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ESTIMATES FOR EIGENVALUES OF AHARONOV–BOHM OPERATORS WITH VARYING POLES AND NON-HALF-INTEGER CIRCULATION

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We study the behavior of eigenvalues of a magnetic Aharonov–Bohm operator with non-half-integer circulation and Dirichlet boundary conditions in a planar domain. As the pole is moving in the interior of the domain, we estimate the rate of the eigenvalue variation in terms of the vanishing order of the limit eigenfunction at the limit pole. We also provide an accurate blow-up analysis for scaled eigenfunctions and prove a sharp estimate for their rate of convergence.

1. Introduction and statement of the main results

An infinitely long, thin solenoid perpendicular to the plane (x_1, x_2) at the point $a = (a_1, a_2) \in \mathbb{R}^2$ produces a point-like magnetic field as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. This magnetic field is a $2\pi\alpha$-multiple of the Dirac delta at a orthogonal to the plane (x_1, x_2) and is generated by the Aharonov–Bohm vector potential

$$A_a(x) = \alpha\left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\};$$

see, e.g., [Adami and Teta 1998; Aharonov and Bohm 1959; Melgaard et al. 2004]. We are interested in the spectral properties of Schrödinger operators with Aharonov–Bohm vector potentials, i.e., of operators

$$(i\nabla + A_a)^2 := -\Delta + 2iA_a \cdot \nabla + |A_a|^2.$$

Since $\operatorname{curl} A_a \equiv 0$ in $\mathbb{R}^2 \setminus \{a\}$, the magnetic field is concentrated at the pole a. If the circulation α is an integer number, then the potential A_a can be gauged away by a phase transformation so that the operator $(i\nabla + A_a)^2$ becomes spectrally equivalent to the standard Laplacian. On the other hand, if $\alpha \notin \mathbb{Z}$, the vector potential A_a cannot be eliminated by gauge transformations and the spectrum of the operator is modified by the presence of the magnetic field: this produces the so-called Aharonov–Bohm effect; i.e., the magnetic potential affects charged quantum particles moving in the region $\Omega \setminus \{a\}$, even if the magnetic field $B_a = \operatorname{curl} A_a$ is zero there.

The dependence on the pole a of the spectrum of the Schrödinger operator $(i\nabla + A_a)^2$ in a bounded domain Ω was investigated in [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017; Bonnaillie-Noël et al. 2014; Noris et al. 2015; Noris and Terracini 2010] under homogeneous Dirichlet boundary conditions. In particular, in [Abatangelo and Felli 2015; 2016] sharp asymptotic estimates for eigenvalues were

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given in the case of half-integer circulation $\alpha \in \mathbb{Z} + \frac{1}{2}$ as the pole a moves towards a fixed point $\bar{a} \in \Omega$; analogous sharp estimates were derived in [Abatangelo et al. 2017] in the case $\bar{a} \in \partial\Omega$. We mention that the continuous dependence of the eigenvalue function on the position of the pole and improved regularity results under simplicity assumptions were established in [Bonnaillie-Noël et al. 2014; Léna 2015] for any value of α (in particular also for non-half-integer circulation); on the other hand, to the best of our knowledge, sharp estimates of the gap of eigenvalues have been investigated only in the case $\alpha \in \mathbb{Z} + \frac{1}{2}$; see [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017; Bonnaillie-Noël et al. 2014; Noris et al. 2015].

The case $\alpha \in \mathbb{Z} + \frac{1}{2}$ studied in the aforementioned papers presents several peculiarities which allow one to approach the problem with a perspective and a technique that are not completely adaptable to a general circulation $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Indeed, if $\alpha \in \mathbb{Z} + \frac{1}{2}$, the problem can be reduced by a gauge transformation to the case $\alpha = \frac{1}{2}$ and, in this case, the eigenfunctions of $(i\nabla + A_a)^2$ can be identified, up to a complex phase, with the antisymmetric eigenfunctions of the Laplace–Beltrami operator on the twofold covering manifold of Ω; see [Helffer et al. 1999; Noris and Terracini 2010]. As a consequence, if $\alpha = \frac{1}{2}$, the magnetic eigenfunctions have an odd number of nodal lines ending at the pole a. It has been proved in [Helffer and Hoffmann-Ostenhof 2013] that the corresponding nodal domains are related to optimal partition problems. We refer to [Bonnaillie-Noël et al. 2009] for related numerical simulations.

The special features characterizing Aharonov–Bohm operators with circulation $\frac{1}{2}$ played a crucial role in [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017; Bonnaillie-Noël et al. 2014; Noris et al. 2015; Noris and Terracini 2010]. In particular, in [Noris et al. 2015] local energy estimates for eigenfunctions near the limit pole are performed by studying an Almgren-type quotient, see [Almgren 1983], which is estimated using a representation formula by Green's functions for solutions to the corresponding Laplace problem on the twofold covering. Moreover, in [Abatangelo and Felli 2015; 2016; Abatangelo et al. 2017] a limit profile vanishing on the special directions determined by the nodal lines of limit eigenfunctions is constructed: this allows one to establish a sharp relation between the asymptotics of the eigenvalue function and the number of nodal lines, which is strongly related to the order of vanishing of the limit eigenfunction.

In this paper we will focus on the case of noninteger and non-half-integer circulation; i.e., we will assume $\alpha \in \mathbb{R} \setminus (\mathbb{Z}/2)$. A reduction to the Laplacian on the twofold covering manifold is no longer available in this case; moreover, magnetic eigenfunctions vanish at the pole a but they do not have nodal lines ending at a (see Proposition 2.1). The lack of the special features of Aharonov–Bohm operators with half-integer circulation described above requires alternative methods and produces a less precise estimate. In particular, in order to estimate the Almgren frequency function, we will give a detailed description of the behavior of eigenfunctions at the pole and we will study the dependence of the coefficients of their asymptotic expansion with respect to the moving pole a, see Lemma 2.2.

By gauge invariance, if $\alpha \in \mathbb{R} \setminus (\mathbb{Z}/2)$ it is not restrictive to assume that

$$\alpha \in (0,1) \setminus \{\tfrac{1}{2}\}. \tag{1-1}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we introduce the functional space $H^{1,a}(\Omega, \mathbb{C})$ as the completion of

$$\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\|\frac{u}{|x-a|}\right\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}. \tag{1-2}$$

The norm (1-2) is equivalent, under condition (1-1), to the norm

$$\big(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})}\big)^{1/2},$$

in view of the Hardy-type inequality proved in [Laptev and Weidl 1999], see also [Alziary et al. 2003] and [Felli et al. 2011, Lemma 3.1 and Remark 3.2],

$$\int_{D_r(a)} |(i\nabla + A_a)u|^2\,dx \geq \big(\min_{j\in\mathbb{Z}} |j-\alpha|\big)^2 \int_{D_r(a)} \frac{|u(x)|^2}{|x-a|^2}\,dx, \tag{1-3}$$

which holds for all $r > 0,\ a \in \mathbb{R}^2$ and $u \in H^{1,a}(D_r(a),\mathbb{C})$. Here we denote by $D_r(a)$ the disk of center a and radius r; we will denote by $D_r := D_r(0)$ the disk with radius r centered at the origin.

It is also worth mentioning the following formulation of the magnetic Hardy inequality proved in [Alziary et al. 2003, Lemma 4.1]: for all $r_1 > r_2 > 0,\ a \in \mathbb{R}^2$, and $u \in H^{1,a}(D_{r_1}(a) \setminus D_{r_2}(a),\mathbb{C})$,

$$\int_{D_{r_1}(a)\setminus D_{r_2}(a)} |(i\nabla + A_a)u|^2\,dx \geq \big(\min_{j\in\mathbb{Z}} |j-\alpha|\big)^2 \int_{D_{r_1}(a)\setminus D_{r_2}(a)} \frac{|u(x)|^2}{|x-a|^2}\,dx. \tag{1-4}$$

We also consider the space $H^{1,a}_0(\Omega,\mathbb{C})$ as the completion of $C^\infty_c(\Omega\setminus\{a\},\mathbb{C})$ with respect to the norm $\|\cdot\|_{H^1_a(\Omega,\mathbb{C})}$, so that

$$H^{1,a}_0(\Omega,\mathbb{C}) = \left\{u \in H^1_0(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\right\}.$$

From classical spectral theory, for every $a \in \Omega$, the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{E_a}$$

admits a diverging sequence of real eigenvalues $\{\lambda^a_k\}_{k\geq 1}$ with finite multiplicity; in the enumeration

$$\lambda^a_1 \leq \lambda^a_2 \leq \cdots \leq \lambda^a_j \leq \cdots,$$

we repeat each eigenvalue as many times as its multiplicity. We are interested in the behavior of the function $a \mapsto \lambda^a_j$ in a neighborhood of a fixed point $\bar{a} \in \Omega$. Up to a translation and a dilation, it is not restrictive to assume that $\bar{a} = 0 \in \Omega$ and $\overline{D}_2 \subset \Omega$.

Let us assume that there exists $n_0 \geq 1$ such that

$$\lambda^0_{n_0} \text{ is simple,} \tag{1-5}$$

and define

$$\lambda_0 = \lambda^0_{n_0} \quad \text{and} \quad \lambda_a = \lambda^a_{n_0}$$

for any $a \in \Omega$. In [Léna 2015, Theorem 1.3] it is proved that

$$\text{if } \lambda_j^0 \text{ is simple, the function } a \mapsto \lambda_j^a \text{ is } \textit{analytic} \text{ in a neighborhood of } 0. \tag{1-6}$$

In particular the function $a \mapsto \lambda_a$ is continuous and, if $a \to 0$, then $\lambda_a \to \lambda_0$; see also [Bonnaillie-Noël et al. 2014]. Let $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C}) \setminus \{0\}$ be an $L^2(\Omega, \mathbb{C})$-normalized eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda_{n_0}^0$, i.e., satisfying

$$\begin{cases} (i\nabla + A_0)^2 \varphi_0 = \lambda_0 \varphi_0 & \text{in } \Omega, \\ \varphi_0 = 0 & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_0(x)|^2 \, dx = 1. \end{cases} \tag{1-7}$$

From [Felli et al. 2011, Theorem 1.3] (see also Proposition 2.1) it is known that

$$\varphi_0 \text{ vanishes at } 0 \text{ with a vanishing order equal to } |\alpha - k| \text{ for some } k \in \mathbb{Z}, \tag{1-8}$$

in the sense that there exist $k \in \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \{0\}$ such that

$$r^{-|\alpha-k|} \varphi_0(r(\cos t, \sin t)) \to \beta \frac{e^{ikt}}{\sqrt{2\pi}} \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \tag{1-9}$$

as $r \to 0^+$ for any $\tau \in (0, 1)$.

Our first result provides an estimate of the rate of convergence of $\lambda_0 - \lambda_a$ in terms of the order of vanishing of φ_0 at 0; in particular we have that higher vanishing orders imply faster convergence of eigenvalues.

Theorem 1.1. *Let $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$ and $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain such that $0 \in \Omega$. Let $n_0 \in \mathbb{N}$ be such that the n_0-th eigenvalue $\lambda_{n_0}^0 = \lambda_0$ of problem (E_0) is simple and let $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ be an associated eigenfunction satisfying* (1-7). *Let $k \in \mathbb{Z}$ be such that $|\alpha - k|$ is the order of vanishing of φ_0 at 0 as in* (1-9). *For $a \in \Omega$, let $\lambda_{n_0}^a = \lambda_a$ be the n_0-th eigenvalue of problem (E_a). Then*

$$|\lambda_a - \lambda_0| = O(|a|^{1 + \lfloor 2|\alpha-k| \rfloor}) \quad \text{as } |a| \to 0,$$

where $\lfloor \cdot \rfloor$ denotes the floor function $\lfloor t \rfloor := \max\{k \in \mathbb{Z} : k \le t\}$.

To prove Theorem 1.1, we will study the quotient

$$\frac{\lambda_0 - \lambda_a}{|a|^{2|\alpha-k|}} \tag{1-10}$$

as a approaches the origin along a straight line $\{tp : t > 0\}$ for any direction

$$p \in \mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| = 1\}.$$

We will prove that, for every $p \in \mathbb{S}^1$, the quotient (1-10) is bounded as $a = |a|p \to 0$. Then (1-6) and the fact that $2|\alpha - k|$ is noninteger imply that the Taylor polynomials of the function $\lambda_0 - \lambda_a$ with center 0 and degree less than or equal to $\lfloor 2|\alpha - k| \rfloor$ vanish, thus yielding the conclusion of Theorem 1.1.

In the case of half-integer circulation $\alpha = \frac{1}{2}$ the special nodal structure of the limit problem allows us to prove instead that the limit

$$\lim_{a=|a|p\to 0} \frac{\lambda_0-\lambda_a}{|a|^{2|\alpha-k|}} = \lim_{a=|a|p\to 0} \frac{\lambda_0-\lambda_a}{|a|^{|1-2k|}}$$

is different from 0 along some special directions p corresponding to tangents to the nodal lines of the limit eigenfunction. As a consequence, the leading term of the Taylor expansion of the eigenvalue variation $\lambda_0 - \lambda_a$ has order exactly $|1-2k|$. That is,

$$\lambda_0 - \lambda_a = P(a) + o(|a|^{|1-2k|}) \quad \text{as } |a| \to 0^+$$

for some homogeneous polynomial $P \not\equiv 0$ of degree $|1-2k|$; see [Abatangelo and Felli 2015, Theorem 1.2]. In [Abatangelo and Felli 2016, Theorem 2] the exact values of all coefficients of the polynomial P are determined, proving that $P(|a|(\cos t, \sin t)) = C_0|a|^{|1-2k|}\cos(|1-2k|(t-t_0))$ for some t_0 and $C_0 > 0$. In particular the leading polynomial P is harmonic.

In this paper we will also describe the behavior of the eigenfunctions as $a \to 0$, proving a blow-up result for scaled eigenfunctions and giving a sharp rate of the convergence to the limit eigenfunction φ_0. In order to state these results more precisely, we need to introduce some notation.

For every $b = (b_1, b_2) = |b|(\cos\vartheta, \sin\vartheta) \in \mathbb{R}^2 \setminus \{0\}$ with $\vartheta \in [0, 2\pi)$, we define the polar angle centered at b, $\theta_b : \mathbb{R}^2 \setminus \{b\} \to [\vartheta, \vartheta + 2\pi)$ as

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{for all } r > 0 \text{ and } t \in [\vartheta, \vartheta + 2\pi), \tag{1-11}$$

and the function $\theta_0^b : \mathbb{R}^2 \setminus \{0\} \to [\vartheta, \vartheta + 2\pi)$ as

$$\theta_0^b(r(\cos t, \sin t)) = t \quad \text{for all } r > 0 \text{ and } t \in [\vartheta, \vartheta + 2\pi). \tag{1-12}$$

We remark that θ_b is discontinuous on the half-line starting at b with slope $\vartheta = \mathrm{Arg}(b)$, whereas θ_0^b is discontinuous on the half-line starting from 0 with the same slope; in particular, the range of θ_0^b depends on $\vartheta = \mathrm{Arg}(b)$. Hence, the difference function $\theta_0^b - \theta_b$ is regular except for the segment

$$\Gamma_b := \{tb : t \in [0, 1]\}. \tag{1-13}$$

For all $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a, i.e., solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_a = \lambda_a \varphi_a & \text{in } \Omega, \\ \varphi_a = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-14}$$

such that the following normalization conditions hold:

$$\int_\Omega |\varphi_a(x)|^2\,dx = 1 \quad \text{and} \quad \int_\Omega e^{i\alpha(\theta_0^a - \theta_a)(x)} \varphi_a(x)\overline{\varphi_0(x)}\,dx \text{ is a positive real number.} \tag{1-15}$$

Using (1-5), (1-7), (1-14), (1-15), and standard elliptic estimates, see, e.g., [Gilbarg and Trudinger 1983, Theorem 8.10], it is easy to prove that

$$\varphi_a \to \varphi_0 \quad \text{in } H^1(\Omega, \mathbb{C}) \text{ and in } C^2_{\mathrm{loc}}(\Omega \setminus \{0\}, \mathbb{C}), \tag{1-16}$$

and

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega, \mathbb{C}). \tag{1-17}$$

To give a precise description of the behavior of the eigenfunction φ_a for a close to 0, we consider a homogeneous scaling of order $|a|^{|\alpha-k|}$ of φ_a along a fixed direction $p \in \mathbb{S}^1$. Theorem 1.2 below gives the convergence of scaled eigenfunctions to a nontrivial limit profile $\Psi_p \in H^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$, which can be characterized as the unique solution to the problem

$$(i\nabla + A_p)^2\Psi_p = 0 \quad \text{in } \mathbb{R}^2 \text{ in a weak } H^{1,p}\text{-sense}, \tag{1-18}$$

satisfying

$$\int_{\mathbb{R}^2\setminus D_1} |(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx < +\infty, \tag{1-19}$$

where $\psi_k : \mathbb{R}^2 \to \mathbb{C}$ is defined as

$$\psi_k(r(\cos t, \sin t)) = r^{|\alpha-k|}\frac{e^{ikt}}{\sqrt{2\pi}}. \tag{1-20}$$

The existence and uniqueness of a limit profile satisfying (1-18) and (1-19) will be proved in Lemma 5.3. We notice that the function ψ_k in (1-20) is the unique (up to a multiplicative constant) $H^{1,0}_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$-solution to $(i\nabla + A_0)^2\psi_k = 0$ in $\mathbb{R}^2$ which is homogeneous of degree $|\alpha - k|$.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, for $p \in \mathbb{S}^1$ and $a = |a|p \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a and satisfying* (1-15). *Let moreover,*

$$\hat{\varphi}_a(x) = \frac{\varphi_a(|a|x)}{|a|^{|\alpha-k|}}.$$

Then

$$\hat{\varphi}_a \to \beta\Psi_p \quad \text{as } |a| \to 0$$

in $H^{1,p}(D_R, \mathbb{C})$ for every $R > 1$, almost everywhere in $\mathbb{R}^2$ and in $C^2_{\text{loc}}(\mathbb{R}^2 \setminus \{p\}, \mathbb{C})$, with $\beta \neq 0$ and $k \in \mathbb{Z}$ being as in (1-9) *and Ψ_p being as in* (1-18)–(1-19).

Finally, we describe the sharp rate of convergence (1-17), which also turns out to depend strongly on the order of vanishing of φ_0 at 0, as stated in the following theorem.

Theorem 1.3. *Under the same assumptions as in Theorems 1.1 and 1.2, for every $p \in \mathbb{S}^1$ there exists $\mathfrak{L}_p > 0$ such that*

$$|a|^{-2|\alpha-k|}\big\|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\big\|^2_{L^2(\Omega,\mathbb{C})} \to |\beta|^2\,\mathfrak{L}_p \quad \text{as } a = |a|p \to 0.$$

We observe that Theorem 1.3 extends to the case of non-half-integer circulation an analogous result obtained in [Abatangelo and Felli 2017] for half-integer circulation.

The main tools in the proof of the above-described results are energy estimates on eigenfunctions obtained by an Almgren-type monotonicity argument and blow-up analysis for scaled eigenfunctions; such a strategy was first developed in [Abatangelo and Felli 2015; Noris et al. 2015] in the half-integer case and is essentially based on the description of the behavior of limit eigenfunctions at the pole through the limit of the Almgren quotient, which is possible in both the cases of half-integer and non-half-integer circulation. On the other hand, in the implementation of this procedure for the non-half-integer case, two main points present deep differences from that of the half-integer case. First of all a reduction to the Laplacian on the twofold covering manifold is no longer available and hence a new strategy has to be developed to prove monotonicity-type formulas: this is the main goal of Section 2, where we derive precise estimates for eigenfunctions on small circles which are needed to prove Lemma 3.1 (whose analogue in the half-integer case can be directly proved using the reduction to the Laplacian on the twofold covering manifold). The second crucial difference arises in the blow-up analysis, more precisely in the construction of the limit profile, which cannot be as explicit as in the half-integer case. This exploits vanishing on the special directions of nodal lines of limit eigenfunctions. In the non-half-integer case, a nontrivial limit profile still exists (see Lemma 5.3) but its description is quite implicit: this is also the reason why the estimate we obtain here in the non-half-integer case is less precise than the estimates of [Abatangelo and Felli 2015; 2016] for half-integer α.

The paper is organized as follows. In Section 2 we perform a detailed description of the behavior of the eigenfunction φ_a near the pole a, which is crucial in Section 3 to prove an Almgren-type monotonicity formula and to derive local energy estimates for eigenfunctions uniformly with respect to the moving pole. In Section 4 we obtain some upper and lower bounds for the difference $\lambda_0 - \lambda_a$ by exploiting the Courant–Fischer minimax characterization of eigenvalues and testing the Rayleigh quotient with suitable competitor functions. Section 5 contains a blow-up analysis for scaled eigenfunctions, which allows us to prove Theorems 1.1 and 1.2. Finally, in Section 6 we prove Theorem 1.3.

Notation. We list below some notation used throughout the paper:

- For all $r > 0$ and $a \in \mathbb{R}^2$, we denote by $D_r(a) = \{x \in \mathbb{R}^2 : |x - a| < r\}$ the disk of center a and radius r.
- For all $r > 0$, we let $D_r = D_r(0)$ and $\mathbb{S}^1 = \partial D_1$.
- ds denotes the arc length on $\partial D_r(a)$.
- For every complex number $z \in \mathbb{C}$, we denote by $\bar{z}$ its complex conjugate.
- For $z \in \mathbb{C}$, we denote its real part by $\mathfrak{Re}\, z$ and its imaginary part by $\mathfrak{Im}\, z$.

2. Local asymptotics of eigenfunctions

The aim of this section is to describe the local asymptotics of eigenfunctions, showing how the coefficients of expansions depend on the pole. This goal is achieved by expanding the angular part of eigenfunctions in Fourier series with respect to the orthonormal basis of $L^2((0, 2\pi), \mathbb{C})$ given by $\{e^{ijt}/\sqrt{2\pi}\}_{j \in \mathbb{Z}}$, see (2-11), and then by estimating the Fourier coefficients (2-13) by means of Gronwall-type lemmas. These estimates will be crucial to developing the monotonicity argument of Section 3, in particular to proving Lemma 3.1 (whose analogue in the half-integer case is obtained in [Noris et al. 2015, Lemma 5.8] with techniques which are not adaptable to the non-half-integer case).

We recall from [Felli et al. 2011] the description of the asymptotics at the singularity of solutions to elliptic equations with Aharonov–Bohm potentials. In the case of Aharonov–Bohm potentials with circulation $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$, such asymptotics is described in terms of eigenvalues and eigenfunctions of the following operator $\mathcal{H}$ acting on 2π-periodic functions

$$\mathcal{H}\psi = -\psi'' + 2i\alpha\psi' + \alpha^2\psi.$$

It is easy to verify that the eigenvalues of $\mathcal{H}$ are $\{(\alpha - j)^2 : j \in \mathbb{Z}\}$; each eigenvalue $(\alpha - j)^2$ has multiplicity 1 and the eigenspace associated is generated by the function $e^{ijt}/\sqrt{2\pi}$. Let us enumerate the eigenvalues $(\alpha - j)^2$ as $\{(\alpha - j)^2 : j \in \mathbb{Z}\} = \{\mu_j : j = 1, 2, \dots\}$ with $\mu_1 < \mu_2 < \mu_3 < \cdots$, so that

$$\mu_1 = \min\{\alpha^2, (1-\alpha)^2\} \tag{2-1}$$

and $\mu_2 = \max\{\alpha^2, (1-\alpha)^2\}$.

Proposition 2.1 [Felli et al. 2011, Theorem 1.3]. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set containing b, $\lambda \in \mathbb{R}$, and $u \in H^{1,b}_0(\Omega, \mathbb{C})$ be a nontrivial weak solution to the problem*

$$(i\nabla + A_b)^2 u = \lambda u \quad \text{in } \Omega;$$

i.e.,

$$\int_\Omega (i\nabla + A_b)u \cdot \overline{(i\nabla + A_b)v}\, dx = \lambda \int_\Omega u\bar{v}\, dx \quad \text{for all } v \in H^{1,b}_0(\Omega, \mathbb{C}).$$

Then there exists $j \in \mathbb{Z}$ such that

$$\lim_{r\to 0^+} \frac{r\int_{D_r(b)} \big(|(i\nabla + A_b)u(x)|^2 - \lambda|u(x)|^2\big)\, dx}{\int_{\partial D_r(b)} |u|^2\, ds} = |\alpha - j|. \tag{2-2}$$

Furthermore, there exists $\beta(b, u, \lambda) \neq 0$ such that

$$r^{-|\alpha - j|} u(b + r(\cos t, \sin t)) \to \beta(b, u, \lambda) \frac{e^{ijt}}{\sqrt{2\pi}} \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \tag{2-3}$$

as $r \to 0^+$ for any $\tau \in (0,1)$.

Let us fix $n \in \mathbb{N}$, $n \geq 1$. For all $a \in \Omega$, let $\varphi_n^a \in H^{1,a}_0(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_n^a, i.e., solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_n^a = \lambda_n^a \varphi_n^a & \text{in } \Omega, \\ \varphi_n^a = 0 & \text{on } \partial\Omega, \end{cases} \tag{2-4}$$

such that

$$\int_\Omega |\varphi_n^a(x)|^2\, dx = 1. \tag{2-5}$$

Since $a \in \Omega \mapsto \lambda_n^a$ admits a continuous extension on $\overline{\Omega}$ as proved in [Bonnaillie-Noël et al. 2014, Theorem 1.1], we have

$$\Lambda_n = \sup_{a\in\Omega} \lambda_n^a \in (0, +\infty). \tag{2-6}$$

Moreover, from (2-4), (2-5), and (1-3) it follows that

$$\{\varphi_n^a\}_{a\in\Omega} \text{ is bounded in } H^1(\Omega,\mathbb{C}), \tag{2-7}$$

which, by (2-4) and classical elliptic regularity theory, implies that, for each $\omega \Subset \Omega \setminus \{0\}$, there exists $\rho_\omega > 0$ such that

$$\{\varphi_n^a\}_{|a|\le\rho_\omega} \text{ is bounded in } C^{2,\sigma}(\omega,\mathbb{C}) \text{ for every } \sigma \in (0,1). \tag{2-8}$$

The following lemma provides a detailed description of the behavior of the Fourier coefficients of the function $t \mapsto \varphi_n^a(a + r(\cos t, \sin t))$ as a is close to 0.

Lemma 2.2. *For $n \ge 1$ fixed and a varying in Ω, let $\varphi_n^a \in H_0^{1,a}(\Omega,\mathbb{C}) \setminus \{0\}$ satisfy* (2-4) *and* (2-5). *For all $j \in \mathbb{Z}$ and $a \in \Omega$, let*

$$v_j^a(r) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \varphi_n^a(a + r(\cos t, \sin t)) e^{-ijt}\, dt. \tag{2-9}$$

Then there exists $\rho_0 > 0$ such that, for all a with $|a| \le \rho_0$, the following properties hold:

(i) *For all $j \in \mathbb{Z}$, we have $v_j^a(r) = O(r^{|\alpha-j|})$ as $r \to 0^+$. In particular, for all $j \in \mathbb{Z}$ and for all $R > 0$ such that $\{x \in \mathbb{R}^2 : |x-a| \le R\} \subset \Omega$, the value*

$$\beta_j^a = \frac{v_j^a(R)}{R^{|\alpha-j|}} + \frac{\lambda_n^a}{2|\alpha-j|} \int_0^R \left(s^{1-|\alpha-j|} - \frac{s^{1+|\alpha-j|}}{R^{2|\alpha-j|}} \right) v_j^a(s)\, ds \tag{2-10}$$

is well-defined and independent of R.

(ii) *For all $j \in \mathbb{Z}$, we have $|\beta_j^a| \le B$ for some $B > 0$ independent of j and a.*

(iii) *For all $j \in \mathbb{Z}$,*

$$v_j^a(r) = r^{|\alpha-j|}\beta_j^a(1 + R_{j,a}(r)) \quad \text{and} \quad (v_j^a)'(r) = |\alpha-j|\beta_j^a r^{|\alpha-j|-1}(1 + \widetilde{R}_{j,a}(r)),$$

where $|R_{j,a}(r)| + |\widetilde{R}_{j,a}(r)| \le \text{const}\, r^2$ for some const > 0 independent of j and a.

(iv) *φ_n^a can be expanded as*

$$\varphi_n^a(a + r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} r^{|\alpha-j|}\beta_j^a(1 + R_{j,a}(r))e^{ijt}$$

with $R_{j,a}(r)$ as in (iii), *where the convergence of the above series is uniform on disks $D_R(a)$ for each $R \in (0,1)$.*

(v) *If we let $\boldsymbol{\nu}(t) = (\cos t, \sin t)$ and $\boldsymbol{\tau}(t) = (-\sin t, \cos t)$, then $(i\nabla + A_a)\varphi_n^a$ can be expanded as*

$$\begin{aligned}(i\nabla + A_a)&\varphi_n^a(a + r(\cos t, \sin t)) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} \beta_j^a r^{|\alpha-j|-1}\big(i|\alpha-j|(1 + \widetilde{R}_{j,a}(r))\boldsymbol{\nu}(t) + (\alpha-j)(1 + R_{j,a}(r))\boldsymbol{\tau}(t)\big)e^{ijt}\end{aligned}$$

with $R_{j,a}(r)$, $\widetilde{R}_{j,a}(r)$ as in (iii), *where the above series converges absolutely in $L^2(D_R(a),\mathbb{C})$ and pointwise in $D_R(a)$ for each $R \in (0,1)$.*

Proof. The functions $\{e^{ijt}/\sqrt{2\pi}\}_{j\in\mathbb{Z}}$ form an orthonormal basis of $L^2((0,2\pi),\mathbb{C})$. Hence, recalling that we are assuming $\overline{D}_2 \subset \Omega$, if $|a|$ is sufficiently small, φ_n^a can be expanded as

$$\varphi_n^a(a+r(\cos t,\sin t)) = \sum_{j\in\mathbb{Z}} v_j^a(r)\frac{e^{ijt}}{\sqrt{2\pi}} \quad \text{in } L^2((0,2\pi),\mathbb{C}) \text{ for all } r\in(0,1], \tag{2-11}$$

where v_j^a is defined in (2-9). Equation (1-14) implies that, for every $j\in\mathbb{Z}$,

$$-(v_j^a)''(r) - \frac{1}{r}(v_j^a)'(r) + \frac{(\alpha-j)^2}{r^2}v_j^a(r) = \lambda_n^a v_j^a(r) \quad \text{for all } r\in(0,1], \tag{2-12}$$

or equivalently

$$-r^{|\alpha-j|-1}\big(r^{1-2|\alpha-j|}(r^{|\alpha-j|}v_j^a)'\big)' = \lambda_n^a v_j^a(r) \quad \text{for all } r\in(0,1].$$

Integrating twice between r and 1, we obtain, for some $c_{1,j}^a, c_{2,j}^a\in\mathbb{C}$,

$$v_j^a(r) = r^{|\alpha-j|}\left(c_{1,j}^a + \lambda_n^a\int_r^1 \frac{s^{-|\alpha-j|+1}}{2|\alpha-j|}v_j^a(s)\,ds\right) + r^{-|\alpha-j|}\left(c_{2,j}^a - \lambda_n^a\int_r^1 \frac{s^{|\alpha-j|+1}}{2|\alpha-j|}v_j^a(s)\,ds\right) \tag{2-13}$$

for all $r\in(0,1]$.

The convergence (2-3) in Proposition 2.1 implies that, for all a,

$$|\varphi_n^a(a+r(\cos t,\sin t))| = O(r^{\sqrt{\mu_1}})$$

as $r\to 0^+$, with μ_1 as in (2-1) (not necessarily uniformly with respect to a). Hence, for every a in a sufficiently small neighborhood of 0, there exists a constant $C(a)>0$ such that, for all $j\in\mathbb{Z}$,

$$|v_j^a(r)| \le C(a)r^{\sqrt{\mu_1}} \quad \text{for all } r\in[0,1]. \tag{2-14}$$

We deduce that each function v_j^a is bounded near 0; hence (2-13) necessarily yields

$$c_{2,j}^a = \lambda_n^a\int_0^1 \frac{s^{|\alpha-j|+1}}{2|\alpha-j|}v_j^a(s)\,ds. \tag{2-15}$$

We can therefore rewrite

$$v_j^a(r) = r^{|\alpha-j|}\left(c_{1,j}^a + \frac{\lambda_n^a}{2|\alpha-j|}\int_r^1 s^{-|\alpha-j|+1}v_j^a(s)\,ds\right) + \frac{\lambda_n^a}{2|\alpha-j|}r^{-|\alpha-j|}\int_0^r s^{|\alpha-j|+1}v_j^a(s)\,ds. \tag{2-16}$$

If $\sqrt{\mu_1}+2\ge|\alpha-j|$, using (2-14) to estimate the right-hand side of (2-16) we obtain the improved estimate $|v_j^a(r)|\le C(j,a)r^{|\alpha-j|}$. Otherwise, if $\sqrt{\mu_1}+2<|\alpha-j|$, we can use (2-14) to estimate the right-hand side of (2-16) to obtain the improved estimate $|v_j^a(r)|\le C(j,a)r^{\sqrt{\mu_1}+2}$ for some constant $C(j,a)>0$ depending on a and j. By iterating the process $m+1$ times, with m the largest natural number such that $\sqrt{\mu_1}+2m<|\alpha-j|$, we obtain $|v_j^a(r)|\le C(j,a)r^{|\alpha-j|}$, possibly for a different constant $C(j,a)$. We deduce that the quantity β_j^a introduced in (2-10) is well-defined. The fact that β_j^a is independent of R is a direct consequence of (2-12) and (2-16). This proves statement (i).

Using the independence of β_j^a with respect to R, we choose $R=1$ in (2-10) and $r=1$ in (2-16) and obtain

$$\beta_j^a = c_{1,j}^a + \frac{\lambda_n^a}{2|\alpha-j|}\int_0^1 s^{-|\alpha-j|+1}v_j^a(s)\,ds, \tag{2-17}$$

so that (2-16) can be rewritten as

$$v_j^a(r) = r^{|\alpha-j|}\biggl(\beta_j^a - \lambda_n^a \int_0^r \frac{s^{-|\alpha-j|+1}}{2|\alpha-j|} v_j^a(s)\,ds\biggr) + \lambda_n^a r^{-|\alpha-j|} \int_0^r \frac{s^{|\alpha-j|+1}}{2|\alpha-j|} v_j^a(s)\,ds. \tag{2-18}$$

From (2-18) it follows that, for all $r \in (0, 1]$,

$$\begin{aligned} r^{-|\alpha-j|}|v_j^a(r)| &\le |\beta_j^a| + \frac{\lambda_n^a}{2|\alpha-j|}\int_0^r s^{-|\alpha-j|}|v_j^a(s)|\,ds + \frac{\lambda_n^a}{2|\alpha-j|} r^{-2|\alpha-j|}\int_0^r s^{2|\alpha-j|}s^{-|\alpha-j|}|v_j^a(s)|\,ds \\ &\le |\beta_j^a| + \frac{\lambda_n^a}{|\alpha-j|}\int_0^r s^{-|\alpha-j|}|v_j^a(s)|\,ds. \end{aligned}$$

Hence the Gronwall lemma applied to the function $r \mapsto r^{-|\alpha-j|}|v_j^a(r)|$ yields that

$$r^{-|\alpha-j|}|v_j^a(r)| \le |\beta_j^a| e^{\lambda_n^a r/|\alpha-j|} \le C|\beta_j^a| \quad \text{for all } r \in (0, 1] \text{ and } j \in \mathbb{Z}, \tag{2-19}$$

where $C = e^{\Lambda_n/\sqrt{\mu_1}}$ is independent of j, a, and r, with μ_1 and Λ_n defined in (2-1) and (2-6) respectively.

From (2-13), (2-9), and (2-8) it follows that

$$|c_{1,j}^a + c_{2,j}^a| = |v_j^a(1)| = \frac{1}{\sqrt{2\pi}}\left|\int_0^{2\pi} \varphi_n^a(a + (\cos t, \sin t))e^{-ijt}\,dt\right| \le \text{const}$$

for some const > 0 independent of j and a; moreover, from (2-15) and (2-5) we deduce that

$$|c_{2,j}^a| \le \frac{\lambda_n^a}{2|\alpha-j|}\int_0^1 s|v_j^a(s)|\,ds \le \frac{\lambda_n^a}{2|\alpha-j|\sqrt{2\pi}}\int_{D_1(a)} |\varphi_n^a|\,dx \le \text{const}$$

for some const > 0 independent of j and a. Hence

$$|c_{1,j}^a| \le \widetilde{C} \tag{2-20}$$

for some $\widetilde{C} > 0$ independent of j and a.

Let $K > 0$ be such that

$$\frac{\Lambda_n C}{2K} < \frac{1}{2}$$

with C being as in (2-19) and Λ_n being as in (2-6). Hence, from (2-6), (2-17), (2-19) and (2-20) it follows that, if $|\alpha - j| > K$, then

$$\tfrac{1}{2}|\beta_j^a| \le \biggl(1 - \frac{\Lambda_n C}{2K}\biggr)|\beta_j^a| \le |c_{1,j}^a| \le \widetilde{C}. \tag{2-21}$$

Let us choose $R_0 \in (0, 1)$ such that

$$\frac{\Lambda_n C R_0^2}{2\sqrt{\mu_1}} < \frac{1}{2}.$$

From (2-10) and (2-19) it follows that, if $|\alpha - j| \le K$,

$$\begin{aligned} R_0^{-K}|v_j^a(R_0)| \ge R_0^{-|\alpha-j|}|v_j^a(R_0)| &= \left|\beta_j^a - \frac{\lambda_n^a}{2|\alpha-j|}\int_0^{R_0}\biggl(s^{1-|\alpha-j|} - \frac{s^{1+|\alpha-j|}}{R_0^{2|\alpha-j|}}\biggr)v_j^a(s)\,ds\right| \\ &\ge |\beta_j^a| - \frac{\Lambda_n C|\beta_j^a|}{2\sqrt{\mu_1}}\int_0^{R_0} 2s\,ds \\ &= |\beta_j^a| - \frac{\Lambda_n C R_0^2}{2\sqrt{\mu_1}}|\beta_j^a| \ge \tfrac{1}{2}|\beta_j^a|. \end{aligned}$$

Since, in view of (2-8), $v_j^a(R_0)$ is bounded uniformly with respect to a and j, we conclude that, for all j such that $|\alpha - j| \le K$, $|\beta_j^a|$ is bounded uniformly with respect to a and j. This, together with (2-21), yields (ii).

From (2-18) and (2-19) it follows that

$$v_j^a(r) = r^{|\alpha - j|}\beta_j^a(1 + R_{j,a}(r)), \tag{2-22}$$

where $|R_{j,a}(r)| \le \text{const}\, r^2$ for some const > 0 independent of j and a, thus proving the first estimate in (iii). Differentiating (2-18) and using the above estimate (2-22), we easily obtain

$$(v_j^a)'(r) = |\alpha - j|\beta_j^a r^{|\alpha-j|-1}(1 + \widetilde{R}_{j,a}(r)),$$

where $|\widetilde{R}_{j,a}(r)| \le \text{const}\, r^2$ for some const > 0 independent of j and a. Hence the proof of (iii) is complete.

From (2-11) and (iii) we have that the series

$$\frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}} r^{|\alpha-j|}\beta_j^a(1 + R_{j,a}(r))e^{ijt}$$

converges in $L^2((0, 2\pi), \mathbb{C})$ to $\varphi_n^a(a + r(\cos t, \sin t))$ for all $r \in (0, 1]$. In view of the estimates obtained in (ii)–(iii), the Weierstrass M-test ensures that the series is uniformly convergent in $D_R(a)$ for every $R \in (0, 1)$, thus proving (iv).

Let $f_j^a(a + r(\cos t, \sin t)) = v_j^a(r)e^{ijt}/\sqrt{2\pi}$. Since

$$(i\nabla + A_a) f_j^a(a + r(\cos t, \sin t)) = \left(i(v_j^a)'(r)\boldsymbol{\nu}(t) + (\alpha - j)\frac{v_j^a(r)}{r}\boldsymbol{\tau}(t)\right)\frac{e^{ijt}}{\sqrt{2\pi}},$$

the above estimates also imply that, for every $R \in (0, 1)$, the series of functions $\sum_j (i\nabla + A_a) f_j^a$ is convergent absolutely in $L^2(D_R(a), \mathbb{C})$ and pointwise in $D_R(a)$ to $(i\nabla + A_a)\varphi_n^a$ for every $R \in (0, 1)$. Hence (v) follows from (iii). □

Corollary 2.3. *Under the same assumptions and with the same notation as in Lemma 2.2, let $R \in (0, 1)$. Then, for all $r \in (0, R)$ and $t \in [0, 2\pi]$,*

$$\varphi_n^a(a + r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}}(r^\alpha\beta_0^a + r^{1-\alpha}\beta_1^a e^{it}) + \mathcal{R}_a(r, t), \tag{2-23}$$

$$\begin{aligned}(i\nabla + A_a)\varphi_n^a(a + r(\cos t, \sin t)) = {}& \frac{1}{\sqrt{2\pi}} r^{\alpha-1}\beta_0^a\alpha(i\boldsymbol{\nu}(t) + \boldsymbol{\tau}(t)) \\ &+ \frac{1}{\sqrt{2\pi}} r^{-\alpha}\beta_1^a(1-\alpha)(i\boldsymbol{\nu}(t) - \boldsymbol{\tau}(t))e^{it} + \widetilde{\mathcal{R}}_a(r, t),\end{aligned} \tag{2-24}$$

where $|\mathcal{R}_a(r, t)| \le \text{const}\, r^{1+\sqrt{\mu_1}}$ and $|\widetilde{\mathcal{R}}_a(r, t)| \le \text{const}\, r^{\sqrt{\mu_1}}$ for some const > 0 independent of a, r, t.

Proof. From part (iv) of Lemma 2.2 we have

$$\varphi_n^a(a + r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}}(\beta_0^a r^\alpha + \beta_1^a r^{1-\alpha}e^{it}) + \mathcal{R}_a(r, t), \quad r \in (0, 1),\ t \in [0, 2\pi],$$

where

$$\mathcal{R}_a(r,t) = \frac{1}{\sqrt{2\pi}}\big(\beta_0^a r^{\alpha} R_{0,a}(r) + \beta_1^a r^{1-\alpha} R_{1,a}(r)e^{it}\big) + \frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|>1}} \beta_j^a r^{|\alpha-j|}(1+R_{j,a}(r))e^{ijt}.$$

Let us fix $R \in (0,1)$. Estimates (ii)–(iii) of Lemma 2.2 imply that, for some const > 0 independent of a, r, t (possibly varying from line to line),

$$|\mathcal{R}_a(r,t)| \le \text{const}\Bigg(r^{\alpha+2} + r^{3-\alpha} + \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\ge 1+\sqrt{\mu_1}}} r^{|\alpha-j|}\Bigg) \le \text{const}\; r^{1+\sqrt{\mu_1}}$$

for all $r \in (0, R)$, thus proving (2-23).

From part (v) of Lemma 2.2 we have

$$(i\nabla + A_a)\varphi_n^a(a + r(\cos t, \sin t)) = \frac{\alpha}{\sqrt{2\pi}}\beta_0^a r^{\alpha-1}(i\boldsymbol{\nu}(t) + \boldsymbol{\tau}(t)) + \frac{1-\alpha}{\sqrt{2\pi}}\beta_1^a r^{-\alpha}(i\boldsymbol{\nu}(t) - \boldsymbol{\tau}(t))e^{it} + \widetilde{\mathcal{R}}_a(r,t),$$

where

$$\begin{aligned}\widetilde{\mathcal{R}}_a(r,t) = {}& \frac{\alpha}{\sqrt{2\pi}}\beta_0^a r^{\alpha-1}\big(i\widetilde{R}_{0,a}(r)\boldsymbol{\nu}(t) + R_{0,a}(r)\boldsymbol{\tau}(t)\big) + \frac{1-\alpha}{\sqrt{2\pi}}\beta_1^a r^{-\alpha}\big(i\widetilde{R}_{1,a}(r)\boldsymbol{\nu}(t) - R_{1,a}(r)\boldsymbol{\tau}(t)\big)e^{it} \\ & + \frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|>1}} \beta_j^a r^{|\alpha-j|-1}\big(i|\alpha-j|(1+\widetilde{R}_{j,a}(r))\boldsymbol{\nu}(t) + (\alpha-j)(1+R_{j,a}(r))\boldsymbol{\tau}(t)\big)e^{ijt}.\end{aligned}$$

From Lemma 2.2(ii)–(iii) we have that, for all $r \in (0, R)$,

$$|\widetilde{\mathcal{R}}_a(r,t)| \le \text{const}\Bigg(r^{\alpha+1} + r^{2-\alpha} + \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\ge 1+\sqrt{\mu_1}}} |\alpha-j|r^{|\alpha-j|-1}\Bigg) \le \text{const}\; r^{\sqrt{\mu_1}}$$

for some const > 0 independent of a, r, t (possibly varying from line to line), thus proving (2-24). □

We now describe some consequences of Lemma 2.2 and Corollary 2.3, which will be needed in Section 3 to prove a monotonicity-type formula.

Lemma 2.4. *Under the same assumptions and with the same notation as in Lemma 2.2, we have*

$$\lim_{\varepsilon\to 0^+}\left\{\left|\frac{1}{2}\int_{\partial D_\varepsilon(a)} |(i\nabla + A_a)\varphi_n^a|^2 x\cdot\nu\, ds\right| + \left|\int_{\partial D_\varepsilon(a)} (i\nabla + A_a)\varphi_n^a \cdot \nu\, \overline{(i\nabla + A_a)\varphi_n^a \cdot x}\, ds\right|\right\} \le 2\alpha(1-\alpha)|a||\beta_0^a||\beta_1^a|.$$

Proof. Let $R \in (0,1)$ be fixed. From (2-24) we have that, for all $r \in (0, R)$,

$$\big|(i\nabla + A_a)\varphi_n^a(a + r(\cos t, r\sin t))\big|^2 = r^{2(\alpha-1)}|\beta_0^a|^2\frac{\alpha^2}{\pi} + r^{-2\alpha}|\beta_1^a|^2\frac{(1-\alpha)^2}{\pi} + \widehat{\mathcal{R}}_a(r,t),$$

where $|\widehat{\mathcal{R}}_a(r,t)| \le \text{const}\; r^{2\sqrt{\mu_1}-1}$ for some const > 0 independent of a, r, t. It follows that

$$\lim_{\varepsilon\to 0^+}\int_{\partial D_\varepsilon(a)} |(i\nabla + A_a)\varphi_n^a|^2 x\cdot\nu\, ds = 0.$$

Moreover, from (2-24) we have

$$
\begin{aligned}
&(i\nabla + A_a)\varphi_n^a(a+\varepsilon(\cos t,\sin t))\cdot \boldsymbol{\nu}(t)\overline{(i\nabla + A_a)\varphi_n^a(a+\varepsilon(\cos t,\sin t))\cdot (a+\varepsilon\boldsymbol{\nu}(t))}\\
&\qquad = \varepsilon^{2(\alpha-1)}|\beta_0^a|^2\frac{\alpha^2}{2\pi}(\boldsymbol{\nu}(t)+i\boldsymbol{\tau}(t))\cdot a + \varepsilon^{-2\alpha}|\beta_1^a|^2\frac{(1-\alpha)^2}{2\pi}(\boldsymbol{\nu}(t)-i\boldsymbol{\tau}(t))\cdot a\\
&\qquad\quad + 2\varepsilon^{-1}\mathfrak{Re}\left(\beta_0^a\bar{\beta}_1^a e^{-it}\frac{\alpha(1-\alpha)}{2\pi}(\boldsymbol{\nu}(t)-i\boldsymbol{\tau}(t))\cdot a\right) + O(\varepsilon^{2\sqrt{\mu_1}-1})
\end{aligned}
$$

as $\varepsilon\to 0^+$, and hence, taking into account that $\int_0^{2\pi} a\cdot\boldsymbol{\nu}(t)\,dt = \int_0^{2\pi} a\cdot\boldsymbol{\tau}(t)\,dt = 0$, we obtain

$$
\lim_{\varepsilon\to 0^+}\int_{\partial D_\varepsilon(a)} (i\nabla + A_a)\varphi_a\cdot \nu\overline{(i\nabla + A_a)\varphi_a\cdot x}\,ds = 2\alpha(1-\alpha)\mathfrak{Re}(\beta_0^a\bar{\beta}_1^a(a_1 - i a_2)),
$$

from which the conclusion follows. □

Lemma 2.5. *For $n\ge 1$ fixed and a varying in Ω, let $\varphi_n^a\in H_0^{1,a}(\Omega,\mathbb{C})\setminus\{0\}$ satisfy (2-4) and (2-5). Let us assume that $\varphi_n^a\to\varphi_n^0$ in $L^2(\Omega,\mathbb{C})$ as $a\to 0$ (or respectively along a sequence $a_\ell\to 0$). Let $k\in\mathbb{Z}$ be such that $|\alpha-k|$ is the order of vanishing of φ_n^0 at 0. For all $j\in\mathbb{Z}$ and $a\in\Omega$, let v_j^a be as in (2-9) and β_j^a be as in (2-10). Then there exists $\rho_0>0$ such that, for all a with $|a|\le\rho_0$ (respectively for $a=a_\ell$ with ℓ sufficiently large), the following properties hold:*

(i) *For all $j\in\mathbb{Z}$, we have $\beta_j^a\to\beta_j^0$ as $a\to 0$ (respectively along the sequence $a_\ell\to 0$).*

(ii) *It holds that*

$$
\int_0^{2\pi}|\varphi_n^a(a+r(\cos t,\sin t))|^2\,dt = \Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{2|\alpha-j|}|\beta_j^a|^2\,|1+R_{j,a}(r)|^2\Bigg) + r^{2|\alpha-k|}|\beta_k^a|^2(1+\widehat{R}_a(r)),
$$

where $|\widehat{R}_a(r)|\le h(r)$ for some function $h(r)$ independent of a such that $h(r)\to 0$ as $r\to 0^+$, and

$$
\varphi_n^a(a+r(\cos t,\sin t)) = \frac{1}{\sqrt{2\pi}}\Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{|\alpha-j|}\beta_j^a(1+R_{j,a}(r))e^{ijt}\Bigg) + \frac{1}{\sqrt{2\pi}}r^{|\alpha-k|}\beta_k^a(e^{ikt}+R_a(r,t)),
$$

where $|R_{j,a}(r)|\le \text{const}\, r^2$ for some const >0 independent of j and a, and $|R_a(r,t)|\le f(r)$ for some function $f(r)$ independent of a and t such that $f(r)\to 0$ as $r\to 0^+$.

(iii) *Let $\boldsymbol{\nu}(t)=(\cos t,\sin t)$ and $\boldsymbol{\tau}(t)=(-\sin t,\cos t)$. It holds that*

$$
\begin{aligned}
&\int_0^{2\pi}\big|(i\nabla + A_a)\varphi_n^a(a+r(\cos t,\sin t))\big|^2\,dt\\
&\quad = \Bigg(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|<|\alpha-k|}} r^{2|\alpha-j|-2}|\beta_j^a|^2\,|\alpha-j|^2\big(|1+R_{j,a}(r)|^2 + |1+\widetilde{R}_{j,a}(r)|^2\big)\Bigg)\\
&\qquad\qquad + r^{2|\alpha-k|-2}|\beta_k^a|^2\,|\alpha-k|^2(1+\widetilde{R}_a(r)),
\end{aligned}
$$

where $|\widetilde{R}_a(r)| \le p(r)$ *for some function* $p(r)$ *independent of* a *such that* $p(r) \to 0$ *as* $r \to 0^+$, *and*

$$(i\nabla + A_a)\varphi_n^a(a + r(\cos t, \sin t)) = \frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z} \\ |\alpha-j|<|\alpha-k|}} r^{|\alpha-j|-1}\beta_j^a\big(i|\alpha-j|\boldsymbol{\nu}(t) + (\alpha-j)\boldsymbol{\tau}(t) + \boldsymbol{R}_{j,a}(r)\big)e^{ijt}$$
$$+ \frac{1}{\sqrt{2\pi}} r^{|\alpha-k|-1}\beta_k^a\Big(\big(i|\alpha-k|\boldsymbol{\nu}(t) + (\alpha-k)\boldsymbol{\tau}(t)\big)e^{ikt} + \widetilde{\boldsymbol{R}}_a(r,t)\Big),$$

where $|\boldsymbol{R}_{j,a}(r,t)| \le \text{const}\, r^2$ *for some positive constant* const > 0 *independent of* j *and* a *and* $|\widetilde{\boldsymbol{R}}_a(r,t)| \le g(r)$ *for some function* $g(r)$ *independent of* a *and* t *such that* $g(r) \to 0$ *as* $r \to 0^+$.

Proof. In order to prove statement (i), we notice that (2-10) evaluated at $R = 1$ provides

$$\beta_j^a = v_j^a(1) + \frac{\lambda_n^a}{2|\alpha-j|}\int_0^1 (s^{1-|\alpha-j|} - s^{1+|\alpha-j|})v_j^a(s)\,ds. \tag{2-25}$$

From Lemma 2.2(ii)–(iii) it follows that, for $|a| \le \rho_0$ with $\rho_0 > 0$ sufficiently small,

$$|v_j^a(r)| \le C' r^{|\alpha-j|} \quad \text{for all } r \in (0,1] \text{ and } j \in \mathbb{Z} \tag{2-26}$$

for some constant $C' > 0$ independent of j, a, and r. Moreover, (2-4), (2-5), the convergence $\varphi_n^a \to \varphi_n^0$ in $L^2(\Omega,\mathbb{C})$, and standard elliptic estimates, see, e.g., [Gilbarg and Trudinger 1983, Theorem 8.10], imply

$$\varphi_n^a \to \varphi_n^0 \quad \text{in } H^1(\Omega,\mathbb{C}) \text{ and } C^2_{\text{loc}}(\Omega\setminus\{0\},\mathbb{C}) \quad \text{as } a \to 0 \text{ (or along the sequence } a_\ell \to 0). \tag{2-27}$$

From (2-25)–(2-27), and the dominated convergence theorem we obtain that, for all $j \in \mathbb{Z}$,

$$\lim_{a\to 0}\beta_j^a = v_j^0(1) + \frac{\lambda_n^0}{2|\alpha-j|}\int_0^1 (s^{1-|\alpha-j|} - s^{1+|\alpha-j|})v_j^0(s)\,ds = \beta_j^0,$$

thus proving (i).

If $k \in \mathbb{Z}$ is such that $|\alpha-k|$ is the order of vanishing of φ_n^0 at 0, from Lemma 2.2(iii) it follows that $\beta_k^0 \ne 0$ and $\beta_j^0 = 0$ for all $j \in \mathbb{Z}$ such that $|\alpha-j| < |\alpha-k|$; in particular, in view of (i), we have $\lim_{a\to 0}\beta_k^a \ne 0$ and hence $\inf_{|a|\le\rho_0}|\beta_k^a| > 0$ for ρ_0 sufficiently small. Then, from Lemma 2.2(iv) and the Parseval identity we deduce that

$$\int_0^{2\pi} |\varphi_n^a(a + r(\cos t,\sin t))|^2\,dt = \sum_{j\in\mathbb{Z}} r^{2|\alpha-j|}|\beta_j^a|^2|1 + R_{j,a}(r)|^2$$
$$= \bigg(\sum_{\substack{j\in\mathbb{Z} \\ |\alpha-j|<|\alpha-k|}} r^{2|\alpha-j|}|\beta_j^a|^2|1 + R_{j,a}(r)|^2\bigg) + r^{2|\alpha-k|}|\beta_k^a|^2(1 + \widehat{R}_a(r)),$$

with

$$\widehat{R}_a(r) = |R_{k,a}(r)|^2 + 2\mathfrak{Re}(R_{k,a}(r)) + \sum_{\substack{j\in\mathbb{Z} \\ |\alpha-j|>|\alpha-k|}} \frac{|\beta_j^a|^2}{|\beta_k^a|^2} r^{2|\alpha-j|-2|\alpha-k|}|1 + R_{j,a}(r)|^2$$

so that the first estimate in (ii) follows from Lemma 2.2(ii)–(iii). From Lemma 2.2(iii) we also deduce that

$$\frac{1}{\sqrt{2\pi}} \sum_{\substack{j\in\mathbb{Z} \\ |\alpha-j|\geq|\alpha-k|}} r^{|\alpha-j|}\beta_j^a(1+R_{j,a}(r))e^{ijt} = \frac{1}{\sqrt{2\pi}} r^{|\alpha-k|}\beta_k^a(e^{ikt}+R_a(r,t)),$$

where $|R_a(r,t)| \leq f(r)$ for some function $f(r)$ independent of a and t such that $f(r) \to 0$ as $r \to 0$. Then the second estimate in (ii) follows from Lemma 2.2(iv).

From Lemma 2.2 (v) and the Parseval identity we deduce that

$$\int_0^{2\pi} \big|(i\nabla+A_a)\varphi_n^a(a+r(\cos t,\sin t))\big|^2\,dt = \sum_{j\in\mathbb{Z}} r^{2|\alpha-j|-2}|\beta_j^a|^2\,|\alpha-j|^2\big(|1+R_{j,a}(r)|^2+|1+\widetilde{R}_{j,a}(r)|^2\big)$$

so that the first estimate in (iii) follows from Lemma 2.2(ii)–(iii) arguing as above. In a similar way, the second estimate in (iii) follows from statements (iii) and (v) of Lemma 2.2. □

Remark 2.6. In the particular case $n = n_0$ with n_0 such that (1-5) holds, the above lemma applies to the family of eigenfunctions $\varphi_a = \varphi_{n_0}^a$ satisfying (1-14) and (1-15). Indeed, in this case (1-16) holds; i.e., the eigenfunctions φ_a converge as $a \to 0^+$ so that the assumptions of Lemma 2.5 are fulfilled. In particular we deduce that, if φ_0 satisfies (1-7)–(1-9) and if φ_a is as in (1-14)–(1-15), then, for a sufficiently close to 0, the vanishing order of φ_a is less than or equal to the vanishing order of φ_0.

Lemma 2.7. *For $n \geq 1$ fixed and a varying in $\Omega\setminus\{0\}$, let $\varphi_n^a \in H_0^{1,a}(\Omega,\mathbb{C})\setminus\{0\}$ satisfy* (2-4) *and* (2-5). *Then there exist $\sigma > 0$ and $C > 0$ such that, for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$,*

$$\frac{1}{|a|}\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx \leq C\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds,$$

$$\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |(i\nabla+A_a)\varphi_n^a|^2\,dx \leq \frac{C}{R^2|a|}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds.$$

Proof. Let us prove the first estimate arguing by contradiction: assume that there exist sequences $R_\ell > 1$ and $a_\ell \in \Omega$ such that $R_\ell|a_\ell| < 1/\ell$ and

$$\frac{1}{|a_\ell|}\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,dx > \ell\int_{\partial D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,ds. \tag{2-28}$$

It is easy to verify that, up to extracting a subsequence, $\varphi_n^{a_\ell} \to \varphi_n^0$ in $L^2(\Omega,\mathbb{C})$ as $\ell \to \infty$ for some $\varphi_n^0 \in H_0^{1,0}(\Omega,\mathbb{C})\setminus\{0\}$ satisfying

$$\begin{cases} (i\nabla+A_0)^2\varphi_n^0 = \lambda_n^0\varphi_n^0 & \text{in } \Omega, \\ \varphi_n^0 = 0 & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_n^0(x)|^2\,dx = 1. \end{cases} \tag{2-29}$$

Let $k \in \mathbb{Z}$ be such that $|\alpha - k|$ is the order of vanishing of φ_n^0 at 0. Then, from Lemma 2.5 (first estimate in (ii)) it follows that, for ℓ sufficiently large,

$$\begin{aligned}\frac{1}{|a_\ell|}\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,dx &= \frac{1}{|a_\ell|}\int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\int_0^{2\pi} |\varphi_n^{a_\ell}(a_\ell+r(\cos t,\sin t))|^2\,dt\biggr)\,dr \\ &\le \frac{2}{|a_\ell|}\int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} r^{2|\alpha-j|}|\beta_j^{a_\ell}|^2\biggr)\,dr \\ &\le \text{const} \sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} (R_\ell|a_\ell|)^{1+2|\alpha-j|}|\beta_j^{a_\ell}|^2\end{aligned}$$

for some positive constant const > 0 independent of ℓ, while

$$\begin{aligned}\int_{\partial D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,ds &= R_\ell|a_\ell|\int_0^{2\pi} \bigl|\varphi_n^{a_\ell}(a_\ell + R_\ell|a_\ell|(\cos t,\sin t))\bigr|^2\,dt \\ &\ge \frac{R_\ell|a_\ell|}{2}\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} (R_\ell|a_\ell|)^{2|\alpha-j|}|\beta_j^{a_\ell}|^2, \end{aligned}\tag{2-30}$$

thus contradicting (2-28) as $\ell \to \infty$.

To prove the second estimate, let us assume by contradiction that there exist sequences $R_\ell > 1$ and $a_\ell \in \Omega$ such that $R_\ell|a_\ell| < 1/\ell$ and

$$\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |(i\nabla + A_{a_\ell})\varphi_n^{a_\ell}|^2\,dx > \frac{\ell}{R_\ell^2|a_\ell|}\int_{\partial D_{R_\ell|a_\ell|}(a_\ell)} |\varphi_n^{a_\ell}|^2\,ds. \tag{2-31}$$

As above we have that, up to extracting a subsequence, $\varphi_n^{a_\ell} \to \varphi_n^0$ in $L^2(\Omega,\mathbb{C})$ as $\ell \to \infty$ for some $\varphi_n^0 \in H_0^{1,0}(\Omega,\mathbb{C}) \setminus \{0\}$ satisfying (2-29). Then, from Lemma 2.5 (first estimate in (iii)) it follows that, for ℓ sufficiently large and for some positive constant const > 0 independent of ℓ,

$$\begin{aligned}&\int_{D_{(R_\ell+1)|a_\ell|}(a_\ell)\setminus D_{R_\ell|a_\ell|}(a_\ell)} |(i\nabla + A_{a_\ell})\varphi_n^{a_\ell}|^2\,dx \\ &\quad= \int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\int_0^{2\pi} |(i\nabla + A_{a_\ell})\varphi_n^{a_\ell}(a_\ell + r(\cos t,\sin t))|^2\,dt\biggr)\,dr \\ &\quad= \int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} r^{2|\alpha-j|-2}|\beta_j^{a_\ell}|^2\,|\alpha-j|^2(2+o(1))\biggr)\,dr \\ &\quad\le 3|\alpha-k|^2\int_{R_\ell|a_\ell|}^{(R_\ell+1)|a_\ell|} r\biggl(\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} r^{2|\alpha-j|-2}|\beta_j^{a_\ell}|^2\biggr)\,dr \le \frac{\text{const}}{R_\ell}\sum_{\substack{j\in\mathbb{Z}\\ |\alpha-j|\le|\alpha-k|}} (R_\ell|a_\ell|)^{2|\alpha-j|}|\beta_j^{a_\ell}|^2,\end{aligned}$$

which, in view of (2-30), contradicts (2-31) as $\ell \to \infty$. □

Remark 2.8. Arguing as in Lemma 2.7, we can also prove the following similar estimate (possibly taking a smaller σ and a larger C if necessary): for all $R>1$ and $a\in\Omega$ such that $0<|a|<\sigma/R$

$$\frac{1}{|a|}\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)}|\varphi_n^a|^2\,dx\le C\int_{\partial D_{(R+1)|a|}(a)}|\varphi_n^a|^2\,ds,$$
$$\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)}|(i\nabla+A_a)\varphi_n^a|^2\,dx\le\frac{C}{R^2|a|}\int_{\partial D_{(R+1)|a|}(a)}|\varphi_n^a|^2\,ds.$$

Lemma 2.9. *For $n\ge 1$ fixed, let φ_n^a be a solution to (2-4)–(2-5). Let $\sigma>0$ and $C>0$ be as in Lemma 2.7 and Remark 2.8. Then, for all $R>2$ and $a\in\Omega$ such that $0<|a|<\sigma/R$,*

$$\left|\int_{\partial D_{R|a|}(0)}|\varphi_n^a|^2\,ds-\int_{\partial D_{R|a|}(a)}|\varphi_n^a|^2\,ds\right|\le\frac{1+6C}{R-2}\int_{\partial D_{R|a|}(a)}|\varphi_n^a|^2\,ds.$$

Proof. We note that

$$\int_{\partial D_{R|a|}(0)}|\varphi_n^a|^2\,ds-\int_{\partial D_{R|a|}(a)}|\varphi_n^a|^2\,ds=\int_{\partial\mathcal{L}_{1,R}^a}|\varphi_n^a|^2\tilde{\nu}\cdot\hat{\nu}\,ds-\int_{\partial\mathcal{L}_{2,R}^a}|\varphi_n^a|^2\tilde{\nu}\cdot(-\hat{\nu})\,ds,\tag{2-32}$$

where

$$\mathcal{L}_{1,R}^a=D_{R|a|}(0)\setminus D_{R|a|}(a),\quad \mathcal{L}_{2,R}^a=D_{R|a|}(a)\setminus D_{R|a|}(0),$$

and

$$\hat{\nu}(x)=\begin{cases}x/|x| & \text{on }\partial D_{R|a|}(0),\\ -(x-a)/|x-a| & \text{on }\partial D_{R|a|}(a),\end{cases}\qquad \tilde{\nu}(x)=\begin{cases}x/|x| & \text{on }\partial D_{R|a|}(0),\\ (x-a)/|x-a| & \text{on }\partial D_{R|a|}(a).\end{cases}$$

We note that $\hat{\nu}$ is the outer unit normal vector on $\partial\mathcal{L}_{1,R}^a$ and $-\hat{\nu}$ is the outer unit normal vector on $\partial\mathcal{L}_{2,R}^a$. By setting $\nu_1(x)=x/|x|$, we can rewrite the right-hand side of (2-32) as

$$\begin{aligned}\int_{\partial\mathcal{L}_{1,R}^a}&|\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds+\int_{\partial\mathcal{L}_{1,R}^a}|\varphi_n^a|^2\nu_1\cdot\hat{\nu}\,ds+\int_{\partial\mathcal{L}_{2,R}^a}|\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds-\int_{\partial\mathcal{L}_{2,R}^a}|\varphi_n^a|^2\nu_1\cdot(-\hat{\nu})\,ds\\ &=\int_{\partial\mathcal{L}_{1,R}^a}|\varphi_n^a|^2\nu_1\cdot\hat{\nu}\,ds-\int_{\partial\mathcal{L}_{2,R}^a}|\varphi_n^a|^2\nu_1\cdot(-\hat{\nu})\,ds\\ &\qquad+\int_{\partial D_{R|a|}(0)}|\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds+\int_{\partial D_{R|a|}(a)}|\varphi_n^a|^2(\tilde{\nu}-\nu_1)\cdot\hat{\nu}\,ds.\end{aligned}\tag{2-33}$$

We observe that

$$(\tilde{\nu}(x)-\nu_1(x))\cdot\hat{\nu}(x)=\begin{cases}0 & \text{on }\partial D_{R|a|}(0),\\ -1+x\cdot(x-a)/(|x|\,|x-a|) & \text{on }\partial D_{R|a|}(a).\end{cases}$$

Moreover, since ν_1 is smooth in $\bar{\mathcal{L}}_{1,R}^a\cup\bar{\mathcal{L}}_{2,R}^a$, we can apply the divergence theorem to the first two terms in the right-hand side of (2-33), thus rewriting the right-hand side of (2-32) as

$$-\int_{\partial D_{R|a|}(a)}|\varphi_n^a|^2\left(1-\frac{x\cdot(x-a)}{|x|\,|x-a|}\right)ds+\int_{\mathcal{L}_{1,R}^a}\operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx-\int_{\mathcal{L}_{2,R}^a}\operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx.\tag{2-34}$$

Estimate of the first term in (2-34). Parametrizing $\partial D_{R|a|}(a)$ as $x = a + R|a|(\cos t, \sin t)$ and writing $a = |a|(\cos\theta_a, \sin\theta_a)$ for some angle $\theta_a \in [0, 2\pi)$, we get

$$\left|1 - \frac{x\cdot(x-a)}{|x|\,|x-a|}\right| = \left|1 - \frac{R + \cos(t-\theta_a)}{(R^2 + 2R\cos(t-\theta_a)+1)^{1/2}}\right| \le \frac{1}{R-1}$$

on $\partial D_{R|a|}(a)$. Therefore,

$$\left|-\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\left(1 - \frac{x\cdot(x-a)}{|x|\,|x-a|}\right) ds\right| \le \frac{1}{R-1}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\, ds. \tag{2-35}$$

Estimate of the second term in (2-34). The second term in (2-34) splits into two parts:

$$\int_{\mathcal{L}^a_{1,R}} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx = \int_{\mathcal{L}^a_{1,R}} \frac{|\varphi_n^a|^2}{|x|}\,dx + \int_{\mathcal{L}^a_{1,R}} 2\mathfrak{Re}(i\varphi_n^a\overline{(i\nabla + A_a)\varphi_n^a}\cdot\nu_1)\,dx.$$

Since $D_{R|a|}(0) \subset D_{(R+1)|a|}(a)$, we have $\mathcal{L}^a_{1,R} \subseteq D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)$. Let $\sigma > 0$ and $C > 0$ be as in Lemma 2.7 and Remark 2.8. Hence by Lemma 2.7 we have that, for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$,

$$\begin{aligned}\left|\int_{\mathcal{L}^a_{1,R}} \frac{|\varphi_n^a|^2}{|x|}\,dx\right| &\le \int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} \frac{|\varphi_n^a|^2}{|x|}\,dx \\ &\le \frac{1}{(R-1)|a|}\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx \le \frac{C}{R-1}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mathcal{L}^a_{1,R}} 2\mathfrak{Re}(i\varphi_n^a\overline{(i\nabla + A_a)\varphi_n^a}\cdot\nu_1)\,dx\right| \\ &\qquad \le 2\left(\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |\varphi_n^a|^2\,dx\right)^{1/2}\left(\int_{D_{(R+1)|a|}(a)\setminus D_{R|a|}(a)} |(i\nabla + A_a)\varphi_n^a|^2\,dx\right)^{1/2} \\ &\qquad \le \frac{2C}{R}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds.\end{aligned}$$

Therefore,

$$\left|\int_{\mathcal{L}^a_{1,R}} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx\right| \le \frac{3C}{R-1}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds \tag{2-36}$$

for all $R > 1$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$.

Estimate of the third term in (2-34). The estimate of the third term can be derived in a similar way, observing that, since $D_{R|a|}(0) \supset D_{(R-1)|a|}(a)$, we have $\mathcal{L}^a_{2,R} \subseteq D_{R|a|}(a)\setminus D_{(R-1)|a|}(a)$, and using Remark 2.8 to obtain

$$\left|\int_{\mathcal{L}^a_{2,R}} \operatorname{div}(|\varphi_n^a|^2\nu_1)\,dx\right| \le \frac{3C}{R-2}\int_{\partial D_{R|a|}(a)} |\varphi_n^a|^2\,ds \tag{2-37}$$

for all $R > 2$ and $a \in \Omega$ such that $0 < |a| < \sigma/R$ (by possibly changing C and σ).

Therefore combining (2-35)–(2-37) we complete the proof. □

3. Monotonicity formula

The aim of this section is to introduce an Almgren-type frequency function and to use it to obtain local estimates of the eigenfunctions in a neighborhood of order $|a|$ of the singularity. In particular, we shall prove that a suitable family of blow-up of the eigenfunctions φ_a is bounded in the magnetic Sobolev space (see Remark 3.8 ahead).

3A. ***Almgren-type frequency function.*** Arguing as in [Abatangelo and Felli 2015, Lemma 3.1], one can easily prove the *Poincaré-type inequality*

$$\frac{1}{r^2}\int_{D_r}|u|^2\,dx \le \frac{1}{r}\int_{\partial D_r}|u|^2\,ds + \int_{D_r}|(i\nabla + A_a)u|^2\,dx, \tag{3-1}$$

which holds for every $r > 0$, $a \in D_r$, and $u \in H^{1,a}(D_r,\mathbb{C})$. Furthermore, defining, for every $b \in D_1$,

$$m_b := \inf_{\substack{v\in H^{1,b}(D_1,\mathbb{C})\\ v\not\equiv 0}} \frac{\int_{D_1}|(i\nabla + A_b)v|^2\,dx}{\int_{\partial D_1}|v|^2\,ds},$$

we have that the infimum m_b is attained and $m_b > 0$. Arguing as in [Abatangelo and Felli 2015], we can prove that $b \mapsto m_b$ is continuous in D_1 and that $m_0 = \sqrt{\mu_1}$, with μ_1 as in (2-1). Therefore a standard dilation argument yields that, for any $\delta \in (0, \sqrt{\mu_1})$, there exists some sufficiently large $\Upsilon_\delta > 1$ such that, for every $r > 0$ and $a \in D_r$ such that $|a|/r < 1/\Upsilon_\delta$,

$$\frac{\sqrt{\mu_1}-\delta}{r}\int_{\partial D_r}|u|^2\,ds \le \int_{D_r}|(i\nabla + A_a)u|^2\,dx \quad \text{for all } u \in H^{1,a}(D_r,\mathbb{C}). \tag{3-2}$$

For $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^2$, $u \in H^{1,b}(D_r,\mathbb{C})$ and $r > |b|$, we define the Almgren-type frequency function as

$$\mathcal{N}(u,r,\lambda,A_b) = \frac{E(u,r,\lambda,A_b)}{H(u,r)},$$

where

$$E(u,r,\lambda,A_b) = \int_{D_r}\big[|(i\nabla + A_b)u|^2 - \lambda|u|^2\big]\,dx \quad \text{and} \quad H(u,r) = \frac{1}{r}\int_{\partial D_r}|u|^2\,ds.$$

For all $1 \le n \le n_0$ and $a \in \Omega$, let $\varphi_n^a \in H_0^{1,a}(\Omega,\mathbb{C})\setminus\{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_n^a, i.e., solving (2-4), such that

$$\int_\Omega |\varphi_n^a(x)|^2\,dx = 1 \qquad \text{and} \qquad \int_\Omega \varphi_n^a(x)\overline{\varphi_\ell^a(x)}\,dx = 0 \quad \text{if } n \neq \ell. \tag{3-3}$$

For $n = n_0$, we choose

$$\varphi_{n_0}^a = \varphi_a,$$

with φ_a as in (1-14)–(1-15). Let

$$\Lambda = \sup_{\substack{a\in\Omega\\ 1\le n\le n_0}} \lambda_n^a \in (0,+\infty).$$

We recall that Λ is finite in view of the continuity result of the eigenvalue function $a \mapsto \lambda_n^a$ in $\overline{\Omega}$ proved in [Bonnaillie-Noël et al. 2014, Theorem 1.1].

Arguing as in [Abatangelo and Felli 2015, Lemma 5.2], we can prove that there exists

$$0 < R_0 < (\Lambda(1+2/\sqrt{\mu_1}))^{-1/2}$$

such that $D_{R_0} \subset \Omega$ and, if $|a| < R_0$,

$$H(\varphi_n^a, r) > 0 \quad \text{for all } r \in (|a|, R_0) \text{ and } 1 \le n \le n_0. \tag{3-4}$$

Furthermore, for every $r \in (0, R_0]$ there exist $C_r > 0$ and $\alpha_r \in (0, r)$ such that

$$H(\varphi_n^a, r) \ge C_r \quad \text{for all } a \text{ with } |a| < \alpha_r \text{ and } 1 \le n \le n_0. \tag{3-5}$$

Thanks to (3-4), the function $r \mapsto N(\varphi_n^a, r, \lambda_n^a, A_a)$ is well-defined in $(|a|, R_0)$. By direct calculations, see [Noris et al. 2015] for details, we can prove that

$$\frac{d}{dr} H(\varphi_n^a, r) = \frac{2}{r} E(\varphi_n^a, r, \lambda_n^a, A_a), \tag{3-6}$$

$$\frac{d}{dr} E(\varphi_n^a, r, \lambda_n^a, A_a) = 2\int_{\partial D_r} |(i\nabla + A_a)\varphi_n^a \cdot \nu|^2 \, ds - \frac{2}{r}\left(M_n^a + \lambda_n^a \int_{D_r} |\varphi_n^a|^2 \, dx\right) \tag{3-7}$$

where

$$M_n^a = \lim_{\varepsilon \to 0^+} \int_{\partial D_\varepsilon(a)} \left(\Re\mathfrak{e}\left((i\nabla + A_a)\varphi_n^a \cdot \nu \overline{(i\nabla + A_a)\varphi_n^a \cdot x}\right) - \tfrac{1}{2}|(i\nabla + A_a)\varphi_n^a|^2 x \cdot \nu\right) ds. \tag{3-8}$$

Lemma 2.9, together with Lemmas 2.2 and 2.4, allow us to give an estimate of the quantity M_n^a defined in (3-8). We notice that the techniques used in [Abatangelo and Felli 2015; Noris et al. 2015] to estimate the term M_n^a for $\alpha = \frac{1}{2}$ were based on the possibility of rewriting the problem as a Laplace equation on the twofold covering; hence it is not possible here to extend such proofs to the case $\alpha \notin \mathbb{Z}/2$ and a new strategy of proof is needed.

Lemma 3.1. *For $n \in \{1, \dots, n_0\}$ and $a \in \Omega$, let φ_n^a be a solution of* (2-4) *satisfying* (3-3). *There exist $\sigma_0 > 0$ and $c_0 > 2$ such that, for every $1 \le n \le n_0$, $R > c_0$ and $a \in \Omega$ such that $|a| < \sigma_0/R$, the quantity M_n^a defined in* (3-8) *satisfies*

$$\frac{|M_n^a|}{H(\varphi_n^a, R|a|)} \le \frac{2\alpha(1-\alpha)}{R - c_0}.$$

Proof. Let us fix $n \in \{1, 2, \dots, n_0\}$ and define, for $|a|$ small and $r \in (0, 1]$,

$$\widetilde{H}(\varphi_n^a, r) = \frac{1}{r}\int_{\partial D_r(a)} |\varphi_n^a|^2 \, ds.$$

From the Parseval identity and Lemma 2.2(iv) it follows that there exists $\sigma_n > 0$ such that, for every $R > 2$ and $a \in \Omega$ such that $|a| < \sigma_n/R$,

$$\begin{aligned}
\widetilde{H}(\varphi_n^a, R|a|) &= \int_0^{2\pi} \left|\varphi_n^a(a + R|a|(\cos t, \sin t))\right|^2 dt = \sum_{j \in \mathbb{Z}} (R|a|)^{2|\alpha - j|} |\beta_j^a|^2 \left|1 + R_{j,a}(R|a|)\right|^2 \\
&\ge (R|a|)^{2\alpha} |\beta_0^a|^2 \left|1 + R_{0,a}(R|a|)\right|^2 + (R|a|)^{2(1-\alpha)} |\beta_1^a|^2 \left|1 + R_{1,a}(R|a|)\right|^2 \\
&\ge \tfrac{1}{2}\left(|\beta_0^a|^2 (R|a|)^{2\alpha} + |\beta_1^a|^2 (R|a|)^{2(1-\alpha)}\right),
\end{aligned} \tag{3-9}$$

where the β_j^a's are the coefficients defined in (2-10) for the eigenfunction φ_n^a (with n fixed). From the elementary inequality $ab \le \frac{1}{2}(a^2+b^2)$, it follows that

$$|\beta_0^a||\beta_1^a||a| = \frac{1}{R}|\beta_0^a|(R|a|)^\alpha |\beta_1^a|(R|a|)^{1-\alpha} \le \frac{1}{2R}\big(|\beta_0^a|^2(R|a|)^{2\alpha} + |\beta_1^a|^2(R|a|)^{2(1-\alpha)}\big). \tag{3-10}$$

Combining (3-9) and (3-10) we obtain

$$\frac{|\beta_0^a||\beta_1^a||a|}{\widetilde{H}(\varphi_n^a, R|a|)} \le \frac{1}{R}. \tag{3-11}$$

Moreover, Lemma 2.4 implies

$$|M_n^a| \le 2\alpha(1-\alpha)|\beta_0^a||\beta_1^a||a|. \tag{3-12}$$

Lemma 2.9 provides some constant c_n (independent of a and R) such that, for a possibly smaller σ_n and for all $R > 2$ and $a \in \Omega$ such that $0 < |a| < \sigma_n/R$,

$$\big|H(\varphi_n^a, R|a|) - \widetilde{H}(\varphi_n^a, R|a|)\big| \le \frac{c_n}{R-2}\widetilde{H}(\varphi_n^a, R|a|). \tag{3-13}$$

Therefore, by combining (3-11)–(3-13), we obtain

$$\frac{|M_n^a|}{H(\varphi_n^a, R|a|)} \le \frac{2\alpha(1-\alpha)}{R}\left(1 + \frac{H(\varphi_n^a, R|a|) - \widetilde{H}(\varphi_n^a, R|a|)}{\widetilde{H}(\varphi_n^a, R|a|)}\right)^{-1} \le \frac{2\alpha(1-\alpha)}{R}\frac{1}{1-\frac{c_n}{R-2}} \le \frac{2\alpha(1-\alpha)}{R-(2+c_n)}$$

for all $R > c_n + 2$ and $a \in \Omega$ such that $0 < |a| < \sigma_n/R$.

The conclusion then follows by repeating the argument for all $n \in \{1, 2, \dots, n_0\}$ and choosing

$$\sigma_0 = \min\{\sigma_n : 1 \le n \le n_0\} \quad \text{and} \quad c_0 = \max\{2 + c_n : 1 \le n \le n_0\}. \qquad \square$$

Lemma 3.2. *For $\delta \in (0, \sqrt{\mu_1}/2)$, let Υ_δ be such that* (3-2) *holds. Let R_0 be as above, $r_0 \le R_0$ and $n \in \{1, \dots, n_0\}$. If $\Upsilon_\delta|a| \le r_1 < r_2 \le r_0$ and φ_n^a is a solution to* (2-4) *satisfying* (3-3), *then*

$$\frac{H(\varphi_n^a, r_2)}{H(\varphi_n^a, r_1)} \ge e^{-\Lambda(2+\sqrt{\mu_1})r_0^2}\left(\frac{r_2}{r_1}\right)^{2(\sqrt{\mu_1}-\delta)}.$$

Proof. Combining (3-1) with (3-2) we obtain that, for every $\Upsilon_\delta|a| < r < r_0$,

$$\begin{aligned}\frac{1}{r^2}\int_{D_r}|\varphi_n^a|^2\,dx &\le \left(1 + \frac{1}{\sqrt{\mu_1}-\delta}\right)\int_{D_r}|(i\nabla + A_a)\varphi_n^a|^2\,dx \\ &\le \left(1 + \frac{2}{\sqrt{\mu_1}}\right)\int_{D_r}|(i\nabla + A_a)\varphi_n^a|^2\,dx.\end{aligned}$$

From above, (3-6) and (3-2), we have that, for every $\Upsilon_\delta|a| < r < r_0$,

$$\begin{aligned}\frac{d}{dr}H(\varphi_n^a, r) &\ge \frac{2}{r}\left(1 - \Lambda r^2\left(1 + \frac{2}{\sqrt{\mu_1}}\right)\right)\int_{D_r}|(i\nabla + A_a)\varphi_n^a|^2\,dx \\ &\ge \frac{2}{r}\left(1 - \Lambda r^2\left(1 + \frac{2}{\sqrt{\mu_1}}\right)\right)(\sqrt{\mu_1}-\delta)H(\varphi_n^a, r),\end{aligned}$$

so that, in view of (3-4),

$$\frac{d}{dr}\log H(\varphi_n^a, r) \ge \frac{2}{r}(\sqrt{\mu_1}-\delta) - 2\Lambda r(2+\sqrt{\mu_1}).$$

Integrating between r_1 and r_2 we obtain the desired inequality. □

Lemma 3.3. *For $n \in \{1,\dots,n_0\}$ and $a \in \Omega$, let φ_n^a be a solution of* (2-4) *satisfying* (3-3). *Let R_0 be as above, σ_0 and $c_0 > 0$ be as in Lemma 3.1 and let $r_0 \le \min\{R_0, \sigma_0\}$. For $\delta \in (0, \sqrt{\mu_1}/2)$, let $\Upsilon_\delta > 1$ be such that* (3-2) *holds. Then, there exists $c_{r_0,\delta} > 0$ such that for all $R > \max\{\Upsilon_\delta, c_0\}$, $|a| < r_0/R$, $R|a| \le r < r_0$ and $n \in \{1,\dots,n_0\}$,*

$$e^{\Lambda r^2/(1-\Lambda r_0^2)}(\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a)+1) \le e^{\Lambda r_0^2/(1-\Lambda r_0^2)}(\mathcal{N}(\varphi_n^a, r_0, \lambda_n^a, A_a)+1) + \frac{c_{r_0,\delta}}{R-c_0}.$$

Proof. By direct calculations, using the expressions for the derivatives of the functions $H(\varphi_n^a, r)$ and $E(\varphi_n^a, r, \lambda_n^a, A_a)$ written in (3-6) and (3-7) and the Cauchy–Schwarz inequality, we obtain

$$\frac{d}{dr}\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a) \ge -\frac{2|M_n^a|}{rH(\varphi_n^a, r)} - \frac{2\lambda_n^a}{rH(\varphi_n^a, r)}\int_{D_r}|\varphi_n^a|^2\,dx. \tag{3-14}$$

By Lemmas 3.2 and 3.1 the first term can be estimated as

$$\begin{aligned}-\frac{2|M_n^a|}{rH(\varphi_n^a, r)} &= -\frac{2|M_n^a|}{rH(\varphi_n^a, R|a|)}\frac{H(\varphi_n^a, R|a|)}{H(\varphi_n^a, r)}\\ &\ge -\frac{4\alpha(1-\alpha)}{R-c_0}e^{\Lambda(2+\sqrt{\mu_1})r_0^2}(R|a|)^{2(\sqrt{\mu_1}-\delta)}r^{-2(\sqrt{\mu_1}-\delta)-1}\end{aligned} \tag{3-15}$$

for all $R > \max\{\Upsilon_\delta, c_0\}$, $|a| < r_0/R$, $R|a| \le r < r_0$ and $n \in \{1,\dots,n_0\}$.

For the second term, the Poincaré inequality (3-1) leads to

$$\frac{1-\Lambda r_0^2}{r^2}\int_{D_r}|\varphi_n^a|^2\,dx \le E(\varphi_n^a, r, \lambda_n^a, A_a) + H(\varphi_n^a, r)$$

for $r < r_0$, which implies

$$-\frac{2r\lambda_n^a}{r^2H(\varphi_n^a, r)}\int_{D_r}|\varphi_n^a|^2\,dx \ge -\frac{2\Lambda r}{1-\Lambda r_0^2}(\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a)+1) \tag{3-16}$$

for $r < r_0$. Using (3-15) and (3-16) we can estimate the right-hand side of (3-14) thus obtaining

$$\begin{aligned}&\frac{d}{dr}(e^{\Lambda r^2/(1-\Lambda r_0^2)}(\mathcal{N}(\varphi_n^a, r, \lambda_n^a, A_a)+1))\\ &\qquad\ge -\frac{4\alpha(1-\alpha)}{R-c_0}e^{\Lambda r_0^2/(1-\Lambda r_0^2)}e^{\Lambda(2+\sqrt{\mu_1})r_0^2}(R|a|)^{2(\sqrt{\mu_1}-\delta)}r^{-2(\sqrt{\mu_1}-\delta)-1}\end{aligned}$$

for all $R|a| \le r < r_0$ with $R > \max\{\Upsilon_\delta, c_0\}$. Integrating between r and r_0 and using the fact that $R|a| \le r < r_0$, we obtain the statement with

$$c_{r_0,\delta} = \frac{2\alpha(1-\alpha)}{\sqrt{\mu_1}-\delta}e^{\Lambda(2+\sqrt{\mu_1})r_0^2+\Lambda r_0^2/(1-\Lambda r_0^2)}.$$ □

Lemma 3.4. *Let φ_a be a solution of* (1-14)–(1-15) *and let k be as in* (1-8)*. For every $\delta \in (0, \sqrt{\mu_1}/2)$, there exist $r_\delta \in (0, R_0)$ and $K_\delta > \Upsilon_\delta$ such that if $R > K_\delta$, $|a| < r_\delta/R$ and $R|a| \le r < r_\delta$, then*

$$\mathcal{N}(\varphi_a, r, \lambda_a, A_a) \le |\alpha - k| + \delta.$$

Proof. From (1-16)–(1-17) it follows that, for every $r < R_0$,

$$\lim_{a\to 0} \mathcal{N}(\varphi_a, r, \lambda_a, A_a) = \mathcal{N}(\varphi_0, r, \lambda_0, A_0).$$

Moreover, from [Felli et al. 2011, Theorem 1.3] we know that, under assumption (1-9),

$$\lim_{r\to 0^+} \mathcal{N}(\varphi_0, r, \lambda_0, A_0) = |\alpha - k|.$$

Then, the proof is a direct consequence of Lemma 3.3; see [Noris et al. 2015, Lemma 7.2; Abatangelo and Felli 2015, Lemma 5.7; Abatangelo et al. 2017, Lemma 5.7] for details. □

3B. *Local energy estimates.*

Corollary 3.5. *For $\delta \in (0, \sqrt{\mu_1}/2)$ let r_δ, K_δ be as in Lemma 3.4 and α_{r_δ} be as in* (3-5)*. Then there exists $C_\delta > 0$ such that*

$$H(\varphi_a, R|a|) \le H(\varphi_a, K_\delta|a|)\left(\frac{R}{K_\delta}\right)^{2(|\alpha-k|+\delta)} \quad \textit{for all } R > K_\delta \textit{ and } |a| < \frac{r_\delta}{R}, \tag{3-17}$$

$$H(\varphi_a, K_\delta|a|) \ge C_\delta |a|^{2(|\alpha-k|+\delta)} \quad \textit{for all } |a| < \min\left\{\frac{r_\delta}{K_\delta}, \alpha_{r_\delta}\right\}, \tag{3-18}$$

$$H(\varphi_a, K_\delta|a|) = O(|a|^{2(\sqrt{\mu_1}-\delta)}) \quad \textit{as } a \to 0. \tag{3-19}$$

Proof. From (3-6), the definition of $\mathcal{N}$, and Lemma 3.4 we have

$$\begin{aligned}\frac{1}{H(\varphi_a, r)}\frac{d}{dr}H(\varphi_a, r) &= \frac{2}{r}\mathcal{N}(\varphi_a, r, \lambda_a, A_a)\\ &\le \frac{2}{r}(|\alpha-k|+\delta) \quad \text{for all } K_\delta|a| \le r < r_\delta \text{ with } |a| < \frac{r_\delta}{K_\delta}\end{aligned}$$

so that estimate (3-17) follows by integration over $[K_\delta|a|, R|a|]$ and estimate (3-18) from integration over $[K_\delta|a|, r_\delta]$ and (3-5). Finally (3-19) is a direct consequence of Lemma 3.2. □

Lemma 3.6. *For $n \in \{1, \dots, n_0\}$ and $a \in \Omega$, let φ_n^a be a solution to* (2-4) *satisfying* (3-3)*. Let $R_0 > 0$ be as in* (3-4)*. For every $\delta \in (0, \sqrt{\mu_1}/2)$, there exist $\widetilde{K}_\delta > 1$ and $\widetilde{C}_\delta > 0$ such that, for all $R > \widetilde{K}_\delta$, $a \in \Omega$ with $R|a| < R_0$, and $n \in \{1, \dots, n_0\}$,*

$$\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_n^a|^2\,dx \le \widetilde{C}_\delta (R|a|)^{2(\sqrt{\mu_1}-\delta)}, \tag{3-20}$$

$$\int_{\partial D_{R|a|}} |\varphi_n^a|^2\,ds \le \widetilde{C}_\delta (R|a|)^{2(\sqrt{\mu_1}-\delta)+1}, \tag{3-21}$$

$$\int_{D_{R|a|}} |\varphi_n^a|^2\,dx \le \widetilde{C}_\delta (R|a|)^{2(\sqrt{\mu_1}-\delta)+2}. \tag{3-22}$$

Proof. By Lemma 3.2 (choosing $r_1 = R|a|$ and $r_2 = R_0$) and the definition of H it follows that

$$\int_{\partial D_{R|a|}} |\varphi_n^a|^2\, ds = R|a| H(\varphi_n^a, R|a|) \le R|a| H(\varphi_n^a, R_0) e^{\Lambda(2+\sqrt{\mu_1})R_0^2} \left(\frac{R|a|}{R_0}\right)^{2(\sqrt{\mu_1}-\delta)}. \tag{3-23}$$

Moreover, from (2-7) and continuous trace embeddings we have $H(\varphi_n^a, R_0) = (1/R_0)\int_{\partial D_{R_0}} |\varphi_n^a|^2\, ds$ is bounded uniformly with respect to a. Hence estimate (3-23) implies (3-21).

From Lemma 3.3 it follows that the frequency $\mathcal{N}$ is bounded in $r = R|a|$ provided R is sufficiently large; hence $E(\varphi_n^a, R|a|, \lambda_n^a, A_a)$ is uniformly estimated by $H(\varphi_n^a, R|a|)$, so that (3-21) and (3-1)–(3-2) yield (3-20). Estimate (3-22) can be proved combining (3-20)–(3-21) with the Poincaré inequality (3-1). We refer to [Abatangelo and Felli 2015, Lemma 5.8] for more details in a related problem. □

Lemma 3.7. *For $a \in \Omega$ let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ be a solution of* (1-14)–(1-15). *For some fixed $\delta \in (0, \sqrt{\mu_1}/2)$, let $K_\delta > \Upsilon_\delta$ be as in Lemma 3.4. Then, for every $R > K_\delta$,*

$$\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx = O(H(\varphi_a, K_\delta|a|)) \qquad \text{as } |a| \to 0^+, \tag{3-24}$$

$$\int_{\partial D_{R|a|}} |\varphi_a|^2\, ds = O(|a| H(\varphi_a, K_\delta|a|)) \qquad \text{as } |a| \to 0^+, \tag{3-25}$$

$$\int_{D_{R|a|}} |\varphi_a|^2\, dx = O(|a|^2 H(\varphi_a, K_\delta|a|)) \qquad \text{as } |a| \to 0^+. \tag{3-26}$$

Proof. The proof follows from the boundedness of the frequency $\mathcal{N}(\varphi_a, R|a|, \lambda_a, A_a)$ established in Lemma 3.4 and by its scaling properties. For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let $K_\delta > \Upsilon_\delta$ and r_δ be as in Lemma 3.4; hence

$$\begin{aligned} N(\varphi_a, R|a|, \lambda_a, A_a) &= \frac{\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx - \lambda_a \int_{D_{R|a|}} |\varphi_a|^2\, dx}{H(\varphi_a, R|a|)} \\ &\le |\alpha - k| + \delta \quad \text{for all } R > K_\delta \text{ and } |a| < \frac{r_\delta}{R}. \end{aligned}$$

Then, by (3-1) and (3-2) it follows that

$$\begin{aligned} \left(1 - \Lambda r_\delta^2\left(1 + \frac{2}{\sqrt{\mu_1}}\right)\right) \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx &\le \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx - \lambda_a \int_{D_{R|a|}} |\varphi_a|^2\, dx \\ &\le H(\varphi_a, R|a|)(|\alpha - k| + \delta). \end{aligned}$$

Then (3-24) follows from (3-17). Estimates (3-25) and (3-26) follow from (3-24) and the Poincaré-type inequalities (3-1) and (3-2). □

Remark 3.8. Let us consider the blow-up family

$$\tilde{\varphi}_a(x) := \frac{\varphi_a(|a|x)}{\sqrt{H(\varphi_a, K_\delta|a|)}}, \tag{3-27}$$

with $K_\delta > \Upsilon_\delta$ as in Lemma 3.4 for some fixed $\delta \in (0, \sqrt{\mu_1}/2)$. By Lemma 3.7 it follows that, for every $p \in \mathbb{S}^1$ fixed, $r_\delta > 0$ as in Lemma 3.4, and $R > K_\delta$, the blow-up family $\{\tilde{\varphi}_a : a = |a|p,\ R|a| < r_\delta\}$ is bounded in $H^{1,p}(D_R, \mathbb{C})$.

4. Estimate on $\lambda_0 - \lambda_a$

The aim of this section is to obtain a bound (both from above and from below) of the eigenvalue variation $\lambda_a - \lambda_0$. These bounds are obtained by considering suitable competitor functions and by plugging them into the Courant–Fischer characterization of λ_a and λ_0:

$$\lambda_a = \min\left\{ \max_{u\in F\setminus\{0\}} \frac{\int_\Omega |(i\nabla + A_a)u|^2\,dx}{\int_\Omega |u|^2\,dx} : F \text{ is a linear subspace of } H_0^{1,a}(\Omega,\mathbb{C}),\ \dim F = n_0 \right\}, \quad (4\text{-}1)$$

$$\lambda_0 = \min\left\{ \max_{u\in F\setminus\{0\}} \frac{\int_\Omega |(i\nabla + A_0)u|^2\,dx}{\int_\Omega |u|^2\,dx} : F \text{ is a linear subspace of } H_0^{1,0}(\Omega,\mathbb{C}),\ \dim F = n_0 \right\}. \quad (4\text{-}2)$$

In Section 4A we construct the competitor function for λ_a. This function is obtained by modifying φ_n^0 in a small neighborhood of a. Since the asymptotics of φ_n^0 is exactly known, this allows us to obtain, in Section 4B, a sharp bound from below of $\lambda_0 - \lambda_a$. The competitor function for λ_0 is constructed in Section 4C, by modifying locally φ_n^a. The energy estimates obtained in Section 3 allow us to obtain a preliminary estimate from above of $\lambda_0 - \lambda_a$ in terms of the quantity $H(\varphi_a, K_\delta|a|)$.

Before proceeding, we find it useful to recall the following technical result, which is proved in [Abatangelo and Felli 2015, Lemma 6.1] and concerns the maximum of quadratic forms depending on the pole $a \to 0$.

Lemma 4.1. *For every $a \in \Omega$, let us consider a quadratic form*

$$Q_a : \mathbb{C}^{n_0} \to \mathbb{R}, \quad Q_a(z_1, z_2, \dots, z_{n_0}) = \sum_{j,n=1}^{n_0} M_{j,n}(a) z_j \bar{z}_n,$$

with $M_{j,n}(a) \in \mathbb{C}$ such that $M_{j,n}(a) = \overline{M_{n,j}(a)}$. Let us assume that there exist $\gamma \in (0, +\infty)$, $a \mapsto \sigma(a) \in \mathbb{R}$ with $\sigma(a) \geq 0$ and $\sigma(a) = O(|a|^{2\gamma})$ as $|a| \to 0^+$, and $a \mapsto \mu(a) \in \mathbb{R}$ with $\mu(a) = O(1)$ as $|a| \to 0^+$, such that the coefficients $M_{j,n}(a)$ satisfy the following conditions:

(i) $M_{n_0,n_0}(a) = \sigma(a)\mu(a)$.

(ii) *For all $j < n_0$, we have $M_{j,j}(a) \to M_j$ as $|a| \to 0^+$ for some $M_j \in \mathbb{R}$, $M_j < 0$.*

(iii) *For all $j < n_0$, we have $M_{j,n_0}(a) = \overline{M_{n_0,j}(a)} = O(|a|^\gamma \sqrt{\sigma(a)})$ as $|a| \to 0^+$.*

(iv) *For all $j, n < n_0$ with $j \neq n$, we have $M_{j,n}(a) = O(|a|^{2\gamma})$ as $|a| \to 0^+$.*

(v) *There exists $M \in \mathbb{N}$ such that $|a|^{(2+M)\gamma} = o(\sigma(a))$ as $|a| \to 0^+$.*

Then

$$\max_{\substack{z\in\mathbb{C}^{n_0}\\ \|z\|=1}} Q_a(z) = \sigma(a)(\mu(a) + o(1)) \quad \textit{as } |a| \to 0^+,$$

where $\|z\| = \|(z_1, z_2, \dots, z_{n_0})\| = \left(\sum_{j=1}^{n_0} |z_j|^2\right)^{1/2}$.

4A. ***Construction of the test functions using φ_n^0.*** Recall that $\varphi_n^0 \in H_0^{1,0}(\Omega, \mathbb{C}) \setminus \{0\}$ is a solution of (2-4), also satisfying (2-5), with $a = 0$. Let R_0 be as in (3-4). For every $R > 1$, $a \in \Omega$ with $|a| < R_0/R$ and $1 \le n \le n_0$ we define

$$w_{n,R,a} = \begin{cases} w_{n,R,a}^{\text{int}} & \text{in } D_{R|a|}, \\ w_{n,R,a}^{\text{ext}} & \text{in } \Omega \setminus D_{R|a|}, \end{cases} \tag{4-3}$$

where

$$w_{n,R,a}^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 \quad \text{in } \Omega \setminus D_{R|a|},$$

and $w_{n,R,a}^{\text{int}}$ is the unique solution to the minimization problem

$$\min\left\{ \int_{D_{R|a|}} |(i\nabla + A_a)u|^2 \, dx : u \in H^{1,a}(D_{R|a|}, \mathbb{C}),\ u = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 \text{ on } \partial D_{R|a|} \right\}.$$

We notice that $w_{n,R,a}^{\text{ext}}$ and $w_{n,R,a}^{\text{int}}$ respectively solve

$$\begin{cases} (i\nabla + A_a)^2 w_{n,R,a}^{\text{ext}} = \lambda_n^0 w_{n,R,a}^{\text{ext}} & \text{in } \Omega \setminus D_{R|a|}, \\ w_{n,R,a}^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 & \text{on } \partial(\Omega \setminus D_{R|a|}) \end{cases} \quad \text{and} \quad \begin{cases} (i\nabla + A_a)^2 w_{n,R,a}^{\text{int}} = 0 & \text{in } D_{R|a|}, \\ w_{n,R,a}^{\text{int}} = e^{i\alpha(\theta_a - \theta_0^a)} \varphi_n^0 & \text{on } \partial D_{R|a|}. \end{cases}$$

As a consequence of Proposition 2.1 we have $\varphi_n^0(x) = O(|x|^{|\alpha - j|})$ as $x \to 0$ for some $j \in \mathbb{Z}$, which implies

$$\varphi_n^0(x) = O(|x|^{\sqrt{\mu_1}}) \quad \text{as } x \to 0, \tag{4-4}$$

since $|\alpha - j| \ge \sqrt{\mu_1}$ for all $j \in \mathbb{Z}$. Furthermore (2-2) implies

$$\begin{aligned} \int_{D_r} |(i\nabla + A_0)\varphi_n^0|^2 \, dx &= \lambda_0 \int_{D_r} |\varphi_n^0|^2 \, dx + \frac{|\alpha - j| + o(1)}{r} \int_{\partial D_r} |\varphi_n^0|^2 \, ds \\ &= O(r^{2\sqrt{\mu_1}}) \quad \text{as } r \to 0^+. \end{aligned} \tag{4-5}$$

From (4-4) and (4-5) we deduce that, for every $R > 1$, $a \in \Omega$ such that $R|a| < R_0$, and $1 \le n \le n_0$,

$$\begin{gathered} \int_{D_{R|a|}} |(i\nabla + A_0)\varphi_n^0|^2 \, dx = O(|a|^{2\sqrt{\mu_1}}), \quad \int_{\partial D_{R|a|}} |\varphi_n^0|^2 \, ds = O(|a|^{2\sqrt{\mu_1}+1}), \\ \text{and} \quad \int_{D_{R|a|}} |\varphi_n^0|^2 \, dx = O(|a|^{2\sqrt{\mu_1}+2}) \quad \text{as } |a| \to 0^+. \end{gathered} \tag{4-6}$$

Using the above estimates (4-6) and the Dirichlet principle (see the proof of [Abatangelo and Felli 2015, Lemma 6.2] for details in the case of half-integer circulation), we obtain that, for every $R > 2$ and $1 \le n \le n_0$,

$$\begin{gathered} \int_{D_{R|a|}} |(i\nabla + A_a) w_{n,R,a}^{\text{int}}|^2 \, dx = O(|a|^{2\sqrt{\mu_1}}), \quad \int_{\partial D_{R|a|}} |w_{n,R,a}^{\text{int}}|^2 \, ds = O(|a|^{2\sqrt{\mu_1}+1}), \\ \text{and} \quad \int_{D_{R|a|}} |w_{n,R,a}^{\text{int}}|^2 \, dx = O(|a|^{2\sqrt{\mu_1}+2}) \quad \text{as } |a| \to 0^+. \end{gathered} \tag{4-7}$$

The above estimates can be made more precise in the case $n = n_0$ in view of (1-9): for every $R > 2$ and $a \in \Omega$ with $R|a| < R_0$,

$$\int_{D_{R|a|}} |(i\nabla + A_0)\varphi_0|^2\,dx = O(|a|^{2|\alpha-k|}), \quad \int_{\partial D_{R|a|}} |\varphi_0|^2\,ds = O(|a|^{2|\alpha-k|+1}),$$
$$\text{and} \quad \int_{D_{R|a|}} |\varphi_0|^2\,dx = O(|a|^{2|\alpha-k|+2}) \quad \text{as } |a| \to 0^+, \tag{4-8}$$

and consequently, in view of the Dirichlet principle,

$$\int_{D_{R|a|}} |(i\nabla + A_a)w^{\text{int}}_{n_0,R,a}|^2\,dx = O(|a|^{2|\alpha-k|}), \quad \int_{\partial D_{R|a|}} |w^{\text{int}}_{n_0,R,a}|^2\,ds = O(|a|^{2|\alpha-k|+1}),$$
$$\text{and} \quad \int_{D_{R|a|}} |w^{\text{int}}_{n_0,R,a}|^2\,dx = O(|a|^{2|\alpha-k|+2}) \quad \text{as } |a| \to 0^+, \tag{4-9}$$

with k as in (1-8). Furthermore, defining

$$W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{|\alpha-k|}} \tag{4-10}$$

for all $R > 2$ and $a \in \Omega$ such that $R|a| < R_0$, (1-9) implies

$$W_a \to \beta\psi_k \quad \text{in } H^{1,0}(D_R, \mathbb{C}), \text{ as } |a| \to 0, \tag{4-11}$$

where ψ_k is defined in (1-20).

4B. *Estimate of the Rayleigh quotient for λ_a.*

Lemma 4.2. *There exists $\mathfrak{c} \in \mathbb{R}$ such that*

$$\lambda_0 - \lambda_a \geq \mathfrak{c}|a|^{2|\alpha-k|} \quad \textit{for all } a \in \Omega,$$

where k is as in (1-8).

Proof. The proof follows along the lines of [Abatangelo and Felli 2015, Lemma 6.7; Abatangelo et al. 2017, Lemma 7.2]. Let $w_{n,R,a}$ be defined in (4-3). Let us fix $R > 2$. By proceeding with a Gram–Schmidt process we define

$$\tilde{w}_{n,a} = \frac{\hat{w}_{n,a}}{\|\hat{w}_{n,a}\|_{L^2(\Omega,\mathbb{C})}}, \quad 1 \leq n \leq n_0,$$

where

$$\hat{w}_{n_0,a} = w_{n_0,R,a},$$
$$\hat{w}_{n,a} = w_{n,R,a} - \sum_{\ell=n+1}^{n_0} c^a_{\ell,n}\hat{w}_{\ell,a}, \quad 1 \leq n \leq n_0 - 1,$$

and

$$c^a_{\ell,n} = \frac{\int_\Omega w_{n,R,a}\bar{\hat{w}}_{\ell,a}\,dx}{\|\hat{w}_{\ell,a}\|^2_{L^2(\Omega,\mathbb{C})}}, \quad 1 \leq n \leq n_0 - 1, \;\; n+1 \leq \ell \leq n_0.$$

From (4-6), (4-7) and an induction argument it follows that, for all ℓ, n such that $1 \le n \le n_0 - 1$ and $n+1 \le \ell \le n_0$,

$$\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{2\sqrt{\mu_1}+2}) \quad \text{and} \quad c^a_{\ell,n} = O(|a|^{2\sqrt{\mu_1}+2}) \tag{4-12}$$

as $|a| \to 0$. Moreover, from (4-8) and (4-9) we have

$$\|\hat{w}_{n_0,a}\|^2_{L^2(\Omega,\mathbb{C})} = \|w_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{2|\alpha-k|+2}) \quad \text{as } |a| \to 0, \tag{4-13}$$

and

$$c^a_{n_0,n} = O(|a|^{|\alpha-k|+\sqrt{\mu_1}+2}) \quad \text{as } |a| \to 0, \text{ for } 1 \le n \le n_0 - 1. \tag{4-14}$$

Since $\dim(\mathrm{span}\{w_{1,R,a}, \dots, w_{n_0,R,a}\}) = n_0$, we have that also $\dim(\mathrm{span}\{\tilde{w}_{1,a}, \dots, \tilde{w}_{n_0,a}\}) = n_0$, and hence from (4-1) we deduce that

$$\lambda_a \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0} |\alpha_n|^2 = 1}} \int_\Omega \left| (i\nabla + A_a)\left(\sum_{n=1}^{n_0} \alpha_n \tilde{w}_{n,a}\right)\right|^2 dx,$$

which leads to

$$\lambda_a - \lambda_0 \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0} |\alpha_n|^2 = 1}} \sum_{n,j=1}^{n_0} \alpha_n \bar{\alpha}_j \, p^a_{n,j}, \tag{4-15}$$

where $p^a_{n,j} = \int_\Omega (i\nabla + A_a)\tilde{w}_{n,a} \cdot \overline{(i\nabla + A_a)\tilde{w}_{j,a}} \, dx - \lambda_0 \delta_{nj}$, with $\delta_{nj} = 1$ if $n = j$ and $\delta_{nj} = 0$ otherwise. Using the estimates above we can now estimate $p^a_{n,j}$. First, using (4-8), (4-9), and (4-13)

$$\begin{aligned} p^a_{n_0,n_0} &= \frac{\lambda_0}{\int_\Omega |w_{n_0,R,a}|^2 \, dx}\left(1 - \int_\Omega |w_{n_0,R,a}|^2 \, dx\right) \\ &\quad + \frac{1}{\int_\Omega |w_{n_0,R,a}|^2 \, dx}\left(\int_{D_{R|a|}} |(i\nabla + A_a) w^{\mathrm{int}}_{n_0,R,a}|^2 \, dx - \int_{D_{R|a|}} |(i\nabla + A_0)\varphi_0|^2 \, dx\right) \\ &= O(|a|^{2|\alpha-k|+2}) + O(|a|^{2|\alpha-k|}) \\ &= |a|^{2|\alpha-k|} O(1) \quad \text{as } |a| \to 0^+. \end{aligned}$$

Next (4-6), (4-7) and (4-12) provide for $n < n_0$

$$\begin{aligned} p^a_{n,n} &= -\lambda_0 + \frac{1}{\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}}\left(\lambda^0_n + \int_{D_{R|a|}} |(i\nabla + A_a) w^{\mathrm{int}}_{n,R,a}|^2 \, dx - \int_{D_{R|a|}} |(i\nabla + A_0)\varphi^0_n|^2 \, dx\right) \\ &\quad + \frac{1}{\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \int_\Omega \left|(i\nabla + A_a)\left(\sum_{\ell=n+1}^{n_0} c^a_{\ell,n} \hat{w}_{\ell,a}\right)\right|^2 dx \\ &\quad - \frac{2}{\|\hat{w}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \mathfrak{Re} \sum_{\ell=n+1}^{n_0} \left\{ \bar{c}^a_{\ell,n} \int_\Omega (i\nabla + A_a) w_{n,R,a} \cdot \overline{(i\nabla + A_a)\hat{w}_{\ell,a}} \, dx \right\} \\ &= (\lambda^0_n - \lambda_0) + o(1), \end{aligned}$$

as $|a| \to 0$. Using (4-6), (4-7), (4-8), (4-9), (4-12) and (4-14), we have that, for all $n < n_0$,

$$p^a_{n,n_0} = \bar{p}^a_{n_0,n} = O(|a|^{\sqrt{\mu_1}+|\alpha-k|}) \quad \text{as } |a| \to 0,$$

while the same estimates imply that, for all $n \neq \ell < n_0$,

$$p^a_{n,\ell} = \bar{p}^a_{\ell,n} = O(|a|^{2\sqrt{\mu_1}}) \quad \text{as } |a| \to 0.$$

Therefore, the quadratic form in (4-15) satisfies the hypothesis of Lemma 4.1 with $\sigma(a) = |a|^{2|\alpha-k|}$, $\gamma = \sqrt{\mu_1}$, $M_j = \lambda^0_j - \lambda_0 < 0$ for $j < n_0$ and $M \in \mathbb{N}$ such that $(2+M)\sqrt{\mu_1} > 2|\alpha - k|$, so that

$$\max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0}\\ \sum_{n=1}^{n_0}|\alpha_n|^2=1}} \sum_{n,j=1}^{n_0} \alpha_n\bar{\alpha}_j\, p^a_{n,j} = |a|^{2|\alpha-k|}O(1) \quad \text{as } |a| \to 0. \qquad \square$$

We notice that Lemma 4.2 does not give any information about the sign of the constant $\mathfrak{c}$.

4C. *Construction of the test functions using φ^a_n.* Let $\varphi^a_n \in H^{1,a}_0(\Omega,\mathbb{C}) \setminus \{0\}$ satisfy (2-4) and (2-5). Let R_0 be as in (3-4), $R > 1$ and $|a| < R_0/R$. For every $1 \le n \le n_0$ we define

$$v_{n,R,a} = \begin{cases} v^{\text{int}}_{n,R,a} & \text{in } D_{R|a|},\\ v^{\text{ext}}_{n,R,a} & \text{in } \Omega \setminus D_{R|a|}, \end{cases}$$

where

$$v^{\text{ext}}_{n,R,a} = e^{i\alpha(\theta^a_0-\theta_a)}\varphi^a_n \quad \text{in } \Omega \setminus D_{R|a|},$$

and $v^{\text{int}}_{n,R,a}$ is the unique solution to the minimization problem

$$\min\left\{\int_{D_{R|a|}} |(i\nabla + A_0)u|^2\,dx : u \in H^{1,0}(D_{R|a|},\mathbb{C}),\ u = e^{i\alpha(\theta^a_0-\theta_a)}\varphi^a_n \text{ on } \partial D_{R|a|}\right\}. \tag{4-16}$$

We notice that $v^{\text{ext}}_{n,R,a}$ and $v^{\text{int}}_{n,R,a}$ respectively solve

$$\begin{cases} (i\nabla + A_0)^2 v^{\text{ext}}_{n,R,a} = \lambda^a_n v^{\text{ext}}_{n,R,a} & \text{in } \Omega \setminus D_{R|a|},\\ v^{\text{ext}}_{n,R,a} = e^{-i\alpha(\theta_a-\theta^a_0)}\varphi^a_n & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

and

$$\begin{cases} (i\nabla + A_0)^2 v^{\text{int}}_{n,R,a} = 0 & \text{in } D_{R|a|},\\ v^{\text{int}}_{n,R,a} = e^{-i\alpha(\theta_a-\theta^a_0)}\varphi^a_n & \text{on } \partial D_{R|a|}. \end{cases} \tag{4-17}$$

The energy estimates in Lemmas 3.6 and 3.7 imply the following estimates for the functions $v^{\text{int}}_{n,R,a}$.

Lemma 4.3. *For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let $\widetilde{K}_\delta$ be as in Lemma 3.6 and R_0 be as in* (3-4). *Let*

$$R > \max\{2, \widetilde{K}_\delta\}$$

and $1 \le n \le n_0$ be fixed. For every $a \in \Omega$ with $|a| < R_0/R$, let $v^{\text{int}}_{n,R,a}$ be defined as in (4-16). *Then*

$$\int_{D_{R|a|}} |(i\nabla + A_0)v^{\text{int}}_{n,R,a}|^2\,dx = O(|a|^{2(\sqrt{\mu_1}-\delta)})$$

$$\int_{D_{R|a|}} |v^{\text{int}}_{n,R,a}|^2\,dx = O(|a|^{2(\sqrt{\mu_1}-\delta)+2}) \quad \text{and} \quad \int_{\partial D_{R|a|}} |v^{\text{int}}_{n,R,a}|^2\,ds = O(|a|^{2(\sqrt{\mu_1}-\delta)+1}) \tag{4-18}$$

as $|a| \to 0^+$.

Proof. The proof follows by combining the Dirichlet principle, a suitable cutting-off procedure, and Lemma 3.6 (see the proof of [Abatangelo and Felli 2015, Lemma 6.2] for details in the case of half-integer circulation). □

Lemma 4.4. *For $R > \max\{2, K_\delta\}$ fixed, with K_δ as in Lemma 3.4, let $v^{\mathrm{int}}_{n_0,R,a}$ be defined as in* (4-16). *Then*

$$\int_{D_{R|a|}} |(i\nabla + A_0)v^{\mathrm{int}}_{n_0,R,a}|^2\,dx = O(H(\varphi_a, K_\delta|a|)), \tag{4-19}$$

$$\int_{D_{R|a|}} |v^{\mathrm{int}}_{n_0,R,a}|^2\,dx = O(|a|^2 H(\varphi_a, K_\delta|a|)), \quad \int_{\partial D_{R|a|}} |v^{\mathrm{int}}_{n_0,R,a}|^2\,ds = O(|a| H(\varphi_a, K_\delta|a|)), \tag{4-20}$$

as $|a| \to 0^+$.

Proof. The proof follows from the estimates of Lemma 3.7, a suitable cutting-off procedure, and the Dirichlet principle; see (4-16). □

Remark 4.5. For all $R > 2$ and $a \in \Omega$ with $|a| < R_0/R$ we consider the blow-up family

$$Z^R_a(x) := \frac{v^{\mathrm{int}}_{n_0,R,a}(|a|x)}{\sqrt{H(\varphi_a, K_\delta|a|)}}, \tag{4-21}$$

with K_δ as in Lemma 3.4 for some fixed $\delta \in (0, \sqrt{\mu_1}/2)$. From Lemma 4.4 it follows that, for every $p \in \mathbb{S}^1$ fixed, $r_\delta > 0$ as in Lemma 3.4, and $R > \max\{K_\delta, 2\}$, the family of functions

$$\{Z^R_a : a = |a|p \in \Omega,\ |a| < r_\delta/R\}$$

is bounded in $H^{1,0}(D_R, \mathbb{C})$.

4D. *Estimate of the Rayleigh quotient for λ_0.* An estimate from above for the limit eigenvalue λ_0 in terms of the approximating eigenvalue λ_a can be obtained by choosing as test functions in (4-2) an orthonormal family constructed starting from the functions $\{v_{n,R,a}\}_{n=1,\dots,n_0}$, as done in the following.

Lemma 4.6. *For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let r_δ, K_δ be as in Lemma 3.4 and α_{r_δ} be as in* (3-5). *Then there exists $\mathfrak{d}_\delta > 0$ such that*

$$\lambda_0 - \lambda_a \le \mathfrak{d}_\delta H(\varphi_a, K_\delta|a|)$$

for all $a \in \Omega$ such that $|a| < \min\{r_\delta/K_\delta, \alpha_{r_\delta}\}$.

Proof. In view of (3-18) it is enough to prove that $\lambda_0 - \lambda_a \le O(H(\varphi_a, K_\delta|a|))$ as $|a| \to 0^+$.

Recall the definition of $v_{n,R,a}$ given at the beginning of Section 4C. Let us fix $R > \max\{2, K_\delta, \widetilde{K}_\delta\}$, with $\widetilde{K}_\delta$ as in Lemma 3.6. As in the proof of Lemma 4.2, we use a Gram–Schmidt process; that is, we define

$$\tilde{v}_{n,a} = \frac{\hat{v}_{n,a}}{\|\hat{v}_{n,a}\|_{L^2(\Omega,\mathbb{C})}}, \quad 1 \le n \le n_0,$$

where

$$\hat{v}_{n_0,a} = v_{n_0,R,a},$$
$$\hat{v}_{n,a} = v_{n,R,a} - \sum_{\ell=n+1}^{n_0} d^a_{\ell,n} \hat{v}_{\ell,a}, \quad 1 \le n \le n_0 - 1,$$

and

$$d^a_{\ell,n} = \frac{\int_\Omega v_{n,R,a}\bar{\hat{v}}_{\ell,a}\,dx}{\|\hat{v}_{\ell,a}\|^2_{L^2(\Omega,\mathbb{C})}}, \quad 1 \le n \le n_0 - 1, \quad n+1 \le \ell \le n_0.$$

From (3-22), (4-18) and an induction argument it follows that, for every $1 \le n \le n_0 - 1$ and $n+1 \le \ell \le n_0$,

$$\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{2(\sqrt{\mu_1}-\delta)+2}) \quad \text{and} \quad d^a_{\ell,n} = O(|a|^{2(\sqrt{\mu_1}-\delta)+2}) \tag{4-22}$$

as $|a| \to 0$. Moreover, from (3-26) and (4-20), we have

$$\|\hat{v}_{n_0,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^2 H(\varphi_a, K_\delta|a|)) \quad \text{as } |a| \to 0, \tag{4-23}$$

and, for $1 \le n \le n_0 - 1$,

$$d^a_{n_0,n} = O(|a|^{\sqrt{\mu_1}-\delta+2}\sqrt{H(\varphi_a, K_\delta|a|)}) \quad \text{as } |a| \to 0. \tag{4-24}$$

Since $\dim(\mathrm{span}\{v_{1,R,a}, \dots, v_{n_0,R,a}\}) = n_0$, we have that also $\dim(\mathrm{span}\{\tilde{v}_{1,a}, \dots, \tilde{v}_{n_0,a}\}) = n_0$, and hence from (4-2) we deduce that

$$\lambda_0 \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0} |\alpha_n|^2 = 1}} \int_\Omega \left| (i\nabla + A_0)\left(\sum_{n=1}^{n_0} \alpha_n \tilde{v}_{n,a}\right)\right|^2 dx,$$

which leads to

$$\lambda_0 - \lambda_a \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{n=1}^{n_0} |\alpha_n|^2 = 1}} \sum_{n,j=1}^{n_0} \alpha_n \bar{\alpha}_j q^a_{n,j}, \tag{4-25}$$

where $q^a_{n,j} = \int_\Omega (i\nabla + A_0)\tilde{v}_{n,a} \cdot \overline{(i\nabla + A_0)\tilde{v}_{j,a}}\,dx - \lambda_a \delta_{nj}$. Using the results above we can now estimate $q^a_{n,j}$. First, using (4-19), (3-24), and (4-23)

$$\begin{aligned} q^a_{n_0 n_0} &= \frac{\lambda_a}{\int_\Omega |v_{n_0,R,a}|^2\,dx}\left(1 - \int_\Omega |v_{n_0,R,a}|^2\,dx\right) \\ &\qquad + \frac{1}{\int_\Omega |v_{n_0,R,a}|^2\,dx}\left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{\mathrm{int}}_{n_0,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\,dx\right) \\ &= H(\varphi_a, K_\delta|a|)O(1), \end{aligned}$$

as $|a| \to 0^+$. Next (4-18), (3-20), (4-22), and the fact that $\lambda^a_n \to \lambda^0_n$ as $|a| \to 0$, provide, for $n < n_0$,

$$\begin{aligned} q^a_{n,n} &= -\lambda_a + \frac{1}{\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}}\left(\lambda^a_n + \int_{D_{R|a|}} |(i\nabla + A_0)v^{\mathrm{int}}_{n,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi^a_n|^2\,dx\right) \\ &\qquad + \frac{1}{\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \int_\Omega \left|(i\nabla + A_0)\left(\sum_{\ell=n+1}^{n_0} d^a_{\ell,n}\hat{v}_{\ell,a}\right)\right|^2 dx \\ &\qquad - \frac{2}{\|\hat{v}_{n,a}\|^2_{L^2(\Omega,\mathbb{C})}} \mathfrak{Re} \sum_{\ell=n+1}^{n_0} \left\{ \bar{d}^a_{\ell,n} \int_\Omega (i\nabla + A_0)v_{n,R,a} \cdot \overline{(i\nabla + A_0)\hat{v}_{\ell,a}}\,dx \right\} \\ &= \lambda^0_n - \lambda_0 + o(1), \end{aligned}$$

as $|a| \to 0$. Now, using (3-20), (3-24), (4-18), (4-19), (4-22), (4-23), and (4-24), we prove that, for all $n < n_0$,

$$q^a_{n,n_0} = \bar{q}^a_{n_0,n} = O\big(|a|^{\sqrt{\mu_1}-\delta}\sqrt{H(\varphi_a, K_\delta|a|)}\big) \quad \text{as } |a| \to 0^+,$$

while the same estimates imply that, for all $n \neq \ell < n_0$,

$$q^a_{n,\ell} = \bar{q}^a_{\ell,n} = O(|a|^{2(\sqrt{\mu_1}-\delta)}), \quad \text{as } |a| \to 0^+.$$

Therefore, the quadratic form in (4-25) satisfies the hypothesis of Lemma 4.1 with $\gamma = \sqrt{\mu_1} - \delta$, $\sigma(a) = H(\varphi_a, K_\delta|a|) = O(|a|^{2\gamma})$ (by (3-19)), $M_j = \lambda^0_j - \lambda_0 < 0$ and M any natural number such that $M > 2(|\alpha-k| - \sqrt{\mu_1} + 2\delta)/(\sqrt{\mu_1} - \delta)$ by Corollary 3.5. Therefore the right-hand side in (4-25) satisfies

$$\max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0}\\ \sum_{n=1}^{n_0}|\alpha_n|^2=1}} \sum_{n,j=1}^{n_0} \alpha_n\bar{\alpha}_j q^a_{n,j} = H(\varphi_a, K_\delta|a|)O(1),$$

as $|a| \to 0^+$. Then the conclusion follows from (4-25). □

4E. ***Energy estimates.***

Corollary 4.7. *For $\delta \in (0, \sqrt{\mu_1}/2)$ fixed, let K_δ be as in Lemma 3.4. Then*

(i) $|\lambda_0 - \lambda_a| = O(1)\max\{H(\varphi_a, K_\delta|a|), |a|^{2|\alpha-k|}\}$ *as* $a \to 0$;

(ii) $|\lambda_0 - \lambda_a| = O((H(\varphi_a, K_\delta|a|))^{|\alpha-k|/(|\alpha-k|+\delta)})$ *as* $a \to 0$.

Proof. Estimate (i) is a direct consequence of Lemmas 4.2 and 4.6. Corollary 3.5 implies

$$|a|^{2|\alpha-k|} = O((H(\varphi_a, K_\delta|a|))^{|\alpha-k|/(|\alpha-k|+\delta)})$$

as $a \to 0$, so that (ii) follows from (i). □

5. Blow-up analysis

In order to obtain a more precise estimate of the order of vanishing of the eigenvalue variation $|\lambda_0 - \lambda_a|$ than Corollary 4.7, we have now to compare the order of $H(\varphi_a, K_\delta|a|)$ with $|a|^{2|\alpha-k|}$. We observe that the estimates obtained so far (in particular Corollary 3.5) are not enough to decide what is the dominant term among $H(\varphi_a, K_\delta|a|)$ and $|a|^{2|\alpha-k|}$. To this aim, our next step is a blow-up analysis for scaled eigenfunctions (3-27) along a fixed direction $p \in \mathbb{S}^1$. In order to identify the limit profile of the blow-up family (3-27), the following energy estimate of the difference between approximating and limit scaled eigenfunctions plays a crucial role.

Let $\mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$ be the completion of $C^\infty_c(\mathbb{R}^2 \setminus \{0\}, \mathbb{C})$ with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})} := \left(\int_{\mathbb{R}^2} \big|(i\nabla + A_0)u(x)\big|^2\,dx\right)^{1/2}.$$

Theorem 5.1 (energy estimates for eigenfunction variation). *Let $p \in \mathbb{S}^1$ be fixed. For some fixed $\delta \in (0, \sqrt{\mu_1}/2)$, let $K_\delta > \Upsilon_\delta$ be as in Lemma 3.4. For every $R > \max\{2, K_\delta\}$ and $a = |a|p \in \Omega$ such* *that* $|a| < R_0/R$, *let* $v_{n_0,R,a}$ *be as in Section 4C. Then*

$$\|v_{n_0,R,a} - \varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} \le C\big(h(p,a,R) + g(p,a,R)\big)\sqrt{H(\varphi_a, K_\delta|a|)},$$

where $C > 0$ *is independent of* a, R, p,

$$h(p,a,R) = \sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left| \int_{\partial D_R} \big(e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla + A_p)\tilde{\varphi}_a - (i\nabla + A_0)Z_a^R\big)\cdot \nu\,\bar{\varphi}\,d\sigma \right|,$$

and, *for* p *and* R *fixed*,

$$h(p,a,R) = O(1) \quad \textit{and} \quad g(p,a,R) = o(1)$$

as $|a| \to 0^+$.

Proof. The proof exploits the invertibility of the differential of the function F defined below, in the spirit of [Abatangelo et al. 2017, Theorem 8.2; Abatangelo and Felli 2015, Theorem 7.2]. Let

$$F : \mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C}) \to \mathbb{R}\times\mathbb{R}\times (H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star,$$
$$(\lambda,\varphi) \mapsto \Big(\|\varphi\|^2_{H_0^{1,0}(\Omega,\mathbb{C})} - \lambda_0,\ \mathfrak{Im}\big(\textstyle\int_\Omega \varphi\bar{\varphi}_0\,dx\big),\ (i\nabla + A_0)^2\varphi - \lambda\varphi\Big).$$

In the above definition, $(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star$ is the real dual space of $H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}) = H_0^{1,0}(\Omega,\mathbb{C})$, which is here meant as a vector space over $\mathbb{R}$ endowed with the norm

$$\|u\|_{H_0^{1,0}(\Omega,\mathbb{C})} = \left(\int_\Omega \big|(i\nabla + A_0)u\big|^2\,dx\right)^{1/2},$$

and $(i\nabla + A_0)^2\varphi - \lambda\varphi \in (H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star$ acts as

$${}_{(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star}\langle (i\nabla + A_0)^2\varphi - \lambda\varphi, u\rangle_{H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C})} = \mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)\varphi\cdot\overline{(i\nabla + A_0)u}\,dx - \lambda\int_\Omega \varphi\bar{u}\,dx\right)$$

for all $\varphi \in H_0^{1,0}(\Omega,\mathbb{C})$. It is easy to prove that the function F is Fréchet-differentiable at (λ_0,φ_0), with differential $dF(\lambda_0,\varphi_0) \in \mathcal{L}(\mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C}), \mathbb{R}\times\mathbb{R}\times(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^*)$ given by

$$dF(\lambda_0,\varphi_0)(\lambda,\varphi) = \left(2\mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)\varphi_0\cdot\overline{(i\nabla + A_0)\varphi}\,dx\right), \mathfrak{Im}\left(\int_\Omega \varphi\bar{\varphi}_0\,dx\right), (i\nabla + A_0)^2\varphi - \lambda_0\varphi - \lambda\varphi_0\right)$$

for every $(\lambda,\varphi) \in \mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C})$. From the simplicity assumption (1-5) it follows that $dF(\lambda_0,\varphi_0)$ is invertible; see [Abatangelo and Felli 2015, Lemma 7.1] for details.

From the definition of $v_{n_0,R,a}$, (1-17), (3-19), (3-24), (4-8), and (4-19) it follows that

$$\begin{aligned}\int_\Omega \big|(i\nabla + A_0)(v_{n_0,R,a} - \varphi_0)\big|^2\,dx &= \int_\Omega \big|e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\big|^2\,dx \\ &\quad - \int_{D_{R|a|}} \big|e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\big|^2\,dx \\ &\quad + \int_{D_{R|a|}} \big|(i\nabla + A_0)(v_{n_0,R,a}^{\mathrm{int}} - \varphi_0)\big|^2\,dx = o(1)\end{aligned}$$

as $|a| \to 0$, so that $v_{n_0,R,a} \to \varphi_0$ in $H^1_0(\Omega,\mathbb{C})$ as $|a| \to 0^+$. Then, from the invertibility of $dF(\lambda_0,\varphi_0)$ we have

$$|\lambda_a - \lambda_0| + \|v_{n_0,R,a} - \varphi_0\|_{H^{1,0}_0(\Omega,\mathbb{C})} \le \|(dF(\lambda_0,\varphi_0))^{-1}\|_{\mathcal{L}(\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star,\mathbb{C}\times H^{1,0}_0(\Omega,\mathbb{C}))} \times \|F(\lambda_a, v_{n_0,R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}(1+o(1)) \quad (5\text{-}1)$$

as $|a| \to 0^+$. We define

$$F(\lambda_a, v_{n_0,R,a}) = (\alpha_a, \beta_a, w_a),$$

where

$$\alpha_a = \|v_{n_0,R,a}\|^2_{H^{1,0}_0(\Omega,\mathbb{C})} - \lambda_0 \in \mathbb{R},$$
$$\beta_a = \mathfrak{Im}\biggl(\int_\Omega v_{n_0,R,a}\bar{\varphi}_0\,dx\biggr) \in \mathbb{R},$$
$$w_a = (i\nabla + A_0)^2 v_{n_0,R,a} - \lambda_a v_{n_0,R,a} \in (H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star.$$

In view of (4-19), (3-24), and Corollary 4.7 we have

$$\begin{aligned}\alpha_a &= \biggl(\int_{D_{R|a|}} |(i\nabla + A_0)v^{\mathrm{int}}_{n_0,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\,dx\biggr) + (\lambda_a - \lambda_0)\\ &= O(H(\varphi_a, K_\delta|a|)) + O((H(\varphi_a, K_\delta|a|))^{|\alpha-k|/(|\alpha-k|+\delta)}) = o(\sqrt{H(\varphi_a, K_\delta|a|)}) \end{aligned}\quad (5\text{-}2)$$

as $|a| \to 0^+$. The normalization condition for the phase in (1-15), together with (4-20), (4-8), and (3-26), yields

$$\begin{aligned}\beta_a &= \mathfrak{Im}\biggl(\int_{D_{R|a|}} v^{\mathrm{int}}_{n_0,R,a}\bar{\varphi}_0\,dx - \int_{D_{R|a|}} e^{i\alpha(\theta^a_0-\theta_a)}\varphi_a\bar{\varphi}_0\,dx + \int_\Omega e^{i\alpha(\theta^a_0-\theta_a)}\varphi_a\bar{\varphi}_0\,dx\biggr)\\ &= \mathfrak{Im}\biggl(\int_{D_{R|a|}} v^{\mathrm{int}}_{n_0,R,a}\bar{\varphi}_0\,dx - \int_{D_{R|a|}} e^{i\alpha(\theta^a_0-\theta_a)}\varphi_a\bar{\varphi}_0\,dx\biggr)\\ &= O(|a|^{2+|\alpha-k|}\sqrt{H(\varphi_a, K_\delta|a|)}) = o(\sqrt{H(\varphi_a, K_\delta|a|)}) \end{aligned}\quad (5\text{-}3)$$

as $|a| \to 0^+$.

For every $a \in \Omega$, the map

$$\mathcal{T}_a : \mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C}) \to \mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C}), \quad \mathcal{T}_a\varphi(x) = \varphi(|a|x),$$

is an isometry of $\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})$.

Since $H^{1,0}_0(\Omega,\mathbb{C})$ is continuously embedded into $\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})$ by trivial extension outside Ω and $\|u\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})} = \|u\|_{H^{1,0}_0(\Omega,\mathbb{C})}$ for every $u \in H^{1,0}_0(\Omega,\mathbb{C})$, we have

$$\begin{aligned}\|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} &= \sup_{\substack{\varphi\in H^{1,0}_0(\Omega,\mathbb{C})\\ \|\varphi\|_{H^{1,0}_0(\Omega,\mathbb{C})}=1}} \biggl|\mathfrak{Re}\biggl(\int_\Omega (i\nabla + A_0)v_{n_0,R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{n_0,R,a}\bar{\varphi}\,dx\biggr)\biggr|\\ &\le \sup_{\substack{\varphi\in \mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \biggl|\mathfrak{Re}\biggl(\int_\Omega (i\nabla + A_0)v_{n_0,R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{n_0,R,a}\bar{\varphi}\,dx\biggr)\biggr|. \end{aligned}\quad (5\text{-}4)$$

For every $\varphi \in \mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})$ we have

$$\begin{aligned}\int_\Omega (i\nabla + A_0)v_{n_0,R,a}\cdot\overline{(i\nabla+A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{n_0,R,a}\bar\varphi\,dx \\ = \int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla+A_a)\varphi_a\cdot\overline{(i\nabla+A_0)\varphi}\,dx - \lambda_a\int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar\varphi\,dx \\ + \int_{D_{R|a|}}(i\nabla+A_0)v_{n_0,R,a}\cdot\overline{(i\nabla+A_0)\varphi}\,dx - \lambda_a\int_{D_{R|a|}} v_{n_0,R,a}\bar\varphi\,dx. \quad (5\text{-}5)\end{aligned}$$

From scaling and integration by parts we have that, letting $\tilde\varphi_a$ be defined in (3-27),

$$\begin{aligned}\int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla+A_a)\varphi_a\cdot\overline{(i\nabla+A_0)\varphi}\,dx - \lambda_a\int_{\Omega\setminus D_{R|a|}} e^{i\alpha(\theta_0^a-\theta_a)}\varphi_a\bar\varphi\,dx \\ = i\sqrt{H(\varphi_a,K_\delta|a|)}\int_{\partial D_R}\overline{\mathcal{T}_a\varphi}\,e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla+A_p)\tilde\varphi_a\cdot\nu\,d\sigma, \quad (5\text{-}6)\end{aligned}$$

where $\nu = x/|x|$ is the outer unit normal vector. In a similar way we have that, defining Z_a^R as in (4-21) and using (4-17),

$$\begin{aligned}\int_{D_{R|a|}}(i\nabla+A_0)v_{n_0,R,a}\cdot\overline{(i\nabla+A_0)\varphi}\,dx - \lambda_a\int_{D_{R|a|}} v_{n_0,R,a}\bar\varphi\,dx \\ = \sqrt{H(\varphi_a,K_\delta|a|)}\Big(-i\int_{\partial D_R}(i\nabla+A_0)Z_a^R\cdot\nu\overline{\mathcal{T}_a\varphi}\,d\sigma - \lambda_a|a|^2\int_{D_R}Z_a^R\overline{\mathcal{T}_a\varphi}\,dx\Big). \quad (5\text{-}7)\end{aligned}$$

Combining (5-4)–(5-7) and recalling that $\mathcal{T}_a$ is an isometry of $\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})$, we obtain

$$(H(\varphi_a,K_\delta|a|))^{-\frac12}\|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} \le h(p,a,R) + \lambda_a|a|^2 \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}}\left|\int_{D_R}Z_a^R\bar\varphi\,dx\right|, \quad (5\text{-}8)$$

where

$$h(p,a,R) = \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}}\left|\int_{\partial D_R}\big(e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla+A_p)\tilde\varphi_a - (i\nabla+A_0)Z_a^R\big)\cdot\nu\,\bar\varphi\,d\sigma\right|.$$

From Remarks 3.8 and 4.5 it follows that, for $R > \max\{2,K_\delta\}$ and $p\in\mathbb{S}^1$ fixed,

$$\big\{\big(e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla+A_p)\tilde\varphi_a - (i\nabla+A_0)Z_a^R\big)\cdot\nu\big\}_{|a|<r_\delta/R} \quad \text{is bounded in } H^{-1/2}(\partial D_R)$$

so that, for p and R fixed, $h(p,a,R) = O(1)$ as $a\to 0$. Moreover, Remark 4.5 implies that, for $R>\max\{2,K_\delta\}$ and $p\in\mathbb{S}^1$ fixed,

$$\sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}}\left|\int_{D_R}Z_a^R\bar\varphi\,dx\right| = O(1) \quad \text{as } |a|\to 0.$$

Hence the conclusion follows from (5-1), (5-2), (5-3), and (5-8). □

The previous theorem allows us to estimate the energy variation of scaled eigenfunctions and improve the results of Corollary 3.5 as follows.

Corollary 5.2. *Let $p \in \mathbb{S}^1$ be fixed. Then*

(*i*) $|a|^{2|\alpha-k|} = O(H(\varphi_a, K_\delta|a|))$ *as* $a = |a|p \to 0$;

(*ii*) *letting* $\tilde{\varphi}_a$ *and* W_a *be as in* (3-27) *and* (4-10), *for every* $R > \max\{2, K_\delta\}$ *it holds that*

$$\int_{(\Omega/|a|)\setminus D_R} \left| (i\nabla + A_p)\left(\tilde{\varphi}_a - e^{i\alpha(\theta_p-\theta_0^p)} W_a \frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}} \right) \right|^2 dx = O(1) \quad \text{as } a = |a|p \to 0. \tag{5-9}$$

Proof. Estimate (5-9) follows from scaling and Theorem 5.1. From (5-9) it follows that

$$\begin{aligned}
&\frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}} \left(\int_{D_{2R}\setminus D_R} \left|(i\nabla + A_0) W_a\right|^2 dx \right)^{1/2} \\
&\qquad = \frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}} \left(\int_{D_{2R}\setminus D_R} \left|(i\nabla + A_p)(e^{i\alpha(\theta_p-\theta_0^p)} W_a)\right|^2 dx \right)^{1/2} \\
&\qquad \le O(1) + \left(\int_{D_{2R}\setminus D_R} \left|(i\nabla + A_p)\tilde{\varphi}_a(x)\right|^2 dx \right)^{1/2}
\end{aligned}$$

as $a = |a|p \to 0$. From Remark 3.8 and (4-11), the above estimate implies (i). □

In the following lemma we prove the existence and uniqueness of the function Ψ_p satisfying (1-18) and (1-19), which will turn out to be the limit of the blow-up family (3-27) as $a \to 0$ along the fixed direction $p \in \mathbb{S}^1$.

Lemma 5.3. *Let $p \in \mathbb{S}^1$. There exists a unique $\Psi_p \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$ satisfying* (1-18) *and* (1-19).

Proof. Let η be a smooth cut-off function such that $\eta \equiv 0$ in D_1 and $\eta \equiv 1$ in $\mathbb{R}^2 \setminus D_R$ for some $R > 1$. Recalling the definition of ψ_k (1-20), we have

$$\begin{aligned}
F &= (i\nabla + A_p)^2 (\eta e^{i\alpha(\theta_p-\theta_0^p)} \psi_k) \\
&= -\Delta\eta e^{i\alpha(\theta_p-\theta_0^p)} \psi_k + 2i\nabla\eta \cdot (i\nabla + A_p)(e^{i\alpha(\theta_p-\theta_0^p)} \psi_k) + \eta(i\nabla + A_p)^2 (e^{i\alpha(\theta_p-\theta_0^p)} \psi_k) \\
&= -\Delta\eta e^{i\alpha(\theta_p-\theta_0^p)} \psi_k + 2i\nabla\eta \cdot (i\nabla + A_p)(e^{i\alpha(\theta_p-\theta_0^p)} \psi_k) \in (\mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C}))^*.
\end{aligned}$$

Here $\mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C})$ is the completion of $C^\infty_c(\mathbb{R}^2 \setminus \{0\}, \mathbb{C})$ with respect to

$$\|u\|_{\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})} = \left(\int_{\mathbb{R}^2} |(i\nabla + A_p)u(x)|^2 dx \right)^{1/2}.$$

By the Lax–Milgram theorem, there exists a unique $g \in \mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C})$ which solves

$$(i\nabla + A_p)^2 g = -F \quad \text{in } (\mathcal{D}^{1,2}_p(\mathbb{R}^2, \mathbb{C}))^*.$$

Then, $\Psi_p = g + \eta e^{i\alpha(\theta_p-\theta_0^p)} \psi_k$ satisfies (1-18) and (1-19), and the existence is proved.

The uniqueness follows from the fact that, if $\Psi_p^1, \Psi_p^2 \in H^{1,p}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$ satisfy (1-18) and (1-19), then

$$(i\nabla + A_p)^2(\Psi_p^1 - \Psi_p^2) = 0 \quad \text{in } (\mathcal{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C}))^*, \tag{5-10}$$

and

$$\int_{\mathbb{R}^2} |(i\nabla + A_p)(\Psi_p^1 - \Psi_p^2)|^2\, dx < +\infty,$$

which, in view of the Hardy inequality (1-3), implies

$$\int_{\mathbb{R}^2} \frac{|\Psi_p^1 - \Psi_p^2|^2}{|x-p|^2}\, dx < +\infty,$$

and hence that $\Psi_p^1 - \Psi_p^2 \in \mathcal{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C})$. Therefore we can test (5-10) with $\Psi_p^1 - \Psi_p^2$ thus concluding that

$$\int_{\mathbb{R}^2} |(i\nabla + A_p)(\Psi_p^1 - \Psi_p^2)|^2\, dx = 0,$$

which implies $\Psi_p^1 \equiv \Psi_p^2$. □

We are now in a position to prove that the scaled eigenfunctions (3-27) converge to a multiple of Ψ_p as $a = |a|p \to 0$.

Lemma 5.4. *Let $p \in \mathbb{S}^1$ and $\delta \in (0, \sqrt{\mu_1}/2)$ be fixed and let $K_\delta > \Upsilon_\delta$ be as in Lemma 3.4. For $a = |a|p \in \Omega$ let $\tilde\varphi_a$ be as in* (3-27). *Then*

$$\tilde\varphi_a \to \frac{\beta}{|\beta|}\left(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\, ds}\right)^{1/2} \Psi_p \quad \text{as } a = |a|p \to 0$$

in $H^{1,p}(D_R, \mathbb{C})$ for every $R > 1$ and in $C^2_{\rm loc}(\mathbb{R}^2 \setminus \{p\}, \mathbb{C})$, where Ψ_p is the function defined in Lemma 5.3. Moreover,

$$\lim_{a=|a|p\to 0} \frac{|a|^{|\alpha-k|}}{\sqrt{H(\varphi_a, K_\delta|a|)}} = \frac{1}{|\beta|}\left(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\, ds}\right)^{1/2}. \tag{5-11}$$

Proof. From Remark 3.8 and Corollary 5.2 it follows that, for every sequence $a_n = |a_n|p$ with $|a_n| \to 0$, there exist a subsequence a_{n_ℓ}, $c \in [0, +\infty)$ and $\widetilde\Phi \in H^{1,p}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$ such that

$$\tilde\varphi_{a_{n_\ell}} \rightharpoonup \widetilde\Phi \text{ weakly in } H^{1,p}(D_R, \mathbb{C}) \text{ as } \ell \to +\infty \quad \text{and} \quad \lim_{\ell\to+\infty} \frac{|a_{n_\ell}|^{|\alpha-k|}}{\sqrt{H(\varphi_{a_{n_\ell}}, K_\delta|a_{n_\ell}|)}} = c$$

for every $R > 1$. Passing to the limit in the equation satisfied by $\tilde\varphi_a$, i.e., $(i\nabla + A_p)^2\tilde\varphi_a = \lambda_a|a|^2\tilde\varphi_a$ in $(1/|a|)\Omega$, we obtain that $\widetilde\Phi$ satisfies

$$(i\nabla + A_p)^2\widetilde\Phi = 0 \quad \text{in } \mathbb{R}^2. \tag{5-12}$$

Moreover, by compact trace embeddings,

$$\frac{1}{K_\delta}\int_{\partial D_{K_\delta}} |\widetilde\Phi|^2\, ds = 1, \tag{5-13}$$

so that $\widetilde\Phi$ is not identically zero. Testing the equation for $\tilde\varphi_a$ with $\tilde\varphi_a$ itself, integrating by parts and exploiting the $C^2_{\rm loc}$-convergence of $\tilde\varphi_a$ in $\mathbb{R}^2 \setminus \{p\}$ (which follows from classic elliptic estimates) we obtain

$\int_{D_R} |(i\nabla + A_p)\tilde{\varphi}_{a_{n_\ell}}|^2\,dx \to \int_{D_R} |(i\nabla + A_p)\widetilde{\Phi}|^2\,dx$ as $\ell \to \infty$ for every $R > 1$. Hence we conclude that, for all $R > 1$, $\tilde{\varphi}_{a_{n_\ell}} \to \widetilde{\Phi}$ strongly in $H^{1,p}(D_R, \mathbb{C})$ as $\ell \to +\infty$.

By the strong $H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$-convergence and recalling (4-11), we can pass to the limit along a_{n_ℓ} in (5-9) to obtain

$$\int_{\mathbb{R}^2 \setminus D_R} \bigl|(i\nabla + A_p)(\widetilde{\Phi} - c\beta e^{i\alpha(\theta_p - \theta_0^p)}\psi_k)\bigr|^2\,dx < +\infty.$$

This implies $c \neq 0$ (and hence $c > 0$), otherwise we would have $\int_{\mathbb{R}^2 \setminus D_R} |(i\nabla + A_p)\widetilde{\Phi}|^2\,dx < +\infty$, which, together with (5-12), implies $\widetilde{\Phi} \equiv 0$, thus contradicting (5-13).

Then Lemma 5.3 and (5-13) provide

$$\widetilde{\Phi} = c\beta\Psi_p \quad \text{and} \quad c = \frac{1}{|\beta|}\left(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2}\right)^{1/2}.$$

Since these limits depend neither on the sequence, nor on the subsequence, the proof is complete. □

Proof of Theorem 1.1. Let $p \in \mathbb{S}^1$. From Corollary 4.7(i) and (5-11) we conclude that

$$\lambda_0 - \lambda_a = O(|a|^{2|\alpha - k|})$$

as $a = |a|p \to 0$. Since the function $a \mapsto \lambda_a$ is analytic in a neighborhood of 0, due to the simplicity of λ_0, see [Léna 2015, Theorem 1.3], and since $2|\alpha - k|$ is noninteger, we have that the Taylor polynomials of the function $\lambda_0 - \lambda_a$ with center 0 and degree less than or equal to $\lfloor 2|\alpha - k| \rfloor$ vanish, thus yielding the conclusion. □

Proof of Theorem 1.2. It is a direct consequence of Lemma 5.4. □

6. Rate of convergence for eigenfunctions

Taking inspiration from [Abatangelo and Felli 2017], we now estimate the rate of convergence of the eigenfunctions. We then take into account the quantity

$$\bigl\|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0\bigr\|_{L^2(\Omega,\mathbb{C})},$$

where $\varphi_a = \varphi_{n_0}^a$ satisfies (1-14), (1-15) and $\varphi_0 = \varphi_{n_0}^0$ satisfies (1-7). We split the argument in two different steps, the first considering the energy variation inside small disks of radius $R|a|$, the second considering the energy variation outside these disks.

Lemma 6.1. *Under the same assumptions as in Theorems 1.1 and 1.2, we have that, for every $p \in \mathbb{S}^1$ and $R > 1$,*

$$\lim_{a = |a|p \to 0} \frac{1}{|a|^{2|\alpha - k|}} \int_{D_{R|a|}} \bigl|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0\bigr|^2\,dx = |\beta|^2\,\mathcal{F}_p(R), \tag{6-1}$$

where

$$\mathcal{F}_p(R) = \int_{D_R} \bigl|(i\nabla + A_p)\Psi_p - e^{i\alpha(\theta_p - \theta_0^p)}(i\nabla + A_0)\psi_k\bigr|^2\,dx,$$

Ψ_p is defined in Lemma 5.3 and ψ_k is as in (1-20). *Moreover,*

$$\mathfrak{L}_p := \lim_{R \to +\infty} \mathcal{F}_p(R) \in (0, +\infty).$$

Proof. We notice that, in view of (1-19), $\mathfrak{L}_p < +\infty$. The proof of (6-1) relies on a change of variables and on the convergences stated in (4-11) and in Theorem 1.2. We have

$$\lim_{R\to+\infty} \mathcal{F}_p(R) = \int_{\mathbb{R}^2} \big|(i\nabla + A_p)\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}(i\nabla + A_0)\psi_k\big|^2\,dx$$
$$= \int_{\mathbb{R}^2\setminus\Gamma_p} \big|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\big|^2\,dx > 0,$$

where Γ_p is defined in (1-13). Indeed, suppose by contradiction that the above limit is zero. Since, for every $r_1 > r_2 > 1$ we have $\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k \in H^{1,p}(D_{r_1}(p)\setminus D_{r_2}(p), \mathbb{C})$, the Hardy inequality (1-4) implies $\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k \equiv 0$ in $\mathbb{R}^2\setminus D_1(p)$. Moreover, since $(i\nabla + A_p)^2(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k) = 0$ in $\mathbb{R}^2\setminus\Gamma_p$, a classical unique continuation principle, see, e.g., [Wolff 1992], implies $\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k \equiv 0$ in $\mathbb{R}^2\setminus\Gamma_p$ necessarily. But this is impossible since, by (1-18) and classical elliptic estimates away from p, Ψ_p is smooth in $\mathbb{R}^2\setminus\{p\}$, whereas $e^{i\alpha(\theta_p-\theta_0^p)}\psi_k$ is discontinuous on $\Gamma_p\setminus\{0\}$ since it is the product of the continuous nonzero function ψ_k and of the discontinuous function $e^{i\alpha(\theta_p-\theta_0^p)}$; see the definitions (1-11), (1-12) and (1-13). □

Before addressing the energy variation outside the disk, it is worthwhile introducing a preliminary result. For all $R > 2$ and $p \in \mathbb{S}^1$, let $z_{p,R}$ be the unique solution to

$$\begin{cases} (i\nabla + A_0)^2 z_{p,R} = 0 & \text{in } D_R, \\ z_{p,R} = e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p & \text{on } \partial D_R. \end{cases} \tag{6-2}$$

From Lemma 5.4 it follows that the family of functions Z_a^R introduced in (4-21) converges in $H^{1,0}(D_R, \mathbb{C})$ to some multiple of $z_{p,R}$.

Lemma 6.2. *Let $p \in \mathbb{S}^1$ and $R > 2$. For $a = |a|p \in \Omega$, let Z_a^R be as in* (4-21). *Then*

$$Z_a^R \to \frac{\beta}{|\beta|}\bigg(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\bigg)^{1/2} z_{p,R}$$

in $H^{1,0}(D_R, \mathbb{C})$ as $|a| \to 0^+$.

Proof. Define

$$\gamma_{p,\delta} = \frac{\beta}{|\beta|}\bigg(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\bigg)^{1/2}.$$

By (4-17) and (6-2) we have that $Z_a^R - \gamma_{p,\delta} z_{p,R}$ solves

$$\begin{cases} (i\nabla + A_0)^2(Z_a^R - \gamma_{p,\delta} z_{p,R}) = 0 & \text{in } D_R, \\ Z_a^R - \gamma_{p,\delta} z_{p,R} = e^{i\alpha(\theta_0^p-\theta_p)}(\tilde{\varphi}_a - \gamma_{p,\delta}\Psi_p) & \text{on } \partial D_R. \end{cases}$$

For $R > 2$, let $\eta_R : \mathbb{R}^2 \to \mathbb{R}$ be a smooth cut-off function such that

$$\eta_R \equiv 0 \quad \text{in } D_{R/2}, \qquad \eta_R \equiv 1 \quad \text{in } \mathbb{R}^2\setminus D_R, \qquad 0 \le \eta_R \le 1. \tag{6-3}$$

Then, by the Dirichlet principle and Lemma 5.4,

$$\begin{aligned}\int_{D_R} &|(i\nabla + A_0)(Z_a^R - \gamma_{p,\delta} z_{p,R})|^2\,dx\\ &\le \int_{D_R} |(i\nabla + A_0)(\eta_R e^{i\alpha(\theta_0^p-\theta_p)}(\tilde\varphi_a - \gamma_{p,\delta}\Psi_p))|^2\,dx\\ &\le 2\int_{D_R} |\nabla\eta_R|^2|\tilde\varphi_a - \gamma_{p,\delta}\Psi_p|^2\,dx + 2\int_{D_R\setminus D_{R/2}} \eta_R^2|(i\nabla + A_p)(\tilde\varphi_a - \gamma_{p,\delta}\Psi_p)|^2\,dx = o(1)\end{aligned}$$

as $a = |a|p \to 0$. Finally, the Hardy-type inequality (1-3) allows us to conclude. □

Lemma 6.3. *Let* $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C})$ *be a solution to* (1-7) *satisfying* (1-5). *Let* $p \in \mathbb{S}^1$. *For* $a = |a|p \in \Omega$, *let* $\varphi_a \in H^{1,a}_0(\Omega,\mathbb{C})$ *satisfy* (1-14)–(1-15). *Then, for all* $R > \max\{2, K_\delta\}$,

$$\left\|e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\right\|^2_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} \le |a|^{2|\alpha-k|} G(p,a,R),$$

where $\lim_{a=|a|p\to 0} G(p,a,R) = G(p,R)$ *for some* $G(p,R)$ *such that*

$$\lim_{R\to+\infty} G(p,R) = 0. \tag{6-4}$$

Proof. Let $R > \max\{2, K_\delta\}$. From Theorem 5.1 and (5-11) we have

$$\begin{aligned}\|e^{i\alpha(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\|_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} &\le \|v_{n_0,R,a} - \varphi_0\|_{H^{1,0}_0(\Omega,\mathbb{C})}\\ &\le C\big(h(p,a,R) + g(p,a,R)\big)|a|^{|\alpha-k|},\end{aligned}$$

where $g(p,a,R) = o(1)$ as $|a| \to 0^+$ and

$$\begin{aligned}h(p,a,R) &= \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\tilde\varphi_a - Z_a^R)\cdot\nu\,\bar\varphi\,d\sigma\right|\\ &\le \text{const}\left\|(i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\tilde\varphi_a - Z_a^R)\cdot\nu - \gamma_{p,\delta}(i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu\right\|_{H^{-1/2}(\partial D_R)}\\ &\qquad + \gamma_{p,\delta} \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar\varphi\,d\sigma\right|,\end{aligned}$$

where

$$\gamma_{p,\delta} = \frac{\beta}{|\beta|}\left(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2\,ds}\right)^{1/2}$$

and the constant const > 0 is independent of a. From Lemmas 5.4 and 6.2 we have

$$(i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\tilde\varphi_a - Z_a^R)\cdot\nu \to \gamma_{p,\delta}(i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu$$

in $H^{-1/2}(\partial D_R)$ as $a = |a|p \to 0$. Therefore $h(p,a,R) \le f(p,a,R)$ with

$$\lim_{a=|a|p\to 0} f(p,a,R) = \gamma_{p,\delta} \sup_{\substack{\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar\varphi\,d\sigma\right|.$$

To complete the proof is then enough to show that

$$\lim_{R\to+\infty} \sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar\varphi\,d\sigma\right| = 0. \tag{6-5}$$

Using an integration by parts we can rewrite

$$\begin{aligned}&\left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar\varphi\,d\sigma\right|\\ &\quad= \left|\int_{\partial D_R} e^{i\alpha(\theta_0^p-\theta_p)}(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\cdot\nu\bar\varphi\,d\sigma + \int_{\partial D_R}(i\nabla + A_0)(\psi_k - z_{p,R})\cdot\nu\bar\varphi\,d\sigma\right|\\ &\quad= \left|-i\int_{\mathbb{R}^2\setminus D_R}(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\cdot\overline{(i\nabla + A_0)\varphi}e^{i\alpha(\theta_0^p-\theta_p)}\,dx\right.\\ &\qquad\qquad\left. + i\int_{D_R}(i\nabla + A_0)(\psi_k - z_{p,R})\cdot\overline{(i\nabla + A_0)\varphi}\,dx\right|,\end{aligned}$$

which implies

$$\begin{aligned}&\sup_{\substack{\varphi\in\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left|\int_{\partial D_R} (i\nabla + A_0)(e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p - z_{p,R})\cdot\nu\,\bar\varphi\,d\sigma\right|\\ &\le \left(\int_{\mathbb{R}^2\setminus D_R}\left|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)\right|^2 dx\right)^{1/2} + \left(\int_{D_R}\left|(i\nabla + A_0)(\psi_k - z_{p,R})\right|^2 dx\right)^{1/2}.\end{aligned} \tag{6-6}$$

The first term in the right-hand side of (6-6) goes to zero as $R\to+\infty$ because of (1-19). To estimate the second term, we consider a test function η_R satisfying (6-3) and the additional property $|\nabla\eta_R|\le 4/R$ in $D_R\setminus D_{R/2}$. Recalling that $\psi_k - z_{p,R}$ satisfies $(i\nabla + A_0)^2(\psi_k - z_{p,R}) = 0$ in D_R with the boundary condition $\psi_k - z_{p,R} = \psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p$ on ∂D_R, the Dirichlet principle and the Hardy inequality (1-4) provide

$$\begin{aligned}&\int_{D_R} |(i\nabla + A_0)(\psi_k - z_{p,R})|^2\,dx\\ &\qquad\le \int_{D_R} |(i\nabla + A_0)(\eta_R(\psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p))|^2\,dx\\ &\qquad\le 2\int_{D_R}|\nabla\eta_R|^2\,|\psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p|^2\,dx + 2\int_{\mathbb{R}^2\setminus D_{R/2}}|(i\nabla + A_0)(\psi_k - e^{i\alpha(\theta_0^p-\theta_p)}\Psi_p)|^2\,dx\\ &\qquad\le \frac{32}{R^2}\int_{D_R\setminus D_{R/2}}|\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k|^2\,dx + 2\int_{\mathbb{R}^2\setminus D_{R/2}}|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx\\ &\qquad\le \frac{32(R+1)^2}{R^2}\int_{D_{R+1}(p)\setminus D_{(R-2)/2}(p)}\frac{|\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k|^2}{|x-p|^2}\,dx\\ &\qquad\qquad + 2\int_{\mathbb{R}^2\setminus D_{R/2}}|(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx\end{aligned}$$

$$\le \frac{32(R+1)^2}{R^2\mu_1}\int_{D_{R+1}(p)\setminus D_{(R-2)/2}(p)} |(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx \\ + 2\int_{\mathbb{R}^2\setminus D_{R/2}} |(i\nabla + A_p)(\Psi_p - e^{i\alpha(\theta_p-\theta_0^p)}\psi_k)|^2\,dx,$$

which goes to zero again thanks to (1-19). Therefore we have obtained (6-5) and the proof is complete. □

Proof of Theorem 1.3. Let $p \in \mathbb{S}^1$ and $\varepsilon > 0$. From Lemma 6.1 and (6-4) there exists some $R_0 > \max\{2, K_\delta\}$ sufficiently large such that

$$|\mathcal{F}_p(R_0) - \mathfrak{L}_p| < \varepsilon \quad \text{and} \quad |G(p, R_0)| < \varepsilon.$$

Moreover, again from Lemmas 6.3 and 6.1 there exists $\rho > 0$ (depending on p, ε, and R_0) such that, if $a = |a|p$ and $|a| < \rho$, then

$$|G(p, a, R_0) - G(p, R_0)| < \varepsilon$$

and

$$\left|\frac{1}{|a|^{2|\alpha-k|}}\int_{D_{R_0|a|}} \big|(i\nabla + A_a)\varphi_a(x) - e^{i\alpha(\theta_a-\theta_0^a)(x)}(i\nabla + A_0)\varphi_0(x)\big|^2\,dx - |\beta|^2\mathcal{F}_p(R_0)\right| < \varepsilon.$$

Therefore, taking into account Lemma 6.3, we have that, for all $a = |a|p$ with $|a| < \rho$,

$$\begin{aligned}
&\left||a|^{-2|\alpha-k|}\int_\Omega \big|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\big|^2\,dx - |\beta|^2\mathfrak{L}_p\right| \\
&\le \left||a|^{-2|\alpha-k|}\int_{D_{R_0|a|}} \big|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\big|^2\,dx - |\beta|^2\mathcal{F}_p(R_0)\right| \\
&\qquad + |a|^{-2|\alpha-k|}\int_{\Omega\setminus D_{R_0|a|}} \big|(i\nabla + A_a)\varphi_a - e^{i\alpha(\theta_a-\theta_0^a)}(i\nabla + A_0)\varphi_0\big|^2\,dx + |\beta|^2\,|\mathfrak{L}_p - \mathcal{F}_p(R_0)| \\
&< \varepsilon + G(p, a, R_0) + |\beta|^2\varepsilon \\
&\le \varepsilon + |G(p, a, R_0) - G(p, R_0)| + |G(p, R_0)| + |\beta|^2\varepsilon = (3 + |\beta|^2)\varepsilon.
\end{aligned}$$

□

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