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COMPLEX ROTATION NUMBERS: BUBBLES AND THEIR INTERSECTIONS

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The construction of complex rotation numbers, due to V. Arnold, gives rise to a fractal-like set "bubbles" related to a circle diffeomorphism. "Bubbles" is a complex analogue to Arnold tongues.

This article contains a survey of the known properties of bubbles, as well as a variety of open questions. In particular, we show that bubbles can intersect and self-intersect, and provide approximate pictures of bubbles for perturbations of Möbius circle diffeomorphisms.

1. Introduction

1.1. ***Complex rotation numbers: Arnold's construction.*** In what follows, $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is an analytic orientation-preserving circle diffeomorphism. Its analytic extension to a small neighborhood of $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}/\mathbb{Z}$ is still denoted by f. $\mathbb{H} \subset \mathbb{C}$ is the open upper half-plane.

The following construction was suggested by V. Arnold [1983, Section 27] in 1978. Given $\omega \in \mathbb{H}/\mathbb{Z}$ and a small positive $\varepsilon \in \mathbb{R}$, one can construct a complex torus $E(f+\omega)$ as the quotient space of a cylinder Π by the action of $f+\omega$:

$$\begin{aligned} \Pi &:= \{z \in \mathbb{C}/\mathbb{Z} \mid -\varepsilon < \operatorname{Im} z < \operatorname{Im}\omega + \varepsilon\}, \\ E(f+\omega) &:= \Pi/(z \sim f(z)+\omega). \end{aligned} \tag{1}$$

For a small positive ε, the quotient space $E(f+\omega)$ is a torus, inherits a complex structure from $\mathbb{C}/\mathbb{Z}$ and does not depend on ε.

Due to the uniformization theorem, for a unique $\tau \in \mathbb{H}/\mathbb{Z}$ there exists a biholomorphism

$$H_\omega : E(f+\omega) \to \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \tag{2}$$

such that H_ω takes $\mathbb{R}/\mathbb{Z} \subset E(f+\omega)$ to a curve homotopic to $\mathbb{R}/\mathbb{Z} \subset \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The number $\tau(f+\omega) := \tau \in \mathbb{H}/\mathbb{Z}$, i.e., the modulus of the complex torus $E(f+\omega)$, is called the *complex rotation number* of $f+\omega$.

In the original Arnold's construction, ω was supposed to be purely imaginary. The above version of this construction was suggested by R. Fedorov. The term "complex rotation number" is due to E. Risler [1999].

The complex rotation number $\tau(f+\omega)$ depends holomorphically on $\omega \in \mathbb{H}/\mathbb{Z}$; see [Risler 1999, Section 2.1, Proposition 2].

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1.2. ***Rotation number and its properties.*** This section lists well-known results on rotation numbers; see [Katok and Hasselblatt 1995, Sections 3.11, 3.12] for more details.

Let f be an orientation-preserving circle homeomorphism, and let $F : \mathbb{R} \to \mathbb{R}$ be its lift to the real line. The limit

$$\operatorname{rot} f = \lim_{n\to\infty} \frac{F^{\circ n}(x)}{n} \bmod 1$$

exists and does not depend on $x \in \mathbb{R}$. It is called the *rotation number* of the circle homeomorphism f.

Rotation number is invariant under continuous conjugations of f. It is rational, $\operatorname{rot} f = p/q$, if and only if f has a periodic orbit of period q. If $\operatorname{rot} f$ is irrational and $f \in C^2(\mathbb{R}/\mathbb{Z})$, then f is continuously conjugate to $z \mapsto z + \operatorname{rot} f$ (Denjoy theorem, see [Katok and Hasselblatt 1995, Section 3.12.1]). We will need the following, much more complicated result.

Definition. A real number ρ is called *Diophantine* if there exist $C, \beta > 0$ such that for all rationals p/q,

$$\left|\rho - \frac{p}{q}\right| \geq \frac{C}{q^{2+\beta}}.$$

Theorem 1 (M. R. Herman [1979], J.-C. Yoccoz [1984]). *If an analytic circle diffeomorphism has a Diophantine rotation number* $\operatorname{rot} f$, *then it is analytically conjugate to* $z \mapsto z + \operatorname{rot} f$.

This motivates the term "complex rotation number" for $\tau(f+\omega)$ above: while a circle diffeomorphism f is conjugate to the rotation $x \mapsto x + \operatorname{rot} f$ on $\mathbb{R}/\mathbb{Z}$, a complex-valued map $f+\omega$ is biholomorphically conjugate to the complex shift $z \mapsto z + \tau(f+\omega)$ in the cylinder $\Pi \subset \mathbb{C}/\mathbb{Z}$.

1.3. ***Steps on the graph of*** $\boldsymbol{\omega \mapsto \operatorname{rot}(f+\omega)}$. Rotation number depends continuously on f in the C^0-topology. In particular, $\operatorname{rot}(f+\omega)$ depends continuously on $\omega \in \mathbb{R}/\mathbb{Z}$; clearly, it (nonstrictly) increases on ω.

Recall that a periodic orbit of a circle diffeomorphism is called *parabolic* if its multiplier is 1, and *hyperbolic* otherwise. If a circle diffeomorphism has periodic orbits, and they are all hyperbolic, then the diffeomorphism is called *hyperbolic*.

Let $I_{p/q} := \{\omega \in \mathbb{R}/\mathbb{Z} \mid \operatorname{rot}(f+\omega) = p/q\}$; from now on, we always assume that p, q are coprime. If for some value of ω the diffeomorphism $f+\omega$ has the rotation number p/q and a *hyperbolic* orbit of period q, then this orbit persists under a small perturbation of ω. In this case, $I_{p/q}$ is a segment of nonzero length. Endpoints of $I_{p/q}$ correspond to diffeomorphisms $f+\omega$ having only parabolic orbits.

In a generic case, the graph of the function $\omega \mapsto \operatorname{rot}(f+\omega)$ contains infinitely many steps, i.e., nontrivial segments $I_{p/q} \times \{p/q\}$, on rational heights.

1.4. ***Rotation numbers as boundary values of a holomorphic function.***

Question 2. Can we find a holomorphic self-map τ on $\mathbb{H}/\mathbb{Z}$ such that its boundary values on $\mathbb{R}/\mathbb{Z}$ coincide with $\omega \mapsto \rho(f+\omega)$?

The answer is No (except for the trivial case $f(x) = x + c$), because the function $\omega \mapsto \rho(f+\omega)$ is locally constant on nonempty intervals $I_{p/q}$, and this is not possible for boundary values of holomorphic functions. In more detail, note that $\mathbb{H}/\mathbb{Z}$ is biholomorphically equivalent to the punctured unit disc $D \setminus \{0\}$, so the map $1/(2\pi i) \ln z : D \setminus \{0\} \to \mathbb{H}/\mathbb{Z}$ conjugates τ to a holomorphic bounded self-map of the punctured unit disc. Clearly, 0 is a removable singularity for this self-map. The following Luzin–Privalov theorem [1925, Section 14, p. 159] shows that such an extension τ does not exist:

Theorem 3 (N. Luzin, J. Privalov). *If a holomorphic function in the unit disc D has finite nontangential limits at all points of $E \subset \partial D$, where E has a nonzero Lebesgue measure, then this function is uniquely defined by these limits.*

This motivates the next question:

Question 4. Can we find a holomorphic self-map on $\mathbb{H}/\mathbb{Z}$ such that its boundary values on $(\mathbb{R}/\mathbb{Z}) \setminus \bigcup I_{p/q}$ coincide with $\omega \mapsto \rho(f+\omega)$?

Remark. The set $(\mathbb{R}/\mathbb{Z}) \setminus \bigcup I_{p/q}$ has nonzero measure due to a result of M. R. Herman [1977, Section 6, p. 287]; so by Theorem 3, such a holomorphic extension must be unique.

The answer to this question is Yes, and this holomorphic function is the complex rotation number $\tau(f+\omega)$. The following theorem is proved in [Buff and Goncharuk 2015]; the proof is based on previous results by E. Risler [1999], V. Moldavskij [2001], Y. Ilyashenko and V. Moldavskij [2003], and N. Goncharuk [2012].

Theorem 5 (X. Buff and N. Goncharuk [2015]). *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving analytic circle diffeomorphism. Then the holomorphic function $\tau(f + \cdot\,) : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ has a continuous extension $\bar\tau(f + \cdot\,) : \overline{\mathbb{H}/\mathbb{Z}} \to \overline{\mathbb{H}/\mathbb{Z}}$. Assume $\omega \in \mathbb{R}/\mathbb{Z}$:*

- *If $\operatorname{rot}(f+\omega)$ is irrational, then $\bar\tau(f+\omega) = \operatorname{rot}(f+\omega)$.*
- *If $\operatorname{rot}(f+\omega)$ is rational and $f+\omega$ has a parabolic periodic orbit, then $\bar\tau(f+\omega) = \operatorname{rot}(f+\omega)$.*
- *If $\operatorname{rot}(f+\omega)$ is rational and $f+\omega$ is hyperbolic on an open interval $\omega \in I \subset \mathbb{R}/\mathbb{Z}$, then $\bar\tau(f+\omega)$ depends analytically on $\omega \in I$ and $\bar\tau(f+\omega) \in \mathbb{H}/\mathbb{Z}$ for $\omega \in I$.*

The extension $\bar\tau(f+\omega)$ is also called the complex rotation number of $f+\omega$. Due to Theorem 5, it is continuous on ω, and coincides with the ordinary rotation number on $\mathbb{R}/\mathbb{Z} \setminus \bigcup I_{p/q}$.

Definition. The image of the segment $I_{p/q} = \{\omega \in \mathbb{R}/\mathbb{Z} \mid \operatorname{rot}(f+\omega) = p/q\}$ under the map $\omega \mapsto \bar\tau(f+\omega)$ is called the *p/q-bubble* of f.

Due to Theorem 5, the p/q-bubble is a union of several analytic curves in the upper half-plane with endpoints at p/q. Each analytic curve corresponds to the interval of hyperbolicity of $f+\omega$, and its endpoints correspond to $f+\omega$ with parabolic orbits.

So, each circle diffeomorphism f gives rise to a "fractal-like" set $\bar\tau(f+\omega)$ (bubbles) in the upper half-plane, containing countably many analytic curves. The picture of bubbles growing from rational points of the real axis was first described by R. Fedorov (oral communication, about 2001), and remained conjecturable until [Goncharuk 2012; Buff and Goncharuk 2015].

The possible shapes of bubbles are not known. The following question is also open.

Question 6. Is the set $\bar\tau(f+\omega)$ self-similar (i.e., is it a fractal set)?

The precise meaning of "self-similarity" in this question is not clear; conjecturably, for certain sequences of rational numbers $\{p_n/q_n\}$, the p_n/q_n-bubbles (when rescaled properly) tend to some limit shape.

1.5. ***Properties of bubbles and the Main Theorem.***

Question 7. Is $\bar{\tau}$ invariant under analytic conjugacies?

The answer is Yes:

Lemma 8. *The complex rotation number $\bar{\tau}$ is invariant under analytic conjugacies*: *for two analytically conjugate circle diffeomorphisms f_1, f_2, we have $\bar{\tau}(f_1) = \bar{\tau}(f_2)$.*

For nonhyperbolic f_1, f_2, their complex rotation numbers coincide with rotation numbers, so this lemma trivially repeats the invariance of rotation numbers under conjugacies. For hyperbolic diffeomorphisms, the proof of this lemma is implicitly contained in [Buff and Goncharuk 2015]; see also Section 5 below.

Note that in general, for conjugate f_1, f_2 and $\omega \in \overline{\mathbb{H}}/\mathbb{Z}$, the numbers $\bar{\tau}(f_1+\omega)$ and $\bar{\tau}(f_2+\omega)$ do not coincide.

Question 9. Is there an explicit formula for $\bar{\tau}(f+\omega)$?

The only case when the author can obtain an explicit formula for $\bar{\tau}(f+\omega)$ is described in the following proposition.

Let $\pi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$ be given by $\pi(z) := \exp(2\pi i z)$.

Proposition 10. *Let F be a Möbius map that preserves the circle $\{|w| = 1 \mid w \in \mathbb{C}\}$. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be given by $f := \pi^{-1} \circ F \circ \pi$. Then f has only a 0-bubble, and this bubble is a vertical segment.*

Proof. First, let us compute $\tau(f+\omega)$ for $\omega \in \mathbb{H}/\mathbb{Z}$.

Put $F_\omega := e^{2\pi i\omega} F$. For $\omega \in \mathbb{H}/\mathbb{Z}$ and small $\varepsilon > 0$, let $E^*(F_\omega)$ be the quotient space of the annulus $\Pi^* := \{1 > |w| > |e^{2\pi i\omega}|\}$ via the map F_ω. Note that the map π induces a biholomorphism of $E(f+\omega)$ to $E^*(F_\omega)$. Indeed, it takes Π to the neighborhood of Π^* and conjugates $f+\omega$ to $F_\omega = \pi \circ (f+\omega) \circ \pi^{-1}$. So $\tau(f+\omega)$ is equal to the modulus of $E^*(F_\omega)$.

The map F_ω is a Möbius map that takes the unit circle to the interior of the unit disc. Let A_ω be its attractor with multiplier $\mu(\omega)$ and R_ω be its repellor. The map $(w - A_\omega)/(w - R_\omega)$ conjugates F_ω to the linear map $w \mapsto \mu(\omega)w$, and thus induces a biholomorphism of $E^*(F_\omega)$ to the complex torus $\mathbb{C}^*/(w \sim \mu(\omega)w)$. The modulus of this torus is equal to $1/(2\pi i)\ln \mu(\omega)$. Finally, $\tau(f+\omega) = 1/(2\pi i)\ln \mu(\omega)$.

Now let us study the boundary values of $\tau(f+\omega)$, i.e., $\bar{\tau}(f+\omega_0) = \lim_{\omega\to\omega_0} \tau(f+\omega)$ for $\omega_0 \in \mathbb{R}/\mathbb{Z}$.

The map F_{ω_0} is a Möbius self-map of the unit circle. If it has two hyperbolic fixed points on the unit circle (i.e., ω_0 is an interior point of I_0), then the multiplier of its attractor, $\mu(\omega_0)$, is real because F_{ω_0} preserves the unit circle. Then

$$\bar{\tau}(f+\omega_0) = \lim_{\omega\to\omega_0} \frac{1}{2\pi i} \ln \mu(\omega) = \frac{1}{2\pi i} \ln \mu(\omega_0) \in i\mathbb{R}.$$

If F_{ω_0} has one parabolic fixed point on the unit circle, then $\lim_{\omega\to\omega_0} \mu(\omega) = 1$, and $\bar{\tau}(f+\omega_0) = 0$. If F_{ω_0} has no fixed points on the unit circle (i.e., $\omega_0 \in (\mathbb{R}/\mathbb{Z}) \setminus I_0$), then it has a unique fixed point A_{ω_0} inside the unit disc and a unique fixed point R_{ω_0} outside it; the Schwarz lemma implies that the multiplier of A_{ω_0}

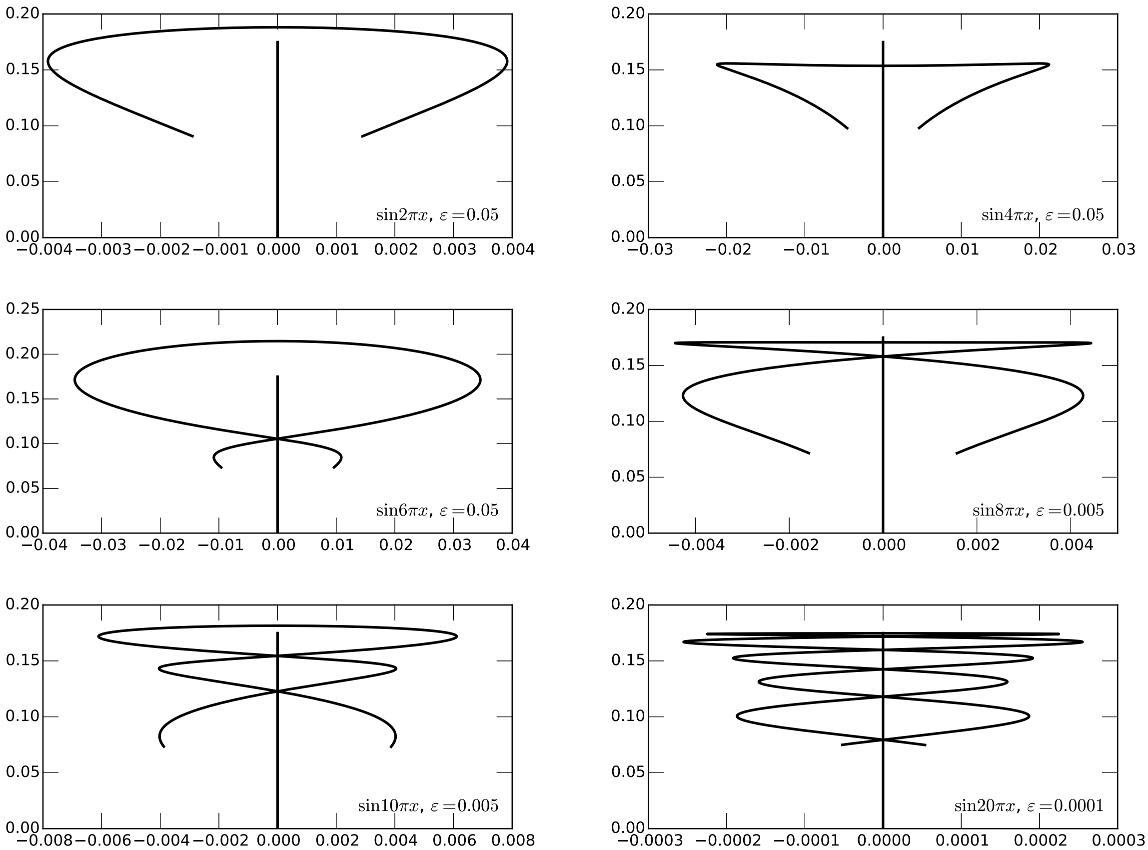

Figure 1. Infinitesimal 0-bubbles for a perturbation of the Möbius map $f = (z+0.5)/(1+0.5z)$ by the map $g = \sin 2\pi n x,\ n = 1, 2, 3, 4, 5, 10$. The pictures are rescaled horizontally. The vertical segment on each picture is the 0-bubble for f.

satisfies $|\mu(\omega_0)| = 1$, so

$$\bar{\tau}(f+\omega_0) = \lim_{\omega\to\omega_0} \frac{1}{2\pi i} \ln \mu(\omega) = \frac{1}{2\pi i} \ln \mu(\omega_0) \in \mathbb{R}/\mathbb{Z}.$$

Finally, the image of I_0 under $\bar{\tau}(f+\cdot)$ belongs to $i\mathbb{R}$, and the image of $(\mathbb{R}/\mathbb{Z}) \setminus I_0$ belongs to $\mathbb{R}/\mathbb{Z}$. We conclude that the only bubble of f is a 0-bubble, and it is a vertical segment. □

Question 11. Is there a way to compute $\bar{\tau}(f+\omega)$ approximately?

In the general case, one can try to implement the construction described in Section 5 as a computer program. The author haven't done this yet. For perturbations of Möbius maps, a simpler approach is described below.

Take a map $f + \varepsilon g$ where f is as in Proposition 10, and g is a trigonometric polynomial. Figure 1 shows infinitesimal 0-bubbles of $f + \varepsilon g$.

Definition. An *infinitesimal 0-bubble* for a perturbation $f+\varepsilon g$ of an analytic circle diffeomorphism f is the image of the segment I_0 for f under the map

$$\omega \mapsto \bar{\tau}(f+\omega)+\varepsilon \cdot \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{\tau}(f+\varepsilon g+\omega),$$

i.e., under the linear approximation to the complex rotation number.

The choice of ε is shown on each picture in Figure 1, but it does not essentially affect the shape of the infinitesimal bubble. In the lower part of bubbles, $(d/d\varepsilon)|_{\varepsilon=0}\bar{\tau}(f+\varepsilon g+\omega)$ tends to infinity. So the linear approximation is not accurate, and this part of infinitesimal bubbles is not shown on the picture.

The following proposition enables us to draw infinitesimal bubbles. Its proof follows the same scheme as the computation in [Risler 1999, Section 2.2.3]; it is postponed until the Appendix.

Proposition 12. *Let f, g be as above. Let γ be a curve in $\mathbb{C}/\mathbb{Z}$ which is close to $\mathbb{R}/\mathbb{Z}$ and passes below the attractor and above the repellor of $f+\omega$, $\omega \in I_0$. Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{\tau}(f+\varepsilon g+\omega)=\int_\gamma \frac{g(z)}{f'(z)}(H'_\omega(z))^2\,dz, \tag{3}$$

where H_ω uniformizes $E(f+\omega)$. As in Proposition 10, one can compute H_ω explicitly. The derivatives in the right-hand side are with respect to z.

For any trigonometric polynomial g (say, $g(x)=\sin 2\pi n x$), the change of variable $w=\pi(z)$ turns the integral (3) into an integral of a rational function along the closed loop $\pi(\gamma)$. We then compute it explicitly via the residue theorem; for $n \geq 3$, the formulas become cumbersome and we use a computer algebra system GiNaC [Bauer et al.; Vollinga 2006] to obtain them. The infinitesimal bubbles thus obtained are shown in Figure 1.

In certain cases, intersections of infinitesimal 0-bubbles for $f+\varepsilon g$ mean that for small ε, the 0-bubbles of $f+\varepsilon g$ intersect as well; see Remark 17 below.

Question 13. Is it true that the map $\omega \mapsto \tau(f+\omega)$ is injective (so that the bubbles belong to the boundary of the set $\{\tau(f+\omega) \mid \omega \in \mathbb{H}/\mathbb{Z}\}$)?

No, see [Buff and Goncharuk 2015, Corollary 16].

Question 14. How large are the bubbles?

In [Buff and Goncharuk 2015, Main Theorem] the authors prove that the p/q-bubble (with coprime p, q) is within a disc of radius $D_f/(4\pi q^2)$ tangent to $\mathbb{R}/\mathbb{Z}$ at p/q, where D_f is the distortion of f,

$$D_f := \int_{\mathbb{R}/\mathbb{Z}} \left|\frac{f''(x)}{f'(x)}\right| dx.$$

Question 15. Can the bubbles intersect or self-intersect?

Here are several results in this direction.

Proposition 16. *If an analytic circle diffeomorphism f is sufficiently close to a rotation in C^2 metrics, its different bubbles do not intersect.*

Proof. We will use the answer to Question 14 above. Suppose that the distortion of f satisfies $D_f < 2\pi$, which holds true if f is C^2-close to a rotation. For each p/q, take the disc of radius $D_f/(4\pi q^2) < 1/(2q^2)$ tangent to $\mathbb{R}/\mathbb{Z}$ at p/q. It is easy to verify that these discs do not intersect for different p/q. As mentioned in the answer to Question 14, the bubbles are within such discs, so they do not intersect as well. □

This proposition does not imply that the bubbles of f are not self-intersecting. This article contains an affirmative answer to Question 15:

Main Theorem. (1) *There exists a circle diffeomorphism f such that its* 0*-bubble is self-intersecting.*

(2) *For each rational p/q, there exists a circle diffeomorphism f such that its* 0*-bubble intersects its p/q-bubble.*

We do not assert that these bubbles intersect transversely; it is possible that they are tangent at a common point.

Remark 17. Let
$$f = \frac{z+0.5}{1+0.5z}$$
be the Möbius map that we chose to draw infinitesimal 0-bubbles. Let $g = \sin 2\pi nx$, $n = 3, 4, 5$, or 10. Using the self-intersections of infinitesimal 0-bubbles for $f+\varepsilon g$, see Figure 1, one may show that for sufficiently small ε, the 0-bubble of $f+\varepsilon g$ is self-intersecting. This provides an alternative proof of the first part of the Main Theorem. Here we sketch this proof.

Let $l_1(\varepsilon)$ and $l_2(\varepsilon)$ be two small intersecting arcs of the infinitesimal 0-bubble for $f+\varepsilon g$. Let $a_\varepsilon, b_\varepsilon$ and $c_\varepsilon, d_\varepsilon$ be the endpoints of $l_1(\varepsilon), l_2(\varepsilon)$ respectively. It is easy to verify that the lengths of the sides and the diagonals of the quadrilateral $a_\varepsilon c_\varepsilon b_\varepsilon d_\varepsilon$ are of order ε, and $l_1(\varepsilon), l_2(\varepsilon)$ are close to these diagonals. The 0-bubble of $f+\varepsilon g$ is $o(\varepsilon)$-close to the infinitesimal 0-bubble for $f+\varepsilon g$, and thus it contains a pair of curves that are $o(\varepsilon)$-close to $l_1(\varepsilon), l_2(\varepsilon)$. This implies that the 0-bubble of $f+\varepsilon g$ is self-intersecting for small ε.

2. Main lemmas

Part 1 of the Main Theorem is based on Lemma 8 and the following lemma.

Lemma 18. *For any hyperbolic analytic circle diffeomorphism f_1 with* $\operatorname{rot} f_1 = 0$ *and any analytic circle diffeomorphism $f_2 \neq \mathrm{id}$, there exists an analytic diffeomorphism f and $\omega \in \mathbb{R}/\mathbb{Z} \setminus \{0\}$ such that f and $f+\omega$ are analytically conjugate to f_1, f_2 respectively.*

This lemma provides a nonrestrictive sufficient condition for two analytic diffeomorphisms to appear (up to analytic conjugacies) in one and the same family of the form $f+\omega$.

Part (2) of the Main Theorem also requires the following lemma, which is interesting in its own right.

Lemma 19. *For any complex number $w \in \mathbb{H}/\mathbb{Z}$ and any natural number m, there exists a hyperbolic circle diffeomorphism f having $2m$ fixed points and the complex rotation number $\bar{\tau}(f) = w$.*

Lemma 8 shows that complex rotation numbers can be used as invariants of analytic classification of families of circle diffeomorphisms; Lemma 19 is a weak version of the realization of these invariants. The following realization question is open:

Question 20. Which holomorphic self-maps of the upper half-plane are realized as $\omega \mapsto \tau(f+\omega)$ for some circle diffeomorphism f?

3. Proof of the Main Theorem modulo Lemmas 18 and 19

3.1. ***Part (1): self-intersecting 0-bubble.*** This part of the Main Theorem does not require Lemma 19.

Fix a hyperbolic circle diffeomorphism f_1 with $\operatorname{rot} f_1 = 0$. Apply Lemma 18 to f_1 and $f_2 = f_1$.

We get a circle diffeomorphism f such that f, $f+\omega$ with $\omega \neq 0 \bmod 1$ are both analytically conjugate to f_1. Due to Lemma 8, $\bar{\tau}(f) = \bar{\tau}(f_1) = \bar{\tau}(f+\omega)$. Note that $\bar{\tau}(f)$, $\bar{\tau}(f+\omega)$ belong to the 0-bubble for f because f, $f+\omega$ have zero rotation number and are hyperbolic.

So the 0-bubble for f passes twice through the point $\bar{\tau}(f_1)$. This completes the proof of the Main Theorem, part (1).

Remark. Using Lemma 19, one can also prove that the 0-bubble may self-intersect at any prescribed point $w \in \mathbb{H}/\mathbb{Z}$. To achieve this, it is sufficient to start with f_1 provided by Lemma 19 such that $\bar{\tau}(f_1) = w$.

3.2. ***Part (2): intersection of 0-bubble and p/q-bubble.*** Take a hyperbolic circle diffeomorphism f_2 with $\operatorname{rot} f_2 = p/q$. Put $w := \bar{\tau}(f_2)$. Using Lemma 19, construct a hyperbolic circle diffeomorphism f_1 with zero rotation number such that $\bar{\tau}(f_1) = w$.

Now, the two circle diffeomorphisms f_1, f_2 satisfy $\operatorname{rot} f_1 = 0$, $\operatorname{rot} f_2 = p/q$ and $\bar{\tau}(f_1) = \bar{\tau}(f_2)$.

Lemma 18 provides us with a circle diffeomorphism f such that f, $f+\omega$ are conjugate to f_1, f_2. Due to Lemma 8, $\bar{\tau}(f) = \bar{\tau}(f_1) = w$ and $\bar{\tau}(f+\omega) = \bar{\tau}(f_2) = w$. The point w belongs to the 0-bubble of f, because $\operatorname{rot} f = \operatorname{rot} f_1 = 0$ and f is hyperbolic, and it also belongs to the p/q-bubble, because $\operatorname{rot}(f+\omega) = \operatorname{rot}(f_2) = p/q$ and $f+\omega$ is hyperbolic. Finally, the 0-bubble and the p/q-bubble for f intersect at w. This completes the proof of the Main Theorem, part (2).

Remark. In a similar way one can prove that the 0-bubble and the p/q-bubble may intersect at any prescribed point $w \in \mathbb{C}/\mathbb{Z}$. This requires an analogue of Lemma 19 for circle diffeomorphisms with nonzero rational rotation numbers; the proof of this analogue repeats the proof of Lemma 19, except for some technical details.

4. Proof of Lemma 18

We say that two circle diffeomorphisms f_1, f_2 have a *Diophantine quotient* if $\operatorname{rot}(f_1 f_2^{-1}) =: \omega$ is Diophantine. Lemma 18 follows from two propositions below.

Proposition 21. *If two analytic circle diffeomorphisms f_1, f_2 have a Diophantine quotient and $\operatorname{rot}(f_1 f_2^{-1}) =: \omega$, then there exists an analytic diffeomorphism f such that f and $f+\omega$ are analytically conjugate to f_1, f_2 respectively.*

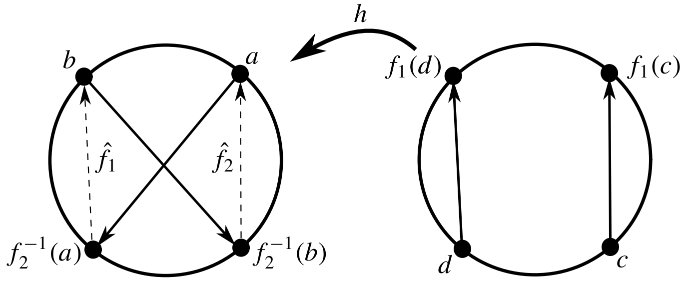

Figure 2. The choice of h that yields $\mathrm{rot}(\hat f_1 f_2^{-1}) = \frac{1}{2}$.

Proposition 22. *Any hyperbolic analytic circle diffeomorphism f_1 with* $\mathrm{rot}\, f_1 = 0$ *is analytically conjugate to a diffeomorphism that has a Diophantine quotient with a given analytic circle diffeomorphism f_2, $f_2 \neq \mathrm{id}$.*

Proof of Proposition 21 . Due to the Herman–Yoccoz theorem (see Theorem 1), in some analytic chart, $f_1 f_2^{-1}$ is the rotation by $\omega = \mathrm{rot}\, f_1 f_2^{-1}$. Let $\tilde f_1$, $\tilde f_2$ be the diffeomorphisms f_1, f_2 in this analytic chart; then $\tilde f_1 \tilde f_2^{-1}(z) = z + \omega$. So $\tilde f_1(z) = \tilde f_2(z) + \omega$, and we can take $f = \tilde f_2$. □

Proof of Proposition 22. Let $\mathcal{A}$ be the set of analytic diffeomorphisms of the form $\hat f_1 = h \circ f_1 \circ h^{-1}$ for all possible analytic orientation-preserving diffeomorphisms h. Then $\mathcal{A}$ is a linearly connected subset of the space of all analytic circle diffeomorphisms, because for each h_1, h_2, we can join h_1 to h_2 by a continuous family of analytic circle diffeomorphisms h^t. Now if we show that the continuous function $\hat f_1 \mapsto \mathrm{rot}(\hat f_1 f_2^{-1})$ on $\mathcal{A}$ takes two distinct values, then it takes all intermediate values, including Diophantine values.

Let us find two maps of the form $\hat f_1 = h \circ f_1 \circ h^{-1}$ such that $\mathrm{rot}(\hat f_1 f_2^{-1})$ attains values 0 and $\frac{1}{2}$:

- $\mathrm{rot}(\hat f_1 f_2^{-1}) = 0$. Choose h such that for some point $a \in \mathbb{R}/\mathbb{Z}$, we have $\hat f_1(a) = f_2(a)$. This is possible, because $f_1 \neq \mathrm{id}$ and $f_2 \neq \mathrm{id}$. Then $\hat f_1 f_2^{-1}(f_2(a)) = f_2(a)$, so $f_2(a)$ is a fixed point for $\hat f_1 f_2^{-1}$, and $\mathrm{rot}(\hat f_1 f_2^{-1}) = 0$.
- $\mathrm{rot}(\hat f_1 f_2^{-1}) = \frac{1}{2}$. Choose two points $a, b \in \mathbb{R}/\mathbb{Z}$ such that these points and their preimages under f_2 are distinct and are ordered in the following way along the circle: $a, b, f_2^{-1}(a), f_2^{-1}(b)$. It is sufficient to take a not fixed and b close to a.

Choose two points $c, d \in \mathbb{R}/\mathbb{Z}$ such that these points and their images under f_1 are distinct and are ordered in the following way along the circle: c, $f_1(c)$, $f_1(d)$, d. It is sufficient to take c and d near an attracting fixed point of f_1, on the different sides with respect to it.

Choose h that takes four points c, $f_1(c)$, $f_1(d)$, d to four points $f_2^{-1}(b)$, a, b, $f_2^{-1}(a)$ (see Figure 2). Then $\hat f_1 = h \circ f_1 \circ h^{-1}$ satisfies $\hat f_1(f_2^{-1}(b)) = a$, $\hat f_1(f_2^{-1}(a)) = b$; hence the point a has period 2 under $\hat f_1 f_2^{-1}$. So $\mathrm{rot}(\hat f_1 f_2^{-1}) = \frac{1}{2}$.

Finally, for some h, the maps $\hat f_1 = h \circ f_1 \circ h^{-1}$ and f_2 have a Diophantine quotient. □

These two propositions imply Lemma 18.

The rest of the article is devoted to the proof of Lemma 19.

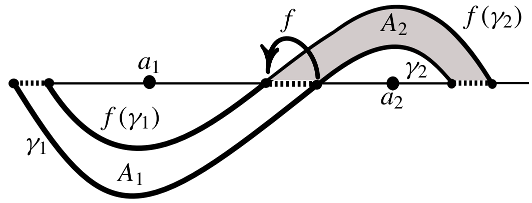

Figure 3. Construction of $\mathcal{E}(f)$.

5. Explicit construction of bubbles

Theorem 5 defines $\bar{\tau}(f+\omega)$, $\omega \in \mathbb{R}/\mathbb{Z}$, as a limit value of the map $\omega \to \tau(f+\omega)$ on the real axis. In this section, we describe $\bar{\tau}(f+\omega)$, $\omega \in I_0$, as a modulus of an explicitly constructed complex torus $\mathcal{E}(f+\omega)$.

This construction was proposed by X. Buff; see [Goncharuk 2012; Buff and Goncharuk 2015] for more details. The key idea of this construction is contained in [Risler 1999], but there it was used in different circumstances.

5.1. ***The complex torus $\mathcal{E}(f)$.*** Let f be a hyperbolic diffeomorphism. Assume that rot $f=0$.

Let a_j, $1 \le j \le 2m$, be its fixed points with multipliers λ_j. We suppose that $0<\lambda_{2j-1}<1<\lambda_{2j}$, i.e., even indices correspond to repellors, and odd indices correspond to attractors. Let $\psi_j : (\mathbb{C}, 0) \to (\mathbb{C}/\mathbb{Z}, a_j)$ be the corresponding linearization charts, i.e., $\psi_j^{-1} \circ f \circ \psi_j(z) = \lambda_j z$, $\psi_j(0)=a_j$, $\psi_j(\mathbb{R}) \subset \mathbb{R}/\mathbb{Z}$, and ψ_j preserve orientation on $\mathbb{R}$. We extend these charts by iterates of f so that the image of ψ_j contains (a_{j-1}, a_{j+1}).

Construct a simple loop $\gamma \subset \mathbb{C}/\mathbb{Z}$ (*le courbe ascendante*, in terms of [Risler 1999]) such that $f(\gamma)$ is above γ in $\mathbb{C}/\mathbb{Z}$. Namely, let $\gamma = \bigcup \gamma_j$; let γ_j have its endpoints on (a_{j-1}, a_j) and (a_j, a_{j+1}); let γ_j be the image of an arc of a circle under ψ_j; let γ_j be above $\mathbb{R}/\mathbb{Z}$ if j is even, and below $\mathbb{R}/\mathbb{Z}$ if j is odd. Since ψ_j conjugates f to $z \mapsto \lambda_j z$, the curve $f(\gamma)$ is above γ in $\mathbb{C}/\mathbb{Z}$.

Let $\widetilde{\Pi} \subset \mathbb{C}/\mathbb{Z}$ be a curvilinear cylinder between γ and $f(\gamma)$ (see Figure 3). Consider the complex torus $\mathcal{E}(f)$ which is the quotient space of a neighborhood of $\widetilde{\Pi}$ by the action of f. Due to the uniformization theorem, there exists $\tau \in \mathbb{H}/\mathbb{Z}$ and a biholomorphism $\widetilde{H}_\omega : \mathcal{E}(f) \to \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ that takes γ to a curve homotopic to $\mathbb{R}/\mathbb{Z}$. Let $\tau(\mathcal{E}(f)) := \tau$ be the modulus of $\mathcal{E}(f)$.

For rot $f=p/q$, the construction of γ should be slightly modified: ϕ_j are linearizing charts of f^q at its fixed points, γ_j are arcs of circles in charts ϕ_j, we let $\gamma = \bigcup \gamma_j$, γ winds above repelling periodic points of f and below attracting periodic points of f, and we choose γ_j so that $f(\gamma)$ is above γ in $\mathbb{C}/\mathbb{Z}$. The rest of the construction is analogous to the case of rot $f=0$.

Theorem 23 [Goncharuk 2012; Buff and Goncharuk 2015, Section 6]. *Let f be a hyperbolic circle diffeomorphism with rational rotation number*; *define $\mathcal{E}(f)$ as above. Then the modulus $\tau(\mathcal{E}(f))$ of the torus $\mathcal{E}(f)$ equals $\bar{\tau}(f)$.*

Due to the construction, $\mathcal{E}(f)$ does not depend on the analytic chart on $\mathbb{R}/\mathbb{Z}$. This implies Lemma 8.

So in order to prove Lemma 19, it is sufficient to find a circle diffeomorphism f with $2m$ fixed points such that $\tau(\mathcal{E}(f))=w$.

5.2. ***Cutting $\mathcal{E}(f)$ by the real line.*** Let $A_j \subset \widetilde{\Pi}$ be the domain bounded by γ_j, $f(\gamma_j)$, and two segments of $\mathbb{R}/\mathbb{Z}$. Note that the complex manifold $\widetilde{A}_j := A_j/f$ is an annulus, and $\widetilde{A}_j \subset \widetilde{\Pi}/f = \mathcal{E}(f)$.

Let $\mathbb{H}^+ = \mathbb{H}$ and $\mathbb{H}^-$ be the upper and the lower half-planes of $\mathbb{C}$ respectively. From now on, we use the notation $A^{\pm}(\lambda)$ for the following standard annulus: $A^{\pm}(\lambda) := \mathbb{H}^{\pm}/(z \sim \lambda z)$. It is easy to see that its modulus is $\pi/|\log \lambda|$.

Remark 24. The linearizing chart ψ_j induces the map from $\widetilde{A}_j$ to the standard annulus $A^+(\lambda_j)$ for even j, and to $A^-(\lambda_j)$ for odd j. This follows from the fact that ψ_j conjugates f to $x \mapsto \lambda_j z$.

This gives a full description of $\mathcal{E}(f)$ in terms of multipliers and transition maps of f: $\mathcal{E}(f)$ is biholomorphically equivalent to the quotient space of the annuli $A^{\pm}(\lambda_j)$, $\operatorname{mod} A^{\pm}(\lambda_j) = \pi/|\log \lambda_j|$, by the transition maps $\psi_{j+1}^{-1} \circ \psi_j$ between linearizing charts of f.

6. Circle diffeomorphisms with prescribed complex rotation numbers

In this section, we prove Lemma 19.

6.1. ***Scheme of the proof.*** Remark 24 above shows that $\mathcal{E}(f)$ can have any modulus, which nearly implies Lemma 19. Indeed, we can obtain a complex torus of an arbitrary modulus by gluing some $2m$ annuli by some maps. We only need to show that there are no restrictions on possible multipliers and transition maps for an analytic circle diffeomorphism. This follows from Theorem 25 below.

The above arguments together with Theorem 25 show that $\mathcal{E}(f)$ can be biholomorphic to a standard torus of any modulus; however, we must also check that this biholomorphism matches the generators, as required by the definition of $\tau(\mathcal{E}(f))$; see Section 5 above. The formal proof of Lemma 19, with the explicit construction of f and the examination of generators, is contained in Section 6.3.

6.2. ***Moduli of analytic classification of hyperbolic circle diffeomorphisms.*** The following theorem is an analytic version of a smooth classification of hyperbolic diffeomorphisms due to G. R. Belitskii [1986, Proposition 2]. The proof is completely analogous, but we provide it for the sake of completeness.

Theorem 25. *Suppose that we are given a tuple of $2m$ real numbers λ_j with $0 < \lambda_{2j-1} < 1 < \lambda_{2j}$, and a tuple of analytic orientation-preserving diffeomorphisms $\psi_{j;j+1} : \mathbb{R}^+ \to \mathbb{R}^-$ such that $\psi_{j;j+1}(\lambda_j z) = \lambda_{j+1}\psi_{j;j+1}(z)$.*

Then there exists an analytic circle diffeomorphism f such that it has $2m$ fixed points with multipliers λ_j, and $\psi_{j;j+1}$ are transition maps between their linearization charts ψ_j: $\psi_{j;j+1} = \psi_{j+1}^{-1} \circ \psi_j$.

Remark. It is also true that such an f is unique up to analytic conjugacy, so the data above is the modulus of an analytic classification of hyperbolic circle diffeomorphisms. Given f, transition maps $\psi_{j;j+1}$ are uniquely defined up to the equivalence

$$(\dots \psi_{j-1;j} \dots) \sim (\dots, a_j \psi_{j-1;j}(z/a_{j-1}), \dots)$$

for some numbers $a_j > 0$; see [Belitskii 1986, Proposition 3].

Proof. Take $2m$ copies of the real axis and glue the j-th to the $(j+1)$-th copy by the map $\psi_{j;j+1} : \mathbb{R}^+ \to \mathbb{R}^-$. We get a one-dimensional C^ω-manifold homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$. It is well known that such manifolds are C^ω-equivalent to $\mathbb{R}/\mathbb{Z}$. Thus there exists a tuple of C^ω charts $\psi_j : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ such that $\psi_{j;j+1} = \psi_{j+1}^{-1} \circ \psi_j$. Due to the equality $\psi_{j;j+1}(\lambda_j z) = \lambda_{j+1}\psi_{j;j+1}(z)$, the maps $\psi_j(\lambda_j \psi_j^{-1}(z))$ glue into the well-defined C^ω circle diffeomorphism f.

Let $a_j = \psi_j(0)$. Note that $f(a_j) = \psi_j(\lambda_j \psi_j^{-1}(a_j)) = \psi_j(\lambda_j \cdot 0) = \psi_j(0) = a_j$, so these points are fixed points of f.

On a segment (a_{j-1}, a_{j+1}), the map ψ_j conjugates $f = \psi_j \circ \lambda_j \psi_j^{-1}$ to $z \mapsto \lambda_j z$, so ψ_j is a linearizing chart of a fixed point a_j, and λ_j is the multiplier of f at a_j. □

6.3. ***Proof of Lemma 19, see Figure 4.*** Recall that our aim is to construct a circle diffeomorphism f with $2m$ hyperbolic fixed points and the complex rotation number w.

Consider the standard elliptic curve $E_w = \mathbb{C}/(\mathbb{Z} + w\mathbb{Z})$; let $\mathbb{R}/\mathbb{Z}$ and $w\mathbb{R}/w\mathbb{Z}$ be its first and second generators respectively. Take $2m$ arbitrary disjoint simple real-analytic loops $\nu_j \subset E_w$ along the second generator. Let $\mathcal{A}_j \subset E_w$ be the annulus between ν_j and ν_{j+1}. Let $\Delta_j = \mathcal{A}_j \cap \mathbb{R}/\mathbb{Z}$; then Δ_j joins boundaries of $\mathcal{A}_j$.

We are going to construct a circle diffeomorphism f with $2m$ fixed points, and a biholomorphism $H : \mathcal{E}(f) \to E_w$ such that $H(\widetilde{A}_j) = \mathcal{A}_j \subset E_w$, where $\widetilde{A}_j$ are the annuli in $\mathcal{E}(f)$ bounded by intervals of $\mathbb{R}/\mathbb{Z}$ as in Section 5.2. This biholomorphism H will take the class of γ in $\mathcal{E}(f)$ to the class of $\mathbb{R}/\mathbb{Z} = \bigcup \Delta_j$ in E_w. This will prove that the modulus of $\mathcal{E}(f)$ equals w.

Uniformize $\mathcal{A}_j$. For each annulus $\mathcal{A}_j$ where j is even, take $\lambda_j > 1$ such that there exists a biholomorphism $\widetilde{\Psi}_j : A^+(\lambda_j) \to \mathcal{A}_j$. For each annulus $\mathcal{A}_j$ where j is odd, take $\lambda_j < 1$ such that there exists a biholomorphism $\widetilde{\Psi}_j : A^-(\lambda_j) \to \mathcal{A}_j$. Each map $\widetilde{\Psi}_j$ extends analytically to a neighborhood of $A^\pm(\lambda_j)$ in $\mathbb{C}^*/(z \sim \lambda_j z)$, because the boundaries of $\mathcal{A}_j$ are real-analytic curves ν_j. Assume that $\widetilde{\Psi}_j^{-1}(\nu_j)$ is the left boundary of $A^\pm(\lambda_j)$; that is, $\widetilde{\Psi}_j^{-1}(\nu_j) = \mathbb{R}^-/(z \sim \lambda_j z)$. Then $\widetilde{\Psi}_j^{-1}(\nu_{j+1}) = \mathbb{R}^+/(z \sim \lambda_j z)$.

Let $\Psi_j : \overline{\mathbb{H}}^\pm \setminus \{0\} \to \mathcal{A}_j$ be the lift of $\widetilde{\Psi}_j$ to the universal cover of $A^\pm(\lambda_j)$; then $\Psi_j(\lambda_j z) = \Psi_j(z)$. For each j, choose one of the preimages $\delta_j = \Psi_j^{-1}(\Delta_j)$. Let $l_j \in \mathbb{R}^-$, $r_j \in \mathbb{R}^+$ be the left and the right endpoints of δ_j respectively. Consider the maps $\psi_{j;j+1} : \mathbb{R}^+ \to \mathbb{R}^-$,

$$\psi_{j;j+1} = \Psi_{j+1}^{-1} \circ \Psi_j,$$

where we choose the branch of Ψ_{j+1}^{-1} so that $\psi_{j;j+1}(r_j) = l_{j+1}$. Note that $\psi_{j;j+1}(\lambda_j z) = \lambda_{j+1}\psi_{j;j+1}(z)$ because $\Psi_j(\lambda_j z) = \Psi_j(z)$.

Now, the complex torus E_w is biholomorphically equivalent to the quotient space of annuli $A^\pm(\lambda_j)$ by the maps $\psi_{j;j+1}$. This, together with Remark 24, motivates the construction of f below.

Construct f and a biholomorphism $H : \mathcal{E}(f) \to E_w$. Use Theorem 25 to construct f with multipliers λ_j and transition maps $\psi_{j;j+1}$.

Let ψ_j be linearization charts of its fixed points; then $\psi_{j;j+1} = \psi_{j+1}^{-1} \circ \psi_j$. Let $\gamma, \mathcal{E}(f), A_j, \widetilde{A}_j$ be defined as in Section 5 for this circle diffeomorphism f.

Consider the tuple of maps $\Psi_j \circ \psi_j^{-1}$ on $A_j \subset \mathbb{C}/\mathbb{Z}$. These maps agree on the boundaries of A_j due to the equality

$$(\Psi_{j+1} \circ \psi_{j+1}^{-1})^{-1} \circ \Psi_j \circ \psi_j^{-1} = \psi_{j+1} \circ \Psi_{j+1}^{-1} \circ \Psi_j \circ \psi_j^{-1} = \psi_{j+1} \circ \psi_{j;j+1} \circ \psi_j^{-1} = \mathrm{id},$$

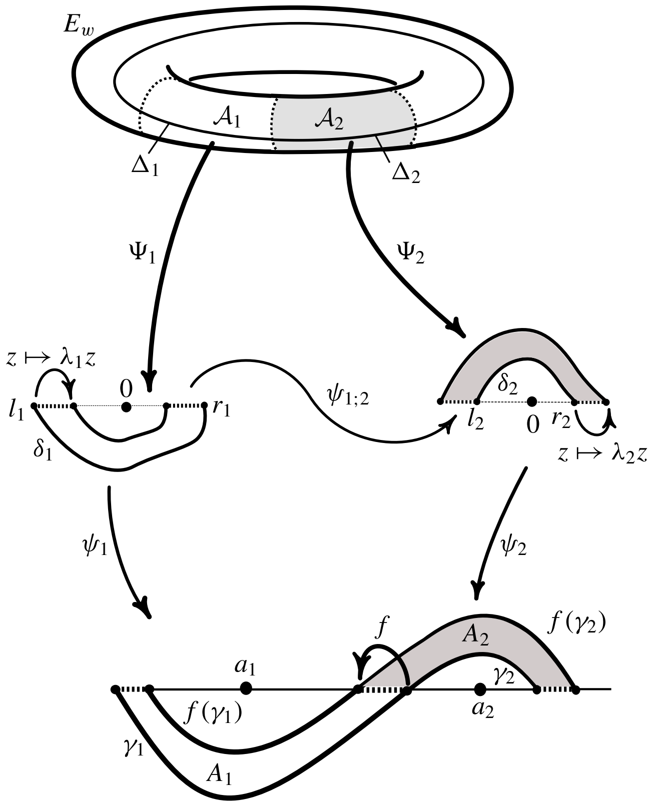

Figure 4. Proof of Lemma 19.

so they define one map on $\widetilde{\Pi}$. They descend to the map $H : \mathcal{E}(f) \to E_w$ because ψ_j conjugates f to $z \mapsto \lambda_j z$ and $\Psi_j(\lambda_j z) = \Psi_j(z)$. Clearly, $H(\widetilde{A}_j) = \mathcal{A}_j$.

H takes the class of γ in $\mathcal{E}(f)$ to the first generator of E_w. Note that the curves $\psi_j(\delta_j)$ have common endpoints $\psi_j(r_j) = \psi_{j+1}(l_{j+1})$ since $\psi_{j;j+1}(r_j) = l_{j+1}$. So $\gamma' := \bigcup \psi_j(\delta_j)$ is a loop in $\mathbb{C}/\mathbb{Z}$ that passes above the attractors $\psi_{2j-1}(0)$ and below the repellors $\psi_{2j}(0)$ of f. So γ' is homotopic to γ in an annular neighborhood of $\mathbb{R}/\mathbb{Z}$ covered by linearizing charts of fixed points; the homotopy does not pass through fixed points. Hence γ' is homotopic to γ in $\mathcal{E}(f)$, i.e., corresponds to the first generator of $\mathcal{E}(f)$.

Finally, $H(\gamma') = \bigcup \Psi_j(\delta_j) = \bigcup \Delta_j = \mathbb{R}/\mathbb{Z} \subset E_w$. This completes the proof of Lemma 19.

Appendix: Derivatives of complex rotation number

In this section we compute $(\partial/\partial\omega)\bar{\tau}(f_\omega)$ for a family of circle diffeomorphisms f_ω. In particular, this yields Proposition 12. The computation is analogous to that of [Risler 1999, Section 2.2.3].

Let f_ω be an analytic family of analytic circle diffeomorphisms. Let $G_\omega := \widetilde{H}_\omega^{-1}$, where $\widetilde{H}_\omega$ rectifies the complex torus $\mathcal{E}(f_\omega)$; see Section 5. Let $\tau(\omega) = \bar{\tau}(f_\omega)$. Then

$$f_\omega(G_\omega(z)) = G_\omega(z + \tau(\omega)) \quad \text{for } z \in G_\omega^{-1}(\gamma). \tag{4}$$

The Ahlfors–Bers theorem implies that the map G_ω, if suitably normalized, depends analytically on ω; see [Risler 1999, Section 2.1, Proposition 2].

Fix $\omega = \omega_0 \in \mathbb{R}/\mathbb{Z}$; in what follows, all derivatives with respect to ω are evaluated at $\omega = \omega_0$, and we will omit the lower indices in f_ω, G_ω etc. Here and below G', f' are derivatives with respect to z; G'_ω, f'_ω, τ'_ω are derivatives with respect to ω.

The following proposition clearly implies Proposition 12.

Proposition 26. *Let f_ω, G_ω be as above. Then*

$$\tau'_\omega = \int_\gamma \frac{f'_\omega(w)}{f'(w)}((G^{-1})'(w))^2\, dw,$$

where all derivatives are evaluated at $\omega = \omega_0$.

Proof. We may and will assume that the curve γ in the construction of $\mathcal{E}(f_\omega)$ does not depend on ω in a small neighborhood of ω_0.

Differentiate (4) with respect to ω:

$$f'_\omega|_{G(z)} + f'|_{G(z)} G'_\omega(z) = G'_\omega(z+\tau) + G'(z+\tau)\tau'_\omega.$$

Express τ'_ω using this equation and the identity $G'(z+\tau) = f'|_{G(z)} G'(z)$ (this is the derivative of (4)). We get

$$\tau'_\omega = \frac{f'_\omega|_{G(z)}}{G'(z+\tau)} + \frac{G'_\omega(z)}{G'(z)} - \frac{G'_\omega(z+\tau)}{G'(z+\tau)}.$$

Integrate this expression along $G^{-1}(\gamma)$. The second and the third summands cancel out because the function $G'_\omega(z)/G'(z)$ is holomorphic. We obtain

$$\tau'_\omega = \int_{G^{-1}(\gamma)} \frac{f'_\omega|_{G(z)}}{G'(z+\tau)}\, dz.$$

Using again $G'(z+\tau) = G'(z) f'|_{G(z)}$ and making the change of variable $w = G(z)$, we get the desired formula. □

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