proposition 5.2. *Suppose (1-7) is satisfied in Q_2. Then under the conditions of Theorem 1.5, we have*

$$[u]^x_{\sigma;Q_{1/2}} \le C\|u\|^x_\alpha + C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{5-1}$$

where $C > 0$ is a constant depending only on d, λ, Λ, ω_a, ω_b, N_0 and σ.

Proof. We will consider three cases separately.

Case 1: $\sigma \in (0,1)$. For $k \in \mathbb{N}$, let v be the solution of

$$\begin{cases} \partial_t v = L(t,0)v + f(t,0) & \text{for } x \in B_{2^{-k}}, \text{and almost every } t \in (-2^{-\sigma k}, 0], \\ v = u & \text{in } ((-2^{-k\sigma}, 0) \times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\} \times B_{2^{-k}}). \end{cases} \tag{5-2}$$

We sketch the proof of the existence of such v as follows. Let $K^\varepsilon(t,0,y)$ and $f^\varepsilon(t,0)$ be the mollifications of $K(t,0,y)$ and $f(t,0)$ in t. Then there exists v^ε satisfying

$$\begin{cases} \partial_t v^\varepsilon = L^\varepsilon(t,0)v^\varepsilon + f^\varepsilon(t,0) & \text{in } Q_{2^{-k}}, \\ v^\varepsilon = u & \text{in } ((-2^{-k\sigma}, 0) \times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\} \times B_{2^{-k}}). \end{cases} \tag{5-3}$$

Since this equation is uniformly elliptic, we have the global uniform Hölder estimate of v^ε, which is independent of ε. Thus, there exists a subsequence converging locally uniformly to a global Hölder continuous function v. On the other hand, by Proposition 5.1, we can reselect a subsequence such that for almost every time, they converge to v locally uniformly in $C^{\sigma+\alpha}_x(B_{2^{-k}})$. Since we have from (5-3) that for all $t \in (-2^{-k\sigma}, 0]$,

$$\begin{cases} v^\varepsilon(t,x) = u(-2^{-k\sigma}, x) + \int_{-2^{-k\sigma}}^t L^\varepsilon(\tau,0)v^\varepsilon(\tau,x)\,d\tau + \int_{-2^{-k\sigma}}^t f^\varepsilon(\tau,0)\,d\tau & \text{in } Q_{2^{-k}}, \\ v^\varepsilon = u & \text{in } ((-2^{-k\sigma}, 0) \times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\} \times B_{2^{-k}}), \end{cases}$$

we can send $\varepsilon \to 0$, using the dominated convergence theorem, to obtain

$$\begin{cases} v(t,x) = u(-2^{-k\sigma}, x) + \int_{-2^{-k\sigma}}^t L(\tau,0)v(\tau,x)\,d\tau + \int_{-2^{-k\sigma}}^t f(\tau,0)\,d\tau & \text{in } Q_{2^{-k}}, \\ v = u & \text{in } ((-2^{-k\sigma}, 0) \times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\} \times B_{2^{-k}}). \end{cases}$$

This proves (5-2). Moreover, we have from the estimates of v^ε in Proposition 5.1 by sending $\varepsilon \to 0$, that

$$[v]^x_{\alpha+\sigma;Q_{2^{-k-1}}} \le C\sum_{j=1}^{\infty} 2^{(k-j)\sigma} M_j + C2^{k\sigma}[v]^x_{\alpha;Q_{2^{-k}}}, \tag{5-4}$$

where $\alpha \in (0,1)$ satisfying $\sigma + \alpha < 1$ and

$$M_j = \sup_{\substack{(t,x),(t,x') \in (-2^{-k\sigma},0) \times B_{2^{j-k}} \\ 0 < |x-x'| < 2^{-k+1}}} \frac{|u(t,x) - u(t,x')|}{|x-x'|^\alpha}.$$

Let $k_0 \geq 1$ be an integer to be specified. From (5-4), we have

$$[v]^x_{\alpha;Q_{2^{-k-k_0}}} \leq C2^{-(k+k_0)\sigma}\sum_{j=1}^{k} 2^{(k-j)\sigma} M_j + C2^{-(k+k_0)\sigma}[u]^x_\alpha + C2^{-k_0\sigma}[v]^x_{\alpha;Q_{2^{-k}}}. \tag{5-5}$$

Let $w := u - v$ which satisfies

$$\begin{cases} \partial_t w = L(t,0)w + C_k & \text{in } Q_{2^{-k}}, \\ \quad w = 0 & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\}\times B_{2^{-k}}), \end{cases}$$

where

$$C_k(t,x) = f(t,x) - f(t,0) + (L(t,x) - L(t,0))u.$$

It is easily seen that

$$\|C_k\|_{L_\infty(Q_{2^{-k}})} \leq \omega_f(2^{-k}) + C\omega_a(2^{-k})\bigg(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg).$$

Then by the Hölder estimate [Dong and Zhang 2016b, Lemma 2.5], we have

$$\begin{aligned}[w]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} &\leq C2^{-k(\sigma-\alpha)}C_k \\ &\leq C2^{-k(\sigma-\alpha)}\bigg[\omega_f(2^{-k}) + \omega_a(2^{-k})\bigg(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg)\bigg].\end{aligned} \tag{5-6}$$

Combining (5-5) and (5-6) yields

$$\begin{aligned} &2^{(k+k_0)(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-k-k_0}}} \\ &\quad\leq C2^{-(k+k_0)\alpha}\sum_{j=1}^{k} 2^{(k-j)\sigma}[u]^x_{\alpha;(-2^{-k\sigma},0)\times B_{2^{j-k}}} \\ &\qquad + C2^{-(k+k_0)\alpha}[u]^x_\alpha + C2^{-k_0\alpha+k(\sigma-\alpha)}[u]^x_{\alpha;(-2^{-k\sigma},0)\times B_{2^{-k}}} + C2^{k_0(\sigma-\alpha)}\omega_f(2^{-k}) \\ &\qquad + C2^{k_0(\sigma-\alpha)}\omega_a(2^{-k})\bigg(\sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg). \end{aligned} \tag{5-7}$$

Let Q^l, $l = \ell_0, \ell_0+1, \dots$, be those in the proof of Proposition 3.2. By translation of the coordinates, from (5-7) we have for $l \geq \ell_0$, $k \geq l+1$,

$$\begin{aligned} &2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\ &\quad\leq C2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{(k-j)\sigma}[u]^x_{\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} + C2^{-(k+k_0)\alpha}[u]^x_\alpha \\ &\qquad + C2^{k_0(\sigma-\alpha)}\bigg[\omega_f(2^{-k}) + \omega_a(2^{-k})\bigg(\sup_{(t_0,x_0)\in Q^{l+1}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)}[u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty}\bigg)\bigg]. \end{aligned} \tag{5-8}$$

Then we take the sum (5-8) in $k = l+1, l+2, \dots$ to obtain

$$\begin{aligned}
&\sum_{k=l+1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\qquad \le C \sum_{k=l+1}^{\infty} 2^{-(k+k_0)\alpha} \sup_{(t_0,x_0)\in Q^l} \sum_{j=0}^{k} 2^{(k-j)\sigma} [u]^x_{\alpha;(t_0-2^{-k\sigma},t_0)\times B_{2^{j-k}}(x_0)} \\
&\qquad\quad + C 2^{-(l+k_0)\alpha}[u]^x_\alpha + C 2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_f(2^{-k}) \\
&\qquad\quad + C 2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_a(2^{-k}) \biggl(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \biggr).
\end{aligned}$$

As before, by switching the order of summations and then replacing k by $k+j$, the first term on the right-hand side is bounded by

$$C 2^{-k_0\alpha} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}.$$

With the above inequality, we have

$$\begin{aligned}
&\sum_{k=l+1}^{\infty} 2^{(k+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-k-k_0}}(t_0,x_0)} \\
&\quad \le C 2^{-k_0\alpha} \sum_{k=0}^{\infty} 2^{k(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} + C 2^{-(l+k_0)\alpha}[u]^x_\alpha + C 2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_f(2^{-k}) \\
&\qquad + C 2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_a(2^{-k}) \biggl(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \biggr).
\end{aligned}$$

The bound above, together with the obvious inequality

$$\sum_{j=0}^{l+k_0} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} \le C 2^{(l+k_0)(\sigma-\alpha)}[u]^x_\alpha,$$

implies

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\qquad \le C 2^{-k_0\alpha} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} \\
&\qquad\quad + C 2^{-(l+k_0)\alpha}[u]^x_\alpha + C 2^{(l+k_0)(\sigma-\alpha)}[u]^x_\alpha + C 2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_f(2^{-k}) \\
&\qquad\quad + C 2^{k_0(\sigma-\alpha)} \sum_{k=l+1}^{\infty} \omega_a(2^{-k}) \biggl(\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + \|u\|_{L_\infty} \biggr).
\end{aligned}$$

By first choosing k_0 sufficiently large and then ℓ_0 sufficiently large (recalling $l \geq \ell_0$), we get

$$\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} \leq \frac{1}{4}\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} + C2^{(l+k_0)(\sigma-\alpha)}\|u\|^x_\alpha + C\sum_{k=1}^{\infty}\omega_f(2^{-k}).$$

This implies

$$\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} [u]^x_{\alpha;Q_{2^{-j}}(t_0,x_0)} \leq C\|u\|^x_\alpha + C\sum_{k=1}^{\infty}\omega_f(2^{-k}),$$

which together with Lemma 2.1(i) gives (5-1).

Case 2: $\sigma \in (1,2)$. For $k \in \mathbb{N}$, let v_M be the solution of

$$\begin{cases} \partial_t v_M = L(t,0)v_M + f(t,0) + b(t,0)Du(t,0) - \partial_t p_0 & \text{in } Q_{2^{-k}}, \\ v_M = g_M & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t=-2^{-k\sigma}\}\times B_{2^{-k}}), \end{cases}$$

where $M \geq 2\|u-p_0\|_{L_\infty(Q_{2^{-k}})}$ is a constant to be specified later,

$$g_M = \max(\min(u-p_0, M), -M),$$

and $p_0 = p_0(t,x)$ is the first-order Taylor's expansion of $u^{(2^{-k})}$ in x at $(t,0)$, and $u^{(2^{-k})}$ is the mollification of u in the x-variable only:

$$u^{(R)}(t,x) = \int_{\mathbb{R}^d} u(t,x-Ry)\zeta(y)\,dy$$

with $\zeta \in C_0^\infty(B_1)$ being a radial nonnegative function with unit integral.

By Proposition 5.1, we have

$$[v_M]^x_{\alpha+\sigma;Q_{2^{-k-1}}} \leq C\sum_{j=1}^{\infty} 2^{(k-j)\sigma}M_j + C2^{k\sigma}[v_M]^x_{\alpha;Q_{2^{-k}}},$$

where $\alpha \in (0, \min\{2-\sigma, (\sigma-1)/2\})$ and

$$M_j = \sup_{\substack{(t,x),(t,x')\in(-2^{-k\sigma},0)\times B_{2^{j-k}} \\ 0<|x-x'|<2^{-k+1}}} \frac{|u(t,x)-p_0(t,x)-u(t,x')+p_0(t,x')|}{|x-x'|^\alpha}.$$

From Lemma 2.4 with $\sigma \in (1,2)$, it follows that for $j > k$, we have

$$M_j \leq C[u]^x_{\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d},$$

and thus,

$$\begin{aligned} [v_M]^x_{\alpha+\sigma;Q_{2^{-k-1}}} &\leq C\sum_{j=1}^{\infty} 2^{(k-j)\sigma}M_j + C2^{k\sigma}[v_M]^x_{\alpha;Q_{2^{-k}}} \\ &\leq C\sum_{j=1}^{k} 2^{(k-j)\sigma}M_j + C[u]^x_{\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{k\sigma}[v_M]^x_{\alpha;Q_{2^{-k}}}. \end{aligned} \tag{5-9}$$

From (5-9), and the mean value formula (recalling that $\alpha < 2-\sigma$),

$$\|v_M - p_1\|_{L_\infty(Q_{2^{-k-k_0}})} \le C2^{-(k+k_0)(\sigma+\alpha)} \sum_{j=1}^{k} 2^{(k-j)\sigma} M_j + C2^{-(k+k_0)(\sigma+\alpha)}[u]^x_{\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{-k\alpha-k_0(\sigma+\alpha)}[v_M]^x_{\alpha;Q_{2^{-k}}},$$

where p_1 is the first-order Taylor's expansion of v_M in x at $(t,0)$. The above inequality, (5-9), and the interpolation inequality imply

$$[v_M - p_1]^x_{\alpha;Q_{2^{-k-k_0}}} \le C2^{-(k+k_0)\sigma} \sum_{j=1}^{k} 2^{(k-j)\sigma} M_j + C2^{-(k+k_0)\sigma}[u]^x_{\alpha;(-2^{-k\sigma},0)\times\mathbb{R}^d} + C2^{-k_0\sigma}[v_M]^x_{\alpha;Q_{2^{-k}}}. \tag{5-10}$$

Next $w_M := g_M - v_M$, which equals $u - p_0 - v_M$ in $Q_{2^{-k}}$, satisfies

$$\begin{cases} \partial_t w_M = L(t,0)w_M + h_M + C_k & \text{in } Q_{2^{-k}}, \\ w_M = 0 & \text{in } ((-2^{-k\sigma},0)\times B^c_{2^{-k}}) \cup (\{t=-2^{-k\sigma}\}\times B_{2^{-k}}), \end{cases}$$

where

$$h_M := L(t,0)(u - p_0 - g_M)$$

and

$$C_k = f - f(t,0) + bDu - b(t,0)Du(t,0) + (L - L(t,0))u.$$

It follows easily that

$$\begin{aligned} |C_k| \le{} & \omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} + \|b\|_{L_\infty} 2^{-k\alpha}[Du]^x_{\alpha;Q_{2^{-k}}} \\ & + C\omega_a(2^{-k}) \sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{t\in(t_0-2^{-j\sigma},t_0)} [u(t,\cdot) - p_{t,x_0}]^x_{\alpha;B_{2^{-j}}(x_0)} \\ & + C\omega_a(2^{-k})(\|Du\|_{L_\infty(Q_{2^{-k}})} + \|u\|_{L_\infty}), \end{aligned}$$

where $p_{t,x_0} = p_{t,x_0}(x)$ is the first-order Taylor's expansion of u with respect to x at (t,x_0). From Lemma 2.3, we obtain

$$\begin{aligned} |C_k| \le{} & \omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} + \|b\|_{L_\infty} 2^{-k\alpha}[Du]^x_{\alpha;Q_{2^{-k}}} \\ & + C\omega_a(2^{-k}) \sup_{(t_0,x_0)\in Q_{2^{-k}}} \sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot) - p]^x_{\alpha;B_{2^{-j}}(x_0)} \\ & + C\omega_a(2^{-k})(\|Du\|_{L_\infty(Q_{2^{-k}})} + \|u\|_{L_\infty}). \end{aligned}$$

By the dominated convergence theorem, it is easy to see that

$$\|h_M\|_{L_\infty(Q_{2^{-k}})} \to 0 \quad \text{as } M \to \infty.$$

Thus, similar to (3-6), choosing M sufficiently large so that

$$\|h_M\|_{L_\infty(Q_{2^{-k}})} \le \tfrac{1}{2}C_k,$$

we have

$$\begin{aligned}
&[w_M]_{\frac{\alpha}{\sigma},\alpha;Q_{2^{-k}}} \\
&\le C2^{-k(\sigma-\alpha)}\Big[\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q_{2^{-k}})}+2^{-k\alpha}[Du]^x_{\alpha;Q_{2^{-k}}} \\
&\qquad +\omega_a(2^{-k})\Big(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{t\in(t_0-2^{-j\sigma},t_0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}+\|u\|_{L_\infty}\Big)\Big].
\end{aligned}\tag{5-11}$$

From Lemma 2.4 (more precisely, its proof)

$$M_j \le C\sup_{t\in(-2^{-k\sigma},0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{j-k}}}.\tag{5-12}$$

From the triangle inequality and Lemma 2.4 with $j=0$,

$$\begin{aligned}
[v_M]^x_{\alpha;Q_{2^{-k}}} &\le [w_M]^x_{\alpha;Q_{2^{-k}}}+[u-p_0]^x_{\alpha;Q_{2^{-k}}} \\
&\le [w_M]^x_{\alpha;Q_{2^{-k}}}+C\sup_{t\in(-2^{-k\sigma},0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-k}}}.
\end{aligned}$$

For $l=1,2,\dots$, let $Q^l=Q_{1-2^{-l}}$. Combining (5-10), (5-11), and (5-12), similar to (5-8), we then get

$$\begin{aligned}
&2^{(k+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\sup_{t\in(t_0-2^{-(k_0+k)\sigma},t_0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-(k_0+k)}}(x_0)} \\
&\quad\le 2^{(k+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^l}[u-p_0-p_1]^x_{\alpha;Q_{2^{-(k_0+k)}}(t_0,x_0)} \\
&\quad\le C2^{-(k+k_0)\alpha}\sup_{(t_0,x_0)\in Q^l}\sum_{j=1}^{k}2^{(k-j)\sigma}\sup_{t\in(t_0-2^{-k\sigma},t_0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{j-k}}(x_0)} \\
&\qquad + C2^{-(k+k_0)\alpha}[u]^x_\alpha+\sup_{(t_0,x_0)\in Q^l}2^{-k\alpha+k_0(\sigma-\alpha)}[Du]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)} \\
&\qquad + C2^{k_0(\sigma-\alpha)}\Big[\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q^{l+1})} \\
&\qquad + \omega_a(2^{-k})\Big(\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\sup_{t\in(t_0-2^{-j\sigma},t_0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}+\|u\|_{L_\infty}\Big)\Big].
\end{aligned}$$

Summing the above inequality in $k=l+1,l+2,\dots$ as before, we obtain

$$\begin{aligned}
&\sum_{k=l+1}^{\infty}2^{(k+k_0)(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^l}\sup_{t\in(t_0-2^{-(k_0+k)\sigma},t_0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-(k_0+k)}}(x_0)} \\
&\le C2^{-k_0\alpha}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\sup_{t\in(t_0-2^{-j\sigma},t_0)}\inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}
\end{aligned}$$

$$+C2^{-(k_0+l)\alpha}[u]^x_\alpha + C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty} 2^{-k\alpha} \sup_{(t_0,x_0)\in Q^l} [Du]^x_{\alpha;Q_{2^{-k}}(t_0,x_0)}$$
$$+C2^{k_0(\sigma-\alpha)}\Bigg[\sum_{k=l+1}^{\infty}\big(\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q^{l+1})}\big)$$
$$+\sum_{k=l+1}^{\infty}\omega_a(2^{-k})\Bigg(\|u\|_{L_\infty}+\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\Bigg)\Bigg], \tag{5-13}$$

and

$$\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}$$
$$\le C2^{-k_0\alpha}\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}$$
$$+C2^{k_0(\sigma-\alpha)}\|u\|_{L_\infty}+C2^{(l+k_0)(\sigma-\alpha)}[u]^x_\alpha+C2^{k_0(\sigma-\alpha)-l\alpha}[Du]^x_{\alpha;Q^{l+1}}$$
$$+C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}\big(\omega_f(2^{-k})+(\omega_b(2^{-k})+\omega_a(2^{-k}))\|Du\|_{L_\infty(Q^{l+1})}\big)$$
$$+C2^{k_0(\sigma-\alpha)}\sum_{k=l+1}^{\infty}\omega_a(2^{-k})\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}.$$

By choosing k_0 and l sufficiently large, and using (2-4) and interpolation inequalities (recalling that $\alpha<(\sigma-1)/2$), we obtain

$$\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}$$
$$\le\frac14\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l+1}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}$$
$$+C2^{(k_0+l)(\sigma-\alpha)}\|u\|^x_\alpha+C\sum_{k=1}^{\infty}\omega_f(2^{-k}).$$

Therefore,

$$\sum_{j=0}^{\infty}2^{j(\sigma-\alpha)}\sup_{(t_0,x_0)\in Q^{l}}\ \sup_{t\in(t_0-2^{-j\sigma},t_0)}\ \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\le C\|u\|^x_\alpha+C\sum_{k=1}^{\infty}\omega_f(2^{-k}), \tag{5-14}$$

which together with Lemma 2.1(ii) (actually the proof of it) gives (5-1).

Case 3: $\sigma=1$. Set

$$B_0(t)=\int_t^0 b(s,0)\,ds$$

and we define

$$\hat{u}(t,x) = u(t, x + B_0(t)), \quad \hat{f}_\beta(t,x) = f_\beta(t, x + B_0(t)), \quad \text{and} \quad \hat{b}(t,x) = b(t, x + B_0(t)).$$

It is easy to see that in Q_δ for some $\delta > 0$,

$$\begin{aligned} \partial_t \hat{u}(t,x) &= (\partial_t u)(t, x + B_0(t)) - b(t,0)\nabla u(t, x + B_0(t)) \\ &= \widehat{L}_\beta \hat{u} + \hat{f}_\beta + (\hat{b} - b(t,0))\nabla \hat{u}, \end{aligned} \tag{5-15}$$

where $\widehat{L}$ is the operator with kernel $a(t, x + B_0(t), y)|y|^{-d-\sigma}$. For $(t,x) \in Q_{2^{-k}}$,

$$|\hat{f}_\beta(t,x) - \hat{f}_\beta(t,0)| \le \omega_f(2^{-k}),$$
$$|\hat{b} - b(t,0)| \le \omega_b((1+N_0)2^{-k}).$$

Furthermore,

$$\|Du\|_{L_\infty} + \|\partial_t u\|_{L_\infty} \le (1+N_0)(\|D\hat{u}\|_{L_\infty} + \|\partial_t \hat{u}\|_{L_\infty}).$$

Therefore, it is sufficient to bound $\hat{u}$. In the rest of the proof, we estimate the solution to (5-15) and abuse the notation to use u instead of $\hat{u}$ for simplicity. By scaling, translation and covering arguments, we also assume u satisfies the equation in Q_2.

The proof is similar to the case $\sigma \in (1,2)$ and we indeed proceed as in the previous case. Take p_0 to be the first-order Taylor's expansion of $u^{(2^{-k})}$ in x at $(t,0)$, where $u^{(2^{-k})}$ is the mollification of u in the x-variable only, as in Case 2. We also assume that the solution v to the equations

$$\begin{cases} \partial_t v = \widehat{L}(t,0)v + \hat{f}(t,0) - \partial_t p_0 & \text{in } Q_{2^{-k}}, \\ v = u - p_0 & \text{in } ((-2^{-k\sigma}, 0) \times B^c_{2^{-k}}) \cup (\{t = -2^{-k\sigma}\} \times B_{2^{-k}}), \end{cases}$$

exists without carrying out another approximation argument. By Proposition 5.1 with $\sigma = 1$ and Lemma 2.4 in [Dong and Zhang 2016a],

$$\begin{aligned} [v]^x_{1+\alpha; Q_{2^{-k-1}}} &\le C \sum_{j=1}^{\infty} 2^{k-j} M_j + C 2^k [v]_{\alpha,\alpha; Q_{2^{-k}}} \\ &\le C \sum_{j=1}^{\infty} 2^{k-j} \sup_{t \in (-2^{-k\sigma}, 0)} \inf_{p \in \mathcal{P}_x} [u(t,\cdot) - p]^x_{\alpha; B_{2^{j-k}}} + C 2^k [v]^x_{\alpha; Q_{2^{-k}}} \\ &\le C \sum_{j=1}^{k} 2^{k-j} \sup_{t \in (-2^{-k\sigma}, 0)} \inf_{p \in \mathcal{P}_x} [u(t,\cdot) - p]^x_{\alpha; B_{2^{j-k}}} + C[u]^x_\alpha + C 2^k [v]^x_{\alpha; Q_{2^{-k}}}. \end{aligned} \tag{5-16}$$

From (5-16) and the interpolation inequality, we obtain

$$\begin{aligned} &[v - p_1]^x_{\alpha; Q_{2^{-k-k_0}}} \\ &\le C 2^{-(k+k_0)} \sum_{j=1}^{k} 2^{k-j} \sup_{t \in (-2^{-k\sigma}, 0)} \inf_{p \in \mathcal{P}_x} [u(t,\cdot) - p]^x_{\alpha; B_{2^{j-k}}} + C 2^{-k_0} [v]^x_{\alpha; Q_{2^{-k}}} + C 2^{-(k+k_0)} [u]^x_\alpha, \end{aligned} \tag{5-17}$$

where p_1 is the first-order Taylor's expansion of v in x at $(t,0)$. Next $w := u - p_0 - v$ satisfies

$$\begin{cases} \partial_t w = \hat{L}(t,0)w + C_k & \text{in } Q_{2^{-k}}, \\ \quad w = 0 & \text{in } ((-2^{-k},0)\times B^c_{2^{-k}}) \cup (\{t=-2^{-k}\}\times B_{2^{-k}}), \end{cases}$$

where by the cancellation property,

$$C_k = \hat{f} - \hat{f}(t,0) + (\hat{b} - b(t,0))\nabla u + (\hat{L} - \hat{L}(t,0))u,$$

so that

$$\begin{aligned}|C_k| \le{}& \omega_f(2^{-k}) + \omega_b((1+N_0)2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} \\ &+ C\omega_a((1+N_0)2^{-k})\Bigg(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)} + \|u\|_{L_\infty}\Bigg).\end{aligned}$$

Clearly, for any $r \ge 0$,

$$\omega_\bullet((1+N_0)r) \le (2+N_0)\omega_\bullet(r).$$

Therefore, similar to (5-11), we have

$$\begin{aligned}&[w]_{\alpha,\alpha;Q_{2^{-k}}} \\ &\le C2^{-k(1-\alpha)}\Bigg[\omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q_{2^{-k}})} \\ &\quad + \omega_a(2^{-k})\Bigg(\sup_{(t_0,x_0)\in Q_{2^{-k}}}\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)} + \|u\|_{L_\infty}\Bigg)\Bigg]. \qquad (5\text{-}18)\end{aligned}$$

From the proof of (2-16) and the triangle inequality,

$$[v]^x_{\alpha;Q_{2^{-k}}} \le [w]^x_{\alpha;Q_{2^{-k}}} + [u-p_0]^x_{\alpha;Q_{2^{-k}}} \le [w]_{\alpha;Q_{2^{-k}}} + C\sup_{t\in(-2^{-k},0)} \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-k}}}.$$

For $l = 1, 2, \ldots$, let $Q^l = Q_{1-2^{-l}}$. Similar to (5-8), by combining (5-17) and (5-18), shifting the coordinates, and using the above inequality, we obtain for $l \ge 1$ and $k \ge l+1$,

$$\begin{aligned}&2^{(k+k_0)(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-(k+k_0)\sigma},t_0)} \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{-k-k_0}}(x_0)} \\ &\le C2^{-(k+k_0)\alpha}\sum_{j=0}^{k} 2^{k-j} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-k\sigma},t_0)} \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha;B_{2^{j-k}}(x_0)} \\ &\quad + C2^{k_0(1-\alpha)}\Bigg[\omega_f(2^{-k}) + \omega_b(2^{-k})\|Du\|_{L_\infty(Q^{l+1})} \\ &\qquad + \omega_a(2^{-k})\Bigg(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x}[u(t,\cdot)-p]^x_{\alpha,B_{2^{-j}}(x_0)} + \|u\|_{L_\infty}\Bigg)\Bigg] \\ &\quad + C2^{-(k+k_0)\alpha}[u]^x_\alpha, \qquad (5\text{-}19)\end{aligned}$$

which by summing in $k=l+1,l+2,\dots$ implies

$$\begin{aligned}
&\sum_{k=l+1}^{\infty} 2^{(k+k_0)(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-(k+k_0)\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-k-k_0}}(x_0)}\\
&\le C2^{-k_0\alpha}\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\\
&\quad +C2^{-(k_0+l)\alpha}[u]^x_\alpha+C2^{k_0(1-\alpha)}\sum_{k=l+1}^{\infty}\omega_f(2^{-k})\\
&\quad +C2^{k_0(1-\alpha)}\sum_{k=l+1}^{\infty}\Big[\omega_b(2^{-k})\|Du\|_{L_\infty(Q^{l+1})}\\
&\qquad\qquad +\omega_a(2^{-k})\Big(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}+\|u\|_{L_\infty}\Big)\Big],
\end{aligned}$$

where for the first term on the right-hand side of (5-19), we replaced j by $k-j$ and switched the order of the summation as before. Therefore,

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\\
&\le C2^{-k_0\alpha}\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\\
&\quad +C2^{(l+k_0)(1-\alpha)}[u]^x_\alpha+C2^{k_0(1-\alpha)}\sum_{k=l+1}^{\infty}\omega_f(2^{-k})\\
&\quad +C2^{k_0(1-\alpha)}\sum_{k=l+1}^{\infty}\omega_b(2^{-k})\|Du\|_{L_\infty(Q^{l+1})}\\
&\quad +C2^{k_0(1-\alpha)}\sum_{k=l+1}^{\infty}\omega_a(2^{-k})\Big(\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}+\|u\|_{L_\infty}\Big).
\end{aligned}$$

Then we choose k_0 and then l sufficiently large, and apply Lemma 2.1(iii) (actually the proof it) to get

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\\
&\quad\le \frac14\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^{l+1}} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)}\\
&\qquad\qquad +C2^{(k_0+l)(1-\alpha)}\|u\|^x_\alpha+C\sum_{k=1}^{\infty}\omega_f(2^{-k}).
\end{aligned}$$

This implies

$$\sum_{j=0}^{\infty} 2^{j(1-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)} \le C\|u\|^x_\alpha + C\sum_{k=1}^{\infty}\omega_f(2^{-k}),$$

from which (5-1) follows. □

Proof of Theorem 1.5. As before, Theorem 1.5 follows from Proposition 5.2 using the argument of freezing the coefficients. We only present the detailed proof of Theorem 1.5 for $\sigma \in (1,2)$. We omit the proof of the case $\sigma \in (0,1]$ since it is almost the same and actually is simpler.

Indeed, the proof here for $\sigma \in (1,2)$ is almost identical to that of Theorem 1.1, so we just sketch it.

Without loss of generality, we assume the equation holds in Q_3.

Step 1: For $k = 1, 2, \dots$, define $Q^k := Q_{1-2^{-k}}$. Let $\eta_k \in C_0^\infty(\hat{Q}^{k+3})$ be a sequence of nonnegative smooth cutoff functions satisfying $\eta \equiv 1$ in Q^{k+2}, $|\eta| \le 1$ in Q^{k+3}, and $\|\partial_t^j D^i \eta_k\|_{L_\infty} \le C2^{k(i+j)}$ for each $i, j \ge 0$. Set $v_k := u\eta_k \in C^{1,\sigma+}$ and notice that in Q^{k+1},

$$\begin{aligned}\partial_t v_k &= \eta_k \partial_t u + \partial_t \eta_k u = \eta_k Lu + \eta_k bDu + \eta_k f + \partial_t \eta_k u\\ &= Lv_k + bDv_k - buD\eta_k + h_k + \eta_k f + \partial_t \eta_k u,\end{aligned}$$

where

$$h_k = \eta_k Lu - Lv_k = \int_{\mathbb{R}^d} \frac{\xi_k(t,x,y)a(t,x,y)}{|y|^{d+\sigma}}\,dy,$$

and

$$\begin{aligned}\xi_k(t,x,y) &= u(t,x+y)(\eta_k(t,x+y)-\eta_k(t,x)) - y\cdot D\eta_k(t,x)u(t,x)\\ &= u(t,x+y)(\eta_k(t,x+y)-\eta_k(t,x)) \quad \text{since } D\eta_k \equiv 0 \text{ in } Q^{k+1}.\end{aligned}$$

We will apply Proposition 5.2 to the equation of v_k in Q^{k+1} and obtain corresponding estimates for v_k in Q^k.

As before, we have $\eta_k f \equiv f$, $\partial_t \eta_k u \equiv 0$, and $buD\eta_k \equiv 0$ in Q^{k+1}. Thus, we only need to estimate the moduli of continuity of h_k in Q^{k+1} with respect to x. The same proof of (3-31) shows that

$$\omega_h(r) := C\left(2^{\sigma k}\|u\|_{L_\infty(Q_3)} + \sum_{j=0}^{\infty} 2^{-j\sigma}\omega_u(2^j)\right)\omega_a(r) + C2^{k\sigma}\omega_u(r) + C2^{k(\sigma+1)}\|u\|_{L_\infty(Q_3)}r. \tag{5-20}$$

As in the proof of Theorem 1.1, by making use of Proposition 5.2 to v_k and interpolation inequalities, an iteration procedure will lead to

$$[u]^x_{\sigma;Q^4} \le C\|u\|_{L_\infty(Q_3)} + C\sum_{j=0}^{\infty}(2^{-j\sigma}\omega_u(2^j) + \omega_u(2^{-j}) + \omega_f(2^{-j})). \tag{5-21}$$

Applying this to the equation of $u(t,x) - u(t,0)$ gives to (1-10).

Finally, since $\|v_1\|^x_\alpha$ is bounded by the right-hand side of (5-21), from (5-14), we see that

$$\sum_{j=0}^{\infty} 2^{j(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^l} \sup_{t\in(t_0-2^{-j\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [u(t,\cdot)-p]^x_{\alpha;B_{2^{-j}}(x_0)} \le C$$

for some large l.

This and (5-13) with u replaced by v_1 and f replaced by $h_1+\eta_1 f+\partial_t\eta_1 u-buD\eta_1$ give

$$\sum_{j=k_1+1}^{\infty} 2^{(j+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{k_1}} \sup_{t\in(t_0-2^{-(j+k_0)\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [v_1-p]^x_{\alpha;B_{2^{-j-k_0}}(x_0)} \le C2^{-k_0\alpha}+C2^{k_0(\sigma-\alpha)}\sum_{j=k_1}^{\infty}(\omega_f(2^{-j})+\omega_a(2^{-j})+\omega_u(2^{-j})+\omega_b(2^{-j})+2^{-j\alpha}).$$

Here we also used (5-20) with $k=1$. Therefore, for any small $\varepsilon>0$, we can find k_0 sufficiently large, then k_1 sufficiently large, depending only on C, σ, N_0, α, ω_a, ω_f, ω_b, and ω_u, such that

$$\sum_{j=k_1+1}^{\infty} 2^{(j+k_0)(\sigma-\alpha)} \sup_{(t_0,x_0)\in Q^{k_1}} \sup_{t\in(t_0-2^{-(j+k_0)\sigma},t_0)} \inf_{p\in\mathcal{P}_x} [v_1-p]^x_{\alpha;B_{2^{-j-k_0}}(x_0)} < \varepsilon.$$

This, together with the fact that $v_1=u$ in $Q_{\frac{1}{2}}$ and the proof of Lemma 2.1(ii), indicates that

$$\sup_{(t_0,x_0)\in Q_{1/2}} [u]^x_{\sigma;Q_r(t_0,x_0)} \to 0 \quad \text{as } r\to 0$$

with a decay rate depending only on d, λ, N_0, Λ, ω_a, ω_f, ω_b, ω_u, and σ. Also, by evaluating (1-7) on both sides and making use of the dominated convergence theorem, we have that $\partial_t u$ is uniformly continuous in x in $Q_{\frac{1}{2}}$ with a modulus of continuity controlled by d, σ, λ, Λ, ω_a, ω_f, ω_u, N_0, ω_b, and $\|u\|_{L_\infty}$.

Hence, the proof of the case when $\sigma\in(1,2)$ is completed. □

Proof of Theorem 1.6. Given the proofs of Theorems 1.3 and 1.4, Theorem 1.6 can be similarly proved (actually simpler), and we omit the details. □

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