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SPECTRAL DISTRIBUTION OF THE FREE JACOBI PROCESS, REVISITED

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We obtain a description for the spectral distribution of the free Jacobi process for any initial pair of projections. This result relies on a study of the unitary operator $RU_tSU_t^*$, where R, S are two symmetries and $(U_t)_{t\geq 0}$ is a free unitary Brownian motion, freely independent from $\{R, S\}$. In particular, for nonnull traces of R and S, we prove that the spectral measure of $RU_tSU_t^*$ possesses two atoms at ± 1 and an L^∞-density on the unit circle $\mathbb{T}$ for every $t > 0$. Next, via a Szegő-type transformation of this law, we obtain a full description of the spectral distribution of $PU_tQU_t^*$ beyond the case where $\tau(P) = \tau(Q) = \frac{1}{2}$. Finally, we give some specializations for which these measures are explicitly computed.

1. Introduction

Let P, Q be two projections in a W^*-probability space $(\mathscr{A}, \tau)$ which are free with $\{U_t, U_t^*, t \geq 0\}$. The present paper is a companion to the series of papers [Collins and Kemp 2014; Demni 2008; Demni 2016; Demni and Hamdi 2018; Demni et al. 2012; Demni and Hmidi 2014] devoted to the study of the spectral distribution, hereafter μ_t, of the self-adjoint-valued process $(X_t := PU_tQU_t^*P)_{t\geq 0}$. Viewed in the compressed algebra $(P\mathscr{A}P, \tau/\tau(P))$, X_t coincides with the so-called free Jacobi process with parameter $(\tau(P)/\tau(Q), \tau(Q))$, introduced by Demni [2008] via free stochastic calculus, as a solution to a free SDE there. Properties of its measure play important roles in free entropy and free information theory; see, e.g., [Hamdi 2017; 2018; Hiai and Ueda 2009; Izumi and Ueda 2015; Voiculescu 1999]. Furthermore, μ_t completely determines the structure of the von Neumann algebra generated by P and $U_tQU_t^*$ for any $t \geq 0$, see, e.g., [Hiai and Ueda 2009; Raeburn and Sinclair 1989], yielding a continuous interpolation from the law of PQP (when $t = 0$) to the free multiplicative convolution of the spectral measures of P and Q separately (when t tends to infinity). Indeed, the pair $(P, U_tQU_t^*)$ tends towards (P, UQU^*) as $t \to \infty$, where U is a Haar unitary free from $\{P, Q\}$. The two projections P and UQU^* are therefore free, see [Nica and Speicher 2006], and hence $\mu_{PUQU^*P} = \mu_P \boxtimes \mu_{UQU^*} = \mu_P \boxtimes \mu_Q$. The Lebesgue decomposition of the last term may be found in [Voiculescu et al. 1992, Example 3.6.7]. More generally, the operators P and $U_tQU_t^*$ are not free for finite t and the process $t \mapsto (P, U_tQU_t^*)$ is known as the free liberation of the pair (P, Q); see [Voiculescu 1999]. When both projections coincide, the series of papers [Demni 2016; Demni and Hamdi 2018; Demni et al. 2012; Demni and Hmidi 2014] aims to determine μ_t for any $t > 0$. In particular, when $P = Q$ and $\tau(P) = \frac{1}{2}$, Demni, Hmidi and the author proved in [Demni et al. 2012, Corollary 3.3] that the measure μ_t possesses a continuous density on

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(0, 1) for $t > 0$ which fits that of the random variable $(I + U_{2t} + (I + U_{2t})^*)/4$. Collins and Kemp [2014] extended this result to the case of two projections P, Q with traces $\frac{1}{2}$. Afterwards this result was partially extended in [Izumi and Ueda 2015] to arbitrary traces. In Proposition 3.1 of that paper, they proved

$$\mu_t = (1 - \min\{\tau(P), \tau(Q)\})\delta_0 + \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_1 + \gamma_t,$$

where γ_t is a positive measure with no atom on (0, 1) for every $t > 0$. In Proposition 3.3 of the same paper, they showed that when $\tau(P) = \tau(Q) = \frac{1}{2}$, this measure coincides with the Szegő transformation of the distribution of UU_t, where U is a unitary random variable determined by the law of PQP. Collins and Kemp [2014, Lemmas 3.2 and 3.6] studied the support of the measure γ_t, for arbitrary traces, and the way in which the edges of this support are propagated, but they were still not able to prove the continuity of γ_t.

The main result proved in this paper is a complete analysis of the spectral distribution of the unitary operator $RU_tSU_t^*$ (hereafter ν_t) for any symmetries $R, S \in \mathscr{A}$ which are free with $\{U_t, U_t^*\}$. In particular, we prove that the measure

$$\nu_t - \tfrac{1}{2}|\tau(R) - \tau(S)|\delta_\pi - \tfrac{1}{2}|\tau(R) + \tau(S)|\delta_0$$

possesses a continuous density κ_t on $\mathbb{T} = (-\pi, \pi]$. Using the relationship between μ_t and ν_t, when $\{P, Q\}$ and $\{R, S\}$ are associated, see [Hamdi 2017, Theorem 4.3], we deduce the regularity of μ_t for any initial projections. In particular, we prove that the measure γ_t possesses a continuous density on [0, 1]:

Theorem 1.1. *Let P, Q be orthogonal projections and U_t a free unitary Brownian motion, freely independent from P, Q. For every $t > 0$, the spectral distribution μ_t of the self adjoint operator $PU_tQU_t^*P$ is given by*

$$\mu_t = (1 - \min\{\tau(P), \tau(Q)\})\delta_0 + \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_1 + \frac{\kappa_t(2\arccos(\sqrt{x}))}{2\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)\,dx.$$

The paper ends with a striking observation on the spectral distribution of $RU_tSU_t^*$ at finite time t when the initial symmetries building it are centered and independent with respected to classical, free, monotone and boolean convolutions. In this respect, we notice that in the case of free independence, ν_t is stationary for all traces of the symmetries, and in the rest of cases, its given by a dilation of the law of U_t for centered symmetries. The result is as follows.

Theorem 1.2. *Let λ_t be the probability distribution of the free unitary Brownian motion U_t and $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$ (considered as a law on $\mathbb{T}$). We denote respectively by $\boxtimes$, $*$, $\uplus$ and $\rhd$ the free, classical, boolean and monotone multiplicative convolutions. Then, for all $t \geq 0$:*

(1) *The measure $(\mu \boxtimes \mu) \boxtimes \lambda_t$ coincides with $\mu \boxtimes \mu$.*

(2) *The push-forward of $(\mu * \mu) \boxtimes \lambda_t$ by the map $z \mapsto z^2$ coincides with the law of U_{2t}.*

(3) *The push-forward of $(\mu \uplus \mu) \boxtimes \lambda_t$ by the map $z \mapsto z^3$ coincides with the law of U_{3t}.*

(4) *The push-forward of $(\mu \rhd \mu) \boxtimes \lambda_t$ by the map $z \mapsto z^4$ coincides with the law of U_{4t}.*

The paper is organized as follows. For sake of completeness, we recall in the next section some preliminaries which gather useful information about the Herglotz transform of probability measures on the unit circle, and the spectral distribution of the free unitary Brownian motion. In Section 3, we fix the basic ideas and notation for the rest of the work presented. In Section 4, we describe the spectral measure ν_t and prove our main result. In the last section, we present explicit computations of the spectral measure ν_t at finite time t when the initial operators are assumed to be centered and classically boolean or monotone independent.

2. Preliminaries

The Herglotz transform. Let $\mathscr{M}_{\mathbb{T}}$ denotes the set of probability measures on the unit circle $\mathbb{T}$. The normalized Lebesgue measure on $\mathbb{T}$ will be denoted by m. The Herglotz transform H_μ of a measure $\mu \in \mathscr{M}_{\mathbb{T}}$ is the analytic function in the unit disc $\mathbb{D}$ defined by the formula

$$H_\mu(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta).$$

This function is related to the moment-generating function of the measure μ

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{z}{\zeta - z}\, d\mu(\zeta), \qquad z \in \mathbb{D},$$

by the simple formula $H_\mu(z) = 1 + 2\psi_\mu(z)$. Since any distribution on the unit circle is uniquely determined by its moments, we deduce that H_μ uniquely determines μ. One of the important applications of H is given in the following result; see, e.g., [Cima et al. 2006, Theorem 1.8.9]:

Theorem 2.1 (Herglotz). *The Herglotz transform sets up a bijection between analytic functions H on $\mathbb{D}$ with $\Re H \geq 0$ and $H(0) > 0$ and the nonzero measures $\mu \in \mathscr{M}_{\mathbb{T}}$.*

For $0 < p < \infty$, let $H^p(\mathbb{D})$ be the space of analytic functions f on $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p\, d\zeta < \infty.$$

For $p = \infty$, let $H^\infty(\mathbb{D})$ denote the Hardy space consisting of all bounded analytic functions on $\mathbb{D}$ with the sup-norm. Let $L^p(\mathbb{T})$ denote the Lebesgue spaces on the circle $\mathbb{T}$ with respect to the normalized Lebesgue measure. The following result proves the existence of a boundary function for all $f \in H^p(\mathbb{D})$.

Theorem 2.2 [Cima et al. 2006, Theorem 1.9.4]. *Let $0 < p \leq \infty$ and $f \in H^p(\mathbb{D})$. Then the boundary function $\tilde{f}(\zeta)$ exists for m-almost all ζ in $\mathbb{T}$ and belongs to $L^p(\mathbb{T})$. Furthermore, the norms of f in $H^p(\mathbb{D})$ and of $\tilde{f}(\zeta)$ in $L^p(\mathbb{T})$ coincide.*

We know, see, e.g., [Cima et al. 2006, Lemma 2.1.11], that $H_\mu \in H^p(\mathbb{D})$ for all $0 < p < 1$; thus $\widetilde{H}_\mu(\zeta)$ exists for m-almost all ζ in $\mathbb{T}$. The density of μ can be recovered then from the boundary values of $\Re H_\mu$ by Fatou's theorem [Cima et al. 2006, Theorem 1.8.6] since $\Re \widetilde{H}_\mu = d\mu/dm$ m-a.e. Note that the atoms of $\mu \in \mathscr{M}_{\mathbb{T}}$ can also be recovered from H_μ by Lebesgue's dominated convergence theorem via

$$\lim_{r \to 1^-} (1 - r) H_\mu(r\zeta) = 2\mu\{\zeta\} \quad \text{for all } \zeta \in \mathbb{T}.$$

Spectral distribution of the free unitary Brownian motion. For $\mu \in \mathscr{M}_{\mathbb{T}}$, let ψ_μ denote its moment-generating function and χ_μ the function $\psi_\mu/(1+\psi_\mu)$. If μ has nonzero mean, we denote by χ_μ^{-1} the inverse function of χ_μ in some neighborhood of zero. In this case the Σ-transform of μ is defined by $\Sigma_\mu(z) = (1/z)\chi_\mu^{-1}(z)$. The spectral distribution λ_t of the free unitary Brownian motion was introduced by Biane [1997a] as the unique probability measure on $\mathbb{T}$ such that its Σ-transform is given by

$$\Sigma_{\lambda_t}(z) = \exp\left(\frac{t}{2}\frac{1+z}{1-z}\right).$$

It is the multiplicative analog of the semicircular distribution. Its moments are the large-size limits of observables of the free Brownian motion (of dimension d) $(U_t^{(d)})_{t\geq 0}$ on the unitary group $\mathscr{U}(d)$:

$$\lim_{d\to\infty}\frac{1}{d}\mathbb{E}(\mathrm{tr}[U_{t/d}^{(d)}]^k) = \int_{\mathbb{T}} \zeta^k \, d\lambda_t(\zeta), \quad k \geq 0.$$

This result was proved independently by Biane [1997a] and Rains [1997], who explicitly calculated these moments:

$$\tau(U_t^k) = e^{-kt/2}\sum_{j=0}^{k-1}\frac{(-t)^j}{j!}\binom{k}{j+1}k^{j-1}, \quad k \geq 0. \tag{2-1}$$

The equality (2-1) can be transformed into the PDE

$$\partial_t H + zH\,\partial_z H = 0, \tag{2-2}$$

with the initial condition $H(0,z) = (1+z)/(1-z)$ for the Herglotz transform $H_{\lambda_{2t}}(z)$; see, e.g., the proof of [Izumi and Ueda 2015, Proposition 3.3]. The measure λ_t is described in [Biane 1997b] from the boundary behavior of the inverse function of $H_{\lambda_t}(z)$ as follows.

Theorem 2.3 [Biane 1997b]. *For every $t > 0$, the measure λ_t has a continuous density ρ_t with respect to the normalized Lebesgue measure on $\mathbb{T}$. Its support is the connected arc $\{e^{i\theta} : |\theta| \leq g(t)\}$ with*

$$g(t) := \tfrac{1}{2}\sqrt{t(4-t)} + \arccos\left(1 - \tfrac{1}{2}t\right)$$

for $t \in [0,4]$, and the whole circle for $t > 4$. The density ρ_t is determined by $\Re h_t(e^{i\theta})$, where $z = h_t(e^{i\theta})$ is the unique solution (*with positive real part*) *to*

$$\frac{z-1}{z+1}e^{zt/2} = e^{i\theta}.$$

3. Notation

We use here the same symbols as in [Hamdi 2017; 2018]. To a given pair of projections P, Q in $\mathscr{A}$ that are independent of $(U_t)_{t\geq 0}$ we associate the symmetries $R = 2P - I$ and $S = 2Q - I$. Set $\alpha = \tau(R)$ and $\beta = \tau(S)$. We sometimes use the notation $a = |\alpha - \beta|/2$ and $b = |\alpha + \beta|/2$ for simplicity. Keep the symbols μ_t and ν_t above. The unit circle is identified with $(-\pi, \pi]$ by $e^{i\theta}$. According to [Hamdi 2017, Section 3], the measure ν_t is connected to μ_t by the formula

$$\nu_t = 2\hat{\mu}_t - \tfrac{1}{2}(2-\alpha-\beta)\delta_\pi - \tfrac{1}{2}(\alpha+\beta)\delta_0, \tag{3-1}$$

where

$$\hat{\mu}_t := \tfrac{1}{2}(\tilde{\mu}_t + (\tilde{\mu}_t|_{(0,\pi)}) \circ j^{-1}) \tag{3-2}$$

is the symmetrization on $(-\pi, \pi)$, with the mapping $j : \theta \in (0, \pi) \mapsto -\theta \in (-\pi, 0)$, of the positive measure $\tilde{\mu}_t(d\theta)$ on $[0, \pi]$ obtained from $\mu_t(dx)$ via the variable change $x = \cos^2(\theta/2)$. Equivalently, we obtain the following relationship between the Herglotz transforms H_{μ_t} and H_{ν_t}:

$$H_{\nu_t}(z) = \frac{z-1}{z+1} H_{\mu_t}\left(\frac{4z}{(1+z)^2}\right) - 2(\alpha+\beta)\frac{z}{z^2-1}; \tag{3-3}$$

see [Hamdi 2017, Corollary 4.2]. The function $H_{\nu_t}(z)$, which we shall denote by $H(t, z)$, is analytic in both variables $z \in \mathbb{D}$ and $t > 0$, see [Collins and Kemp 2014, Theorem 1.4], and solves the PDE

$$\partial_t H + zH\,\partial_z H = \frac{2z(\alpha z^2 + 2\beta z + \alpha)(\beta z^2 + 2\alpha z + \beta)}{(1-z^2)^3}, \tag{3-4}$$

see [Hamdi 2017, Proposition 2.3]. Let

$$K(t, z) := \sqrt{H(t, z)^2 - \left(a\frac{1-z}{1+z} + b\frac{1+z}{1-z}\right)^2}. \tag{3-5}$$

The PDE (3-4) is then transformed into

$$\partial_t K + zH(t, z)\,\partial_z K = 0.$$

Note that steady state solution $K(\infty, z)$ is the constant $\sqrt{1-(a+b)^2}$; see [Hamdi 2017, Remark 3.3]. The ordinary differential equations (ODEs for short) of the characteristic curves associated with this PDE are

$$\begin{cases} \partial_t \phi_t(z) = \phi_t(z) H(t, \phi_t(z)), & \phi_0(z) = z, \\ \partial_t[K(t, \phi_t(z))] = 0. \end{cases} \tag{3-6}$$

The second ODE of (3-6) implies that $K(t, \phi_t(z)) = K(0, z)$, while the first one is nothing but the radial Loewner ODE, see [Lawler 2005, Theorem 4.14], which defines a unique family of conformal transformations ϕ_t from some region $\Omega_t \subset \mathbb{D}$ onto $\mathbb{D}$ with $\phi_t(0) = 0$ and $\partial_z \phi_t(0) = e^t$. Moreover, from [Lawler 2005, Remark 4.15], ϕ_t is invertible from Ω_t onto $\mathbb{D}$ and it has a continuous extension to $\mathbb{T} \cap \bar{\Omega}_t$ by [Hamdi 2018, Proposition 2.1]. Integrating the first ODE in (3-6), we get

$$\phi_t(z) = z \exp\left(\int_0^t H(s, \phi_s(z))\,ds\right).$$

Let us define

$$h_t(r, \theta) = 1 - \int_0^t \frac{1-|\phi_s(re^{i\theta})|^2}{-\ln r} \int_{\mathbb{T}} \frac{1}{|\xi - \phi_s(re^{i\theta})|^2}\,d\nu_s(\xi)\,ds,$$

so that

$$\ln|\phi_t(re^{i\theta})| = \ln r + \Re\int_0^t H(s, \phi_s(re^{i\theta}))\,ds = (\ln r) h_t(r, \theta). \tag{3-7}$$

Define $R_t : [-\pi, \pi] \to [0, 1]$ as

$$R_t(\theta) = \sup\{r \in (0, 1) : h_t(r, \theta) > 0\},$$

and let

$$I_t = \{\theta \in [-\pi, \pi] : h_t(\theta) < 0\},$$

where $h_t(\theta) = \lim_{r\to 1^-} h_t(r, \theta) \in \mathbb{R} \cup \{-\infty\}$; see the fact given under Lemma 3.2 in [Hamdi 2018]. The next result gives a description of Ω_t and its boundary.

Proposition 3.1 [Hamdi 2018, Proposition 3.3]. *For any* $t > 0$, *we have*:

(1) $\Omega_t = \{re^{i\theta} : h_t(r, e^{i\theta}) > 0\}$.

(2) $\partial\Omega_t \cap \mathbb{D} = \{re^{i\theta} : h_t(r, e^{i\theta}) = 0$ *and* $\theta \in I_t\}$.

(3) $\partial\Omega_t \cap \mathbb{T} = \{e^{i\theta} : h_t(r, e^{i\theta}) = 0$ *and* $\theta \in [-\pi, \pi] \setminus I_t\}$.

In closing, we recall the following result which will be of use later on; see the proof of Theorem 1.1 in [Hamdi 2018].

Lemma 3.2 [Hamdi 2018]. *For every* $t > 0$, *the function* $K(t, \cdot)$ *has a continuous extension to the unit circle* $\mathbb{T}$.

4. Analysis of spectral distributions of $RU_tSU_t^*$

In this section, we shall prove Theorem 1.1. To this end, we start by giving a description of the spectral measure ν_t of $RU_tSU_t^*$ for any $t > 0$, and deriving a formula for its density. We notice that from the asymptotic freeness of R and $U_tSU_t^*$, the measure ν_t converges weakly as $t \to \infty$, see [Hamdi 2017, Proposition 2.6], to

$$\nu_\infty = a\delta_\pi + b\delta_0 + \frac{\sqrt{-(\cos\theta - r_+)(\cos\theta - r_-)}}{2\pi|\sin\theta|}\mathbf{1}_{(\theta_-,\theta_+)\cup(-\theta_+,-\theta_-)}\,d\theta, \tag{4-1}$$

with $r_\pm = -\alpha\beta \pm \sqrt{(1-\alpha^2)(1-\beta^2)}$ and $\theta_\pm = \arccos r_\pm$. The following theorem asserts that an analogous result holds for finite t.

Theorem 4.1. *For every* $t > 0$, *the measure* $\nu_t - a\delta_\pi - b\delta_0$ *is absolutely continuous with respect to the normalized Lebesgue measure on* $\mathbb{T} = (-\pi, \pi]$. *Moreover, its density* κ_t *at the point* $e^{i\theta}$ *is equal to the real part of*

$$\sqrt{[K(t, e^{i\theta})]^2 + (a+b)^2 - 1 - \frac{(\cos\theta - r_+)(\cos\theta - r_-)}{\sin^2\theta}}.$$

Proof. Define the function

$$L(t, z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z}(\nu_t - a\delta_\pi - b\delta_0)(d\theta) = H(t, z) - a\frac{1-z}{1+z} - b\frac{1+z}{1-z}.$$

The real part of this function is nothing but the Poisson integral of the measure $\nu_t - a\delta_\pi - b\delta_0$. Using (3-5) and multiplying by the conjugate, we get

$$L(t,z) = \frac{K(t,z)^2}{\sqrt{K(t,z)^2 + \left(a\frac{1-z}{1+z} + b\frac{1+z}{1-z}\right)^2} + a\frac{1-z}{1+z} + b\frac{1+z}{1-z}}$$
$$= \frac{(1-z^2)K(t,z)^2}{\sqrt{[(1-z^2)K(t,z)]^2 + [a(1-z)^2 + b(1+z)^2]^2} + a(1-z)^2 + b(1+z)^2}.$$

Note that $K(t,z)$ extends continuously to $\mathbb{T}$ by Lemma 3.2. The denominator of the above expression does not vanish on the closed unit disc and

$$z \mapsto (1-z^2)^2 K(t,z)^2 + [a(1-z)^2 + b(1+z)^2]^2 = (1-z^2)H(t,z)^2$$

does not take negative values. These together imply that $L(t,z)$ has a continuous extension on the boundary $\mathbb{T}$. Hence, by uniqueness of the Herglotz representation (see Theorem 2.1), the measure $\nu_t - a\delta_\pi - b\delta_0$ is absolutely continuous with respect to the Haar measure in $\mathbb{T}$ and its density is given by

$$\Re\left[H(t,e^{i\theta}) - a\frac{1-e^{i\theta}}{1+e^{i\theta}} - b\frac{1+e^{i\theta}}{1-e^{i\theta}}\right] = \Re\sqrt{[K(t,e^{i\theta})]^2 + \left[a\frac{1-e^{i\theta}}{1+e^{i\theta}} - b\frac{1+e^{i\theta}}{1-e^{i\theta}}\right]^2}$$
$$= \Re\sqrt{[K(t,e^{i\theta})]^2 - [a\tan(\theta/2) - b\cot(\theta/2)]^2}.$$

To complete the proof, we need only show that

$$[a\tan(\theta/2) - b\cot(\theta/2)]^2 = 1 - (a+b)^2 + \frac{(\cos\theta - r_+)(\cos\theta - r_-)}{\sin^2\theta}$$

or equivalently that

$$(1-a^2-b^2)\sin^2\theta - a^2\sin^2\theta\tan^2(\theta/2) - b^2\sin^2\theta\cot^2(\theta/2) = -(\cos\theta - r_+)(\cos\theta - r_-).$$

Working from the left-hand side and using the identities

$$\sin^2\theta = 1 - \cos^2\theta, \quad \sin^2\theta\tan^2(\theta/2) = (1-\cos\theta)^2, \quad \sin^2\theta\cot^2(\theta/2) = (1+\cos\theta)^2,$$

we get

$$(1-a^2-b^2)(1-\cos^2\theta) - a^2(1-\cos\theta)^2 - b^2(1+\cos\theta)^2.$$

Rearranging these terms, we obtain

$$-\cos^2\theta + 2(a^2-b^2)\cos\theta - 2(a^2+b^2) + 1.$$

So, by substituting the equalities $\alpha\beta = b^2 - a^2$ and $\alpha^2 + \beta^2 = 2(a^2+b^2)$, we obtain the required formula:

$$-\cos^2\theta - 2\alpha\beta\cos\theta + 1 - \alpha^2 - \beta^2 = -(\cos\theta - r_+)(\cos\theta - r_-). \qquad \square$$

Remark 4.2. We can prove directly that κ_t is an L^∞-density. In fact, by (3-5), we have

$$K(t,z)^2 = H(t,z)^2 - \left(a\frac{1-z}{1+z} + b\frac{1+z}{1-z}\right)^2 = L(t,z)\left(L(t,z) + 2a\frac{1-z}{1+z} + 2b\frac{1+z}{1-z}\right).$$

Then

$$(\Re L(t,z))^2 \le \Re L(t,z)\Re\left(L(t,z) + 2a\frac{1-z}{1+z} + 2b\frac{1+z}{1-z}\right) \le |K(t,z)^2|.$$

But, the function $K(t,z)$ is analytic in $\mathbb{D}$ and extends continuously to $\mathbb{T}$. It becomes then of Hardy class $H^\infty(\mathbb{D})$, and hence the density of $\nu_t - a\delta_\pi - b\delta_0$ belongs to $L^\infty(\mathbb{T})$ by [Koosis 1998, Theorem on p. 15].

Proposition 4.3. *The support of ν_t is a subset of $\{\phi_t(R_t(\theta)e^{i\theta}) : \theta \in I_t\}$.*

Proof. By (3-7), we have

$$\int_0^t \Re H(s, \phi_s(R_t(\theta)e^{i\theta}))\, ds = -\ln R_t(\theta),$$

where we used the fact that $\ln|\phi_t(R_t(\theta)e^{i\theta})| = 0$ due to the equality $|\phi_t(R_t(\theta)e^{i\theta})| = 1$. Then, by continuity of $s \mapsto \Re H(s, \phi_s(R_t(\theta)e^{i\theta}))$ on $[0,t]$, we deduce that the assertion $\Re H(t, \phi_t(R_t(\theta)e^{i\theta})) > 0$ yields $R_t(\theta) \neq 1$. Finally, by the definition of $R_t(\theta)$ and I_t, we have

$$\{\theta : R_t(\theta) \neq 1\} = \{\theta : \exists\, r_0 \in (0,1),\ h_t(r_0, e^{i\theta}) = 0\} = \{\theta : h_t(\theta) < 0\} = I_t. \qquad \square$$

We now proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. By (3-1), we have

$$\nu_t - a\delta_\pi - b\delta_0 = 2\big[\hat{\mu}_t - (1 - \min\{\tau(P), \tau(Q)\})\delta_\pi - \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_0\big].$$

This measure is absolutely continuous with respect to the normalized Lebesgue measure $d\theta/(2\pi)$ on $\mathbb{T} = (-\pi, \pi]$, by Theorem 4.1, and its density is given by the function κ_t. Hence, (3-2) implies

$$\big(\tilde{\mu}_t - (1 - \min\{\tau(P), \tau(Q)\})\delta_\pi - \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_0\big)(d\theta) = \kappa_t(\theta)\frac{d\theta}{2\pi}, \qquad \theta \in [0, \pi],$$

and so the desired assertion holds via the variable change $\theta = 2\arccos(\sqrt{x})$. $\square$

Remark 4.4. It is worth noting that the spectral distribution ν_t is stationary for all traces of the symmetries, when the initial operators R and S are free. Actually, by Proposition 2.5 in [Hamdi 2017], we have

$$H(0,z) = \sqrt{1 + 4z\left(\frac{b^2}{(1-z)^2} - \frac{a^2}{(1+z)^2}\right)},$$

so that

$$K(0,z) = \sqrt{H(0,z)^2 - \left(a\frac{1-z}{1+z} + b\frac{1+z}{1-z}\right)^2} = \sqrt{1 - (a+b)^2}.$$

Hence, for every $z \in \mathbb{D}$ and $t \ge 0$, we have $K(t,z) = K(0, \phi_t^{-1}(z)) = \sqrt{1-(a+b)^2}$, and therefore ν_t coincides with the measure ν_∞.

The above fact can be explained directly by use of the sequence of moments

$$m_n(t) := \tau[(PU_tQU_t^*P)^n], \quad n \geq 1.$$

In fact, we can prove by induction on n that $m_n(t)$ becomes stationary when P and Q are free. Recall from [Demni et al. 2012] that $m_n(t)$ satisfy the infinite system of ODEs

$$\partial_t m_1(t) = -m_1(t) + \tau[P]\tau[Q], \tag{4-2}$$

$$\partial_t m_n(t) = -nm_n(t) + n\sum_{k=1}^{n-1} m_{n-k}(t)(m_{k-1}(t) - m_k(t)), \quad n \geq 2, \tag{4-3}$$

with $m_0(t) = \tau[P] + \tau[Q]$. When $n = 1$, (4-2) can be solved explicitly and gives $m_1(t) = \tau[P]\tau[Q] + e^{-t}(m_1(0) - \tau[P]\tau[Q])$. Since $m_1(0) = \tau[PQ] = \tau[P]\tau[Q]$ by freeness, we get $m_1(t) = m_1(0)$. For $n \geq 2$, we note that the moments

$$c_n := m_n(0) = \tau[(PQ)^n]$$

satisfy

$$c_n = \sum_{k=1}^{n-1} c_{n-k}(c_{k-1} - c_k).$$

Assume that $m_k(t) = c_k$ holds up to level $n - 1$. Then, the ODE (4-3) can be written in the form

$$\partial_t m_n(t) = -nm_n(t) + nc_n,$$

with solution the constant c_n. Thus, μ_t (and therefore ν_t) is stationary.

5. Special cases

We present here some specializations for which the measure ν_t (and hence μ_t) is explicitly determined.

Centered initial operators. That is, $\tau(R) = \tau(S) = 0$ or $a = b = 0$. In this case, the PDE (3-4) can be rewritten as

$$\partial_t H + zH\,\partial_z H = 0,$$

and the measure ν_t becomes identical to the probability distribution of UU_{2t}, where U is a free unitary whose distribution is ν_0; see [Izumi and Ueda 2015, Proposition 3.3] or [Hamdi 2017, Remark 4.7]. Hence, the measure ν_t is given by the multiplicative free convolution $\nu_0 \boxtimes \lambda_{2t}$, studied in [Zhong 2015]. The density of this measure and its support are explicitly computed in Theorem 3.8 and Corollary 3.9 of that paper. In particular, when ν_0 is a Dirac mass at 1 (on the unit circle), the Herglotz transforms $H(t, z)$ of ν_t satisfy the PDE

$$\partial_t H + zH\,\partial_z H = 0, \quad H(0, z) = \frac{1+z}{1-z}.$$

Then it follows from the uniqueness of the solution of (2-2) that $H(t, z) = H_{\lambda_{2t}}(z)$, and by uniqueness of the Herglotz representation, ν_t coincides with the law λ_{2t} of U_{2t}. Hence, by Theorem 2.3 the density of ν_t is given by the formula $\kappa_t(\omega) = \rho_{2t}(\omega)$ and the support is the full unit circle for $t > 2$ and the set $\{e^{i\theta} : |\theta| < g(2t)\}$ for $t \in [0, 2]$.

In the rest of the paper, we illustrate how the family of measures $(\nu_t)_{t \geq 0}$ provides a continuous interpolation between freeness and different type of independence.

Classically independent initial operators. In this case, the measure ν_t is considered as a t-free convolution which interpolates between classical independence and free independence; see [Benaych-Georges and Lévy 2011]. Let R, S be two independent symmetries. From the facts given above Lemma 5.4 in [Hamdi 2017], we have

$$H(0, z) = 1 + 2 \sum_{n \geq 1} \tau(R^n)\tau(S^n)z^n = \frac{1 + z^2 + 2z\tau(R)\tau(S)}{1 - z^2}.$$

In particular, when $\tau(R) = \tau(S) = 0$, the function $H(t, z)$ satisfies the PDE

$$\partial_t H + zH\,\partial_z H = 0, \qquad H(0, z) = \frac{1 + z^2}{1 - z^2},$$

and hence, by (2-2), it coincides with $H_{\lambda_{4t}}(z^2)$. We retrieve then the result obtained in [Benaych-Georges and Lévy 2011, Theorem 3.6]: for any $t \geq 0$, the push-forward of ν_t by the map $z \mapsto z^2$ coincides with the law of U_{4t}. In particular, the density of ν_t is given by $\kappa_t(\omega) = \rho_{4t}(\omega^2)$ for any ω in the unit circle and the support is the full unit circle for $t > 1$ and the set $\{e^{i\theta} : |\theta| < g(4t)/2\}$ for $t \in [0, 1]$.

Boolean independent initial operators. To a given probability measure μ on the unit circle, we keep the same notation ψ_μ, H_μ and χ_μ as in Section 2. Let $\mu_1, \mu_2 \in \mathscr{M}_{\mathbb{T}}$ and set $F_\mu(z) = (1/z)\chi_\mu(z)$. Then the multiplicative boolean convolution $\mu = \mu_1 \mathbin{\underline{\circledtimes}} \mu_2$ is uniquely determined by

$$F_\mu(z) = F_{\mu_1}(z) F_{\mu_2}(z);$$

see [Hamdi 2015; Franz 2008] for more details. Then, for boolean independent symmetries R, S with law $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$, we have

$$\psi_\mu(z) = \frac{z^2}{1 - z^2}, \qquad \chi_\mu(z) = z^2, \qquad F_\mu(z) = z$$

and therefore $F_{\mu \mathbin{\underline{\circledtimes}} \mu}(z) = F_\mu(z)^2 = z^2$. It follows that

$$\psi_{\mu \mathbin{\underline{\circledtimes}} \mu}(z) = \frac{z^3}{1 - z^3} \quad \text{and} \quad H_{\mu \mathbin{\underline{\circledtimes}} \mu}(z) = \frac{1 + z^3}{1 - z^3}.$$

Hence, by (2-2) the Herglotz transform $H(t, z)$ of ν_t and $H_{\lambda_{6t}}(z^3)$ solve the same PDE with the initial condition $H(0, z) = (1 + z^3)/(1 - z^3)$. By uniqueness, it follows that the push-forward of ν_t by the map $z \mapsto z^3$ coincides with the law of U_{6t} for any $t \geq 0$. In particular, we have $\kappa_t(\omega) = \rho_{6t}(\omega^3)$ for any ω in the unit circle and ν_t is supported in the full unit circle for $t > \frac{2}{3}$ and the set $\{e^{i\theta} : |\theta| < g(6t)/3\}$ for $t \in \left[0, \frac{2}{3}\right]$.

Monotone independent initial operators. For $\mu_1, \mu_2 \in \mathscr{M}_{\mathbb{T}}$, the multiplicative monotone convolution $\mu = \mu_1 \triangleright \mu_2$ is uniquely determined by

$$\chi_\mu(z) = \chi_{\mu_1}(\chi_{\mu_2}(z));$$

see [Hamdi 2015; Franz 2006] for more details. Here, we shall compute the measure ν_t for monotone independent symmetries R, S with law $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$. As usual, we have

$$\psi_\mu(z) = \frac{z^2}{1-z^2}, \qquad \chi_\mu(z) = z^2,$$

and then $\chi_{\mu \triangleright \mu}(z) = \chi_\mu(\chi_\mu(z)) = z^4$. Hence,

$$\psi_{\mu \triangleright \mu}(z) = \frac{z^4}{1-z^4} \quad \text{and} \quad H_{\mu \triangleright \mu}(z) = \frac{1+z^4}{1-z^4}.$$

It follows that $H(t, z) = H_{\lambda_{8t}}(z^4)$ by uniqueness. Thus, the push-forward of ν_t by the map $z \mapsto z^4$ coincides with the law of U_{8t} for any $t \geq 0$. In particular, we have $\kappa_t(\omega) = \rho_{8t}(\omega^4)$ for any ω in the unit circle and ν_t is supported in the full unit circle for $t > \frac{1}{2}$ and the set $\{e^{i\theta} : |\theta| < g(8t)/4\}$ for $t \in \left[0, \frac{1}{2}\right]$.

Finally, we recall (see the first subsection above) that $\nu_t = \nu_0 \boxtimes \lambda_{2t}$ for centered initial operators R, S (i.e., $\tau(R) = \tau(S) = 0$). Hence, the discussions so far can be summarized in Theorem 1.2.

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