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INVARIANT MEASURE AND LONG TIME BEHAVIOR OF REGULAR SOLUTIONS OF THE BENJAMIN–ONO EQUATION

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The Benjamin–Ono equation describes the propagation of internal waves in a stratified fluid. In the present work, we study large time dynamics of its regular solutions via some probabilistic point of view. We prove the existence of an invariant measure concentrated on $C^\infty(\mathbb{T})$ and establish some qualitative properties of this measure. We then deduce a recurrence property of regular solutions and other corollaries using ergodic theorems. The approach used in this paper applies to other equations with infinitely many conservation laws, such as the KdV and cubic Schrödinger equations in one dimension. It uses the fluctuation-dissipation-limit approach and relies on a *uniform* smoothing lemma for stationary solutions to the damped-driven Benjamin–Ono equation.

1. Introduction

The problem and statement of the main result. The Benjamin–Ono (BO) equation

$$\partial_t u + H\,\partial_x^2 u + u\,\partial_x u = 0 \tag{1-1}$$

describes the propagation of internal waves in a stratified fluid. The operator H in the equation is the Hilbert transform; it can be defined in the Fourier setting as the multiplier given by $-\mathrm{i}\,\mathrm{sgn}$ (see the Appendix). We assume that $u(t,x)$ is a real-valued function, $t \in \mathbb{R}_+$ and x belongs to the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. In this setting, existence and uniqueness of solution hold in any Sobolev space H^s for $s \geq 0$ (see, e.g., [Molinet 2008; Molinet and Pilod 2012] for its global well-posedness in $L^2(\mathbb{T})$). In the present paper, we use only the well-posedness of the problem in Sobolev spaces $H^s(\mathbb{T})$ with $s \geq 2$, so we refer the reader to [Abdelouhab et al. 1989].

In $L^2 := L^2(\mathbb{T})$, the well-posedness of (1-1) generates a topological dynamical system (DS) (L^2, ϕ_t), where ϕ_t is the flow of the equation. We are concerned with the description of the long time behavior of this dynamical system.

Given a Borel measure μ on L^2, we say that μ is invariant for (L^2, ϕ_t) if for any Borel set A of L^2 we have

$$\mu(\phi_t^{-1} A) = \mu(A) \quad \text{for all } t.$$

When such a measure exists, the triple (L^2, ϕ_t, μ) is called a measurable dynamical system (MDS). If in addition μ is finite, then we have very important information on the dynamics. Indeed the Poincaré recurrence theorem states that the dynamics is recurrent; that is, μ-almost every orbit returns in any

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neighborhood of its origin in finite time. The well-known Von Neumann and Birkhoff ergodic theorems also apply to give more information on the long time behavior of the system. Our aim here is to construct such a measure, which will contribute to improving the understanding of the behavior of the solutions of (1-1).

Matsuno [1984] derived (at least formally) infinitely many conservation laws for the BO equation (1-1). They have the form

$$E_n(u) = \|u\|_n^2 + R_n(u), \quad n \in \tfrac{1}{2}\mathbb{N}, \tag{1-2}$$

where $\|\cdot\|_n$ stands for the homogeneous Sobolev norm of order n and R_n is a lower-order term.

In [Tzvetkov and Visciglia 2013; 2014; 2015; Deng 2015; Deng et al. 2015], the authors constructed a sequence of invariant Gaussian-type measures $\{\mu_n\}$ for (L^2, ϕ_t) satisfying

$$\mu_n \text{ is concentrated on } H^s(\mathbb{T}) \text{ for } s < n - \tfrac{1}{2}, \tag{$*$}$$

$$\mu_n(H^{n-\frac{1}{2}}(\mathbb{T})) = 0. \tag{$**$}$$

Formally, μ_n is defined as a renormalization of

$$d\mu_n(u) = e^{-E_n(u)}\, du = e^{-R_n(u)} e^{-\|u\|_n^2}\, du,$$

where $E_n(u)$ and $R_n(u)$ are the quantities given in (1-2). These authors constructed a Gaussian interpretation of the expression $e^{-\|u\|_n^2}\, du$ on the concerned spaces and proved that $e^{-R_n(u)}$ is an integrable density. In view of these results, there is an MDS for (1-1) in any Sobolev space and then its large time dynamics is described keeping in mind the theorems mentioned above. However, these results do not apply to infinitely smooth solutions; indeed by the property $(**)$ we have

$$\mu_n(C^\infty(\mathbb{T})) = 0 \quad \text{for all } n.$$

In the present work, we construct a measurable dynamical system for (1-1) on the space $C^\infty(\mathbb{T})$. Naturally, the Dirac measure at 0 is not the desired measure; although it is invariant under the flow of the BO equation, it gives only trivial information. More generally, to get substantial information on the system we have to also avoid singular measures. Another example of such a measure is the one concentrated on a stationary solution. Notice that measures μ_n discussed above verify the following "consistency" property: every set of full μ_n-measure is dense in $\dot{H}^{(n-\frac{1}{2})^-}$. Concerning the space C^∞, an obstruction to the construction of an invariant Gaussian-type measure is the nonexistence of a conservation law compatible with the regularity of that space. In particular, the approach used in the construction of the measures μ_n above does not seem to apply.

Another method allowing the construction of invariant measures (a priori not of Gaussian type) for PDEs was developed in [Kuksin 2004; Kuksin and Shirikyan 2004] in the context of Euler and Schrödinger equations, respectively. It is based on a fluctuation-dissipation (FD) argument and consists of adding to the equation appropriately normalized damping and stochastic terms, constructing an invariant measure for the resulting problem, and passing to the limit. But, a priori, the obstruction encountered in the Gaussian-type measure approach still remains in the FD approach because the underlying regularization is of Sobolev order and not C^∞. The idea in the present work is to exploit the regularization inherent in this approach with the use of an infinite subsequence of the Benjamin–Ono conservation laws to reach the C^∞-regularity.

In order to bring out a key preliminary result, we give the following stochastic set up: consider the diffusion problem (also called the stochastic Benjamin–Ono–Burgers (BOB) equation)

$$\partial_t u + H\,\partial_x^2 u + u\,\partial_x u = \alpha\,\partial_x^2 u + \sqrt{\alpha}\eta, \quad t > 0,\ x \in \mathbb{T}, \tag{1-3}$$

where η is a stochastic force and $\alpha \in (0, 1)$ is a viscosity parameter. In fact the problem (1-1) is the limit as $\alpha \to 0$ of (1-3). A probabilistic global well-posedness for (1-3) is proved in Section 3. Moreover, in Section 4, we establish the existence of stationary solutions[1] for this equation. We present now the following smoothing property for stationary solutions:

Lemma 1.1. *Suppose that the noise η is sufficiently regular in space. Let u_α be a stationary solution to (1-3) such that*

$$\mathbb{E}\|u_\alpha(t)\|^p < \infty \quad \textit{for all } p \geq 2. \tag{1-4}$$

Then

$$\mathbb{E}\|u_\alpha(t)\|_n^2 < \infty \quad \textit{for all } n \geq 1. \tag{1-5}$$

Moreover, if (1-4) holds uniformly in α then so does (1-5).

The proof of this lemma relies on a combination of deterministic and probabilistic estimations based on the conservation laws of (1-1).

We prove in Section 4 that any stationary solution to (1-3) satisfies (1-4) uniformly in α. Then, from (1-5) we conclude that stationary solutions to (1-3) are concentrated on C^∞. Passing to the limit as the viscosity goes to 0, we find the main result of this paper (Theorems 5.3, 6.1, 6.3 and 6.6):

Theorem 1.2. *There is a probability measure μ invariant under the flow of the BO equation (1-1) defined on $H^3(\mathbb{T})$ and such that*

$$\mu(C^\infty(\mathbb{T})) = 1.$$

Moreover, μ satisfies the following properties:

(1) *For any integer n, we have*

$$0 < \int_{H^3} \|u\|_n^2\,\mu(du) < \infty.$$

(2) *There are constants $\sigma, C > 0$ such that for any $R > 0$*

$$\mu(u \in H^3, \|u\| \geq R) \leq Ce^{-\sigma R^2}.$$

(3) *There is an infinite sequence of conservation laws of the form (1-2) whose laws under μ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

(4) *The measure μ is of at least 2-dimensional nature in the sense that any compact set of Hausdorff dimension smaller than 2 has μ-measure 0.*

In fact, we expect infinite-dimensionality of the measure constructed here as in [Kuksin 2008; Kuksin and Shirikyan 2012] concerning the 2-dimensional Euler equations. To show this property in the context of the Benjamin–Ono equation, we have to prove some algebraic independence of the gradients of the

[1] Solutions to (1-3) whose laws are invariant along the time.

conservation laws. In the present work, we face a technical difficulty in establishing such an independence for an arbitrary number of conservation laws. We propose a proof inspired by [Kuksin 2008; Kuksin and Shirikyan 2012] which works for the (at least) 2-dimensionality. Then the infinite-dimensionality of μ remains an open question.

We deduce the following result by applying the Poincaré recurrence theorem.

Corollary 1.3. *For μ-almost all w in $C^\infty(\mathbb{T})$, there is a sequence $\{t_k\}$ increasing to infinity such that*

$$\lim_{k\to\infty} \|S_{t_k} w - w\|_n = 0 \quad \text{for any } n \geq 0.$$

Here S_t denotes the flow of the Benjamin–Ono equation (1-1) on $H^3(\mathbb{T})$.

In the construction of such a measure, we use the control of Sobolev norms provided by the infinite sequence of conservation laws. The KdV and cubic 1-dimensional NLS equations have infinitely many conservation laws whose structure is similar to (1-2) and our approach applies to these equations. Notice that an infinite sequence of invariant Gaussian-type measures of increasing regularity was constructed for KdV and cubic 1-dimensional NLS equations in [Zhidkov 2001a; 2001b]; we give then a kind of extension of this work to the $C^\infty(\mathbb{T})$ space. However, the Benjamin–Ono equation is more difficult than these equations because of the weakness of its dispersion compared to KdV and the presence of a derivative in its nonlinearity compared to NLS. Then, here, we confine ourselves to the study of the BO equation, which is less understood.

Let us briefly discuss an equation having infinitely many conservation laws but which is not admissible to the approach developed here. Consider the nonviscous Burgers' equation

$$\partial_t u + u\,\partial_x u = 0. \tag{1-6}$$

It is easy to check that an infinite sequence of conservation laws is given by the quantities

$$L_p(u) = \int u^p, \quad p \geq 1.$$

Our approach does not apply to (1-6). This is due to its lack of dispersion which breaks the control of Sobolev's norms.

Notation. • Let A and B be two positive quantities, we write

$$A \lesssim B$$

if there is a universal constant $\lambda \geq 0$ such that $A \leq \lambda B$.

• For a real number r, we denote by r^+ (resp. r^-) the quantity $r + \epsilon$ (resp. $r - \epsilon$), where ϵ is a positive number close enough to 0, while $r_+ := \max(r, 0)$.

• $\mathbb{Z}$ denotes the set of nonzero integers.

• $\dot{H}(\mathbb{T}) = \{u \in L^2(\mathbb{T}) \mid \int_{\mathbb{T}} u(x)\,dx = 0\}$.

• $\dot{H}^s(\mathbb{T}) = \{u \in \dot{H}(\mathbb{T}) \mid D^s u \in \dot{H}(\mathbb{T})\}$, and D^s is the s-th derivative of u, where $s \geq 0$.

• The $\dot{H}^s$-norm is denoted by $\|\cdot\|_s$ when $s > 0$ and the L^2-norm is denoted by $\|\cdot\|$.

• For a functional $A(u)$, we denote the first and second derivatives of A by $A'(u,v) := \partial_u A(u,v) = \partial A|_u(v)$ and $A''(u,v) := \partial_u^2 A(u,v) = \partial^2 A|_u(v,v)$.

• The sequence $\{e_n \mid n \in \mathbb{Z}\}$ is given by

$$e_n(x) = \begin{cases} \sin(nx)/\sqrt{\pi} & \text{for } n > 0, \\ \cos(nx)/\sqrt{\pi} & \text{for } n < 0 \end{cases}$$

and forms an orthonormal basis of $\dot{H}(\mathbb{T})$.

• $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space and $\mathscr{F}_t$ is a right-continuous filtration augmented with respect to $(\mathscr{F}, \mathbb{P})$. Given a sequence of real numbers $\{\lambda_n\}$ and a sequence of independent real standard Brownian motions $\{\beta_n(t)\}$ adapted to $\mathscr{F}_t$, we set

$$\zeta(t,x) = \sum_{n\in\mathbb{Z}} \lambda_n \beta_n(t) e_n(x), \tag{1-7}$$

$$\eta(t,x) = \frac{d}{dt}\zeta(t,x), \tag{1-8}$$

$$A_s = \sum_{n\in\mathbb{Z}} \lambda_n^2 n^{2s}. \tag{1-9}$$

Some stochastic results. The theorem and lemma below are useful ingredients in our work; we refer to [Karatzas and Shreve 1991] for their proofs.

Theorem 1.4 (Doob's optional theorem). *Let x_t be a continuous $\mathscr{F}_t$-martingale and $\tau \le \sigma$ be two $\mathscr{F}_t$-stopping times which are almost surely finite. Then*

$$\mathbb{E}x_\tau = \mathbb{E}x_\sigma = \mathbb{E}x_0. \tag{1-10}$$

Lemma 1.5. *Let x_t be a continuous random process which is adapted to $\mathscr{F}_t$. Then $x_t(\omega)$ is adapted to $\mathscr{F}_t$.*

Stochastic convolution. Let B be an operator on a separable Hilbert space H with which we endow a Hilbert basis $\{e_m\}_{m\in\mathbb{Z}}$. Suppose that $\{e_m\}$ are eigenvectors of B whose associated eigenvalues are $\{b_m\} \subset \mathbb{C}$, and moreover $|b_m| \to \infty$ as $m \to \infty$. Suppose that

$$V_t(B) := \sum_{m\in\mathbb{Z}} \lambda_m^2 \frac{|1 - e^{2tb_m}|}{2|b_m|} < \infty \quad \text{for all } t \ge 0; \tag{1-11}$$

then the quantity (which is called stochastic convolution)

$$\Theta_t(B) := \int_0^t e^{(t-s)B}\, d\zeta(s,x) := \sum_{m\in\mathbb{Z}} \lambda_m \left(\int_0^t e^{(t-s)b_m}\, d\beta_m(s) \right) e_m(x), \quad t \ge 0, \tag{1-12}$$

is well-defined in H. In fact $\Theta_t(B)$ is a continuous Gaussian process in H: for all $t \ge 0$, we have $\Theta_t(B) \sim \mathscr{N}_H(0, V_t(B))$.

Remark 1.6. If $\mathrm{Re}(b_m) < 0$, then the sequence $\{|1 - e^{2tb_m}|/|2b_m|\}$ is bounded (even uniformly in t); therefore $\Theta_t(B)$ is well-defined in H as soon as $\sum_m \lambda_m^2 < \infty$.

The concrete case that is studied in this paper is the Hilbert space $L^2(\mathbb{T})$ and an operator of type $-\partial_x^2$ (more exactly $B = -(H - \alpha)\,\partial_x^2$). With the use of the Itô isometry, we have

$$\mathbb{E}\|\Theta_t(B)\|_s^2 = \sum_{m\in\mathbb{Z}} m^{2s}\lambda_m^2 \frac{|1 - e^{2tb_m}|}{2|b_m|}, \quad b_m = -(\mathrm{i}\,\mathrm{sgn}(m) + \alpha)m^2. \tag{1-13}$$

Then $\mathrm{Re}(b_m) < 0$ for any $m \in \mathbb{Z}$. Therefore $\Theta_t(B) \in H^s$ almost surely as soon as

$$\sum_{m\in\mathbb{Z}} m^{2s}\lambda_m^2 < \infty. \tag{1-14}$$

In that case, as a Gaussian random variable in H^s, the stochastic convolution Θ_t verifies the Fernique theorem; that is, there is a constant c_s such that

$$\mathbb{E}e^{c_s\|\Theta_t\|_{H^s}^2} < \infty \quad \text{for all } t \geq 0. \tag{1-15}$$

Stochastic well-posedness and the Itô property relative to a Gelfand triple. Let us consider the following stochastic PDE:

$$du_t = (Lu + f(u))\,dt + d\zeta, \tag{1-16}$$

where L is a differential operator, f is a function possibly nonlinear in u and ζ is a Brownian motion defined as in (1-7).

Definition 1.7. Let $s \in \mathbb{R}$. Equation (1-16) is said to be stochastically (globally) well-posed in H^s if for all $T > 0$ the following properties hold:

(1) For any random variable u_0 in H^s which is independent of $\mathscr{F}_t$, we have, for almost all $\omega \in \Omega$,
   (a) (existence) There exists $u := u^\omega \in \Lambda_T(s) := C(0, T; H^s) \cap L^2(0, T; H^{s+1})$ satisfying the relation
   $$u(t) = u_0 + \int_0^t (Lu_s + f(u_s))\,ds + \zeta(t) \quad \text{for all } t \in [0, T] \tag{1-17}$$
   in H^{s-1}. We denote this solution by $u(t, u_0) := u^\omega(t, u_0)$.
   (b) (uniqueness) If $u_1, u_2 \in \Lambda_T(s)$ are two solutions in the sense of (1-17), then $u_1 \equiv u_2$ on $[0, T]$.

(2) (continuity with respect to initial data) For almost all ω, we have
$$\lim_{u_0 \to u_0'} u(\cdot, u_0) = u(\cdot, u_0') \quad \text{in } \Lambda_T(s), \tag{1-18}$$
where u_0 and u_0' are deterministic data in H^s.

(3) The process $(\omega, t) \mapsto u^\omega(t)$ is adapted to the filtration $\sigma(u_0, \mathscr{F}_t)$.

Remark 1.8. In what follows we call (H^{s-1}, H^s, H^{s+1}) a Gelfand triple. The process u_t described in Definition 1.7 satisfies the following properties:

- Considered as a process in H^s, it is progressively measurable with respect to $\sigma(u_0, \mathscr{F}_t)$; this follows from the continuity of u_t and Lemma 1.5.
- It satisfies the Feller property, being continuous in t and with respect to initial data.
- It is a Markov process: Set

$$P_t(w, \Gamma) := \mathbb{P}(u(t, w) \in \Gamma \mid u(0) = w);$$

then P_t satisfies the so-called Chapman–Kolmogorov relation. Let us write down the corresponding Markov semigroups:

$$\mathfrak{P}_t f(v) = \int_{H^s} f(w) P_t(v, dw), \quad C_b(H^s) \to C_b(H^s), \tag{1-19}$$

$$\mathfrak{P}_t^* \mu(\Gamma) = \int_{H^s} \mu(dw) P_t(w, \Gamma), \quad \mathfrak{p}(H^s) \to \mathfrak{p}(H^s). \tag{1-20}$$

Here, $C_b(H^s)$ is the space of bounded continuous functions on H^s, and $\mathfrak{p}(H^s)$ is the set of probability measures on H^s. These maps satisfy the duality relation

$$(\mathfrak{P}_t f, \mu) = (f, \mathfrak{P}_t^* \mu). \tag{1-21}$$

Now, let us introduce the following definition:

Definition 1.9. We say that (1-16) has the Itô property on the Gelfand triple (H^{s-1}, H^s, H^{s+1}) if

(1) it is stochastically well-posed on H^s;

(2) the process $h := Lu + f(u)$ is $\mathscr{F}_t$-adapted and

$$\mathbb{P}\left(\int_0^t (\|u(r)\|_{s+1}^2 + \|h(r)\|_{s-1}^2)\, dr < \infty \;\middle|\; \text{for all } t > 0 \right) = 1, \quad \sum_{m \in \mathbb{Z}} m^{2s} \lambda_m^2 < \infty. \tag{1-22}$$

Remark 1.10. Our definition of the Itô property is different from what we find in some literature. But the interest of our choice is that part (2) gathers "good" properties of a process allowing us to apply a version of the Itô formula proved in Section A.7 (Theorem A.7.5 and Corollary A.7.6) of [Kuksin and Shirikyan 2012]. Below, we present that formula.

Theorem 1.11 [Kuksin and Shirikyan 2012, Section A.7]. *Let $F \in C^2(H^s, \mathbb{R})$ be a functional which is uniformly continuous, together with its first two derivatives, on any ball of H^s. Suppose that F satisfies the following conditions*:

(1) *There is a function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|\nabla_u F(u, v)| \le K(\|u\|_s) \|u\|_{s+1} \|v\|_{s-1}, \quad u \in H^{s+1},\ v \in H^{s-1}. \tag{1-23}$$

(2) *For any sequence $\{w_k\} \subset H^{s+1}$ converging toward $w \in H^{s+1}$ and any $v \in H^{s-1}$, we have*

$$\nabla_u F(w_k, v) \to \nabla_u F(w, v) \quad \text{as } k \to \infty. \tag{1-24}$$

(3)
$$\sum_{m\in\mathbb{Z}} a_m^2 \mathbb{E}\int_0^t |\nabla_u F(u,e_m)|^2\,ds<\infty \quad \text{for all } t>0. \tag{1-25}$$

Then we have

$$F(u(t)) = F(u(0)) + \int_0^t \left(\partial_u F(u(s),f(s)) + \frac{1}{2}\sum_{m\in\mathbb{Z}} \partial_u^2 F(u(s),g_m)\right) ds + \sum_{m\in\mathbb{Z}} \int_0^t \partial_u F(u(s),g_m)\,d\beta_m(s). \tag{1-26}$$

In particular,

$$\mathbb{E}F(u(t)) = \mathbb{E}F(u(0)) + \int_0^t \mathbb{E}\left(\partial_u F(u(s),f(s)) + \frac{1}{2}\sum_{m\in\mathbb{Z}} \partial_u^2 F(u(s),g_m)\right) ds. \tag{1-27}$$

If one omits (1-25), then we have the formula (1-26) in which t is replaced by the stopping time $t\wedge\tau_n$, where

$$\tau_n = \inf\{t\ge 0 \mid \|u(t)\|_s > n\}, \quad n\in\mathbb{N}, \tag{1-28}$$

with the convention $\inf\varnothing = +\infty$.

2. Deterministic estimates

Conservation laws. Following [Tzvetkov and Visciglia 2014], we define the following subsets of $C^\infty(\mathbb{T})$:

$$\mathscr{P}_1 = \{\partial_x^\alpha u \mid \partial_x^\alpha H u,\ \alpha\in\mathbb{N}\},$$
$$\mathscr{P}_2 = \left\{(\partial_x^{\alpha_1} Z_1 u)(\partial_x^{\alpha_2} Z_2 u) \mid \alpha_i\in\mathbb{N},\ Z_i\in\{\mathrm{Id}, H\}\right\}.$$

Let us define in a generic manner the sets $\mathscr{P}_n$, $n\ge 3$, containing the functions of the form

$$p_n(u) = \prod_{i=1}^k Z_i(p_{j_i}(u)), \quad \text{where} \quad Z_i\in\{\mathrm{Id}, H\}, \quad \sum_1^k j_i = n, \quad p_{j_i}\in\mathscr{P}_{j_i}, \quad 2\le k\le n, \quad j_i<n. \tag{2-1}$$

To a function $p_n(u)$ of the form (2-1) we associate the function

$$\tilde{p}_n(u) = \prod_{i=1}^k p_{j_i}(u), \tag{2-2}$$

and we set the quantities

$$S(p(u)) = \sum_{i=1}^n \alpha_i, \quad M(p(u)) = \max_{1\le i\le n} \alpha_i.$$

The following is a description given in [Tzvetkov and Visciglia 2014] for the integer-order remainder terms:

$$R_n(u) = \sum_{\substack{p(u)\in\mathscr{P}_3\\ \tilde{p}(u)=u\,\partial_x^{n-1}u\,\partial_x^n u}} c_n(p)\int p(u) + \sum_{\substack{p(u)\in\mathscr{P}_j, j=3,\dots,2n+2\\ S(p(u))=2n-j+2\\ M(p(u))\le n-1}} c_n(p)\int p(u), \tag{2-3}$$

where $c_n(p)$ are some constants. The first three integer-order conservation laws are

$$E_0(u) = \int u^2,$$

$$E_1(u) = \int (\partial_x u)^2 + \frac{3}{4}\int u^2 H\,\partial_x u + \frac{1}{8}\int u^4,$$

$$E_2(u) = \int (\partial_x^2 u)^2 - \frac{5}{4}\int \Big((\partial_x u)^2 H\,\partial_x u + 2\,\partial_x^2 u H\,\partial_x u\Big) + \frac{5}{16}\int \Big(5u^2(\partial_x u)^2 + u^2(H\,\partial_x u)^2 + 2uH(\partial_x u)H(u\,\partial_x u)\Big) + \int \Big(\frac{5}{32}u^4 H(\partial_x u) + \frac{5}{24}u^3 H(u\,\partial_x u)\Big) + \frac{1}{48}\int u^6.$$

Estimates. Let us give some properties for the integer-order conservation laws of the Benjamin–Ono equation.

Lemma 2.1. *For any integer $n \geq 1$, there are $c_n^-, c_n^+ > 0$ such that for all u in $H^n(\mathbb{T})$*

$$\tfrac{1}{2}\|u\|_n^2 - c_n^-\|u\|^{2n+2} \leq E_n(u) \leq 2\|u\|_n^2 + c_n^+\|u\|^{2n+2}. \tag{2-4}$$

Lemma 2.2. *For all $\epsilon > 0$, there is $C_\epsilon > 0$ such that for all u in $H^{n+1}(\mathbb{T})$*

$$E_n'(u, \partial_x^2 u) \leq (-2+\epsilon)\|u\|_{n+1}^2 + C_\epsilon\|u\|(1+\|u\|)^{b_n},$$

where b_n depends only on n.

Remark 2.3. Since the L^2-norm is preserved by (1-1) we can deduce from (2-4) and the arguments of the proof of Lemma 2.2, by adding appropriate polynomials of $\|u\|$, new conservation laws $E_n^*(u)$ and $\widetilde{E}_n(u)$ satisfying

$$0 \leq \|u\|_n^2 \leq E_n^*(u), \quad 0 \leq \|u\|_n^2 \leq \widetilde{E}_n'(u, u).$$

Inequalities (2-4) can be established using arguments similar to those of the proof of Lemma 2.2.

Proof of Lemma 2.2. Taking into account of the properties of the Hilbert transform such as continuity on H^s and L^p ($s \geq 0$, $p \in]1, \infty[$), we can neglect its effect for our purpose and just consider the functions

$$R_n^1(u) = \int u\,\partial_x^{n-1} u\,\partial_x^n u,$$

$$R_n^{2,j}(u) = \int \prod_{i=1}^{j} \partial_x^{\alpha_i} u, \quad j = 3, \ldots, 2n+2, \ \sum_{i=1}^{j} \alpha_i = 2n+2-j.$$

Here $R_n^1(u)$ corresponds to the first term of (2-3) and the second term of (2-3) can be estimated considering the quantities $R_n^{2,j}(u)$. Set

$$R_n^0 = \|u\|_n^2.$$

Estimates concerning R_n^0.

$$\partial_u R_n^0(u, \partial_x^2 u) = -2\|u\|_{n+1}^2. \tag{2-5}$$

Estimates concerning R_n^1.

$$\begin{aligned}\partial_u R_n^1(u,\partial_x^2 u) &= \int \partial_x^2 u\,\partial_x^{n-1}u\,\partial_x^n u + \int u\,\partial_x^{n+1}u\,\partial_x^n u + \int u\,\partial_x^{n-1}u\,\partial_x^{n+2}u\\ &= \int \partial_x^2 u\,\partial_x^{n-1}u\,\partial_x^n u - \int \partial_x u\,\partial_x^{n-1}u\,\partial_x^{n+1}u\\ &= 2\int \partial_x^2 u\,\partial_x^{n-1}u\,\partial_x^n u + \int \partial_x u(\partial_x^n u)^2 = I + II.\end{aligned}$$

Let $\gamma_i,\ i=1,2,3$, be three positive numbers satisfying $\sum_{i=1}^3 \frac{1}{\gamma_i}=1$. We apply the generalized Hölder formula with them to find

$$|I| \le \|\partial_x^2 u\|_{L^{\gamma_1}}\,\|\partial_x^{n-1}u\|_{L^{\gamma_2}}\,\|\partial_x^n u\|_{L^{\gamma_3}}.$$

By the embedding inequality $\|\cdot\|_{L^{\gamma_i}} \lesssim \|\cdot\|_{\frac12-\frac{1}{\gamma_i}}$, we get

$$|I| \lesssim \|u\|_{\frac52-\frac{1}{\gamma_i}}\,\|u\|_{-\frac12-\frac{1}{\gamma_i}+n}\,\|u\|_{\frac12-\frac{1}{\gamma_i}+n}.$$

Now interpolate between L^2 and H^{n+1} to find

$$|I| \le C_1\|u\|_{n+1}^{d_1}\,\|u\|^{3-d_1},$$

where

$$d_1 = \frac{2n+3}{2(n+1)} < 2.$$

One can establish the same control (with the same d_1) for $|II|$ by remarking that

$$|II| \lesssim \|u\|_1\,\|\partial_x^n u\|_{L^4}^2 \lesssim \|u\|_1\,\|u\|_{n+\frac14}^2 \lesssim \|u\|_{n+1}^{\frac{(n+1-1)+2(n+1-n-1/4)}{n+1}}\,\|u\|^c.$$

Then for suitable b_1

$$|\partial_u R_n^1(u,\partial_x^2 u)| \le \epsilon\|u\|_{n+1}^2 + C_\epsilon^1\|u\|^{b_1}. \tag{2-6}$$

Estimates concerning $R_n^{2,j}$.

$$\partial_u R_n^{2,j}(u,\partial_x^2 u) = \int \prod_{i=1}^{j}\partial_x^{\alpha_i}u, \quad j=3,\dots,2n+2,$$

where $\sum_{i=1}^j \alpha_i = 2n-j+4$ and $\max_{1\le i\le j}\alpha_i \le n+1$.

We follow two complementary cases:

Case 1: $\max_{1\le i\le j}\alpha_i \le n$. Let (γ_i) be j real numbers such that $\sum_{i=1}^j \frac{1}{\gamma_i}=1$. Then the generalized Hölder formula combined with usual interpolation inequalities shows

$$|\partial_u R_n^{2,j}(u,\partial_x^2 u)| \le C\prod_{i=1}^{j}\|u\|_{\kappa_i},$$

where $\kappa_i = \frac12 - \frac{1}{\gamma_i} + \alpha_i$. Then

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \le C \prod_{i=1}^{j} \|u\|^{\frac{n+1-\kappa_i}{n+1}} \|u\|_{n+1}^{\frac{\kappa_i}{n+1}}.$$

We remark now that

$$\sum_{i=1}^{j} \kappa_i = \sum_{1}^{j} \Big(\frac{1}{2} - \frac{1}{\gamma_i} + \alpha_i \Big) = 2n + 3 - \frac{j}{2}.$$

Then

$$\sum_{i=1}^{j} \frac{\kappa_i}{n+1} = \frac{2n + 3 - \frac{j}{2}}{n+1} < 2.$$

Thus for suitable b_2,

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \le \epsilon \|u\|_{n+1}^2 + C_\epsilon^2 \|u\|^{b_2}.$$

Case 2: $\alpha_1 = n+1$. Then $\sum_{i=2}^{j} \alpha_i = n - j + 3 \le n$, and we have

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \le \|u\|_{n+1} \Big(\int \prod_{i=2}^{j} |\partial_x^{\alpha_i} u|^2 \Big)^{\frac{1}{2}}.$$

Take again (γ_i) such that $\sum_{i=2}^{j} \frac{1}{\gamma_i} = 1$. Then

$$\begin{aligned} |\partial_u R_n^{2,j}(u, \partial_x^2 u)| &\le \|u\|_{n+1} \prod_{i=2}^{j} \|\partial_x^{\alpha_i} u\|_{L^{2\gamma_i}} \\ &\le \|u\|_{n+1} \prod_{i=2}^{j} \|u\|_{\kappa_i}, \quad \kappa_i = \frac{1}{2} - \frac{1}{2\gamma_i} + \alpha_i, \\ &\le \|u\|_{n+1} \prod_{i=2}^{j} \|u\|^{\frac{n+1-\kappa_i}{n+1}} \|u\|_{n+1}^{\frac{\kappa_i}{n+1}}. \end{aligned}$$

Since $\sum_{i=2}^{j} \kappa_i = n + 2 - \frac{j}{2} \le n + \frac{1}{2}$, we have $\frac{1}{n+1} \sum_{i=2}^{j} \kappa_i < 1$ and the existence of a suitable b_3 such that

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \le \epsilon \|u\|_{n+1}^2 + C_\epsilon^3 \|u\|^{b_3}. \tag{2-7}$$

Combining (2-5), (2-6) and (2-7) with a good choice of ϵ, we have the claim. □

3. IVP of the stochastic BOB equation

Consider the initial value problem concerning the stochastic BOB equation (1-3)

$$\begin{cases} \partial_t u + H\, \partial_x^2 u + u\, \partial_x u = \alpha\, \partial_x^2 u + \sqrt{\alpha}\eta, \quad t > 0, \\ u|_{t=0} = u_0. \end{cases} \tag{3-1}$$

Recall that, for $s \ge 0$,

$$A_s = \sum_{m \in \mathbb{Z}} m^{2s} \lambda_m^2.$$

These quantities measure the regularity in space of the noise. Namely,

$$A_s < +\infty \iff \eta(t,\cdot) \in \dot{H}^s.$$

Stochastic well-posedness, well-structuredness.

Proposition 3.1. *Let $s \geq 2$ be an integer. Suppose A_s is finite. Then the problem (3-1) is stochastically globally well-posed in $\dot{H}^s(\mathbb{T})$ in the sense of Definition 1.7.*

In order to prove the existence result in Proposition 3.1, we split the problem (3-1) as follows:

- A linear stochastic problem:

$$\begin{cases} \partial_t z_\alpha + H\,\partial_x^2 z_\alpha = \alpha\,\partial_x^2 z_\alpha + \sqrt{\alpha}\eta, \quad t > 0, \\ z_\alpha|_{t=0} = 0. \end{cases} \tag{3-2}$$

- A nonlinear deterministic problem:

$$\begin{cases} \partial_t v + H\,\partial_x^2 v + (v + z_\alpha)\,\partial_x(v + z_\alpha) = \alpha\,\partial_x^2 v, \quad t > 0, \\ v|_{t=0} = u_0. \end{cases} \tag{3-3}$$

  Here z_α is a realization of a solution of (3-2).

For z_α and v respective solutions of (3-2) and (3-3), it is easy to see that $u = v + z_\alpha$ is a solution of (3-1). The linear problem (3-2) is solved by the stochastic convolution (see the subsection on page 1845)

$$z_\alpha(t) = \sqrt{\alpha}\int_0^t e^{-(t-s)(H-\alpha)\,\partial_x^2}\,d\zeta(s) =: \sqrt{\alpha}z(t). \tag{3-4}$$

Remark that, as defined, the function z still depends on α. But all its Sobolev norms are uniformly controlled with respect to α; this justifies that abuse of notation.

If, for some $s \geq 0$, A_s is finite, then we have for all $T > 0$

$$z \in \Lambda_T(s) := C([0,T], \dot{H}^s(\mathbb{T})) \cap L^2([0,T], \dot{H}^{s+1}(\mathbb{T})) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{3-5}$$

Uniqueness of solution for the problem (3-2) is obtained by standard arguments. Moreover, if we suppose A_n finite, we can apply the Itô formula to the $\dot{H}^n$-norms (which are preserved by the linear Benjamin–Ono equation) to find that

$$\mathbb{E}\|z_\alpha\|_n^2 + 2\alpha\int_0^t \mathbb{E}\|z_\alpha\|_{n+1}^2\,ds = \alpha A_n t. \tag{3-6}$$

Denoting by z^m the projection (z, e_m), we have that

$$z^m(t) = \lambda_m \int_0^t e^{m^2(t-s)(\mathrm{i}\,\mathrm{sgn}(m)-\alpha)}\,d\beta_m(s).$$

Since the function $s \to e^{m^2(t-s)(\mathrm{i}\,\mathrm{sgn}(m)-\alpha)}$ is C^1, we employ a usual (stochastic) integration by parts formula to obtain

$$z^m(t) = \lambda_m\beta_m(t) + m^2(\mathrm{i}\,\mathrm{sgn}(m) - \alpha)\lambda_m \int_0^t e^{m^2(t-s)(\mathrm{i}\,\mathrm{sgn}(m)-\alpha)}\beta(s)s\,ds.$$

Then we arrive at

$$\sup_{t\in[0,T]}|z^m(t)|^2 \le 2\lambda_m^2[1+(1-\alpha)^2m^4T^2]\sup_{t\in[0,T]}|\beta_m(t)|^2 \le 2\lambda_m^2[1+m^4T^2]\sup_{t\in[0,T]}|\beta_m(t)|^2.$$

After summing in m, we arrive at

$$\sup_{t\in[0,T]}\|z(t)\|^2 \lesssim_T \sup_{t\in[0,T]}\|\zeta(t)\|_2^2.$$

More generally, for any m such that A_{m+2} is finite, we have

$$\sup_{t\in[0,T]}\|z(t)\|_m^2 \lesssim_T \sup_{t\in[0,T]}\|\zeta(t)\|_{m+2}^2,$$

and finally

$$\sup_{t\in[0,T]}\|z_\alpha(t)\|_m^2 \lesssim_T \alpha \sup_{t\in[0,T]}\|\zeta(t)\|_{m+2}^2. \tag{3-7}$$

Proposition 3.2. *Let $s\ge 2$ be an integer, and suppose $A_s<\infty$. Let u_0 be a random variable in $\dot H^s(\mathbb{T})$ independent of $\mathscr{F}_t$. Then for any $T>0$, for a.e. ω, the nonlinear problem (3-3) associated to u_0 admits a solution in $\Lambda_T(s)$. Moreover the process solution is adapted to $\sigma(u_0,\mathscr{F}_t)$.*

Proposition 3.2 is proved combining the two paragraphs below:

A priori estimates. The following lemma is proved using the first three integer-order (modified) conservation laws $E_n^*(u)$ of the Remark 2.3, its proof is presented in the Appendix.

Lemma 3.3. *For any $T>0$, for almost any realization of z we have the following a priori estimates for the nonlinear problem (3-3):*

$$\sup_{t\in[0,T]}\|v(t)\|_i^2+\alpha\int_0^T\|v(t)\|_{i+1}^2\,dt \le C(T,\|u_0\|_i,\|z\|_{L^\infty(0,T;H^i)}),\quad i=0,1,2, \tag{3-8}$$

where C does not depend on $\alpha\in(0,1)$.

Since $H^2(\mathbb{T})$ is continuously embedded in $C^1(\mathbb{T})$, we infer:

Corollary 3.4. *For any $T>0$, for almost any realization of z, and for any initial datum $u_0\in H^2$, a solution v to (3-3) satisfies*

$$\sup_{t\in[0,T]}\|\partial_x v(t)\|_{L^\infty} \le C(T,\|u_0\|_2,\|z\|_{L^\infty(0,T;H^2)}), \tag{3-9}$$

where C does not depend on $\alpha\in(0,1)$.

Lemma 3.5. *For any $T>0$, for any integer $s>2$, and for almost any realization of z we have the higher-order a priori estimate for (3-3)*

$$\sup_{t\in[0,T]}\|v(t)\|_s^2+\alpha\int_0^T\|v(t)\|_{s+1}^2\,dt \le C(T,\|u_0\|_s,\|z\|_{L^\infty(0,T;H^s)}), \tag{3-10}$$

where C does not depend on $\alpha\in(0,1)$.

Before giving the proof of the estimate (3-10), let us prove the following commutator estimate:

Lemma 3.6. *Let $s \geq 3$ be an integer and v be in H^{s+1}. We have*

$$\|[\partial_x^s, v]\partial_x v\| \lesssim \|v\|_2 \, \|v\|_s, \tag{3-11}$$

where $[\partial_x^s, v]\partial_x v = \partial_x^s(v\,\partial_x v) - v\,\partial_x^s(\partial_x v)$.

Proof. By the Leibniz rule we have

$$[\partial_x^s, v]\partial_x v = \sum_{k=1}^{s} \binom{s}{k} \partial_x^k v\, \partial_x^{s+1-k} v.$$

We separate the above sum into three general terms:

(1) We have $k \in \{1, s\}$ if and only if the general term is $\partial_x v \partial_x^s v$. By using the embedding $H^1 \subset L^\infty$, we have the inequality

$$\|\partial_x v\, \partial_x^s v\| \leq \|v\|_2 \, \|v\|_s.$$

(2) We have $k \in \{2, s-1\}$ if and only if the general term is $\partial_x^2 v\, \partial_x^{s-1} v$. We have (always by $H^1 \subset L^\infty$)

$$\|\partial_x^2 v\, \partial_x^{s-1} v\| \leq \|v\|_2 \, \|v\|_s.$$

(3) When $s \geq 5$ we have the last situation, which is $3 \leq k \leq s-2$; we have then $3 \leq s+1-k \leq s-2$ as well. We estimate the corresponding general term as follows:

$$\|\partial_x^k v\, \partial_x^{s+1-k}\| \leq \|v\|_{k+1} \, \|v\|_{s+1-k} \lesssim \|v\|_2^{\frac{s-k-1}{s-2}} \, \|v\|_s^{\frac{k-1}{s-2}} \, \|v\|_2^{\frac{k-1}{s-2}} \, \|v\|_s^{\frac{s-k-1}{s-2}} = \|v\|_2 \, \|v\|_s.$$

We complete the proof after taking a weighted sum of these terms. □

Proof of the estimate (3-10). We recall the nonlinear equation satisfied by v

$$\partial_t v + H\,\partial_x^2 v - \alpha\,\partial_x^2 v = -v\,\partial_x v - \partial_x(v z_\alpha) - \tfrac{1}{2}\partial_x z_\alpha^2.$$

Then for an integer $s > 2$, we have

$$(\partial_x^s v, \partial_x^s \partial_t v) + \alpha(\partial_x^{s+1} v, \partial_x^{s+1} v) = -(\partial_x^s v, \partial_x^s(v\,\partial_x v)) \underbrace{-(\partial_x^s v, \partial_x^{s+1}(v z_\alpha))}_{=+(\partial_x^{s+1} v, \partial_x^s(v z_\alpha))} + \tfrac{1}{2}(\partial_x^{s+1} v, \partial_x^s z_\alpha^2).$$

Therefore

$$\tfrac{1}{2}\partial_t \|v\|_s^2 + \alpha \|v\|_{s+1}^2 = I + II + III.$$

Using the commutator estimate (3-11) and the algebra structure of $H^s(\mathbb{T})$, we have

$$\begin{aligned} |I| &= |(\partial_x^s v, \partial_x^s(v\,\partial_x v) - v\partial_x^s \partial_x v) + (\partial_x^s v, v\,\partial_x^s \partial_x v)| \\ &= |(\partial_x^s v, [\partial_x^s, v]\partial_x v) - \tfrac{1}{2}(\partial_x v, |\partial_x^s v|^2)| \\ &\lesssim \|v\|_s^2 \, \|v\|_2. \end{aligned}$$

By Cauchy–Schwarz and the algebra structure of H^s, we have

$$|II| + |III| \leq \tfrac{1}{2}\alpha \|v\|_{s+1}^2 + C_1 \|v\|_s^2 \, \|z\|_s^2 + C_2 \alpha \|z\|_s^4,$$

where C_1 and C_2 depend only on s. It remains to combine the Gronwall lemma with (3-8) to get the claim. □

Local and global existence for the nonlinear problem (3-3). Let $s \geq 2$. For a positive T the space $\Lambda_T(s)$ is endowed with the norm defined by

$$\|u\|_{\Lambda_T(s)} = \sup_{t\in[0,T]} \left(e^{-\frac{t}{T}} \left\{ \|u(t)\|_s^2 + \alpha \int_0^t \|u(r)\|_{s+1}^2 \, dr \right\} \right)^{\frac{1}{2}}. \tag{3-12}$$

Let $R > 0$; denote by B_R the ball in H^s of center 0 and radius R.

Remark 3.7. The factor $e^{-\frac{t}{T}}$ in (3-12) is introduced just for convenience in the computations. The norm defined in (3-12) is actually equivalent to the one without that factor.

Proposition 3.8. *Let $s \geq 2$ and $\alpha \in (0,1)$. For all $R > 0$, there is $T_R > 0$ such that for any u_0 in $B_{\frac{R}{2}}$, the nonlinear problem (3-3) has a unique solution in $\Lambda_{T_R}(s)$.*

Remark 3.9. We combine the local existence of Proposition 3.8, Lemma 3.3, and estimate (3-10) to get the global existence for (3-3).

Proof of Proposition 3.8. Let us look for a fixed point of the map

$$\mathfrak{F}v = e^{-t(H-\alpha)\partial_x{}^2} u_0 - \int_0^t e^{-(t-s)(H-\alpha)\partial_x{}^2} (z_\alpha + v)\, \partial_x (z_\alpha + v)\, ds.$$

We proceed as follows:

Step 1: We prove that for any $R > 0$, there is $T > 0$ such that the ball $B_{T,s}$ of $\Lambda_T(s)$ centered at 0 and of radius R satisfies $\mathfrak{F}(B_{T,s}) \subset B_{T,s}$ if $\|u_0\|_s \leq \frac{1}{2}R$:

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|\mathfrak{F}v\|_s^2 &= -(\partial_t D^s \mathfrak{F}v, D^s \mathfrak{F}(v)) \\
&= -((H-\alpha)D^{s+1}\mathfrak{F}(v), D^{s+1}\mathfrak{F}(v)) + \frac{1}{2}(D^s(z_\alpha+v)^2, D^{s+1}\mathfrak{F}(v)) \\
&\geq \alpha\|\mathfrak{F}(v)\|_{s+1}^2 - \frac{1}{2}\|z_\alpha + v\|_s^2 \, \|\mathfrak{F}(v)\|_{s+1} \\
&\geq \alpha\|\mathfrak{F}(v)\|_{s+1}^2 - \frac{\alpha}{2}\|\mathfrak{F}(v)\|_{s+1}^2 - \frac{C}{\alpha}(\|z_\alpha\|_s^4 + \|v\|_s^4).
\end{aligned}$$

Then there is an universal constant $c > 0$ such that

$$\frac{d}{dt}\|\mathfrak{F}(v)\|_s^2 + \alpha\|\mathfrak{F}(v)\|_{s+1}^2 \leq \frac{c}{\alpha} e^{\frac{2t}{T}} (R^4 + \|z_\alpha\|_{\Lambda_T(s)}^4).$$

Thus, after integration with respect to t, we find

$$\|\mathfrak{F}(v)\|_s^2 + \alpha\int_0^t \|\mathfrak{F}(v)\|_{s+1}^2 \, ds \leq \|u_0\|_s^2 + \frac{\tilde{c}T}{\alpha} e^{\frac{t}{T}} (R^4 + \|z_\alpha\|_{\Lambda_T(s)}^4).$$

Multiplying the last relation by $e^{-\frac{t}{T}}$, it remains to choose T small enough so that we obtain the claimed result.

Step 2: We now prove that $\mathfrak{F}$ is a contraction on the ball constructed above. We have

$$\partial_t \mathfrak{F}v = -\{(v+z_\alpha)\,\partial_x(v+z_\alpha) + (H-\alpha)\,\partial_x{}^2\mathfrak{F}v\}.$$

Then for v_1 and v_2 in $\Lambda_T(s)$, we have

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\|\mathfrak{F}v_1-\mathfrak{F}v_2\|_s^2 &= -(\partial_t D^s(\mathfrak{F}v_1-\mathfrak{F}v_2), D^s(\mathfrak{F}v_1-\mathfrak{F}v_2))\\ &= (D^s(F_z(v_1)-F_z(v_2)), D^{s+1}(\mathfrak{F}v_1-\mathfrak{F}v_2)) + \alpha\|\mathfrak{F}v_1-\mathfrak{F}v_2\|_{s+1}^2,\end{aligned}$$

where

$$F_z(v) = \tfrac{1}{2}(z_\alpha+v)^2.$$

We show easily that

$$\|D^s(F_z(v_1)-F_z(v_2))\|^2 \le C(s)\|v_1-v_2\|_s^2(\|v_1+v_2\|_s^2+\|z_\alpha\|_s^2).$$

This allows us to get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathfrak{F}v_1-\mathfrak{F}v_2\|_s^2 + \frac{\alpha}{2}\|\mathfrak{F}v_1-\mathfrak{F}v_2\|_{s+1}^2 &\le \frac{C(s)}{\alpha}\|v_1-v_2\|_s^2(\|v_1+v_2\|_s^2+\|z_\alpha\|_s^2)\\ &\le e^{\frac{t}{T}}\frac{C(s)(4R^2+\|z\|_{\Lambda_T(s)}^2)}{\alpha}\|v_1-v_2\|_{\Lambda_T(s)}^2.\end{aligned}$$

After integration in t, we find

$$\|\mathfrak{F}v_1-\mathfrak{F}v_2\|_s^2 + \alpha\int_0^t \|\mathfrak{F}v_1-\mathfrak{F}v_2\|_{s+1}^2\,ds \le Te^{\frac{t}{T}}\frac{C(s)(4R^2+\|z_\alpha\|_{\Lambda_T(s)}^2)}{\alpha}\|v_1-v_2\|_{\Lambda_T(s)}^2.$$

We multiply this inequality by $e^{-\frac{t}{T}}$; the T found in the first step can be decreased if necessary to give a contraction.

We conclude by using the fixed point theorem. □

Remark 3.10. By definition, v is $\sigma(u_0,\mathscr{F}_t)$-adapted. Then the process $u=v+z_\alpha$ is continuous and $\sigma(u_0,\mathscr{F}_t)$-adapted. Thanks to Lemma 1.5, the process u is progressively measurable with respect to that filtration.

End of the proof of Proposition 3.1, the well-posedness of (1-3). Let u_1 and u_2 be two solutions of (1-3) starting respectively at $u_{1,0}$ and $u_{2,0}$, and set $w=u_1-u_2$; then the problem solved by w is

$$\begin{cases}\partial_t w + (H-\alpha)\,\partial_x^2 w + w\,\partial_x w + \partial_x(wu_2) = 0,\\ w|_{t=0} = u_{1,0}-u_{2,0} =: w_0.\end{cases}$$

Using the arguments of the proof of (3-10), we show

$$\sup_{t\in[0,T]}\|w(t)\|_s^2 + \alpha\int_0^T \|w(r)\|_{s+1}^2\,dr \le C\big(\alpha,T,\|\partial_x w\|_{L^\infty(0,T;L^\infty)},\|u_2\|_{L^\infty(0,T;H^s)}\big)\|w_0\|_s^2.$$

Hence follow the uniqueness and the continuity with respect to initial data. □

The stochastic well-posedness that we just established combined with the estimates (3-8) and the equation (3-10) implies the following:

Proposition 3.11. *Let $j \geq 2$. Suppose A_j finite. Then (1-3) is well-structured on the Gelfand triple (H^{j-1}, H^j, H^{j+1}) in the sense of Definition 1.9.*

Probabilistic estimates and proof of Lemma 1.1.

Exponential control of the L^2-norm.

Proposition 3.12. *Let $p \geq 1$. Then the functional $E_0^p(u) = \|u\|^{2p}$ satisfies the conditions of Theorem 1.11 on the Gelfand triple (H^{-1}, L^2, H^1).*

Proof. Thanks to the polynomial nature of $E_0^p(u)$ on L^2, the uniform continuity on bounded sets and the conditions (1-23) and (1-24) follow easily. We confine ourself to the proof of (1-25). The argument we use, to this end, is the following: As we have already shown, the solution of (1-3) can be represented as the sum of a linear part and a nonlinear part. Now we will show that the nonlinear part can be controlled by the initial datum and an "exponential of the averaged linear part". On the other hand, we show that the linear part is exponentially controlled; then we get the needed control on the initial solution u.

Control of the nonlinear part v. In this part we prove that for all $r, \epsilon > 0$ and $p \geq 1$

$$\|v(r)\|^{2p} \leq e^{f(r,\epsilon,p)} e^{\frac{2\epsilon p}{r}\int_0^r \|\partial_x z_\alpha\|_{L^\infty}^2 \, ds} \left(\|u_0\|^2 + \int_0^r \|z_\alpha\|_1^4 \, ds \right)^p, \tag{3-13}$$

where $f(r, \epsilon, p) = \frac{p}{4}\left(2r + \frac{r^2}{\epsilon}\right)$. Indeed, multiplying (3-3) by v and integrating in x, one obtains

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 + \alpha\|v\|_1^2 &= -(v, \partial_x(vz_\alpha)) - (v, z_\alpha \, \partial_x z_\alpha) \\
&= \frac{1}{2}[(v, v\partial_x z_\alpha) + (v, \partial_x z_\alpha^2)] \\
&\leq \frac{1}{2}[\|v\| \, \|vz_\alpha\| + \|v\| \, \|z_\alpha\|_1^2] \\
&\leq \frac{r}{8\epsilon}\|v\|^2 + \frac{\epsilon}{r}\|v\|^2 \, \|\partial_x z_\alpha\|_{L^\infty}^2 + \frac{1}{4}\|v\|^2 + \frac{1}{4}\|z_\alpha\|_1^4.
\end{aligned}$$

Then we use the Gronwall lemma, choose $t = r$ and take the resulting inequality to the power p to arrive at the claim.

Exponential control of the linear part. Now, the linear part of the solution satisfies the estimate

$$\mathbb{E} e^{\frac{\epsilon}{t}\int_0^t \|z_\alpha\|_2^2 \, ds} \leq 3, \tag{3-14}$$

where $\epsilon > 0$ is small enough. Indeed, by applying the Itô formula to $\|z\|_2^{2p}$ for $p \geq 1$ we have

$$\mathbb{E}\|z_\alpha\|_2^{2p} \leq \frac{A_1^p p^p}{\kappa^p}. \tag{3-15}$$

Integrating in t, we find

$$\mathbb{E}\left(\frac{1}{t}\int_0^t \|z_\alpha\|_2^{2p} \, ds \right) \leq \frac{A_1^p p^p}{\kappa^p}.$$

Thanks to Jensen's inequality, we infer

$$\mathbb{E}\left(\frac{1}{t}\int_0^t \|z_\alpha\|_2^2\,ds\right)^p \le \frac{A_1^p p^p}{\kappa^p}.$$

Now, let $0<\epsilon\le \kappa/(2A_1e)$; then we have

$$\mathbb{E}\frac{\left(\frac{\epsilon}{t}\int_0^t \|z_\alpha\|_2^2\,ds\right)^p}{p!} \le \frac{p^p}{2^p e^p\, p!}.$$

We recall that for any integer $p>0$, we have that $p!\ge \left(\frac{p}{e}\right)^p$; then we arrive at the claimed result.

Control of the quadratic variation of $E_0^p(u)$. We have that

$$\begin{aligned}\sum_{m\ge 0} a_m^2\mathbb{E}\int_0^t |\partial_u(E_0^p)(u,e_m)|^2\,ds &\lesssim_p \sum_{m\in\mathbb{Z}} a_m^2\mathbb{E}\int_0^t \|u\|^{4(p-1)}|(u,e_m)|^2\,ds\\ &\lesssim_p \mathbb{E}\int_0^t \|u\|^{4p-2}\,ds \lesssim_p \mathbb{E}\int_0^t (\|v\|^{4p-2}+\|z_\alpha\|^{4p-2})\,ds.\end{aligned}$$

Set $q=4p-2$; one sees, with the use of the estimate (3-15) (or just by invoking the Fernique theorem), that

$$\mathbb{E}\int_0^t \|z\|_2^q\,ds<\infty \quad \text{for any } t\ge 0.$$

Now we use the estimate (3-13); then, for any $\epsilon>0$,

$$\mathbb{E}\int_0^t \|v_s\|^q\,ds \le \int_0^t e^{f(s,\epsilon,q)}\mathbb{E}\left[e^{\frac{\epsilon q}{s}\int_0^s \|\partial_x z_\alpha\|_{L^\infty}^2 dr}\left(\|u_0\|^2+\int_0^s \|z_\alpha\|_{L1}^4\,dr\right)^q\right]ds.$$

Then for any $\delta>0$, we use the Young inequality to find

$$\mathbb{E}\int_0^t \|u_s\|^q\,ds \lesssim \int_0^t e^{f(s,\epsilon,q)}\mathbb{E}\left[e^{\frac{q(1+\delta)\epsilon}{\delta s}\int_0^s \|\partial_x z_\alpha\|_{L^\infty}^2 dr}+\underbrace{\left(\|u_0\|^2+\int_0^s \|z_\alpha\|_1^4\,dr\right)^{q(1+\delta)}}_{R_{q,\delta(s)}}\right]ds.$$

One uses the estimate (3-15) to bound $\mathbb{E}R_{q,\delta}(s)$ by $C_{q,\delta}(1+s^{q(1+\delta)})$. On the other hand, for any $\delta>0$ we choose $\epsilon>0$ small enough so that one can use the estimate (3-14) and the embedding $H^2\subset L^\infty$ to get the bound

$$\mathbb{E}e^{\frac{2p(1+\delta)\epsilon}{\delta s}\int_0^s \|\partial_x z_\alpha\|_{L^\infty}^2 dr}\le 3.$$

Then we get

$$\mathbb{E}\int_0^t \|v_s\|^{2p}\,ds \lesssim \int_0^t e^{f(s,\epsilon,p)}(1+s^{q(1+\delta)})\,ds<\infty \quad \text{for all } t\ge 0. \qquad \square$$

Proposition 3.13. *Let u be the solution of* (3-1):

(1) *Suppose that $\mathbb{E}E_0(u_0)<\infty$; then*

$$\mathbb{E}E_0(u)+2\alpha\int_0^t \mathbb{E}\|u(s)\|_1^2\,ds=\mathbb{E}E_0(u_0)+\alpha A_0 t. \tag{3-16}$$

(2) *Let $p > 1$. Suppose that $\mathbb{E}E_0^p(u_0) < \infty$; then*

$$\mathbb{E}E_0^p(u) \le e^{-p\alpha t}\mathbb{E}E_0^p(u_0) + p^p A_0^p. \tag{3-17}$$

Proof. The identity (3-16) is easily proven by applying the Itô formula to the conservation law $E_0(u)$. Let us prove (3-17):

For $p > 1$, we apply the Itô formula to $E_0^p(u)$ to find

$$dE_0^p(u) = pE_0^{p-1}(u)dE_0(u) + \frac{\alpha p(p-1)}{2}E_0^{p-2}(u)\sum_{m\in\mathbb{Z}}\lambda_m^2|E_0'(u,e_m)|^2dt.$$

Taking the expectation, we get

$$\mathbb{E}E_0^p(u) + \mathbb{E}\int_0^t f_\alpha(u(s))\,ds = \mathbb{E}E_0^p(u_0),$$

where

$$f_\alpha(u) = 2p\alpha E_0^{p-1}(u)\|u\|_1^2 - \alpha pE_0^{p-1}(u)A_0 - \frac{\alpha p(p-1)}{2}E_0^{p-2}(u)\sum_{m\in\mathbb{Z}}\lambda_m^2|E_0'(u,e_m)|^2.$$

Let us set

$$Q = pE_0^{p-1}(u)A_0 + \frac{p(p-1)}{2}E_0^{p-2}(u)\sum_{m\in\mathbb{Z}}\lambda_m^2|E_0'(u,e_m)|^2.$$

Remarking that

$$\sum_{m\in\mathbb{Z}}\lambda_m^2|E_0'(u,e_m)|^2 \le 2A_0E_0(u),$$

we get, with the use of the Young inequality, the estimate

$$Q \le \epsilon E_0^p(u) + \frac{p^{2p}}{\epsilon^{p-1}}A_0^p.$$

On the other hand

$$p\alpha E_0^{p-1}(u)\|u\|_1^2 \ge p\alpha E_0^p(u).$$

Choosing $\epsilon = p$, we see that

$$\mathbb{E}f_\alpha(u) \ge p\alpha\mathbb{E}E_0^p(u) - p^{p+1}A_0^p\alpha.$$

Then

$$\mathbb{E}E_0^p(u) + p\alpha\int_0^t \mathbb{E}E_0^p(u(s))\,ds \le \mathbb{E}E_0^p(u_0) + p^{p+1}A_0^p\alpha t.$$

Gronwall's lemma gives the claimed result. □

Control of higher-order Sobolev norms. The polynomial nature of the Benjamin–Ono conservation laws E_j allows to establish the following result:

Proposition 3.14. *Let $j \ge 1$, then the functional E_j satisfies the conditions (1-23) and (1-24) of Theorem 1.11 on the triple (H^{j-1}, H^j, H^{j+1}).*

In view of this result the "stopping time" version of the Itô formula (1-26) applies to the functionals E_j.

Theorem 3.15. *Let $j \geq 1$ be an integer. Suppose A_j is finite. There are $\theta_j > 0$, $\gamma_j > 0$ such that for any solution u of (3-1) in H^2 issued from $u_0 \in H^2$ which satisfies $\mathbb{E}E_j(u_0) < \infty$, we have*

$$\mathbb{E}E_j(u) + \alpha \int_0^t \mathbb{E}\|u\|_{j+1}^2 \, ds \leq \mathbb{E}E_j(u_0) + \alpha A_j \left(t + c_j \int_0^t \mathbb{E}\|u\|_j^2 \, ds + \gamma_j \int_0^t \mathbb{E}\|u\|(1+\|u\|)^{\theta_j} \, ds \right), \quad (3\text{-}18)$$

where c_j depends only on j.

Proof. The fact that $E_j(u)$ is preserved by the BO equation translates into

$$\partial_u E_j(u, -H\,\partial_x^2 u - u\,\partial_x u) = 0.$$

Setting the Markov time $\tau_n = \inf\{t \geq 0, \|u(t)\|_j > n\}$ and applying the Itô formula (1-26), we get

$$E_j(u(t\wedge\tau_n)) = E_j(u_0) + \alpha \int_0^{t\wedge\tau_n} \left(\partial_u E_j(u, \partial_x^2 u) + \frac{1}{2} \sum_{m\in\mathbb{Z}} \lambda_m^2 \, \partial_u^2 E_j(u, e_m) \right) ds + \sum_{m\in\mathbb{Z}} \lambda_m \int_0^{t\wedge\tau_n} \partial_u E_j(u, e_m) \, d\beta_m(s).$$

Then by the Doob optional stopping theorem, Theorem 1.4, we have

$$\mathbb{E}E_j(u(t\wedge\tau_n)) = \mathbb{E}E_j(u_0) + \alpha\mathbb{E} \int_0^{t\wedge\tau_n} \left(\partial_u E_j(u, \partial_x^2 u) + \frac{1}{2} \sum_{m\in\mathbb{Z}} \lambda_m^2 \, \partial_u^2 E_j(u, e_m) \right) ds.$$

Using the monotone convergence theorem, we arrive at

$$\mathbb{E}E_j(u(t)) = \mathbb{E}E_j(u_0) + \alpha\mathbb{E} \int_0^{t} \left(\partial_u E_j(u, \partial_x^2 u) + \frac{1}{2} \sum_{m\in\mathbb{Z}} \lambda_m^2 \, \partial_u^2 E_j(u, e_m) \right) ds.$$

By Lemma 2.2, we have

$$\partial_u E_j(u, \partial_x^2 u) \leq -\|u\|_{j+1}^2 + P_j(\|u\|), \quad (3\text{-}19)$$

where P_j is the polynomial of Lemma 2.2. Following the arguments of the proof of Lemma 2.2, we establish that

$$|\partial_u^2 E_j(u, e_m)| \leq c_j m^{2j} (\|u\|_j^2 + Q_j(\|u\|)), \quad (3\text{-}20)$$

where $Q_j(r) = q_j r(1+r)^{k_j}$, q_j and k_j depend only on j. Then take the expectation and combine (3-19) with (3-20) to get the claim. □

Now we are able to give the proof of Lemma 1.1.

Proof of Lemma 1.1. Let u be a stationary solution to (1-3) which satisfies the integrability assumption (1-4), and suppose that A_j is finite for any j. Recall the estimate

$$\mathbb{E}E_j(u) \leq \mathbb{E}\|u\|_j^2 + c_n^+ \mathbb{E}\|u\|^{2j+2}. \quad (3\text{-}21)$$

Then using the integrability assumption (1-4), we see that $\mathbb{E}E_j(u)$ is finite as soon as $\mathbb{E}\|u\|_j^2 < \infty$.

Note that, by the stationarity of u, the estimates (3-18) become (under the assumption that $\mathbb{E}E_j(u)$ is finite)

$$\mathbb{E}\|u\|_{j+1}^2 \le A_j[1 + c_j\mathbb{E}\|u\|_j^2 + \gamma_j\mathbb{E}\|u\|(1+\|u\|)^{\theta_j}] \tag{3-22}$$

since the distribution do not depend on t. We are going to argue by induction. Note that the needed induction property is given by the combination of (3-22) and (3-21) because they give at the same time the finiteness of $\mathbb{E}E_j(u)$ and the control of $\mathbb{E}\|u\|_{j+1}^2$ as soon as $\mathbb{E}\|u\|_j^2$ is finite. Moreover if (1-4) holds uniformly in α then so does $\mathbb{E}\|u\|_{j+1}^2$ once the control on $\mathbb{E}\|u\|_j^2$ is uniform in α. It remains to prove the initial step, namely $\mathbb{E}\|u\|_1^2$ is finite and does not depend on α. But using again the integrability assumption at the order $p = 2$, the stationarity of u combined with the estimate (3-16) gives

$$\mathbb{E}\|u\|_1^2 = \frac{A_0}{2}. \qquad \square$$

4. Stationary measures for the viscous problem

Consider the stochastic BOB problem (1-3) posed on $\dot{H}^2(\mathbb{T})$. By the estimates (3-16), (3-17) and Theorem 3.15, we have

$$\mathbb{E}E_0(u) + 2\alpha\int_0^t \mathbb{E}\|u\|_1^2\,ds = \mathbb{E}E_0(u_0) + \alpha A_0 t,$$

$$\mathbb{E}E_0^p(u) \le e^{-p\alpha t}\mathbb{E}E_0^p(u_0) + C_p A_0^p,$$

$$\mathbb{E}E_1(u) + \alpha\int_0^t \mathbb{E}\|u\|_2^2\,ds \le \mathbb{E}E_1(u_0) + \alpha\left(A_1 t + c_1\int_0^t \mathbb{E}\|u\|_1^2\,ds + \int_0^t \mathbb{E}W_1(\|u\|)\,ds\right),$$

$$\mathbb{E}E_2(u) + \alpha\int_0^t \mathbb{E}\|u\|_3^2\,ds \le \mathbb{E}E_2(u_0) + \alpha\left(A_2 t + c_2\int_0^t \mathbb{E}\|u\|_2^2\,ds + \int_0^t \mathbb{E}W_2(\|u\|)\,ds\right),$$

where W_1 and W_2 are the polynomials resulting from the estimate (3-18); their expectation is controlled using the second estimate. Now suppose $u_0 = 0$ almost surely; then by an induction argument, we get

$$\mathbb{E}E_2(u) + \alpha\int_0^t \mathbb{E}\|u\|_3^2\,ds \le \alpha C t,$$

where C is universal. Now in view of Remark 2.3, we can suppose $E_n(u) \ge 0$ (indeed, adding $c\|u\|^6$ to $E_2(u)$ we find a similar estimate). Then

$$\frac{1}{t}\int_0^t \mathbb{E}\|u\|_3^2\,ds \le C, \tag{4-1}$$

where C is, in particular, independent of t. Denote by $\lambda_\alpha(t)$ the law of the solution $u(t)$ to (1-3) starting at 0, and consider the time average

$$\bar{\lambda}_\alpha(t) = \frac{1}{t}\int_0^t \lambda_\alpha(s)\,ds.$$

Using the estimate (4-1), we show

$$\int_{H^2} \|u\|_3^2 \, \bar{\lambda}_\alpha(t)(du) \le C. \tag{4-2}$$

Then by the Chebyshev inequality we have

$$\bar{\lambda}_\alpha(t)(\{\|u\|_3 > R\}) \le \frac{C}{R^2} \quad \text{for any } R > 0.$$

Thus the compactness of the embedding $H^3(\mathbb{T}) \subset H^2(\mathbb{T})$ combined with the Prokhorov theorem implies that the family $\{\lambda_\alpha(t) \mid t > 0\}$ is compact with respect to the weak topology of H^2. Then for any α we denote by μ_α an accumulation point at infinity of the above family. The classical Bogoliubov–Krylov argument implies that μ_α is a stationary measure for (1-3). Passing to the limit $t \to \infty$ in (4-2) (using an approximation argument), we see that $\mu_\alpha(H^3) = 1$ for any α. We summarize these results in the following statement:

Proposition 4.1. *For any $\alpha \in (0,1)$, the stochastic BOB equation (1-3) posed in $H^2(\mathbb{T})$ has a stationary measure μ_α concentrated on $H^3(\mathbb{T})$.*

Theorem 4.2. *Let $\alpha \in (0,1)$. Suppose that A_n is finite for any n. Then any stationary measure μ_α of the problem (1-3) posed in $\dot{H}^2(\mathbb{T})$ satisfies*

$$\int_{H^2(\mathbb{T})} \|u\|_1^2 \, \mu_\alpha(du) = \frac{A_0}{2}, \tag{4-3}$$

$$\int_{H^2(\mathbb{T})} \|u\|^{2p} \, \mu_\alpha(du) \le p^p A_0^p \quad \textit{for any } 1 \le p < \infty, \tag{4-4}$$

$$\int_{H^2(\mathbb{T})} \|u\|_n^2 \, \mu_\alpha(du) \le D_n \quad \textit{for any } n \ge 2, \tag{4-5}$$

where, for any n, D_n does not depend on (t, α).

Proof. It suffices to prove (4-4) since then the estimate (4-5) follows from Lemma 1.1. We combine (3-16) and the stationarity of u to get (4-3). Let us prove (4-4).

To this end, let $R > 0$. Consider a C^∞-function χ_R satisfying

$$\chi_R(u) = \begin{cases} 1 & \text{if } \|u\|_2 \le R, \\ 0 & \text{if } \|u\|_2 > R + 1. \end{cases}$$

Let $p \ge 1$; we have

$$\int_{H^2} E_0^p(u) \, \chi_R(u) \, \mu_\alpha(du) = \int_{H^2} \mathbb{E}\{E_0^p(u(t,v)) \, \chi_R(u(t,v))\} \mu_\alpha(dv), \tag{4-6}$$

where $u(\cdot, v)$ is the solution of (1-3) starting at v. We pass to the limit $t \to \infty$ in the right-hand side of (4-6), and using (3-17) (u is in the ball of size R) and the stationarity of μ_α, we find

$$\int_{H^2} E_0^p(u) \, \chi_R(u) \, \mu_\alpha(du) \le p^p A_0^p.$$

Now Fatou's lemma allows to conclude. □

Corollary 4.3. *Let $\alpha \in (0,1)$. Suppose $A_n < \infty$ for any n. Then any stationary measure μ_α for the stochastic BOB problem (1-3) posed in $\dot{H}^2(\mathbb{T})$ is concentrated on $C^\infty(\mathbb{T})$.*

Proof. Let $n > 2$. Combining the estimate (4-5) and the Chebyshev inequality we find

$$\mu_\alpha(\{u \in H^2 \mid \|u\|_n \geq R\}) \leq \frac{D_n}{R^2}.$$

Setting $B_n(0,R)$ to be the ball in H^n of center 0 and radius R, we have

$$\int_{H^2} \mathbb{1}_{B_n(0,R)}(u)\, \mu_\alpha(du) = \mu_\alpha(B_n(0,R)) \geq 1 - \frac{D_n}{R^2}.$$

Passing to the limit on R (with the use of the Lebesgue convergence theorem), we get

$$\mu_\alpha(H^n(\mathbb{T})) = 1.$$

Thus

$$1 = \mu_\alpha\left(\bigcap_{n>2} H^n(\mathbb{T})\right) = \mu_\alpha(C^\infty(\mathbb{T})). \qquad \square$$

5. Invariant measure for the BO equation

In this section, $S_t : H^3(\mathbb{T}) \to H^3(\mathbb{T})$, $t \geq 0$, denotes the flow of the Benjamin–Ono equation (1-1). The map $S_{t,\alpha} : H^3 \to H^3$ denotes the one of the stochastic Benjamin–Ono–Burgers equation (1-3). We denote by ϕ_t, ϕ_t^*, $\phi_{t,\alpha}$, $\phi_{t,\alpha}^*$ the associated Markov semigroups, respectively. We suppose in what follows that $A_n < \infty$ for any $n > 0$.

Some convergence results of the stochastic BOB equation to the BO equation.

Lemma 5.1. *For any $T > 0$. For any $w \in H^3(\mathbb{T})$, we have, $\mathbb{P}$-almost surely,*

$$\sup_{t\in[0,T]} \|S_{t,\alpha} w - S_t w\|_2 \to 0 \quad \text{as } \alpha \to 0.$$

Proof. We write

$$\|S_{t,\alpha} w - S_t w\|_2 = \|v + z_\alpha - S_t w\|_2 \leq \|v - S_t w\|_2 + \|z_\alpha\|_2,$$

where

$$z_\alpha(t) = \sqrt{\alpha} \int_0^t e^{-(t-s)(H-\alpha)\,\partial_x^2}\, d\zeta(s) = \sqrt{\alpha} z(t)$$

and v is the solution of

$$\partial_t v + H\,\partial_x^2 v + (v + z_\alpha)\,\partial_x(v + z_\alpha) = \alpha\,\partial_x^2 v, \tag{5-1}$$

$$v_{t=0} = w. \tag{5-2}$$

Thanks to the estimate (3-7), we have that $\sup_{t\in[0,T]} \|z_\alpha\|_2 = \sqrt{\alpha} \sup_{t\in[0,T]} \|z\|_2$, where the quantity $\sup_{t\in[0,T]} \|z\|_2$ does not depend on α. Setting $h = v - S_t w$, we have

$$\sup_{t\in[0,T]} \|S_{t,\alpha} w - S_t w\|_2 \leq \sup_{t\in[0,T]} \|h\|_2 + \sqrt{\alpha} \sup_{t\in[0,T]} \|z\|_2.$$

We claim that $\sup_{t\in[0,T]}\|h\|_2 = O(\sqrt{\alpha})$. Indeed using the estimate (3-10) and the H^3-conservation law, we show that

$$\|h\|_2^3 \le c\|h\|\,\|h\|_3^2 \le C(T, \|w\|_{L^\infty(0,T;H^3)}, \|z\|_{H^3})\|h\|.$$

Taking the difference between (5-1) and the BO equation (1-1), we see that h satisfies

$$\partial_t h + H\,\partial_x^2 h + h\,\partial_x h = -\partial_x(hS_t w) - \partial_x(vz_\alpha) - z_\alpha\,\partial_x z_\alpha.$$

We multiply the above equation by h and we integrate on $\mathbb{T}$ to get

$$\partial_t\|h\|^2 = \tfrac{1}{2}(h^2, \partial_x S_t w) - (h, \partial_x(vz_\alpha)) - \tfrac{1}{2}(h, \partial_x z_\alpha^2).$$

By the Cauchy–Schwarz inequality and the algebra structure of H^1 we find

$$\begin{aligned}\partial_t\|h\|^2 &\le \tfrac{1}{2}\|h\|^2\,\|\partial_x S_t w\|_{L^\infty} + \tfrac{1}{2}\|h\|^2 + C\,\|v\|_1^2\,\|z_\alpha\|_1^2 + \tfrac{1}{4}\|h\|^2 + \tfrac{1}{4}\|z_\alpha\|_1^4\\ &\le \tfrac{1}{2}\|h\|^2\big(\|\partial_x S_t w\|_{L^\infty} + \tfrac{3}{2}\big) + C\alpha \sup_{t\in[0,T]}\|v\|_1^2 \sup_{t\in[0,T]}\|z\|_1^2 + \tfrac{1}{4}\alpha^2 \sup_{t\in[0,T]}\|z\|_1^4.\end{aligned}$$

Using the H^2-conservation law, we control $\|S_t w\|_{L^\infty(0,T;H^{3/2+})}$ (which does not depend on α) and $\|v\|_{L^\infty(0,T;H^1)}$ (see the estimate (3-8)). It remains to apply the Gronwall lemma to get the claim. □

Lemma 5.2. *For all $T, R, r > 0$, we have*

$$\sup_{w\in B(0,R)}\ \sup_{t\in[0,T]}\ \mathbb{E}\big[\|S_{t,\alpha}w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L^\infty(0,T;H^2)}\le r\}}\big] = O_{R,r,T}(\sqrt{\alpha}).$$

Here $B(0,R)$ is the ball in $H^3(\mathbb{T})$ of center 0 and radius R.

Proof.

$$\begin{aligned}\mathbb{E}\big[\|S_{t,\alpha}w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L^\infty(0,T;H^2)}\le r\}}\big] &= \int_\Omega \|S_{t,\alpha}w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L^\infty(0,T;H^2)}\le r\}}(\omega)\,d\mathbb{P}(\omega)\\ &\le \int_\Omega [\|h\|_2 + r\sqrt{\alpha}]\mathbb{1}_{\{\|z\|_{L^\infty(0,T;H^2)}\le r\}}(\omega)\,d\mathbb{P}(\omega),\end{aligned}$$

where $h = v - S_t w$ as before. The arguments of the proof of Lemma 5.1 allow to see that $\sup_{t\in[0,T]}\|h\|_2 \le C_{R,r,T}\sqrt{\alpha}$. This gives the claimed result. □

An accumulation point for the viscous stationary measures. In what follows we denote by $M(H^3)$ the space of probability measures on H^3.

Theorem 5.3. *For any sequence $(\alpha_k)_{k\in\mathbb{N}} \subset (0,1)$ converging to 0 as $k\to\infty$, there is a subsequence $\alpha_{r(k)}$ and $\mu \in M(H^3)$ such that*

- $\lim_{k\to\infty}\mu_{\alpha_{r(k)}} = \mu$ *in the weak topology of H^3,*
- *μ is invariant under the flow of the Benjamin–Ono equation in $H^3(\mathbb{T})$,*
- *μ is concentrated on $C^\infty(\mathbb{T})$,*

- *μ satisfies*

$$\int_{H^3(\mathbb{T})} \|u\|_1^2 \, \mu(du) = \frac{A_0}{2}, \tag{5-3}$$

$$\int_{H^3(\mathbb{T})} \|u\|^{2p} \, \mu(du) \le p^p A_0^p \quad \textit{for any } 1 \le p < \infty, \tag{5-4}$$

$$\int_{H^3(\mathbb{T})} \|u\|_n^2 \, \mu(du) < \infty \quad \textit{for } n \ge 2. \tag{5-5}$$

Proof. The proof consists in the following four steps:

(1) Existence of an accumulation point μ. The estimate (4-5) with $n = 4$ implies the tightness of the sequence of measures (μ_α) in $H^3(\mathbb{T})$ and, by the Prokhorov theorem, the existence of the claimed accumulation point μ on $H^3(\mathbb{T})$.

(2) Invariance of μ under the Benjamin–Ono flow. Denote by $(\mu_{\alpha_k})_{k\in\mathbb{N}}$ a subsequence of (μ_α) converging to μ (with $\lim_{k\to\infty} \alpha_k = 0$); to simplify the notations we write μ_k instead. The corresponding flow and Markov semigroup will be denoted $S_{t,k}$ and $\phi_{t,k}$.

The following diagram represents the idea of proof of the invariance of μ:

$$\begin{array}{ccc} \phi_{t,k}^* \mu_k & \overset{(I)}{=\!=} & \mu_k \\ {\scriptstyle (III)}\downarrow & & \downarrow{\scriptstyle (II)} \\ \phi_t^* \mu & \overset{(IV)}{=\!=} & \mu \end{array}$$

The equality (I) is the invariance of μ_k by $\phi_{t,k}$, and (II) is proved above. Then (IV) is proved once (III) is checked.

Let f be a real bounded Lipschitz function on $H^2(\mathbb{T})$. Without loss of generality assume that f is bounded by 1. Then

$$\begin{aligned} (\phi_{k,t}^* \mu_k, f) - (\phi_t^* \mu, f) &= (\mu_k, \phi_{t,k} f) - (\mu, \phi_t f) \\ &= (\mu_k, \phi_{t,k} f - \phi_t f) - (\mu - \mu_k, \phi_t f) \\ &= A - B. \end{aligned}$$

The term B converges to 0 as $k \to \infty$ by the weak convergence of (μ_k) to μ. And for any $R > 0$

$$\begin{aligned} |A| &\le \int_{H^3} \mathbb{E}|f(S_{t,k}w) - f(S_t w)| \, \mu_k(dw) \\ &= \int_{B(0,R)} \mathbb{E}|f(S_{t,k}w) - f(S_t w)| \, \mu_k(dw) + \int_{H^3 \setminus B(0,R)} \mathbb{E}|f(S_{t,k}w) - f(S_t w)| \, \mu_k(dw) \\ &= A_1 + A_2. \end{aligned}$$

Recalling that f is bounded by 1, we get by the Chebyshev inequality

$$A_2 \le 2\mu_k(H^3 \setminus B(0, R)) \le \frac{C}{R^2}, \tag{5-6}$$

where C is finite and does not depend on k (estimate (4-5)). Denote by $L_t^\infty H_x^2$ the space $L^\infty(0,T;H^2)$. Let $r>0$. We have

$$A_1=\int_{B(0,R)}\mathbb{E}\big[|f(S_{t,k}w)-f(S_tw)|\mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2}\le r\}}\big]\,\mu_k(dw)$$
$$+\int_{B(0,R)}\mathbb{E}\big[|f(S_{t,k}w)-f(S_tw)|\mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2}> r\}}\big]\,\mu_k(dw)=A_{1,1}+A_{1,2}.$$

As before, since f is bounded by 1, we use (3-6) and Chebyshev's inequality to get

$$A_{1,2}\le\frac{C_T}{r^2}.$$

On the other hand, since f is Lipschitz on H^2, we have

$$A_{1,1}\le C_f\int_{B(0,R)}\mathbb{E}\big[\|S_{t,k}w-S_tw\|_2\mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2}\le r\}}\big]\,\mu_k(dw)$$
$$\le C_f\sup_{w\in B(0,R)}\mathbb{E}\big[\|S_{t,k}w-S_tw\|_2\mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2}\le r\}}\big],$$

where C_f is the Lipschitz constant of f.

According to Lemma 5.2, we find

$$A_{1,1}\le C_{f,R,r,T}\sqrt{\alpha_k}.$$

Finally, we arrive at

$$|A|\le C_{f,R,r,T}\sqrt{\alpha_k}+\mathrm{Const}(T)\left(\frac{1}{r^2}+\frac{1}{R^2}\right),$$

where Const does not depend on k. We get the desired result after passing to the limits in the order

$$k\to\infty,\quad R,r\to\infty.$$

(3) The estimates for the measure μ. Denoting by χ_R a bump function on the ball $B(0,R)$ of $H^3(\mathbb{T})$, by (4-3) we have

$$\int_{H^3}\chi_R(v)\|v\|_1^2\,\mu_k(dv)\le\frac{A_0}{2}.$$

Passing to the limit $k\to\infty$ we find

$$\int_{H^3}\chi_R(v)\|v\|_1^2\,\mu(dv)\le\frac{A_0}{2}.$$

Then Fatou's lemma gives

$$\mathbb{E}\|u\|_1^2=\int_{H^3}\|v\|_1^2\,\mu(dv)\le\frac{A_0}{2}.\tag{5-7}$$

We proceed similarly to show (5-4) and (5-5).

Now we write

$$\frac{A_0}{2}=\int_{B(0,R)}\|v\|_1^2\,\mu_k(dv)+\int_{H^3\setminus B(0,R)}\|v\|_1^2\,\mu_k(dv).$$

We use the Cauchy–Schwarz and Chebyshev inequalities to show that

$$\int_{H^3\setminus B(0,R)} \|u\|_1^2\,\mu_k(du) = \int_{H^3} \|u\|_1^2 \mathbb{1}_{\|u\|_3>R}(u)\,\mu_k(du)$$
$$\leq \left(\int_{H^3} \|u\|_1^4\,\mu_k(du)\right)^{\frac{1}{2}} (\mu_k(\|u\|_3 > R))^{\frac{1}{2}} \leq \frac{\sqrt{\mathbb{E}[\|u\|_3^2]\mathbb{E}[\|u\|_1^4]}}{R}.$$

We can control $\mathbb{E}[\|u\|_1^4]$ and $\mathbb{E}[\|u\|_3^2]$ uniformly in k combining interpolation inequalities and the estimates (5-4) and (5-5). Then there is a constant $C > 0$ independent of k such that

$$\frac{A_0}{2} - \frac{C}{R} \leq \int_{H^3} \chi_R(v)\|v\|_1^2\,\mu_k(dv).$$

We find (5-3) after passing to the limits in the order

$$k \to \infty, \quad R \to \infty,$$

and combining this with (5-7).

(4) The measure μ is concentrated on $C^\infty(\mathbb{T})$. This immediately follows from the estimates (5-5) with use of the arguments of the proof of Corollary 4.3. □

6. Qualitative properties of the measure

Absolute continuity of some observables with respect to the Lebesgue measure. The following result is inspired by [Shirikyan 2011; Kuksin and Shirikyan 2012], where the local time concept is used to deduce nondegeneracy properties of measures constructed for the nonlinear Schrödinger and Euler equations.

Theorem 6.1. *Suppose that $\lambda_m \neq 0$ for all m. Then for any integer $n \geq 1$, there are constants b_n and c_n such that the distribution of the observable $\widetilde{E}_n(u) := E_n(u) + c_n\|u\|^2(1+\|u\|^2)^{b_n}$ under μ has a density with respect to the Lebesgue measure on $\mathbb{R}$.*

For the proof of Theorem 6.2 below, we refer the reader to [Shirikyan 2011] and the proof of Theorem 5.2.12 of [Kuksin and Shirikyan 2012], where the authors prove similar results in the case of the nonlinear Schrödinger and Euler equations respectively.

Theorem 6.2. *The measure μ constructed in Theorem 5.3 satisfies the following nondegeneracy properties*:

(1) *Let $\lambda_m \neq 0$ for at least two indices. Then μ has no atom at 0 and*

$$\mu(\{u \in C^\infty \mid \|u\| \leq \delta\}) \leq C\sqrt{A_0}\gamma^{-1}\delta \quad \text{for all } \delta > 0, \tag{6-1}$$

where $\gamma = \inf\{A_0 - \lambda_m^2 \mid m \in \mathbb{Z}\}$ and C is a universal constant.

(2) *Let $\lambda_m \neq 0$ for all indices. Then there is an increasing continuous function $h(r)$ vanishing at $r = 0$ such that*

$$\mu(\{u \in C^\infty(\mathbb{T}) \mid \|u\| \in \Gamma\}) \leq h(\ell(\Gamma)) \tag{6-2}$$

for any Borel set $\Gamma \subset \mathbb{R}$, where ℓ stands for the Lebesgue measure on $\mathbb{R}$.

Proof of Theorem 6.1. We prove the claim for the stationary measures in the case $\alpha > 0$, with uniform bounds in α. Then we can pass to the limit $\alpha \to 0$ to obtain the desired result using the Portmanteau theorem. First we apply the Itô formula to $\widetilde{E}_n(u)$:

$$\widetilde{E}_n(u(t)) = \widetilde{E}_n(u(0)) + \alpha \int_0^t A(s)\, ds + \sqrt{\alpha} \sum_{m\in\mathbb{Z}} \lambda_m \int_0^t \widetilde{E}_n'(u, e_m)\, d\beta_m(s),$$

where

$$A(s) = \partial_u \widetilde{E}_n(u, \partial_x^2 u) + \frac{1}{2} \sum_{m\in\mathbb{Z}} \lambda_m^2\, \partial_u^2 \widetilde{E}_n(u, e_m).$$

Denote by $\Lambda_t(a, \omega)$ its local time which reads (see the identity (A.45) of [Kuksin and Shirikyan 2012])

$$\begin{aligned}\Lambda_t(a,\omega) = (\widetilde{E}_n(u(t)) - a)_+ - (\widetilde{E}_n(u(0)) - a)_+ - \alpha \int_0^t A(s) \mathbb{1}_{(a,+\infty)}(\widetilde{E}_n(u))\, ds \\ - \sqrt{\alpha} \sum_{m\in\mathbb{Z}} \lambda_m \int_0^t \mathbb{1}_{(a,+\infty)}(\widetilde{E}_n(u)) \widetilde{E}_n'(u, e_m)\, d\beta_m(s).\end{aligned}$$

Using the stationarity of u, we infer that

$$\mathbb{E}\Lambda_t(a) = -\alpha t \mathbb{E}[A(0) \mathbb{1}_{(a,+\infty)}(\widetilde{E}_n(u))]. \tag{6-3}$$

Now using the (local time) identity (A.44) of [Kuksin and Shirikyan 2012] with the function $\mathbb{1}_\Gamma$, we get

$$2\int_\Gamma \Lambda_t(a)\, da = \alpha \sum_{m\in\mathbb{Z}} \lambda_m^2 \int_0^t \mathbb{1}_\Gamma(\widetilde{E}_n(u)) \widetilde{E}_n'(u, e_m)^2\, ds.$$

The stationarity of u gives again

$$2\int_\Gamma \mathbb{E}\Lambda_t(a)\, da = \alpha t \sum_{m\in\mathbb{Z}} \lambda_m^2 \mathbb{E}[\mathbb{1}_\Gamma(\widetilde{E}_n(u)) \widetilde{E}_n'(u, e_m)^2]. \tag{6-4}$$

Comparing (6-3) and (6-4), we find

$$\sum_{m\in\mathbb{Z}} \lambda_m^2 \mathbb{E}[\mathbb{1}_\Gamma(\widetilde{E}_n(u)) \widetilde{E}_n'(u, e_m)^2] \le 2\lambda(\Gamma)\mathbb{E}|A(0)| \le C\ell(\Gamma). \tag{6-5}$$

Recall now the form of $\widetilde{E}_n(u)$:

$$\widetilde{E}_n(u) = \|u\|_n^2 + R_n(u) + P_n(\|u\|^2),$$

where

$$P_n(r) = c_n r(1 + r)^{b_n}.$$

Then

$$\widetilde{E}_n'(u, v) = 2(D^n u, D^n v) + R_n'(u, v) + 2(u, v) P_n'(\|u\|^2).$$

Recalling Remark 2.3, we have

$$\widetilde{E}_n'(u, u) \ge \|u\|_n^2. \tag{6-6}$$

Now we define the operator H_n so that

$$\widetilde{E}_n'(u,v) = (H_n u, v).$$

Therefore

$$\begin{aligned}(H_n u, u) &= \sum_{m\in\mathbb{Z}} u_m (H_n u, e_m) = \sum_{|m|\le N} u_m (H_n u, e_m) + \sum_{|m|>N} u_m (H_n u, e_m)\\ &\le \frac{\|u\|}{\underline{\lambda}_N}\Big(\sum_{|m|\le N} \lambda_m^2 (H_n u, e_m)^2\Big)^{\frac{1}{2}} + \|H_n u\|\Big(\sum_{|m|>N} u_m^2\Big)^{\frac{1}{2}}\\ &\le \frac{\|u\|_1}{\underline{\lambda}_N}\Big(\sum_{m\in\mathbb{Z}} \lambda_m^2 \widetilde{E}_n'(u,e_m)^2\Big)^{\frac{1}{2}} + \|H_n u\|\frac{\|u\|_1}{N},\end{aligned}$$

where $\underline{\lambda}_N = \min\{\lambda_m \mid |m|\le N\} > 0$ for any $N > 0$. We take into account (6-6) and consider u belonging to

$$K_\epsilon = \Big\{v \;\Big|\; \|v\| \ge \epsilon,\ \|H_n v\| \le \frac{1}{\epsilon}\Big\}.$$

We get

$$\sum_{m\in\mathbb{Z}} \lambda_m^2 \widetilde{E}_n'(u,e_m)^2 \ge \underline{\lambda}_N^2 \Big(\epsilon - \frac{1}{N\epsilon}\Big)^2.$$

The integer N can be chosen to depend on ϵ so that we have

$$\alpha(\epsilon) := \underline{\lambda}_N^2 \Big(\epsilon - \frac{1}{N\epsilon}\Big)^2 > 0.$$

Then, by (6-5)

$$\mu(\{u \mid \widetilde{E}_n(u) \in \Gamma\} \cap K_\epsilon) \le \frac{C}{\alpha(\epsilon)}\ell(\Gamma).$$

Consider now the complementary set

$$K_\epsilon^c = \Big\{u \;\Big|\; \|u\| < \epsilon \text{ or } \|H_n u\| > \frac{1}{\epsilon}\Big\}$$

Since

$$\mathbb{E}\|H_n u\| \le \text{Const.}$$

Using the Chebyshev inequality, we find

$$\mu_\alpha\Big(\Big\{u \;\Big|\; \|H_n u\| > \frac{1}{\epsilon}\Big\}\Big) \le \text{Const}\ \epsilon.$$

By Theorem 6.2, we have that

$$\mu_\alpha(\{u \mid \|u\| < \epsilon\}) \le C\epsilon.$$

Finally we write

$$\mu_\alpha(\{u \mid \widetilde{E}_n(u) \in \Gamma\}) \le \mu(\{u \mid \widetilde{E}_n(u) \in \Gamma\} \cap K_\epsilon) + \mu(K_\epsilon^c) \le \frac{C_1}{\alpha(\epsilon)}\ell(\Gamma) + C_2\epsilon.$$

This, combined with the Portmanteau theorem, proves the absolute continuity of $\widetilde{E}_n(u)$ under μ with respect to the Lebesgue measure on $\mathbb{R}$. □

About the dimension of the measure μ. This subsection is inspired by [Kuksin 2008; Kuksin and Shirikyan 2012], where it was proved that the invariant measures constructed for the Euler equation are not concentrated on a countable union of finite-dimensional compact sets. The proof relies on a Krylov estimate (see Section A.9 of [Kuksin and Shirikyan 2012]) for Itô processes. Roughly speaking, this estimate provides an inequality of the type (6-5) for multidimensional processes. Namely, for a d-dimensional stationary Itô process

$$y_t = y_0 + \int_0^t x_s\, ds + \sum_{j=1}^{\infty} \int_0^t \theta_j(s)\, d\beta_j(s),$$

define the nonnegative $d \times d$-matrix σ with entries

$$\sigma_{m,n} = \sum_{j=1}^{\infty} \theta_j^m \theta_j^n,$$

where θ_j^i is the i-th component of the d-vector θ_j. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function. Then the Krylov estimate is

$$\mathbb{E} \int_0^1 f(y_t)(\det \sigma_t)^{\frac{1}{d}}\, dt \le C_d |f|_d \mathbb{E} \int_0^1 |x_t|\, dt, \tag{6-7}$$

where $|\cdot|_d$ stands for the L^d-norm and C_d is a constant that only depends on d.

In our context the independence needed to make the Krylov estimate successful leads to solving nonlinear differential equations with order increasing with the size of the underlying vector (process). This is due to the structure of the BO conservation laws and represents a technical difficulty as discussed in the Introduction, while in the Euler case the components of this vector can be chosen to satisfy this independence. We bypass the equation mentioned above in the 2-dimensional case by splitting suitably the phase space.

Theorem 6.3. *The measure μ is of at least 2-dimensional nature in the sense that any compact set of Hausdorff dimension smaller than 2 has μ-measure 0.*

Before proving Theorem 6.3, we describe the general framework.

We use the following splitting of $H^2(\mathbb{T})$:

$$H^2(\mathbb{T}) = O \cup O^c,$$

where

$$O := \left\{ u \;\middle|\; \int u^2 H\, \partial_x^2 u = 0 \right\}. \tag{6-8}$$

Consider the functionals on $\dot{H}^1(\mathbb{T})$ defined by

$$F_j(u) = \frac{1}{j+1} \int u^{j+1}, \quad j = 1, 2.$$

Remark that F_1 is preserved by the BO equation. Now for u a solution of (1-1), we have that

$$\partial_t F_2(u) = 0 \quad \text{on } O.$$

Therefore the vector $F(u) = (F_1(u), F_2(u))$ is constant on O for any solution u of the BO equation.

On the other hand, consider the following BO conservation laws

$$E_0(u) = \int u^2, \quad E_{\frac{1}{2}}(u) = \int uH\,\partial_x u + \frac{1}{3}\int u^3.$$

Set the following preserved vector

$$E(u) = (E_0(u), E_{\frac{1}{2}}(u)).$$

$E(u)$ is in particular constant on O^c for the solutions of (1-1).

Let μ_1 and μ_2 be two measures. We write $\mu_1 \lhd \mu_2$ if there is a continuous increasing function f vanishing at 0 such that

$$\mu_1(\cdot) \le f(\mu_2(\cdot)).$$

This implies the absolute continuity of μ_1 with respect to μ_2. For ν a probability measure on H^2, we define

$$\nu^O(\cdot) = \nu(\cdot \cap O), \quad \nu^{O^c}(\cdot) = \nu(\cdot \cap O^c),$$

where O is the set described before.

Proposition 6.4. *Suppose $\lambda_m \neq 0$ for all $m \in \mathbb{Z}$, then*

(1) $F_*\mu_\alpha^O \lhd \ell_2$, *where* $F = (F_1, F_2)$,

(2) $E_*\mu_\alpha^{O^c} \lhd \ell_2$, *where* $E = (E_0, E_{\frac{1}{2}})$.

The functions describing the absolute continuity do not depend on α, and ℓ_2 is the Lebesgue measure on $\mathbb{R}^2$.

Proof of Theorem 6.3. Let W be an open set of H^2. Clearly

$$W = (W \cap O) \cup (W \backslash O).$$

By Proposition 6.4, we have

$$\mu_\alpha(W) \le f(\ell_2(F(W \cap O))) + g(\ell_2(E(W \backslash O))),$$

where f and g are the functions describing the absolute continuity established in Proposition 6.4. Using the Portmanteau theorem, we get

$$\mu(W) \le f(\ell_2(F(W \cap O))) + g(\ell_2(E(W \backslash O))), \tag{6-9}$$

and by the regularity of μ and ℓ_2 the estimate (6-9) holds for any bounded Borelian set W.

When W is a compact set of Hausdorff dimension $\mathscr{H}(W) < 2$. It is clear that E and F are Lipschitz on any compact set. Since the Lipschitz maps do not increase the Hausdorff dimension, we have the right-hand side of (6-9) is equal to 0, then so is the left-hand side. □

Proof of Proposition 6.4. The proof consists of two steps:

(1) Absolute continuity uniformly in α of μ on the set O. The first and second derivatives of the functionals $F_j(u)$ are

$$F_j'(u, v) = \int u^j v, \quad F_j''(u, v) = j\int u^{j-1}v^2.$$

Then applying the Itô formula to F_j, we find

$$F_j(u) = F_j(u(0)) + \int_0^t A_j(s)\,ds + \sqrt{\alpha}\sum_{m\in\mathbb{Z}}\lambda_m\int_0^t (u^j, e_m)\,d\beta_m(s), \quad j=1,2,$$

where

$$A_j = -(u^j, H\,\partial_x^2 u - \alpha\,\partial_x^2 u) + j\,\frac{\alpha}{2}\sum_{m\in\mathbb{Z}}\lambda_m^2(u^{j-1}, e_m^2).$$

On the set O, we have $(u^j, H\,\partial_x^2 u)=0,\ j=1,2$. Then recalling estimate on $\mathbb{E}\|u\|_2^2$ (Theorem 5.3), we get

$$\mathbb{E}|A_j| \le \alpha\,\text{Const}, \tag{6-10}$$

where Const does not depend on α.

We consider the 2×2-matrix $\sigma(u),\ u\in O$, with entries

$$\sigma_{k,l}(u) = \sum_{m\in\mathbb{Z}}\lambda_m^2(u^k, e_m)(u^l, e_m), \quad k,l=1,2.$$

It is clear that σ is nonnegative. It follows from the Krylov estimate (6-7) with the use of the function $\mathbb{1}_\Gamma$, Γ being a Borel set of $\mathbb{R}^2$, that

$$\mathbb{E}\big[(\det(\sigma(u)))^{\frac{1}{2}}\mathbb{1}_\Gamma(F)\big] \le C\ell_2(\Gamma), \tag{6-11}$$

where ℓ_2 is the Lebesgue measure on $\mathbb{R}^2$ and C does not depend on α.

Now define the map

$$D : \dot{H}^1(\mathbb{T}) \to \mathbb{R}_+,$$
$$u \mapsto \det(\sigma(u)).$$

We remark that D is continuous since it is the composition of continuous maps. We have the following:

Lemma 6.5. *Suppose $\lambda_m \neq 0$ for all $m\in\mathbb{Z}$; then*

$$D(u)=0 \implies u\equiv 0.$$

Proof. Suppose there is a nonzero vector $\gamma=(\gamma_1,\gamma_2)\in\mathbb{R}^2$ such that

$$\gamma\sigma(u)\gamma^T = 0,$$

then

$$\sum_{m\in\mathbb{Z}}\lambda_m^2\Big(\sum_{j=1}^2\gamma_j(u^j, e_m)\Big)^2 = 0.$$

Since $\lambda_m\neq 0$ for all $m\neq 0$, we infer that

$$\sum_{j=1}^2\gamma_j u^j \equiv \text{Const},$$

which is possible only if $u\equiv 0$, taking into account that $\int u = 0$. □

Now define the set

$$J_\epsilon = \Big\{\|u\|_1^2 \ge \epsilon,\ \|u\|_2^2 \le \frac{1}{\epsilon}\Big\} \subset H^2(\mathbb{T}).$$

$J_\epsilon \cap O$ is a compact set in $H^1(\mathbb{T})$ not containing 0; then by the continuity of D, we have $D(J_\epsilon \cap O)$ is a compact set in $\mathbb{R}_+$ not containing 0. Then there is $c_\epsilon > 0$ such that $D(u) \geq c_\epsilon$ for all $u \in J_\epsilon \cap O$. Using the same splitting argument as in the proof of Theorem 6.1, we arrive at the claimed result.

(2) Absolute continuity uniformly in α of μ on the set O^c. We follow the construction above to set a 2×2-matrix M with entries

$$M_{k,l}(u) = \sum_{m \in \mathbb{Z}} \lambda_m^2 B_k(u) B_l(u), \quad k, l = 1, 2,$$

where

$$B_1 = E_0'(u, e_m) \quad \text{and} \quad B_2 = E_{\frac{1}{2}}'(u, e_m).$$

It follows from the Krylov estimate (6-7) that

$$\mathbb{E}\big[(\det(M(u)))^{\frac{1}{2}} \mathbb{1}_\Gamma(E)\big] \leq C \ell_2(\Gamma),$$

where C does not depend on α thanks to the preservation of E_0 and $E_{\frac{1}{2}}$ by the BO flow.

Now $\det M(u) = 0$ only if there is a nonzero vector $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ such that

$$\gamma_1 u + \gamma_2 (2H\, \partial_x u + u^2) \equiv \text{Const}.$$

Note that if $\gamma_2 = 0$, we have that $u \equiv 0$ since $\int u = 0$; therefore $u \in O$. Now we suppose that $\gamma_2 \neq 0$. We differentiate with respect to x to find

$$\gamma_1 \partial_x u + \gamma_2 (2H\, \partial_x^2 u + 2u\, \partial_x u) \equiv 0.$$

Therefore, multiplying by u^p for $p > 0$ and integrating in x, we find

$$\int u^p H\, \partial_x^2 u = 0,$$

and in particular u belongs to the set O. Then on O^c we have $\det(M(u)) \neq 0$. We can follow the same splitting argument with the use of the splitting set J_ϵ defined in the first part to get the result. □

A Gaussian decay property for the measure μ. Here we establish a large deviation bound for the measure μ.

Theorem 6.6. *The measure μ constructed in Theorem 5.3 satisfies*

$$\mathbb{E} e^{\sigma \|u\|^2} < \infty, \tag{6-12}$$

where $\sigma = (aeA_0)^{-1}$ for arbitrary $a > 1$. In particular, for any $r > 0$

$$\mu(\{u \in C^\infty \mid \|u\| > r\}) \leq C e^{-\sigma r^2},$$

where the constant C does not depend on r.

Proof. Recall the estimate (5-4):

$$\mathbb{E}\|u\|^{2p} \le p^p A_0^p.$$

Then

$$\mathbb{E}(\sqrt{\sigma}\|u\|)^{2p} \le \sigma^p p^p A_0^p = \frac{p^p}{a^p e^p}.$$

Now, with use of the Stirling formula, we have

$$\frac{\mathbb{E}(\sqrt{\sigma}\|u\|)^{2p}}{p!} \le \frac{p^p}{p!\,a^p e^p} \sim_{p\to\infty} \frac{1}{a^p\sqrt{2\pi p}}.$$

Since $a > 1$, we have that the series

$$\sum_{p\ge 1} \frac{\mathbb{E}(\sqrt{\sigma}\|u\|)^{2p}}{p!}$$

is convergent, and we are led to (6-12). The other claim is obtained after combining (6-12) with the Chebyshev inequality. □

Remark 6.7. We obtain in the same way the result of Theorem 6.6 for the viscous measures uniformly in α.

Appendix

Proof of Lemma 3.3. Note first that for a solution v of the nonlinear equation (3-3), we have

$$\partial_t E_n(v) = E_n'(v, \partial_t v) = \alpha E_n'(v, \partial_x^2 v) - \sqrt{\alpha} E_n'(v, \partial_x(vz)) - \alpha \tfrac{1}{2} E_n'(v, \partial_x(z^2)), \quad n = 0, 1, 2. \tag{A-1}$$

The E_n are the first three conservation laws of the BO equation.

The case $n = 0$: $E_0'(v, w) = 2\int vw$. Applying (A-1), we get

$$\begin{aligned}
\partial_t E_0(v) + 2\alpha\|v\|_1^2 &= 2\sqrt{\alpha}(v, \partial_x(vz)) + \alpha(v, \partial_x z^2) \\
&= \sqrt{\alpha}(v^2, \partial_x z) + \alpha(v, \partial_x z^2) \\
&\le \sqrt{\alpha}\|z\|_{\frac{3+}{2}}\|v\|^2 + c\alpha\|v\|\,\|z\|_1^2 \\
&\le \sqrt{\alpha}\|z\|_{\frac{3+}{2}}\|v\|^2 + c\alpha(1 + \|v\|^2)\|z\|_1^2.
\end{aligned}$$

Note that $\|z(\cdot)\|_{\frac{3+}{2}}$ is bounded uniformly in α for almost all realizations and in t (on $[0, T]$) by continuity. Then with the use of the Gronwall inequality we get

$$\sup_{t\in[0,T]} \|v(t)\|^2 + 2\alpha\int_0^T \|v(t)\|_1^2\,dt \le C(T, \omega, \|v_0\|). \tag{A-2}$$

The case $n = 1$: Recall that

$$E_1(u) = \int (\partial_x u)^2 + \frac{3}{4}\int u^2 H\,\partial_x u + \frac{1}{8}\int u^4.$$

Then

$$E_1'(v, w) = -2(\partial_x^2 v, w) + \underbrace{\tfrac{3}{2}(vH\partial_x v, w) + \tfrac{3}{4}(v^2, H\partial_x w) + \tfrac{1}{2}(v^3, w)}_{R_1'(v,w)}.$$

It was already shown that (see the more general estimates (2-7) and (2-6))

$$|R_1'(v,\partial_x^2 v)| \le \epsilon\|v\|_2^2 + C_\epsilon\|v\|^c.$$

Then

$$\alpha E_1'(v,\partial_x^2 v) \le -(2-\epsilon)\alpha\|v\|_2^2 + C_\epsilon\alpha\|v\|^c.$$

Taking into account some properties of H, it suffices to treat $(vH\,\partial_x v, w) + (v^3, w)$ instead of $R_1'(v,w)$ for our purpose.

Now

$$\begin{aligned}
\sqrt{\alpha}|(\partial_x^2 v, \partial_x(vz))| &\le C\sqrt{\alpha}\|v\|_2\,\|v\|_1\,\|z\|_1\\
&\le \epsilon\alpha\|v\|_2^2 + C_\epsilon\|v\|_1^2\,\|z\|_1^2 \le \epsilon\alpha\|v\|_2^2 + C_{T,\epsilon,\omega}\|v\|_1^2,\\
\sqrt{\alpha}|(vH\,\partial_x v, \partial_x(vz))| &= \sqrt{\alpha}|(\partial_x(vH\partial_x v), vz)|\\
&\le C\sqrt{\alpha}\|v\|_1\,\|v\|_2\,\|v\|\,\|z\|_{\frac{1+}{2}}\\
&\le \epsilon\alpha\|v\|_2^2 + C_\epsilon\|v\|_1^2\,\|v\|^2\,\|z\|_{\frac{1+}{2}}^2 \le \epsilon\alpha\|v\|_2^2 + C_{T,\epsilon,\omega}\|v\|_1^2,\\
\sqrt{\alpha}|(v^3, \partial_x(vz))| &\le C\sqrt{\alpha}\|v\|_{L^6}^3\,\|v\|_1\,\|z\|_1\\
&\le C\sqrt{\alpha}\|v\|_{\frac{1}{3}}^3\,\|v\|_1\,\|z\|_1\\
&\le C\sqrt{\alpha}\|v\|^2\,\|v\|_1^2\,\|z\|_1 \le \sqrt{\alpha}C_{T,\omega}\|v\|_1^2.
\end{aligned}$$

To summarize, we have

$$\sqrt{\alpha}E_1'(v,\partial_x(vz)) \le \epsilon\|v\|_2^2 + C_{T,\epsilon,\omega}\|v\|_1^2.$$

To estimate the last term, we compute

$$\begin{aligned}
\alpha|(\partial_x^2 v, \partial_x z^2)| &\le C\alpha\|v\|_2\,\|z\|_1^2\\
&\le \epsilon\alpha\|v\|_2^2 + \alpha C_\epsilon\|z\|_1^4,\\
\alpha|(vH\,\partial_x v, \partial_x z^2)| &\le C\alpha\|v\|_2\,\|v\|_1\,\|z\|_{\frac{1}{4}}^2\\
&\le \epsilon\alpha\|v\|_2^2 + \alpha C_{T,\epsilon,\omega}\|v\|_1^2,\\
\alpha|(v^3, \partial_x z^2)| &\le C\alpha\|v\|^2\,\|v\|_1\,\|z\|_1^2\\
&\le \epsilon\alpha\|v\|_1^2 + \alpha C_{T,\epsilon,\omega}.
\end{aligned}$$

To conclude, we can choose ϵ so that

$$E_1(v) + \alpha\int_0^t \|v(r)\|_2^2\,dr \le E_1(v_0) + C_{T,\omega}^1\int_0^t \|v(r)\|_1^2\,dr + C_{T,\omega}^2 t.$$

Recalling the inequality (2-4) and (A-2) we have

$$\|v\|_1^2 + 2\alpha\int_0^t \|v(r)\|_2^2\,dr \le E_1(v_0) + C_{T,\omega}^0 + C_{T,\omega}^1\int_0^t \|v(r)\|_1^2\,dr + C_{T,\omega}^2 t.$$

With the use of the Gronwall lemma, we arrive at

$$\sup_{t\in[0,T]} \|v(t)\|_1^2 + 2\alpha \int_0^T \|v(t)\|_2^2\, dt \le C_{T,\omega}(\|v_0\|_1).$$

The case $n = 2$: Recall that

$$\begin{aligned} E_2(u) = \int (\partial_x^2 u)^2 - \frac{5}{4}\int \big((\partial_x u)^2 H\,\partial_x u + 2\partial_x^2 u H\,\partial_x u\big) \\ + \frac{5}{16}\int \big(5u^2(\partial_x u)^2 + u^2(H\,\partial_x u)^2 + 2uH(\partial_x u)H(u\,\partial_x u)\big) \\ + \int \Big(\frac{5}{32}u^4 H(\partial_x u) + \frac{5}{24}u^3 H(u\,\partial_x u)\Big) + \frac{1}{48}\int u^6. \end{aligned}$$

The form of $E_2(v)$ combined with some properties of H allows us to reduce to the treatment of the quantity

$$R_2(v) = \|v\|_2^2 + \int (\partial_x v)^3 + (\partial_x^2 v, H\,\partial_x v) + (v^2, (\partial_x v)^2) + (v^4, H\,\partial_x v) + \int v^6.$$

Then

$$\begin{aligned} R_2'(v,w) &= 2(\partial_x^2 v, \partial_x^2 w) + 3((\partial_x v)^2, \partial_x w) + 2(\partial_x^2 v, H\,\partial_x w) + 2(vw, (\partial_x v)^2) \\ &\quad + 2(v^2\partial_x v, \partial_x w) + 4(v^3 H\,\partial_x v, w) + (v^4, H\,\partial_x w) + 6(v^5, w) \\ &= 2(\partial_x^2 v, \partial_x^2 w) + R_3'(v,w). \end{aligned}$$

It was already shown in the proof of Lemma 2.2 (see estimates (2-7) and (2-6)) that

$$|R_3'(v, \partial_x^2 v)| \le \epsilon \|v\|_3^2 + C_\epsilon \|v\|^c$$

for some constants $c, C_\epsilon > 0$. Now we have

$$2\alpha(\partial_x^2 v, \partial_x^2(\partial_x^2 v)) = -2\alpha\|v\|_3^2.$$

Then

$$\alpha E_2'(v, \partial_x^2 v) \le -(2-\epsilon)\alpha\|v\|_3^2 + \alpha C_\epsilon \|v\|^c.$$

Now

$$\begin{aligned} \sqrt{\alpha}|(\partial_x^2 v, \partial_x^2(\partial_x(vz)))| &\le C\sqrt{\alpha}\|v\|_3 \|v\|_2 \|z\|_2 \le \epsilon\alpha\|v\|_3^2 + C_{T,\epsilon,\omega}\|v\|_2^2, \\ \sqrt{\alpha}|((\partial_x v)^2, \partial_x^2(vz))| &\le C_{T,\omega}\sqrt{\alpha}\|v\|_{\frac{5}{4}}^2 \|v\|_2 \le C_{T,\omega}\sqrt{\alpha}\|v\|_{\frac{1}{4}} \|v\|_2^2 \le C_{T,\omega}\sqrt{\alpha}\|v\|_2^2, \\ \sqrt{\alpha}|(\partial_x^2(vz), H\,\partial_x^2 v)| &\le \sqrt{\alpha}C\|v\|_2^2 \|z\|_2 \le \sqrt{\alpha}C_{T,\omega}\|v\|_2^2, \\ \sqrt{\alpha}|(v(\partial_x v)^2, \partial_x(vz))| &\le C\sqrt{\alpha}\|v\| \|z\|_{\frac{1}{2}+} \|v\|_2^2 \le C_{T,\omega}\sqrt{\alpha}\|v\|_2^2, \\ \sqrt{\alpha}|(v^2\partial_x v, \partial_x^2(vz))| &\le \sqrt{\alpha}C\|v\|_1^3 \|v\|_2 \|z\|_2 \le C_{T,\omega}\sqrt{\alpha}\|v\|_2^2, \\ \sqrt{\alpha}|(v^3 H\partial_x v, \partial_x(vz))| &\le C\sqrt{\alpha}\|v\|_1^3 \|v\|_2 \|z\|_{\frac{1}{2}+} \|v\| \le \sqrt{\alpha}C_{T,\omega}\|v\|_2^2, \\ \sqrt{\alpha}|(v^4, H\,\partial_x^2(vz))| &\le \sqrt{\alpha}C\|v\|_1^5 \|z\|_1 \le \sqrt{\alpha}C_{T,\omega}, \\ \sqrt{\alpha}|(v^5, \partial_x(vz))| &\le C\sqrt{\alpha}\|v\|_1^5 \|v\| \|z\|_{\frac{1}{2}+} \le \sqrt{\alpha}C_{T,\omega}. \end{aligned}$$

The estimates concerning the term $\partial_x z^2$ are easier because they do not contain v. Finally, using the same argument as before (in the case of $E_1(v)$), we arrive at the claimed result.

The periodic Hilbert transform. We present in this section a definition of the Hilbert transform in the periodic setting and establish some of its elementary properties. Recall that the sequence defined by

$$e_n(x) = \begin{cases} \sin(nx)/\sqrt{\pi} & \text{if } n < 0, \\ \cos(nx)/\sqrt{\pi} & \text{if } n > 0, \end{cases}$$

forms a Hilbertian basis of $\dot{H}(\mathbb{T})$; let us denote this basis by $\mathscr{B}$. We define the Hilbert transform on $\mathscr{B}$ by

$$He_n(x) = \operatorname{sgn}(n)\, e_{-n}(x),$$

where

$$\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } p > 0, \\ 0 & \text{if } p = 0, \\ -1 & \text{if } p < 0. \end{cases}$$

We first remark that H defines an isometry on $\dot{H}$.

Proposition A.1. *Let $f, g \in \dot{H}(\mathbb{T})$. Then*

$$H^2 f = -f, \tag{A-3}$$

$$\int_{\mathbb{T}} Hf = 0, \tag{A-4}$$

$$(g, Hf) = -(Hg, f), \tag{A-5}$$

$$\widehat{HF}_0(p) = -\mathrm{i}\operatorname{sgn}(p)\hat{f}_0(p), \tag{A-6}$$

where $\hat{h}_0$ denotes the complex Fourier coefficient of a function h, defined below.

Define now the Fourier coefficients associated to a function f in $\dot{H}$:

$$\hat{f}_1(n) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{T}} \cos(nx) f(x)\, dx,$$
$$\hat{f}_2(n) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{T}} \sin(nx) f(x)\, dx.$$

The function f is represented in $\mathscr{B}$ as follows:

$$f(x) = \sum_{n>0} \big(\hat{f}_1(n) e_n(x) - \hat{f}_2(n) e_{-n}(x)\big). \tag{A-7}$$

Hence the Hilbert transform of f can be expressed as

$$Hf(x) = \sum_{n>0} \big(\hat{f}_1(n) e_{-n}(x) + \hat{f}_2(n) e_n(x)\big). \tag{A-8}$$

The complex Fourier coefficient is defined by

$$\hat{f}_0(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ipx} f(x)\, dx. \tag{A-9}$$

The relationship between the three Fourier coefficients of f is

$$\hat{f}_0(p) = \frac{\hat{f}_1(p) - \mathrm{i}\,\mathrm{sgn}(p)\hat{f}_2(p)}{\sqrt{2}}. \tag{A-10}$$

Proof of Proposition A.1. Equation (A-4) follows immediately from (A-8). Now from (A-7) and (A-8), we can easily deduce that

$$H^2 f(x) = -\sum_{n>0} \big(\hat{f}_1(n)e_n(x) - \hat{f}_2(n)e_{-n}(x)\big) = -f(x).$$

and (A-3) is shown.

From (A-8), we infer that

$$\widehat{HF}_1(p) = -\hat{f}_2(p), \quad \widehat{HF}_2(p) = \hat{f}_1(p).$$

Thus using (A-10), we can write

$$\begin{aligned}\widehat{HF}_0(p) &= \frac{-\hat{f}_2(p) - \mathrm{i}\,\mathrm{sgn}(p)\hat{f}_1(p)}{\sqrt{2}} \\ &= \frac{-\mathrm{i}\,\mathrm{sgn}(p)(\hat{f}_1(p) - \mathrm{i}\,\mathrm{sgn}(p)\hat{f}_2(p))}{\sqrt{2}} = -\mathrm{i}\,\mathrm{sgn}(p)\hat{f}_0(p),\end{aligned}$$

and we have arrived at (A-6).

To prove (A-5), we compute

$$\begin{aligned}(g, Hf) &= \sum_{n>0} \hat{f}_1(n) \int_{\mathbb{T}} g(x)e_{-n}(x)\, dx + \sum_{n>0} \hat{f}_2(n) \int_{\mathbb{T}} g(x)e_n(x)\, dx \\ &= -\sum_{n>0} \hat{f}_1(n)\hat{g}_2(n) + \sum_{n>0} \hat{g}_1(n)\hat{f}_2(n) \\ &= -\sum_{n>0} \hat{g}_2(n) \int_{\mathbb{T}} f(x)e_n(x)\, dx - \sum_{n>0} \hat{g}_1(n) \int_{\mathbb{T}} f(x)e_{-n}(x)\, dx \\ &= -\int_{\mathbb{T}} f(x) \sum_{n>0} (\hat{g}_1(n)e_{-n}(x) + \hat{g}_2(n)e_n(x)) \\ &= -(Hg, f).\end{aligned}$$

□

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