ANALYSIS & PDE

Volume 11 No. 8 2018

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ANALYSIS AND PDE
Vol. 11, No. 8, 2018
dx.doi.org/10.2140/apde.2018.11.2049

MONOTONICITY OF NONPLURIPOLAR PRODUCTS AND COMPLEX MONGE–AMPÈRE EQUATIONS WITH PRESCRIBED SINGULARITY

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We establish the monotonicity property for the mass of nonpluripolar products on compact Kähler manifolds, and we initiate the study of complex Monge–Ampère-type equations with prescribed singularity type. Using the variational method of Berman, Boucksom, Guedj and Zeriahi we prove existence and uniqueness of solutions with small unbounded locus. We give applications to Kähler–Einstein metrics with prescribed singularity, and we show that the log-concavity property holds for nonpluripolar products with small unbounded locus.

1. Introduction and main results

Let X be a compact Kähler manifold of complex dimension n, and let θ be a smooth closed real $(1, 1)$-form on X such that $\{\theta\}$ is big. Broadly speaking, the purpose of this article is threefold. First, we develop the potential theory of nonpluripolar products without any restrictions on the singularity type by combining techniques of Witt Nyström [2017] and previous work of the authors [Darvas et al. 2018]. Second, given $\phi \in \mathrm{PSH}(X, \theta)$, we introduce and study the spaces $\mathcal{E}(X, \theta, \phi)$ and $\mathcal{E}^1(X, \theta, \phi)$, generalizing the content of [Boucksom et al. 2010] to the relative framework. These latter spaces contain potentials that are slightly more singular than ϕ, and satisfy a (relative) full mass/finite energy condition. Lastly, with sufficient potential theory developed, we focus on the variational study of the complex Monge–Ampère equation

$$(\theta + i\partial\bar{\partial}u)^n = f\omega^n, \tag{1}$$

where $f \geq 0$, $f \in L^p(\omega^n)$, $p > 1$, and the singularity type of $u \in \mathrm{PSH}(X, \theta)$ is the same as that of ϕ. As it will turn out, this equation is well-posed only for potentials ϕ with a certain type of "model" singularity, which includes the case of analytic singularities, and we provide existence of unique solutions with small unbounded locus. As we will see, on the right-hand side of (1) one may even consider more general (nonpluripolar) Radon measures.

When θ is a Kähler form, $f > 0$ is smooth, and $\phi = 0$, the above equation was solved (with smooth solutions) by Yau [1978], see also [Aubin 1978], resolving the famous Calabi conjecture. Using both a priori estimates and pluripotential theory, this result was later extended in many different directions; see [Kołodziej 1998; 2003; Guedj and Zeriahi 2007; Boucksom et al. 2010; Berman et al. 2013; Berman

MSC2010: primary 32Q15, 32U05, 32W20; secondary 32Q20.
Keywords: Monge–Ampère equation, variational approach, pluripotential theory.

2013; Phong and Sturm 2014]. Our approach seems to unify all existing works (in the compact setting), under the theme of solutions with arbitrary prescribed (model) singularity type.

At the end of the paper, we give applications of our results to singular Kähler–Einstein metrics and establish the log-concavity property for certain nonpluripolar products. Other applications will be treated in a sequel.

Though we will work in the general framework of big cohomology classes throughout the paper, we note that all our results seem to be new in the particular case of Kähler classes as well.

Monotonicity of nonpluripolar products and relative finite energy. Unless otherwise specified, we fix a background Kähler structure (X, ω) for the remainder of the paper.

We say that a potential $u \in L^1(X, \omega^n)$ is θ-plurisubharmonic (θ-psh) if locally u is the difference of a plurisubharmonic and a smooth function, and $\theta_u := \theta + i\partial\bar{\partial}u \geq 0$ in the sense of currents. The set of θ-psh potentials is denoted by $\mathrm{PSH}(X, \theta)$. We say that $\{\theta\}$ is *pseudoeffective* if $\mathrm{PSH}(X, \theta)$ is nonempty. Along these lines, $\{\theta\}$ is *big* if $\mathrm{PSH}(X, \theta - \varepsilon\omega)$ is nonempty for some $\varepsilon > 0$.

If u and v are two θ-psh functions on X, then u is said to be *less singular* than v if $v \leq u + C$ for some $C \in \mathbb{R}$. We say that u has the same singularities as v if u is less singular than v, and v is less singular than u. This defines an equivalence relation on $\mathrm{PSH}(X, \theta)$ whose equivalence classes are the *singularity types* $[u]$, $u \in \mathrm{PSH}(X, \theta)$.

Given closed positive $(1, 1)$-currents $T_1 := \theta^1_{u_1}, \ldots, T_p := \theta^p_{u_p}$, where the θ^j are closed smooth real $(1, 1)$-forms, generalizing the construction of [Bedford and Taylor 1987] in the local setting, it was shown in [Boucksom et al. 2010] that one can define the *nonpluripolar product* of these currents:

$$\theta^1_{u_1} \wedge \cdots \wedge \theta^p_{u_p} := \langle T_1 \wedge \cdots \wedge T_p \rangle.$$

The resulting positive (p, p)-current does not charge pluripolar sets and it is *closed*. For a θ-psh function u, the *nonpluripolar complex Monge–Ampère measure* of u is simply $\theta^n_u := \theta_u \wedge \cdots \wedge \theta_u$.

It was recently proved by Witt Nyström [2017, Theorem 1.2] that the complex Monge–Ampère mass of θ-psh potentials decreases as the singularity type increases. Our main result about monotonicity of nonpluripolar products generalizes this result to the case of different cohomology classes $\{\theta^j\}$, fully proving what was conjectured by Boucksom, Eyssidieux, Guedj and Zeriahi (see the comments after [Boucksom et al. 2010, Theorem 1.16] in which they prove that the result holds for potentials with small unbounded locus):

Theorem 1.1. *Let θ^j, $j \in \{1, \ldots, n\}$, be smooth closed real $(1, 1)$-forms on X. Let $u_j, v_j \in \mathrm{PSH}(X, \theta^j)$ such that u_j is less singular than v_j for all $j \in \{1, \ldots, n\}$. Then*

$$\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} \geq \int_X \theta^1_{v_1} \wedge \cdots \wedge \theta^n_{v_n}.$$

To prove the above theorem, we first need to generalize the main convergence theorems of Bedford–Taylor theory [1987]; see also [Xing 1996; 2009]. This is done collectively in the next result, further elaborated in Theorem 2.3 below:

Theorem 1.2. *Let θ^j, $j \in \{1, \dots, n\}$, be smooth closed real $(1,1)$-forms on X. Suppose that we have $u_j, u_j^k \in \mathrm{PSH}(X, \theta^j)$ such that $u_j^k \to u_j$ in capacity as $k \to \infty$, and*

$$\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} \geq \limsup_{k\to\infty} \int_X \theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k}. \tag{2}$$

Then $\theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k} \to \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n}$ in the weak sense of measures.

We recall that a sequence $\{u_k\}_k$ converges in capacity to u if for any $\delta > 0$ we have

$$\lim_{k\to\infty} \mathrm{Cap}_\omega\{|u_k - u| \geq \delta\} = 0,$$

where Cap_ω is the Monge–Ampère capacity associated to ω; see [Guedj and Zeriahi 2017, Definition 4.23].

We note that condition (2) is necessary in this generality, even in the Kähler case. Indeed, if $u \in \mathrm{PSH}(X, \omega)$ is a pluricomplex Green potential, then the cut-offs $u_j := \max(u, -j) \in \mathrm{PSH}(X, \omega)$ satisfy $u_j \searrow u$. However, $\int_X \omega^n_{u_j} = \int_X \omega^n > 0$ for all j, and $\int_X \omega^n_u = 0$; hence $\omega^n_{u_j}$ cannot converge to ω^n_u weakly.

As noted above, Theorem 1.2 generalizes classical theorems of Bedford and Taylor (when u_j^k, u_j are uniformly bounded) and also results from [Boucksom et al. 2010] (when u_j^k, u_j have full mass). In both of these cases, there are severe restrictions on the singularity class of the potentials u_j^k, u_j. On the other hand, the above theorem shows that there is no need for restrictions on singularity type of the potentials involved. Instead, one needs only a semicontinuity condition on the total masses.

To develop the variational approach to (1), with the above general results in hand, we initiate the study of relative full mass/relative finite energy currents. Let $\phi \in \mathrm{PSH}(X, \theta)$. We say that $v \in \mathrm{PSH}(X, \theta)$ has *full mass relative* to ϕ ($v \in \mathcal{E}(X, \theta, \phi)$) if v is more singular than ϕ and $\int_X \theta^n_v = \int_X \theta^n_\phi$. In our investigation of these classes, the following well-known envelope constructions will be of great help:

$$\psi \quad\to\quad P_\theta(\psi, \phi),\ P_\theta[\psi](\phi),\ P_\theta[\psi] \in \mathrm{PSH}(X, \theta) \quad \text{where } \psi \in \mathrm{PSH}(X, \theta).$$

These were introduced by Ross and Witt Nyström [2014] in their construction of geodesic rays, building on ideas of Rashkovskii and Sigurdsson [2005] in the local setting. Due to the frequency of these operators appearing in this work, we choose to follow slightly different notations. The starting point is the "rooftop envelope"

$$P_\theta(\psi, \phi) := \sup\{v \in \mathrm{PSH}(X, \theta) \mid v \leq \min(\psi, \phi)\}.$$

This allows us to introduce

$$P_\theta[\psi](\phi) := \big(\lim_{C\to\infty} P_\theta(\psi + C, \phi)\big)^*,$$

and it is easy to see that $P_\theta[\psi](\phi)$ only depends on the singularity type of ψ. When $\phi = 0$ or $\phi = V_\theta$, we will simply write $P_\theta[\psi] := P_\theta[\psi](0) = P_\theta[\psi](V_\theta)$, and we refer to this potential as the *envelope of the singularity type* $[\psi]$.

Using the techniques of our recent work [Darvas et al. 2018], we can give a generalization of [Darvas 2017, Theorem 3], paralleling [Darvas et al. 2018, Theorem 1.2]. This result characterizes membership in $\mathcal{E}(X, \theta, \phi)$ solely in terms of singularity type:

Theorem 1.3. *Suppose* $\phi \in \mathrm{PSH}(X, \theta)$ *and* $\int_X \theta_\phi^n > 0$. *The following are equivalent*:

(i) $u \in \mathcal{E}(X, \theta, \phi)$.

(ii) ϕ *is less singular than* u, *and* $P_\theta[u](\phi) = \phi$.

(iii) ϕ *is less singular than* u, *and* $P_\theta[u] = P_\theta[\phi]$.

Without the *nonzero mass* condition $\int_X \theta_\phi^n > 0$ this characterization cannot hold (see Remark 3.3). The equivalence between (i) and (iii) in the above theorem shows that $P_\theta[u]$ is the same potential for any $u \in \mathcal{E}(X, \theta, \phi)$, and is equal to $P_\theta[\phi]$. Given this and the inclusion $\mathcal{E}(X, \theta, \phi) \subset \mathcal{E}(X, \theta, P_\theta[\phi])$, one is tempted to consider only potentials ϕ in the image of the operator $\psi \to P_\theta[\psi]$, when studying the classes of relative full mass $\mathcal{E}(X, \theta, \phi)$. These potentials seemingly play the same role as V_θ, the potential with minimal singularities from [Boucksom et al. 2010]. Implementation of this idea will be further motivated by the results of the next subsection.

In addition to the above result, we also establish analogs of many classical results for $\mathcal{E}(X, \theta, \phi)$, like the comparison, maximum and domination principles. Some of these are routine, while others, like the domination principle, require new techniques and a more involved analysis compared to the existing literature (see Proposition 3.11).

Complex Monge–Ampère equations with prescribed singularity. With the potential theoretic tools developed, we focus on solving (1). A simple minded example shows that this equation is not well-posed for arbitrary $\phi \in \mathrm{PSH}(X, \theta)$ (see the introduction of Section 4). Instead, one needs to consider only potentials ϕ that are fixed points of the operator $\psi \to P_\theta[\psi]$, i.e., $\psi = P_\theta[\psi]$. Such potentials ψ will be called *model potentials*, and their singularity types $[\psi]$ will be called *model-type singularities*. In this direction we have the following result:

Theorem 1.4. *Suppose* $\phi \in \mathrm{PSH}(X, \theta)$ *has small unbounded locus, and* $\phi = P_\theta[\phi]$. *Let* $f \in L^p(\omega^n)$, $p > 1$ *such that* $f \geq 0$ *and* $\int_X f\omega^n = \int_X \theta_\phi^n > 0$. *Then the following hold*:

(i) *There exists* $u \in \mathrm{PSH}(X, \theta)$, *unique up to a constant, such that* $[u] = [\phi]$ *and*

$$\theta_u^n = f\omega^n. \tag{3}$$

(ii) *For any* $\lambda > 0$ *there exists a unique* $v \in \mathrm{PSH}(X, \theta)$ *such that* $[v] = [\phi]$ *and*

$$\theta_v^n = e^{\lambda v} f\omega^n. \tag{4}$$

That ϕ has *small unbounded locus* means that ϕ is locally bounded outside a closed complete pluripolar set $A \subset X$. It will be interesting to see if this condition is simply technical, or otherwise necessary. This seemingly extra condition on ϕ does have some benefits. Indeed, since in this setting solutions are locally bounded on $X \setminus A$, one can interpret (3) and (4) in the following simple way: u and v satisfy (3) and (4) on $X \setminus A$, in the sense of Bedford and Taylor.

Remark 1.5. As argued in Theorem 4.34, if (3) can be solved for all $f \in L^p(X)$, $p > 1$, (with the constraint $[u] = [\phi]$) then ϕ *must have* model-type singularity. Consequently, our choice of ϕ in the above theorem is not ad hoc, but truly natural!

In our study of the above equations, we will start with a much more general context. In particular, we will show in Theorems 4.28 and 4.23 below that instead of $f\omega^n$, one can consider, on the right-hand side of (3) and (4), nonpluripolar measures, thereby generalizing [Boucksom et al. 2010, Theorems A, D].

Remark 1.6. Naturally, $V_\theta = P_\theta[V_\theta]$, but our reader may wonder if there are other interesting enough potentials with model-type singularity. We believe this to be the case, as evidenced below:

- By Theorem 3.12 below, $P_\theta[\psi] = P_\theta[P_\theta[\psi]]$ for any $\psi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\psi^n > 0$. In particular, $P_\theta[\psi]$ is a model potential, giving an abundance of potentials with model-type singularity.
- By Proposition 4.35 below, if $\psi \in \mathrm{PSH}(X,\theta)$ has small unbounded locus, and $\theta_\psi^n/\omega^n \in L^p(\omega^n)$, $p > 1$, with $\int_X \theta_\psi^n > 0$, then ψ has model-type singularity.
- All *analytic singularity types* (those that can be locally written as $c \log\bigl(\sum_j |f_j|^2\bigr) + g$, where the f_j are holomorphic, $c > 0$ and g is smooth) are of model type [Ross and Witt Nyström 2014, Remark 4.6; Rashkovskii and Sigurdsson 2005]; see also Proposition 4.36. In particular, discrete logarithmic singularity types are of model type, making a connection with pluricomplex Green currents [Coman and Guedj 2009; Phong and Sturm 2014; Rashkovskii and Sigurdsson 2005].
- By [Ross and Witt Nyström 2014; Darvas 2017; Darvas et al. 2018], potentials with model-type singularity naturally arise as degenerations along geodesic rays and in particular along test configurations.

Complex Monge–Ampère equations with bounded/minimally singular solutions have been intensely studied in the past; see [Kołodziej 1998; 2003; Guedj and Zeriahi 2007; Boucksom et al. 2010; Berman et al. 2013], to name only a few works in a fast expanding literature. To our knowledge, in the compact case, only [Phong and Sturm 2014] discusses at length solutions that are not "minimally singular", without severe restrictions on the right-hand side of the equation. They treat the case of solutions to (3) with isolated algebraic singularities in the Kähler case, with a view toward constructing pluricomplex Green currents on X. Given the specific setting, [Phong and Sturm 2014, Theorem 3] obtains more precise regularity estimates compared to ours, using blowup techniques. In our general framework better estimates are likely not possible. However, for smooth f, we suspect that away from the singularity locus our solution u should be as regular as ϕ (up to order 2). For a general result on the regularity of certain model potentials we refer to [Ross and Witt Nyström 2017, Theorem 1.1].

Lastly, let us mention that in [Berman 2013, Section 4] solutions to complex Monge–Ampère equations with divisorial singularity type are used in the construction/approximation of geodesic rays corresponding to certain test configurations. In Section 5 of the same work, Berman speculates that solutions with more general singularity type should allow for better understanding of degenerations along test configurations/geodesic rays, and we believe our treatise will lead to more results of this flavor.

In addition to the results in the compact setting mentioned above, finding singular/unbounded solutions to the related Dirichlet problem on domains in $\mathbb{C}^n$, or more generally on compact manifolds with boundary, was studied by a number of authors. We only mention [Lempert 1983; Bedford and Demailly 1988; Guan 1998; Phong and Sturm 2010a; 2010b] to highlight a few works in a fast expanding literature.

Applications. Solutions of complex Monge–Ampère equations are linked to existence of special Kähler metrics. In particular, we can think of the solution to (3) as a potential with prescribed singularity type and prescribed Ricci curvature in the philosophy of the Calabi–Yau theorem. As an immediate application of our solution to (4) we obtain existence of singular *Kähler–Einstein* (KE) metrics with prescribed singularity type on Kähler manifolds of general type. An analogous result also holds on Calabi–Yau manifolds as well, via solutions of (3).

Corollary 1.7. *Let X be a smooth projective variety of general type ($K_X > 0$) and let h be a smooth Hermitian metric on K_X with $\theta := \Theta(h) > 0$. Suppose also that $\phi \in \mathrm{PSH}(X, \theta)$ is a model potential, has small unbounded locus and $\int_X \theta_\phi^n > 0$. Then there exists a unique singular KE metric $he^{-\phi_{\mathrm{KE}}}$ on K_X ($\theta^n_{\phi_{\mathrm{KE}}} = e^{\phi_{\mathrm{KE}}+f_\theta}\theta^n$, where f_θ is the Ricci potential of θ satisfying $\mathrm{Ric}\,\theta = \theta + i\partial\bar\partial f_\theta$), with $\phi_{\mathrm{KE}} \in \mathrm{PSH}(X, \theta)$ having the same singularity type as ϕ.*

As another application we confirm the log-concavity conjecture [Boucksom et al. 2010, Conjecture 1.23] in the case of currents with potentials having small unbounded locus:

Theorem 1.8. *Let $T_1, \dots, T_n$ be positive closed $(1, 1)$-currents on a compact Kähler manifold X. Assume that each T_j has a potential with small unbounded locus. Then*

$$\int_X \langle T_1 \wedge \cdots \wedge T_n\rangle \geq \left(\int_X \langle T_1^n\rangle\right)^{1/n} \cdots \left(\int_X \langle T_n^n\rangle\right)^{1/n}.$$

Possible future directions. It is well known that, for $\lambda < 0$, (4) does not always have a solution. More importantly, solvability of this equation is tied together with existence of KE metrics on Fano manifolds. It would be interesting to see if the techniques of [Darvas and Rubinstein 2017] apply to give characterizations for existence of KE metrics with prescribed singularity type in terms of energy properness.

By [Darvas 2017; Darvas et al. 2018] the geometry of geodesic rays and properties of (relative) full mass potentials seems to be intimately related. In a future work we will explore this avenue further, by introducing a metric geometry on the space of singularity types, via the constructions of [Darvas 2017; Darvas et al. 2018]. By understanding the metric properties of this space, we hope to study degenerations of singularity types along complex Monge–Ampère equations.

Organization of the paper. Most of our notation and terminology carries over from [Darvas et al. 2018], and we refer the reader to the introductory sections of this work. In Section 2 we prove Theorems 1.1 and 1.2. In Section 3 we develop the theory of the relative full mass classes $\mathcal{E}(X, \theta, \phi)$ and we exploit properties of envelopes to prove Theorem 1.3. In Section 4 we generalize the variational methods of [Berman et al. 2013] to prove Theorem 1.4. Finally, Theorem 1.8 is proved in Section 5.

2. The monotonicity property and convergence of nonpluripolar products

To begin, from the main result of [Witt Nyström 2017] we deduce the following proposition:

Proposition 2.1. *Let θ^j, $j \in \{1, \dots, n\}$, be smooth closed real $(1,1)$-forms on X whose cohomology classes are pseudoeffective. Let $u_j, v_j \in \mathrm{PSH}(X, \theta^j)$ be such that u_j has the same singularity type as v_j, $j \in \{1, \dots, n\}$. Then*

$$\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} = \int_X \theta^1_{v_1} \wedge \cdots \wedge \theta^n_{v_n}.$$

The proof of this result uses the arguments in [Boucksom et al. 2010, Corollary 2.15].

Proof. First we note that we can assume that the classes $\{\theta^j\}$ are in fact big. Indeed, if this is not the case we can just replace each θ^j with $\theta^j + \varepsilon\omega$, and using the multilinearity of the nonpluripolar product [Boucksom et al. 2010, Proposition 1.4] we can let $\varepsilon \to 0$ at the end of our argument to conclude the statement for pseudoeffective classes.

For each $t \in \Delta = \{t = (t_1, \dots, t_n) \in \mathbb{R}^n \mid t_j > 0\}$ consider $u_t := \sum_j t_j u_j$, $v_t := \sum_j t_j v_j$ and $\theta^t := \sum_j t_j \theta^j$. Clearly, $\{\theta^t\}$ is big, and u_t has the same singularities as v_t. Hence it follows from [Witt Nyström 2017, Theorem 1.2] that $\int_X (\theta^t_{u_t})^n = \int_X (\theta^t_{v_t})^n$ for all $t \in \Delta$. On the other hand, using multilinearity of the nonpluripolar product again [Boucksom et al. 2010, Proposition 1.4], we see that both $t \to \int_X (\theta^t_{u_t})^n$ and $t \to \int_X (\theta^t_{v_t})^n$ are homogeneous polynomials of degree n. Our last identity forces all the coefficients of these polynomials to be equal, giving the statement of our result. □

We recall a classical convergence theorem from Bedford–Taylor theory. We refer to [Guedj and Zeriahi 2017, Theorem 4.26] for a proof of this result, which is merely a slight generalization of [Xing 1996, Theorem 1].

Proposition 2.2. *Let $\Omega \subset \mathbb{C}^n$ be an open set. Suppose $\{f_j\}_j$ are uniformly bounded quasicontinuous functions which converge in capacity to another quasicontinuous function f on Ω. Let $\{u^j_1\}_j, \{u^j_2\}_j, \dots, \{u^j_n\}_j$ be uniformly bounded plurisubharmonic functions on Ω, converging in capacity to $u_1, u_2, \dots, u_n$ respectively. Then we have the following weak convergence of measures*:

$$f_j i\partial\bar{\partial} u^j_1 \wedge i\partial\bar{\partial} u^j_2 \wedge \cdots \wedge i\partial\bar{\partial} u^j_n \to f i\partial\bar{\partial} u_1 \wedge i\partial\bar{\partial} u_2 \wedge \cdots \wedge i\partial\bar{\partial} u_n.$$

The following lower-semicontinuity property of nonpluripolar products will be key in the sequel:

Theorem 2.3. *Let θ^j, $j \in \{1, \dots, n\}$, be smooth closed real $(1,1)$-forms on X whose cohomology classes are big. Suppose that for all $j \in \{1, \dots, n\}$ we have $u_j, u^k_j \in \mathrm{PSH}(X, \theta^j)$ such that $u^k_j \to u_j$ in capacity as $k \to \infty$. Then for all positive bounded quasicontinuous functions χ we have*

$$\liminf_{k \to \infty} \int_X \chi \theta^1_{u^k_1} \wedge \cdots \wedge \theta^n_{u^k_n} \geq \int_X \chi \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n}.$$

If additionally,

$$\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} \geq \limsup_{k \to \infty} \int_X \theta^1_{u^k_1} \wedge \cdots \wedge \theta^n_{u^k_n}, \tag{5}$$

then $\theta^1_{u^k_1} \wedge \cdots \wedge \theta^n_{u^k_n} \to \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n}$ in the weak sense of measures on X.

Proof. Set $\Omega := \bigcap_{j=1}^{n} \mathrm{Amp}(\theta^j)$ and fix an open relatively compact subset U of Ω. Then the functions V_{θ^j} are bounded on U. We now use a classical idea in pluripotential theory. Fix $C > 0$, $\varepsilon > 0$ and consider

$$f_j^{k,C,\varepsilon} := \frac{\max(u_j^k - V_{\theta^j} + C, 0)}{\max(u_j^k - V_{\theta^j} + C, 0) + \varepsilon}, \quad j = 1, \dots, n, \quad k \in \mathbb{N}^*,$$

and

$$u_j^{k,C} := \max(u_j^k, V_{\theta^j} - C).$$

Observe that for C, j fixed, the functions $u_j^{k,C} \geq V_{\theta^j} - C$ are uniformly bounded in U (since V_{θ^j} is bounded in U) and converge in capacity to u_j^C as $k \to \infty$. Moreover, $f_j^{k,C,\varepsilon} = 0$ if $u_j^k \leq V_{\theta^j} - C$. By locality of the nonpluripolar product we can write

$$f^{k,C,\varepsilon} \chi \theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k} = f^{k,C,\varepsilon} \chi \theta^1_{u_1^{k,C}} \wedge \cdots \wedge \theta^n_{u_n^{k,C}},$$

where $f^{k,C,\varepsilon} = f_1^{k,C,\varepsilon} \cdots f_n^{k,C,\varepsilon}$. For each fixed C, ε, the functions $f^{k,C,\varepsilon}$ are quasicontinuous, uniformly bounded (with values in $[0, 1]$) and converge in capacity to $f^{C,\varepsilon} := f_1^{C,\varepsilon} \cdots f_n^{C,\varepsilon}$, where $f_j^{C,\varepsilon}$ is defined by

$$f_j^{C,\varepsilon} := \frac{\max(u_j - V_{\theta^j} + C, 0)}{\max(u_j - V_{\theta^j} + C, 0) + \varepsilon}.$$

With the information above we can apply Proposition 2.2 to get

$$f^{k,C,\varepsilon} \chi \theta^1_{u_1^{k,C}} \wedge \cdots \wedge \theta^n_{u_n^{k,C}} \to f^{C,\varepsilon} \chi \theta^1_{u_1^C} \wedge \cdots \wedge \theta^n_{u_n^C} \quad \text{as } k \to \infty,$$

in the weak sense of measures on U. In particular since $0 \leq f^{k,C,\varepsilon} \leq 1$ we have

$$\liminf_{k\to\infty} \int_X \chi \theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k} \geq \liminf_{k\to\infty} \int_U f^{k,C,\varepsilon} \chi \theta^1_{u_1^{k,C}} \wedge \cdots \wedge \theta^n_{u_n^{k,C}} \geq \int_U f^{C,\varepsilon} \chi \theta^1_{u_1^C} \wedge \cdots \wedge \theta^n_{u_n^C}.$$

Now, letting $\varepsilon \to 0$ and then $C \to \infty$, by definition of the nonpluripolar product we obtain

$$\liminf_{k\to\infty} \int_X \chi \theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k} \geq \int_U \chi \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n}.$$

Finally, letting U increase to Ω and noting that the complement of Ω is pluripolar we conclude the proof of the first statement of the theorem.

To prove the last statement, we set $\mu_k := \theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k}$ and $\mu := \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n}$. Note that the total mass of these measures is bounded by $\int_X \theta^1 \wedge \cdots \wedge \theta^n$ [Boucksom et al. 2010, Definition 1.17]. As a result, by the Banach–Alaoglu theorem, it suffices to show that any cluster point of $\{\mu_k\}_k$ coincides with μ. Let ν be such a cluster point and assume (after extracting a subsequence) that μ_k converges weakly to ν. Condition (5) implies that $\nu(X) \leq \mu(X)$. It suffices to argue that $\nu \geq \mu$, which is a consequence of the first statement, thus finishing the proof. □

Now we move on to the monotonicity of nonpluripolar products:

Theorem 2.4. *Let θ^j, $j \in \{1, \dots, n\}$, be smooth closed real $(1, 1)$-forms on X whose cohomology classes are pseudoeffective. Let $u_j, v_j \in \mathrm{PSH}(X, \theta^j)$ be such that u_j is less singular than v_j for all $j \in \{1, \dots, n\}$. Then*

$$\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} \geq \int_X \theta^1_{v_1} \wedge \cdots \wedge \theta^n_{v_n}.$$

Proof. By the same reason as in Proposition 2.1, we can assume that the classes $\{\theta^j\}$ are in fact big. For each $t > 0$ we set $v_j^t := \max(u_j - t, v_j)$ for $j = 1, \ldots, n$. Observe that the v_j^t converge decreasingly to v_j as $t \to \infty$. In particular, by [Guedj and Zeriahi 2005, Proposition 3.7] the convergence holds in capacity. As v_j^t and u_j have the same singularity type, it follows from Proposition 2.1 that

$$\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} = \int_X \theta^1_{v_1^t} \wedge \cdots \wedge \theta^n_{v_n^t}.$$

Letting $t \to \infty$, the first part of Theorem 2.3 allows us to conclude the argument. □

Remark 2.5. We note that condition (5) in Theorem 2.3 is automatically satisfied if $u_j^k \nearrow u_j$ a.e. as $k \to \infty$. Indeed, in this case $u_j^k \to u_j$ in capacity, see [Guedj and Zeriahi 2017, Proposition 4.25], and by Theorem 2.4 we have $\int_X \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} \geq \limsup_k \int_X \theta^1_{u_1^k} \wedge \cdots \wedge \theta^n_{u_n^k}$.

On the other hand, if $u_j^k, u_j \in \mathcal{E}(X, \theta^j)$, by Corollary 3.2 below, it follows that (5) is again automatically satisfied. Moreover, in the next section we will show that this last property holds for potentials of relative full mass as well (see Corollary 3.15), giving Theorem 2.4 a more broad spectrum of applications.

3. Pluripotential theory with relative full mass

3A. ***Nonpluripolar products of relative full mass.*** Suppose θ^j, $j \in \{1, \ldots, n\}$, are smooth closed real $(1,1)$-forms on X with $\{\theta^j\}$ pseudoeffective. Let $\phi_j, \psi_j \in \mathrm{PSH}(X, \theta^j)$ be such that ϕ_j is less singular than ψ_j. We say that $\theta^1_{\psi_1} \wedge \cdots \wedge \theta^n_{\psi_n}$ has full mass with respect to $\theta^1_{\phi_1} \wedge \cdots \wedge \theta^n_{\phi_n}$, denoted as $(\psi_1, \ldots, \psi_n) \in \mathcal{E}(X, \theta^1_{\phi_1}, \ldots, \theta^n_{\phi_n})$, if

$$\int_X \theta^1_{\psi_1} \wedge \cdots \wedge \theta^n_{\psi_n} = \int_X \theta^1_{\phi_1} \wedge \cdots \wedge \theta^n_{\phi_n}.$$

By Theorem 2.4, in general we only have that the left-hand side is less than the right-hand side in the above identity.

In the particular case when the potentials involved are from the same cohomology class $\{\theta\}$, and $\phi, \psi \in \mathrm{PSH}(X, \theta)$ with ϕ less singular than ψ along with $\int_X \theta^n_\phi = \int_X \theta^n_\psi$, we simply write $\psi \in \mathcal{E}(X, \theta, \phi)$, and say that ψ has *full mass relative to* θ^n_ϕ. When $\phi = V_\theta$, we recover the well-known concept of full mass currents from the literature; see [Boucksom et al. 2010].

As a consequence of Theorem 2.3, we prove a criterion for testing membership in $\mathcal{E}(X, \theta^1_{\phi_1}, \ldots, \theta^n_{\phi_n})$:

Proposition 3.1. *Let θ^j, $j \in \{1, \ldots, n\}$, be smooth closed real $(1,1)$-forms on X with cohomology classes that are pseudoeffective. For all $j \in \{1, \ldots, n\}$ we choose $\phi_j, \psi_j \in \mathrm{PSH}(X, \theta^j)$ such that ϕ_j is less singular than ψ_j. If $P_{\theta^j}[\psi_j](\phi_j) = \phi_j$ then $(\psi_1, \ldots, \psi_n) \in \mathcal{E}(X, \theta^1_{\phi_1}, \ldots, \theta^n_{\phi_n})$.*

Proof. If $P_{\theta^j}[\psi_j](\phi_j) = \phi_j$, then $v_j^C := P_{\theta^j}(\psi_j + C, \phi_j) \nearrow \phi_j$ a.e. as $C \to \infty$. Theorem 2.3 and Remark 2.5 then imply

$$\lim_{C \to \infty} \int_X \theta_{v_1^C} \wedge \cdots \wedge \theta_{v_n^C} = \int_X \theta_{\phi_1} \wedge \cdots \wedge \theta_{\phi_n}.$$

As $P_{\theta^j}(\psi_j + C, \phi_j)$ has the same singularity type as ψ_j for any C, the result follows from Proposition 2.1. □

As a result of this simple criterion, we obtain that condition (5) in Theorem 2.3 is satisfied if the potentials u_j^k, u_j are from $\mathcal{E}(X,\theta^j)$:

Corollary 3.2. *Let θ^j, $j \in \{1,\dots,n\}$, be smooth closed real $(1,1)$-forms on X with cohomology classes that are pseudoeffective. If $\psi_j \in \mathcal{E}(X,\theta^j)$, $j\in\{1,\dots,n\}$, then*

$$\int_X \theta^1_{\psi_1}\wedge\cdots\wedge\theta^n_{\psi_n} = \int_X \theta^1_{V_{\theta^1}}\wedge\cdots\wedge\theta^n_{V_{\theta^n}},$$

or equivalently, $(\psi_1,\dots,\psi_n)\in\mathcal{E}(X,\theta^1_{V_{\theta^1}},\dots,\theta^n_{V_{\theta^n}})$.

Proof. By [Darvas et al. 2018, Theorem 1.2] we have $P_{\theta^j}[\psi_j] := P_{\theta^j}[\psi_j](V_{\theta^j}) = V_{\theta^j}$. Hence Proposition 3.1 yields the conclusion. □

Remark 3.3. Unfortunately, the reverse direction in Proposition 3.1 does not hold in general. Indeed, let $X=\mathbb{CP}^1\times\mathbb{CP}^1$ with $\theta=\pi_1^*\omega_{FS}+\pi_2^\star\omega_{FS}$, where π_1,π_2 are the projections to the first and second components respectively.

Consider $\phi(z,w) := u(z)+v(w)\in \mathrm{PSH}(X,\theta)$, where $u,v\le 0$ satisfy $\omega_{FS}+i\partial\bar\partial u=\delta_{z_0}$ and $\omega_{FS}+i\partial\bar\partial v=\delta_{w_0}$, where $\delta_{z_0},\delta_{w_0}$ are Dirac masses for some $z_0,w_0\in\mathbb{CP}^1$. Clearly, $\int_X\theta_\phi^2=\int_X\theta^2_{\pi_2^*v}=0$, and since $\phi\le\pi_2^*v$, we have $\phi\in\mathcal{E}(X,\theta,\pi_2^*v)$.

On the other hand, we know that ϕ has the same Lelong number as $P_\theta[\phi]$ [Darvas et al. 2018, Theorem 1.1]. As $P_\theta[\phi](\pi_2^*v)\le P_\theta[\phi]$, it follows however that $P_\theta[\phi](\pi_2^*v)\lneq\pi_2^*v$, since at some points of $\mathbb{CP}^1\times\mathbb{CP}^1$ the Lelong number of π_2^*v is zero, but the Lelong number of ϕ is nonzero.

As we will see below (Theorem 3.14), a partial converse of Proposition 3.1 is still possible under the assumption of nonvanishing total mass.

In the remaining part of this subsection we prove basic properties of nonpluripolar products with relative full mass, which will be used later in this work.

Lemma 3.4. *Suppose $\phi_j,\psi_j\in\mathrm{PSH}(X,\theta^j)$. Then $(\psi_1,\dots,\psi_n)\in\mathcal{E}(X,\theta^1_{\phi_1},\dots,\theta^n_{\phi_n})$ if and only if ϕ_j is less singular than ψ_j and*

$$\int_{\bigcup_j\{\psi_j\le\phi_j-k\}}\theta^1_{\max(\psi_1,\phi_1-k)}\wedge\cdots\wedge\theta^n_{\max(\psi_n,\phi_n-k)}\to 0\quad \textit{as } k\to\infty.$$

Proof. If ϕ_j is less singular than ψ_j, then $\max(\psi_j,\phi_j-k)$ has the same singularity type as ϕ_j. Consequently, Proposition 2.1 gives

$$\begin{aligned}\int_X\theta^1_{\phi_1}\wedge\cdots\wedge\theta^n_{\phi_n} &= \int_X\theta^1_{\max(\psi_1,\phi_1-k)}\wedge\cdots\wedge\theta^n_{\max(\psi_n,\phi_n-k)}\\ &= \int_{\bigcap_j\{\psi_j>\phi_j-k\}}\theta^1_{\psi_1}\wedge\cdots\wedge\theta^n_{\psi_n}+\int_{\bigcup_j\{\psi_j\le\phi_j-k\}}\theta^1_{\max(\psi_1,\phi_1-k)}\wedge\cdots\wedge\theta^n_{\max(\psi_n,\phi_n-k)}.\end{aligned}$$

Since $\int_{\bigcap_j\{\psi_j>\phi_j-k\}}\theta^1_{\psi_1}\wedge\cdots\wedge\theta^n_{\psi_n}\to\int_X\theta^1_{\psi_1}\wedge\cdots\wedge\theta^n_{\psi_n}$ as $k\to\infty$, the equivalence of the lemma follows after we take the limit $k\to\infty$ in the above identity. □

As a consequence of this last lemma and the locality of the nonpluripolar product with respect to the plurifine topology we obtain the uniform estimate

$$\lim_{k\to\infty}\left|\int_B \theta^1_{\psi_1}\wedge\cdots\wedge\theta^n_{\psi_n}-\int_B \theta^1_{\max(\psi_1,\phi_1-k)}\wedge\cdots\wedge\theta^n_{\max(\psi_n,\phi_n-k)}\right| \le 2\int_{\bigcup_j\{\psi_j\le\phi_j-k\}}\theta^1_{\max(\psi_1,\phi_1-k)}\wedge\cdots\wedge\theta^n_{\max(\psi_n,\phi_n-k)}\to 0$$

for any Borel set $B\subset X$ and $(\psi_1,\dots,\psi_n)\in\mathcal{E}(X,\theta^1_{\phi_1},\dots,\theta^n_{\phi_n})$.

Lastly, we note the *partial comparison principle* for nonpluripolar products of relative full mass, generalizing a result of [Dinew 2009b]:

Proposition 3.5. *Suppose* $\phi_k,\psi_k\in\mathrm{PSH}(X,\theta^k)$, $k=1,\dots,j\le n$, *and* $\phi\in\mathrm{PSH}(X,\theta)$. *Assume that* $(u,\dots,u,\psi_1,\dots,\psi_j),(v,\dots,v,\psi_1,\dots,\psi_j)\in\mathcal{E}(X,\theta_\phi,\dots,\theta_\phi,\theta_{\phi_1},\dots,\theta_{\phi_j})$. *Then*

$$\int_{\{u<v\}}\theta^{n-j}_v\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\le\int_{\{u<v\}}\theta^{n-j}_u\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}.$$

Proof. The proof follows the argument of [Boucksom et al. 2010, Proposition 2.2] with a vital ingredient from Theorem 2.4.

Since $\max(u,v)$ is more singular than ϕ and ψ_k is more singular than ϕ_k for $k=1,\dots,j$, it follows from the assumption and Theorem 2.4 that

$$\begin{aligned}\int_X\theta^{n-j}_\phi\wedge\theta^1_{\phi_1}\wedge\cdots\wedge\theta^j_{\phi_j}&=\int_X\theta^{n-j}_v\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\\&\le\int_X\theta^{n-j}_{\max(u,v)}\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\\&\le\int_X\theta^{n-j}_\phi\wedge\theta^1_{\phi_1}\wedge\cdots\wedge\theta^j_{\phi_j}.\end{aligned}$$

Hence the inequalities above are in fact equalities. By locality of the nonpluripolar product we then can write

$$\begin{aligned}&\int_X\theta^{n-j}_{\max(u,v)}\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\\&\quad\ge\int_{\{u>v\}}\theta^{n-j}_u\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}+\int_{\{v>u\}}\theta^{n-j}_v\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\\&\quad=\int_X\theta^{n-j}_u\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}-\int_{\{u\le v\}}\theta^{n-j}_u\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}+\int_{\{v>u\}}\theta^{n-j}_v\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\\&\quad=\int_X\theta^{n-j}_{\max(u,v)}\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}-\int_{\{u\le v\}}\theta^{n-j}_u\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}+\int_{\{v>u\}}\theta^{n-j}_v\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}.\end{aligned}$$

We thus get

$$\int_{\{u<v\}}\theta^{n-j}_v\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}\le\int_{\{u\le v\}}\theta^{n-j}_u\wedge\theta^1_{\psi_1}\wedge\cdots\wedge\theta^j_{\psi_j}.$$

Replacing u with $u+\varepsilon$ in the above inequality, and letting $\varepsilon\searrow 0$, by the monotone convergence theorem we arrive at the result. $\square$

In the next subsection, after we explore the class $\mathcal{E}(X,\theta,\phi)$, we will give a partial comparison principle specifically for this class, as a corollary of the above general proposition. Here we only note the following trivial consequence:

Corollary 3.6. *Suppose $\phi \in \mathrm{PSH}(X,\theta)$ and assume that $u, v \in \mathcal{E}(X,\theta,\phi)$. Then*

$$\int_{\{u<v\}} \theta_v^n \le \int_{\{u<v\}} \theta_u^n.$$

Note that the above result generalizes [Boucksom et al. 2010, Corollary 2.3].

3B. ***The envelope $P_\theta[\phi]$ and the class $\mathcal{E}(X,\theta,\phi)$.*** Let θ be a smooth closed real $(1,1)$-form on X which represents a big class and fix $\phi \in \mathrm{PSH}(X,\theta)$ such that $\phi \le 0$. In this short subsection we focus on the relative full mass class $\mathcal{E}(X,\theta,\phi)$.

Based on our previous findings, one wonders if the following set of potentials has a maximal element:

$$F_\phi := \left\{ v \in \mathrm{PSH}(X,\theta) \;\middle|\; \phi \le v \le 0 \text{ and } \int_X \theta_v^n = \int_X \theta_\phi^n \right\}.$$

In other words, does there exist a *least singular* potential that is less singular than ϕ but has the same full mass as ϕ. As we will see, if $\int_X \theta_\phi^n > 0$, this is indeed the case; moreover this maximal potential is equal to $P_\theta[\phi]$ (Theorem 3.12).

Linking the envelope $P_\theta[\phi]$ to the class $\mathcal{E}(X,\theta,\phi)$, observe that $\phi \le P_\theta[\phi] \le 0$ and $\int_X \theta^n_{P_\theta[\phi]} = \int_X \theta_\phi^n$; in particular $P_\theta[\phi] \in F_\phi$ and $\phi \in \mathcal{E}(X,\theta,P_\theta[\phi])$. Indeed, since $P_\theta(\phi + C, 0) \nearrow P_\theta[\phi](0) = P_\theta[\phi]$ a.e. as $C \to \infty$, using Theorems 2.4 and 2.3 we can conclude that $\int_X \theta^n_{P_\theta[\phi]} = \int_X \theta_\phi^n$.

In our study, we will need the following preliminary result, providing an estimate for the complex Monge–Ampère operator of rooftop envelopes, which builds on recent progress in [Guedj et al. 2017]:

Lemma 3.7. *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. If $P_\theta(\varphi,\psi) \not\equiv -\infty$ then*

$$\theta^n_{P_\theta(\varphi,\psi)} \le \mathbb{1}_{\{P_\theta(\varphi,\psi)=\varphi\}}\theta_\varphi^n + \mathbb{1}_{\{P_\theta(\varphi,\psi)=\psi\}}\theta_\psi^n.$$

Proof. For each $t > 0$ we set $\varphi_t := \max(\varphi, V_\theta - t)$, $\psi_t := \max(\psi, V_\theta - t)$ and $v_t := P_\theta(\varphi_t, \psi_t)$. Set $v := P_\theta(\phi, \psi)$. Since φ_t, ψ_t have minimal singularities, it follows from [Guedj et al. 2017, Lemma 4.1] that

$$\theta_{v_t}^n \le \mathbb{1}_{\{v_t=\varphi_t\}}\theta_{\varphi_t}^n + \mathbb{1}_{\{v_t=\psi_t\}}\theta_{\psi_t}^n. \tag{6}$$

For $C > 0$ we introduce

$$G_C := \{v > V_\theta - C\}, \quad v^C := \max(v, V_\theta - C), \quad \text{and} \quad v_t^C := \max(v_t, V_\theta - C).$$

Since $P_\theta(\varphi,\psi) \le \varphi, \psi, v_t$, we have $G_C \subset \{V_\theta - C < \varphi\} \cap \{V_\theta - C < \psi\} \cap \{V_\theta - C < v_t\}$. For arbitrary $A > 0$ and $t > C$, this inclusion allows us to build on (6) and write

$$\begin{aligned}
\mathbb{1}_{G_C}\theta^n_{v_t^C} = \mathbb{1}_{G_C}\theta^n_{v_t} &\le \mathbb{1}_{\{v_t=\varphi_t\}\cap G_C}\theta^n_{\varphi_t} + \mathbb{1}_{\{v_t=\psi_t\}\cap G_C}\theta^n_{\psi_t} \\
&\le \mathbb{1}_{\{v_t=\varphi_t\}\cap\{\varphi>V_\theta-t\}}\theta^n_{\varphi_t} + \mathbb{1}_{\{v_t=\psi_t\}\cap\{\psi>V_\theta-t\}}\theta^n_{\psi_t} \\
&= \mathbb{1}_{\{v_t=\varphi_t\}\cap\{\varphi>V_\theta-t\}}\theta^n_{\varphi} + \mathbb{1}_{\{v_t=\psi_t\}\cap\{\psi>V_\theta-t\}}\theta^n_{\psi} \le e^{A(v_t-\varphi_t)}\theta_\varphi^n + e^{A(v_t-\psi_t)}\theta_\psi^n.
\end{aligned} \tag{7}$$

To proceed, we want to prove that

$$\liminf_{t\to\infty} \mathbb{1}_{G_C}\theta^n_{v^C_t} \geq \mathbb{1}_{G_C}\theta^n_{v^C}. \tag{8}$$

More precisely, alluding to the Banach–Alaoglu theorem, we want to show that any weak limit of $\{\mathbb{1}_{G_C}\theta^n_{v^C_t}\}_t$ is greater than $\mathbb{1}_{G_C}\theta^n_{v^C}$.

Let $U := \mathrm{Amp}(\theta)$. The potential V_θ is locally bounded on U; hence so are v^C_t and v^C. To obtain (8), we employ an idea from the proof of Theorem 2.3. For $\varepsilon > 0$ consider

$$f_\varepsilon := \frac{\max(v - V_\theta + C, 0)}{\max(v - V_\theta + C, 0) + \varepsilon},$$

and observe that $f_\varepsilon \geq 0$ is quasicontinuous on X. Moreover, the f_ε increase pointwise to $\mathbb{1}_{G_C}$ as ε goes to zero. Since $v^C_t \searrow v^C$ as $t \to \infty$, from [Guedj and Zeriahi 2017, Theorem 4.26] it follows that $f_\varepsilon\theta^n_{v^C_t}|_U \to f_\varepsilon\theta^n_{v^C}|_U$ weakly. Using this we can write

$$\liminf_{t\to\infty} \mathbb{1}_{G_C}\theta^n_{v^C_t}|_U \geq \lim_{t\to\infty} f_\varepsilon\theta^n_{v^C_t}|_U = f_\varepsilon\theta^n_{v^C}|_U.$$

Since $X \setminus U$ is pluripolar, we let $\varepsilon \to 0$ and use the monotone convergence theorem to conclude (8).

Now, letting $t \to \infty$ in (7), the estimate in (8) allows us to conclude that

$$\mathbb{1}_{G_C}\theta^n_{\max(P_\theta(\varphi,\psi), V_\theta - C)} \leq e^{A(P_\theta(\varphi,\psi)-\varphi)}\theta^n_\varphi + e^{A(P_\theta(\varphi,\psi)-\psi)}\theta^n_\psi.$$

Letting $C \to \infty$, and later $A \to \infty$, we arrive at the conclusion. □

We prove in the following that the nonpluripolar complex Monge–Ampère measure of $P_\theta[\psi](\chi)$ has bounded density with respect to θ^n_χ. This plays a crucial role in the sequel.

Theorem 3.8. *Let $\psi, \chi \in \mathrm{PSH}(X, \theta)$ be such that ψ is more singular than χ. Then $\theta^n_{P_\theta[\psi](\chi)} \leq \mathbb{1}_{\{P_\theta[\psi](\chi)=\chi\}}\theta^n_\chi$. In particular, $\theta^n_{P_\theta[\psi]} \leq \mathbb{1}_{\{P_\theta[\psi]=0\}}\theta^n$.*

This result can be thought of as a regularity result for the envelope $P_\theta[\psi](\chi)$. For a more precise regularity result on such envelopes in the particular case of potentials with algebraic singularities we refer to [Ross and Witt Nyström 2017, Theorem 1.1].

Proof. Without loss of generality we can assume that $\psi, \chi \leq 0$. For each $t > 0$ we consider $P_\theta(\psi + t, \chi)$. Since ψ is more singular than χ, we note that $P_\theta(\psi + t, \chi)$ has the same singularity type as ψ and $P_\theta(\psi + t, \chi) \nearrow P_\theta[\psi](\chi)$ a.e. It follows from Lemma 3.7 that

$$\theta^n_{P_\theta(\psi+t,\chi)} \leq \mathbb{1}_{\{P_\theta(\psi+t,\chi)=\psi+t\}}\theta^n_\psi + \mathbb{1}_{\{P_\theta(\psi+t,\chi)=\chi\}}\theta^n_\chi.$$

Since $\{P_\theta(\psi + t, \chi) = \psi + t\} \subset \{\psi + t \leq \chi\} \subset \{\psi + t \leq V_\theta\}$, and the latter decreases to a pluripolar set, the first term on the right-hand side above goes to zero as $t \to \infty$. For the second term, we observe that $\{P_\theta(\psi + t, \chi) = \chi\} \subset \{P_\theta[\psi](\chi) = \chi\}$. Hence, applying Theorem 2.3, the result follows.

For the last statement, we can apply the above argument to $\chi := V_\theta$, and note that from [Berman 2013, (1.2)], see also [Darvas et al. 2018, Theorem 2.6 (arXiv version); Guedj et al. 2017, Proposition 5.2], it follows that $\theta^n_{V_\theta} \leq \mathbb{1}_{\{V_\theta=0\}}\theta^n$. □

Using the above result, we can establish a *noncollapsing property* for the class of potentials with the same singularity type as ϕ, when $\theta_\phi^n(X) > 0$:

Corollary 3.9. *Assume that* $\phi \in \mathrm{PSH}(X,\theta)$ *is such that* $\int_X \theta_\phi^n > 0$. *If* U *is a Borel subset of* X *with positive Lebesgue measure, then there exists* $\psi \in \mathrm{PSH}(X,\theta)$ *having the same singularity type as* ϕ *such that* $\theta_\psi^n(U) > 0$.

Proof. It follows from [Boucksom et al. 2010, Theorems A, B] that there exists $h \in \mathrm{PSH}(X,\theta)$ with minimal singularities such that $\theta_h^n = c\mathbb{1}_U\omega^n$ for some normalization constant $c > 0$. For $C > 0$ consider $\varphi_C := P_\theta(\phi + C, h)$ and note that φ_C has the same singularities as ϕ. It follows from Lemma 3.7 that

$$\theta_{\varphi_C}^n \le \mathbb{1}_{\{\varphi_C = \phi + C\}}\theta_\phi^n + \mathbb{1}_{\{\varphi_C = h\}}\theta_h^n \le \mathbb{1}_{\{\phi + C \le h\}}\theta_\phi^n + c\mathbb{1}_{\{\varphi_C = h\}\cap U}\omega^n.$$

Since θ_ϕ^n is nonpluripolar, we have that $\lim_{C\to\infty}\int_{\{\phi+C\le h\}}\theta_\phi^n = 0$. Thus for $C > 0$ big enough, by the above estimate we have

$$\int_{X\setminus U}\theta_{\varphi_C}^n \le \int_{\{\phi+C\le h\}}\theta_\phi^n < \int_X \theta_{\varphi_C}^n,$$

where in the last inequality we used the fact that $\int_X \theta_{\varphi_C}^n = \int_X \theta_\phi^n > 0$. This implies that $\int_U \theta_{\varphi_C}^n > 0$ for big enough $C > 0$, finishing the argument. □

A combination of Corollary 3.9 and [Witt Nyström 2017, Corollary 4.2] immediately gives the following version of the domination principle, making the conclusion of the latter corollary more precise:

Corollary 3.10. *Assume that* $u, v \in \mathrm{PSH}(X,\theta)$, *$u$ is less singular than v and* $\int_X \theta_u^n > 0$. *If* $u \ge v$ *a.e. with respect to* θ_u^n, *then* $u \ge v$ *on* X.

Proof. Assume by contradiction that $\{u < v\} \subseteq X$ has positive Lebesgue measure. Then, by Corollary 3.9 we can ensure that there exists $\psi \in \mathrm{PSH}(X,\theta)$ having the same singularity type as u such that $\theta_\psi^n(\{u < v\}) > 0$. On the other hand, since $\theta_u^n(\{u < v\}) = 0$, [Witt Nyström 2017, Corollary 4.2] gives that $\theta_\psi^n(\{u < v\}) = 0$, which is a contradiction. □

The noncollapsing mass condition $\int_X \theta_u^n > 0$ is trivially seen to be necessary. We now give the version of the *domination principle* for the relative full mass class $\mathcal{E}(X,\theta,\phi)$:

Proposition 3.11. *Suppose* $\phi \in \mathrm{PSH}(X,\theta)$ *satisfies* $\int_X \theta_\phi^n > 0$ *and* $u, v \in \mathcal{E}(X,\theta,\phi)$. *If* $\theta_u^n(\{u < v\}) = 0$ *then* $u \ge v$.

Proof. First, assume that v is less singular than u. In view of Corollary 3.9 it suffices to prove that $\theta_h^n(\{u < v\}) = 0$ for all $h \in \mathrm{PSH}(X,\theta)$ with the same singularity type as u. Let h be such a potential, and after possibly adding a constant, we can assume that $h \le u, v$. We claim that for each $t \in (0,1)$, $(1-t)v + th \in \mathcal{E}(X,\theta,\phi)$. Indeed, since $(1-t)v + th$ is less singular than u, and more singular than v, by Theorem 2.4 we can write

$$\int_X \theta_u^n \le \int_X \theta_{(1-t)v+th}^n \le \int_X \theta_v^n.$$

The comparison principle (Corollary 3.6) allows us then to write

$$t^n \int_{\{u<(1-t)v+th\}} \theta_h^n \leq \int_{\{u<(1-t)v+th\}} \theta_{(1-t)v+th}^n \leq \int_{\{u<v\}} \theta_u^n = 0.$$

Since $0 = \theta_h^n(\{u < (1-t)v + th\}) \nearrow \theta_h^n(\{u < v\})$ as $t \to 0$, it follows that $\theta_h^n(\{u < v\}) = 0$.

For the general case, we observe that $\theta_u^n(\{u < v\}) = \theta_u^n(\{u < \max(u, v)\})$, and the first step implies $u \geq \max(u, v) \geq v$. □

Next we show that F_ϕ, the set of potentials introduced in the beginning of this subsection, has a very specific maximal element:

Theorem 3.12. *Assume that $\phi \in \mathrm{PSH}(X, \theta)$ satisfies $\int_X \theta_\phi^n > 0$ and $\phi \leq 0$. Then*

$$P_\theta[\phi] = \sup_{v \in F_\phi} v.$$

In particular, $P_\theta[\phi] = P_\theta[P_\theta[\phi]]$.

As remarked in the beginning of the subsection, $P_\theta[\phi] \in F_\phi$; hence by the above result $P_\theta[\phi]$ is the maximal element of F_ϕ.

Proof. Let $u \in F_\phi$. By Theorem 3.8 we have

$$\theta_{P_\theta[\phi]}^n(\{P_\theta[\phi] < u\}) \leq \mathbb{1}_{\{P_\theta[\phi]=0\}}\theta^n(\{P_\theta[\phi] < u\}) \leq \mathbb{1}_{\{P_\theta[\phi]=0\}}\theta^n(\{P_\theta[\phi] < 0\}) = 0.$$

As $\phi \leq u$, and $\int_X \theta_\phi^n = \int_X \theta_u^n$, by Theorems 2.4 and 2.3 we have

$$\int_X \theta_{P_\theta[\phi]}^n = \int_X \theta_\phi^n = \int_X \theta_u^n = \int_X \theta_{P_\theta[u]}^n > 0.$$

Consequently, $P_\theta[\phi], u \in \mathcal{E}(X, \theta, P_\theta[u])$ and Proposition 3.11 now ensures that $P_\theta[\phi] \geq u$; hence $P_\theta[\phi] \geq \sup_{v \in F_\phi} v$. As $P_\theta[\phi] \in F_\phi$, it follows that $P_\theta[\phi] = \sup_{v \in F_\phi} v$.

For the last statement notice that $P_\theta[\phi] = \sup_{v \in F_\phi} v \geq \sup_{v \in F_{P_\theta[\phi]}} v = P_\theta[P_\theta[\phi]]$, since $F_\phi \supset F_{P_\theta[\phi]}$. The reverse inequality is trivial. □

Remark 3.13. The assumption $\int_X \theta_\phi^n > 0$ is necessary in the above theorem. Indeed, in the setting of Remark 3.3, it can be seen that $P_\theta[\phi] \lneq \sup_{h \in F_\phi} h$, as the potential on the right-hand side is greater than $\pi_2^* v$, since $\pi_2^* v \in F_\phi$.

As a consequence of this last result, we obtain the following characterization of membership in $\mathcal{E}(X, \theta, \phi)$, providing a partial converse to Proposition 3.1:

Theorem 3.14. *Suppose $\phi \in \mathrm{PSH}(X, \theta)$ with $\int_X \theta_\phi^n > 0$ and $\phi \leq 0$. The following are equivalent*:

(i) *$u \in \mathcal{E}(X, \theta, \phi)$.*

(ii) *ϕ is less singular than u, and $P_\theta[u](\phi) = \phi$.*

(iii) *ϕ is less singular than u, and $P_\theta[u] = P_\theta[\phi]$.*

As a consequence of the equivalence between (i) and (iii), we see that the potential $P_\theta[u]$ stays the same for all $u \in \mathcal{E}(X, \theta, \phi)$; i.e., it is an invariant of this class. In particular, since $\mathcal{E}(X, \theta, \phi) \subset \mathcal{E}(X, \theta, P_\theta[\phi])$, by the last statement of Theorem 3.12, it seems natural to only consider potentials ϕ that are in the image of the operator $\psi \to P_\theta[\psi]$, when studying classes of relative full mass $\mathcal{E}(X, \theta, \phi)$. What is more, in the next section it will be clear that considering such a ϕ is not just more natural, but also necessary when trying to solve complex Monge–Ampère equations with prescribed singularity.

Proof. Assume that (i) holds. By Theorem 3.8 it follows that $P_\theta[u](\phi) \geq \phi$ a.e. with respect to $\theta^n_{P_\theta[u](\phi)}$. Proposition 3.11 gives $P_\theta[u](\phi) = \phi$; hence (ii) holds.

Suppose (ii) holds. We can assume that $u \leq \phi \leq 0$. Then $P_\theta[u] \geq P_\theta[u](\phi) = \phi$. By the last statement of the previous theorem, this implies

$$P_\theta[u] = P_\theta[P_\theta[u]] \geq P_\theta[\phi].$$

As the reverse inequality is trivial, (iii) follows.

Lastly, assume that (iii) holds. By Theorems 2.4 and 2.3 it follows that $\int_X \theta^n_u = \int_X \theta^n_{P_\theta[u]} = \int_X \theta^n_{P_\theta[\phi]} = \int_X \theta^n_\phi$; hence (i) holds. □

Corollary 3.15. *Suppose $\phi \in \mathrm{PSH}(X, \theta)$ such that $\int_X \theta^n_\phi > 0$. Then $\mathcal{E}(X, \theta, \phi)$ is convex. Moreover, given $\psi_1, \dots, \psi_n \in \mathcal{E}(X, \theta, \phi)$ we have*

$$\int_X \theta^{s_1}_{\psi_1} \wedge \cdots \wedge \theta^{s_n}_{\psi_n} = \int_X \theta^n_\phi, \tag{9}$$

where $s_j \geq 0$ are integers such that $\sum_{j=1}^n s_j = n$.

Proof. Let $u, v \in \mathcal{E}(X, \theta, \phi)$ and fix $t \in (0, 1)$. It follows from Theorem 3.14 that $P_\theta[v](\phi) = P_\theta[u](\phi) = \phi$. This implies

$$P_\theta[tv + (1-t)u](\phi) \geq t P_\theta[v](\phi) + (1-t) P_\theta[u](\phi) = \phi.$$

As the reverse inequality is trivial, another application of Theorem 3.14 gives $tv + (1-t)u \in \mathcal{E}(X, \theta, \phi)$.

We now prove the last statement. Since $\mathcal{E}(X, \theta, \phi)$ is convex, given $\psi_1, \dots, \psi_n \in \mathcal{E}(X, \theta, \phi)$ we know that any convex combination $\psi := \sum_{j=1}^n s_j \psi_j$ with $0 \leq s_j \leq 1$ and $\sum_j s_j = n$, belongs to $\mathcal{E}(X, \theta, \phi)$. Hence

$$\int_X \Big(\sum_j s_j \theta_{\psi_j}\Big)^n = \int_X \theta^n_\psi = \int_X \theta^n_\phi = \int_X \Big(\sum_j s_j \theta_\phi\Big)^n.$$

As a result, we have an identity of two homogeneous polynomials of degree n. Therefore all the coefficients of these polynomials have to be equal, giving (9). □

Lastly, we provide another corollary, in the spirit of the partial comparison principle from Proposition 3.5:

Corollary 3.16. *Suppose $\phi \in \mathrm{PSH}(X, \theta)$ with $\int_X \theta^n_\phi > 0$. Assume that $u, v, \psi_1, \dots, \psi_j \in \mathcal{E}(X, \theta, \phi)$ for some $j \in \{0, \dots, n\}$. Then*

$$\int_{\{u<v\}} \theta^{n-j}_v \wedge \theta_{\psi_1} \wedge \cdots \wedge \theta_{\psi_j} \leq \int_{\{u<v\}} \theta^{n-j}_u \wedge \theta_{\psi_1} \wedge \cdots \wedge \theta_{\psi_j}.$$

Proof. The conclusion follows immediately from (9) together with Proposition 3.5. □

4. Complex Monge–Ampère equations with prescribed singularity type

Let θ be a smooth closed real $(1,1)$-form on X such that $\{\theta\}$ is big and $\phi \in \mathrm{PSH}(X,\theta)$. By $\mathrm{PSH}(X,\theta,\phi)$ we denote the set of θ-psh functions that are more singular than ϕ. We say that $v \in \mathrm{PSH}(X,\theta,\phi)$ has *relatively minimal singularities* if v has the same singularity type as ϕ. Clearly, $\mathcal{E}(X,\theta,\phi) \subset \mathrm{PSH}(X,\theta,\phi)$.

Let μ be a nonpluripolar positive measure on X such that $\mu(X) = \int_X \theta_\phi^n > 0$. Our aim is to study existence and uniqueness of solutions to the following equation of complex Monge–Ampère type:

$$\theta_\psi^n = \mu, \quad \psi \in \mathcal{E}(X,\theta,\phi). \tag{10}$$

It is not hard to see that this equation does not have a solution for arbitrary ϕ. Indeed, suppose for the moment that $\theta = \omega$, and choose $\phi \in \mathcal{E}(X,\omega) := \mathcal{E}(X,\omega,0)$ unbounded. It is clear that $\mathcal{E}(X,\omega,\phi) \subsetneq \mathcal{E}(X,\omega,0)$. By [Boucksom et al. 2010, Theorem A], the (trivial) equation $\omega_\psi^n = \omega^n$, $\psi \in \mathcal{E}(X,\omega,0)$, is *only* solved by potentials ψ that are constant over X; hence we cannot have $\psi \notin \mathcal{E}(X,\omega,\phi)$.

This simple example suggests that we need to be more selective in our choice of ϕ to make (10) well-posed. As it turns out, the natural choice is to take ϕ such that $P_\theta[\phi] = \phi$, as suggested by our study of currents of relative full mass in the previous subsection. Therefore, for the rest of this section we ask that ϕ additionally satisfies

$$\phi = P_\theta[\phi]. \tag{11}$$

Such a potential ϕ is called a model potential, and $[\phi]$ is a *model-type singularity*. As $V_\theta = P_\theta[V_\theta]$, one can think of such ϕ as generalizations of V_θ, the potential with minimal singularity from [Boucksom et al. 2010]. We refer to Remark 1.6 for natural constructions of model-type singularities.

As a technical assumption, we will ask that ϕ has additionally *small unbounded locus*; i.e., ϕ is locally bounded outside a closed pluripolar set $A \subset X$. This will be needed to carry out arguments involving integration by parts in the spirit of [Boucksom et al. 2010].

One wonders if maybe model-type potentials (those that satisfy (11)) always have small unbounded locus. Sadly, this is not the case, as the following simple example shows. Suppose θ is a Kähler form, and $\{x_j\}_j \subset X$ is a dense countable subset. Also let $v_j \in \mathrm{PSH}(X,\theta)$ be such that $v_j < 0$, $\int_X v_j \theta^n = 1$, and v_j has a positive Lelong number at x_j. Then $\psi = \sum_j (1/2^j) v_j \in \mathrm{PSH}(X,\theta)$ has positive Lelong numbers at all x_j. As we have argued in [Darvas et al. 2018, Theorem 1.1], the Lelong numbers of $P_\theta[\psi]$ are the same as those of ψ; hence the model-type potential $P_\theta[\psi]$ cannot have small unbounded locus.

The following convergence result is important in our later study, and it can be implicitly found in the arguments of [Boucksom et al. 2010], as well as other works:

Lemma 4.1. *Let $u_k, u_k^j \in \mathrm{PSH}(X,\theta,\phi)$ and $C > 0$ such that*

$$-C \le u_k^j - \phi \le C$$

for all $j \in \mathbb{N}$ and $k \in \{1,\dots,n\}$. Assume also that $u_k^j \to u_k$, $k \in \{1,\dots,n\}$, in capacity. Suppose also that f, f_j are uniformly bounded, quasicontinuous, such that $f_j \to f$ in capacity. Then $f_j \theta_{u_1^j} \wedge \dots \wedge \theta_{u_n^j} \to f \theta_{u_1} \wedge \dots \wedge \theta_{u_n}$ weakly.

Proof. Let $A \subset X$ be closed pluripolar such that $\{\phi = -\infty\} \subset A$. We set $\mu_j := \theta_{u_1^j} \wedge \cdots \wedge \theta_{u_n^j}$, and $\mu := \theta_{u_1} \wedge \cdots \wedge \theta_{u_n}$. Fix a continuous function χ on X, $\varepsilon > 0$ and U an open relatively compact subset of $X \setminus A$ such that $\mu(X \setminus U) \leq \varepsilon$. Fix V a slightly larger open subset of $X \setminus A$ such that $U \Subset V \Subset X \setminus A$. Fix ρ a continuous nonnegative function on X which is supported in V and is identically 1 in U. Since all functions u_k^j are uniformly bounded in V (along with u_k) it follows from [Guedj and Zeriahi 2017, Theorem 4.26] that $\chi f_j \mu_j$ converges weakly to $\chi f \mu$ in V. Also, Bedford–Taylor theory gives that μ_j converges weakly to μ in V. Thus $\liminf_j \mu_j(U) \geq \mu(U)$; hence $\limsup_j \mu_j(X \setminus U) \leq \mu(X \setminus U) \leq \varepsilon$ since $\mu_j(X) = \mu(X)$. Since χ, ρ, f_j, f are uniformly bounded it follows that $\limsup_j \int_{X\setminus U} \rho|\chi f_j|\mu_j$, $\limsup_j \int_{X\setminus U} |\chi f_j|\mu_j$, $\int_{X\setminus U} \rho|\chi f|\mu$, $\int_{X\setminus U} |\chi f|\mu$ are all bounded by $C\varepsilon$ for some uniform constant $C > 0$. On the other hand, since $\chi f_j \mu_j$ converges weakly to $\chi f \mu$ in V and $\rho = 0$ outside V, we have

$$\lim_j \int_X \rho \chi f_j \, d\mu_j = \int_X \rho \chi f \, d\mu.$$

Thus,

$$\limsup_j \left| \int_X \chi f_j \, d\mu_j - \int_X \chi f \, d\mu \right| \leq \limsup_j \left| \int_X \rho \chi f_j \, d\mu_j - \int_X \rho \chi f \, d\mu \right| + 4C\varepsilon.$$

It then follows that

$$\limsup_j \left| \int_X \chi f_j \, d\mu_j - \int_X \chi f \, d\mu \right| \leq C'\varepsilon.$$

Letting $\varepsilon \to 0$ we arrive at the conclusion. $\square$

4A. *The relative Monge–Ampère capacity.* We introduce the *relative Monge–Ampère capacity* of a Borel set $B \subset X$:

$$\mathrm{Cap}_\phi(B) := \sup\left\{ \int_B \theta_\psi^n \;\middle|\; \psi \in \mathrm{PSH}(X,\theta),\ \phi \leq \psi \leq \phi + 1 \right\}.$$

Note that in the Kähler case a related notion of capacity was studied in [Di Nezza and Lu 2015; 2017]. In the case when $\phi = V_\theta$ we recover the Monge–Ampère capacity used in [Boucksom et al. 2010, Section 4.1]. As is well known, the (generalized) Monge–Ampère capacity and the global relative extremal functions play a vital role in establishing uniform estimates for complex Monge–Ampère equations; see [Kołodziej 1998; Boucksom et al. 2010; Di Nezza and Lu 2015; 2017]. Along these lines the capacity Cap_ϕ will play a crucial role in proving the regularity part of Theorem 1.4.

Lemma 4.2. *The relative Monge–Ampère capacity* Cap_ϕ *is inner regular*; *i.e.*,

$$\mathrm{Cap}_\phi(E) = \sup\{\mathrm{Cap}_\phi(K) \mid K \subset E,\ K \textit{ is compact}\}.$$

Proof. By definition, $\mathrm{Cap}_\phi(E) \geq \mathrm{Cap}_\phi(K)$ for any compact set $K \subset E$. Fix $\varepsilon > 0$. There exists $u \in \mathrm{PSH}(X,\theta)$ such that $\phi \leq u \leq \phi + 1$ and

$$\int_E \theta_u^n \geq \mathrm{Cap}_\phi(E) - \varepsilon.$$

Since θ_u^n is an inner regular Borel measure it follows that there exists a compact set $K \subset E$ such that $\int_K \theta_u^n \geq \int_E \theta_u^n - \varepsilon \geq \mathrm{Cap}_\phi(E) - 2\varepsilon$. Hence $\mathrm{Cap}_\phi(K) \geq \mathrm{Cap}_\phi(E) - 2\varepsilon$. Letting $\varepsilon \to 0$ and taking the supremum over all the compact sets $K \subset E$, we arrive at the conclusion. □

By definition, $\mathrm{Cap}_\theta(B) \leq \mathrm{Cap}_\theta(X) = \int_X \theta_\phi^n$. Next we note that if $\mathrm{Cap}_\phi(B) = 0$ then B is a very "small" set:

Lemma 4.3. *Let $B \subset X$ be a Borel set. Then $\mathrm{Cap}_\phi(B) = 0$ if and only if B is pluripolar.*

Proof. Fix ω Kähler with $\omega \geq \theta$. Recall that a Borel subset $E \subset X$ is pluripolar if and only if $\mathrm{Cap}_\omega(E) = 0$; see [Guedj and Zeriahi 2005, Corollary 3.11], which goes back to [Bedford and Taylor 1982].

If B is pluripolar then $\mathrm{Cap}_\phi(B) = 0$ by definition. Conversely, assume that $\mathrm{Cap}_\phi(B) = 0$. If B is nonpluripolar then $\mathrm{Cap}_\omega(B) > 0$. Since Cap_ω is inner regular [Berman et al. 2013, Remark 1.7], there exists a compact subset K of B such that $\mathrm{Cap}_\omega(K) > 0$. In particular K is nonpluripolar; hence the global extremal function of (K, ω), $V^*_{\omega,K}$, is bounded from above (i.e., it is not identically ∞) by [Guedj and Zeriahi 2017, Theorem 9.17]. Since $\omega \geq \theta$ we have $V^*_{\theta,K} \leq V^*_{\omega,K}$; hence $V^*_{\theta,K}$ is also bounded from above.

We recall that $\theta^n_{V^*_{\theta,K}}$ is supported on K [Guedj and Zeriahi 2017, Theorem 9.17], and we consider $u_t := P_\theta(\phi + t, V^*_{\theta,K})$, $t > 0$. By the argument of Corollary 3.9 there exists $t_0 > 0$ big enough such that $\psi := u_{t_0} \in \mathrm{PSH}(X, \theta)$ has the same singularity type as ϕ and $\int_K \theta_\psi^n > 0$. We can assume that $\phi \leq \psi \leq \phi + C$ for some $C > 0$. If $C \leq 1$ then ψ is a candidate in the definition of $\mathrm{Cap}_\phi(B)$; hence $\mathrm{Cap}_\phi(B) > 0$, which is a contradiction. In case $C > 1$, then $(1 - 1/C)\phi + (1/C)\psi$ is a candidate in the definition of $\mathrm{Cap}_\phi(K)$; hence

$$\mathrm{Cap}_\phi(B) \geq \mathrm{Cap}_\phi(K) \geq \int_K \theta^n_{(1-1/C)\phi+(1/C)\psi} > \frac{1}{C^n}\int_K \theta_\psi^n > 0,$$

a contradiction. □

4A1. *The ϕ-relative extremal function.* Recall that ϕ has small unbounded locus; i.e., ϕ is locally bounded outside a closed complete pluripolar subset $A \subset X$. Recall that by $\mathrm{PSH}(X, \theta, \phi)$ we denote the set of all θ-psh functions which are more singular than ϕ.

Let E be a Borel subset of X. The relative extremal function of (E, ϕ, θ) is defined as

$$h_{E,\phi} := \sup\{u \in \mathrm{PSH}(X, \theta, \phi) \mid u \leq \phi - 1 \text{ on } E,\ u \leq 0 \text{ on } X\}.$$

Lemma 4.4. *Let E be a Borel subset of X and $h_{E,\phi}$ be the relative extremal function of (E, ϕ, θ). Then $h^*_{E,\phi}$ is a θ-psh function such that $\phi - 1 \leq h^*_{E,\phi} \leq \phi$. Moreover, $\theta^n_{h^*_{E,\phi}}$ vanishes on $\{h^*_{E,\phi} < 0\} \setminus \bar{E}$.*

Proof. Since $\phi - 1$ is a candidate defining $h_{E,\phi}$, it follows that $\phi - 1 \leq h_{E,\phi} \leq h^*_{E,\phi}$. Any $u \in \mathrm{PSH}(X, \theta, \phi)$ with $u \leq 0$ is a candidate of $P_\theta(\phi + C, 0)$ for some $C \in \mathbb{R}$. By Theorem 3.12 we get $u \leq P_\theta[\phi] = \phi$; hence $h^*_{E,\phi} \leq \phi$.

By the above, $h^*_{E,\phi}$ is locally bounded outside the closed pluripolar set A, and a standard balayage argument, see, e.g., [Bedford and Taylor 1976; Guedj and Zeriahi 2005, Proposition 4.1; Berman et al. 2013, Lemma 1.5], gives that $\theta^n_{h^*_{E,\phi}}$ vanishes in $\{h^*_{E,\phi} < 0\} \setminus \bar{E}$. □

Theorem 4.5. *If K is a compact subset of X and $h := h^*_{K,\phi}$ then*

$$\mathrm{Cap}_\phi(K) = \int_K \theta_h^n = \int_X (\phi - h)\theta_h^n.$$

Proof. Set $h := h^*_{K,\phi}$ and observe that $h+1$ is a candidate defining Cap_ϕ. Since θ_h^n puts no mass on the set $\{h < \phi\} \setminus K$ and $h = \phi - 1$ on K modulo a pluripolar set, we thus get

$$\mathrm{Cap}_\phi(K) \geq \int_K \theta_h^n = \int_X (\phi - h)\theta_h^n.$$

Now let u be a θ-psh function such that $\phi - 1 \leq u \leq \phi$. For a fixed $\varepsilon \in (0, 1)$ set $u_\varepsilon := (1-\varepsilon)u + \varepsilon\phi$. Since $h = \phi - 1$ on K modulo a pluripolar set and $\phi - 1 \leq u_\varepsilon$ it follows that $K \subset \{h < u_\varepsilon\}$ modulo a pluripolar set. By the comparison principle we then get

$$(1-\varepsilon)^n \int_K \theta_u^n \leq \int_{\{h<u_\varepsilon\}} \theta_{u_\varepsilon}^n \leq \int_{\{h<u_\varepsilon\}} \theta_h^n = \int_K \theta_h^n,$$

where in the last equality we use the fact that θ_h^n vanishes in $\{h < 0\} \setminus K$. Since u was taken arbitrarily, letting $\varepsilon \to 0$ we obtain $\mathrm{Cap}_\phi(K) \leq \int_K \theta_h^n$. This together with the previous step gives the result. □

Corollary 4.6. *If (K_j) is a decreasing sequence of compact sets then*

$$\mathrm{Cap}_\phi(K) = \lim_{j\to\infty} \mathrm{Cap}_\phi(K_j),$$

where $K := \bigcap_j K_j$. In particular, for any compact set K we have

$$\mathrm{Cap}_\phi(K) = \inf\{\mathrm{Cap}_\phi(U) \mid K \subset U \subset X,\ U \text{ is open in } X\}.$$

Proof. Let $h_j := h^*_{K_j,\phi}$ be the relative extremal function of (K_j, ϕ). Then (h_j) increases almost everywhere to $h \in \mathrm{PSH}(X,\theta)$, which satisfies $\phi - 1 \leq h \leq \phi$, since $\phi - 1 \leq h_j \leq \phi$.

Next we claim that $\theta_h^n(\{h < 0\} \setminus K) = 0$. Indeed, for $m \in \mathbb{N}$ fixed and for each $j > m$ we have that $\{h < 0\} \setminus K_m \subset \{h_j < 0\} \setminus K_j$ and by Lemma 4.4,

$$\theta_{h_j}^n(\{h_j < 0\} \setminus K_j) = 0.$$

Using the continuity of the Monge–Ampère measure along monotone sequences (Theorem 2.3 and Remark 2.5) we have that $\theta_{h_j}^n$ converges weakly to θ_h^n. Since $\{h < 0\} \setminus K_m$ is open, it follows that

$$\theta_h^n(\{h < 0\} \setminus K_m) \leq \liminf_{j\to\infty} \theta_{h_j}^n(\{h < 0\} \setminus K_m) = 0.$$

The claim follows as $m \to \infty$. It then follows from Theorem 4.5 and Lemma 4.1 that

$$\lim_{j\to\infty} \mathrm{Cap}_\phi(K_j) = \lim_{j\to\infty} \int_X (\phi - h_j)\theta_{h_j}^n = \int_X (\phi - h)\theta_h^n = \int_K \theta_h^n \leq \mathrm{Cap}_\phi(K).$$

As the reverse inequality is trivial, the first statement follows.

To prove the last statement, let (K_j) be a decreasing sequence of compact sets such that K is contained in the interior of K_j for all j. Then by the first part of the corollary we have

$$\begin{aligned}\mathrm{Cap}_\phi(K) = \lim_{j\to\infty} \mathrm{Cap}_\phi(K_j) &\geq \lim_{j\to\infty} \mathrm{Cap}_\phi(\mathrm{Int}(K_j)) \\ &\geq \inf\{\mathrm{Cap}_\phi(U) \mid K \subset U \subset X,\ U \text{ is open in } X\},\end{aligned}$$

and hence equality. □

Corollary 4.7. *If U is an open subset of X then*

$$\mathrm{Cap}_\phi(U) = \int_X (\phi - h_{U,\phi})\theta^n_{h_{U,\phi}}.$$

Proof. Let (K_j) be an increasing sequence of compact subsets of U such that $\bigcup K_j = U$. For each j we set $h_j := h^*_{K_j,\phi}$. By Theorem 4.5 we have

$$\mathrm{Cap}_\phi(K_j) = \int_X (\phi - h_j)\theta^n_{h_j}.$$

Since h_j decreases to $h_{U,\phi}$, it follows from Lemma 4.1 that the right-hand side above converges to $\int_X (\phi - h_{U,\phi})\theta^n_{h_{U,\phi}}$. Moreover, by the argument of Lemma 4.2 we have $\lim_j \mathrm{Cap}_\phi(K_j) = \mathrm{Cap}_\phi(U)$; hence the result follows. □

4A2. *The global ϕ-extremal function.* For a Borel set $E \subset X$, we define the global *ϕ-extremal function* of (E, ϕ, θ) by

$$V_{E,\phi} := \sup\{\psi \in \mathrm{PSH}(X,\theta,\phi) \mid \psi \leq \phi \text{ on } E\}.$$

We then introduce the *relative Alexander–Taylor capacity* of E,

$$T_\phi(E) := \exp(-M_\phi(E)), \quad \text{where } M_\phi(E) := \sup_X V^*_{E,\phi}.$$

Paralleling Lemma 4.3, we have the following result:

Lemma 4.8. *Let $E \subset X$ be a Borel set. If $M_\phi(E) = \infty$, then E is pluripolar.*

Proof. Let ω be a Kähler form such that $\omega \geq \theta$. By definition we have

$$V_{E,\phi} \leq V_{E,\omega} := \sup\{\psi \in \mathrm{PSH}(X,\omega) \mid \psi \leq 0 \text{ on } E\}.$$

This clearly implies $M_\phi(E) \leq \sup_X V^*_{E,\omega}$, and so by assumption we know that $\sup_X V^*_{E,\omega} = \infty$. It then follows from [Guedj and Zeriahi 2005, Theorem 5.2] that E is pluripolar. □

If $M_\phi(E) < \infty$ then $V^*_{E,\phi} \in \mathrm{PSH}(X,\theta)$, and standard arguments give that $\theta^n_{V^*_{E,\phi}}$ does not charge $X \setminus \bar{E}$; see [Guedj and Zeriahi 2005, Theorem 5.2; 2017, Theorem 9.17]. Now, we claim that

$$\phi \leq V^*_{E,\phi} \leq P_\theta[\phi] + M_\phi(E) = \phi + M_\phi(E). \tag{12}$$

The first inequality simply follows by definition, since $\phi \leq 0$ is a candidate in the definition of $V_{E,\phi}$. If $M_\phi(E) = \infty$ then the second inequality holds trivially. Assume that $M_\phi(E) < \infty$. The inequality then holds, since $V^*_{E,\phi} - M_\phi(E) \leq 0$, and each candidate potential ψ in the definition of $V^*_{E,\phi}$ is more singular than ϕ; i.e., $\psi - M_\phi(E)$ is a candidate in the definition of $P_\theta(\phi + C, 0)$ for some $C > 0$. Finally, the last identity follows from Theorem 3.12.

In particular, since ϕ has small unbounded locus, so does the upper semicontinuous regularization $V^*_{E,\phi}$. Also, from (12) we deduce that if $M_\phi(E) < \infty$, the θ-psh functions $V^*_{E,\phi}$ and ϕ have the same singularity type; hence Proposition 2.1 ensures that

$$\int_X \theta^n_{V^*_{E,\phi}} = \int_X \theta^n_\phi.$$

The Alexander–Taylor and Monge–Ampère capacities are related by the following estimates:

Lemma 4.9. *Suppose $K \subset X$ is a compact subset and $\mathrm{Cap}_\phi(K) > 0$. Then we have*

$$1 \le \left(\frac{\int_X \theta^n_\phi}{\mathrm{Cap}_\phi(K)} \right)^{1/n} \le \max(1, M_\phi(K)).$$

Proof. The first inequality is trivial. We now prove the second inequality. Note that we can assume that $M_\phi(K) < \infty$, since otherwise the inequality is trivially satisfied. We then consider two cases. If $M_\phi(K) \le 1$, then $V^*_{K,\phi} \le \phi + 1$; hence $V^*_{K,\phi}$ is a candidate in the definition of $\mathrm{Cap}_\phi(K)$. Since $\theta^n_{V^*_{K,\theta}}$ is supported on K, we thus have

$$\mathrm{Cap}_\phi(K) \ge \int_K \theta^n_{V^*_{K,\phi}} = \int_X \theta^n_{V^*_{K,\phi}} = \int_X \theta^n_\phi,$$

and the desired inequality holds in this case.

If $M := M_\phi(K) \ge 1$, then by (12) we have $\phi \le M^{-1} V^*_{K,\phi} + (1 - M^{-1})\phi \le \phi + 1$, and by the definition of the relative capacity we can write

$$\mathrm{Cap}_\phi(K) \ge \int_K \theta^n_{M^{-1}V^*_{K,\phi}+(1-M^{-1})\phi} \ge \frac{1}{M^n} \int_K \theta^n_{V^*_{K,\phi}} = \frac{1}{M^n} \int_X \theta^n_{V^*_{K,\phi}} = \frac{1}{M^n} \int_X \theta^n_\phi,$$

implying the desired inequality. □

4B. *The relative finite energy class $\mathcal{E}^1(X, \theta, \phi)$.* To develop the variational approach to (10), we need to understand the relative version of the Monge–Ampère energy, and its bounded locus $\mathcal{E}^1(X, \theta, \phi)$.

For $u \in \mathcal{E}(X, \theta, \phi)$ with relatively minimal singularities, we define the Monge–Ampère energy of u relative to ϕ as

$$\mathrm{I}_\phi(u) := \frac{1}{n+1} \sum_{k=0}^{n} \int_X (u - \phi)\theta^k_u \wedge \theta^{n-k}_\phi.$$

In the next theorem we collect basic properties of the Monge–Ampère energy:

Theorem 4.10. *Suppose $u, v \in \mathcal{E}(X, \theta, \phi)$ have relatively minimal singularities. The following hold*:

(i) $\mathrm{I}_\phi(u) - \mathrm{I}_\phi(v) = 1/(n+1) \sum_{k=0}^n \int_X (u - v)\theta^k_u \wedge \theta^{n-k}_v$.

(ii) *If $u \le \phi$ then, $\int_X (u - \phi)\theta^n_u \le \mathrm{I}_\phi(u) \le 1/(n+1) \int_X (u - \phi)\theta^n_u$.*

(iii) *I_ϕ is nondecreasing and concave along affine curves. Additionally, the estimates $\int_X (u - v)\theta^n_u \le \mathrm{I}_\phi(u) - \mathrm{I}_\phi(v) \le \int_X (u - v)\theta^n_v$ hold.*

Proof. Since ϕ has small unbounded locus, it is possible to repeat the arguments of [Boucksom et al. 2010, Proposition 2.8] almost word for word. As a courtesy to the reader, the detailed proof is presented here.

To start, we note that the nonpluripolar products appearing in our arguments are simply the mixed Monge–Ampère measures defined in the sense of [Bedford and Taylor 1976] on $X \setminus A$, where A is a closed complete pluripolar subset of X such that ϕ is locally bounded on $X \setminus A$ (consequently, u and v are locally bounded on $X \setminus A$). Since $u - v$ is globally bounded on X, we can perform integration by parts in our arguments below, via [Boucksom et al. 2010, Theorem 1.14].

For any fixed $k \in \{0, \dots, n-1\}$, set $T = \theta_u^k \wedge \theta_v^{n-k-1}$. Using integration by parts [Boucksom et al. 2010, Theorem 1.14], we can write

$$
\begin{aligned}
\int_X (u-v)\theta_u^k \wedge \theta_v^{n-k} &= \int_X (u-v)(\theta + i\partial\bar{\partial} v) \wedge T \\
&= \int_X (u-v) i\partial\bar{\partial}(v-u) \wedge T + \int_X (u-v) i\partial\bar{\partial} u \wedge T + \int_X (u-v)\theta \wedge T \\
&= \int_X (v-u) i\partial\bar{\partial}(u-v) \wedge T + \int_X (u-v)\theta_u \wedge T \\
&\geq \int_X (u-v)\theta_u \wedge T = \int_X (u-v)\theta_u^{k+1} \wedge \theta_v^{n-k-1},
\end{aligned} \tag{13}
$$

where in the last inequality we used that

$$
\int_X (-\varphi) i\partial\bar{\partial}\varphi \wedge T = i \int_X \partial\varphi \wedge \bar{\partial}\varphi \wedge T \geq 0
$$

with $\varphi := u - v$. This shows in particular that the sequence $k \mapsto \int_X (u-\phi)\theta_u^k \wedge \theta_\phi^{n-k}$ is nonincreasing in k, verifying (ii).

Now we compute the derivative of $f(t) := \mathrm{I}_\phi(u_t)$, $t \in [0,1]$, where $u_t := tu + (1-t)v$. By the multilinearity property of the nonpluripolar product we see that $f(t)$ is a polynomial in t. Using again integration by parts [Boucksom et al. 2010, Theorem 1.14], one can check the following formula:

$$
\begin{aligned}
f'(t) &= \frac{1}{n+1}\left(\sum_{k=0}^{n} \int_X (u-v)\theta_{u_t}^k \wedge \theta_\phi^{n-k} + \sum_{k=1}^{n} \int_X k(u_t - \phi) i\partial\bar{\partial}(u-v) \wedge \theta_{u_t}^{k-1} \wedge \theta_\phi^{n-k} \right) \\
&= \frac{1}{n+1}\left(\sum_{k=0}^{n} \int_X (u-v)\theta_{u_t}^k \wedge \theta_\phi^{n-k} + \sum_{k=1}^{n} \int_X k(u-v)(\theta_{u_t} - \theta_\phi) \wedge \theta_{u_t}^{k-1} \wedge \theta_\phi^{n-k} \right) = \int_X (u-v)\theta_{u_t}^n.
\end{aligned}
$$

Computing one more derivative, we arrive at

$$
f''(t) = n \int_X (u-v) i\partial\bar{\partial}(u-v) \wedge \theta_{u_t}^{n-1} = -ni \int_X \partial(u-v) \wedge \bar{\partial}(u-v)\theta_{u_t}^{n-1} \leq 0.
$$

This shows that I_ϕ is concave along affine curves.

Now, the function $t \mapsto f'(t)$ is continuous on $[0,1]$, thanks to the convergence property of the Monge–Ampère operator (see Lemma 4.1). It thus follows that

$$
\mathrm{I}_\phi(u_1) - \mathrm{I}_\phi(u_0) = \int_0^1 f'(t)\,dt = \int_0^1 \int_X (u-v)\theta_{u_t}^n \,dt.
$$

Using the multilinearity of the nonpluripolar product again, we get

$$\int_0^1 \int_X (u-v)\theta^n_{u_t}\,dt = \sum_{k=0}^n \left(\int_0^1 \binom{n}{k} t^k (1-t)^{n-k}\,dt\right) \int_X (u-v)\theta^k_u \wedge \theta^{n-k}_v$$
$$= \frac{1}{n+1}\sum_{k=0}^n \int_X (u-v)\theta^k_u \wedge \theta^{n-k}_v.$$

This verifies (i), and another application of (13) finishes the proof of (iii). □

Lemma 4.11. *Suppose $u_j, u \in \mathcal{E}(X,\theta,\phi)$ have relatively minimal singularities such that u_j decreases to u. Then $\mathrm{I}_\phi(u_j)$ decreases to $\mathrm{I}_\phi(u)$.*

Proof. From Theorem 4.10(iii) it follows that $|\mathrm{I}_\phi(u_j) - \mathrm{I}_\phi(u)| = \mathrm{I}_\phi(u_j) - \mathrm{I}_\phi(u) \le \int_X (u_j - u)\theta^n_u$. An application of the dominated convergence theorem finishes the argument. □

We can now define the Monge–Ampère energy for arbitrary $u \in \mathrm{PSH}(X,\theta,\phi)$ using a familiar formula:

$$\mathrm{I}_\phi(u) := \inf\{\mathrm{I}_\phi(v) \mid v \in \mathcal{E}(X,\theta,\phi),\ v \text{ has relatively minimal singularities, and } u \le v\}.$$

Lemma 4.12. *If $u \in \mathrm{PSH}(X,\theta,\phi)$ then $\mathrm{I}_\phi(u) = \lim_{t\to\infty} \mathrm{I}_\phi(\max(u,\phi - t))$.*

Proof. It follows from the above definition that $\mathrm{I}_\phi(u) \le \lim_{t\to\infty} \mathrm{I}_\phi(\max(u,\phi-t))$. Assume now that $v \in \mathrm{PSH}(X,\theta,\phi)$ is such that $u \le v$, and v has the same singularity type as ϕ (i.e., v is a candidate in the definition of $\mathrm{I}_\phi(u)$). Then for t large enough we have $\max(u,\phi-t) \le v$; hence the other inequality follows from the monotonicity of I_ϕ. □

We let $\mathcal{E}^1(X,\theta,\phi)$ denote the set of all $u \in \mathrm{PSH}(X,\theta,\phi)$ such that $\mathrm{I}_\phi(u)$ is finite. As a result of Lemma 4.12 and Theorem 4.10(iii) we observe that I_ϕ is nondecreasing in $\mathrm{PSH}(X,\theta,\phi)$. Consequently, $\mathcal{E}^1(X,\theta,\phi)$ is stable under the max operation; moreover, we have the following familiar characterization of $\mathcal{E}^1(X,\theta,\phi)$:

Lemma 4.13. *Suppose $u \in \mathrm{PSH}(X,\theta,\phi)$. Then $u \in \mathcal{E}^1(X,\theta,\phi)$ if and only if $u \in \mathcal{E}(X,\theta,\phi)$ and $\int_X (u-\phi)\theta^n_u > -\infty$.*

Proof. We can assume that $u \le \phi$. For each $C > 0$ we set $u^C := \max(u,\phi - C)$. If $\mathrm{I}_\phi(u) > -\infty$ then by the monotonicity property we have $\mathrm{I}_\phi(u^C) \ge \mathrm{I}_\phi(u)$. Since $u^C \le \phi$, an application of Theorem 4.10(ii) gives that $\int_X (u^C - \phi)\theta^n_{u^C} \ge -A$ for all C, for some $A > 0$. From this we obtain that

$$\int_{\{u \le \phi - C\}} \theta^n_{u^C} \le \frac{A}{C} \to 0$$

as $C \to \infty$. Hence it follows from Lemma 3.4 that $u \in \mathcal{E}(X,\theta,\phi)$. Moreover by the plurifine property of the nonpluripolar product we have

$$\int_X (u^C - \phi)\theta^n_{u^C} \le \int_{\{u > \phi - C\}} (u-\phi)\theta^n_u.$$

Letting $C \to \infty$ we see that $\int_X (u-\phi)\theta^n_u > -A$.

To prove the reverse statement, assume that $u \in \mathcal{E}(X,\theta,\phi)$ and $\int_X (u-\phi)\theta_u^n > -\infty$. For each $C > 0$, since θ_u^n and $\theta_{u^C}^n$ have the same mass and coincide in $\{u > \phi - C\}$, it follows that $\int_{\{u\le\phi-C\}} \theta_{u^C}^n = \int_{\{u\le\phi-C\}} \theta_u^n$. From this we deduce that

$$\int_X (u^C-\phi)\theta_{u^C}^n = -\int_{\{u\le\phi-C\}} C\theta_u^n + \int_{\{u>\phi-C\}} (u-\phi)\theta_u^n = \int_X (u-\phi)\theta_u^n > -A.$$

It thus follows from Theorem 4.10(ii) that $\mathrm{I}_\phi(u^C)$ is uniformly bounded. Finally, it follows from Lemma 4.12 that $\mathrm{I}_\phi(u^C) \searrow \mathrm{I}_\phi(u)$ as $C \to \infty$, finishing the proof. □

We finish this subsection with a series of small results listing various properties of the class $\mathcal{E}^1(X,\theta,\phi)$:

Lemma 4.14. *Assume that (u_j) is a sequence in $\mathcal{E}^1(X,\theta,\phi)$ decreasing to $u \in \mathcal{E}^1(X,\theta,\phi)$. Then $\mathrm{I}_\phi(u_j)$ decreases to $\mathrm{I}_\phi(u)$.*

Proof. Without loss of generality we can assume that $u_j \le \phi$ for all j. For each $C > 0$ we set $u_j^C := \max(u_j, \phi - C)$ and $u^C := \max(u, \phi - C)$. Note that u_j^C, u^C have the same singularities as ϕ. Then Lemma 4.11 ensures that $\lim_j \mathrm{I}_\phi(u_j^C) = \mathrm{I}_\phi(u^C)$. The monotonicity of I_ϕ gives now that $\mathrm{I}_\phi(u) \le \lim_j \mathrm{I}_\phi(u_j) \le \lim_j \mathrm{I}_\phi(u_j^C) = \mathrm{I}_\phi(u^C)$. Letting $C \to \infty$, the result follows. □

Lemma 4.15. *Assume that (u_j) is a decreasing sequence in $\mathcal{E}^1(X,\theta,\phi)$ such that $\mathrm{I}_\phi(u_j)$ is uniformly bounded. Then the limit $u := \lim_j u_j$ belongs to $\mathcal{E}^1(X,\theta,\phi)$ and $\mathrm{I}_\phi(u_j)$ decreases to $\mathrm{I}_\phi(u)$.*

Proof. We can assume that $u_j \le \phi$ for all j. Since $\mathrm{I}_\phi(u_j) \le \int_X (u_j - \phi)\theta_\phi^n$, $\mathrm{I}_\phi(u_j)$ is uniformly bounded and θ_ϕ^n has bounded density with respect to ω^n, it follows that $\int_X u_j\omega^n$ is uniformly bounded; hence $u \not\equiv -\infty$.

By continuity along decreasing sequences (Lemma 4.14) we have $\lim_{j\to\infty} \mathrm{I}_\phi(\max(u_j, \phi - C)) = \mathrm{I}_\phi(\max(u, \phi - C))$. It follows that $\mathrm{I}_\phi(\max(u, \phi - C))$ is uniformly bounded. Lemma 4.12 then ensures that $\mathrm{I}_\phi(u)$ is finite; i.e., $u \in \mathcal{E}^1(X,\theta,\phi)$. □

Corollary 4.16. *I_ϕ is concave along affine curves in $\mathrm{PSH}(X,\theta,\phi)$. In particular, the set $\mathcal{E}^1(X,\theta,\phi)$ is convex.*

Proof. Let $u, v \in \mathrm{PSH}(X,\theta,\phi)$ and $u_t := tu + (1-t)v,\ t \in (0,1)$. If one of u, v is not in $\mathcal{E}^1(X,\theta,\phi)$ then the conclusion is obvious. So, we can assume that both u and v belong to $\mathcal{E}^1(X,\theta,\phi)$. For each $C > 0$ we set $u_t^C := t\max(u, \phi - C) + (1-t)\max(v, \phi - C)$. By Theorem 4.10(iii), $t \to \mathrm{I}_\phi(u_t^C)$ is concave. Since u_t^C decreases to u_t as $C \to \infty$, Lemma 4.15 gives the conclusion. □

4C. ***The variational method.*** Recall that ϕ is a θ-psh function with small unbounded locus such that $\phi = P_\theta[\phi]$, and $\int_X \theta_\phi > 0$. For this subsection we additionally normalize our class so that $\int_X \theta_\phi^n = 1$.

We adapt the variational method of [Berman et al. 2013] to solve the complex Monge–Ampère equations in our more general setting:

$$\theta_u^n = e^{\lambda u}\mu, \quad u \in \mathcal{E}(X,\theta,\phi), \tag{14}$$

where $\lambda \ge 0$ and μ is a positive nonpluripolar measure on X. If $\lambda = 0$ then we also assume that $\mu(X) = 1$, which is a necessary condition for the equation to be solvable.

We introduce the following functionals on $\mathcal{E}^1(X, \theta, \phi)$:

$$F_\lambda(u) := F_{\lambda,\mu}(u) := \mathrm{I}_\phi(u) - L_{\lambda,\mu}(u), \quad u \in \mathcal{E}^1(X, \theta, \phi),$$

where $L_{\lambda,\mu}(u) := (1/\lambda)\int_X e^{\lambda u}\, d\mu$ if $\lambda > 0$ and $L_\mu(u) := L_{0,\mu}(u) := \int_X (u - \phi)\, d\mu$. Note that when $\lambda > 0$, F_λ is finite on $\mathcal{E}^1(X, \theta, \phi)$. It is no longer the case if $\lambda = 0$, in which case we will restrict ourselves to the following set of measures. For each constant $A \geq 1$ we let $\mathcal{M}_A$ denote the set of all probability measures μ on X such that

$$\mu(E) \leq A \cdot \mathrm{Cap}_\phi(E) \quad \text{for all Borel subsets } E \subset X.$$

Lemma 4.17. *$\mathcal{M}_A$ is a compact convex subset of the set of probability measures on X.*

Proof. The convexity is obvious. We now prove that $\mathcal{M}_A$ is closed. Assume that $(\mu_j) \subset \mathcal{M}_A$ is a sequence converging weakly to a probability measure μ. Then for any open set U we have

$$\mu(U) \leq \liminf_j \mu_j(U) \leq A\, \mathrm{Cap}_\phi(U).$$

Now, let $K \subset X$ be a compact subset. Taking the infimum over all open sets $U \supset K$ in the above inequality, it follows from Corollary 4.6 that $\mu(K) \leq A\, \mathrm{Cap}_\phi(K)$. Since μ and Cap_ϕ are inner regular (Lemma 4.2) it follows that the inequality holds for all Borel sets, finishing the proof. □

Lemma 4.18. *If $\mu \in \mathcal{M}_A$ then $F_{0,\mu}$ is finite on $\mathcal{E}^1(X, \theta, \phi)$. Moreover, there is a constant $B > 0$ depending on A such that for all $u \in \mathrm{PSH}(X, \theta, \phi)$ with $\sup_X u = 0$ we have*

$$\int_X (u - \phi)^2\, d\mu \leq B(|\mathrm{I}_\phi(u)| + 1).$$

The proof given below is inspired by [Berman et al. 2013, Lemma 2.9].

Proof. Fix $u \in \mathrm{PSH}(X, \theta, \phi)$ such that $\sup_X u = 0$. By considering $u_k := \max(u, \phi - k)$ and then letting $k \to \infty$, we can assume that $u - \phi$ is bounded. We first prove that

$$\int_1^\infty t\, \mathrm{Cap}_\phi(u < \phi - 2t)\, dt \leq C(-\mathrm{I}_\phi(u) + 1) \tag{15}$$

for some uniform constant $C := C(n) > 0$.

Indeed, for each $t > 1$ we set $u_t := t^{-1}u + (1 - t^{-1})\phi$. We also fix $\psi \in \mathrm{PSH}(X, \theta)$ such that $\phi - 1 \leq \psi \leq \phi$. Observe that $u_t, \psi \in \mathcal{E}(X, \theta, \phi)$ and that the following inclusions hold:

$$(u < \phi - 2t) \subset (u_t < \psi - 1) \subset (u < \phi - t), \quad t > 1.$$

It thus follows that

$$\theta_\psi^n(u < \phi - 2t) \leq \theta_\psi^n(u_t < \psi - 1) \leq \theta_{u_t}^n(u_t < \psi - 1) \leq \theta_{u_t}^n(u < \phi - t), \tag{16}$$

where in the second inequality we used the comparison principle (see Corollary 3.6). Expanding $\theta_{u_t}^n$ we see that

$$\theta_{u_t}^n \leq C t^{-1} \sum_{k=1}^n \theta_u^k \wedge \theta_\phi^{n-k} + \theta_\phi^n \quad \text{for all } t > 1, \tag{17}$$

for a uniform constant $C = C(n)$. Since θ_ϕ^n has bounded density with respect to Lebesgue measure (see Theorem 3.8), using [Guedj and Zeriahi 2017, Theorem 2.50] we infer that

$$\theta_\phi^n(u < \phi - t) \leq A \int_{\{u \leq -t\}} \omega^n \leq A e^{-at} \tag{18}$$

for some uniform constants $a, A > 0$ depending only on n, ω, X. Combining (18) with (16) and (17) we get that

$$\begin{aligned}
\int_1^\infty t\theta_\psi^n(u < \phi - 2t)\,dt &\leq \int_1^\infty t\theta_{u_t}^n(u < \phi - t)\,dt \\
&\leq C \int_1^\infty \sum_{k=0}^n \theta_u^k \wedge \theta_\phi^{n-k}(u < \phi - t)\,dt + \int_1^\infty t\theta_\phi^n(u < \phi - t)\,dt \\
&\leq C(n+1)|\mathrm{I}_\phi(u)| + C'.
\end{aligned}$$

Taking the supremum over all candidates $\psi + 1$ we arrive at

$$\int_1^\infty t\,\mathrm{Cap}_\phi(u < \phi - 2t)\,dt \leq C(n+1)|\mathrm{I}_\phi(u)| + C',$$

proving (15). Finally, we can write

$$\begin{aligned}
\int_X (u-\phi)^2\,d\mu = 2\int_0^\infty t\mu(u < \phi - t)\,dt &\leq 4 + 8\int_1^\infty t\mu(u < \phi - 2t)\,dt \\
&\leq 4 + 8\int_1^\infty At\,\mathrm{Cap}_\phi(u < \phi - 2t)\,dt \leq B(|\mathrm{I}_\phi(u)| + 1),
\end{aligned}$$

where $B > 0$ is a uniform constant depending on n, C, C'. □

Observe that Lemma 4.18 together with Hölder's inequality give that $F_{0,\mu}$ is finite on $\mathcal{E}^1(X, \theta, \phi)$ whenever $\mu \in \mathcal{M}_A$ for some $A \geq 1$. Indeed

$$\int_X |u - \phi|\,d\mu \leq \left(\int_X (u-\phi)^2\,d\mu\right)^{1/2} \mu(X)^{1/2} \leq C(|\mathrm{I}_\phi(u)|^{1/2} + 1) \tag{19}$$

for a suitable $C > 0$.

4C1. *Maximizers are solutions.*

Proposition 4.19. $\mathrm{I}_\phi : \mathcal{E}^1(X, \theta, \phi) \to \mathbb{R}$ *is upper semicontinuous with respect to the weak L^1 topology of potentials.*

Proof. Assume that (u_j) is a sequence in $\mathcal{E}^1(X, \theta, \phi)$ converging in L^1 to $u \in \mathcal{E}^1(X, \theta, \phi)$. We can assume that $u_j \leq 0$ for all j. For each $k, \ell \in \mathbb{N}$ we set $v_{k,\ell} := \max(u_k, \dots, u_{k+\ell})$. As $\mathcal{E}^1(X, \theta, \phi)$ is stable under the max operation, we have $v_{k,\ell} \in \mathcal{E}^1(X, \theta, \phi)$.

Moreover $v_{k,\ell} \nearrow \varphi_k := (\sup_{j \geq k} u_j)^*$; hence by the monotonicity property we get $\mathrm{I}_\phi(\varphi_k) \geq \mathrm{I}_\phi(v_{k,\ell}) \geq \mathrm{I}_\phi(u_k) > -\infty$. As a result, $\varphi_k \in \mathcal{E}^1(X, \theta, \phi)$. By Hartogs' lemma, $\varphi_k \searrow u$ as $k \to \infty$. By Lemma 4.14 it follows that $\mathrm{I}_\phi(\varphi_k)$ decreases to $\mathrm{I}_\phi(u)$. Thus, using the monotonicity of I_ϕ we get $\mathrm{I}_\phi(u) = \lim_{k\to\infty} \mathrm{I}_\phi(\varphi_k) \geq \limsup_{k\to\infty} \mathrm{I}_\phi(u_k)$, finishing the proof. □

Next we describe the first-order variation of I_ϕ, shadowing a result from [Berman and Boucksom 2010]:

Proposition 4.20. *Let $u \in \mathcal{E}^1(X,\theta,\phi)$ and χ be a continuous function on X. For each $t > 0$ set $u_t := P_\theta(u+t\chi)$. Then $u_t \in \mathcal{E}^1(X,\theta,\phi)$, $t \mapsto \mathrm{I}_\phi(u_t)$, is differentiable, and its derivative is given by*

$$\frac{d}{dt}\mathrm{I}_\phi(u_t) = \int_X \chi\theta^n_{u_t}, \qquad t \in \mathbb{R}.$$

Proof. Note that $u + t\inf_X \chi$ is a candidate in each envelope; hence $u + t\inf_X \chi \le u_t$. The monotonicity of I_ϕ now implies that $u_t \in \mathcal{E}^1(X,\theta,\phi)$.

As the singularity type of each u_t is the same, we can apply Lemma 4.21 below and conclude

$$\int_X (u_{t+s}-u_t)\theta^n_{u_{t+s}} \le \mathrm{I}_\phi(u_{t+s}) - \mathrm{I}_\phi(u_t) \le \int_X (u_{t+s}-u_t)\theta^n_{u_t}.$$

It follows from [Darvas et al. 2018, Proposition 2.13] that $\theta^n_{u_t}$ is supported on $\{u_t = u+t\chi\}$. We thus have

$$\int_X (u_{t+s}-u_t)\theta^n_{u_t} = \int_X (u_{t+s}-u-t\chi)\theta^n_{u_t} \le \int_X s\chi\theta^n_{u_t},$$

since $u_{t+s} \le u+(t+s)\chi$. Similarly we have

$$\int_X (u_{t+s}-u_t)\theta^n_{u_{t+s}} = \int_X (u+(t+s)\chi-u_t)\theta^n_{u_{t+s}} \ge \int_X s\chi\theta^n_{u_{t+s}}.$$

Since u_{t+s} converges uniformly to u_t as $s \to 0$, by Theorem 2.3 it follows that $\theta^n_{u_{t+s}}$ converges weakly to $\theta^n_{u_t}$. As χ is continuous, dividing by $s > 0$ and letting $s \to 0^+$ we see that the right derivative of $\mathrm{I}_\phi(u_t)$ at t is $\int_X \chi\theta^n_{u_t}$. The same argument applies for the left derivative. □

Lemma 4.21. *Suppose $u, v \in \mathcal{E}^1(X,\theta,\phi)$ have the same singularity type. Then*

$$\int_X (u-v)\theta^n_u \le \mathrm{I}_\phi(u) - \mathrm{I}_\phi(v) \le \int_X (u-v)\theta^n_v.$$

Proof. First, note that these estimates hold for $u^C := \max(u,\phi-C)$ and $v^C := \max(v,\phi-C)$, by Theorem 4.10(iii). It is easy to see that $u^C - v^C$ is uniformly bounded and converges to $u-v$. Also, by the comments after Lemma 3.4 it follows that the measures $\theta^n_{v^C}$ converge uniformly to θ^n_v (not just weakly!). Putting these last two facts together, the dominated convergence theorem gives

$$\left|\int_X (u^C-v^C)\theta^n_{v^C} - \int_X (u-v)\theta^n_v\right| \le \left|\int_X (u^C-v^C)(\theta^n_{v^C}-\theta^n_v)\right| + \left|\int_X (u^C-v^C)\theta^n_v - \int_X (u-v)\theta^n_v\right| \to 0$$

as $C \to \infty$. A similar convergence statement holds for the left-hand side of our double estimate as well, and using Lemma 4.12, the result follows. □

Theorem 4.22. *Assume that $L_{\lambda,\mu}$ is finite on $\mathcal{E}^1(X,\theta,\phi)$ and $u \in \mathcal{E}^1(X,\theta,\phi)$ maximizes $F_{\lambda,\mu}$ on $\mathcal{E}^1(X,\theta,\phi)$. Then u solves (14).*

Proof. First, let's assume that $\lambda \neq 0$. Let χ be an arbitrary continuous function on X and set $u_t := P_\theta(u+t\chi)$. It follows from Proposition 4.20 that $u_t \in \mathcal{E}^1(X,\theta,\phi)$ for all $t \in \mathbb{R}$, that the function

$$g(t) := \mathrm{I}_\phi(u_t) - L_{\lambda,\mu}(u+t\chi)$$

is differentiable on $\mathbb{R}$, and its derivative is given by $g'(t) = \int_X \chi\theta^n_{u_t} - \int_X \chi e^{\lambda(u+t\chi)}\, d\mu$. Moreover, as $u_t \le u + t\chi$, we have $g(t) \le F_{\lambda,\mu}(u_t) \le \sup_{\mathcal{E}^1(X,\theta,\phi)} F_{\lambda,\mu} = F(u) = g(0)$. This means that g attains a maximum at 0; hence $g'(0) = 0$. Since χ was taken to be arbitrary, it follows that $\theta^n_u = e^{\lambda u}\mu$. When $\lambda = 0$, similar arguments give the conclusion. □

4C2. *The case* $\lambda > 0$. Having computed the first-order variation of the Monge–Ampère energy, we establish the following existence and uniqueness result.

Theorem 4.23. *Assume that* μ *is a positive nonpluripolar measure on* X *and* $\lambda > 0$*. Then there exists a unique* $\varphi \in \mathcal{E}^1(X,\theta,\phi)$ *such that*

$$\theta^n_\varphi = e^{\lambda\varphi}\mu. \tag{20}$$

Proof. We use the variational method as above; see also [Darvas et al. 2018]. It suffices to treat the case $\lambda = 1$ as the other cases can de done similarly. Consider

$$F(u) := \mathrm{I}_\phi(u) - \int_X e^u\, d\mu, \quad u \in \mathcal{E}^1(X,\theta,\phi).$$

Let (φ_j) be a sequence in $\mathcal{E}^1(X,\theta,\phi)$ such that $\lim_j F(\varphi_j) = \sup_{\mathcal{E}^1(X,\theta,\phi)} F > -\infty$. We claim that $\sup_X \varphi_j$ is uniformly bounded from above. Indeed, assume that it were not the case. Then by relabeling the sequence we can assume that $\sup_X \varphi_j$ increases to ∞. By the compactness property [Guedj and Zeriahi 2005, Proposition 2.7] it follows that the sequence $\psi_j := \varphi_j - \sup_X \varphi_j$ converges in $L^1(X,\omega^n)$ to some $\psi \in \mathrm{PSH}(X,\theta)$ such that $\sup_X \psi = 0$. In particular $\int_X e^\psi\, d\mu > 0$. It thus follows that

$$\int_X e^{\varphi_j}\, d\mu = e^{\sup_X \varphi_j} \int_X e^{\psi_j}\, d\mu \ge c e^{\sup_X \varphi_j} \tag{21}$$

for some positive constant c. Note also that $\psi_j \le \phi$ since $\psi_j \in \mathcal{E}(X,\theta,\phi)$ and $\psi_j \le 0$ and ϕ is the maximal function with these properties (see Theorem 3.12). It then follows that

$$\mathrm{I}_\phi(\varphi_j) = \mathrm{I}_\phi(\psi_j) + \sup_X \varphi_j \le \sup_X \varphi_j. \tag{22}$$

From (21) and (22) we arrive at

$$\lim_{j\to\infty} F(\varphi_j) \le \lim_{j\to\infty} (\sup_X \varphi_j - c e^{\sup_X \varphi_j}) = -\infty,$$

which is a contradiction. Thus $\sup_X \varphi_j$ is bounded from above as claimed. Since $F(\varphi_j) \le \mathrm{I}_\phi(\varphi_j) \le \sup_X \varphi_j$, it follows that $\mathrm{I}_\phi(\varphi_j)$ and hence $\sup_X \varphi_j$ is also bounded from below. It follows again from [Guedj and Zeriahi 2005, Proposition 2.7] that a subsequence of φ_j (still denoted by φ_j) converges in $L^1(X,\omega^n)$ to some $\varphi \in \mathrm{PSH}(X,\theta)$. Since I_ϕ is upper semicontinuous it follows that $\varphi \in \mathcal{E}^1(X,\theta,\phi)$. Moreover, by continuity of $u \mapsto \int_X e^u\, d\mu$ we get that $F(\varphi) \ge \sup_{\mathcal{E}^1(X,\theta,\phi)} F$. Hence φ maximizes F on $\mathcal{E}^1(X,\theta,\phi)$. Now Theorem 4.22 shows that φ solves the desired complex Monge–Ampère equation. The next lemma address the uniqueness question. □

Lemma 4.24. *Let* $\lambda > 0$*. Assume that* $\varphi \in \mathcal{E}(X,\theta,\phi)$ *is a solution of* (20) *while* $\psi \in \mathcal{E}(X,\theta,\phi)$ *satisfies* $\theta^n_\psi \ge e^{\lambda\psi}\mu$*. Then* $\varphi \ge \psi$ *on* X*.*

Proof. By the comparison principle for the class $\mathcal{E}(X,\theta,\phi)$ (Corollary 3.6) we have

$$\int_{\{\varphi<\psi\}} \theta_\psi^n \le \int_{\{\varphi<\psi\}} \theta_\varphi^n.$$

As φ is a solution and ψ is a subsolution to (20) we also have

$$\int_{\{\varphi<\psi\}} e^{\lambda\psi}\,d\mu \le \int_{\{\varphi<\psi\}} \theta_\psi^n \le \int_{\{\varphi<\psi\}} \theta_\varphi^n = \int_{\{\varphi<\psi\}} e^{\lambda\varphi}\,d\mu \le \int_{\{\varphi<\psi\}} e^{\lambda\psi}\,d\mu.$$

It follows that all inequalities above are equalities; hence $\varphi \ge \psi$ μ-almost everywhere on X. Since $\mu = e^{-\lambda\varphi}\theta_\varphi^n$, it follows that $\theta_\varphi^n(\{\varphi<\psi\}) = 0$. By the domination principle (Proposition 3.11) we get that $\varphi \ge \psi$ everywhere on X. □

4C3. *The case* $\lambda = 0$.

Theorem 4.25. *Assume that $\mu \in \mathcal{M}_A$ for some $A \ge 1$. Then there exists $u \in \mathcal{E}^1(X,\theta,\phi)$ such that $\theta_u^n = \mu$.*

Proof. In view of Theorem 4.22 it suffices to find a maximizer in $\mathcal{E}^1(X,\theta,\phi)$ of the functional $F := F_{0,\mu}$ defined by

$$F(u) := \mathrm{I}_\phi(u) - \int_X (u-\phi)\,d\mu, \quad u \in \mathcal{E}^1(X,\theta,\phi).$$

Note that $F(u)$ is finite for all $u \in \mathcal{E}^1(X,\theta,\phi)$ since $\mu \in \mathcal{M}_A$ (see Lemma 4.18). Let (u_j) be a sequence in $\mathcal{E}^1(X,\theta,\phi)$ such that $\sup_X u_j = 0$ and $F(u_j)$ increases to $\sup_{\mathcal{E}^1(X,\theta,\phi)} F > -\infty$. By the compactness property [Guedj and Zeriahi 2005], a subsequence of (u_j) converges to $u \in \mathrm{PSH}(X,\theta,\phi)$, and $\sup_X u = 0$. Moreover, since $\mu \in \mathcal{M}_A$, by (19) we have

$$F(u_j) \le \mathrm{I}_\phi(u_j) + C|\mathrm{I}_\phi(u_j)|^{1/2} + C \quad \text{for all } j.$$

It thus follows that $\mathrm{I}_\phi(u_j)$ is uniformly bounded. Since I_ϕ is upper semicontinuous it follows that $u \in \mathcal{E}^1(X,\theta,\phi)$. Also, since $\int_X (u_j-\phi)^2\,d\mu$ is uniformly bounded (Lemma 4.18) it follows from the same arguments as [Guedj and Zeriahi 2017, Lemma 11.5] that $\int_X (u_j-\phi)\,d\mu$ converges to $\int_X (u-\phi)\,d\mu$. Since I_ϕ is upper semicontinuous, we obtain that $F(u) \ge \limsup_j F(u_j)$. Hence u maximizes F on $\mathcal{E}^1(X,\theta,\phi)$, and the result follows. □

Lemma 4.26. *If μ is a positive nonpluripolar measure on X and $A \ge 1$ then there exists $\nu \in \mathcal{M}_A$ and $0 \le f \in L^1(X,\nu)$ such that $\mu = f\nu$.*

The short proof given below is due to Cegrell [1998].

Proof. It follows from Lemma 4.17 that $\mathcal{M}_A$ is a convex compact subset of $\mathcal{M}(X)$, the space of probability measures on X. It follows from [König and Seever 1969, Lemma 1] that we can write

$$\mu = \nu + \sigma,$$

where ν, σ are nonnegative Borel measures on X such that ν is absolutely continuous with respect to an element in $\mathcal{M}_A$ and σ is singular with respect to any element of $\mathcal{M}_A$; i.e., $\sigma \perp m$ for any $m \in \mathcal{M}_A$. It then follows from [Rainwater 1969, Theorem] that σ is supported on a Borel set E such that $m(E) = 0$ for all $m \in \mathcal{M}_A$. If u is a candidate defining the capacity $\mathrm{Cap}_\phi(E)$, then clearly $\theta_u^n \in \mathcal{M}_A$; hence $\int_E \theta_u^n = 0$. It follows that $\mathrm{Cap}_\phi(E) = 0$; hence by Lemma 4.3 E is pluripolar. Therefore, $\sigma = 0$ since μ does not charge pluripolar sets. □

To prove the main existence result in this subsection we also need the following lemma. The argument uses the locality of nonpluripolar Monge–Ampère measures with respect to the plurifine topology, and is identical to the proof of [Guedj and Zeriahi 2007, Corollary 1.10].

Lemma 4.27. *Assume that ν is a positive nonpluripolar Borel measure on X and $u, v \in \mathrm{PSH}(X, \theta)$. If $\theta_u^n \geq \nu$ and $\theta_v^n \geq \nu$ then $\theta_{\max(u,v)}^n \geq \nu$.*

Theorem 4.28. *Assume that μ is a positive nonpluripolar measure on X such that $\mu(X) = \int_X \theta_\phi^n$. Then there exists $u \in \mathcal{E}(X, \theta, \phi)$ (unique up to a constant) such that $\theta_u^n = \mu$.*

Proof. It follows from Lemma 4.26 that $\mu = f\nu$, where $\nu \in \mathcal{M}_1$ and $0 \leq f \in L^1(X, \nu)$. For each j it follows from Theorem 4.25 that there exists $u_j \in \mathcal{E}^1(X, \theta, \phi)$ such that $\sup_X u_j = 0$ and

$$\theta_{u_j}^n = c_j \min(f, j)\nu.$$

Here, c_j is a normalization constant and $c_j \to 1$ as $j \to \infty$. We can assume that $1 \leq c_j \leq 2$ for all j. By compactness [Guedj and Zeriahi 2017, Proposition 8.5], a subsequence of (u_j) converges in $L^1(X, \omega^n)$ to $u \in \mathrm{PSH}(X, \theta, \phi)$ with $\sup_X u = 0$. We will show that $u \in \mathcal{E}(X, \theta, \phi)$. For each $k \in \mathbb{N}$ we set $v_k := (\sup_{j \geq k} u_j)^*$. Then $v_k \in \mathcal{E}^1(X, \theta, \phi)$ and (v_k) decreases pointwise to u. For each k fixed, and for all $j > k$ we have $\theta_{u_j}^n \geq \min(f, k)\nu$. Thus for all $\ell \in \mathbb{N}$ it follows from Lemma 4.27 that $\theta_{w_{k,\ell}}^n \geq \min(f, k)\nu$, where $w_{k,\ell} := \max(u_k, \ldots, u_{k+\ell})$. Since $(w_{k,\ell})$ increases almost everywhere to v_k as $\ell \to \infty$, it follows from Theorem 2.3 and Remark 2.5 that

$$\theta_{v_k}^n \geq \min(f, k)\nu.$$

Thus for each $C > 0$, setting $v_k^C := \max(v_k, V_\theta - C)$, using the plurifine property of the Monge–Ampère measure and observing that $\{u > V_\theta - C\} \subseteq \{v_k > V_\theta - C\}$, we have

$$\theta_{v_k^C}^n \geq \mathbb{1}_{\{v_k > V_\theta - C\}} \theta_{v_k}^n \geq \mathbb{1}_{\{v_k > V_\theta - C\}} \min(f, k)\nu \geq \mathbb{1}_{\{u > V_\theta - C\}} \min(f, k)\nu.$$

Since (v_k^C) decreases to $u^C := \max(u, V_\theta - C)$ and $v_k^C, u^C \in \mathcal{E}(X, \theta)$, it follows from Theorem 2.3 that $\theta_{v_k^C}^n$ converges weakly to $\theta_{u^C}^n$; hence

$$\theta_{u^C}^n \geq \mathbb{1}_{\{u > V_\theta - C\}} \mu.$$

Since μ is nonpluripolar, by letting $C \to \infty$ it follows that

$$\theta_u^n = \lim_{C \to \infty} \mathbb{1}_{\{u > V_\theta - C\}} \theta_{u^C}^n \geq \lim_{C \to \infty} \mathbb{1}_{\{u > V_\theta - C\}} \mu = \mu.$$

Moreover by [Witt Nyström 2017, Theorem 1.2] the total mass of θ_u^n is smaller than $\int_X \theta_\phi^n = \mu(X)$ since $u \leq \phi$. Hence $\int_X \theta_\phi^n = \mu(X) = \int_X \theta_u^n$. It thus follows that $u \in \mathcal{E}(X, \theta, \phi)$ and $\theta_u^n = \mu$. Uniqueness is addressed in the next theorem. □

Theorem 4.29. *Assume $u, v \in \mathcal{E}(X, \theta, \phi)$ are such that $\theta_u^n = \theta_v^n$. Then $u - v$ is constant.*

The proof of this uniqueness result rests on the adaptation of the mass concentration technique of Kołodziej and Dinew [2009b] to our more general setting; see also [Boucksom et al. 2010; Dinew and Lu 2015]. The arguments carry over almost verbatim, but as a courtesy to the reader we provide a detailed account.

Proof. Set $\mu := \theta_u^n = \theta_v^n$. We will prove that there exists a constant C such that μ is supported on $\{u = v + C\}$. This will allow us to apply the domination principle (Proposition 3.11) to ensure the conclusion. Assume that it is not the case. Arguing exactly as in [Boucksom et al. 2010, Section 3.3] we can assume that $0 < \mu(U) < \mu(X) = \int_X \theta_\phi^n$ and $\mu(\{u = v\}) = 0$, where $U := \{u < v\}$. Let $c > 1$ be a normalization constant such that $\int_{\{u<v\}} c^n \, d\mu = \mu(X)$. It follows from Theorem 4.28 that there exists $h \in \mathcal{E}(X, \theta, \phi)$, $\sup_X h = 0$, such that $\theta_h^n = c^n \mathbb{1}_U \mu$. In particular, $h \le \phi$. For each $t \in (0, 1)$ we set $U_t := \{(1-t)u + t\phi < (1-t)v + th\}$ and note that, since $h \le \phi$, the sets U_t increase as $t \to 0^+$ to $U \setminus \{h = -\infty\}$.

By the mixed Monge–Ampère inequalities [Boucksom et al. 2010, Proposition 1.11], which go back to [Dinew 2009a; Kołodziej 2003], we have

$$\theta_u^{n-1} \wedge \theta_h \ge \mathbb{1}_U c\mu, \quad \theta_u^k \wedge \theta_v^{n-k} \ge \mu, \quad k = 0, \dots, n. \tag{23}$$

Moreover, since $u, v, h \in \mathcal{E}(X, \theta, \phi)$, it follows from Corollary 3.15 that all the above nonpluripolar products have the same mass. Consequently, $\theta_u^k \wedge \theta_v^{n-k} = \mu$, $k = 0, \dots, n$. Using the partial comparison principle (Proposition 3.5) we can write

$$\int_{U_t} \theta_u^{n-1} \wedge \theta_{(1-t)v+th} \le \int_{U_t} \theta_u^{n-1} \wedge \theta_{(1-t)u+t\phi}.$$

Expanding, and using the fact that $\theta_u^n = \theta_u^{n-1} \wedge \theta_v$ we get

$$\int_{U_t} \theta_u^{n-1} \wedge \theta_h \le \int_{U_t} \theta_u^{n-1} \wedge \theta_\phi. \tag{24}$$

Combining (23) and (24) we have $c\mu(U_t) \le \int_{U_t} \theta_u^{n-1} \wedge \theta_h \le \int_{U_t} \theta_u^{n-1} \wedge \theta_\phi$. Letting $t \to 0$, and noting that μ is nonpluripolar (hence μ puts no mass on the set $\{h = -\infty\}$) we obtain

$$c\mu(U) \le \int_U \theta_u^{n-1} \wedge \theta_\phi.$$

Now, applying the same arguments for $V := \{u > v\}$ we obtain

$$b\mu(V) \le \int_V \theta_u^{n-1} \wedge \theta_\phi,$$

where $b > 1$ is a constant such that $b^n \mu(V) = \mu(X)$. Using that $\mu(\{u = v\}) = 0$, we can sum up the last two inequalities and obtain

$$0 < \min(b, c)\mu(X) \le \int_X \theta_u^{n-1} \wedge \theta_\phi = \mu(X),$$

where the last equality follows again from Corollary 3.15. This is a contradiction since $\min(b, c) > 1$. □

4D. ***Regularity of solutions.*** Recall that we work with $\phi \in \mathrm{PSH}(X, \theta)$ with small unbounded locus such that $P_\theta[\phi] = \phi$, and $\int_X \theta_\phi^n > 0$. Let $f \in L^p(\omega^n)$ with $f \geq 0$. In the previous subsection we have shown that the equation

$$\theta_\psi^n = f\omega^n, \quad \psi \in \mathcal{E}^1(X, \theta, \phi),$$

has a unique solution. In this subsection we will show that this solution has the same singularity type as ϕ. This generalizes [Boucksom et al. 2010, Theorem B], which treats the particular case of solutions with minimal singularities in a big class. Analogous results will be obtained for the solutions of (20) as well.

Our arguments will closely follow the path laid out in [Boucksom et al. 2010, Section 4.1], which builds on fundamental work of Kołodziej [1998; 2003] in the Kähler case. As we shall see, the fact that ϕ has model-type singularity plays a vital role in making sure that the methods of [Boucksom et al. 2010] work in our more general context as well.

We first prove that any measure with $L^{1+\varepsilon}$, $\varepsilon > 0$, density is dominated by the relative capacity:

Proposition 4.30. *Let $f \in L^p(\omega^n)$, $p > 1$, with $f \geq 0$. Then there exists $C > 0$ depending only on θ, ω, p and $\|f\|_{L^p}$ such that*

$$\int_E f\omega^n \leq \frac{C}{\left(\int_X \theta_\phi^n\right)^2} \cdot \mathrm{Cap}_\phi(E)^2$$

for all Borel sets $E \subset X$.

Proof. Since Cap_ϕ is inner regular we can assume that E is compact. Thanks to Lemma 4.8 we can also assume that $M_\phi(E) < \infty$.

We introduce $\nu_\theta := \sup_{T,x} \nu(T, x)$, where $x \in X$, T is any closed positive $(1, 1)$-current cohomologous with θ, and $\nu(T, x)$ denotes the Lelong number of T at x. As a result, the uniform version of Skoda's integrability theorem [Guedj and Zeriahi 2017, Theorem 2.50] yields a constant $C > 0$, only depending on θ and ω such that $\int_X \exp(-\nu_\theta^{-1}\psi)\omega^n \leq C$ for all $\psi \in \mathrm{PSH}(X, \theta)$ with $\sup_X \psi = 0$. Applying this to $V^*_{E,\phi} - M_\phi(E)$ we get

$$\int_X \exp(-\nu_\theta^{-1} V^*_{E,\phi})\omega^n \leq C \cdot \exp(-\nu_\theta^{-1} M_\phi(E)).$$

On the other hand, $V^*_{E,\phi} \leq 0$ on E a.e. with respect to Lebesgue measure; hence

$$\mathrm{Vol}_\omega(E) := \int_E \omega^n \leq C \cdot \exp(-\nu_\theta^{-1} M_\phi(E)). \tag{25}$$

An application of Hölder's inequality gives

$$\int_E f\omega^n \leq \|f\|_{L^p} \mathrm{Vol}_\omega(E)^{(p-1)/p}. \tag{26}$$

At this point we may assume that $M_\phi(E) \geq 1$. Indeed, if this were not the case, then Lemma 4.9 would imply that $\mathrm{Cap}_\phi(E) = \int_X \theta_\phi^n$, yielding the desired estimate of the proposition. Putting together Lemma 4.9,

(25) and (26) we get

$$\int_E f\omega^n \le C^{p-1/p}\cdot \|f\|_{L^p}\cdot \exp\left(-\frac{p-1}{p\nu_\theta}\left(\frac{\mathrm{Cap}_\phi(E)}{\int_X \theta_\phi^n}\right)^{-1/n}\right).$$

The result now follows, as $\exp(-t^{-1/n}) = O(t^2)$ when $t \to 0_+$. □

Before we state the main result of this subsection, we need one last lemma, which is a simple consequence of our comparison principle:

Lemma 4.31. *Let $u \in \mathcal{E}(X,\theta,\phi)$. Then for all $t > 0$ and $\delta \in (0,1]$ we have*

$$\mathrm{Cap}_\phi\{u < \phi - t - \delta\} \le \frac{1}{\delta^n}\int_{\{u<\phi-t\}} \theta_u^n.$$

Proof. Let $\psi \in \mathrm{PSH}(X,\theta,\phi)$ be such that $\phi \le \psi \le \phi + 1$. In particular, note that $\psi \in \mathcal{E}(X,\theta,\phi)$. We then have

$$\{u < \phi - t - \delta\} \subset \{u < \delta\psi + (1-\delta)\phi - t - \delta\} \subset \{u < \phi - t\}.$$

Since $\delta^n\theta_\psi^n \le \theta^n_{\delta\psi+(1-\delta)\phi}$, u has relative full mass and $\mathcal{E}(X,\theta,\phi)$ is convex, Corollary 3.6 yields

$$\delta^n\int_{\{u<\phi-t-\delta\}}\theta_\psi^n \le \int_{\{u<\delta\psi+(1-\delta)\phi-t-\delta\}}\theta^n_{\delta\psi+(1-\delta)\phi} \le \int_{\{u<\delta\psi+(1-\delta)\phi-t-\delta\}}\theta_u^n \le \int_{\{u<\phi-t\}}\theta_u^n.$$

Since ψ is an arbitrary candidate in the definition of Cap_ϕ, the proof is complete. □

We arrive at the main results of this subsection:

Theorem 4.32. *Suppose $\phi = P_\theta[\phi]$ has small unbounded locus and $\int_X \theta_\phi^n > 0$. Let also $\psi \in \mathcal{E}(X,\theta,\phi)$ with $\sup_X \psi = 0$. If $\theta_\psi^n = f\omega^n$ for some $f \in L^p(\omega^n)$, $p > 1$, then ψ has the same singularity type as ϕ; more precisely,*

$$\phi - C\left(\|f\|_{L^p}, p, \omega, \theta, \int_X \theta_\phi^n\right) \le \psi \le \phi.$$

Proof. To begin, we introduce the function

$$g(t) := (\mathrm{Cap}_\phi\{\psi < \phi - t\})^{1/n}, \quad t \ge 0.$$

We will show that $g(M) = 0$ for some M under control. By Lemma 4.3 we will then have $\psi \ge \phi - M$ a.e. with respect to ω^n, which then implies $\psi \ge \phi - M$ on X.

Since $\theta_\psi^n = f\omega^n$, it follows from Proposition 4.30 and Lemma 4.31 that

$$g(t+\delta) \le \frac{C^{1/n}}{\delta} g(t)^2, \quad t > 0, \quad 0 < \delta < 1.$$

Consequently, we can apply [Eyssidieux et al. 2009, Lemma 2.3] to conclude that $g(M) = 0$ for $M := t_0 + 2$. As an important detail, the constant $t_0 > 0$ has to be chosen so that

$$g(t_0) < \frac{1}{2C^{1/n}}.$$

On the other hand, Lemma 4.31 (with $\delta = 1$) implies

$$g(t+1)^n \le \int_{\{\psi<\phi-t-1\}} f\omega^n \le \frac{1}{t+1}\int_X |\phi-\psi| f\omega^n \le \frac{1}{t+1}\|f\|_{L^p}(\|\psi\|_{L^q}+\|\phi\|_{L^q}),$$

where in the last estimate we used Hölder's inequality with $q = p/(p-1)$. Since ψ and ϕ both belong to the compact set of θ-psh functions normalized by $\sup_X u = 0$, their L^q norms are bounded by an absolute constant only depending on θ, ω and p. Consequently, it is possible to choose t_0 to be only dependent on $\|f\|_{L^p}$, θ, ω, $\int_X \theta_\phi^n$ and p, finishing the proof. □

Corollary 4.33. *Suppose $\phi = P_\theta[\phi]$ has small unbounded locus and $\int_X \theta_\phi^n > 0$. If $\lambda > 0$ and, $\psi \in \mathcal{E}(X,\theta,\phi)$, $\theta_\psi^n = e^{\lambda\psi} f\omega^n$ for some $f \in L^p(\omega^n)$, $p > 1$, then ψ has the same singularity type as ϕ.*

Proof. Since ψ is bounded from above on X and $\lambda > 0$, it follows that $e^{\lambda\psi} f \in L^p(X,\omega^n)$, $p > 1$. The result follows from Theorem 4.32. □

4E. *Naturality of model-type singularities and examples.* Our readers may still wonder if our choice of model potentials is a natural one in the discussion of complex Monge–Ampère equations with prescribed singularity. We hope to address the doubts in the next result.

Theorem 4.34. *Suppose $\psi \in \mathrm{PSH}(X,\theta)$ has small unbounded locus and the equation*

$$\theta_u^n = f\omega^n$$

has a solution $u \in \mathrm{PSH}(X,\theta)$ with the same singularity type as ψ for all $f \in L^\infty$, $f \ge 0$, satisfying $\int_X \theta_\psi^n = \int_X f\omega^n > 0$. Then ψ has model-type singularity.

Proof. Our simple proof follows the guidelines of the example described in the beginning of Section 4. Indeed, suppose that $[\psi]$ is not of model type. Then $P_\theta[\psi]$ is strictly less singular than ψ, but of course $\mathcal{E}(X,\theta,\psi) \subset \mathcal{E}(X,\theta,P_\theta[\psi])$, as $\int_X \theta_\psi^n = \int_X \theta_{P_\theta[\psi]}^n$.

By Theorem 3.8, there exists $g \in L^\infty$ such that $\theta^n_{P_\theta[\psi]} = g\omega^n$. By the uniqueness theorem (Theorem 4.29), $P_\theta[\psi]$ is the only solution of this last equation inside $\mathcal{E}(X,\theta,P_\theta[\psi])$.

Since $\mathcal{E}(X,\theta,\psi) \subset \mathcal{E}(X,\theta,P_\theta[\psi])$, but $P_\theta[\psi] \notin \mathcal{E}(X,\theta,\psi)$, we get that $\theta_u^n = g\omega^n$ cannot have any solution that has the same singularity type as ψ. □

Next we point out a simple way to construct model singularity types:

Proposition 4.35. *Suppose that $\psi \in \mathrm{PSH}(X,\theta)$ has small unbounded locus and $\theta_\psi^n = f\omega^n$ for some $f \in L^p(\omega^n)$, $p > 1$, with $\int_X f\omega^n > 0$. Then ψ has model-type singularity.*

Proof. We first observe that $\psi \in \mathcal{E}(X,\theta,P_\theta[\psi])$. Since θ_ψ^n has L^p density with $p > 1$, it thus follows from Theorem 4.32 that $\psi - P_\theta[\psi]$ is bounded on X; hence $[\psi] = [P_\theta[\psi]]$, implying that ψ has model-type singularity. □

Using this simple proposition, one can show that all analytic singularity types are of model type, which was previously known to be true using algebraic methods; see [Ross and Witt Nyström 2014; Rashkovskii and Sigurdsson 2005]:

Proposition 4.36. *Suppose $\psi \in \mathrm{PSH}(X,\theta)$ has analytic singularity type; i.e., ψ can be locally written as $c\log\big(\sum_j |f_j|^2\big)+g$, where f_j are holomorphic, $c>0$ and g is smooth. Then $[\psi]$ is of model type.*

Proof. We can assume that our fixed Kähler form ω satisfies $\omega \geq 2\theta$. Since $P_\theta[\psi] \leq P_\omega[\psi]$, it suffices to prove that $\psi - P_\omega[\psi]$ is globally bounded on X. In fact we will prove the following stronger result:

$$\rho := \frac{\omega_\psi^n}{\omega^n} \in L^p(\omega^n) \quad \text{for some } p>1. \tag{27}$$

As $\omega/2 \geq \theta$ it follows that $\int_X \omega_\psi^n \geq 2^{-n}\int_X \omega^n > 0$; hence Proposition 4.35 will imply that $\psi - P_\omega[\psi]$ is globally bounded on X.

We now prove (27). Since X is compact it suffices to prove that there exists a small open neighborhood U around a given point $x \in X$ (which will be fixed) such that $\rho \in L^p(U, dV)$ for some $p>1$. Since ψ has analytic singularities we can find a holomorphic coordinate chart Ω around x such that

$$\psi = c\log\sum_{j=1}^N |f_j|^2 + g$$

in a neighborhood of Ω, where $c>0$ is a constant, f_j are holomorphic functions in Ω and g is a smooth real-valued function in Ω. Let $A>0$ be large enough so that $(A-1)\omega + i\partial\bar\partial g \geq 0$ in Ω.

In $X\setminus\{\psi=-\infty\}$, since ψ is smooth we can write $\omega_\psi^n = \rho\omega^n$, where $\rho \geq 0$ is smooth. We extend ρ to be 0 over the set $\{\psi=-\infty\}$. Then $\rho\omega^n$ is the nonpluripolar Monge–Ampère measure of ψ with respect to ω as follows from [Boucksom et al. 2010]; hence

$$\int_\Omega \rho\omega^n \leq \int_X \rho\omega^n \leq \int_X \omega^n.$$

Similarly we can write $(A\omega + i\partial\bar\partial\psi)^n = \rho_A\omega^n$ in $\Omega\setminus\{\psi=-\infty\}$, where $0 \leq \rho_A \in L^1(\Omega, dV)$.

Now, we carry out the computation in $\Omega\setminus\{\psi=-\infty\}$. For notational convenience we set $h := \sum_{j=1}^N |f_j|^2$, $\varphi := \log\sum_{j=1}^N |f_j|^2$ and we compute $i\partial\bar\partial\varphi$:

$$i\partial\bar\partial\varphi = \frac{\sum_{j=1}^N i\partial f_j \wedge \overline{\partial f_j}}{h} - \frac{i\big(\sum_{j=1}^N \bar f_j \partial f_j\big)\wedge\big(\sum_{j=1}^N f_j\overline{\partial f_j}\big)}{h^2}.$$

For each $1 \leq j < k \leq N$ we set $\alpha_{j,k} := f_j\partial f_k - f_k\partial f_j$. Then we obtain

$$i\partial\bar\partial\varphi = h^{-2}\sum_{j<k} i\alpha_{j,k}\wedge\bar\alpha_{j,k}. \tag{28}$$

Let $C>0$ be large enough such that $C^{-1}\beta \leq A\omega + i\partial\bar\partial g \leq C\beta$ in Ω, where β is the standard Kähler form in $\mathbb{C}^n$. For each $\ell = 0,\dots,n$, set $\gamma_l := (i\partial\bar\partial\varphi)^\ell \wedge \beta^{n-\ell}$. Then there exists a constant $B>1$ (depending on $c, C>0$) such that in $\Omega\setminus\{\psi=-\infty\}$ one has

$$\frac{1}{B}\sum_{\ell=0}^n \gamma_\ell = \frac{1}{B}\sum_{\ell=0}^n (i\partial\bar\partial\varphi)^\ell\wedge\beta^{n-\ell} \leq (A\omega + i\partial\bar\partial\psi)^n \leq B\sum_{p=0}^n (i\partial\bar\partial\varphi)^\ell\wedge\beta^{n-\ell} = B\sum_{\ell=0}^n \gamma_\ell. \tag{29}$$

By the definition of $\alpha_{j,k}$ it follows that the $(\ell,0)$-forms $\alpha_{j_1,k_1}\wedge\cdots\wedge\alpha_{j_\ell,k_\ell}$ are of the type $\sum F_k\,dz_{I_k}$, where $|I_k|=\ell$, and each F_k is holomorphic in Ω. By the above identity in (28), each γ_ℓ is the sum of (n,n)-forms of type $|F|^2h^{-2\ell}\beta^n$, where F is holomorphic in Ω. By the first estimate in (29) it follows that for each ℓ,

$$\int_\Omega |F|^2h^{-2\ell}\beta^n \le B\int_\Omega \rho_A\omega^n<\infty;$$

hence $|F|^2e^{-2\ell\log h}$ is integrable in Ω. From the resolution of Demailly's strong openness conjecture [2001] due to Guan and Zhou [2015] (see also [Hiep 2014] for an alternative proof) it follows that each $|F|^2h^{-2\ell}$ is in $L^p(U,dV)$ for some $p>1$ and a smaller neighborhood $U\subset\Omega$ of x. Finally, from the second estimate in (29) we see that $\omega^n_\psi/\omega^n\in L^p(U,dV)$, which what we wanted. □

5. Log-concavity of nonpluripolar products

Theorem 5.1. *Let $T_1,\dots,T_n$ be positive $(1,1)$-currents on a compact Kähler manifold X. Assume that each T_j has potential with small unbounded locus. Then*

$$\int_X\langle T_1\wedge\cdots\wedge T_n\rangle \ge \left(\int_X\langle T_1^n\rangle\right)^{1/n}\cdots\left(\int_X\langle T_n^n\rangle\right)^{1/n}.$$

Proof. We can assume that the classes of T_j are big and their masses are nonzero. Otherwise the right-hand side of the inequality to be proved is zero. Consider smooth closed real $(1,1)$-forms θ^j, and $u_j\in\operatorname{PSH}(X,\theta^j)$ with small unbounded locus such that $T_j=\theta^j_{u_j}$.

For each $j=1,\dots,n$, Theorem 4.28 ensures that there exists a normalizing constant $c_j>0$ and $\varphi_j\in\mathcal{E}(X,\theta^j,P_\theta[u_j])$ such that $(\theta^j_{\varphi_j})^n=c_j\omega^n$.

We can assume that $\int_X\omega^n=1$; thus we can write

$$c_j=\int_X(\theta^j_{\varphi_j})^n=\int_X(\theta^j_{P_\theta[u_j]})^n=\int_X(\theta^j_{u_j})^n=\int_X\langle T_j^n\rangle.$$

A combination of Proposition 2.1 and Theorem 2.3 then gives

$$\int_X\theta^1_{\varphi_1}\wedge\cdots\wedge\theta^n_{\varphi_n}=\int_X\theta^1_{P_\theta[u_1]}\wedge\cdots\wedge\theta^n_{P_\theta[u_n]}=\int_X\theta^1_{u_1}\wedge\cdots\wedge\theta^n_{u_n}=\int_X\langle T_1\wedge\cdots\wedge T_n\rangle.$$

An application of [Boucksom et al. 2010, Proposition 1.11] gives that $\theta^1_{\varphi_1}\wedge\cdots\wedge\theta^n_{\varphi_n}\ge c_1^{1/n}\cdots c_n^{1/n}\omega^n$. The result follows after we integrate this estimate. □

Acknowledgements

Darvas was partially supported by BSF grant 2012236 and NSF grant DMS–1610202. Di Nezza was supported by a Marie Skłodowska Curie individual fellowship 660940–KRF–CY (MSCA–IF).

We would like to thank Robert Berman, Mattias Jonsson and László Lempert for their insightful comments that improved the presentation of the paper. We warmly thank the referees for many suggestions improving the presentation of the paper.

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Received 27 Jun 2017. Revised 4 Feb 2018. Accepted 10 Apr 2018.

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APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 11 No. 8 2018