ANALYSIS & PDE

Volume 12 No. 1 2019

msp

Analysis & PDE

msp.org/apde

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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ANALYSIS AND PDE
Vol. 12, No. 1, 2019
dx.doi.org/10.2140/apde.2019.12.1

BOUNDARY BEHAVIOR OF SOLUTIONS TO THE PARABOLIC p-LAPLACE EQUATION

BENNY AVELIN, TUOMO KUUSI AND KAJ NYSTRÖM

We establish boundary estimates for nonnegative solutions to the p-parabolic equation in the degenerate range $p > 2$. Our main results include new parabolic intrinsic Harnack chains in cylindrical NTA domains together with sharp boundary decay estimates. If the underlying domain is $C^{1,1}$-regular, we establish a relatively complete theory of the boundary behavior, including boundary Harnack principles and Hölder continuity of the ratios of two solutions, as well as fine properties of associated boundary measures. There is an intrinsic waiting-time phenomenon present which plays a fundamental role throughout the paper. In particular, conditions on these waiting times rule out well-known examples of explicit solutions violating the boundary Harnack principle.

1. Introduction and results

This paper is devoted to a study of the boundary behavior of nonnegative solutions to the p-parabolic equation, in the degenerate range $p > 2$. We restrict the analysis to space-time cylinders $\Omega_T = \Omega \times (0, T)$, $T > 0$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, i.e., an open and connected set. Given p, $1 < p < \infty$, fixed, recall that the p-parabolic equation is

$$\partial_t u - \Delta_p u := \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0. \tag{1-1}$$

In the special case $p = 2$, the p-parabolic equation coincides with the heat equation, and in this case we refer to [Kemper 1972; Salsa 1981], and also [Fabes et al. 1984; 1986; 1999; Fabes and Safonov 1997; Garofalo 1984; Nyström 1997] concerning the boundary behavior of nonnegative solutions. Key results established in these works, in the context of Lipschitz-cylinders Ω_T, include Carleson-type estimates, the relation between the associated parabolic measure and the Green's function, the backward-in-time Harnack inequality, boundary Harnack principles (local and global) and Hölder continuity up to the boundary of quotients of nonnegative solutions vanishing on the lateral boundary.

On the contrary for $p \neq 2$, $1 < p < \infty$, much less is known concerning these problems and we refer the reader to [Avelin 2016; Avelin et al. 2016; Kuusi et al. 2014] for accounts of the current literature. For a relatively complete picture in the case of nonlinear parabolic operators with linear growth, we refer to [Nyström et al. 2015]. However, it is also important to mention that there is interesting and related recent literature devoted to the asymptotic and pointwise behavior of solutions to nonlinear diffusion

MSC2010: primary 35K20; secondary 35K65, 35B65.
Keywords: p-parabolic equation, degenerate, intrinsic geometry, waiting time phenomenon, intrinsic Harnack chains, boundary Harnack principle, p-stability.

equations on bounded domains; see [Stan and Vázquez 2013], and also [Bonforte and Vázquez 2015] for the porous medium-type equations.

Considering nonnegative solutions to the p-parabolic equation, for p in the degenerate range $p > 2$, it is a priori not clear to what extent and in what sense the above-mentioned results can hold. Indeed, on the one hand we have to account for the lack of homogeneity of the p-parabolic equation, and on the other hand we have to account for the fact that in the degenerate regime the phenomenon of finite speed propagation is present. As a matter of fact, simple examples show that in this case there are, compared to the case $p = 2$, much more delicate waiting-time phenomena to take into account.

To discuss the aspects of the waiting time phenomena further, we here first briefly describe some by now classical results in the case $p = 2$; see [Fabes et al. 1986; Salsa 1981]. Assume that Ω is, say, a Lipschitz domain, that $x_0 \in \partial\Omega$ and let $A_\pm := (a_r(x_0), t_0 \pm r^2)$, where $a_r(x_0)$ is an interior point of Ω with distance to the boundary comparable to r. Assume also that u and v are nonnegative caloric functions in Ω_T, i.e., functions satisfying (1-1) with $p = 2$ in Ω_T, vanishing continuously on $(\partial\Omega \cap B_r(x_0)) \times (t_0 - r^2, t_0 + r^2)$, where $B_r(x_0) \subset \mathbb{R}^n$ is the standard Euclidean ball of radius r and centered at $x_0 \in \mathbb{R}^n$. Then

$$c^{-1}\frac{u(A_-)}{v(A_+)} \le \frac{u(x,t)}{v(x,t)} \le c\frac{u(A_+)}{v(A_-)}, \tag{1-2}$$

for a universal constant c, whenever $(x,t) \in (\Omega \cap B_{r/2}(x_0)) \times (t_0 - (r/2)^2, t_0 + (r/2)^2)$. However, in general an estimate like (1-2) dramatically fails in the case $p \neq 2$. To see this, recall the following two classical solutions, see, e.g., [Avelin et al. 2016], in the case when $\Omega := \mathbb{R}^{n-1} \times \{x_n : x_n > 0\}$:

$$u(x,t) = c_p(T-t)^{-1/(p-2)} x_n^{p/(p-2)}, \quad v(x,t) = x_n. \tag{1-3}$$

In view of the examples in (1-3) it is not clear under what conditions the boundary Harnack principle in (1-2) could hold. Let us make a few observations. When defining u as in (1-3), we see that the larger we take T, the longer the solution u exists and the smaller its pointwise values become at a fixed time $t < T$. If we wish to show an estimate as in (1-2), we need to be able to rule out examples like u in (1-3); see also some other examples in [Avelin et al. 2016]. We do this by simply requiring, for $(x,t) \in \Omega_T$ fixed, that

$$T - t > C_0 u(x,t)^{2-p} d(x, \partial\Omega)^p \tag{1-4}$$

for a large enough constant C_0. It is easily seen that the solution u in (1-3) does not satisfy (1-4) at any point $(x,t) \in \Omega_T$ if we require $C_0 \ge c_p^{p-2}$.

In this context, and for the p-parabolic equation, it is here natural to make a link to the by-now classical method of *intrinsic scaling* due to [DiBenedetto 1993]. The intrinsic scalings define the canonical geometry in which weak solutions to the p-parabolic equation become homogenized in a sense to be made precise. Indeed, in this geometry we consider, instead of the standard parabolic cylinders, intrinsically time-scaled cylinders of the type

$$\begin{aligned} Q_r^{\lambda,+}(x,t) &:= B_r(x) \times (t, t + \lambda^{2-p} r^p), \\ Q_r^{\lambda,-}(x,t) &:= B_r(x) \times (t - \lambda^{2-p} r^p, t), \end{aligned} \qquad \lambda := u(x,t).$$

These kinds of intrinsic cylinders appear naturally in the context of Harnack inequalities, oscillation reduction estimates, and decay estimates, and define the correct geometry in our setting.

The main goal of this paper is to study to what extent the theory developed in [Fabes et al. 1986; Salsa 1981] generalizes to the case $p > 2$, under suitable intrinsic conditions. We have already seen that it rules out the pathological examples like u in (1-3). In fact, we prove that (1-4) is a sufficient condition for developing a rather general theory concerning the boundary behavior of nonnegative solutions to the p-parabolic equation. For instance, (1-4) allows us to prove a counterpart of (1-2) valid for $2 < p < \infty$; see Theorem 9.4 below.

1A. ***Summary of results.*** We will now give an informal summary of our results. The precise statements can be found in the body of the paper.

Harnack chains. Fundamental tools in the study of the boundary behavior of nonnegative solutions are the Harnack inequality and Harnack chains. Harnack chains allow one to relate the value of nonnegative solutions at different space-time points in the domain. For $p = 2$ the Harnack inequality is homogeneous and, roughly, to control the values of the solution in a ball of size r requires a waiting time comparable to r^2. For $p > 2$ we have to use an intrinsic version of the Harnack inequality [DiBenedetto 1993; DiBenedetto et al. 2012]. In particular, the intrinsic Harnack inequality states, see Theorem 3.1, that if we have a nonnegative solution u to the p-parabolic equation in Ω_T, with $(x,t) \in \Omega_T$, and $Q_{4r}^{u(x,t)/c_h,-}(x,t) \subset \Omega_T$, then

$$u(x,t) \le C_h \inf_{y \in B_r(x)} u\left(y, t + \left[\frac{c_h}{u(x,t)}\right]^{p-2} r^p\right),$$

provided that $t + [c_h/u(x,t)]^{p-2} r^p < T$. The intrinsic waiting time required in this Harnack inequality is consistent with the condition stated in (1-4). In Section 3 we develop a sequence of Harnack chain estimates and the goal of that section is twofold. First, we want to establish estimates applicable in cylindrical NTA domains; see Definition 2.1. Second, we want to establish a p-stable counterpart of the sharp Harnack chain estimate proved by Salsa [1981, Theorem C], which in the case $p = 2$ can be written as

$$u(y,s) \le u(x,t) \exp\left[C\left(\frac{|x-y|^2}{t-s} + \frac{t-s}{k} + 1\right)\right], \tag{1-5}$$

where $k = \min\{1, s, d(x,\partial\Omega)^2, d(y,\partial\Omega)^2\}$, $s < t$, and $(x,t), (y,s) \in \Omega_T$. To do this we develop Harnack chains based on the weak Harnack inequality proved in [Kuusi 2008], see Theorem 3.2 below, valid for supersolutions to the p-parabolic equation. As truncations of our solutions are supersolutions to (1-1), we are able to control the waiting times more precisely by adjusting the levels at which the solutions are truncated. This is in sharp contrast to the Harnack chains developed in [Avelin 2016; Avelin et al. 2016] for which there is very little control over the waiting time. Our approach to Harnack chains has at least three advantages. First, it allows us to construct Harnack chains starting from the measure $u(x,t_0)\,\mathrm{d}x$ at the initial time t_0. Second, it allows us to develop a flexible Carleson estimate, see Section 4, generalizing the one in [Avelin et al. 2016] and which, in addition, remains valid in the context of time-independent NTA cylinders. We note that although the Carleson estimate proved in [Avelin 2016] is valid in the setting of time-independent NTA cylinders, a difference compared to the results in this paper is that the Carleson estimate proved there is not p-stable as $p \to 2$. Third, we develop a version of (1-5) which is p-stable in the sense that we recover (1-5) as $p \to 2$. To establish this Gaussian-type behavior for $p > 2$ is technically rather involved.

Estimates of associated boundary measures. In the study of the boundary behavior of quasilinear equations of p-Laplace type, certain Riesz measures supported on the boundary and associated to nonnegative solutions vanishing on a portion of the boundary are important; see [Lewis and Nyström 2007; 2010]. These measures are nonlinear generalizations of the harmonic measure relevant in the study of the Laplace equation and the Green's function. Corresponding measures can also be associated to solutions to the p-parabolic equation. Indeed, let u be a nonnegative solution in Ω_T, assume that u is continuous on the closure of Ω_T, and that u vanishes on $\partial_p \Omega_T \cap Q$ with some open set Q. Extending u to be zero in $Q \setminus \Omega_T$, it is straightforward to see that u is a continuous weak subsolution to (1-1) in Q. Using this, one can conclude that there exists a unique locally finite positive Borel measure μ, supported on $S_T \cap Q$, such that

$$\int_Q u\, \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t - \int_Q |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t = \int_Q \phi \, \mathrm{d}\mu \tag{1-6}$$

whenever $\phi \in C_0^\infty(Q)$. In Section 5 we establish, in cylindrical NTA domains, both upper and lower bounds for the measure μ in terms of u. If Ω is smooth, then $\mathrm{d}\mu = |\nabla u|^{p-1} \, \mathrm{d}H^{n-1} \, \mathrm{d}t$. Based on this, the lower bound established on μ can be interpreted as a nondegeneracy estimate, close to the boundary, of the solution. Our proof of the lower bound for the measure μ is a modification of the elliptic proof; see for example [Avelin and Nyström 2013; Kilpeläinen and Zhong 2003]. However, our proof is genuinely nonlinear, it applies to much more general operators of p-parabolic type, and the result seems to be new already in the case $p = 2$.

A "complete theory" in $C^{1,1}$-domains. We establish a "complete theory" concerning the boundary behavior of nonnegative solutions in Ω_T in the case when Ω is a $C^{1,1}$-domain. As comprehensive literature is missing, we in Sections 8 and 9 develop both a local, as well as a global, theory of boundary behavior in $C^{1,1}$-cylinders. In the global setting we are able, as in [Fabes et al. 1986] with corresponding estimates in the case $p = 2$, to give a rather complete picture. For nonnegative solutions vanishing on the lateral boundary, our results include a global boundary Harnack principle and Hölder continuity of ratios. On the other hand, in the local setting we prove a new intrinsic local boundary Harnack principle. In the context of $C^{1,1}$-cylinders we are also able to show that the boundary measure in (1-6) is mutually absolutely continuous with respect to the surface measure in a suitably chosen intrinsic geometry. The results in Sections 8 and 9 are obtained by combining Harnack chains and Carleson estimates with explicit barrier constructions from Section 6 and decay estimates from Section 7.

2. Notation and preliminaries

Points in $\mathbb{R}^{n+1}$ are denoted by $x = (x_1, \ldots, x_n, t)$. Given a set $E \subset \mathbb{R}^n$, let $\overline{E}$, ∂E, $\operatorname{diam} E$, E^c, E°, denote the closure, boundary, diameter, complement and interior of E, respectively. Let $\cdot$ denote the standard inner product on $\mathbb{R}^n$, let $|x| = (x \cdot x)^{1/2}$ be the Euclidean norm of x, and let $\mathrm{d}x$ be the Lebesgue

n-measure on $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$, let $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Given $E, F \subset \mathbb{R}^n$, let $d(E, F)$ be the Euclidean distance from E to F. When $E = \{y\}$, we write $d(y, F)$. For simplicity, we define "sup" to be the essential supremum and "inf" to be the essential infimum. If $O \subset \mathbb{R}^n$ is open and $1 \le q \le \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, and both f and ∇f are q-th power integrable on O. Let

$$\|f\|_{W^{1,q}(O)} = \|f\|_{L^q(O)} + \||\nabla f|\|_{L^q(O)}$$

be the norm in $W^{1,q}(O)$, where $\|\cdot\|_{L^q(O)}$ denotes the usual Lebesgue q-norm in O. We define $C_0^\infty(O)$ to be the set of infinitely differentiable functions with compact support in O and we let $W_0^{1,q}(O)$ denote the closure of $C_0^\infty(O)$ in the norm $\|\cdot\|_{W^{1,q}(O)}$. We define $W_{\text{loc}}^{1,q}(O)$ in the standard way. By $\nabla\cdot$ we denote the divergence operator. Given $t_1 < t_2$ we denote by $L^q(t_1, t_2, W^{1,q}(O))$ the space of functions such that for almost every t, $t_1 \le t \le t_2$, the function $x \to u(x,t)$ belongs to $W^{1,q}(O)$ and

$$\|u\|_{L^q(t_1,t_2,W^{1,q}(O))} := \left(\int_{t_1}^{t_2}\int_O \left(|u(x,t)|^q + |\nabla u(x,t)|^q\right) \mathrm{d}x\, \mathrm{d}t\right)^{1/q} < \infty.$$

The spaces $L^q(t_1, t_2, W_0^{1,q}(O))$ and $L_{\text{loc}}^q(t_1, t_2, W_{\text{loc}}^{1,q}(O))$ are defined analogously. Finally, for $I \subset \mathbb{R}$, we define $C(I; L^q(O))$ as the space of functions such that $t \to \|u(t,\cdot)\|_{L^q(O)}$ is continuous whenever $t \in I$. We define $C_{\text{loc}}(I; L_{\text{loc}}^q(O))$ analogously.

2A. *Weak solutions.* Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, i.e., a connected open set. For $t_1 < t_2$, we let $\Omega_{t_1,t_2} := \Omega \times (t_1, t_2)$. Given p, $1 < p < \infty$, we say that u is a weak solution to

$$\partial_t u - \Delta_p u = 0 \tag{2-1}$$

in Ω_{t_1,t_2} if $u \in L_{\text{loc}}^p(t_1, t_2, W_{\text{loc}}^{1,p}(\Omega))$ and

$$\int_{\Omega_{t_1,t_2}} (-u\, \partial_t\phi + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{2-2}$$

whenever $\phi \in C_0^\infty(\Omega_{t_1,t_2})$. If u is a weak solution to (2-1) in the above sense, then we will often refer to u as being p-parabolic in Ω_{t_1,t_2}. For $p \in (2, \infty)$ we have by the parabolic regularity theorem, see [DiBenedetto 1993], that any p-parabolic function u has a locally Hölder continuous representative. In particular, in the following we will assume that $p \in (2, \infty)$ and any solution u is continuous. If (2-2) holds with "$=$" replaced by "$\ge$" ("$\le$") for all $\phi \in C_0^\infty(\Omega_{t_1,t_2})$, $\phi \ge 0$, then we will refer to u as a weak supersolution (subsolution).

2B. *Geometry.* We here state the geometrical notions used throughout the paper.

Definition 2.1. A bounded domain Ω is called nontangentially accessible (NTA) if there exist $M \ge 2$ and r_0 such that the following are fulfilled:

(1) *Corkscrew condition:* for any $w \in \partial\Omega$, $0 < r < r_0$, there exists a point $a_r(w) \in \Omega$ such that

$$M^{-1}r < |a_r(w) - w| < r, \quad d(a_r(w), \partial\Omega) > M^{-1}r.$$

(2) $\mathbb{R}^n \setminus \Omega$ satisfies (1).

(3) *Uniform condition:* if $w \in \partial\Omega,\ 0 < r < r_0$, and $w_1, w_2 \in B_r(w) \cap \Omega$, then there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1,\ \gamma(1) = w_2$, such that
  (a) $H^1(\gamma) \le M|w_1 - w_2|$,
  (b) $\min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \le M\, d(\gamma(t), \partial\Omega)$ for all $t \in [0, 1]$.

We choose this definition as it very useful when we explicitly construct the parabolic Harnack chains in Section 3; see specifically Theorem 3.5. The values M and r_0 will be called the NTA constants of Ω. For more on the notion of NTA domains we refer to [Jerison and Kenig 1982].

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω satisfies the ball condition with radius $r_0 > 0$ if for each point $y \in \partial\Omega$ there exist points $x^+ \in \Omega$ and $x^- \in \Omega^c$ such that $B_{r_0}(x^+) \subset \Omega$, $B_{r_0}(x^-) \subset \Omega^c$, $\partial B_{r_0}(x^+) \cap \partial\Omega = \{y\} = \partial B_{r_0}(x^-) \cap \partial\Omega$, and such that the points $x^+(y)$, $x^-(y)$, y are collinear for each $y \in \partial\Omega$.

Remark 2.3. It is easy to see that a domain satisfying the ball condition with radius $r_0 > 0$ is an NTA domain with a constant M and r_0. In particular, we may canonically choose

$$a_r(x_0) := x_0 + \frac{r}{2}\frac{x^+ - x_0}{|x^+ - x_0|},$$

since the direction given by $(x^+ - x_0)/|x^+ - x_0|$ is unique. The exterior corkscrew point is defined analogously.

Remark 2.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then Ω is a $C^{1,1}$ domain if and only if it satisfies the ball condition. For a proof of this fact, see for example [Aikawa et al. 2007, Lemma 2.2].

2C. ***The continuous Dirichlet problem.*** Assuming that Ω is a bounded NTA domain one can prove, see [Björn et al. 2015; Kilpeläinen and Lindqvist 1996], that all points on the parabolic boundary

$$\partial_p\Omega_T = S_T \cup (\overline{\Omega} \times \{0\}), \quad S_T = \partial\Omega \times [0, T],$$

of the cylinder Ω_T are regular for the Dirichlet problem for (2-1). In particular, for any $f \in C(\partial_p\Omega_T)$, there exists a unique Perron solution $u = u_f^{\Omega_T} \in C(\overline{\Omega}_T)$ to the Dirichlet problem $\partial_t u - \Delta_p u = 0$ in Ω_T and $u = f$ on $\partial_p\Omega_T$.

3. Harnack chains

In this section we prove a sequence of results concerning intrinsic Harnack chains. Forward-in-time chains describe the diffusion with an appropriate waiting time. On the other hand, backward-in-time chains say that if the solution has existed for a long enough time, the future values will control the values from the past as well. Throughout the section we let $\Omega \subset \mathbb{R}^n$ be a bounded domain and given $T > 0$ we let $\Omega_T = \Omega \times (0, T)$.

3A. ***Local Harnack inequalities.*** We here collect two estimates from the literature. The following theorem can be found in [DiBenedetto 1993; DiBenedetto et al. 2008; 2012].

Theorem 3.1. *Let u be a nonnegative p-parabolic function in Ω_T, let $(x_0,t_0)\in\Omega_T$ and assume that $u(x_0,t_0)>0$. There exist positive constants c_h and C_h, depending only on p,n, such that if $B_{4r}(x_0)\subset\Omega$ and*

$$(t_0-\theta(4r)^p,t_0+\theta(4r)^p)\subset(0,T],$$

where $\theta=(c_h/u(x_0,t_0))^{p-2}$, then

$$u(x_0,t_0)\le C_h\inf_{B_r(x_0)}u(\cdot,t_0+\theta r^p).$$

The constants c_h and C_h are stable as $p\to 2$ and deteriorate as $p\to\infty$ in the sense that $c_h(p),C_h(p)\to\infty$ as $p\to\infty$.

The next theorem is instead valid for nonnegative weak supersolutions. See [Kuusi 2008] for the proof.

Theorem 3.2. *Let u be a nonnegative weak supersolution in $B_{4r}(x_0)\times(0,T)$. There exist constants $C_i\equiv C_i(p,n)$, $i=1,2$, such that*

$$\fint_{B_r(x_0)}u(x,t_1)\,\mathrm{d}x\le\frac12\left(\frac{C_1r^p}{T-t_1}\right)^{1/(p-2)}+C_2\inf_Q u$$

for almost every $0<t_1<T$, where $Q:=B_{2r}(x_0)\times(t_1+T_1/2,t_1+T_1)$, and

$$T_1=\min\left\{T-t_1,C_1r^p\left(\fint_{B_r(x_0)}u(x,t_1)\,\mathrm{d}x\right)^{2-p}\right\}.$$

In particular, if $T_1<T-t_1$, then

$$\fint_{B_r(x_0)}u(x,t_1)\,\mathrm{d}x\le 2C_2\inf_Q u.$$

3B. ***Forward Harnack chains.*** We begin by describing a simple Harnack chain for weak supersolutions.

Lemma 3.3 (weak forward Harnack chains). *Let $\Omega\subset\mathbb{R}^n$ be a domain and let $T>0$. Let x,y be two points in Ω and assume that there exist a sequence of balls $\{B_{4r}(x_j)\}_{j=0}^k$ such that $x_0=x$, $x_k=y$, $B_{4r}(x_j)\subset\Omega$ for all $j=0,\dots,k$ and $x_{j+1}\in B_r(x_j)$, $j=0,\dots,k-1$. Assume that u is a continuous nonnegative weak supersolution in Ω_T with*

$$\bar\Lambda:=\fint_{B_r(x_0)}u(x,t_0)\,\mathrm{d}x>0.$$

There exist constants $\bar c_i\equiv\bar c_i(p,n)>1$, $i\in\{1,2\}$, such that if

$$t_0+\tau_k\bar\Lambda^{2-p}r^p<T,\quad \tau_k:=\bar c_1\sum_{j=0}^k\bar c_2^{\,j(p-2)},$$

then

$$\fint_{B_r(x)}u(x,t_0)\,\mathrm{d}x\le\bar c_2^{\,k+1}\inf_{z\in B_{2r}(y)}u(z,t_0+\tau_k\bar\Lambda^{2-p}r^p).$$

Furthermore, the constants $\bar c_i$, $i\in\{1,2\}$, are stable as $p\to 2^+$. In particular, when $p=2$, we have $\tau_k=\bar c_1(k+1)$ with $\bar c_1=\bar c_1(n)$.

Proof. Using Theorem 3.2 we first get

$$\frac{\bar{\Lambda}}{2C_2} \leq \inf_{z \in B_{2r}(x_0)} u(z, t_1), \quad t_1 := t_0 + C_1 \bar{\Lambda}^{2-p} r^p. \tag{3-1}$$

Define then

$$u_j := \min(u, \Lambda_j), \quad \Lambda_j := (2C_2)^{-j} \bar{\Lambda}, \quad t_{j+1} := t_j + C_1 \Lambda_j^{2-p} r^p$$

for $j = \{1, \dots, k-1\}$. Assume inductively that for $t_{i+1} \leq T$ we have

$$B_r(x_i) \subset B_{2r}(x_{i-1})$$

and

$$u_i(z, t_i) = \Lambda_i \quad \text{for } z \in B_r(x_i)$$

hold for $i \in \{0, \dots, j\}$. For $j = 1$ this is certainly the case as we see from (3-1). Since u_j is a nonnegative weak supersolution, Theorem 3.2 gives us

$$\Lambda_{j+1} = \frac{\Lambda_j}{2C_2} \leq \inf_{z \in B_{2r}(x_j)} u_j(z, t_{j+1}),$$

and hence also

$$u_{j+1}(z, t_{j+1}) = \Lambda_{j+1} \quad \text{for } z \in B_r(x_{j+1}).$$

This proves the induction step. By the construction,

$$\inf_{z \in B_r(y)} u_k(z, t_k) = \Lambda_k$$

holds. Thus, applying Theorem 3.2 one more time we get

$$\inf_{z \in B_{2r}(y)} u(z, \bar{t}) \geq (2C_2)^{-(k+1)} \bar{\Lambda},$$

with

$$\bar{t} := t_0 + C_1 \sum_{j=0}^{k} (2C_2)^{j(p-2)} \bar{\Lambda}^{2-p} r^p.$$

Setting $\bar{c}_1 = C_1$ and $\bar{c}_2 = 2C_2$ completes the proof of the lemma. □

For p-parabolic functions we have the following pointwise version of Lemma 3.3.

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $T > 0$. Let x, y be two points in Ω and assume that there exists a sequence of balls $\{B_{4r}(x_j)\}_{j=0}^k$ such that $x_0 = x$, $x_k = y$, $B_{4r}(x_j) \subset \Omega$ for all $j = 0, \dots, k$, and $x_{j+1} \in B_r(x_j)$, $j = 0, \dots, k-1$. Assume that u is a nonnegative p-parabolic function in Ω_T and assume that $u(x, t_0) > 0$. There exist constants $c \equiv c(p, n)$ and $c_1 \equiv c_1(p, n, k) > 1$ such that if*

$$t_0 - \left(\frac{c_h}{u(x, t_0)}\right)^{p-2} (4r)^p > 0, \quad t_0 + c_1(k) u(x, t_0)^{2-p} r^p < T,$$

then

$$u(x, t_0) \leq c^k \inf_{z \in B_r(y)} u(z, t_0 + c_1(k) u(x, t_0)^{2-p} r^p).$$

Furthermore, c_1 *satisfies the estimate*

$$\tilde{c}_1 k \le c_1 \le \tilde{c}_1 (k+1) c^{(k+1)(p-2)},$$

with $\tilde{c}_1 \equiv \tilde{c}_1(p,n)$ *and* $\tilde{c}_1(p,n) \to \tilde{c}_1(n)$ *as* $p \to 2$.

Proof. After applying Theorem 3.1 once, the result follows from Lemma 3.3. □

We next focus on cylindrical NTA domains. The first theorem, Theorem 3.5, holds for weak supersolutions and shows how to bound the values of a solution at points close to the boundary using pointwise interior values. A remarkable fact of the proof is that the waiting time is explicitly defined and, as $p \to 2$, it gives a supersolution version of [Salsa 1981, Theorem C], as alluded to in the Introduction; see (1-5). The proof uses heavily the assumptions on NTA domains and iterations of Lemma 3.3.

Theorem 3.5. *Let* $\Omega \subset \mathbb{R}^n$ *be an NTA domain with constants* M *and* r_0, *let* $x_0 \in \partial\Omega$, $T > 0$ *and* $0 < r < r_0$. *Let* x, y *be two points in* $\Omega \cap B_r(x_0)$ *such that*

$$\varrho := d(x, \partial\Omega) \le r \quad \text{and} \quad d(y, \partial\Omega) \ge \frac{r}{4}.$$

Assume that u *is a nonnegative continuous weak supersolution in* Ω_T, *and assume that*

$$\Lambda := \fint_{B_{\varrho/4}(x)} u(z, t_0)\, \mathrm{d}z > 0. \tag{3-2}$$

Let $\delta \in (0,1]$. *Then there exist positive constants* $c_i \equiv c_i(M, p, n)$, $i \in \{1,2,3\}$, *such that if* $t_0 + \tau < T$, *where*

$$\tau := \delta^{p-1} \left(c_2^{-1/\delta} \left(\frac{r}{\varrho} \right)^{-c_3/\delta} \Lambda \right)^{2-p} r^p,$$

then

$$\fint_{B_{\varrho/4}(x)} u(z, t_0)\, \mathrm{d}z \le c_1^{1/\delta} \left(\frac{r}{\varrho} \right)^{c_3/\delta} \inf_{z \in B_{r/16}(y)} u(z, t_0 + \tau).$$

Furthermore, the constants c_i, $i \in \{1,2,3\}$, *are stable as* $p \to 2^+$.

Proof. We split the proof into three steps.

Step 1: parametrization of the curve connecting x *and* y. According to the uniform condition (3) in Definition 2.1, we can find a rectifiable curve γ connecting x and y such that $\gamma(0) = x$, $\gamma(1) = y$, and

(a) $H^1(\gamma) \le M|w_1 - w_2|$,

(b) $\min\{H^1(\gamma([0,t])), H^1(\gamma([t,1]))\} \le M\, d(\gamma(t), \partial\Omega)$ for all $t \in [0,1]$.

We call a ball $B \subset \Omega$ *admissible* if $4B \subset \Omega$ and is thus eligible for the Harnack inequality. Our goal in this step is to construct a sequence of admissible balls covering the curve γ. In the following we may, without loss of generality, assume that $H^1(\gamma([0,1])) > 2^{-4}r$. We define $\hat{t}_1, \hat{t}_2 \in (0,1)$ such that $H^1(\gamma([0,\hat{t}_1])) = 2^{-5}r$ and $H^1(\gamma([\hat{t}_2, 1])) = 2^{-5}r$. The technical part will be in the interval $(0, \hat{t}_1)$. To continue we choose k as the integer which satisfies $2^{-k}r \in (\varrho/16, \varrho/8]$. We define sequence of real numbers $\{s_j\}$ through

$$H^1(\gamma([0,s_j])) = 2^{-k+j} H^1(\gamma([0,\hat{t}_1])) := 2^{-k-5+j} r, \quad s_0 = 0.$$

Then, for any $s \in [s_j, s_{j+1})$, (a) and (b) imply

$$d(\gamma(s), \partial\Omega) \geq \frac{2^{-k+j-5}r}{M}$$

and

$$H^1(\gamma([s_j, s_{j+1}])) = 2^{-k+j-5} r.$$

Thus, defining

$$\varrho_j := N^{-1} 2^{-k+j-5} r, \quad N \in \mathbb{N}, \quad N \geq 2^9 M,$$

we see that the piece $\gamma([s_j, s_{j+1}])$ can be covered with N admissible balls of the type $B^{i,j} := B_{\varrho_j}(y_{i,j})$ such that $y_{i,j} \in \gamma([s_j, s_{j+1}])$ for $i \in \{1, \dots, N\}$, $y_{i,j-1} \in B^{i,j}$ and $\gamma([s_j, s_{j+1}]) \subset \cup_i B^{i,j}$. Finally, we observe that the middle piece of the curve $\gamma([\hat{t}_1, \hat{t}_2])$, due to the definitions of $\hat{t}_1, \hat{t}_2$ together with (b) can be covered with MN admissible balls of size r/N. Moreover we can cover the end piece $\gamma([\hat{t}_2, 1])$ with N admissible balls of size r/N since $\gamma([\hat{t}_2, 1]) \subset B_{r/16}(y) \subset \Omega$. At this point, we consider $N \in \mathbb{N}$ to be a free parameter such that $N \geq 2^9 M$.

Step 2: iteration via Harnack estimates. Let now Λ be as in (3-2). Theorem 3.2 implies that if

$$t_0 + C_1 \Lambda^{2-p} \varrho_0^p < T,$$

then

$$\inf_{z \in B_{\varrho_0}(x_0)} u(z, t_1) \geq \frac{1}{2C_2} \Lambda, \quad t_1 := t_0 + C_1 \Lambda^{2-p} \varrho_0^p.$$

Let

$$\Lambda_1 = \sigma \Lambda, \quad \sigma \in (0, (2C_2)^{-1}].$$

Defining thus $u_1 := \min(u, \Lambda_1)$, we obtain by Lemma 3.3 (see also its proof) that there exist constants $\bar{c}_1 \equiv \bar{c}_1(p,n)$ and $\bar{c}_2 \equiv \bar{c}_2(p,n)$, such that if

$$t_2 := t_1 + \tau_N \Lambda_1^{2-p} \varrho_1^p < T, \quad \tau_N := \bar{c}_1 \sum_{j=0}^{N-1} \bar{c}_2^{j(p-2)} \in [\bar{c}_1 N, \bar{c}_1 N \bar{c}_2^{N(p-2)}), \tag{3-3}$$

then

$$\inf_{z \in B_{\varrho_2}(\gamma(s_2))} u(z, t_2) = \inf_{z \in B_{2\varrho_1}(\gamma(s_2))} u(z, t_2) \geq \frac{\Lambda_1}{\bar{c}_2^N} =: \Lambda_2.$$

Define

$$\Lambda_{j+1} := \bar{c}_2^{-(j+1)N} \Lambda_1, \quad u_j := \min(u, \Lambda_j), \quad j \in \mathbb{N},$$

let $\hat{k} := k + M + 1$, and let

$$t_{j+1} := \begin{cases} t_j + \tau_N \Lambda_j^{2-p} \varrho_j^p & \text{if } j \in \{1, \dots, k\}, \\ t_j + \tau_N \Lambda_j^{2-p} (r/N)^p & \text{if } j \in \{k+1, \dots, \hat{k}\}. \end{cases}$$

Iterating Lemma 3.3 it follows by induction that

$$u(y, t_{\hat{k}+1}) \geq \Lambda_{\hat{k}+1}. \tag{3-4}$$

Step 3: waiting time. Let us now analyze the waiting time $t_{\hat{k}+1}$, which we want to show to be precisely $t_0 + \tau$ by a suitable choice of Λ_1. We have

$$\begin{aligned} t_{\hat{k}+1} &= t_1 + C_1 \Lambda^{2-p} \varrho_0^p + \tau_N \Bigg[\sum_{j=1}^{k} \Lambda_j^{2-p} \varrho_j^p + \sum_{j=k+1}^{\hat{k}} \Lambda_j^{2-p} \varrho_j^p \Bigg] \\ &= t_0 + C_1 \Lambda^{2-p} \varrho_0^p + N^{-p} \tau_N \Lambda_1^{2-p} r^p \Bigg[2^{-kp} \sum_{j=0}^{k} (2^p \bar{c}_2^{(p-2)N})^j \qquad (3\text{-}5) \\ &\qquad\qquad + \bar{c}_2^{(p-2)kN} \sum_{j=0}^{M} (\bar{c}_2^{(p-2)N})^j \Bigg] \qquad (3\text{-}6) \\ &=: t_0 + \boldsymbol{T} \Lambda^{2-p} r^p. \end{aligned}$$

We can write the sum in (3-5) as

$$2^{-kp} \sum_{j=0}^{k} (2^p \bar{c}_2^{(p-2)N})^j = 2^p \frac{\bar{c}_2^{(p-2)(k+1)N} - 2^{-(k+1)p}}{2^p \bar{c}_2^{(p-2)N} - 1},$$

while the sum in (3-6) can be estimated similarly to τ_N, see (3-3):

$$\bar{c}_2^{(p-2)kN} \sum_{j=0}^{M} (\bar{c}_2^{(p-2)N})^j \in (M \bar{c}_2^{(p-2)kN}, M \bar{c}_2^{(p-2)(k+M)N}].$$

Hence, recalling the definitions of τ_N and $\boldsymbol{T}$, we get after some straightforward estimation that

$$\tfrac{1}{2} \bar{c}_1 \sigma^{2-p} N^{1-p} \left(\frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 N} \leq \boldsymbol{T} \leq 2 \bar{c}_3 \sigma^{2-p} N^{1-p} \bar{c}_3^{(p-2)N} \left(\frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 N} \tag{3-7}$$

for new constants $\bar{c}_3, \bar{c}_4$ depending only on p, n, M. We now choose $N = \tilde{c}/\delta$ and let $\tilde{c}$ be a degree of freedom. First note that choosing $\sigma_1 = (2C_2)^{-1}$, then choosing $c_3 = \tilde{c} c_4$ and $c_2 = [\bar{c}_3]^{\tilde{c}}$, for a large enough $\tilde{c} = \tilde{c}(p, n, M)$ we have

$$2 \bar{c}_3 (2C_2)^{p-2} \tilde{c}^{1-p} \bar{c}_3^{(p-2)\tilde{c}/\delta} \left(\frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 \tilde{c}/\delta} < c_2^{(p-2)/\delta} \left(\frac{r}{\varrho} \right)^{(p-2)c_3/\delta}. \tag{3-8}$$

Second we see that choosing $\sigma_2 = c_5^{-1/\delta}$ for large enough $c_5 = c_5(p, n, M)$ the following holds:

$$\tfrac{1}{2} \bar{c}_1 \sigma_2^{2-p} \tilde{c}^{1-p} \left(\frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 \tilde{c}/\delta} > c_2^{(p-2)/\delta} \left(\frac{r}{\varrho} \right)^{(p-2)c_3/\delta}. \tag{3-9}$$

With (3-8) and (3-9) and (3-7) at hand we see that there is a choice of $\sigma \in [\sigma_2, \sigma_1]$ such that

$$\boldsymbol{T} = \delta^{p-1} c_2^{(p-2)/\delta} \left(\frac{r}{\varrho} \right)^{(p-2)c_3/\delta},$$

and thus we have proved $t_{\hat{k}+1} = t_0 + \boldsymbol{T} \Lambda^{2-p} r^p = t_0 + \tau$. This together with (3-4) finishes the proof by taking suitably large c_1 in the statement. □

For p-parabolic functions we have the following pointwise version of Theorem 3.5.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and r_0, let $x_0 \in \partial\Omega$, $T > 0$ and $0 < r < r_0$. Let x, y be two points in $\Omega \cap B_r(x_0)$ such that*

$$\varrho := d(x, \partial\Omega) \le r \quad \textit{and} \quad d(y, \partial\Omega) \ge \frac{r}{4}.$$

Assume that u is a nonnegative p-parabolic function in Ω_T, and assume that $u(x, t_0)$ is positive. Let $\delta \in (0, 1]$. Then there exist constants $c_i \equiv c_i(M, p, n)$, $i \in \{1, 2, 3\}$, such that if

$$t_0 - (c_h/u(x, t_0))^{p-2}(\delta\varrho)^p > 0, \quad t_0 + \tau < T,$$

with

$$\tau := \delta^{p-1}\left(c_2^{-1/\delta}\left(\frac{r}{\varrho}\right)^{-c_3/\delta} u(x, t_0)\right)^{2-p} r^p,$$

then

$$u(x, t_0) \le c_1^{1/\delta}\left(\frac{r}{\varrho}\right)^{c_3/\delta} \inf_{z \in B_{r/16}(y)} u(z, t_0 + \tau).$$

Furthermore, the constants c_i, $i \in \{1, 2, 3\}$, are stable as $p \to 2^+$.

Proof. Applying Theorem 3.1 once, we see that the theorem follows from Theorem 3.5. □

3C. *Backward Harnack chains.* The philosophy of the forward Harnack chains in Section 3B is that the data at the starting point will start to diffuse according to the intrinsic Harnack inequality. The finite speed of diffusion forces the waiting time to blow up if we wish to spread our information in an infinite chain. In the backward Harnack chains that we develop in this section the philosophy is reversed. Instead of looking to the future we look to the past. This means that if the value of the solution at a point (y, s) is, say, 1, then we ask the question: how large can the values in the past be without violating the fact that the solution is 1 at (y, s).

We start with the weak version of the backward Harnack chains, valid for weak supersolutions.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and r_0, let $x_0 \in \partial\Omega$, $T > 0$, and $0 < r < r_0$. Let x, y be two points in $\Omega \cap B_r(x_0)$ such that*

$$\varrho := d(x, \partial\Omega) \le r \quad \textit{and} \quad d(y, \partial\Omega) \ge \frac{r}{4}.$$

Assume that u is a nonnegative continuous weak supersolution in Ω_T, and assume that $u(y, s)$ is positive. Let $\delta \in (0, 1]$. Then there exist positive constants $C_i \equiv C_i(p, n)$ and $c_i \equiv c_i(p, n, M)$, $i \in \{4, 5\}$, such that if $s \in (\tau, T)$ and

$$t \in [s - \tau, s - \delta^{p-1}\tau],$$

with

$$\tau := C_4[C_5 u(y, s)]^{2-p} r^p$$

then

$$\fint_{B_{\varrho/4}(x)} u(z, t)\, \mathrm{d}z \le c_4^{1/\delta}\left(\frac{r}{\varrho}\right)^{c_5/\delta} u(y, s). \tag{3-10}$$

Furthermore, the constants c_i, C_i, $i \in \{4, 5\}$, are stable as $p \to 2^+$.

Remark 3.8. If we assume that $u \in C([0,T); L^2(\Omega))$ then we can replace $s \in (\tau, T)$ with $s \in [\tau, T)$. That is, the chain can be taken all the way to the initial time, i.e., $t = 0$.

Proof. After scaling, we may assume that $u(y,s) = 1$. Assume now the contrary to (3-10), i.e.,

$$\fint_{B_{\varrho/4}(x)} u(z,t)\,\mathrm{d}z > H\left(\frac{r}{\varrho}\right)^{c_5/\delta} \tag{3-11}$$

for constants c_5, H to be fixed. Then Theorem 3.5 implies that with

$$\tilde{\tau} := \tilde{\delta}^{p-1}\left(c_2^{-1/\tilde{\delta}}\left(\frac{r}{\varrho}\right)^{-c_3/\tilde{\delta}} \fint_{B_{\varrho/4}(x)} u(z,t)\,\mathrm{d}z\right)^{2-p} r^p$$

we get

$$\fint_{B_{\varrho/4}(x)} u(z,t)\,\mathrm{d}z \le c_1^{1/\tilde{\delta}}\left(\frac{r}{\varrho}\right)^{c_3/\tilde{\delta}} \inf_{z \in B_{r/16}(y)} u(z, t+\tilde{\tau}) \tag{3-12}$$

with constants $c_i \equiv c_i(p,n,M)$, $i \in \{1,2,3\}$, and $\tilde{\delta} \in (0,\delta]$. Now we have an upper bound for $\tilde{\tau}$ by means of (3-11) as follows:

$$\tilde{\tau} \le \tilde{\delta}^{p-1}\left(c_2^{-1/\tilde{\delta}}\left(\frac{r}{\varrho}\right)^{-c_3/\tilde{\delta}} H\left(\frac{r}{\varrho}\right)^{c_5/\delta}\right)^{2-p} r^p = \tilde{\delta}^{p-1}(c_2^{-1/\tilde{\delta}}H)^{2-p} r^p \le \delta^{p-1}\tau,$$

provided that

$$H \ge C_5 c_2^{1/\tilde{\delta}}, \quad \tilde{\delta} := \delta \min\{1, C_4\}^{1/(p-1)}, \quad c_5 := \frac{c_3}{\min\{1, C_4\}^{1/(p-1)}}. \tag{3-13}$$

Therefore we have

$$t + \tilde{\tau} \le s.$$

Observe that both C_4 and C_5 are still to be fixed. Thus we need to carry the information from the time $t+\tilde{\tau}$ up to s. To this end, connecting (3-11) and (3-12) with the choices in (3-13) leads to

$$H c_1^{-1/\tilde{\delta}} < \inf_{z \in B_{r/16}(y)} u(z, t+\tilde{\tau}). \tag{3-14}$$

Truncate u as

$$\tilde{u} = \min(4C_2, u),$$

and take

$$H := c_4^{1/\delta}, \quad c_4 := \max\{4C_2 c_1, C_5 c_2\}^{1/\min\{1,C_4\}^{1/(p-1)}}, \tag{3-15}$$

where C_2 is as in Theorem 3.2. Then $\tilde{u}$ is a continuous weak supersolution, and we have by (3-13) and (3-15) and (3-14) that

$$\fint_{B_{\tilde{r}}(y)} \tilde{u}(z, t+\tilde{\tau})\,\mathrm{d}z = 4C_2, \quad \tilde{r} \in \left(0, \frac{r}{16}\right].$$

Applying thus the forward-in-time weak Harnack estimate in Theorem 3.2 gives

$$4C_2 \le 2C_2 \inf_{z \in B_{2\tilde{r}}(y)} \tilde{u}(z, t+\tilde{\tau}+C_1(4C_2)^{2-p}\tilde{r}^p),$$

provided that

$$t+\tilde{\tau}+C_1(4C_2)^{2-p}\tilde{r}^p<T.$$

Choosing $C_4=16^{-p}C_1$ and $C_5=4C_2$, we can always find $\tilde{r}\le r/16$ such that

$$t+\tilde{\tau}+C_1(4C_2)^{2-p}\tilde{r}^p=s,$$

and hence

$$2\le\inf_{z\in B_{2\tilde{r}}(y)}\tilde{u}(z,s).$$

This gives a contradiction since we assumed that $u(y,s)=1$, and thus the proof is complete. □

The following theorem is the corresponding result for weak solutions, where we use pointwise information in the past instead of information in mean. The main difference this imposes on the assumptions in Theorem 3.7 is that we have to require the solution to have lived for a certain amount of time, which is precisely the price we have to pay if we wish to control pointwise values in the past.

Theorem 3.9. *Let $\Omega\subset\mathbb{R}^n$ be an NTA domain with constants M and r_0, let $x_0\in\partial\Omega$, $T>0$, and $0<r<r_0$. Let x,y be two points in $\Omega\cap B_r(x_0)$ such that*

$$\varrho:=d(x,\partial\Omega)\le r\quad\textit{and}\quad d(y,\partial\Omega)\ge\frac{r}{4}.$$

Assume that u is a nonnegative p-parabolic function in Ω_T, and assume that $u(y,s)$ is positive. Let $\delta\in(0,1]$. Then there exist positive constants $C_i\equiv C_i(p,n)$ and $c_i\equiv c_i(p,n,M)$, $i\in\{4,5\}$, such that if $s<T$ and

$$\max\left\{\left(\frac{c_4^{1/\delta}}{c_h}\left(\frac{r}{\varrho}\right)^{c_5/\delta}u(y,s)\right)^{2-p}(\delta\varrho)^p,\,s-\tau\right\}\le t\le s-\delta^{p-1}\tau,\tag{3-16}$$

with

$$\tau:=C_4[C_5u(y,s)]^{2-p}r^p,$$

then

$$u(x,t)\le c_4^{1/\delta}\left(\frac{r}{\varrho}\right)^{c_5/\delta}u(y,s).$$

Furthermore, the constants c_i,C_i, $i\in\{4,5\}$, are stable as $p\to2^+$.

Proof. To prove the lemma we follow the same outline as the proof of Theorem 3.7, but instead of assuming (3-11) we assume the contrary assumption

$$u(x,t)>H\left(\frac{r}{\varrho}\right)^{c_5/\delta}$$

for some constants c_5, H to be fixed. Applying Theorem 3.6 instead of Theorem 3.5 we get

$$u(x,t)\le c_1^{1/\tilde{\delta}}\left(\frac{r}{\varrho}\right)^{c_3/\tilde{\delta}}\inf_{z\in B_{r/16}(y)}u(z,t+\tilde{\tau}).$$

Note that it is the usage of Theorem 3.6 which requires (3-16). The proof now follows repeating the remaining part of the proof of Theorem 3.7 essentially verbatim. □

4. Carleson estimate

In this section we prove, using the improved Harnack chain estimate in Theorem 3.9, a flexible Carleson estimate valid in cylindrical NTA domains. Versions of the Carleson estimate were originally proved, for equations of p-parabolic type, in [Avelin et al. 2016] in Lipschitz cylinders, and in [Avelin 2016] in certain time-dependent space-time domains. We begin with a standard oscillation decay lemma valid for weak subsolutions; see for example [DiBenedetto 1993].

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and r_0. Let u be a nonnegative, continuous weak subsolution in Ω_T. Let $(x_0, t_0) \in S_T$, $0 < r < r_0$,*

$$Q_r^\lambda(x_0, t_0) := (B_r(x_0) \cap \Omega) \times (t_0 - \lambda^{2-p} r^p, t_0) \subset \Omega_T,$$

and assume that u vanishes continuously on $S_T \cap Q_r^\lambda(x_0, t_0)$ and that

$$\sup_{Q_r^\lambda(x_0,t_0)} u \le \lambda.$$

Then there is a constant σ depending only on p, n, M such that $Q_{\sigma r}^{\lambda/2}(x_0, t_0) \subset Q_r^\lambda(x_0, t_0)$ and

$$\sup_{Q_{\sigma r}^{\lambda/2}(x_0,t_0)} u \le \frac{\lambda}{2}.$$

In particular, we have

$$\sup_{Q_{\sigma^j r}^{2^{-j}\lambda}(x_0,t_0)} u \le 2^{-j}\lambda$$

for any $j \in \mathbb{N}$.

The following theorem is usually referred to as a Carleson estimate. We want to point out that, compared to [Avelin et al. 2016], not only does it hold for cylindrical NTA domains, but also the formulation is more flexible for applications. In particular, we are able to adjust the waiting time, the height of the cylinder, and the distance to the initial boundary. All these parameters influence the constant in the inequality and a Gaussian-type behavior is proved to be present.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and r_0. Let u be a nonnegative, weak solution in Ω_T. Let $(x, t) \in S_T$ and $0 < r < r_0$. Assume that $u(a_r(x), t) > 0$ and let*

$$\tau = \frac{C_4}{4}[C_5 u(a_r(x), t)]^{2-p} r^p,$$

where C_4 and C_5, both depending on p, n, are as in Theorem 3.9. Assume that $t > (\delta_1^{p-1} + \delta_2^{p-1} + 2\delta_3^{p-1})\tau$ for $0 < \delta_1 \le \delta_3 \le 1$, $\delta_2 \in (0, 1)$, and that for a given $\lambda \ge 0$, the function $(u-\lambda)_+$ vanishes continuously on $S_T \cap B_r(x) \times (t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau, t - \delta_1^{p-1}\tau)$ from Ω_T. Then there exist constants $c_i \equiv c_i(M, p, n)$, $i \in \{6, 7\}$, such that

$$\sup_Q u \le \left(\frac{c_6}{\delta_3}\right)^{c_7/\delta_1} u(a_r(x), t) + \lambda,$$

where $Q := B_r(x) \times (t - (\delta_1^{p-2} + \delta_2^{p-1})\tau, t - \delta_1^{p-1}\tau)$. Furthermore, the constants c_i, $i \in \{6, 7\}$, are stable as $p \to 2^+$.

Remark 4.3. Note that in the case $p = 2$, Theorem 4.2 for $\lambda = 0$ is equivalent to the estimate above by linearity. However, for $p > 2$ this result extends the ones in [Avelin 2016; Avelin et al. 2016] if $\lambda > 0$.

Proof. By scaling the function u we can assume that $u(a_r(x), t) = 1$, and replacing λ with its scaled version. Consider the boxes

$$\widetilde{Q}_\nu := (\Omega \cap B_r(x)) \times (t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau, t - \nu^{p-1}\tau), \quad \nu \in \{\delta_1, \delta_3\},$$

and define also

$$Q_r^\theta(x, t) := (\Omega \cap B_r(x)) \times (t - \theta^{2-p} r^p, t), \quad \theta > 0.$$

Observe that with the choice of τ, we have by Theorem 3.9, for any $(y, s) \in \widetilde{Q}_\nu$ with $d(y, \partial\Omega) \leq r$ and $\nu \in \{\delta_1, \delta_3\}$, that

$$u(y, s) \leq c_8^{1/\nu} \left(\frac{r}{d(y, \partial\Omega)} \right)^{c_9/\nu} \tag{4-1}$$

holds with c_8, c_9 depending only on p, n, M (apply Theorem 3.9 with the choice $\delta := \nu \min\{1, C_4/2\}$ in order to guarantee the conditions in (3-16)).

We proceed by induction via a contradiction assumption. Assume that $P_0 = (x_0, t_0) \in Q$, where Q is as in the statement, is such that $u(P_0) > H + \lambda$ for some large H to be fixed. Assume then that we find inductively points $P_j = (x_j, t_j) \in \widetilde{Q}_{\hat{\delta}_j}$, where $\hat{\delta}_j = \delta_3$ if $t_j \leq t_0 - \delta_3^{p-1}\tau/2$ and $\hat{\delta}_j = \delta_1$ if $t_j > t_0 - \delta_3^{p-1}\tau/2$ for any $j \in \mathbb{N}$. Set $r_j := d(x_j, \partial\Omega)$ and let $x_j' \in \partial\Omega$ be such that $r_j = |x_j - x_j'|$. Assume inductively that

$$u(P_j) > 2^j H + \lambda \quad \text{and} \quad t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau < t_j \leq t_{j-1} \leq t_j + (2^j H)^{2-p} (r_{j-1}/\sigma)^p \tag{4-2}$$

hold for all $j \in \{1, \dots, k\}$, where $\sigma \equiv \sigma(p, n, M) \in (0, 1)$ is as in Lemma 4.1. We then want to show that for large enough H this continues to hold for $j = k + 1$ as well.

To show the induction step, observe that (4-1) and the induction (4-2) imply

$$2^j H + \lambda < u(P_j) \leq c_8^{1/\hat{\delta}_j} \left(\frac{r}{r_j} \right)^{c_9/\hat{\delta}_j} \quad \Longrightarrow \quad r_j \leq (2^j H c_8^{-1/\hat{\delta}_j})^{-\hat{\delta}_j/c_9}.$$

Fixing

$$H := \left(\frac{4 c_8^{p/c_9} c_9}{p\sigma^p \log 2} \frac{\tau}{r^p \delta_3^p} \right)^{c_9/[p\delta_1]} =: \left(\frac{c_6}{\delta_3} \right)^{c_7/\delta_1},$$

we have after simple manipulations that

$$r_j^p \leq \sigma^p \frac{\delta_3^{p-1}\tau}{4} 2^{-j\hat{\delta}_j p/c_9} \frac{\delta_3 p \log 2}{c_9} \leq \sigma^p \frac{\delta_3^{p-1}\tau}{4} \frac{2^{-j\hat{\delta}_j p/c_9}}{\sum_{j=0}^\infty 2^{-j\delta_3 p/c_9}}. \tag{4-3}$$

In particular,

$$(2^{k+1} H)^{2-p} \left(\frac{r_k}{\sigma} \right)^p \leq \left(\frac{r_k}{\sigma} \right)^p \leq \frac{\delta_3^{p-1}\tau}{4}.$$

Hence we set $K_1 := Q^{2^k H}_{r_k}(x'_k, t_k)$ and $K_2 := Q^{2^{k+1} H}_{r_k/\sigma}(x'_k, t_k)$, and we deduce, by the induction assumption and the estimate in the previous display, that $K_2 \subset \Omega_T$. Now, if

$$\sup_{K_2}(u-\lambda)_+ \le 2^{k+1} H,$$

then, using that $(u-\lambda)_+$ is a weak subsolution, Lemma 4.1 would imply

$$\sup_{K_1} u \le 2^k H + \lambda,$$

which is a contradiction since $P_k \in \overline{K}_1$. Thus there is $P_{k+1} \in K_2$ such that

$$u(P_{k+1}) > 2^{k+1} H + \lambda.$$

By the definition of K_2 and P_{k+1}, we must have

$$t_{k+1} \le t_k \le t_{k+1} + (2^{k+1} H)^{2-p} (r_k/\sigma)^p.$$

Therefore we are left to show that $t_{k+1} \ge t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau$ in order to prove the induction step.

To this end, let now $\hat{k} \le k+1$ be the largest integer such that $t_0 - t_{\hat{k}} \le \delta_3^{p-1}\tau/2$. We may without loss of generality assume that $\hat{k} < k+1$, since otherwise $t_{k+1} \ge t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau$, because $t_0 > t - (\delta_1^{p-1} + \delta_2^{p-1})\tau$. Now (4-3) and (4-2), together with the fact that $\hat{\delta}_j = \delta_3$ for $j > \hat{k}$, give

$$\begin{aligned} t_0 - t_{k+1} &= (t_0 - t_{\hat{k}}) + (t_{\hat{k}} - t_{k+1}) \\ &\le \frac{\delta_3^{p-1}\tau}{2} + \sum_{j=\hat{k}}^{k} (t_j - t_{j+1}) \\ &\le \frac{\delta_3^{p-1}\tau}{2} + (2^{\hat{k}+1} H)^{2-p} \left(\frac{r_{\hat{k}}}{\sigma}\right)^p + \sum_{j=\hat{k}+1}^{k} (2^{j+1} H)^{2-p} \left(\frac{r_j}{\sigma}\right)^p \\ &\le \frac{\delta_3^{p-1}\tau}{2} + \frac{\delta_3^{p-1}\tau}{4} + \frac{\delta_3^{p-1}\tau}{4} \left[\sum_{j=0}^{\infty} 2^{-j\delta_3 p/c_9}\right]^{-1} \sum_{j=\hat{k}+1}^{k} 2^{-j\delta_3 p/c_9} < \delta_3^{p-1}\tau. \end{aligned} \tag{4-4}$$

Therefore, since $t_0 > t - (\delta_1^{p-1} + \delta_2^{p-1})\tau$, we have

$$t - t_{k+1} = t - t_0 + t_0 - t_k < (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau,$$

which was to be proven. Hence we have concluded the proof of the induction step. As a consequence, we have constructed a sequence of points $P_j = (x_j, t_j) \in \widetilde{Q}_{\delta_1}$ such that $d(x_j, \partial\Omega) \to 0$ and $u(P_j) \to \infty$ as $j \to \infty$. This violates the assumed continuity of $(u-\lambda)_+$ in the neighborhood of $S_T \cap \overline{\widetilde{Q}}_{\delta_1}$, giving the contradiction. Hence,

$$\sup_{Q} u \le H + \lambda,$$

completing the proof of the theorem. □

5. Estimating the boundary-type Riesz measure

In this section we establish, in NTA cylinders, upper and lower bounds for the measure μ defined in (1-6).

5A. *Upper estimate on μ.* We will first provide an upper bound on the measure. The proof relies on the Carleson estimate in Theorem 4.2 and the following standard Caccioppoli-type estimate; see [DiBenedetto 1993].

Lemma 5.1. *Let u be a nonnegative weak subsolution in Ω_T, and $\phi \in C_0^\infty(\Omega \times (t_1, T))$ with $\phi \geq 0$. Then*

$$\int_{t_1}^{t_2}\int_\Omega |\nabla u|^p \phi^p \,\mathrm{d}x\,\mathrm{d}t \leq C\biggl(\int_{t_1}^{t_2}\int_\Omega u^p |\nabla \phi|^p \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega u^2 (\phi_t)_+ \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t\biggr)$$

for $C = C(p,n)$.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and r_0. Let $0 < r \leq r_0$ and let u be a weak nonnegative solution in Ω_T. Fix a point $x_0 \in \partial\Omega$ and define*

$$\tau = \frac{C_4}{16}[C_5 u(a_r(x_0), t_0)]^{2-p} r^p,$$

where C_4 and C_5, both depending on p, n, are as in Theorem 3.9. Let $0 < \delta \leq \tilde{\delta} \leq 1$ and assume that $t_0 > 5\tilde{\delta}^{p-1}\tau$ and that u vanishes continuously on $S_T \cap \bigl(B_r(x) \times (t_0 - 4\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau)\bigr)$ from Ω_T. Then there is a constant $C \equiv C(p,n)$ and $c_8 \equiv c_8(p,n,M)$ such that

$$\frac{\mu(Q)}{r^n} \leq C\left(\frac{c_6}{\tilde{\delta}}\right)^{c_8/\delta} u(a_r(x_0), t_0),$$

where μ is the measure from (1-6),

$$Q := B_{r/2}(x_0) \times (t_0 - 2\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau),$$

and c_6 is from Theorem 4.2. Furthermore, the constants C, c_8, are stable as $p \to 2^+$.

Proof. After scaling, we may assume that $u(a_r(x_0), t_0) = 1$. Let

$$\widehat{Q} = B_r(x_0) \times (t_0 - 3\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau),$$

and observe, by our assumptions, that Theorem 4.2 implies

$$\sup_{\widehat{Q}} u \leq \left(\frac{c_6}{\tilde{\delta}}\right)^{c_7/\delta} =: \Lambda. \tag{5-1}$$

As in the construction of the measure μ in (1-6), we see that extending u to the entire cylinder $\widehat{Q}$ as zero, we obtain a weak subsolution in $\widehat{Q}$. Take a cut-off function $\phi \in C^\infty(\widehat{Q})$ vanishing on $\partial_p \widehat{Q}$ such that $0 \leq \phi \leq 1$, ϕ is 1 on Q, and

$$|\nabla \phi| < \frac{4}{r} \quad \text{and} \quad (\phi_t)_+ < \frac{4}{[\tilde{\delta}^{p-1}\tau]}.$$

Then by (1-6), the definition of ϕ and Hölder's inequality we get

$$\begin{aligned}\int_{\widehat{Q}} \phi^p \,\mathrm{d}\mu &\le \int_{\widehat{Q}} |\nabla u|^{p-1} |\nabla \phi| \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{4}{r} \int_{\widehat{Q}} |\nabla u|^{p-1} \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{4}{r} |\widehat{Q}|^{1/p} \biggl(\int_{\widehat{Q}} |\nabla u|^p \phi^p \,\mathrm{d}x\,\mathrm{d}t \biggr)^{(p-1)/p} + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Now using Lemma 5.1 and (5-1) we see that

$$\begin{aligned}\mu(Q) &\le C \frac{|\widehat{Q}|^{1/p}}{r} \biggl(\int_{\widehat{Q}} u^p |\nabla \phi|^p + u^2 (\phi_t)_+ \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t \biggr)^{(p-1)/p} + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \frac{|\widehat{Q}|^{1/p}}{r} \biggl(|\widehat{Q}| \biggl(\frac{\Lambda^p}{r^p} + \frac{\Lambda^2}{\tilde{\delta}^{p-1} \tau} \biggr) \biggr)^{(p-1)/p} + C |\widehat{Q}| \frac{\Lambda}{\tilde{\delta}^{p-1} \tau} \\ &\le C \frac{|\widehat{Q}|}{\tilde{\delta}^{p-1} r^p} \Lambda^{p-1}.\end{aligned}$$

After scaling back, this can be rewritten in the homogeneous form

$$\frac{\mu(Q_r)}{r^n} \le C \biggl(\frac{c_6}{\tilde{\delta}} \biggr)^{(p-1)c_7/\delta} u(a_r(x_0), t_0),$$

completing the proof with $c_8 = (p-1)c_7$. □

5B. ***Lower estimate on μ.*** We next prove the lower bound for the measure μ.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and r_0, and let u be a weak nonnegative solution in Ω_T. Fix a point $(x_0, t_0) \in \partial\Omega \times (0, T]$, and define $A_r^- = (a_{r/2}(x_0), t_0)$ for $0 < r < r_0$. There exist C, τ_0, τ_1, all depending only on p, n, M, such that if*

$$\bigl(t_0 - \tau_0 u(A_r^-)^{2-p} r^p, t_0 + (\tau_0 + \tau_1) u(A_r^-)^{2-p} r^p\bigr) \subset (0, T),$$

and if u vanishes continuously on $S_T \cap \bigl(B_r(x) \times (t_0, t_0 + (\tau_0 + \tau_1) u(A_r^-)^{2-p} r^p)\bigr)$ from Ω_T, then

$$u(A_r^-) \le C \frac{\mu(Q)}{r^n},$$

where μ is the measure from (1-6) *and*

$$Q := B_r(x_0) \times \bigl(t_0 + \tau_0 u(A_r^-)^{2-p} r^p, t_0 + (\tau_0 + \tau_1) u(A_r^-)^{2-p} r^p\bigr).$$

Furthermore, the constants C, τ_0, τ_1, are stable as $p \to 2^+$.

To prove Theorem 5.3 we first consider the model problem in Lemma 5.4, and we prove that the measure associated to this model problem is bounded from below by a constant. Returning to Theorem 5.3, we then apply the intrinsic Harnack inequality to obtain a lower bound on the function such that by the comparison principle the solution v in Lemma 5.4 is below our solution u. The result then follows by the fact that the corresponding measures are ordered according to Lemma 5.5, a fact easily realized if the domain is smooth, as in this case the measure is just the modulus of the gradient to the power $p-1$.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^n$ be an NTA domain with constants M and $r_0 = 2$. There exist constants C, T_M, both depending on p, n, M, such that if v is a continuous solution to the problem*

$$\begin{cases} v_t - \Delta_p v = 0 & \text{in } (\Omega \cap B_2(0)) \times (0, T_0), \\ v = 0 & \text{on } \partial(\Omega \cap B_2(0)) \times [0, T_0), \\ v = \chi_{B_{1/(4M)}(a_1(0))} & \text{on } (\Omega \cap B_2(0)) \times \{0\}, \end{cases}$$

then

$$\mu_v(B_2(0) \times (0, T_0)) \geq \frac{1}{C}.$$

Furthermore, the constants C, T_M are stable as $p \to 2^+$.

Proof. To begin with, extend v as zero to the rest of $Q = B_2(0) \times (0, T_0)$, i.e., set $v \equiv 0$ in $(B_2(0) \setminus \Omega) \times (0, T_0)$, and let μ_v be the associated measure as in (1-6). Let then h be the solution to the problem

$$\begin{cases} h_t - \Delta_p h = 0 & \text{in } Q, \\ h = w & \text{on } \partial_p Q. \end{cases}$$

We observe that the supremum of h and v, which is 1, is attained at the bottom of the cylinder. Let us now recall the decay estimate in Lemma 4.1, which implies that for $Q_r^\lambda(0, t_0) := (\Omega \cap B_r(0)) \times (t_0 - \lambda^{2-p} r^p, t_0)$ we have

$$\sup_{Q^{2^{-j}}_{\sigma^j}(0,t_0)} v \leq 2^{-j} \tag{5-2}$$

for $j \in \mathbb{N}$ provided that $t_0 \in [1, T_0]$ and $T_0 > 1$. On the other hand, Lemma 3.3 gives us

$$1 = \fint_{B_{(4M)^{-1}}(a_1(0))} h(x, 0)\, \mathrm{d}x \leq C \inf_{z \in B_{(2M)^{-1}}(0)} u(z, \tau),$$

with τ and C depending on p, n, M. We then apply Theorem 3.2 in order to get

$$1 \leq \widehat{C} \inf_{\widehat{Q}} h, \quad \widehat{Q} := B_{1/M}(0) \times \left(\frac{T_0}{2}, T_0\right), \tag{5-3}$$

by properly choosing T_0 by means of τ and C to be larger than 2. We then choose large enough $j^\star \in \mathbb{N}$ so that

$$2^{-j^\star} \leq \frac{1}{2\widehat{C}} \quad \text{and} \quad \sigma^j \leq \frac{1}{M}.$$

Then, sliding t_0 along $(1, T_0]$ in (5-2), we obtain by combining (5-2) and (5-3) that there is $r_1 = r_1(p, n, M)$ such that

$$\inf_{\widetilde{Q}} (h - v) \geq \frac{1}{2\widehat{C}} =: \epsilon, \quad \widetilde{Q} := B_{r_1}(0) \times \left(\frac{T_0}{2}, T_0\right),$$

where $\epsilon \equiv \epsilon(p, n, M)$.

Let us now define $\phi = \min\{h - v, \epsilon\}$, which is vanishing on $\partial_p Q$ and is ϵ on $\widetilde{Q}$. Then from the weak formulation of h and v we get

$$\int_{Q\cap\{t<\tau\}} (h-v)_t\,\phi + (|\nabla h|^{p-2}\nabla h - |\nabla v|^{p-2}\nabla v)\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \int_{Q\cap\{t<\tau\}} \phi \,\mathrm{d}\mu_w.$$

For the time term, we integrate to obtain

$$\int_{Q\cap\{t<\tau\}} (h-v)_t\,\phi \,\mathrm{d}x\,\mathrm{d}t = \int_{Q\cap\{t<\tau\}} \partial_t\left(\int_0^{h-v} \min(s,\epsilon)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t \ge \frac{1}{2}\int_{B_2(0)} \phi^2(x,\tau)\,\mathrm{d}x.$$

We also have the elementary inequality

$$(|\nabla h|^{p-2}\nabla h - |\nabla v|^{p-2}\nabla v)\cdot\nabla\phi \ge \frac{1}{C}|\nabla\phi|^p,$$

since $p > 2$. Hence by combining the last three displays we arrive at

$$\sup_{0<t<T_0}\int_{B_1(0)} \phi^2(x,t)\,\mathrm{d}x + \int_Q |\nabla\phi|^p \,\mathrm{d}x\,\mathrm{d}t \le C\epsilon\mu_v(Q).$$

Using the parabolic Sobolev inequality [DiBenedetto 1993, Corollary I.3.1] we obtain

$$\begin{aligned}
\epsilon^p|\widetilde{Q}| &\le \int_Q \phi^p \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\left(\sup_{0<t<T_0}\int_{B_2(0)} \phi^p(x,t)\,\mathrm{d}x + \int_Q |\nabla\phi|^p \,\mathrm{d}x\,\mathrm{d}t\right) \\
&\le C\left(\epsilon^{p-2}\sup_{0<t<T_0}\int_{B_2(0)} \phi^2(x,t)\,\mathrm{d}x + \int_Q |\nabla\phi|^p \,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}$$

Hence we see that

$$1 \le C\,\mu_v(Q)$$

with a constant $C \equiv C(p,n,M)$ through the dependencies of ϵ, r_1, T_0. □

The next lemma provides a comparison estimate for the measures. If two solutions are ordered, then the corresponding measures will be ordered as well.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let u and v be weak solutions in $(\Omega\cap B_r(0))\times(0,T)$ such that $u \ge v \ge 0$ and both vanish continuously on the lateral boundary $(\partial\Omega\cap B_r(0))\times(0,T)$. Then*

$$\mu_v \le \mu_u \quad \text{in } B_r(0)\times(0,T)$$

in the sense of measures.

Proof. To show this, consider the test function $\phi = \min(1, (u-v-\epsilon)_+/\epsilon)\psi$, where ψ is nonnegative and belongs to $C_0^\infty(Q)$ with $Q = B_r(0)\times(0,T)$. Obviously ϕ is supported in $(\Omega\cap B_r(0))\times(0,T)$, because both u and v vanish continuously on the lateral boundary $(\partial\Omega\cap B_r(0))\times(0,T)$. Since both u and v are weak solutions, we have, by extending them both by zero in $(B_r(0)\setminus\Omega)\times(0,T)$, that

$$\int_Q (u-v)_t\,\phi \,\mathrm{d}x\,\mathrm{d}t + \int_Q (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5-4}$$

Let us first treat the time term. Integrating by parts we get

$$\begin{aligned}\int_Q (u-v)_t\,\phi\,\mathrm{d}x\,\mathrm{d}t &= \int_Q \partial_t\biggl(\int_0^{u-v}\min\Bigl(1,\frac{(s-\epsilon)_+}{\epsilon}\Bigr)\,\mathrm{d}s\biggr)\psi\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_Q\biggl(\int_0^{u-v}\min\Bigl(1,\frac{(s-\epsilon)_+}{\epsilon}\Bigr)\,\mathrm{d}s\biggr)\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\to -\int_Q (u-v)\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t \qquad (5\text{-}5)\end{aligned}$$

as $\epsilon \to 0$. To treat the elliptic term, we begin by noting that

$$\nabla\phi = \nabla\psi\,\min\Bigl(1,\frac{(u-v-\epsilon)_+}{\epsilon}\Bigr) + \frac{1}{\epsilon}\psi\nabla(u-v)\chi_{U_\epsilon}, \qquad (5\text{-}6)$$

where $U_\epsilon := \{u-v>\epsilon\}\cap(\Omega\cap B_r(0))\times(0,T)$. The second term in (5-6) will give rise to a positive term in (5-4); hence we discard it and obtain the inequality

$$\begin{aligned}&\int_Q (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\ge \int_Q (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\psi\,\min\Bigl(1,\frac{(u-v-\epsilon)_+}{\epsilon}\Bigr)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\to \int_Q (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \qquad (5\text{-}7)\end{aligned}$$

as $\epsilon\to 0$ by dominated convergence. Combining the convergence in (5-5) and (5-7) with (5-4) we arrive at the inequality

$$-\int_Q (u-v)\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t + \int_Q (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \le 0 \qquad (5\text{-}8)$$

after sending $\epsilon\to 0$. Since the nonnegative function $\psi \in C_0^\infty(Q)$ is arbitrary, (5-8) finishes the proof after recalling the definitions of μ_u and μ_v. □

We now have all the technical tools to complete the proof of Theorem 5.3.

Proof of Theorem 5.3. Let u be as in Theorem 5.3 with $A_r^- := (a_{r/2}(x_0), t_0)$. Applying the Harnack estimate in Theorem 3.6 yields for a constant $C = C(p,n,M)$

$$u(A_r^-) \le \widetilde{C} \inf_{y\in B_{r/(8M)}(a_{r/2}(x_0))} u(y, t_0 + \tau_0 u(A_r^-)^{2-p} r^p), \qquad \tau_0 := \frac{C}{(2M)^p},$$

since

$$t_0 - \tau_0 u(A_r^-)^{2-p} r^p > 0.$$

Consider the scaled solution

$$\hat{u}(x,t) = \frac{1}{\lambda} u\Bigl(x_0 + \frac{r}{2}x, t_0 + \tau_0 u(A_r^-)^{2-p} r^p + \lambda^{2-p}\Bigl(\frac{r}{2}\Bigr)^p t\Bigr), \qquad \lambda := \frac{u(A_r^-)}{\widetilde{C}},$$

so that

$$\inf_{y\in B_{1/(4M)}(a_1(0))} \hat{u}(y,0) \geq 1,$$

and Ω is mapped to $\widehat{\Omega}$ and $(x_0, t_0+\tau_0 u(A_r^-)^{2-p} r^p)$ to $(0,0)$. The comparison principle shows that the function v defined in Lemma 5.4 satisfies $v \leq \hat{u}$ in $(\widehat{\Omega}\cap B_2(0))\times(0,T_0)$ provided that

$$t_0+\tau_0 u(A_r^-)^{2-p} r^p + \lambda^{2-p}\Big(\frac{r}{2}\Big)^p T_0 < T.$$

Thus we choose $\tau_1 := \widetilde{C}^{p-2} 2^{-p} T_0$ in the statement. Applying Lemmas 5.4 and 5.5

$$\mu_{\hat{u}}(B_2(0)\times(0,T_0)) \geq \mu_v(B_2(0)\times(0,T_0)) \geq \frac{1}{C}.$$

Scaling back to u gives us the result. □

6. Construction of barriers

In this section we construct the barriers that will serve as the starting point for the estimates of the decay rate of the solutions. The upper barrier in Lemma 6.4 is based on the function constructed in [DiBenedetto et al. 1991, Theorem 4.1]. However the subsolution constructed in Lemma 6.1 seems to be new and allows us to obtain p-stable estimates from below on the decay rate.

Lemma 6.1. *Let $T=(n^{p-1}p)^{-1}$ and for $a\in(0,1)$, let*

$$\varrho_0 := \min\left\{\frac{ap}{n(p-2)}+1, 2\right\}^{(p-1)/p}.$$

Then the function h,

$$h(x,t) = g(|x|,t) - g(\varrho_0,t), \qquad g(r,t) = \left[1-\frac{p-2}{p^{p/(p-1)}}\frac{r^{p/(p-1)}-1}{t^{1/(p-1)}}\right]_+^{(p-1)/(p-2)},$$

is a classical subsolution in $(B_{\varrho_0}(0)\setminus\overline{B_1(0)})\times(0,T)$ satisfying the boundary conditions

$$\begin{cases} h=0 & \text{on } \partial B_{\varrho_0}(0)\times(0,T),\\ h=0 & \text{on } (B_{\varrho_0}(0)\setminus\overline{B_1(0)})\times\{0\},\\ h=1-g(\varrho_0,t) & \text{on } \partial B_1(0)\times(0,T).\end{cases} \tag{6-1}$$

Furthermore, $x\mapsto h(x,t)$ is a radially decreasing function satisfying

$$\inf_{1\leq|x|\leq\varrho_0} |\nabla h(x,T)| \geq n\exp\Big(-\frac{n}{p}\Big)(1-a)^{n/p},$$

and $h(x,t)\equiv h_p(x,t)$ tends to

$$\exp\Big(-\frac{|x|^2-1}{4t}\Big) - \exp\Big(-\frac{1}{4t}\Big)$$

as $p\to 2^+$.

Remark 6.2. Note that the function in Lemma 6.1 is not continuous up to the boundary at the corner $\partial B_1(0) \times \{0\}$. However, the limsup as we approach a point on this piece from the inside of $D := (B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$ is 1 for h. This implies, see [Kilpeläinen and Lindqvist 1996, Lemma 4.4], that if we have a weak supersolution u in D, staying above the boundary conditions in (6-1) in liminf sense, and staying above 1 on the corner $\partial B_1(0) \times \{0\}$ again in the liminf sense, then u will be above h in D.

Proof. Let h and ϱ_0 be as stated. By construction, the boundary conditions for h are in force. To verify that h is a classical subsolution in $(B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$, we first compute

$$
\begin{aligned}
\nabla g(|x|, t) &= -\frac{x}{|x|}\left[\frac{1}{p}\frac{|x|}{t} g(|x|, t)\right]^{1/(p-1)}, \\
|\nabla g(|x|, t)|^{p-2}\nabla g(|x|, t) &= -\frac{1}{p}\frac{x}{t} g(|x|, t), \\
-\Delta_p g(|x|, t) &= \left[\frac{n}{pt} g(|x|, t)^{(p-2)/(p-1)} - p^{-p/(p-1)}\frac{|x|^{p/(p-1)}}{t^{p/(p-1)}}\right] g(|x|, t)^{1/(p-1)}, \\
\partial_t g(r, t) &= p^{-p/(p-1)}\frac{r^{p/(p-1)} - 1}{t^{p/(p-1)}} g(r, t)^{1/(p-1)}.
\end{aligned}
$$

Observing that $\partial_t h(x, t) \le \partial_t g(|x|, t)$, it is enough to verify $g_t - \Delta_p g \ge 0$ in $(B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$ for $g > 0$. Assuming $g > 0$, we see, since $g(|x|, t) \le 1$ for $|x| > 1$, that

$$
\begin{aligned}
\frac{(h_t - \Delta_p h)(x, t)}{g(|x|, t)^{1/(p-1)}} &\le \left[p^{-p/(p-1)}\frac{|x|^{p/(p-1)} - 1}{t^{p/(p-1)}} + \frac{n}{pt} g(|x|, t)^{(p-2)/(p-1)} - p^{-p/(p-1)}\frac{|x|^{p/(p-1)}}{t^{p/(p-1)}}\right] \\
&\le \frac{1}{pt}\left[n - \left(\frac{1}{pt}\right)^{1/(p-1)}\right].
\end{aligned}
$$

Since $0 < t < T = (n^{p-1} p)^{-1}$ we have $h_t - \Delta_p h \le 0$ for $|x| > 1$ and $t \in (0, T)$. Note also that our choices of parameters are stable as $p \to 2$. Next, by yet another explicit calculation we obtain

$$
\inf_{1 \le |x| \le \varrho_0} |\nabla h(x, T)| \ge \left(\frac{1}{pT} g(\varrho_0, T)\right)^{1/(p-1)} = n g(\varrho_0, T)^{1/(p-1)}.
$$

To complete the proof we need to estimate $g(\varrho_0, T)$ from below. To do this we note that

$$
g(\varrho_0, T)^{1/(p-1)} = \left[1 - \frac{n(p-2)}{p}(\varrho_0^{p/(p-1)} - 1)\right]^{1/(p-2)},
$$

and we consider two cases. First, if $\varrho_0 = 2^{(p-1)/p}$, then $ap \ge n(p-2)$ and

$$
g(\varrho_0, T)^{1/(p-1)} = (1 - s)^{b/s}, \quad s = \frac{n(p-2)}{p}, \quad b = \frac{n}{p}.
$$

Furthermore, for $\eta \in [0, 1)$ we have

$$
\begin{aligned}
(1-\eta)^{b/\eta} &= \exp\biggl(-\frac{b}{\eta}\sum_{k=1}^{\infty}\frac{\eta^k}{k}\biggr) = \exp\biggl(-b\sum_{k=1}^{\infty}\frac{\eta^{k-1}}{k}\biggr) = \exp\biggl(-b-b\sum_{k=1}^{\infty}\frac{\eta^k}{k+1}\biggr) \\
&\geq \exp\biggl(-b-b\sum_{k=1}^{\infty}\frac{\eta^k}{k}\biggr) = e^{-b}(1-\eta)^b. \qquad (6\text{-}2)
\end{aligned}
$$

Since $ap \geq n(p-2)$ implies $s \leq a < 1$, we can apply (6-2) to get

$$g(\varrho_0, T)^{1/(p-1)} \geq \exp(-b)(1-s)^b \geq \exp(-b)(1-a)^b.$$

Second, if $\varrho_0 < 2^{(p-1)/p}$, then $ap < n(p-2)$ and using (6-2) we get

$$g(\varrho_0, T) = (1-a)^{(p-1)/(p-2)} \geq (1-a)^{b/a} \geq \exp(-b)(1-a)^b.$$

Collecting the results of the two cases completes the proof of the lemma. □

Remark 6.3. Note that we could, as in [Stan and Vázquez 2013], instead of the function in Lemma 6.1 use the Barenblatt fundamental solution together with the barriers from [Bidaut-Véron 2009] to establish a version of Lemma 6.1. However, this would result in a construction which is not p-stable.

In the next lemma we construct a certain supersolution to be used in the subsequent arguments.

Lemma 6.4. *Let $T, H > 0$ be given degrees of freedom. Let*

$$k \in (0, k_0], \quad k_0 := \min\biggl\{\frac{p-1}{n}, T^{1/(p-1)}H^{(p-2)/(p-1)}\biggr\}.$$

There exists a classical supersolution $\tilde{h}$ in

$$N = \{(x,t) : 1 < |x| < 1+k,\ 0 < t < T\}$$

such that

$$
\begin{cases}
\tilde{h} \geq 0 & \text{in } \partial B_1(0) \times (0, T], \\
\tilde{h} \geq H & \text{on } (\overline{B_{1+k}(0)} \setminus B_1(0)) \times \{0\}, \\
\tilde{h} \geq H & \text{on } \partial B_{1+k}(0) \times [0, T],
\end{cases}
$$

and such that

$$\tilde{h}(x, T) \leq \frac{H \exp(2)}{k}(|x|-1) \qquad (6\text{-}3)$$

whenever $x \in B_{1+k}(0) \setminus B_1(0)$.

Proof. This type of construction was originally carried out in [DiBenedetto et al. 1991, Theorem 4.1] and we here include a proof for completeness. Let

$$v(x,t) = \exp\biggl(\frac{t-T}{T} - \frac{|x|-1}{k}\biggr),$$

and let

$$\tilde{h}(x,t) = \widetilde{H}(1-v(x,t)), \quad \widetilde{H} = H\exp(2),$$

accordingly. Then

$$\partial_t \tilde{h}(x,t) = -\frac{\widetilde{H}}{T} v(x,t) \quad \text{and} \quad \nabla \tilde{h}(x,t) = \frac{\widetilde{H} x}{k|x|} v(x,t).$$

Observe also that $\widetilde{H} v \geq H$ for all $(x,t) \in N$ and that $\tilde{h}$ satisfies the boundary conditions. We now show that $\tilde{h}$ is a classical supersolution in N. Indeed, by a straightforward calculation we see that

$$\begin{aligned} \tilde{h}_t - \Delta_p \tilde{h} &= -\frac{\widetilde{H}}{T} v + \widetilde{H}^{p-1} v^{p-1} \Big((p-1) k^{-p} - k^{1-p} \frac{n-1}{|x|} \Big) \\ &\geq \frac{\widetilde{H}}{T} v \big(-1 + T H^{p-2} k_0^{1-p} + T H^{p-2} k^{1-p} ((p-1) k_0^{-1} - n) \big) \geq 0 \end{aligned}$$

whenever $(x,t) \in N$. Finally, since

$$\sup_{1<|x|<1+k} |\nabla \tilde{h}(x,T)| \leq \frac{\widetilde{H}}{k} \quad \text{and} \quad \tilde{h}(x,T) = 0 \quad \text{for all } x \in \partial B_1(0),$$

we obtain the upper bound for $\tilde{h}(x,T)$ as well. □

7. Decay estimates and a change of variables

In this section we prove a lower bound (Lemma 7.2) and an upper bound (Lemma 7.4) on the decay of solutions. The following lemma, which is a change of variables, will be used in the proof of our decay estimates. The proof of the lemma follows from [Kuusi 2008, Lemma 3.5].

Lemma 7.1. *Let $u = u(x,t)$ be p-parabolic in $\Omega \times (T_0, T_1)$. Let $C > 0$ be given and let*

$$f(\tau) = \frac{1}{C(p-2)} \big(\exp(C(p-2)\tau) - 1 \big), \quad g(\eta) = (C(p-2)\eta + 1)^{1/(p-2)}$$

for $p > 2$, and

$$f(\tau) = \tau, \quad g(\eta) = \exp(C\eta)$$

for $p = 2$. Let $w(x,\tau) = g(f(\tau))\, u(x, f(\tau))$. Then $w(x,\tau)$ is a (weak) solution to the equation,

$$\partial_\tau w = \Delta_p w + C w$$

in $\Omega \times (\tau_0, \tau_1)$, where $f(\tau_i) = T_i,\ i \in \{0,1\}$.

7A. *A lower bound on the decay.* Using the classical subsolution constructed in Lemma 6.1 and the change of variable outlined in Lemma 7.1 we here prove the following lemma, which describes the optimal decay rate from below after a certain intrinsic waiting time. This lemma will be crucial when proving global $C^{1,1}$-estimates, see Section 8, and when proving the local $C^{1,1}$-estimates, see Section 9.

Lemma 7.2. *Let $0 < \varrho \leq r/4$ and let $g \in L^2(B_r(x_0))$ be a nonnegative function satisfying*

$$\fint_{B_\varrho(x_0)} g \, \mathrm{d}x \geq \lambda > 0.$$

Assume that $\hat h \in C([t_0,\infty); L^2(B_r(x_0)))$ is a weak nonnegative solution solving the Cauchy problem

$$\begin{cases} \hat h_t - \Delta_p \hat h = 0 & \text{in } B_r(x_0)\times(t_0,\infty),\\ \hat h = g & \text{on } B_r(x_0)\times\{t_0\}. \end{cases}$$

Then there exist constants $c_i \equiv c_i(\varrho/r, n, p)$, $i \in \{1,2\}$, such that

$$\hat h(x,t) \ge \frac{\lambda}{c_1}\left(c_1(p-2)\frac{t-t_0}{\lambda^{2-p}r^p}+1\right)^{-1/(p-2)}\frac{d(x,\partial B_r(x_0))}{r}$$

whenever $(x,t) \in B_r(x_0)\times(t_0+c_2\lambda^{2-p}r^p,\infty)$. Furthermore, the constants c_i, $i\in\{1,2\}$, are stable as $p\to 2^+$.

Proof. After scaling we may assume that $x_0 = 0,\ t_0 = 0,\ r = 1$, and $\lambda = 1$. Let

$$\varrho_0 := \min\left\{\frac{1}{2}\frac{p}{n(p-2)}+1, 2\right\}^{(p-1)/p}.$$

Applying Lemma 3.3 we find a time $t^\star \equiv t^\star(n,p,\varrho/r)$ and a constant $c^\star \equiv c^\star(n,p,\varrho/r)$ such that

$$\hat h(x,t^\star) \ge \frac{1}{c^\star} \quad \text{for all } x \in B_{1/\varrho_0}(0).$$

Set $\bar h(x,t) = c^\star \hat h(x/\varrho_0, t^\star + t[c^\star]^{p-2}/\varrho_0^p)$ and let $w(x,\tau) = g(f(\tau))\,\bar h(x,f(\tau))$, where g and f are defined as in Lemma 7.1. Then $w(x,\tau)$ is a nonnegative weak solution to the equation

$$\partial_\tau w = \Delta_p w + Cw$$

in $B_{\varrho_0}(0)\times(0,\infty)$ and $w(x,0)\ge 1$ for all $x\in B_1(0)$. In particular, w is a weak supersolution in $B_{\varrho_0}(0)\times(0,\infty)$. Now, Theorem 3.2, [Kuusi 2008, Corollary 3.6], Lemma 7.1, and [Kuusi 2008, Proposition 3.1] imply that we have, for a new constant $\bar c \equiv \bar c(n,p) > 1$,

$$w(x,\tau) \ge \frac{1}{\bar c}, \quad (x,\tau)\in B_1(0)\times(0,\infty), \tag{7-1}$$

provided we choose C large enough in the definitions of f and g in Lemma 7.1. Consider $\hat\tau \ge 0$ arbitrary, let h be the classical subsolution of Lemma 6.1 and let T be as in Lemma 6.1. Then, simply using the intrinsic scaling, the comparison principle, and (7-1) we see that

$$w(x,\tau) \ge \frac{1}{\bar c} h(x, \bar c^{2-p}(\tau-\hat\tau)) \tag{7-2}$$

whenever $(x,\tau)\in (B_{\varrho_0}(0)\setminus B_1(0))\times(\hat\tau, \hat\tau + \bar c^{p-2}T)$. Since $\hat\tau\ge 0$ is arbitrary we get from (7-2) and Lemma 6.1 that there is a $c\equiv c(n,p)$ such that

$$w(x,\tau) \ge \frac{1}{c} d(x,\partial B_{\varrho_0}(0)), \quad (x,\tau)\in B_{\varrho_0}(0)\times(\bar c^{p-2}T,\infty). \tag{7-3}$$

To complete the proof, it suffices to rephrase (7-3) in terms of $\hat h(x,t)$. □

7B. ***An upper bound on the decay.*** Working with solutions vanishing on the entire lateral boundary, we will make use of the following decay estimates.

Lemma 7.3. *Let $u \in C([0,T); L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$ be a nonnegative weak subsolution in $\Omega \times (0,T)$. Then there exists a constant $c = c(n, p, |\Omega|)$ such that*

$$\sup_{\Omega} u(\,\cdot\,,t) \le \frac{c}{t^{n/\sigma}} \left(\fint_{\Omega} u(x,0)\,\mathrm{d}x \right)^{p/\sigma}, \quad \sigma = n(p-2)+p.$$

The constant c is stable as $p \to 2^+$.

Proof. See [Kuusi and Parviainen 2009, Lemma 3.24] and use the L^1-contractivity (test with the function $\min(1, u/\epsilon)$) and the comparison principle. □

In the following lemma we describe the optimal decay of the supremum of a solution vanishing on the entire lateral boundary; it follows from an iterative rescaling and comparison together with the decay estimate in Lemma 7.3.

Lemma 7.4. *Let $u \in C([0,T); L^2(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega))$ be a nonnegative weak subsolution in $\Omega \times (0,\infty)$. Then there exist constants $c_i \equiv c_i(p,n,|\Omega|)$, $i \in \{1,2\}$, such that the following holds. Let*

$$\bar{\Lambda} := \fint_{\Omega} u(x,0)\,\mathrm{d}x.$$

Then

$$\sup_{\Omega} u(\,\cdot\,,t) \le c_1 \left((p-2) \frac{\bar{\Lambda}^{p-2}}{c_1} t + 1 \right)^{-1/(p-2)} \bar{\Lambda}$$

whenever $t > c_2 \bar{\Lambda}^{2-p}$. The constants c_i, $i \in \{1,2\}$, are stable as $p \to 2^+$.

Proof. Let $w = w(x,t)$ solve the Dirichlet problem

$$\begin{cases} w_t - \Delta_p w = 0 & \text{in } \Omega \times (0,\infty), \\ w = 0 & \text{on } \partial\Omega \times [0,\infty) \end{cases} \tag{7-4}$$

and assume that

$$\fint_{\Omega} w(x,0)\,\mathrm{d}x \le 1. \tag{7-5}$$

Applying Lemma 7.3 to w we see that

$$\sup_{\Omega} w(\,\cdot\,,t) \le \frac{c}{t^{n/\sigma}}$$

for some $c = c(n,p,|\Omega|)$, and for all $t > 0$. In particular, there exists $t^\star = t^\star(n,p,|\Omega|) > 0$ such that

$$\sup_{\Omega} w(\,\cdot\,,t^\star) \le \tfrac{1}{2}. \tag{7-6}$$

To prove the lemma we will now use (7-6) in an iterative argument. In particular, consider the function

$$w_1(x,t) := \bar{\Lambda}^{-1} u(x, \bar{\Lambda}^{2-p} t).$$

Then w_1 is a solution to (7-4) satisfying (7-5). Hence we have by (7-6) that

$$w_1(x, t^\star) \le \tfrac{1}{2},$$

which after scaling back becomes

$$u(x, \bar\Lambda^{2-p} t^\star) \le 2^{-1}\bar\Lambda \quad \text{whenever } x \in \Omega.$$

Next, consider the function

$$w_2(x,t) := (2^{-1}\bar\Lambda)^{-1} u(x, \bar\Lambda^{2-p} t^\star + (2^{-1}\bar\Lambda)^{2-p} t),$$

which again satisfies (7-4) and (7-5). Applying (7-6) to the function w_2 we deduce, by elementary manipulations, that

$$\sup_\Omega u(\,\cdot\,, (\bar\Lambda^{2-p} + (2^{-1}\bar\Lambda)^{2-p}) t^\star) \le 2^{-2}\bar\Lambda.$$

Proceeding inductively we deduce that

$$\sup_\Omega u(\,\cdot\,, t_j) \le 2^{-j}\bar\Lambda,$$

where

$$t_j := t^\star \bar\Lambda^{2-p} \sum_{k=0}^{j-1} 2^{k(p-2)}, \quad j \in \{1, 2, \dots\}.$$

To complete the argument consider $t \in (t_1, \infty)$ and let j be the largest j such that $t_j \le t$. Then, by the comparison principle and by construction,

$$\sup_{x\in\Omega} u(\,\cdot\,, t) \le 2^{-j}\bar\Lambda$$

and

$$t_j \le t < t_{j+1}.$$

Since

$$2^{-j} = \left(\frac{(2^{p-2}-1)}{t^\star}\bar\Lambda^{p-2} t_j + 1\right)^{-1/(p-2)},$$

and $2^{p-2} - 1 \ge \log(2)(p-2)$, by retracing the argument we derive the conclusion of the lemma. Furthermore, the constants c_i, $i \in \{1, 2\}$, are stable as $p \to 2^+$. In particular, we see that

$$\big((p-2)c_1^{-1}(n, p, |\Omega|)\bar\Lambda^{p-2} t + 1\big)^{-1/(p-2)} \to \exp(-c_1^{-1}(n, 2, |\Omega|)t). \qquad \square$$

8. Global estimates in $C^{1,1}$-domains

In this section we combine the optimal decay estimate established in Lemma 8.1 together with the barrier function in Lemma 6.4 to obtain the sharp decay estimate from above. Note that taking the initial data to be $+\infty$ allows us to see that this is sharp with respect to the so-called "friendly giant"; see for example [Kuusi et al. 2016].

Lemma 8.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain satisfying the ball condition with radius r_0. Let $u \in C([0,\infty); L^2(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega))$ be a nonnegative p-parabolic function in $\Omega \times (0,\infty)$. Let*

$$\bar\Lambda := \fint_\Omega u(x,0)\,\mathrm{d}x.$$

Then there exists $C_i \equiv C_i(\operatorname{diam}\Omega/r_0, n, p)$, $i \in \{1,2\}$, such that

$$u(x,t) \le C_1 \bar\Lambda((p-2)C_1^{-1}\bar\Lambda^{p-2}t+1)^{-1/(p-2)} d(x,\partial\Omega)$$

whenever $t > C_2\bar\Lambda^{2-p}$. Furthermore, the constants C_i, $i \in \{1,2\}$, are stable as $p \to 2^+$.

Proof. By scaling we can without loss of generality assume that $\bar\Lambda = 1$. Let $\hat x_0 \in \partial\Omega$ be an arbitrary point. Assume, for simplicity, that $a^-_{2r_0}(\hat x_0) = 0$, where $a^-_{2r_0}(\hat x_0)$ is the exterior corkscrew point as in Remark 2.3. Consider an arbitrary number $\hat t$ such that $\hat t > c_2$ (where c_2 is from Lemma 7.4). Then, using Lemma 7.4 we see that

$$u(x,t) \le c_1\left(\frac{p-2}{c_1}\hat t + 1\right)^{1/(2-p)} =: \bar\Lambda_u(\hat t), \quad x \in \Omega,\ \ t \ge \hat t. \tag{8-1}$$

Construct the function $\tilde h$ in Lemma 6.4 with the choices $T = \bar\Lambda_u^{2-p}$ and $H = \bar\Lambda_u$; then k_0 from Lemma 6.4 simplifies to

$$k_0 = \min\left\{\bar\Lambda_u^{(2-p)/(p-1)}\bar\Lambda_u^{(p-2)/(p-1)}, \frac{p-1}{n}\right\} = \min\left\{1, \frac{p-1}{n}\right\}.$$

Consider now the function $\hat h$ defined as

$$\hat h(x,t) = \tilde h\left(\frac{x}{r_0}, \frac{t-\hat t}{r_0^p}\right). \tag{8-2}$$

The function $\hat h$ is a supersolution in

$$N := (B_{(1+1/k_0)r_0} \setminus \bar B_{r_0}) \times (\hat t, \hat t + \bar\Lambda_u^{2-p} r_0^p).$$

Thus, the comparison principle, Definition 2.2, (8-1) and (8-2) imply that

$$u(x,t) \le \hat h(x,t) \quad \text{in } N \cap \Omega_\infty.$$

Next, using the upper estimate (6-3) from Lemma 6.4 we see that

$$u(x, \hat t + \bar\Lambda_u^{2-p} r_0^p) \le C\bar\Lambda_u\left(\left|\frac{x}{r_0}\right| - 1\right) \tag{8-3}$$

for a constant $C = C(n,p)$. As $\hat x_0 \in \partial\Omega$ is arbitrary, we see, using (8-3) and (8-1), that

$$u(x, \hat t + \Lambda_u^{2-p} r_0^p) \le C d(x,\partial\Omega)$$

for a new constant $C \equiv C(\operatorname{diam}\Omega/r_0, n, p)$. Furthermore, as $\hat t > c_2$ is arbitrary, we see that if $t > \bar\Lambda_u(c_2)^{2-p} r_0^p + c_2 := C_2$, then

$$u(x,t) \le C_1((p-2)C_1^{-1}t+1)^{-1/(p-2)} d(x,\partial\Omega)$$

for a constant $C_1 \equiv C_1(\operatorname{diam}\Omega/r_0, n, p) > 1$. □

The next lemma provides the corresponding lower bound.

Lemma 8.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected $C^{1,1}$-domain satisfying the ball condition with radius r_0. Let $u \in C([0,\infty); L^2(\Omega))$ be a nonnegative p-parabolic function in $\Omega \times (0,\infty)$. Suppose that there is a ball $B_{4r_1}(x_1) \subset \Omega$, $r_1 \in (0, r_0)$, such that*

$$\bar{\Lambda} := \fint_{B_{r_1}(x_1)} u(x,0)\,\mathrm{d}x > 0.$$

Then there exist $C_i \equiv C_i((\operatorname{diam}\Omega)/r_0, r_1/r_0, n, p)$, $i \in \{3,4\}$, such that

$$u(x,t) \geq \frac{\bar{\Lambda}}{C_3}((p-2)C_3\bar{\Lambda}^{p-2}t+1)^{-1/(p-2)} d(x,\partial\Omega)$$

whenever $x \in \Omega$ and $t > C_4\bar{\Lambda}^{2-p}$. Furthermore, the constants C_i, $i \in \{3,4\}$, are stable as $p \to 2^+$.

Proof. After scaling and translating we may assume that $\bar{\Lambda} = 1$, $x_1 = 0$ and $r_0 = 1$. Note that with these assumptions we have

$$\fint_{B_{r_1}(0)} u(x,0)\,\mathrm{d}x = 1. \tag{8-4}$$

Define now the set

$$\Omega^\delta := \{x \in \Omega : d(x,\Omega) > \delta\}.$$

Since Ω is connected and satisfies the ball condition with radius 1, we also obtain that Ω^δ is connected for $\delta \in \left(0, \frac{1}{2}\right]$ and thus any two points in Ω^δ can be connected by a Harnack chain of balls of size $\delta/4$ and with length depending only on n, p, $\operatorname{diam}\Omega$, and δ. By using Lemma 3.3 and (8-4) we then find positive constants $c^\star$ and $t^\star$, both depending only on n, p, $\operatorname{diam}\Omega$, and r_1, such that

$$\inf_{x \in \Omega^{1/2}} u(x,t^\star) \geq \frac{1}{c^\star}.$$

Lemma 7.2 then proves the result whenever $x \in \Omega^1$. Next, let $y \in \Omega \setminus \Omega^1$ and let $y^\star \in \partial\Omega$ be such that $d(y,\partial\Omega) = |y-y^\star|$. Since $d(y,\partial\Omega) \leq 1$ and since the direction is unique (see Remark 2.3) we have that $y = a_{2d(y,\partial\Omega)}(y^\star)$. With this at hand we can consider the point $a_2(y^\star)$ (which is collinear with $y, y^\star$) satisfying

$$\fint_{B_{1/4}(a_2(y^\star))} u(x,t^\star)\,\mathrm{d}x \geq \frac{1}{c^\star}.$$

Applying Lemma 7.2 we see that

$$u(x,t) \geq \frac{1}{c^\star c_1}\left(c_1(p-2)[c^\star]^{p-2}(t-t^\star)+1\right)^{-1/(p-2)} d(x,\partial B_1(a_2(y^\star)))$$

whenever $x \in B_1(a_2(y^\star))$, $t > t^\star$. Applying this for $x = y$ completes the proof. □

Remark 8.3. Note that our tools are too rough to obtain the lower bound in Lemma 8.2 independent of the distribution of the initial data. To remedy this, we assume that the initial data is positive in a region away from the boundary.

In the next theorem we combine the results of Lemmas 8.1 and 8.2 to obtain an elliptic-type global Harnack estimate.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain satisfying the ball condition with radius r_0. Let $u \in C([0,\infty); L^2(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega))$ be a nonnegative p-parabolic function in $\Omega \times (0,\infty)$. Let*

$$\bar{\Lambda} := \fint_\Omega u(x,0)\,\mathrm{d}x.$$

Assume that

$$\operatorname{supp} u(\,\cdot\,,0) \subset \Omega^\delta = \{x \in \Omega : d(x,\partial\Omega) > \delta\}.$$

Then there are constants $c_i \equiv c_i((\operatorname{diam}\Omega)/r_0, \delta/r_0, n, p)$, $i \in \{1,2\}$, such that

$$c_1{}^{-1} \le \frac{u(x, t+\epsilon\bar{\Lambda}^{2-p})}{u(x,t)} \le c_1$$

whenever $\epsilon \in [0,1]$, $x \in \Omega$ and $t \ge c_2\bar{\Lambda}^{2-p}$. Furthermore, the constants c_i, $i \in \{1,2\}$, are stable as $p \to 2^+$.

Proof. In the following we will use the constants C_i, $i \in \{1,\dots,4\}$, introduced in Lemmas 8.1 and 8.2. Let $t_0 = \max\{C_2, C_4\}\bar{\Lambda}^{2-p}$ and consider $t \ge t_0$ and $\epsilon \in (0,1)$. Then, using Lemmas 8.1 and 8.2 we see that

$$\frac{u(x,t+\epsilon\bar{\Lambda}^{2-p})}{u(x,t)} \le C_1C_3\left(\frac{(p-2)C_1^{-1}(t\bar{\Lambda}^{p-2}+\epsilon)+1}{(p-2)C_2t\bar{\Lambda}^{p-2}+1}\right)^{-1/(p-2)},$$

$$\frac{u(x,t+\epsilon\bar{\Lambda}^{2-p})}{u(x,t)} \ge \frac{1}{C_1C_3}\left(\frac{(p-2)C_3(t\bar{\Lambda}^{p-2}+\epsilon)+1}{(p-2)C_1^{-1}t\bar{\Lambda}^{p-2}+1}\right)^{-1/(p-2)}.$$

Theorem 8.5 follows from this by elementary manipulations. We omit further details. □

In the next theorem we use Lemmas 8.1 and 8.2 together with $C^{1,\alpha}$ estimates for weak solutions to obtain a global boundary Harnack principle as well as Hölder continuity of ratios of solutions. The intrinsic time interval ensures that the estimate is p-stable.

Theorem 8.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain satisfying the ball condition with radius r_0. Let $u, v \in C([0,\infty); L^2(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega))$ be nonnegative p-parabolic functions in $\Omega \times (0,\infty)$. Let*

$$\bar{\Lambda}_u = \fint_\Omega u(x,0)\,\mathrm{d}x, \quad \bar{\Lambda}_v = \fint_\Omega v(x,0)\,\mathrm{d}x.$$

Assume that the initial data is distributed as follows:

$$\operatorname{supp} u(\,\cdot\,,0),\ \operatorname{supp} v(\,\cdot\,,0) \subset \Omega^\delta = \{x \in \Omega : d(x,\partial\Omega) > \delta\}.$$

Then there exists $\bar{C}_1 \equiv \bar{C}_1((\operatorname{diam}\Omega)/r_0, \delta/r_0, n, p)$ such that if $\bar{C}_1 \le T_- \le T_+$ satisfy

$$T_- \min\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p} \le T_+ \max\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p}, \tag{8-5}$$

the following holds. There exist $\bar C_i \equiv \bar C_i((\operatorname{diam}\Omega)/r_0, \delta/r_0, T_-, T_+, n, p),\ i \in \{2,3\}$, *such that*

$$\bar C_2^{-1} \frac{\bar\Lambda_u}{\bar\Lambda_v} \le \frac{u(x,t)}{v(x,t)} \le \bar C_2 \frac{\bar\Lambda_u}{\bar\Lambda_v} \tag{8-6}$$

whenever

$$(x,t) \in D := \Omega \times \big(T_- \min\{\bar\Lambda_u, \bar\Lambda_v\}^{2-p},\ T_+ \max\{\bar\Lambda_u, \bar\Lambda_v\}^{2-p}\big).$$

Furthermore, there exists an exponent $\sigma \equiv \sigma(n,p) \in (0,1)$ *such that*

$$\left|\frac{u(x,t)}{v(x,t)} - \frac{u(y,s)}{v(y,s)}\right| \le \bar C_3 \frac{\bar\Lambda_u}{\bar\Lambda_v}\big(|x-y| + \max\{\bar\Lambda_u, \bar\Lambda_v\}^{(2-p)/p}|t-s|^{1/p}\big)^\sigma \tag{8-7}$$

whenever $(x,t), (y,s) \in D$. *The constants* $\bar C_i,\ i \in \{1,2,3\}$, *and* σ *are stable as* $p \to 2^+$.

Proof. In the following we will let C_1, C_2, C_3 and C_4 be the constants in Lemmas 8.1 and 8.2. We begin by proving (8-6). Indeed, using Lemmas 8.1 and 8.2 we see that

$$\frac{\bar\Lambda_u}{\bar\Lambda_v}\frac{1}{C_1C_3}\left(\frac{(p-2)C_1^{-1}\bar\Lambda_v^{p-2}t+1}{(p-2)C_3\bar\Lambda_u^{p-2}t+1}\right)^{1/(p-2)} \le \frac{u(x,t)}{v(x,t)}, \tag{8-8}$$

and

$$\frac{u(x,t)}{v(x,t)} \le \frac{\bar\Lambda_u}{\bar\Lambda_v}C_1C_3\left(\frac{(p-2)C_3\bar\Lambda_v^{p-2}t+1}{(p-2)C_1^{-1}\bar\Lambda_u^{p-2}t+1}\right)^{1/(p-2)} \tag{8-9}$$

whenever $t \ge \max\{C_2, C_4\}\min\{\bar\Lambda_u, \bar\Lambda_v\}^{2-p}$. In particular we see that if $T_- > \max\{C_2, C_4\}$, and if T_-, T_+ satisfy (8-5), then (8-6) holds with a constant $\bar C_2$ depending only on $(\operatorname{diam}\Omega)/r_0$, δ/r_0, T_-, T_+, n, p.

To proceed, consider the rescaled p-parabolic functions

$$\tilde u(x,t) = \frac{u(x, \bar\Lambda_u^{2-p}t)}{\bar\Lambda_u}, \quad \tilde v(x,t) = \frac{v(x, \bar\Lambda_v^{2-p}t)}{\bar\Lambda_v}.$$

Using Lemma 7.4 for $w \in \{\tilde u, \tilde v\}$ we get for $t > c_2$ that

$$\sup_{x\in\Omega} w(x,t) \le c_1((p-2)c_1^{-1}t+1)^{-1/(p-2)} \le c_1,$$

where c_1, c_2 are as in Lemma 7.4. Thus we can apply [Lieberman 1993, Theorem 0.1] to conclude that there exist C and σ, depending on Ω, p and n, such that, for $w \in \{\tilde u, \tilde v\}$,

$$|\nabla w(x,t) - \nabla w(\tilde x, \tilde t)| \le C(|x-\tilde x| + |t-\tilde t|^{1/p})^\sigma$$

whenever $(x,t), (\tilde x, \tilde t) \in \Omega \times (c_2, \infty)$. In particular, arguing as in [Kuusi et al. 2014, (3.31), p. 2717], we have, for $w \in \{\tilde u, \tilde v\}$,

$$\left|\frac{w(x,t)}{d(x,\partial\Omega)} - \frac{w(\tilde x,\tilde t)}{d(\tilde x,\partial\Omega)}\right| \le C(|x-\tilde x| + |t-\tilde t|^{1/p})^\sigma \tag{8-10}$$

whenever $(x,t),(\tilde{x},\tilde{t}) \in \Omega \times (c_2,\infty)$. Scaling back to the original u, v, (8-10) becomes, with $w \in \{u,v\}$,

$$\left|\frac{w(x,t)}{d(x,\partial\Omega)} - \frac{w(\tilde{x},\tilde{t})}{d(\tilde{x},\partial\Omega)}\right| \le \bar{\Lambda}_w C(|x-\tilde{x}| + \bar{\Lambda}_w^{(p-2)/p}|t-\tilde{t}|^{1/p})^\sigma \tag{8-11}$$

whenever $(x,t),(\tilde{x},\tilde{t}) \in \Omega \times (c_2\bar{\Lambda}_w^{p-2},\infty)$. Next, using the identity

$$\begin{aligned}\frac{u(x,t)}{v(x,t)} &- \frac{u(y,s)}{v(y,s)}\\ &= \frac{d(x,\partial\Omega)}{v(x,t)}\left(\frac{u(x,t)}{d(x,\partial\Omega)} - \frac{u(y,s)}{d(y,\partial\Omega)}\right) + \frac{u(y,s)}{v(y,s)}\frac{d(x,\partial\Omega)}{v(x,t)}\left(\frac{v(y,s)}{d(y,\partial\Omega)} - \frac{v(x,t)}{d(x,\partial\Omega)}\right),\end{aligned} \tag{8-12}$$

assuming that $s,t \in (T_- \min\{\bar{\Lambda}_u,\bar{\Lambda}_v\}^{2-p}, T_+ \max\{\bar{\Lambda}_u,\bar{\Lambda}_v\}^{2-p})$ and that $T_- > c_2$ (where c_2 is from Lemma 7.4), we can apply Lemmas 8.1 and 8.2 together with (8-6) and (8-11) in the identity (8-12) to obtain (8-7). This completes the proof of Theorem 8.5. □

Remark 8.6. Considering the estimates (8-8) and (8-9) in the proof of Theorem 8.5, we see that the nonlinearities dominate for large values of t. In particular, there exists a p-unstable constant $C \equiv C((\operatorname{diam}\Omega/r_0),\delta/r_0,n,p)$ such that

$$C^{-1} \le \lim_{t\to\infty}\frac{u(x,t)}{v(x,t)} \le C$$

whenever $x \in \Omega$. Note that C is independent of the initial data. This does not happen when $p = 2$ and the effect is purely nonlinear.

Remark 8.7. Note that to prove Theorem 8.5 we rely on estimates established in Lemmas 8.1 and 8.2, instead of relying on the comparison principle, the Carleson estimate and the Harnack inequality, as in [Fabes et al. 1986, Theorem 2.1]. This is why our estimates from below depend on the distribution of the initial data. Furthermore, this falls fairly short of the result in [Fabes et al. 1986], but nonetheless we provide a p-stable version of the phenomena involved in our case.

9. Local estimates in $C^{1,1}$-domains

In this section the main focus is to develop an intrinsic version of the boundary Harnack principle (1-2); see Section 9B. To do this, we first prove an upper and a lower decay rate estimate in the next section.

9A. ***Upper and lower bound on the decay.*** We begin with the upper bound, which follows by combining the barrier function from Lemma 6.4 together with the Carleson estimate Theorem 4.2. In the following, M will denote the NTA constant of the $C^{1,1}$ domain Ω; see Remark 2.3.

Lemma 9.1. *Let u be a nonnegative solution in Ω_T, where Ω is a $C^{1,1}$ domain satisfying the ball condition with radius r_0. Let $x_0 \in \partial\Omega$ and $0 < r \le r_0$. Let $0 < \delta \le \tilde{\delta} \le 1$. Assume that $u(a_r(x_0),t_0) > 0$ for a fixed $t_0 \in (0,T)$ and let*

$$\tau = \frac{C_4}{16}[C_5 u(a_r(x_0),t_0)]^{2-p} r^p,$$

where C_4 and C_5, both depending on p, n, are as in Theorem 3.9 and with $t_0 > 5\tilde{\delta}^{p-1}\tau$. Assume furthermore that u vanishes continuously on $S_T \cap \big(B_r(x_0) \times (t_0 - 4\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau)\big)$. Then there exist constants $c_i \equiv c_i(M, p, n)$, $i \in \{8, 9\}$, such that

$$\sup_Q u \leq \left(\frac{c_8}{\tilde{\delta}}\right)^{c_9/\delta} \frac{d(x_0, \partial\Omega)}{r} u(a_r(x_0), t),$$

where $Q := (B_r(x_0) \cap \Omega) \times (t_0 - 2\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau)$. Furthermore, the constants c_i, $i \in \{8, 9\}$, are stable as $p \to 2^+$.

Proof. Without loss of generality, we may after scaling assume that $u(a_r(x_0), t_0) = 1$ and $r = 4$. Applying Lemma 6.4 and Theorem 4.2, with

$$k := \min\left\{\frac{p-1}{n}, \tilde{\delta}\left(\frac{C_4}{16} C_5^{2-p}\right)^{1/(p-1)}, 1\right\}, \quad H := \left(\frac{c_6}{\tilde{\delta}}\right)^{c_7/\delta},$$

with constants as in Theorem 4.2 we get

$$u(x, t) \leq H$$

whenever $(x, t) \in (B_4(x_0) \cap \Omega) \times (t_0 - 3\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau)$. The comparison function, indexed by its initial time s_0 and center point y_0, is

$$\hat{h}_{y_0, s_0}(x, t) = \tilde{h}(x - y_0, t - s_0)$$

with $T = \tilde{\delta}^{p-1}\tau$, where $\tilde{h}$ is from Lemma 6.4. Let $y \in \partial\Omega \cap B_1(x_0)$ and consider $y_0 = a_2(y)$, an outer corkscrew point as in Remark 2.3. Then, by the comparison principle

$$u(x, t) \leq \hat{h}_{y_0, s_0}(x, t), \tag{9-1}$$

in

$$(\Omega \cap [B_{1+k}(y_0) \setminus B_1(y_0)]) \times (s_0, s_0 + T)$$

whenever

$$(s_0, s_0 + T) \subset (t_0 - 3\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau).$$

From Lemma 6.4 and (9-1) we have the estimate

$$u(x, t) \leq \frac{H \exp(2)}{k} (|y_0 - x| - 1)$$

whenever

$$(x, t) \in (\Omega \cap [B_{1+k}(y_0) \setminus B_1(y_0)]) \times (t_0 - 2\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau),$$

and $y_0 \in \partial\Omega \cap B_1(x_0)$. From this the result follows by scaling back. □

The following lemma establishes a local lower bound on the decay, by combining the barrier from Lemma 6.1 and the Harnack estimates in Theorem 3.6.

Lemma 9.2. *Let u be a nonnegative solution in Ω_T, where Ω is a $C^{1,1}$ domain satisfying the ball condition with radius r_0. Let $x_0 \in \partial\Omega$ and let $0 < r < r_0$ be fixed. Let $A^- = (a_r(x_0), t_0)$, $\theta_- = u(A^-)^{2-p}$ and $t_0 \in (0, T)$. There exist constants $c_i \equiv c_i(M, p, n)$, $i \in \{3, 4\}$, such that if*

$$\theta_- r^p < t_0 \quad \textit{and} \quad t_0 + 2c_4\theta_- r^p < T,$$

then

$$\frac{1}{c_3}\frac{d(x, \partial\Omega)}{r} u(A^-) \leq u(x, t)$$

for $x \in B_r(x_0) \cap \Omega$ and $t_0 + c_4\theta_- r^p < t < t_0 + 2c_4\theta_- r^p$. Furthermore, the constants c_i, $i \in \{3, 4\}$, are stable as $p \to 2^+$.

Proof. Set $\lambda_- := u(A^-)$ and consider the scaled solution

$$v(y, s) = \frac{1}{\lambda_-} u(x_0 + ry, t_0 + s\lambda_-^{2-p} r^p).$$

Set also $\widetilde{\Omega} := \{y : x_0 + ry \in \Omega\}$ so that $0 \in \partial\widetilde{\Omega}$. For the new function v we have the following situation: defining $A_v^- = ((a_r(x_0) - x_0)/r, 0)$,

$$v(A_v^-) = 1, \quad d(A^-, \partial\widetilde{\Omega}) = 1,$$

and v is a solution in $\widetilde{\Omega} \times (-1, \widetilde{T})$, where $\widetilde{T} := (T - t_0)\lambda_-^{p-2} r^{-p}$. Since $0 < r < r_0/4$ we know that $\widetilde{\Omega}$ satisfies the ball condition with radius 4. To continue, consider the set

$$D = \{y \in \widetilde{\Omega} : d(y, B_1(0) \cap \partial\widetilde{\Omega}) = d(y, \partial\widetilde{\Omega}) = 1\}.$$

Note that $D \subset B_2(0) \cap \widetilde{\Omega}$ and that $\sup\{d(a_1(y_0), y) : y \in D\} \leq 2$ for any $y_0 \in \partial\widetilde{\Omega} \cap B_1(0)$. We obtain from the Harnack chain estimate in Theorem 3.6 (applied with $\delta = \min\{c_h^{(2-p)/p}, 1\}$ where c_h is from Theorem 3.6) that there is a $\tilde{\tau}_1 > 0$ depending only on n, p, M such that

$$v(x, \tilde{\tau}_1) \geq \frac{1}{\tilde{c}_1}$$

whenever $x \in \left\{y : d(y, D) < \frac{1}{4}\right\}$ provided $\widetilde{T} > \tilde{\tau}_1$. Using Lemma 7.2 (applied with $r = 1$, $\varrho = \frac{1}{4}$, $g = v(\cdot, \tilde{\tau}_1)$, $x_0 = \tilde{y} \in D$, $t_0 = \tilde{\tau}_1$) for all points $\tilde{y} \in D$, we get

$$v(y, t) \geq \frac{1}{\tilde{c}_1 c_1}\left(c_1\tilde{c}_1^{2-p}(p-2)(t - \tilde{\tau}_2) + c_1 c_2(p-2) + 1\right)^{-1/(p-2)} d(y, \widetilde{\Omega}) \tag{9-2}$$

whenever $(y, t) \in (\widetilde{\Omega} \cap B_1(0)) \times (\tilde{\tau}_2, \widetilde{T})$, with $\tilde{\tau}_2 = \tilde{\tau}_1 + c_2\tilde{c}_1^{p-2}$ provided $\widetilde{T} > \tilde{\tau}_2$. Going back to Ω and u gives us the result provided $\widetilde{T} > 2\tilde{\tau}_2$. □

Combining Lemmas 9.1 and 9.2 we obtain the joint estimate.

Theorem 9.3. *Let u be a nonnegative solution in Ω_T, where Ω is a $C^{1,1}$ domain satisfying the ball condition with radius r_0. Let $x_0 \in \partial\Omega$, $t_0 \in (0, T)$, and let $0 < r < r_0$ be fixed. Let $A^- = (a_r(x_0), t_0)$ and $\theta_- = u(A^-)^{2-p}$. There exist constants $c_i \equiv c_i(M, p, n)$, $i \in \{5, 6\}$, such that if*

$$\theta_- r^p < t_0 \quad \textit{and} \quad t_0 + 2c_4\theta_- r^p < T,$$

then for $A^+ = (a_r(x_0), t_0 + 2c_4\theta_- r^p)$ *and* $\theta_+ = c_6^{-1} u(A^+)^{2-p}$ *(where* c_4 *is from Lemma 9.2), we have*

$$5\theta_+ \le \theta_-.$$

Furthermore, if u *vanishes continuously on*

$$S_T \cap \big(B_r(x_0) \times (t_0 + [2c_4\theta_- - 5\theta_+]r^p, t_0 + [2c_4\theta_- - \theta_+]r^p)\big),$$

then

$$\frac{1}{c_5}\frac{d(x, \partial\Omega)}{r} u(A^-) \le u(x,t) \le c_5 \frac{d(x,\partial\Omega)}{r} u(A^+)$$

for $x \in B_r(x_0) \cap \Omega$ *and* $t_0 + [2c_4\theta_- - 2\theta_+]r^p < t < t_0 + [2c_4\theta_- - \theta_+]r^p$. *Furthermore, the constants* c_i, $i \in \{5, 6\}$, *are stable as* $p \to 2^+$.

Proof. Rescale u as in the proof of Lemma 9.2; let also $\tilde{\tau}_2$ be as in the proof of Lemma 9.2. Thus

$$\frac{1}{c_3} d(x, \partial\Omega) \le v(x,t)$$

holds for $(x,t)(B_r(x_0) \cap \Omega) \times (\tilde{\tau}_2, 2\tilde{\tau}_2)$. Define $\tau_+ = 2\tilde{\tau}_2$ and consider $A_+ = (a_1(0), \tau_+)$. Then using (9-2) we get

$$\begin{aligned} v(A_+)^{2-p} &\le \left[\frac{1}{\tilde{c}_1 c_1}\big(c_1 \tilde{c}_1^{2-p}(p-2)(t-\tilde{\tau}_2) + c_1c_2(p-2)+1\big)^{-1/(p-2)}\right]^{2-p} \\ &= \frac{1}{(\tilde{c}_1c_1)^{2-p}}[c_1\tilde{c}_1^{2-p}(p-2) + c_1c_2(p-2)+1] \\ &=: \tilde{c}_2. \end{aligned}$$

We will now apply Lemma 9.1 with ($r = 1$, $\delta^{p-1} = \min\{(16/(5\tilde{c}_2C_4))C_5^{p-2}, 1\}$ and $\tilde{\delta} = \delta$, with C_4, C_5 from Lemma 9.1). Doing this we see that

$$v(x,t) \le \tilde{c}_3 d(x, \partial\widetilde{\Omega}) v(A_+)$$

whenever $(x,t) \in (B_1(0) \cap \Omega) \times (\tau_+ - v(A_+)^{2-p}, \tau_+ - v(A_+)^{2-p}/2)$. Going back to Ω and u gives us the result. □

9B. *Local boundary Harnack estimate.* We are now ready to state and prove our local boundary Harnack principle; consult Figure 1 for a schematic of the geometry.

Theorem 9.4. *Let* u, v *be two nonnegative solutions in* Ω_T, *where* Ω *is a* $C^{1,1}$*-domain satisfying the ball condition with radius* r_0. *Let* $x_0 \in \partial\Omega$, $t_0 \in (0,T)$, *and let* $0 < r < r_0$ *be fixed. Let* $A_- = (a_r(x_0), t_0)$ *and assume that* $u(A_-) = v(A_-)$. *Let the constants* c_i, $i \in \{4, 5, 6\}$ *be as in Lemmas 9.2 and 9.3. Let* $\theta_- = u(A_-)^{2-p}$, *and assume*

$$\theta_- r^p < t_0 \quad \textit{and} \quad t_0 + 2c_4\theta_- r^p < T.$$

Set

$$A_+ = (a_r(x_0), t_0 + 2c_4\theta_- r^p), \quad \theta_{+,u} = c_6^{-1}u(A_+)^{2-p}.$$

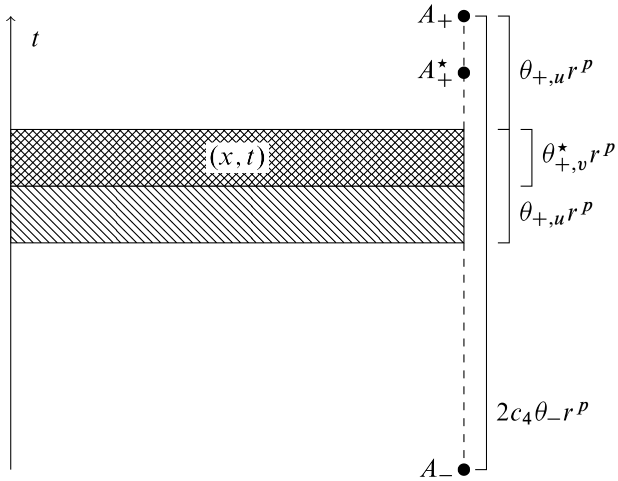

Figure 1. The boxes ▧ and ▩ denote the regions where the right- and left-hand sides of Theorem 9.4 hold respectively.

Assume that $v(A_+) \geq u(A_+)$. Then there exists a time $t_+^\star$, depending on v, satisfying

$$t_+^\star \in (t_0 + (2c_4\theta_- - \theta_{+,u})r^p, t_0 + 2c_4\theta_- r^p),$$
$$A_+^\star = (a_r(x_0), t_+^\star), \quad \theta_{+,v}^\star = c_6^{-1} v(A_+^\star)^{2-p}$$

such that the following holds. If both u and v vanish continuously on

$$S_T \cap \big(B_r(x_0) \times (t_0 + [2c_4\theta_- - 5\theta_{+,u}]r^p, t_0 + [2c_4\theta_- - \theta_{+,u}]r^p)\big),$$

then

$$\frac{1}{c_5^2}\frac{u(A_-)}{v(A_+^\star)} \leq \frac{u(x,t)}{v(x,t)} \leq c_5^2 \frac{u(A_+)}{v(A_-)}$$

whenever (x,t) belongs to the set

$$(B_r(x_0) \cap \Omega) \times \big(t_0 + [2c_4\theta_- - (\theta_{+,v}^\star + \theta_{+,u})]r^p,\ t_0 + [2c_4\theta_- - \theta_{+,u}]r^p\big).$$

Remark 9.5. It should be noted that we cannot control the time $t_+^\star$ except which interval it lies in; it is a purely intrinsic parameter. Furthermore note that Theorem 9.4 is equivalent to the boundary Harnack principle (1-2) when $p = 2$.

Proof. Let $c_i,\ i \in \{3, \dots, 6\}$, be as in Lemmas 9.2 and 9.3. By the assumptions, we know that $\theta_{+,u} \geq \theta_{+,v} := c_6^{-1} v(A_+)^{2-p}$. We then obtain by Theorem 9.3, for $x \in B_r(x_0) \cap \Omega$ and $t_0 + (2c_4\theta_- - 2\theta_{+,u})r^p < t < t_0 + (2c_4\theta_- - \theta_{+,u})r^p$, that

$$\frac{1}{c_5}\frac{d(x,\partial\Omega)}{r} u(A_-) \leq u(x,t) \leq c_5 \frac{d(x,\partial\Omega)}{r} u(A_+). \tag{9-3}$$

Using Lemma 9.2 for $x \in B_r(x_0) \cap \Omega$ and $t_0 + c_4\theta_- r^p < t < t_0 + 2c_4\theta_- r^p$ we get that

$$\frac{1}{c_3}\frac{d(x,\partial\Omega)}{r} v(A_-) \le v(x,t). \tag{9-4}$$

Now let $t_+ = t_0 + 2c_4\theta_- r^p$ and let $t_+^\star$ be a time to be fixed such that $t_+ - \theta_{+,u} r^p < t_+^\star \le t_+$. First note that if $t_+^\star = t_+$ we have for $\theta_{+,v}^\star = c_6^{-1}(v(a_r(x_0), t_+^\star))^{2-p}$,

$$t_+^\star - \theta_{+,v}^\star r^p \ge t_+ - \theta_{+,u} r^p;$$

furthermore, if $t_+^\star = t_+ - \theta_+^u r^p$ then

$$t_+^\star - \theta_{+,v}^\star r^p < t_+ - \theta_{+,u} r^p.$$

Thus by continuity there is a largest $t_+^\star$ such that

$$t_+^\star - \theta_{+,v}^\star r^p = t_+ - \theta_{+,u} r^p.$$

With $t_+^\star$ at hand we now apply Lemma 9.1 (with the same $\delta, \tilde{\delta}$ as in the proof of Theorem 9.3) combining it with (9-4) to get

$$\frac{1}{c_5}\frac{d(x,\partial\Omega)}{r} v(A_-) \le v(x,t) \le c_5 \frac{d(x,\partial\Omega)}{r} v(A_+^\star) \tag{9-5}$$

for $x \in B_r(x_0) \cap \Omega$ and $t_0 + (2c_4\theta_- - (\theta_{+,u} + \theta_{+,v}^\star))r^p < t < t_0 + (2c_4\theta_- - \theta_{+,u})r^p$. Combining (9-3) and (9-5) we have completed the proof. □

9C. *Boundary measures in $C^{1,1}$-domains.* We conclude the section by describing the fine properties of the boundary measure defined in (1-6). The theorem below says that the induced measure is mutually absolutely continuous with respect to the surface measure of S_T.

Theorem 9.6. *Under the hypotheses of Theorem 9.3,*

$$0 < \liminf_{\varrho \to 0} \frac{\mu_u(Q_\varrho(x,t))}{\varrho^{n+1}} \le \limsup_{\varrho \to 0} \frac{\mu_u(Q_\varrho(x,t))}{\varrho^{n+1}} < +\infty,$$

where $Q_\varrho(x,t) := B_\varrho(x) \times (t - \varrho^2, t)$, whenever $(x,t) \in V$,

$$V := (\partial\Omega \cap B_r(x_0)) \times \big(t_0 + (2c_4\theta_- - 2\theta_+)r^p, t_0 + (2c_4\theta_- - \theta_+)r^p\big).$$

In particular, μ_u is mutually absolutely continuous with respect to the surface measure of S_T on V.

Proof. By Theorem 9.3 we have

$$\lambda_- d(x,\partial\Omega) \le u(x,t) \le \lambda_+ d(x,\partial\Omega), \quad \lambda_\pm := c_5^{\pm 1}\frac{u(A_r^\pm)}{r} \tag{9-6}$$

whenever $(x,t) \in Q$, with

$$Q := (\Omega \cap B_r(x_0)) \times \big(t_0 + (2c_4\theta_- - 2\theta_+)r^p, t_0 + (2c_4\theta_- - \theta_+)r^p\big)$$

and $\theta_\pm$ as in Theorem 9.3. We now pick a point $(y,s) \in S_T \cap \partial_p Q$. Choose ϱ small enough so that $U_\varrho(y,s) \cap (\Omega \times \mathbb{R})$ is contained in Q, where

$$U_\varrho(y,s) := B_\varrho(y) \times (s - \tilde{\tau}\varrho^2, s + \tilde{\tau}\varrho^2),$$

$\tilde{\tau} := \lambda_-^{2-p} \max\{8C_4 C_5^{2-p}, 2(\tau_0 + \tau_1)\}$ and C_4, C_5 and τ_0, τ_1 are as in Theorems 5.2 and 5.3. After a simple covering argument using (9-6), and Theorems 5.2 and 5.3, we find a constant $C \equiv C(p, n, M, \lambda_\pm)$ such that

$$\frac{1}{C} \le \frac{\mu_u(U_{\varrho/2}(y,s))}{\varrho^{n+1}} \le C.$$

Taking a possibly larger C, and a smaller ϱ, this actually implies

$$\frac{1}{C} \le \frac{\mu_u(Q_\varrho(y,s))}{\varrho^{n+1}} \le C$$

uniformly for small enough ϱ. This proves the statement. □

Remark 9.7. Note that in the same region V as in Theorem 9.6 we have that the measure is doubling. Moreover note that Lemma 9.2 implies a Hopf-type result on this boundary cylinder V; thus together with the fact that solutions are $C^{1,\alpha}$ up to the boundary, we get that the logarithm of the normal derivative on the boundary is Hölder continuous. Now arguing as in [Avelin et al. 2011, (1.7)–(1.10)] we get, for $(x_0, t_0) \in V$ given, and $\epsilon \in (0,1)$, that

$$\lim_{\varrho \to 0} \frac{\mu_u(Q_{\epsilon\varrho}(x_0,t_0))}{\mu_u(Q_\varrho(x_0,t_0))} = \epsilon^{n+1}.$$

In particular, the measure μ_u is asymptotically optimal doubling.

Acknowledgment

Avelin was supported by the Swedish Research Council, grant no. 637-2014-6822. Kuusi was supported by the Academy of Finland, grant no. 258000.

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Received 28 Oct 2015. Accepted 10 Apr 2018.

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ANALYSIS AND PDE
Vol. 12, No. 1, 2019
dx.doi.org/10.2140/apde.2019.12.43

ON ASYMPTOTIC DYNAMICS FOR L^2 CRITICAL GENERALIZED KDV EQUATIONS WITH A SATURATED PERTURBATION

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We consider the L^2 critical gKdV equation with a saturated perturbation: $\partial_t u+(u_{xx}+u^5-\gamma u|u|^{q-1})_x=0$, where $q>5$ and $0<\gamma\ll 1$. For any initial data $u_0\in H^1$, the corresponding solution is always global and bounded in H^1. This equation has a family of solutions, and our goal is to classify the dynamics near solitons. Together with a suitable decay assumption, there are only three possibilities: (i) the solution converges asymptotically to a solitary wave whose H^1 norm is of size $\gamma^{-2/(q-1)}$ as $\gamma\to 0$; (ii) the solution is always in a small neighborhood of the modulated family of solitary waves, but blows down at $+\infty$; (iii) the solution leaves any small neighborhood of the modulated family of the solitary waves.

This extends the classification of the rigidity dynamics near the ground state for the unperturbed L^2 critical gKdV (corresponding to $\gamma=0$) by Martel, Merle and Raphaël. However, the blow-down behavior (ii) is completely new, and the dynamics of the saturated equation cannot be viewed as a perturbation of the L^2 critical dynamics of the unperturbed equation. This is the first example of classification of the dynamics near the ground state for a saturated equation in this context. The cases of L^2 critical NLS and L^2 supercritical gKdV, where similar classification results are expected, are completely open.

1. Introduction

1A. ***Setting of the problem.*** Let us consider the following Cauchy problem:

$$\begin{cases}\partial_t u+(u_{xx}+u^5-\gamma u|u|^{q-1})_x=0, & (t,x)\in[0,T)\times\mathbb{R},\\ u(0,x)=u_0(x)\in H^1(\mathbb{R}),\end{cases}\tag{gKdV$_\gamma$}$$

with $q>5$ and $0<\gamma\ll 1$.

MSC2010: primary 35Q53; secondary 35B20, 35B40, 37K40.
Keywords: gKdV, L^2-critical, saturated perturbation, dynamics near ground state, blow down.

The equation has two conservation laws, i.e., the mass and the energy:

$$M(u(t)) = \int u(t)^2 = M_0,$$
$$E(u(t)) = \frac{1}{2}\int u_x(t)^2 - \frac{1}{6}\int u(t)^6 + \frac{\gamma}{q+1}\int |u(t)|^{q+1} = E_0.$$

We can see that the solution of (gKdV$_\gamma$) is always global in time and bounded in H^1. First of all, (gKdV$_\gamma$) is locally well-posed in H^1 due to [Kato 1983; Kenig, Ponce and Vega 1993]; i.e., for any $u_0 \in H^1$, there exists a unique strong solution in $C([0,T), H^1)$ with either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T} \|u_x(t)\|_{L^2} = +\infty$. Since $\gamma > 0$, $q > 5$, the mass and energy conservation laws ensure that, for all $t \in [0,T)$,

$$\|u_x(t)\|_{L^2}^2 \lesssim |E_0| + \gamma^{-\frac{4}{q-5}} M_0 < +\infty,$$

so $T = +\infty$ and $u(t)$ is always bounded in H^1.

This equation does not have a standard scaling rule, but has the following pseudoscaling rule: for all $\lambda_0 > 0$, if $u(t,x)$ is a solution to (gKdV$_\gamma$), then

$$u_{\lambda_0}(t,x) = \lambda_0^{-\frac{1}{2}} u(\lambda_0^{-3}t, \lambda_0^{-1}x) \tag{1-1}$$

is a solution to

$$\begin{cases} \partial_t v + (v_{xx} + v^5 - \lambda_0^{-m}\gamma v|v|^{q-1})_x = 0, & (t,x) \in [0, \lambda_0^{-3}T) \times \mathbb{R}, \\ v(0,x) = \lambda_0^{-\frac{1}{2}} u_0(\lambda_0^{-1}x) \in H^1(\mathbb{R}), \end{cases}$$

with

$$m = \tfrac{1}{2}(q-5) > 0. \tag{1-2}$$

The pseudoscaling rule (1-1) leaves the L^2 norm of the initial data invariant.

There is a special class of solutions. We first introduce the ground state Q_ω for $0 \le \omega < \omega^* \ll 1$, which is the unique radial nonnegative solution with exponential decay to the ODE[1]

$$Q_\omega'' - Q_\omega + Q_\omega^5 - \omega Q_\omega |Q_\omega|^{q-1} = 0.$$

Then for all $\lambda_0 > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$ with $\lambda_0^{-m}\gamma < \omega^*$, the following is a solution to (gKdV$_\gamma$):

$$u(t,x) = \lambda_0^{-\frac{1}{2}} Q_{\lambda_0^{-m}\gamma}(\lambda_0^{-1}(x-x_0) - \lambda_0^{-3}(t-t_0)).$$

A solution of this type is called a *solitary wave* solution.

1B. ***On the critical problem with saturated perturbation.*** The saturated perturbation was first introduced for the nonlinear Schrödinger (NLS)

$$i\partial_t u + \Delta u + g(|u|^2)u = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d. \tag{NLS}$$

[1] The existence of such Q_ω was proved in [Berestycki and Lions 1983, Section 6], but in this paper we will give an alternative proof for the existence.

In many applications, the leading-order approximation of the nonlinearity, $g(s)$, is the power nonlinearity; i.e., $g(s) = \pm s^\sigma$. For example, $g(s) = s$ leads to the focusing cubic NLS equation, which appears in many contexts.

But such approximation may lead to nonphysical predictions. For example, from [Fibich 2015; Merle and Raphaël 2005; Merle, Raphaël and Szeftel 2010; Sulem and Sulem 1999], for NLS with critical or supercritical focusing nonlinearities (i.e., $g(s) = s^\sigma$ with $\sigma d \geq 2$), blow up may occur. However, this contradicts the experiments in the optical settings [Josserand and Rica 1997], which shows that there is no "singularity" and the solution always remains bounded.

One way to correct this model is to replace the power nonlinearities by saturated nonlinearities. A typical example[2] is $g(s) = s^\sigma - \gamma s^{\sigma+\delta}$, with $\delta > 0$, $\gamma > 0$. Similar to (gKdV$_\gamma$), in this case any H^1 solution to (NLS) is global in time and bounded in H^1.

On the other hand, the saturated perturbation is also related to the problem of continuation after blow up time. These kinds of problems arising in physics are poorly understood even at a formal level. One approach is to consider the solution $u_\varepsilon(t)$ to the following critical NLS with saturated perturbation:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u - \varepsilon |u|^q u = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases}$$

where

$$\frac{4}{d} < q < \frac{4}{d-2}.$$

Suppose the solution $u(t)$ to the unperturbed NLS (i.e., $\varepsilon = 0$) with initial data u_0, blows up in finite time $T < +\infty$. Then, it is easy to see that for all $\varepsilon > 0$, the solution $u_\varepsilon(t)$ exists globally in time, and for all $t < T$

$$\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t) \quad \text{in } H^1.$$

Now, we may consider the limit

$$\lim_{\varepsilon \to 0} u_\varepsilon(t), \quad t > T,$$

to see whether the limiting function exists and in what sense it satisfies the critical NLS. Such a construction for blow-up solutions using the Virial identity was given by Merle [1992a]. An alternative way to construct the approximate solution $u_\varepsilon(t)$ can also be found in [Merle 1989; 1992b; Merle, Raphaël and Szeftel 2013], but this only holds for very special cases. General constructions of this type are mostly open. In all cases, the asymptotic behavior of the approximate solution $u_\varepsilon(t)$ is crucial in the analysis.

Therefore, the asymptotic dynamics of dispersive equations with a saturated perturbation becomes a natural question.

1C. *Results for L^2 critical gKdV equations.* Let us recall some results for the following L^2 critical gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \tag{gKdV}$$

[2] See [Glasner and Allen-Flowers 2016; Marzuola, Raynor and Simpson 2010] for other kinds of saturated perturbations.

This equation is L^2 critical since, for all $\lambda > 0$,

$$u_\lambda(t,x) = \lambda^{-\frac{1}{2}} u(\lambda^{-3}t, \lambda^{-1}x)$$

is still a solution to (gKdV) and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$.

There is a special class of solutions, i.e., the solitary waves, which is given by

$$u(t,x) = \frac{1}{\lambda_0^{1/2}} Q\left(\frac{x - x_0 - \lambda_0^{-2}(t-t_0)}{\lambda_0}\right),$$

with

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}, \quad Q'' - Q + Q^5 = 0.$$

The function Q is called the ground state.

From variational arguments [Weinstein 1982], we know that if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution to (gKdV) is global in time and bounded in H^1, while for $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, blow up may occur. The blow up dynamics for solutions with slightly supercritical mass

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^* \tag{1-3}$$

has been developed in a series papers [Martel and Merle 2002a; 2002b; 2002c; Merle 2001]. In particular, they prove the existence of blow up solutions with negative energy, and give a specific description of the blow up dynamics and the formation of singularities.

Martel, Merle and Raphaël [2014; 2015a; 2015b] give an exclusive study of the asymptotic dynamics near the ground state Q.

More precisely, consider the initial data set

$$\mathcal{A}_{\alpha_0} = \left\{ u_0 \in H^1 \;\middle|\; u_0 = Q + \varepsilon_0,\ \|\varepsilon_0\|_{H^1} < \alpha_0,\ \int_{y>0} y^{10}\varepsilon_0^2 < 1 \right\},$$

and the L^2 tube around the solitary wave family

$$\mathcal{T}_{\alpha^*} = \left\{ u_0 \in H^1 \;\middle|\; \inf_{\lambda_0 > 0,\, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{1/2}} Q\left(\frac{x - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

Theorem 1.1. *For $0 < \alpha_0 \ll \alpha^* \ll 1$ and $u_0 \in \mathcal{A}_{\alpha_0}$, let $u(t)$ be the corresponding solution to* (gKdV), *and $0 < T \leq +\infty$ be the maximal lifetime. Then one of the following scenarios occurs*:

Blow up: The solution $u(t)$ blows up in finite time $0 < T < +\infty$ with

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T - t}, \quad \ell(u_0) > 0.$$

In addition, for all $t < T$, we have $u(t) \in \mathcal{T}_{\alpha^*}$.

Soliton: The solution is global, and for all $t < T = +\infty$, we have $u(t) \in \mathcal{T}_{\alpha^*}$. In addition, there exist a constant $\lambda_\infty > 0$ and a C^1 function $x(t)$ such that

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot +x(t)) \to Q \quad \text{in } H^1_{\text{loc}} \quad \text{as } t \to +\infty,$$
$$|\lambda_\infty - 1| \lesssim \delta(\alpha_0), \quad x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as } t \to +\infty.$$

Exit: *For some finite time* $0 < t^* < T$, *we have* $u(t^*) \notin \mathcal{T}_{\alpha^*}$.

Moreover, the blow-up and exit scenarios are stable by small perturbation in $\mathcal{A}_{\alpha_0}$.

Martel, Merle, Nakanishi and Raphaël [2016] proved that the initial data in $\mathcal{A}_{\alpha_0}$ which corresponds to the soliton regime is a codimension-one threshold submanifold between blow up and exit.

Theorem 1.2. *Let*

$$\mathcal{A}_{\alpha_0}^{\perp} = \left\{ \varepsilon_0 \in H^1 \;\middle|\; \|\varepsilon_0\|_{H^1} < \alpha_0,\ \int_{y>0} y^{10}\varepsilon_0^2 < 1,\ (\varepsilon_0, Q) = 0 \right\}.$$

Then there exist $\alpha_0 > 0,\ \beta_0 > 0$, *and a* C^1 *function* A,

$$\mathcal{A}_{\alpha_0}^{\perp} \to (-\beta_0, \beta_0),$$

such that for all $\gamma_0 \in \mathcal{A}_{\alpha_0}^{\perp}$ *and* $a \in [-\beta_0, \beta_0]$, *the solution of* (gKdV) *corresponding to* $u_0 = (1+a)Q + \gamma_0$ *satisfies*

- *the soliton regime if* $a = A(\gamma_0)$;
- *the blow-up regime if* $a > A(\gamma_0)$;
- *the exit regime if* $a < A(\gamma_0)$.

In particular, let

$$\mathcal{Q} = \{u_0 \in H^1 \mid \text{there exists } \lambda_0, x_0 \text{ such that } u_0 = \lambda_0^{-\frac{1}{2}} Q(\lambda_0^{-1}(x - x_0))\}.$$

Then there exists a small neighborhood $\mathcal{O}$ *of* $\mathcal{Q}$ *in* $H^1 \cap L^2(y_+^{10} dy)$ *and a codimension-one* C^1 *submanifold* $\mathcal{M}$ *of* $\mathcal{O}$ *such that* $\mathcal{Q} \subset \mathcal{M}$ *and for all* $u_0 \in \mathcal{O}$ *the corresponding solution of* (gKdV) *is in the soliton regime if and only if* $u_0 \in \mathcal{M}$.

1D. ***Statement of the main result.*** The aim of this paper is to classify the dynamics of (gKdV$_\gamma$) near the ground state Q for (gKdV), when γ is small enough. The main idea is that the defocusing term $\gamma u|u|^{q-1}$ has weaker nonlinear effect than the focusing term u^5. So, we may expect that (gKdV$_\gamma$) has similar separation behavior as (gKdV), when γ is small.

More precisely, we fix a small universal constant $\omega^* > 0$ (to ensure the existence of the ground state Q_ω), and then introduce the following L^2 tube around Q_γ:

$$\mathcal{T}_{\alpha^*,\gamma} = \left\{ u_0 \in H^1 \;\middle|\; \inf_{\lambda_0 > 0,\, \lambda_0^{-m}\gamma < \omega^*,\, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{1/2}} Q_{\lambda_0^{-m}\gamma}\left(\frac{x - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

Theorem 1.3 (dynamics in $\mathcal{A}_{\alpha_0}$). *For all* $q > 5$, *there exists a constant* $0 < \alpha^*(q) \ll 1$ *such that if* $0 < \gamma \ll \alpha_0 \ll \alpha^* < \alpha^*(q)$, *then for all* $u_0 \in \mathcal{A}_{\alpha_0}$ *the corresponding solution* $u(t)$ *to* (gKdV$_\gamma$) *has one and only one of the following behaviors*:

Soliton: *For all $t \in [0,+\infty)$, we have $u(t) \in \mathcal{T}_{\alpha^*,\gamma}$. Moreover, there exist a constant $\lambda_\infty \in (0,+\infty)$ and a C^1 function $x(t)$ such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot +x(t)) \to Q_{\lambda_\infty^{-m}\gamma} \quad \text{in } H^1_{\text{loc}} \quad \text{as } t \to +\infty, \tag{1-4}$$

$$x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as } t \to +\infty. \tag{1-5}$$

Blow down: *For all $t \in [0,+\infty)$, we have $u(t) \in \mathcal{T}_{\alpha^*,\gamma}$. Moreover, there exist two C^1 functions $\lambda(t)$ and $x(t)$ such that*

$$\lambda^{\frac{1}{2}}(t) u(t, \lambda(t) \cdot +x(t)) \to Q \quad \text{in } H^1_{\text{loc}} \quad \text{as } t \to +\infty; \tag{1-6}$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}} \quad \text{as } t \to +\infty, \tag{1-7}$$

Exit: *There exists $0 < t_\gamma^* < +\infty$ such that $u(t_\gamma^*) \notin \mathcal{T}_{\alpha^*,\gamma}$.*

There exist solutions associated to each regime. Moreover, the soliton and exit regimes are stable under small perturbation in $\mathcal{A}_{\alpha_0}$.

Comments. (1) *Classification of the flow near the ground state.* Theorem 1.3 gives a detailed description of the flow near the ground state Q_γ of (gKdV$_\gamma$). This kind of problem has attracted considerable attention, especially for dispersive equations. See for example, [Nakanishi and Schlag 2011; 2012a; 2012b] for Klein–Gordon and mass-supercritical nonlinear Schrödinger equations; [Fibich, Merle and Raphaël 2006; Merle and Raphaël 2003; 2004; 2005; 2005; 2006; Raphaël 2005; Merle, Raphaël and Szeftel 2013] for mass-critical nonlinear Schrödinger equations; [Martel, Merle and Raphaël 2014; 2015a] for L^2 critical gKdV equations; [Kenig and Merle 2006; Duyckaerts and Merle 2009] for energy-critical nonlinear Schrödinger equations; [Kenig and Merle 2008; Duyckaerts and Merle 2008; Krieger, Nakanishi and Schlag 2013; 2014] for energy-critical wave equations; and [Collot, Merle and Raphaël 2017] for energy-critical nonlinear heat equations. Note that the fact that the blow-down regime near the ground state is a codimension-one threshold submanifold of initial data in $\mathcal{A}_{\alpha_0}$ could be proved much as in [Martel, Merle, Nakanishi and Raphaël 2016].

(2) *Asymptotic stability of solitons for* (gKdV$_\gamma$). Since the soliton regime is open, Theorem 1.3 also implies the asymptotic stability of the soliton Q_γ for (gKdV$_\gamma$) under some suitable decay assumption. Recall that from [Martel and Merle 2001], the soliton Q for the unperturbed critical gKdV equation is not stable in H^1.

(3) *Blow-down behaviors.* Theorem 1.3 shows that a saturated perturbation may lead to some chaotic behaviors (i.e., the blow-down behaviors), which does not seem to appear in the unperturbed case. Examples of solutions with a blow-down behavior were also found in [Donninger and Krieger 2013] for energy-critical wave equations. While for mass-critical NLS, the blow-down behavior can be obtained as the pseudoconformal transformation of the log-log regime.[3] However, Theorem 1.3 is the first time that this type of blow-down behavior is obtained in the context of a saturated perturbation. Furthermore, in

[3] See [Merle, Raphaël and Szeftel 2013, (1.16)], for example.

Theorem 1.3, the blow-down regime is a codimension-one threshold between two stable ones, which is in contrast with the mass-critical nonlinear Schrödinger case, where the blow-down regime is stable.

Now we consider the case when $\gamma \to 0$. As we mentioned before, the defocusing term $\gamma u|u|^{q-1}$ has weaker nonlinear effect than the focusing term u^5. So the results in Theorem 1.3 are expected to be a perturbation of the one in Theorem 1.1.

More precisely, we have:

Theorem 1.4. *Let us fix a nonlinearity $q > 5$, and choose $0 < \alpha_0 \ll \alpha^* < \alpha^*(q)$ as in Theorem 1.3. For all $u_0 \in \mathcal{A}_{\alpha_0}$, let $u(t)$ be the corresponding solution of* (gKdV), *and $u_\gamma(t)$ be the corresponding solution of* (gKdV_γ). *Then we have*:

(1) *If $u(t)$ is in the blow-up regime defined in Theorem 1.1, then there exists $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$ such that if $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$, then $u_\gamma(t)$ is in the soliton regime defined in Theorem 1.3. Moreover, there exist constants $d_i = d_i(u_0, q) > 0$, $i = 1, 2$, such that*

$$d_1 \gamma^{\frac{2}{q-1}} \leq \lambda_\infty \leq d_2 \gamma^{\frac{2}{q-1}}, \tag{1-8}$$

where λ_∞ is the constant defined in (1-4).

(2) *If $u(t)$ is in the exit regime defined in Theorem 1.1, then there exists $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$ such that if $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$, then $u_\gamma(t)$ is in the exit regime defined in Theorem 1.3.*

Remark 1.5. We can see from Theorem 1.4 that (gKdV_γ) is a perturbation of (gKdV) as $\gamma \to 0$: the soliton regime of (gKdV_γ) "converges" to the blow-up regime of (gKdV), and the exit regime "converges" to the exit regime of (gKdV).

Remark 1.6. Theorem 1.4 is the first result of this type for nonlinear dispersive equations. One may also expect similar results for the critical NLS or the slightly supercritical gKdV cases. But they are still completely open.

Indeed, for critical NLS, Malkin [1993] predicted a similar asymptotic behavior for the solution to the saturated problem of critical NLS in the log-log region. However, due to the different structures of NLS and gKdV, it seems hard to apply the strategy in this paper to the NLS case.

While for the slightly supercritical gKdV case, the stable self-similar blow-up dynamics is well-studied in [Lan 2016]. But, due to the fact that the self-similar profile constructed in [Koch 2015, Theorem 3] is not in the energy space H^1, we have to choose a suitable cut-off as an approximation of this profile. As a consequence, this generates some error terms that are hard to control, which makes it impossible to consider the saturated problem in this case. However, Strunk [2014] proved the local well-posedness result for supercritical gKdV in a space that contains the self-similar profile, which provides an alternative option for the saturated problems.

1E. *Notation.* For $0 \leq \omega < \omega^* \ll 1$, we let $\mathcal{Q}_\omega$ be the unique nonnegative radial solution with exponential decay to the ODE

$$\mathcal{Q}_\omega'' - \mathcal{Q}_\omega + \mathcal{Q}_\omega^5 - \omega \mathcal{Q}_\omega |\mathcal{Q}_\omega|^{q-1} = 0. \tag{1-9}$$

For simplicity, we define $Q = Q_0$. Recall that we have

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}.$$

We also introduce the linearized operator at Q_ω:

$$L_\omega f = -f'' + f - 5Q_\omega^4 f + q\omega |Q_\omega|^{q-1} f.$$

Similarly, we define $L = L_0$.

Next, we introduce the scaling operator

$$\Lambda f = \tfrac{1}{2} f + y f'.$$

Then, for a given small constant α^*, we denote by $\delta(\alpha^*)$ a generic small constant with

$$\lim_{\alpha^* \to 0} \delta(\alpha^*) = 0.$$

Finally, we denote the L^2 scalar product by

$$(f, g) = \int f(x) g(x)\, dx.$$

1F. *Outline of the proof.*

1F1. *Decomposition of the flow.* We are searching for solutions of the form

$$u(t,x) \sim \frac{1}{\lambda(t)^{1/2}} Q_{b(t),\omega(t)}\left(\frac{x - x(t)}{\lambda(t)}\right),$$
$$\omega = \frac{\gamma}{\lambda^m}, \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{\lambda_s}{\lambda} = -b, \quad \frac{x_s}{\lambda} = 1,$$

which lead to the modified self-similar equation

$$b\Lambda Q_{b,\omega} + (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega} |Q_{b,\omega}|^{q-1})' = 0. \tag{1-10}$$

Formal computations show that b and ω must satisfy the condition

$$b_s + 2b^2 + c_0 \omega_s = 0,$$

where $c_0 = c_0(q) > 0$ is a universal constant.

Combining all the above, we get the formal finite-dimensional system

$$\begin{cases} \dfrac{ds}{dt} = \dfrac{1}{\lambda^3}, \quad \dfrac{\lambda_s}{\lambda} = -b, \quad \dfrac{x_s}{\lambda} = 1, \\ b_s + 2b^2 + c_0\omega_s = 0, \quad \omega = \dfrac{\gamma}{\lambda^m}. \end{cases} \tag{1-11}$$

By standard computations, it is easy to see that (1-11) has the following behavior. Let

$$L_0 = \frac{b(0)}{\lambda^2(0)} + \frac{m c_0 \gamma}{(m+2)\lambda^{m+2}(0)}.$$

We have:

(1) If $L_0 > 0$, then
$$b(t) \to 0, \quad \lambda(t) \to \left(\frac{m\gamma c_0}{(m+2)L_0}\right)^{\frac{1}{m+2}}, \quad x(t) \sim \left(\frac{(m+2)L_0}{m\gamma c_0}\right)^{\frac{2}{m+2}} t$$
as $t \to +\infty$, which corresponds to the soliton regime.

(2) If $L_0 = 0$, then
$$b(t) \to 0, \quad \lambda(t) \to +\infty, \quad x(t) \to +\infty$$
as $t \to +\infty$, which corresponds to the blow-down regime.

(3) If $L_0 < 0$, then
$$b(t) \to -\infty, \quad \lambda(t) \to +\infty$$
as $t \to +\infty$, which corresponds to the exit regime.

1F2. *Modulation theory.* Our first step is to find a solution to (1-10). But for our analysis, it is enough to consider a suitable approximation:[4]
$$Q_{b,\omega}(y) = \mathcal{Q}_\omega(y) + b\chi(|b|^\beta y) P_\omega(y).$$

As long as the solution remains in $\mathcal{T}_{\alpha^*,\gamma}$, we can introduce the geometrical decomposition
$$u(t) = \frac{1}{\lambda(t)^{1/2}}[Q_{b(t),\omega(t)} + \varepsilon(t)]\left(\frac{x - x(t)}{\lambda(t)}\right),$$
with $\omega(t) = \gamma/\lambda(t)^m$ and the error term satisfies some orthogonality conditions. Then the equations of the parameters are roughly speaking of the form
$$\frac{\lambda_s}{\lambda} + b = \frac{dJ_1}{ds} + O(\|\varepsilon\|^2_{H^1_{\text{loc}}}),$$
$$b_s + 2b^2 + c_0\omega_s = \frac{dJ_2}{ds} + O(\|\varepsilon\|^2_{H^1_{\text{loc}}}),$$
with
$$|J_i| \lesssim \|\varepsilon\|_{H^1_{\text{loc}}} + \int_{y>0} |\varepsilon|.$$

Therefore, a L^1 control on the right is needed, otherwise J_i will perturb the formal system (1-11).

1F3. *Monotonicity formula.* Our next step is to derive a control for $\|\varepsilon\|_{H^1_{\text{loc}}}$. Similar to [Martel, Merle and Raphaël 2014, Proposition 3.1], we introduce the nonlinear functional
$$\mathcal{F} \sim \int (\psi\varepsilon_y^2 + \varphi\varepsilon^2 - 5\varepsilon^2 Q^4_{b,\omega}\psi + q\omega\varepsilon^2 |Q_{b,\omega}|^{q-1}\psi)$$

[4] See Section 2A for more details.

for some well-chosen weight functions (ψ, φ), which decay exponentially to the left and grow polynomially on the right. We will see from the choice of the orthogonality condition that the leading quadratic term of $\mathcal{F}$ is coercive:

$$\mathcal{F} \gtrsim \|\varepsilon\|^2_{H^1_{\text{loc}}}.$$

Most importantly, we have the monotonicity formula

$$\frac{d}{ds}\left(\frac{\mathcal{F}}{\lambda^{2j}}\right) + \frac{\|\varepsilon\|^2_{H^1_{\text{loc}}}}{\lambda^{2j}} \lesssim \frac{\omega^2 b^2 + b^4}{\lambda^{2j}}$$

for $j = 0, 1$. This formula is crucial in all three cases.

1F4. *Rigidity.* The selection of the dynamics depends on:

(1) For all t,

$$\left|b(t) + \frac{mc_0}{m+2}\omega(t)\right| \lesssim \|\varepsilon(t)\|^2_{H^1_{\text{loc}}} + b^2(t) + \omega^2(t).$$

(2) For some $t_1^* < T = +\infty$,

$$b(t_1^*) + \frac{mc_0}{m+2}\omega(t_1^*) \gg \|\varepsilon(t)\|^2_{H^1_{\text{loc}}} + b^2(t) + \omega^2(t).$$

(3) For some $t_1^* < T = +\infty$,

$$-b(t_1^*) - \frac{mc_0}{m+2}\omega(t_1^*) \gg \|\varepsilon(t)\|^2_{H^1_{\text{loc}}} + b^2(t) + \omega^2(t).$$

We will see that in the first case we have for all t,

$$|b(t)| \sim \omega(t) \gg \|\varepsilon(t)\|^2_{H^1_{\text{loc}}},$$

and in the second case we have

$$\omega(t) \gg |b(t)| \gg \|\varepsilon(t)\|^2_{H^1_{\text{loc}}}$$

for $t > t_2^* \geq t_1^*$ as long as $u(t)$ remains in $\mathcal{T}_{\alpha^*,\gamma}$. While in the third case, we have

$$-b(t) \gg \omega(t) \gg \|\varepsilon(t)\|^2_{H^1_{\text{loc}}}$$

for $t > t_1^*$ as long as $u(t)$ remains in $\mathcal{T}_{\alpha^*,\gamma}$. Then reintegrating the modulation equations, we will see that these three cases correspond to the blow-down, soliton and exit regimes respectively.

Moreover, the condition on $b(t_1^*)$ and $\omega(t_1^*)$ which determines the soliton and exit regimes is an open condition to the initial data due to the continuity of the flow. On the other hand, it is easy to construct solutions, which belongs to the soliton and exit regimes respectively. Since, the initial data set $\mathcal{A}_{\alpha_0}$ is connected, we can see that there exist solutions corresponding to the blow-down regime.

1F5. *Proof of Theorem 1.4.* The proof of Theorem 1.4 is based on the fact that the separation condition for (gKdV$_\gamma$) is close to the separation condition for (gKdV) when $\gamma \to 0$. Then Theorem 1.4 follows immediately from a modified H^1 perturbation theory.[5]

[5] See [Killip, Kwon, Shao, and Visan 2012, Theorem 3.1] for the standard L^2 perturbation theory.

2. Nonlinear profile and decomposition of the flow

We will introduce the nonlinear profile and the geometrical decomposition similar to the one in [Martel, Merle and Raphaël 2014], which turns out to lead to the desired rigidity dynamics.

2A. ***Structure of the linearized operator L_ω.*** Denote by $\mathcal{Y}$ the set of smooth functions f such that for all $k \in \mathbb{N}$ there exist $r_k > 0$, $C_k > 0$ with

$$|\partial_y^k f(y)| \le C_k(1+|y|)^{r_k} e^{-|y|}. \tag{2-1}$$

Let us first recall some results about the linearized operator L.

Lemma 2.1 (properties of L [Martel and Merle 2001; Martel, Merle and Raphaël 2014; Weinstein 1985]). *The self-adjoint operator L (recall that we use the notation $L = L_0$, which was introduced in Section 1E) in L^2 has the following properties*:

(1) *Eigenfunction*: $LQ^3 = -8Q^3$, $LQ' = 0$, $\ker L = \{aQ' | a \in \mathbb{R}\}$.

(2) *Scaling*: $L(\Lambda Q) = -2Q$.

(3) *For any function $f \in L^2$ orthogonal to Q', there exists a unique $g \in H^2$ such that $Lg = f$ with $(g, Q') = 0$. Moreover, if f is even, then g is even, and if f is odd, then g is odd.*

(4) *If $f \in L^2$ such that $Lf \in \mathcal{Y}$, then $f \in \mathcal{Y}$.*

(5) *Coercivity*: *For all $f \in H^1$, if $(f, Q^3) = (f, Q') = 0$, then $(Lf, f) \ge (f, f)$. Moreover, there exists $\kappa_0 > 0$ such that for all $f \in H^1$,*

$$(Lf, f) \ge \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0}[(f, Q)^2 + (f, \Lambda Q)^2 + (f, y\Lambda Q)^2].$$

Proposition 2.2 (nonlocalized profiles [Martel, Merle and Raphaël 2014, Proposition 2.2]). *There exists a unique function P with $P' \in \mathcal{Y}$ such that*

$$(LP)' = \Lambda Q, \qquad \lim_{y\to-\infty} P(y) = \frac{1}{2}\int Q, \qquad |P(y)| \lesssim e^{-\frac{y}{2}} \quad \textit{for } y > 0, \tag{2-2}$$

$$(P, Q) = \frac{1}{16}\left(\int Q\right)^2, \quad (P, Q') = 0. \tag{2-3}$$

Now for the ground state $\mathcal{Q}_\omega$ and the linearized operator L_ω, we have the following properties:

Lemma 2.3. *For $0 < \omega < \omega^* \ll 1$, we have*:

(1) *Null space*: $\ker L_\omega = \{a\mathcal{Q}_\omega' \mid a \in \mathbb{R}\}$.

(2) *Pseudoscaling rule*: $L_\omega(\Lambda \mathcal{Q}_\omega) = -2\mathcal{Q}_\omega + \frac{1}{2}(q-5)\omega \mathcal{Q}_\omega^q$.

(3) *For any function $f \in L^2$ orthogonal to $\mathcal{Q}_\omega'$, there exists a unique $g \in H^2$ such that $L_\omega g = f$ with $(g, \mathcal{Q}_\omega') = 0$. Moreover, if f is even, then g is even, and if f is odd, then g is odd.*

(4) *If $f \in L^2$ such that $L_\omega f \in \mathcal{Y}$, then $f \in \mathcal{Y}$.*

(5) *Let $Z_\omega = \partial \mathcal{Q}_\omega/\partial\omega$. Then $Z_\omega \in \mathcal{Y}$, and $L_\omega Z_\omega = -\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1}$.*

(6) *Coercivity*: *There exists a* $\kappa_0 > 0$ *such that for all* $f \in H^1$,

$$(L_\omega f, f) \geq \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0}[(f, \mathcal{Q}_\omega)^2 + (f, \Lambda \mathcal{Q}_\omega)^2 + (f, y\Lambda \mathcal{Q}_\omega)^2].$$

Proof. Part (1) follows from the same arguments as the proof of [Weinstein 1985, Proposition 2.8; 1986, Proposition 3.2]. Part (2) follows from direct computation. Part (3) is a direct corollary of (1), while for (4), from standard elliptic theory, we know that f is smooth and bounded. So we have $Lf \in \mathcal{Y}$, and from Lemma 2.1, we have $f \in \mathcal{Y}$.

Now we turn to the proof of (5). Differentiating (1-9), we obtain $L_\omega Z_\omega = -\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1}$. Since $\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} \in \mathcal{Y}$, if we can show that $Z_\omega \in L^2$, then we have $Z_\omega \in \mathcal{Y}$. To do this, we introduce the map

$$F : H_e^2 \times \mathbb{R} \mapsto L_e^2, \quad (u, \omega) \mapsto -u'' + u - u^5 + \omega u|u|^{q-1},$$

where H_e^2 and L_e^2 are the Banach spaces consisting of all H^2 and L^2 functions, respectively, which are even. Since $H^2(\mathbb{R})$ is continuously embedded into $L^\infty(\mathbb{R})$, the map F is well-defined.

We claim that there exists a small $\omega^* > 0$ such that if $0 \leq \omega < \omega^*$, then there exists a unique $u(\omega) \in H_e^2$ such that $F(u(\omega), \omega) = 0$. Since we have $F(Q, 0) = 0$, from implicit function theory, it only remains to show that the Fréchet derivative with respect to u, i.e., $\partial F/\partial u|_{(Q,0)} \in \mathcal{L}(H_e^2, L_e^2)$, is invertible and continuous. But it is easy to see that

$$\left.\frac{\partial F}{\partial u}\right|_{(Q,0)} = L,$$

which is invertible and continuous due to part (3) of Lemma 2.1. Hence, we obtain the existence of such $u(\omega)$. Moreover, since F is continuously differentiable with respect to both u and ω, we have $u(\omega)$ is continuously differentiable with respect to ω. In particular, we have $\partial u/\partial\omega \in H_e^2$. But from the uniqueness of $u(\omega)$, we must have $u(\omega) = \mathcal{Q}_\omega$. As a consequence, we have $Z_\omega = \partial \mathcal{Q}_\omega/\partial\omega = \partial u/\partial\omega \in H_e^2$, which concludes the proof of (5).

Finally, (6) follows immediately from a perturbation argument for part (5) of Lemma 2.1. More precisely, since $\mathcal{Q}_\omega$ is C^1 with respect to ω, we have, for all $f \in H^1$,

$$(L_\omega f, f) = (Lf, f) + O(\omega)\|f\|_{H^1}^2,$$

and

$$(f, \mathcal{Q}_\omega)^2 + (f, \Lambda \mathcal{Q}_\omega)^2 + (f, y\Lambda \mathcal{Q}_\omega)^2 = (f, Q)^2 + (f, \Lambda Q)^2 + (f, y\Lambda Q)^2 + O(\omega)\|f\|_{H^1}^2.$$

Together with part (5) of Lemma 2.1, we conclude the proof of part (6) of Lemma 2.3, which finishes the proof of Lemma 2.3. □

Proposition 2.4. *For* $0 < \omega < \omega^* \ll 1$, *there exists a smooth function* P_ω, *with* $P'_\omega \in \mathcal{Y}$, *such that*

$$(L_\omega P_\omega)' = \Lambda \mathcal{Q}_\omega, \quad \lim_{y \to -\infty} P_\omega(y) = \frac{1}{2}\int \mathcal{Q}_\omega, \tag{2-4}$$

$$(P_\omega, \mathcal{Q}'_\omega) = 0, \quad (P_\omega, \mathcal{Q}_\omega) = \frac{1}{16}\left(\int Q\right)^2 + F(\omega), \tag{2-5}$$

where F is a C^1 function with $F(0)=0$. Moreover there exist constants $C_0, C_1, \dots$, independent of ω, such that

$$|P_\omega(y)| + \left|\frac{\partial P_\omega}{\partial \omega}(y)\right| \le C_0 e^{-\frac{y}{2}} \quad \textit{for all } y > 0, \tag{2-6}$$

$$|P_\omega(y)| + \left|\frac{\partial P_\omega}{\partial \omega}(y)\right| \le C_0 \quad \textit{for all } y \in \mathbb{R}, \tag{2-7}$$

$$|\partial_y^k P_\omega(y)| \le C_k e^{-\frac{|y|}{2}} \quad \textit{for all } k \in \mathbb{N}_+,\ y \in \mathbb{R}. \tag{2-8}$$

Proof. The proof of Proposition 2.4 is almost parallel to Proposition 2.2. We look for a solution of the form $P_\omega = \widetilde{P}_\omega - \int_y^{+\infty} \Lambda \mathcal{Q}_\omega$. The function $y \to \int_y^{+\infty} \Lambda \mathcal{Q}_\omega$ is bounded and decays exponentially as $y \to +\infty$. Then, P_ω solves (2-4) if and only if $\widetilde{P}_\omega$ solves

$$(L_\omega \widetilde{P}_\omega)' = \Lambda \mathcal{Q}_\omega + \left(L_\omega \int_y^{+\infty} \Lambda \mathcal{Q}_\omega\right)' = R_\omega',$$

where

$$R_\omega = (\Lambda \mathcal{Q}_\omega)' - 5\mathcal{Q}_\omega^4 \int_y^{+\infty} \Lambda \mathcal{Q}_\omega + q\omega |\mathcal{Q}_\omega|^{q-1} \int_y^{+\infty} \Lambda \mathcal{Q}_\omega.$$

Note that $R_\omega \in \mathcal{Y}$. Since $(\Lambda \mathcal{Q}_\omega, \mathcal{Q}_\omega) = 0$ and $L_\omega \mathcal{Q}_\omega' = 0$, we have $(R_\omega, \mathcal{Q}_\omega') = -(R_\omega', \mathcal{Q}_\omega) = 0$. Then from Lemma 2.3, there exists a unique $\widetilde{P}_\omega \in \mathcal{Y}$, orthogonal to $\mathcal{Q}_\omega'$, such that $L_\omega \widetilde{P}_\omega = R_\omega$. Then $P_\omega = \widetilde{P}_\omega - \int_y^{+\infty} \Lambda \mathcal{Q}_\omega$ satisfies (2-4) with $(P_\omega, \mathcal{Q}_\omega') = 0$ and $\lim_{y \to -\infty} P_\omega(y) = \frac{1}{2}\int \mathcal{Q}_\omega$. Moreover, we have

$$\begin{aligned} 2\int P_\omega \mathcal{Q}_\omega &= -\int (L_\omega P_\omega)\Lambda \mathcal{Q}_\omega + O(\omega) = \int \Lambda \mathcal{Q}_\omega \int_y^{+\infty} \Lambda \mathcal{Q}_\omega + O(\omega) \\ &= \frac{1}{2}\left(\int \Lambda \mathcal{Q}_\omega\right)^2 + O(\omega) = \frac{1}{8}\left(\int Q\right)^2 + O(\omega). \end{aligned}$$

Let

$$F(\omega) = (P_\omega, \mathcal{Q}_\omega) - \frac{1}{16}\left(\int Q\right)^2.$$

Then $F(0) = 0$.

Next we claim that $\partial \widetilde{P}_\omega / \partial \omega \in \mathcal{Y}$. Let us differentiate the equation $L_\omega \widetilde{P}_\omega = R_\omega$ to get

$$L_\omega\left(\frac{\partial \widetilde{P}_\omega}{\partial \omega}\right) = \frac{\partial R_\omega}{\partial \omega} - 20 Z_\omega \mathcal{Q}_\omega^3 \widetilde{P}_\omega + q(q-1)\omega Z_\omega \mathcal{Q}_\omega |\mathcal{Q}_\omega|^{q-3} \widetilde{P}_\omega + q|\mathcal{Q}_\omega|^{q-1} \widetilde{P}_\omega. \tag{2-9}$$

Since $Z_\omega \in \mathcal{Y}$, it is easy to check that $\partial R_\omega / \partial \omega \in \mathcal{Y}$. So Lemma 2.3 implies $\partial \widetilde{P}_\omega / \partial \omega \in \mathcal{Y}$.

Now it only remains to prove (2-6)–(2-8). But from [Berestycki and Lions 1983, Section 6], there exist constants $M_0, M_1, \dots$, independent of ω, such that for all $k \in \mathbb{N}$, $y \in \mathbb{R}$,

$$|\partial_y^k \mathcal{Q}_\omega(y)| \le M_k e^{-\frac{2|y|}{3}}.$$

Together with (2-9) and the construction of P_ω, we obtain (2-6)–(2-8). It is easy to see that (2-6)–(2-8) also imply $F \in C^1$. □

Now, we proceed to a simple localization of the profile to avoid the nontrivial tail on the left. Let χ be a smooth function with $0 \le \chi \le 1$, $\chi' \ge 0$, $\chi(y) = 1$ if $y > -1$, and $\chi(y) = 0$ if $y < -2$. We fix

$$\beta = \tfrac{3}{4}, \tag{2-10}$$

and define the localized profile

$$\chi_b(y) = \chi(|b|^\beta y), \quad Q_{b,\omega}(y) = \mathcal{Q}_\omega + b\chi_b(y)P_\omega(y). \tag{2-11}$$

Lemma 2.5 (localized profiles). *For $|b| < b^* \ll 1$, $0 < \omega < \omega^* \ll 1$, we have*:

(1) *Estimates on Q_b: For all $y \in \mathbb{R}$, $k \in \mathbb{N}$,*

$$|Q_{b,\omega}(y)| \lesssim e^{-|y|} + |b|(\mathbf{1}_{[-2,0]}(|b|^\beta y) + e^{-\frac{|y|}{2}}), \tag{2-12}$$

$$|\partial_y^k Q_{b,\omega}(y)| \lesssim e^{-|y|} + |b|e^{-\frac{|y|}{2}} + |b|^{1+k\beta}\mathbf{1}_{[-2,-1]}(|b|^\beta y), \tag{2-13}$$

where $\mathbf{1}_I$ denotes the characteristic function of the interval I.

(2) *Equation of $Q_{b,w}$: Let*

$$-\Psi_{b,\omega} = b\Lambda Q_{b,\omega} + (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})'. \tag{2-14}$$

Then, for all $y \in \mathbb{R}$,

$$-\Psi_{b,\omega} = b^2((10\mathcal{Q}_\omega^3 P_\omega^2)_y + \Lambda P_\omega) - \tfrac{1}{2}b^2(1-\chi_b)P_\omega + O\big(|b|^{1+\beta}\mathbf{1}_{[-2,-1]}(|b|^\beta y) + b^2(\omega + |b|)e^{-\frac{|y|}{2}}\big). \tag{2-15}$$

Moreover, we have

$$|\partial_y \Psi_{b,\omega}(y)| \lesssim |b|^{1+2\beta}\mathbf{1}_{[-2,-1]}(|b|^\beta y) + b^2 e^{-\frac{|y|}{2}}. \tag{2-16}$$

(3) *Mass and energy properties of $Q_{b,\omega}$:*

$$\left|\int Q^2_{b,\omega} - \left(\int \mathcal{Q}^2_\omega + 2b\int P_\omega \mathcal{Q}_\omega\right)\right| \lesssim |b|^{2-\beta}, \tag{2-17}$$

$$|E(Q_{b,\omega})| \lesssim |b| + \omega. \tag{2-18}$$

Proof. The proof of (1) follows immediately from the definition of $Q_{b,\omega}$ and Proposition 2.4. For (2), let us expand $Q_{b,\omega} = \mathcal{Q}_\omega + b\chi_b P_\omega$ in the expression of $\Psi_{b,\omega}$; using the fact that

$$\mathcal{Q}''_\omega - \mathcal{Q}_\omega + \mathcal{Q}^5_\omega - \omega\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} = 0, \quad (L_\omega P_\omega)' = \Lambda\mathcal{Q}_\omega,$$

we have

$$\begin{aligned}-\Psi_{b,\omega} &= b(1-\chi_b)\Lambda\mathcal{Q}_\omega + b\big(\chi'''_b P_\omega + 3\chi''_b P'_\omega + 2\chi'_b P''_\omega - \chi'_b P_\omega + 5\chi'_b \mathcal{Q}_\omega P_\omega - q\omega\chi'_b|\mathcal{Q}_\omega|^{q-1}P_\omega\big) \\ &\quad + b^2\big((10\mathcal{Q}^3_\omega\chi^2_b P^2_\omega)_y + P_\omega\Lambda\chi_b + \chi_b y P'_\omega\big) \\ &\quad + b^3(10\mathcal{Q}^2_\omega\chi^3_b P^3_\omega)_y + b^4(5\mathcal{Q}_\omega\chi^4_b P^4_\omega)_y + b^5(\chi^5_b P^5_\omega)_y \\ &\quad - \omega\big((\mathcal{Q}_\omega + b\chi_b P_\omega)|\mathcal{Q}_\omega + b\chi_b P_\omega|^{q-1} - \mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} - qb\chi_b P_\omega|\mathcal{Q}_\omega|^{q-1}\big)_y.\end{aligned}$$

We keep track of all terms up to b^2. Then (2-15) and (2-16) follow from the construction of the profile $Q_{b,\omega}$.

Finally, for (3), we have

$$\int \chi_b^2 P_\omega^2 \lesssim |b|^{-\beta}.$$

Then (2-17) follows from

$$\int Q_{b,\omega}^2 = \int \mathcal{Q}_\omega^2 + 2b \int \chi_b P_\omega \mathcal{Q}_\omega + b^2 \int \chi_b^2 P_\omega^2.$$

While for (2-18), since $E(\mathcal{Q}_\omega) = O(\omega)$, we have

$$|E(Q_{b,\omega})| \lesssim |b| + |E(\mathcal{Q}_\omega)| \lesssim |b| + \omega.$$

□

2B. ***Geometrical decomposition and modulation estimates.*** In this paper we consider H^1 solutions to (gKdV$_\gamma$) a priori in the modulates tube $\mathcal{T}_{\alpha^*,\gamma}$ of functions near the soliton manifold. More precisely:

Lemma 2.6. *Assume that there exist* $(\lambda_1(t), x_1(t)) \in ((\gamma/\omega^*)^{1/m}, +\infty) \times \mathbb{R}$ *and* $\varepsilon_1(t)$ *such that for all* $t \in [0, t_0)$, *the solution* $u(t)$ *to* (gKdV$_\gamma$) *satisfies*

$$u(t,x) = \frac{1}{\lambda_1^{1/2}(t)} [\mathcal{Q}_{\omega_1(t)} + \varepsilon_1(t)] \left(\frac{x - x_1(t)}{\lambda_1(t)} \right), \tag{2-19}$$

with, for all $t \in [0, t_0)$,

$$\omega_1(t) + \|\varepsilon_1(t)\|_{L^2} \leq \kappa \ll 1, \tag{2-20}$$

where

$$\omega_1(t) = \frac{\gamma}{\lambda_1^m(t)}.$$

Then we have:

(1) *There exist continuous functions* $(\lambda(t), x(t), b(t)) \in (0, +\infty) \times \mathbb{R}^2$ *such that for all* $t \in [0, t_0)$,

$$\varepsilon(t, y) = \lambda^{\frac{1}{2}}(t) u(t, \lambda(t) y + x(t)) - Q_{b(t),\omega(t)} \tag{2-21}$$

satisfies the orthogonality conditions

$$(\varepsilon(t), \mathcal{Q}_{\omega(t)}) = (\varepsilon(t), \Lambda \mathcal{Q}_{\omega(t)}) = (\varepsilon(t), y \Lambda \mathcal{Q}_{\omega(t)}) = 0, \tag{2-22}$$

where

$$\omega(t) = \frac{\gamma}{\lambda^m(t)}.$$

Moreover,

$$\omega(t) + \|\varepsilon(t)\|_{L^2} + |b(t)| + \left| 1 - \frac{\lambda_1(t)}{\lambda(t)} \right| \lesssim \delta(\kappa), \tag{2-23}$$

$$\|\varepsilon(0)\|_{H^1} \lesssim \delta(\|\varepsilon_1(0)\|_{H^1}). \tag{2-24}$$

(2) *The parameters and error term depend continuously on the initial data. Consider a family of solutions* $u_n(t)$, *with* $u_{0,n} \in \mathcal{A}_{\alpha_0}$, *and* $u_{0,n} \to u_0$ *in* H^1 *as* $n \to +\infty$. *Let* $(\lambda_n(t), b_n(t), x_n(t), \varepsilon_n(t))$ *be the corresponding geometrical parameters and error terms of* $u_n(t)$. *Suppose the geometrical decompositions of* $u_n(t)$ *and* $u(t)$ *hold on* $[0, T_0]$ *for some* $T_0 > 0$. *Then for all* $t \in [0, T_0]$, *we have*

$$(\lambda_n(t), b_n(t), x_n(t), \varepsilon_n(t)) \xrightarrow{\mathbb{R}^3 \times H^1} (\lambda(t), b(t), x(t), \varepsilon(t)) \tag{2-25}$$

as $n \to +\infty$.

Proof. Lemma 2.6 is a standard consequence of the implicit function theorem. We leave the proof for Appendix A. □

Remark 2.7. Similar arguments have also been used in [Martel and Merle 2002a, Lemma 1; 2002c, Lemma 1; Martel, Merle and Raphaël 2014, Lemma 2.5; Merle 2001, Lemma 2] etc.

Remark 2.8. The smallness of $\omega(t)$ ensures that $\mathcal{Q}_{\omega(t)}$ and $Q_{b(t),\omega(t)}$ are both well-defined.

2C. *Modulation equation.* In the framework of Lemma 2.6, we introduce the rescaled variables (s, y)

$$y = \frac{x - x(t)}{\lambda(t)}, \quad s = \int_0^t \frac{1}{\lambda^3(\tau)}\, d\tau. \tag{2-26}$$

Then, we have the following properties:

Proposition 2.9. *Assume for all* $t \in [0, t_0)$,

$$\omega(t) + \|\varepsilon(t)\|_{L^2} + \int \varepsilon_y^2 e^{-\frac{3|y|}{2(q-2)}}\, dy \le \kappa \tag{2-27}$$

for some small universal constant $\kappa > 0$. *Then the functions* $(\lambda(s), x(s), b(s))$ *are all* C^1 *and the following hold*:

(1) *Equation of* ε: *For all* $s \in [0, s_0)$,

$$\varepsilon_s - (L_\omega \varepsilon)_y + b\Lambda\varepsilon = \left(\frac{\lambda_s}{\lambda} + b\right)(\Lambda Q_{b,\omega} + \Lambda\varepsilon) + \left(\frac{x_s}{\lambda} - 1\right)(Q_{b,\omega} + \varepsilon)_y - b_s \frac{\partial Q_{b,\omega}}{\partial b} - \omega_s \frac{\partial Q_{b,\omega}}{\partial \omega} + \Psi_{b,\omega} - (R_b(\varepsilon))_y - (R_{\mathrm{NL}}(\varepsilon))_y, \tag{2-28}$$

where

$$\Psi_{b,\omega} = -b\Lambda Q_{b,\omega} - (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})', \tag{2-29}$$

$$R_b(\varepsilon) = 5(Q^4_{b,\omega} - \mathcal{Q}^4_\omega)\varepsilon - q\omega(|Q_{b,\omega}|^{q-1} - |\mathcal{Q}_\omega|^{q-1})\varepsilon, \tag{2-30}$$

$$R_{\mathrm{NL}}(\varepsilon) = (\varepsilon + Q_{b,\omega})^5 - 5Q^4_{b,\omega}\varepsilon - Q^5_{b,\omega} - \omega\big[(\varepsilon + Q_{b,\omega})|\varepsilon + Q_{b,\omega}|^{q-1} - q\varepsilon|Q_{b,\omega}|^{q-1} - Q_{b,\omega}|Q_{b,\omega}|^{q-1}\big]. \tag{2-31}$$

(2) *Estimate induced by the conservation laws*: *For* $s \in [0, s_0)$,

$$\|\varepsilon\|_{L^2} \lesssim |b|^{\frac{1}{4}} + \omega^{\frac{1}{2}} + \left|\int u_0^2 - \int Q^2\right|^{\frac{1}{2}}, \tag{2-32}$$

$$\frac{\|\varepsilon_y\|^2_{L^2}}{\lambda^2} \lesssim \frac{1}{\lambda^2}\left(\omega + |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right) + \gamma \frac{\|\varepsilon_y\|^{m+2}_{L^2}}{\lambda^{m+2}} + |E_0|. \tag{2-33}$$

(3) *H^1 modulation equation*: *For all $s \in [0, s_0)$,*

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right| \lesssim \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|(\omega+|b|), \tag{2-34}$$

$$|b_s|+|\omega_s| \lesssim (\omega+|b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|\right]+\int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-35}$$

(4) *L^1 control on the right*: *Assume uniformly L^1 control on the right*; *that is*, *for all $t \in [0, t_0)$,*

$$\int_{y>0} |\varepsilon(t)| \lesssim \delta(\kappa). \tag{2-36}$$

Then the quantities J_1 and J_2 below are well-defined. Moreover, *we have*:

(a) *Law of λ*: *Let*

$$\rho_1(y)=\frac{4}{\left(\int Q\right)^2}\int_{-\infty}^{y} \Lambda Q, \quad J_1(s)=(\varepsilon(s), \rho_1), \tag{2-37}$$

where Q is the ground state for (gKdV). *Then we have*

$$\left|\frac{\lambda_s}{\lambda}+b-2\left((J_1)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_1\right)\right| \lesssim (\omega+|b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|\right]+\int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-38}$$

(b) *Law of b*: *Let*

$$\rho_2=\frac{16}{\left(\int Q\right)^2}\left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}\Lambda Q+P-\frac{1}{2}\int Q\right)-8\rho_1, \quad J_2(s)=(\varepsilon(s), \rho_2), \tag{2-39}$$

where P was introduced in Proposition 2.2. Then we have

$$\left|b_s+2b^2+\omega_s G'(\omega)+b\left((J_2)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right)\right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}}+(\omega+|b|)b^2, \tag{2-40}$$

where $G \in C^2$ with $G(0)=0$, $G'(0)=c_0>0$, for some universal constant c_0.

(c) *Law of b/λ^2*: *Let*

$$\rho=4\rho_1+\rho_2 \in \mathcal{Y}, \quad J(s)=(\varepsilon(s), \rho). \tag{2-41}$$

Then we have

$$\left|\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)+\frac{b}{\lambda^2}\left(J_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J\right)+\frac{\omega_s G'(\omega)}{\lambda^2}\right| \lesssim \frac{1}{\lambda^2}\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\right). \tag{2-42}$$

Remark 2.10. The proof of Proposition 2.9 follows almost the same procedure as [Martel, Merle and Raphaël 2014, Lemma 2.7]. It is important that there is no a priori assumption on the upper bound of $\lambda(t)$. This fact ensures that Proposition 2.9 can be used in all three regimes.[6]

Proof. (1) Equation (2-28) follows by direct computation from the equation of $u(t)$.

[6] We will see in Section 4 that we can't expect any (finite) upper bound on the scaling parameter $\lambda(t)$ in both the blow-down and exit cases.

(2) We write down the mass conservation law

$$\int Q_{b,\omega}^2 - \int Q^2 + \int \varepsilon^2 + 2(\varepsilon, Q_{b,\omega}) = \int u_0^2 - \int Q^2. \tag{2-43}$$

From (2-17) and the orthogonality condition (2-22), we have

$$\int \varepsilon^2 \lesssim |b| + \omega + |b|^{1-\beta}\|\varepsilon\|_{L^2} + \left|\int u_0^2 - \int Q^2\right|.$$

Then (2-32) follows from $\beta = \frac{3}{4}$.

Similarly, we use the energy conservation law and (2-18) to obtain

$$\begin{aligned}
2\lambda^2 E_0 &= 2E(Q_{b,\omega}) - 2\int \varepsilon (Q_{b,\omega})_{yy} + \int \varepsilon_y^2 - \frac{1}{3}\int [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6] \\
&\quad + \frac{2\omega}{q+1}\int [|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1}] \\
&= O(|b|+\omega) + \int \varepsilon_y^2 - 2\int \varepsilon[(Q_{b,\omega} - \mathcal{Q}_\omega)_{yy} + (Q_{b,\omega}^5 - \mathcal{Q}_\omega^5) + \omega(Q_{b,\omega}|Q_{b,\omega}|^{q-1} - \mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1})] \\
&\quad - \frac{1}{3}\int [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \\
&\quad + \frac{2\omega}{q+1}\int [|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}].
\end{aligned}$$

We estimate all terms in the above identity. By the definition of $Q_{b,\omega}$, we have

$$\begin{aligned}
&\left|\int \varepsilon[(Q_{b,\omega} - \mathcal{Q}_\omega)_{yy} + (Q_{b,\omega}^5 - \mathcal{Q}_\omega^5) + \omega(Q_{b,\omega}|Q_{b,\omega}|^{q-1} - \mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1})]\right| \\
&\qquad \lesssim |b|\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|^{1+2\beta}\int_{-2|b|^{-\beta}\le y\le 0} |\varepsilon| \\
&\qquad \lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}}.
\end{aligned}$$

For the nonlinear term, we use the Gagliardo–Nirenberg inequality to estimate

$$\begin{aligned}
\left|\int [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5]\right| &\lesssim \int \varepsilon^2 \mathcal{Q}_\omega^4 + \int \varepsilon^6 + |b|\int \varepsilon^2 \\
&\lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b| + \|\varepsilon\|_{L^2}^4 \|\varepsilon_y\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
\left|\omega\int [|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\right| &\lesssim \omega\left(|b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int |\varepsilon|^{q+1}\right) \\
&\lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \frac{\gamma}{\lambda^m}\|\varepsilon\|_{L^2}^{\frac{q+3}{2}}\|\varepsilon_y\|_{L^2}^{m+2} \\
&\lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \gamma\frac{\|\varepsilon_y\|_{L^2}^{m+2}}{\lambda^m}.
\end{aligned}$$

Collecting all the estimates above, we obtain (2-33).

(3) Let us differentiate the orthogonality conditions

$$(\varepsilon(t), \Lambda Q_{\omega(t)}) = (\varepsilon(t), y\Lambda Q_{\omega(t)}) = 0.$$

Note that

$$\frac{d}{ds}(\varepsilon, \Lambda Q_\omega) = (\varepsilon_s, \Lambda Q_\omega) + \omega_s(\varepsilon, \Lambda Z_\omega),$$

where $Z_\omega = \partial Q_\omega/\partial\omega \in \mathcal{Y}$. So we have

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon, L_\omega(\Lambda Q_\omega)')}{\|\Lambda Q_\omega\|_{L^2}^2}\right| + \left|\left(\frac{x_s}{\lambda}-1\right)-\frac{(\varepsilon, L_\omega(y\Lambda Q_\omega)')}{\|\Lambda Q_\omega\|_{L^2}^2}\right| \\ \lesssim \left(\left|\frac{\lambda_s}{\lambda}+b\right| + \left|\frac{x_s}{\lambda}-1\right| + |b|\right) \times \left(\omega + |b| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}\right) \\ + |b_s| + |\omega_s| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int \varepsilon^5 e^{-\frac{9|y|}{10}} + \int |\varepsilon|^q e^{-\frac{9|y|}{10}}.\end{aligned}$$

For the nonlinear term, we use Sobolev embedding and the a priori smallness (2-27),

$$\begin{aligned}\|\varepsilon e^{-\frac{|y|}{4}}\|_{L^\infty}^2 &\le \|\varepsilon e^{-\frac{3|y|}{4(q-2)}}\|_{L^\infty}^2 \\ &\lesssim \int (\partial_y\varepsilon^2 + \varepsilon^2) e^{-\frac{3|y|}{4(q-2)}} \\ &\ll 1,\end{aligned}$$

to estimate

$$\int \varepsilon^5 e^{-\frac{9|y|}{10}} + \int |\varepsilon|^q e^{-\frac{9|y|}{10}} \lesssim (\|\varepsilon e^{-\frac{|y|}{4}}\|_{L^\infty}^3 + \|\varepsilon e^{-\frac{3|y|}{4(q-2)}}\|_{L^\infty}^{q-2}) \int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-44}$$

Here we use the basic fact that $q > 5$.

For ω_s, we have

$$\omega_s = -m\omega\frac{\lambda_s}{\lambda} = m\omega b - m\omega\left(\frac{\lambda_s}{\lambda}+b\right). \tag{2-45}$$

The above estimates imply

$$\left|\frac{\lambda_s}{\lambda}+b\right| + \left|\frac{x_s}{\lambda}-1\right| \lesssim (\omega+|b|)|b| + |b_s| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} \tag{2-46}$$

and

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon, L_\omega(\Lambda Q_\omega)')}{\|\Lambda Q_\omega\|_{L^2}^2}\right| + \left|\left(\frac{x_s}{\lambda}-1\right)-\frac{(\varepsilon, L_\omega(y\Lambda Q_\omega)')}{\|\Lambda Q_\omega\|_{L^2}^2}\right| \\ \lesssim (\omega+|b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|\right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}.\end{aligned} \tag{2-47}$$

Next, let us differentiate the relation $(\varepsilon, \mathcal{Q}_\omega) = 0$ to obtain

$$\begin{aligned}
0 = (\varepsilon, \mathcal{Q}_\omega)_s &= (\varepsilon_s, \mathcal{Q}_\omega) + \left(\varepsilon, \omega_s \frac{\partial \mathcal{Q}_\omega}{\partial \omega}\right) \\
&= \omega_s \left(\varepsilon, \frac{\partial \mathcal{Q}_\omega}{\partial \omega}\right) - (\varepsilon, L_\omega(\mathcal{Q}_\omega')) - b(\Lambda\varepsilon, \mathcal{Q}_\omega) \\
&\quad + \left(\frac{\lambda_s}{\lambda} + b\right)[(\Lambda Q_{b,\omega}, \mathcal{Q}_\omega) + (\Lambda\varepsilon, \mathcal{Q}_\omega)] + \left(\frac{x_s}{\lambda} - 1\right)[(Q_{b,\omega}', \mathcal{Q}_\omega) + (\varepsilon', \mathcal{Q}_\omega)] \\
&\quad - b_s[(P_\omega \chi_b, \mathcal{Q}_\omega) + (\beta y \chi_b', \mathcal{Q}_\omega)] - \omega_s \left(\mathcal{Q}_\omega, \frac{\partial Q_{b,\omega}}{\partial \omega}\right) + (\Psi_{b,\omega}, \mathcal{Q}_\omega) + (R_b(\varepsilon) + R_{\mathrm{NL}}(\varepsilon), \mathcal{Q}_\omega'). \quad (2\text{-}48)
\end{aligned}$$

Substituting the facts

$$\begin{gathered}
(P_\omega \chi_b, \mathcal{Q}_\omega) + (\beta y \chi_b', \mathcal{Q}_\omega) = (P_\omega, \mathcal{Q}_\omega) + O(b^{10}) \sim 1, \\
L_\omega \mathcal{Q}_\omega' = 0, \quad (\mathcal{Q}_\omega, \Lambda \mathcal{Q}_\omega) = (\mathcal{Q}_\omega, \mathcal{Q}_\omega') = (\varepsilon, \Lambda \mathcal{Q}_\omega) = 0, \\
|(R_b(\varepsilon) + R_{\mathrm{NL}}(\varepsilon), \mathcal{Q}_\omega')| \lesssim (\omega + |b|)\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + \int \varepsilon^2 e^{-\frac{|y|}{10}}
\end{gathered}$$

and (2-15), (2-16), (2-44), (2-45) into (2-48), we obtain

$$|b_s| \lesssim \left(\omega + |b| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}\right)\left(\left|\frac{\lambda_s}{\lambda} + b\right| + \left|\frac{x_s}{\lambda} - 1\right|\right) + (\omega + |b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|\right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \quad (2\text{-}49)$$

Combining (2-45), (2-46) and (2-49), we get (2-34) and (2-35).

(4) First, we claim the sharp equation

$$\begin{aligned}
b_s + 2b^2 + \omega_s G'(\omega) - \frac{16b}{\left(\int Q\right)^2}\left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, P Q^3 Q')\right]& \\
= O\left(b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right)& \quad (2\text{-}50)
\end{aligned}$$

holds. To prove this, we take the scalar product of (2-28) with $\mathcal{Q}_\omega$. We keep track of all terms up to b^2.

First, from (2-15), we have

$$\begin{aligned}
(\Psi_{b,\omega}, \mathcal{Q}_\omega) &= -b^2((10 P_\omega^2 \mathcal{Q}_\omega^3)_y + \Lambda P_\omega, \mathcal{Q}_\omega) + O(b^2(|b| + \omega)) \\
&= -b^2((10 P^2 Q^3)_y + \Lambda P, Q) + O(b^2(|b| + \omega)) \\
&= -\tfrac{1}{8} b^2 \|Q\|_{L^1}^2 + O(b^2(|b| + \omega)), \quad (2\text{-}51)
\end{aligned}$$

where for the last step we use the computation

$$\begin{aligned}
(\Lambda P, Q) = -(P, \Lambda Q) = -(P, (LP)') &= (P, (P'' - P + 5Q^4 P)') \\
&= (P, P''' - P') + 10 \int Q^3 Q' P^2,
\end{aligned}$$

and from Proposition 2.2, we obtain

$$((10P^2Q^3)_y+\Lambda P, Q)=\tfrac{1}{2}\lim_{y\to-\infty}P^2=\tfrac{1}{8}\|Q\|_{L^1}^2.$$

Next, from Proposition 2.4, we have

$$\begin{aligned}\left(b_s\frac{\partial Q_{b,\omega}}{\partial b},\mathcal{Q}_\omega\right)&=b_s((\chi_b+\beta y\chi_b')P_\omega,\mathcal{Q}_\omega)=b_s(P_\omega,\mathcal{Q}_\omega)+O(b^{10})\\&=\tfrac{1}{16}b_s\|Q\|_{L^1}^2+F(\omega)b_s+O(b^{10}),\end{aligned}\tag{2-52}$$

where F is the C^1 function introduced in Proposition 2.4. From Lemma 2.3, we have

$$\begin{aligned}(Z_\omega,\mathcal{Q}_\omega)&=-\frac{1}{2}(L_\omega Z_\omega,\Lambda\mathcal{Q}_\omega)+O(\omega)=\frac{1}{2}\int(\Lambda\mathcal{Q}_\omega)\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1}+O(\omega)\\&=\frac{q-1}{4(q+1)}\int|\mathcal{Q}_\omega|^{q+1}+O(\omega)>0.\end{aligned}$$

Then from (2-35), we have

$$\left(\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega},\mathcal{Q}_\omega\right)=\omega_s\frac{1}{2}\frac{\partial\|\mathcal{Q}_\omega\|_{L^2}^2}{\partial\omega}+O(|b\omega_s|)=\omega_s\widetilde{G}'(\omega)+O\left(b^2(\omega+|b|)+\int\varepsilon^2e^{-\frac{|y|}{10}}\right),\tag{2-53}$$

with $\widetilde{G}(\omega)=\frac{1}{2}(\|\mathcal{Q}_\omega\|_{L^2}^2-\|Q\|_{L^2}^2)$. It is easy to check $\widetilde{G}(0)=0$, $\widetilde{G}\in C^1$, and

$$\widetilde{G}'(0)=(Z_\omega,\mathcal{Q}_\omega)|_{\omega=0}=\frac{q-1}{4(q+1)}\int|Q|^{q+1}>0.$$

Next, from Proposition 2.4 we have

$$|(Q_{b,\omega}'+\varepsilon_y,\mathcal{Q}_\omega)|\lesssim\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|(\mathcal{Q}_\omega',\mathcal{Q}_\omega)|+|(P_\omega',\mathcal{Q}_\omega)|+b^{10},$$

which together with (2-34) implies

$$\left|\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}'+\varepsilon_y,\mathcal{Q}_\omega)\right|\lesssim b^2(\omega+|b|)+\int\varepsilon^2e^{-\frac{|y|}{10}}.\tag{2-54}$$

For the small linear term, we have

$$\begin{aligned}\int R_b(\varepsilon)\mathcal{Q}_\omega'&=20b\int P_\omega\mathcal{Q}_\omega^3\mathcal{Q}_\omega'\varepsilon+|b|(\omega+|b|)O\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}\\&=20b\int PQ^3Q'\varepsilon+|b|(\omega+|b|)O\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}.\end{aligned}\tag{2-55}$$

Since the nonlinear term can be estimated with the help of (2-44), we then have

$$\begin{aligned}b_s+\frac{2b^2+\omega_s\widetilde{G}'(\omega)}{1+H(\omega)}-\frac{16}{(1+H(\omega))(\int Q)^2}\left[(\Lambda Q_{b,\omega},\mathcal{Q}_\omega)\left(\frac{\lambda_s}{\lambda}+b\right)+20b(\varepsilon,PQ^3Q')\right]\\=O\left(b^2(\omega+|b|)+\int\varepsilon^2e^{-\frac{|y|}{10}}\right),\end{aligned}$$

where

$$H(\omega)=\frac{16}{\left(\int Q\right)^2}F(\omega).$$

From (2-47) we have

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon,L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}\right|\lesssim(\omega+|b|)\left[\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+|b|\right]+\int\varepsilon^2e^{-\frac{|y|}{10}}.$$

Moreover, we have

$$|(\Lambda Q_{b,\omega},\mathcal{Q}_\omega)-b(\Lambda P,Q)|\lesssim b^{10}+|b(\Lambda P,Q)-b(\Lambda P_\omega,\mathcal{Q}_\omega)|\lesssim|b|(\omega+|b|).$$

We then conclude that

$$b_s+\frac{2b^2+\omega_s\widetilde{G}'(\omega)}{1+H(\omega)}-\frac{16b}{(1+H(\omega))(\int Q)^2}\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon,L(\Lambda Q)')+20(\varepsilon,PQ^3Q')\right]$$
$$=O\left(b^2(\omega+|b|)+\int\varepsilon^2e^{-\frac{|y|}{10}}\right).\quad(2\text{-}56)$$

Finally, since $H\in C^1$, $H(0)=0$, it is enough to check that the function

$$G(\omega)=\int_0^\omega\frac{\widetilde{G}'(x)}{1+H(x)}\,dx$$

satisfies $G\in C^2$, $G(0)=0$, $G'(0)=c_0>0$. Then, (2-56) implies (2-50) immediately.

Now, we turn to the proof of (2-38), (2-40) and (2-42). For all $f\in\mathcal{Y}$, independent of s, $\left(\varepsilon,\int_{-\infty}^y f\right)$ is well-defined due to (2-36). Moreover, we have

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon,\int_{-\infty}^y f\right)&=-(\varepsilon,L_\omega f)+\left(\frac{\lambda_s}{\lambda}+b\right)\left(\Lambda Q_{b,\omega},\int_{-\infty}^y f\right)+\frac{\lambda_s}{\lambda}\left(\Lambda\varepsilon,\int_{-\infty}^y f\right)\\&\quad-\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon,f)-\left(b_s\frac{\partial Q_{b,\omega}}{\partial b}+\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega},\int_{-\infty}^y f\right)\\&\quad+\left(\Psi_{b,\omega},\int_{-\infty}^y f\right)+(R_b(\varepsilon)+R_{\mathrm{NL}}(\varepsilon),f).\end{aligned}$$

Using (2-34), (2-35), (2-44) and Proposition 2.4, we have

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon,\int_{-\infty}^y f\right)&=-(\varepsilon,Lf)+\left(\frac{\lambda_s}{\lambda}+b\right)\left(\Lambda Q,\int_{-\infty}^y f\right)+\left(\frac{x_s}{\lambda}-1\right)(f,Q)\\&\quad-\frac{1}{2}\frac{\lambda_s}{\lambda}\left(\varepsilon,\int_{-\infty}^y f\right)+O\left((|b|+\omega)\left[\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}\right]\right)\\&\quad+O((|b|+\omega)|b|)+O\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right).\end{aligned}\quad(2\text{-}57)$$

Proof of (2-38). We apply (2-57) to $f = \Lambda Q$, using the facts

$$L\Lambda Q = -2Q, \quad \Bigl(\Lambda Q, \int_{-\infty}^{y} \Lambda Q\Bigr) = \frac{1}{8}\Bigl(\int Q\Bigr)^2, \quad \Bigl(Q', \int_{-\infty}^{y} \Lambda Q\Bigr) = 0,$$

to obtain

$$2(J_1)_s = \frac{16(\varepsilon, Q)}{\left(\int Q\right)^2} + \Bigl(\frac{\lambda_s}{\lambda} + b\Bigr) - \frac{\lambda_s}{\lambda} J_1 + O\biggl((|b| + \omega)\Bigl[\Bigl(\int \varepsilon^2 e^{-\frac{|y|}{10}}\Bigr)^{\frac{1}{2}} + |b|\Bigr] + \int \varepsilon^2 e^{-\frac{|y|}{10}}\biggr).$$

Then (2-38) follows immediately from the orthogonality condition (2-22).

Proof of (2-40). We apply (2-57) to $f = \rho_2'$. Then from Lemma 2.1 and Proposition 2.2, we have

$$\begin{aligned}(\Lambda Q, \rho_2) &= \frac{16}{\left(\int Q\right)^2}\Bigl(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}}\Lambda Q + P - \frac{1}{2}\int Q, \Lambda Q\Bigr) - \frac{32}{\left(\int Q\right)^2}\Bigl(\Lambda Q, \int_{-\infty}^{y} \Lambda Q\Bigr) \\ &= \frac{16}{\left(\int Q\right)^2}[(\Lambda P, Q) + (\Lambda Q, P)] + \frac{4\|Q\|_{L^1}^2}{\left(\int Q\right)^2} - \frac{16}{\left(\int Q\right)^2}\Bigl(\int \Lambda Q\Bigr)^2 = 0,\end{aligned}$$

and

$$(\rho', Q) = \frac{16}{\left(\int Q\right)^2}\Bigl(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}}(\Lambda Q)' + P', Q\Bigr) - 8(\rho_1', Q).$$

Next, from

$$\begin{aligned}L(P') &= (LP)' + 20Q'Q^3P \\ &= \Lambda Q + 20Q'Q^3P\end{aligned}$$

and the orthogonality condition $(\varepsilon, \Lambda Q_\omega) = 0$, we have

$$\begin{aligned}(\varepsilon, L\rho_2') &= \frac{16}{\left(\int Q\right)^2}\Bigl(\varepsilon, L\Bigl[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}}(\Lambda Q)' + P'\Bigr]\Bigr) - 8(\varepsilon, L\rho_1') \\ &= \frac{16}{\left(\int Q\right)^2}\Bigl[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q')\Bigr] + O(\omega)\Bigl(\int \varepsilon^2 e^{-\frac{|y|}{10}}\Bigr)^{\frac{1}{2}}.\end{aligned}$$

Substituting all the above estimates into (2-57) with $f = \rho_2'$, we obtain

$$\begin{aligned}(J_2)_s = -\frac{16}{\left(\int Q\right)^2}\Bigl[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q')\Bigr] - \frac{1}{2}\frac{\lambda_s}{\lambda} J_2 \\ + O\biggl((|b| + \omega)\Bigl[\Bigl(\int \varepsilon^2 e^{-\frac{|y|}{10}}\Bigr)^{\frac{1}{2}} + |b|\Bigr] + \int \varepsilon^2 e^{-\frac{|y|}{10}}\biggr). \quad \text{(2-58)}\end{aligned}$$

Then (2-40) follows from (2-50) and (2-58).

Proof of (2-42). From (2-38) and (2-42),

$$\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)=\frac{b_s+2b^2}{\lambda^2}-\frac{2b}{\lambda^2}\left(\frac{\lambda_s}{\lambda}+b\right)$$
$$=-\frac{b}{\lambda^2}\left[(J_2)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right]-\frac{2b}{\lambda^2}\left[2(J_1)_s+\frac{\lambda_s}{\lambda}J_1\right]-\frac{\omega_s G'(\omega)}{\lambda^2}+O\left(\frac{1}{\lambda^2}\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\right)\right)$$
$$=-\frac{b}{\lambda^2}\left[J_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J\right]-\frac{\omega_s G'(\omega)}{\lambda^2}+O\left(\frac{1}{\lambda^2}\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\right)\right),$$

which is exactly (2-42).

Finally, it is easy to check that $\lim_{|y|\to+\infty}\rho(y)=0$, which implies $\rho\in\mathcal{Y}$. □

3. Monotonicity formula

We will introduce the monotonicity tools developed in [Martel and Merle 2002c; Martel, Merle and Raphaël 2014]. This is the key technical argument of the analysis for solutions near the soliton.

3A. ***Pointwise monotonicity.*** Let $(\varphi_i)_{i=1,2}$, $\psi\in C^\infty(\mathbb{R})$ be such that

$$\varphi_i(y)=\begin{cases} e^y & \text{for } y<-1,\\ 1+y & \text{for } -\frac{1}{2}<y<\frac{1}{2},\\ y^i & \text{for } y>2,\end{cases}\qquad \varphi'(y)>0 \text{ for all } y\in\mathbb{R}, \tag{3-1}$$

$$\psi(y)=\begin{cases} e^{2y} & \text{for } y<-1,\\ 1 & \text{for } y>-\frac{1}{2},\end{cases}\qquad \psi'(y)\ge 0 \text{ for all } y\in\mathbb{R}. \tag{3-2}$$

Let $B>100$ be a large universal constant to be chosen later. We then define the weight function

$$\psi_B(y)=\psi\left(\frac{y}{B}\right),\quad \varphi_{i,B}(y)=\varphi\left(\frac{y}{B}\right), \tag{3-3}$$

and the weighted Sobolev norm of ε

$$\mathcal{N}_i(s)=\int\left(\varepsilon_y^2(s,y)\psi_B(y)+\varepsilon^2(s,y)\varphi_{i,B}(y)\right)dy,\quad i=1,2, \tag{3-4}$$

$$\mathcal{N}_{i,\mathrm{loc}}(s)=\int\varepsilon^2(s,y)\varphi'_{i,B}(y)\,dy,\quad i=1,2. \tag{3-5}$$

Then we have the following monotonicity:

Proposition 3.1 (monotonicity formula). *There exist universal constants* $\mu>0$, $B=B(q)>100$ *and* $0<\kappa\ll 1$ *such that the following holds. Let* $u(t)$ *be a solution of* (gKdV$_\gamma$) *satisfying* (2-20) *on* $[0,t_0]$, *and hence the geometrical decomposition* (2-21) *holds on* $[0,t_0]$. *Let* $s_0=s(t_0)$, *and assume the following a priori bounds hold for all* $s\in[0,s_0]$:

(H1) *Scaling-invariant bound:*

$$\omega(s)+|b(s)|+\mathcal{N}_2(s)+\|\varepsilon(s)\|_{L^2}+\omega(s)\|\varepsilon_y(s)\|_{L^2}^m\le\kappa. \tag{3-6}$$

(H2) *Bound related to H^1 scaling:*

$$\frac{\omega(s)+|b(s)|+\mathcal{N}_2(s)}{\lambda^2(s)} \le \kappa. \tag{3-7}$$

(H3) *L^2-weighted bound on the right:*

$$\int_{y>0} y^{10}\varepsilon^2(s,y)\,dy \le 50\Big(1+\frac{1}{\lambda^{10}(s)}\Big). \tag{3-8}$$

We define the Lyapounov functionals for $(i,j)\in\{1,2\}^2$ as

$$\mathcal{F}_{i,j} = \int\Big(\varepsilon_y^2\psi_B + (1+\mathcal{J}_{i,j})\varepsilon^2\varphi_{i,B} - \tfrac{1}{3}\psi_B[(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \\ + \frac{2\omega}{q+1}[|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\psi_B\Big), \tag{3-9}$$

with[7]

$$\mathcal{J}_{i,j} = (1-J_1)^{-4(j-1)-2i} - 1. \tag{3-10}$$

Then the following estimates hold on $[0,s_0]$:

(1) *Scaling-invariant Lyapounov control*: *for $i=1,2$,*

$$\frac{d\mathcal{F}_{i,1}}{ds} + \mu\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B} \lesssim_B b^2(\omega^2+b^2). \tag{3-11}$$

(2) *H^1-scaling Lyapounov control*: *for $i=1,2$,*

$$\frac{d}{ds}\Big(\frac{\mathcal{F}_{i,2}}{\lambda^2}\Big) + \frac{\mu}{\lambda^2}\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B} \lesssim_B \frac{b^2(\omega^2+b^2)}{\lambda^2}. \tag{3-12}$$

(3) *Coercivity and pointwise bounds*: *there hold for all $(i,j)\in\{1,2\}^2$,*

$$\mathcal{N}_i \lesssim \mathcal{F}_{i,j} \lesssim \mathcal{N}_i, \tag{3-13}$$

$$|J_i| + |\mathcal{J}_{i,j}| \lesssim \mathcal{N}_2^{\frac{1}{2}}. \tag{3-14}$$

Remark 3.2. The proof of Proposition 3.1 is almost the same as that of [Martel, Merle and Raphaël 2014, Proposition 3.1]. The only difference here is the additional terms involving ω.

Remark 3.3. Similar to Proposition 2.9, we do not assume any a priori control on the upper bound of $\lambda(s)$ so that the monotonicity formula can be used in all three cases.

Remark 3.4. As mentioned in [Martel, Merle and Raphaël 2014, Proposition 3.1], the weight function ψ decays faster than φ_i on the left. As a result, $\mathcal{N}_2$ and $\mathcal{F}_{i,j}$ do not control $\int \varepsilon_y^2\varphi'_{i,B}$ (see Remark 3.5 of that paper for more details).

[7] Recall that J_1 was defined in (2-37).

Proof of Proposition 3.1. The proofs of (3-13) and (3-14) are exactly the same as [Martel, Merle and Raphaël 2014, Proposition 3.1]. We only need to prove (3-11) and (3-12). To do this, we compute directly to obtain that, for all $(i,j)\in\{1,2\}^2$,

$$\lambda^{2(j-1)}\left(\frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}}\right)_s = f_1+f_2+f_3+f_4+f_5, \tag{3-15}$$

where

$$\begin{aligned}
f_1 &= 2\int\left(\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon\right)(-(\psi_B\varepsilon)_y+\varepsilon\varphi_{i,B}-\psi_B\Delta_{b,\omega}(\varepsilon)),\\
f_2 &= 2\int\left(\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon\right)\varepsilon\mathcal{J}_{i,j}\varphi_{i,B},\\
f_3 &= 2\frac{\lambda_s}{\lambda}\int\Lambda\varepsilon(-(\psi_B\varepsilon_y)_y+(1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B}-\psi_B\Delta_{b,\omega}(\varepsilon))+(\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j},\\
f_4 &= -2\int\psi_B(Q_{b,\omega})_s[\Delta_{b,\omega}-5\varepsilon Q_{b,\omega}^4+q\omega\varepsilon|Q_{b,\omega}|^{q-1}],\\
f_5 &= \frac{2\omega_s}{q+1}\int[|Q_{b,\omega}+\varepsilon|^{q+1}-|Q_{b,\omega}|^{q+1}-(q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\psi_B,\\
\Delta_{b,\omega}(\varepsilon) &= (Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5-\omega(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}+\omega Q_{b,\omega}|Q_{b,\omega}|^{q-1}.
\end{aligned}$$

Our goal is to show that for some $\mu_0>0$,

$$\frac{d}{ds}f_1 \le -\mu_0\int((\varepsilon^2+\varepsilon^2)\varphi_{i,B}'+\varepsilon_{yy}^2\psi_B')+Cb^2(\omega^2+b^2), \tag{3-16}$$

$$\left|\frac{d}{ds}f_k\right| \le \frac{\mu_0}{10}\int((\varepsilon^2+\varepsilon^2)\varphi_{i,B}'+\varepsilon_{yy}^2\psi_B')+Cb^2(\omega^2+b^2)\quad\text{for } k=2,3,4,5. \tag{3-17}$$

The following properties will be used several times in this paper:[8]

$$|\varphi_i'''(y)|+|\varphi_i''(y)|+|\psi'''(y)|+|y\psi'(y)|+|\psi(y)|\lesssim\varphi_i'\lesssim\varphi_i\,\text{for all } y\in\mathbb{R}, \tag{3-18}$$

$$e^{|y|}(\psi(y)+|\psi'(y)|)\lesssim\varphi_i'\sim\varphi_i\quad\text{for all } y\in\left(-\infty,\tfrac{1}{2}\right], \tag{3-19}$$

$$\mathcal{N}_{1,\mathrm{loc}}\lesssim\mathcal{N}_{2,\mathrm{loc}}\lesssim\mathcal{N}_1\lesssim\mathcal{N}_2,\qquad\int\varepsilon^2\varphi_{1,B}\,dy\lesssim\mathcal{N}_{2,\mathrm{loc}}, \tag{3-20}$$

$$\int_{y>0}y^2\varepsilon^2(s)\lesssim\left(1+\frac{1}{\lambda^{\frac{10}{9}}(s)}\right)\mathcal{N}_{2,\mathrm{loc}}^{\frac{8}{9}}(s). \tag{3-21}$$

Control of f_1. First, we rewrite f_1 using the equation of ε in the form

$$\begin{aligned}
\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon &= (-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon))_y\\
&\quad+\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_{b,\omega}+\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon)_y-b_s\frac{\partial Q_{b,\omega}}{\partial b}-\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}+\Psi_{b,\omega},
\end{aligned}\tag{3-22}$$

[8] See [Martel, Merle and Raphaël 2014, Section 3] for more details.

where $-\Psi_{b,\omega} = b\Lambda Q_{b,\omega} + (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})_y$. This yields

$$f_1 = f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5},$$

with

$$f_{1,1} = 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),$$

$$f_{1,2} = 2\Big(\frac{\lambda_s}{\lambda} + b\Big)\int \Lambda Q_{b,\omega}(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),$$

$$f_{1,3} = 2\Big(\frac{x_s}{\lambda} - 1\Big)\int (Q_{b,\omega} + \varepsilon)_y(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),$$

$$f_{1,4} = -2\int \Big(b_s\frac{\partial Q_{b,\omega}}{\partial b} + \omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}\Big)(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),$$

$$f_{1,5} = 2\int \Psi_{b,\omega}(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)).$$

For the term $f_{1,1}$, we integrate by parts to obtain a more manageable formula:

$$f_{1,1} = 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))\psi_B + 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-\psi'_B\varepsilon_y + \varepsilon(\varphi_B - \psi_B)).$$

We compute these terms separately. First, we have

$$\begin{aligned} 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y&(-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))\psi_B \\ &= -\int \psi'_B(-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))^2 \\ &= -\int \psi'_B\big([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2\big) - \int \psi'_B(-\varepsilon_{yy} + \varepsilon)^2 \\ &= -\Big[\int \psi'_B(\varepsilon_{yy}^2 + 2\varepsilon_y^2) + \varepsilon^2(\psi'_B - \psi'''_B)\Big] - \int \psi'_B\big([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2\big). \end{aligned}$$

Next, we integrate by parts to obtain

$$\begin{aligned} -2\int &(\Delta_{b,\omega}(\varepsilon))_y(\varphi_{i,B} - \psi_B)\varepsilon \\ &= -\frac{1}{3}\int (\varphi_{i,B} - \psi_B)'\big([(Q_{b,\omega} + \varepsilon)^6 - Q^6_{b,\omega} - 6\varepsilon Q^5_{b,\omega}] - 6\varepsilon[(Q_{b,\omega} + \varepsilon)^5 - Q^5_{b,\omega}]\big) \\ &\quad - 2\int (\varphi_{i,B} - \psi_B)(Q_{b,\omega})_y[(Q_{b,\omega} + \varepsilon)^5 - Q^5_{b,\omega} - 5\varepsilon Q^4_{b,\omega}] \\ &\quad + \frac{2\omega}{q+1}\int (\varphi_{i,B} - \psi_B)'\big([|Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}] \\ &\qquad - (q+1)\varepsilon[(Q_{b,\omega} + \varepsilon)|Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\big) \\ &\quad + 2\omega\int (\varphi_{i,B} - \psi_B)(Q_{b,\omega})_y\big[(Q_{b,\omega} + \varepsilon)|Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}|^{q-1} - q\varepsilon|Q_{b,\omega}|^{q-1}\big], \end{aligned}$$

and

$$2\int(-\varepsilon_{yy}+\varepsilon)_y(-\psi_B'\varepsilon_y+\varepsilon(\varphi_{i,B}-\psi_B))$$
$$=-2\Big[\int\psi_B'\varepsilon_{yy}^2+\int\varepsilon_y^2\big(\tfrac{3}{2}\varphi_{i,B}'-\tfrac{1}{2}\psi_B'-\tfrac{1}{2}\psi_B'''\big)+\int\varepsilon^2\big(\tfrac{1}{2}(\varphi_B-\psi_B)'-\tfrac{1}{2}(\varphi_B-\psi_B)'''\big)\Big].$$

Finally, by direct expansion, we have

$$\int(\Delta_{b,\omega}(\varepsilon))_y\psi_B'\varepsilon_y=5\int\psi_B'\varepsilon_y\big((Q_{b,\omega})_y[(Q_{b,\omega}+\varepsilon)^4-Q_{b,\omega}^4]+\varepsilon_y(Q_{b,\omega}+\varepsilon)^4\big)$$
$$-q\omega\int\psi_B'\varepsilon_y\big((Q_{b,\omega})_y[|Q_{b,\omega}+\varepsilon|^{q-1}-|Q_{b,\omega}|^{q-1}]+\varepsilon_y|Q_{b,\omega}+\varepsilon|^{q-1}\big).$$

Collecting all the estimates above, we have

$$f_{1,1}=I+II,$$

where

$$\begin{aligned}I=&-\int[3\psi_B'\varepsilon_{yy}^2+(3\varphi_{i,B}'+\psi_B'-\psi_B''')\varepsilon_y^2+(\varphi_{i,B}'-\varphi_{i,B}''')\varepsilon^2]\\&-2\int\big[\tfrac{1}{6}(Q_{b,\omega}+\varepsilon)^6-Q_{b,\omega}^6-6\varepsilon Q_{b,\omega}^5-\varepsilon[(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5]\big](\varphi_{i,B}'-\psi_B')\\&+2\int[(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5-5\varepsilon Q_{b,\omega}^4](Q_{b,\omega})_y(\psi_B-\varphi_{i,B})\\&+10\int\psi_B'\varepsilon_y\big((Q_{b,\omega})_y[(Q_{b,\omega}+\varepsilon)^4-Q_{b,\omega}^4]+\varepsilon_y(Q_{b,\omega}+\varepsilon)^4\big)\\&-\int\psi_B'\big([-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon)]^2-(-\varepsilon_{yy}+\varepsilon)^2\big)\\=&\,I_1+I_2+I_3+I_4+I_5,\end{aligned}$$

and

$$\begin{aligned}II=&\,2\omega\int\Big[\frac{|Q_{b,\omega}+\varepsilon|^{q+1}-|Q_{b,\omega}|^{q+1}-(q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}}{q+1}\\&\qquad-\varepsilon[(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}-Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\Big](\varphi_{i,B}'-\psi_B')\\&-2\omega\int[(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}-Q_{b,\omega}|Q_{b,\omega}|^{q-1}-q\varepsilon|Q_{b,\omega}|^{q-1}](Q_{b,\omega})_y(\psi_B-\varphi_{i,B})\\&-2q\omega\int\psi_B'\varepsilon_y\big((Q_{b,\omega})_y[|Q_{b,\omega}+\varepsilon|^{q-1}-|Q_{b,\omega}|^{q-1}]+\varepsilon_y|Q_{b,\omega}+\varepsilon|^{q-1}\big).\end{aligned}$$

For I_k, $k=1,2,3,4$, we can use the same strategy as in [Martel, Merle and Raphaël 2014, Proposition 3.1] to obtain

$$\sum_{k=1}^{4}I_k\le-\mu_1\int(\varepsilon_{yy}^2\psi_B'+\varepsilon_y^2\varphi_{i,B}'+\varepsilon^2\varphi_{i,B}')+Cb^4 \tag{3-23}$$

for some universal constant $\mu_1>0$.

The idea is to split the integral into three parts. We denote by $I_k^<$, $I_k^\sim$ and $I_k^>$ the integration on $y < -\frac{B}{2}$, $|y| \le \frac{B}{2}$ and $y > \frac{B}{2}$, respectively, for $k = 1, 2, 3, 4$.

On the region $y < -\frac{B}{2}$, using the weighted Sobolev bound introduced in [Merle 2001, Lemma 6; Martel, Merle and Raphaël 2014, Proposition 3.1],

$$\|\varepsilon^2 \sqrt{\varphi'_{i,B}}\|^2_{L^\infty} \lesssim \|\varepsilon\|^2_{L^2} \left(\int \varepsilon_y^2 \varphi'_{i,B} + \int \varepsilon^2 \frac{(\varphi''_{i,B})^2}{\varphi'_B} \right) \lesssim \delta(\kappa) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}, \tag{3-24}$$

we have

$$\begin{aligned} I_2^< + I_3^< + I_4^< &\lesssim_B \int (\varepsilon^6 + \varepsilon^5 Q'_{b,\omega} + \varepsilon_y^2 \varepsilon^4) \varphi'_{i,B} + \|Q_{b,\omega}\|_{L^\infty(y<-\frac{B}{2})} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \\ &\lesssim \delta(\kappa) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}. \end{aligned}$$

Hence we have

$$\sum_{k=1}^4 I_k^< \le -\mu_2 \int_{y<-\frac{B}{2}} (\varepsilon_{yy}^2 \psi'_B + \varepsilon_y^2 \varphi'_{i,B} + \varepsilon^2 \varphi'_{i,B}) \tag{3-25}$$

for some $\mu_2 > 0$.

For the region $|y| \le \frac{B}{2}$, we have

$$\sum_{k=1}^4 I_k^\sim = -\frac{1}{B} \int_{|y|<\frac{B}{2}} (3\varepsilon_y^2 + \varepsilon^2 - 5Q^4 \varepsilon^2 + 20 y Q' Q^3 \varepsilon^2) + O\left(\int_{|y|<\frac{B}{2}} (|b| + \omega) \varepsilon^2 + \varepsilon^6 \right).$$

We then introduce the following coercivity lemma:

Lemma 3.5 [Martel, Merle and Raphaël 2014, Lemma 3.4]. *There exist $B_0 > 100$, $\mu_3 > 0$ such that, for all $\varepsilon \in H^1$ and $B > B_0$, we have*

$$\int_{|y|<\frac{B}{2}} (3\varepsilon_y^2 + \varepsilon^2 - 5Q^4 \varepsilon^2 + 20 y Q' Q^3 \varepsilon^2) \ge \mu_3 \int_{|y|<\frac{B}{2}} \varepsilon_y^2 + \varepsilon^2 - \frac{1}{B} \int \varepsilon^2 e^{-\frac{|y|}{2}}.$$

The above lemma implies immediately that

$$\sum_{k=1}^4 I_k^\sim \le -\mu_2 \int_{|y|<\frac{B}{2}} (\varepsilon_{yy}^2 \psi'_B + \varepsilon_y^2 \varphi'_{i,B} + \varepsilon^2 \varphi'_{i,B}), \tag{3-26}$$

while for the region $y > \frac{B}{2}$, we have $\psi'_B = \psi'''_B \equiv 0$. We also have

$$\|\varepsilon\|^2_{L^\infty(y>\frac{B}{2})} \lesssim \|\varepsilon\|^2_{H^1(y>\frac{B}{2})} \lesssim \mathcal{N}_2 \le \delta(\kappa).$$

Hence, we have

$$\sum_{k=2}^4 I_k^> \lesssim [\|Q_{b,\omega}\|_{L^\infty(y>\frac{B}{2})} + \|\varepsilon\|_{L^\infty(y>\frac{B}{2})}] \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B},$$

which implies

$$\sum_{k=1}^4 I_k^> \le -\mu_2 \int_{y>\frac{B}{2}} (+\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}. \tag{3-27}$$

Combining (3-25)–(3-27), we obtain (3-23).

Now we turn to the estimate of I_5. We have

$$|I_5| \lesssim \int \psi_B'(|\varepsilon_{yy}| + |\varepsilon| + |\varepsilon|^5 + \omega|\varepsilon|^q)(|\varepsilon|^5 + \omega|\varepsilon|^q + |Q_{b,\omega}\varepsilon|)$$
$$\leq \frac{\mu_1}{100}\int (\varepsilon_{yy}^2 + \varepsilon^2)\psi_B' + C(\mu_1)\biggl(\int Q_{b,\omega}^2\varepsilon^2\psi_B' + \int \varepsilon^{10}\psi_B' + \omega^2\int |\varepsilon|^{2q}\psi_B'\biggr). \tag{3-28}$$

Combining (3-24) and the hypothesis (H1), we have

$$\int Q_{b,\omega}^2\varepsilon^2\psi_B' \lesssim \|Q_{b,\omega}\|_{L^\infty(y<-\frac{B}{2})}^2 \int \varepsilon^2\varphi_{i,B}' \leq \frac{\mu_1}{500}\int (\varepsilon_{yy}^2 + \varepsilon^2)\psi_B', \tag{3-29}$$

$$\int \varepsilon^{10}\psi_B' \lesssim \|\varepsilon^2(\psi_B')^{\frac14}\|_{L^\infty}^4 \int \varepsilon^2 \lesssim \delta(\kappa)\int (\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}' + \biggl(\int \varepsilon_y^2(\psi_B')^{\frac12}\biggr)^2, \tag{3-30}$$

and

$$\omega^2\int |\varepsilon|^{2q}\psi_B' \lesssim \omega^2\|\varepsilon^2(\psi_B')^{\frac14}\|_{L^\infty}^4 \int |\varepsilon|^{2q-8} \lesssim \omega^2\|\varepsilon^2(\psi_B')^{\frac14}\|_{L^\infty}^4 \|\varepsilon\|_{L^2}^{q-3}\|\varepsilon_y\|_{L^2}^{q-5}$$
$$\lesssim \delta(\kappa)\int (\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}' + \biggl(\int \varepsilon_y^2(\psi_B')^{\frac12}\biggr)^2,$$

where we use the fact that $\omega\|\varepsilon_y\|_{L^2}^m \leq \kappa$ for the last inequality.

From $((\psi')^{\frac12})'' \lesssim \varphi_i'$ and (H1), we have

$$\biggl(\int \varepsilon_y^2(\psi_B')^{\frac12}\biggr)^2 = \biggl(-\int \varepsilon\varepsilon_{yy}(\psi_B')^{\frac12} + \frac12\int \varepsilon^2((\psi_B')^{\frac12})''\biggr)^2$$
$$\lesssim \int \varepsilon^2 \int \varepsilon_{yy}^2\psi_B' + \biggl(\int \varepsilon^2\varphi_{i,B}'\biggr)^2$$
$$\lesssim \delta(\kappa)\int (\varepsilon_{yy}^2\psi_B' + \varepsilon^2\varphi_{i,B}'). \tag{3-31}$$

Substituting (3-29)–(3-31) into (3-28), we have

$$|I_5| \lesssim \frac{\mu_1}{50}\biggl(\int \varepsilon_{yy}^2\psi_B' + \int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'\biggr). \tag{3-32}$$

Now, we turn to the estimate of II. We write II in the form

$$II = II^< + II^>,$$

where $II^<$ and $II^>$ correspond to the integration on $y < -\frac{B}{2}$ and $y > -\frac{B}{2}$ respectively.

For $II^<$, using the fact that $\psi_B' \sim (\varphi_{i,B}')^2$ for $y < -\frac{B}{2}$, we have

$$|II^<| \lesssim \omega\biggl(\int_{y<-\frac{B}{2}} (|\varepsilon|^{q+1} + |Q_{b,\omega}|^{q-1}\varepsilon^2)\varphi_{i,B}' + \int_{y<-\frac{B}{2}} |Q_{b,\omega}'|(|\varepsilon|^q + \varepsilon^2)\varphi_{i,B}\biggr)$$
$$+\omega\int_{y<-\frac{B}{2}} \psi_B'|\varepsilon_y|\bigl(|\varepsilon|^{q-1} + |Q_{b,\omega}|^{q-2}|\varepsilon| + |\varepsilon_y||\varepsilon|^{q-1} + |\varepsilon_y||Q_{b,\omega}|^{q-1}\bigr)$$
$$\leq C(\mu_1)\omega\biggl(\int \varphi_{i,B}'(|\varepsilon|^{q+1} + |\varepsilon|^q) + \int \psi_B'(\varepsilon_y^2|\varepsilon|^{q-1} + |\varepsilon_y||\varepsilon|^{q-1})\biggr) + \frac{\mu_1}{500}\int_{y<-\frac{B}{2}} (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'.$$

We use (H1)–(H3) and the Gagliardo–Nirenberg inequality to estimate these terms separately. First, we have

$$\begin{aligned}\omega\int |\varepsilon|^{q+1}\varphi'_{i,B} &\lesssim \omega\|\varepsilon^2(\varphi'_{i,B})^{\frac{1}{2}}\|^2_{L^\infty}\int|\varepsilon|^{q-3}\\ &\lesssim \omega\Big(\int\varepsilon^2\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\Big)(\|\varepsilon_y\|_{L^2}^{\frac{q-5}{2}}\|\varepsilon\|_{L^2}^{\frac{q-1}{2}})\\ &\lesssim (\omega\|\varepsilon_y\|^m_{L^2})\Big(\int\varepsilon^2\Big)^{\frac{q+3}{4}}\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B},\end{aligned}$$

and

$$\begin{aligned}\omega\int|\varepsilon|^q\varphi'_{i,B}&\lesssim\omega\big\|\varepsilon^2(\varphi'_{i,B})^{\frac{1}{2}}\big\|^{\frac{3}{2}}_{L^\infty}\big\||\varepsilon|^{\frac{1}{2}}(\varphi'_{i,B})^{\frac{1}{4}}\big\|_{L^4}\big\||\varepsilon|^{\frac{2q-7}{2}}\big\|_{L^{4/3}}\\ &\lesssim\omega\Big(\int\varepsilon^2\Big)^{\frac{3}{4}}\Big(\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\Big)\|\varepsilon\|_{L^2}^{\frac{q-2}{2}}\|\varepsilon_y\|_{L^2}^{\frac{q-5}{2}}\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}.\end{aligned}$$

From $\psi'\lesssim(\varphi_i')^2$ and (3-31), we also have

$$\begin{aligned}\omega\int\psi'_B\varepsilon_y^2|\varepsilon|^{q-1}&\lesssim\omega\|\varepsilon^2(\psi'_B)^{\frac{1}{4}}\|^2_{L^\infty}\|\varepsilon\|^{q-5}_{L^\infty}\int\varepsilon_y^2(\psi'_B)^{\frac{1}{2}}\\ &\lesssim(\omega\|\varepsilon_y\|^m_{L^2})\|\varepsilon\|^{m+2}_{L^2}\Big(\int(\varepsilon^2+\varepsilon_y^2)(\psi'_B)^{\frac{1}{2}}\Big)\int\varepsilon_y^2(\psi'_B)^{\frac{1}{2}}\\ &\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+\Big(\int\varepsilon_y^2(\psi'_B)^{\frac{1}{2}}\Big)^2\\ &\lesssim\delta(\kappa)\Big(\int\varepsilon_{yy}^2\psi'_B+\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\Big),\end{aligned}$$

and

$$\begin{aligned}\omega\int\psi'_B|\varepsilon_y||\varepsilon|^{q-1}&\lesssim\omega\|\varepsilon^2(\psi'_B)^{\frac{1}{4}}\|^{\frac{3}{2}}_{L^\infty}\|\varepsilon_y(\psi'_B)^{\frac{1}{4}}\|_{L^2}\||\varepsilon|^{q-4}\|_{L^2}\\ &\lesssim(\omega\|\varepsilon_y\|^m_{L^2})\|\varepsilon\|^{m+2}_{L^2}\Big(\int(\varepsilon^2+\varepsilon_y^2)(\psi'_B)^{\frac{1}{2}}\Big)\Big(\int\varepsilon_y^2(\psi'_B)^{\frac{1}{2}}\Big)^{\frac{1}{2}}\\ &\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+\Big(\int\varepsilon_y^2(\psi'_B)^{\frac{1}{2}}\Big)^{\frac{3}{2}}\\ &\le\frac{\mu_1}{1000}\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+C(\mu_1)\Big(\int\varepsilon_y^2(\psi'_B)^{\frac{1}{2}}\Big)^2\\ &\le\frac{\mu_1}{500}\Big(\int\varepsilon_{yy}^2\psi'_B+\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\Big).\end{aligned}$$

In conclusion, we have

$$|II^<|\le\frac{\mu_1}{50}\Big(\int\varepsilon_{yy}^2\psi'_B+\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\Big).\tag{3-33}$$

For $II^{>}$, we know that $\psi_B' \equiv 0$ for $y > -\frac{B}{2}$. Using Sobolev embedding, we have

$$\|\varepsilon\|^2_{L^\infty(y>-\frac{B}{2})} \lesssim \|\varepsilon\|_{L^2(y>-\frac{B}{2})}\|\varepsilon_y\|_{L^2(y>-\frac{B}{2})} \le \mathcal{N}_2 \le 1.$$

Thus, we have

$$|II^{>}| \lesssim \omega\left(\int \varepsilon^2\varphi_{i,B}' + \int (Q_{b,\omega})_y\varepsilon^2\varphi_{i,B}\right) \lesssim_B \delta(\kappa)\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}'. \tag{3-34}$$

Combining (3-23), (3-32), (3-33) and (3-34), we have

$$f_{1,1} \le -\mu_0\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}' + Cb^4 \tag{3-35}$$

for some universal constant $\mu_0 > 0$.

Now, let us deal with $f_{1,2}$. It is easy to see that

$$f_{1,2} = \tilde{I} + \widetilde{II},$$

where

$$\tilde{I} = 2\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda Q_{b,\omega}\big(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B[(Q_{b,\omega}+\varepsilon)^5 - Q_{b,\omega}^5]\big),$$
$$\widetilde{II} = 2\omega\left(\frac{\lambda_s}{\lambda}+b\right)\int \Lambda Q_{b,\omega}\psi_B((Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}^{q-1}|).$$

The term $\tilde{I}$ can be estimated by the same argument as in [Martel, Merle and Raphaël 2014, Propostion 3.1]. Thus, we have

$$|\tilde{I}| \le \frac{\mu_0}{500}\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}' + Cb^2(\omega^2+b^2).$$

We mention here that the modulation estimate (2-34) in this paper is slightly different from [Martel, Merle and Raphaël 2014, (2.29)]; i.e., there is an additional term "$\omega|b|$" on the right-hand side of (2-34). This additional term results in the appearance of the term "ω^2b^2" on the right-hand side of the above inequality.

While for $\widetilde{II}$, we have

$$|\widetilde{II}| \lesssim \omega\left|\frac{\lambda_s}{\lambda}+b\right|\left(B^{\frac{1}{2}}\mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}} + \int |\varepsilon|^q\psi_B\right).$$

Using (2-34) and the strategy for $f_{1,1}$, we have

$$|\widetilde{II}| \le \frac{\mu_0}{500}\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}' + Cb^2(\omega^2+b^2).$$

A similar argument can be applied to $f_{1,k}$, $k = 3, 4, 5$. Together with (3-35), we conclude the proof of (3-16).

Control of f_2. For f_2, we integrate by parts, using (3-22) to get

$$f_2 = 2\mathcal{J}_{i,j}\int \varepsilon\varphi_{i,B}\Big[(-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon))_y + \Big(\frac{\lambda_s}{\lambda}+b\Big)\Lambda Q_{b,\omega} + \Big(\frac{x_s}{\lambda}-1\Big)(Q_{b,\omega}+\varepsilon)_y - b_s\frac{\partial Q_{b,\omega}}{\partial b} - \omega_s\frac{\partial Q_{b,\omega}}{\partial \omega} + \Psi_{b,\omega}\Big].$$

We integrate by parts, estimating all terms like we did for f_1. Together with

$$|\mathcal{J}_{i,j}| \lesssim |J_1| \lesssim \mathcal{N}_2^{\frac{1}{2}} \lesssim \delta(\kappa),$$

we have

$$|f_2| \lesssim \delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B} + b^2(\omega^2+b^2). \tag{3-36}$$

Control of f_3. Recall that

$$f_3 = 2\frac{\lambda_s}{\lambda}\int \Lambda\varepsilon\big(-(\psi_B\varepsilon_y)_y + (1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)\big) + (\mathcal{J}_{i,j})_s\int \varphi_{i,B}\varepsilon^2 - 2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j}.$$

Integrating by parts,[9] we have

$$f_3 = \hat{I} + \widehat{II},$$

where

$$\begin{aligned}\hat{I} &= \frac{\lambda_s}{\lambda}\int[(2-2(j-1))\psi_B - y\psi'_B]\varepsilon_y^2\\ &\quad - \frac{1}{3}\frac{\lambda_s}{\lambda}\int[(2-2(j-1))\psi_B - y\psi'_B][(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5]\\ &\quad + 2\frac{\lambda_s}{\lambda}\int \psi_B\Lambda Q_b[(Q_{b,\omega}+\varepsilon)^5 - Q_{b,\omega}^5 - 5\varepsilon Q_{b,\omega}^4]\\ &\quad + (\mathcal{J}_{i,j})_s\int \varepsilon^2\varphi_{i,B} - \frac{\lambda_s}{\lambda}(1+\mathcal{J}_{i,j})\int y\varphi'_{i,B}\varepsilon^2 - 2(j-1)\frac{\lambda_s}{\lambda}(1+\mathcal{J}_{i,j})\int \varepsilon^2\varphi_{i,B},\end{aligned}$$

and

$$\begin{aligned}\widehat{II} &= \frac{2\omega}{q+1}\frac{\lambda_s}{\lambda}\int\Big[\Big(\frac{q+3}{q-1} - 2(j-1)\Big)\psi_B - y\psi'_B\Big]\times\big[|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}\big]\\ &\quad - 2\omega\frac{\lambda_s}{\lambda}\int \psi_B\Lambda Q_b\big[(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}|^{q-1} - q\varepsilon|Q_{b,\omega}|^{q-1}\big].\end{aligned}$$

Similarly, we can use the same strategy as in [Martel, Merle and Raphaël 2014, Proposition 3.1] to estimate I, which leads to

$$|\hat{I}| \lesssim \delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B} + b^2(\omega^2+b^2). \tag{3-37}$$

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[9] See [Martel, Merle and Raphaël 2014, Proposition 3.1, Step 5] and [Lan 2016, (5.22)] for more details.

More precisely, we can rewrite $\hat{I}$ as

$$\begin{aligned}\hat{I} = \frac{\lambda_s}{\lambda}\int [2(2-j)\psi_B - y\psi_B']\varepsilon_y^2 \\ -\frac{1}{3}\frac{\lambda_s}{\lambda}\int [2(2-j)\psi_B - y\psi_B'] \times [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \\ +2\frac{\lambda_s}{\lambda}\int \psi_B \Lambda Q_b[(Q_{b,\omega}+\varepsilon)^5 - Q_{b,\omega}^5 - 5\varepsilon Q_{b,\omega}^4] \\ +\frac{1}{i}\left[(\mathcal{J}_{i,j})_s - 2(j-1)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right]\int (i\varphi_{i,B} - y\varphi_{i,B}')\varepsilon^2 \\ +\frac{1}{i}\left[(\mathcal{J}_{i,j})_s - (2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right]\int y\varphi_{i,B}'\varepsilon^2 \\ = \hat{I}_1 + \hat{I}_2,\end{aligned}$$

where

$$\hat{I}_2 = \frac{1}{i}\left[(\mathcal{J}_{i,j})_s - (2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right]\int y\varphi_{i,B}'\varepsilon^2.$$

We also denote by $\hat{I}_k^<$, $\hat{I}_k^\sim$ and $\hat{I}_k^>$, $k=1,2$, the integration over $y < -\frac{B}{2}$, $|y| < \frac{B}{2}$ and $y > \frac{B}{2}$ respectively.

For integration over $|y| < \frac{B}{2}$, the estimate is straightforward, we have

$$|\hat{I}_1^\sim| + |\hat{I}_2^\sim| \lesssim \delta(\kappa)\int_{|y|<\frac{B}{2}} (\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.$$

While for $y < -\frac{B}{2}$, using (2-34), we have

$$\begin{aligned}|\hat{I}_1^<| + |\hat{I}_2^<| &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\int_{y<-\frac{B}{2}} (\psi_B + |y|\varphi_{i,B}' + \varphi_{i,B})(\varepsilon_y^2+\varepsilon^2) + |y|\varphi_{i,B}'\varepsilon^2 \\ &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\left[\int_{y<-\frac{B}{2}} \varepsilon_y^2\varphi_{i,B}' + \int_{y<-\frac{B}{2}} |y|\varphi_{i,B}'\varepsilon^2\right] \\ &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\left[\int \varepsilon_y^2\varphi_{i,B}' + \left(\int_{y<-\frac{B}{2}} y^{100}e^{\frac{y}{B}}\varepsilon^2\right)^{\frac{1}{100}}\left(\int_{y<-\frac{B}{2}} e^{\frac{y}{B}}\varepsilon^2\right)^{\frac{99}{100}}\right] \\ &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}}) \times \left(\int \varepsilon_y^2\varphi_{i,B}' + \mathcal{N}_{i,\mathrm{loc}}^{\frac{99}{100}}\right) \le \delta(\kappa)\int (\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}' + Cb^4.\end{aligned}$$

Now, for $y > \frac{B}{2}$, we first have

$$i\varphi_{i,B} - y\varphi_{i,B}' = 0$$

for all $y > B$. Hence

$$|\hat{I}_1^>| \lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\int (\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}' \lesssim \delta(\kappa)\int (\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'.$$

Next, for $\hat{I}_2^>$, we know from (2-38) that

$$\left|(\mathcal{J}_{i,j})_s - (2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right| = \frac{4(j-1)+2i}{(1-J_1)^{4(j-1)+2i+1}}\left|(J_1)_s - \frac{\lambda_s}{2\lambda}(1-J_1)\right| \lesssim |b| + \mathcal{N}_{i,\mathrm{loc}}.$$

Together with (3-7), (3-8) and (3-21), we have

$$\begin{aligned}|\hat{I}_2^{>}| &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\left(1 + \frac{1}{\lambda^{\frac{10}{9}}}\right)\mathcal{N}_{i,\mathrm{loc}}^{\frac{8}{9}} \\ &\lesssim |b|(1 + \delta(\kappa)|b|^{-\frac{5}{9}})\mathcal{N}_{i,\mathrm{loc}}^{\frac{8}{9}} + \mathcal{N}_{i,\mathrm{loc}}(1 + \delta(\kappa)\mathcal{N}_{i,\mathrm{loc}}^{-\frac{5}{9}})\mathcal{N}_{i,\mathrm{loc}}^{\frac{8}{9}} \\ &\lesssim \delta(\kappa)\int (\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}' + b^2(\omega^2 + b^2).\end{aligned}$$

Combining the above estimates, we obtain (3-37).

Finally, for $\widehat{II}$, from the fact

$$\psi_B + \left|\left(\frac{q+3}{q-1} - 2(j-1)\right)\psi_B - y\psi_B'\right| \lesssim_B \varphi_{i,B}',$$

we have

$$|\widehat{II}| \lesssim \omega\left|\frac{\lambda_s}{\lambda}\right|\int (|\varepsilon|^{q+1} + |\varepsilon|^q + \varepsilon^2)\varphi_{i,B}'.$$

Using $|\lambda_s/\lambda| \lesssim \delta(\kappa)$ and the strategy for $f_{1,1}$, we have

$$|\widehat{II}| \lesssim \delta(\kappa)\int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'.$$

In conclusion, we have

$$|f_3| \lesssim \delta(\kappa)\int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}' + b^2(\omega^2 + b^2). \tag{3-38}$$

Control of f_4. From (2-5) and (3-12), we have

$$|(Q_{b,\omega})_s| \lesssim |b_s|\left|\frac{\partial Q_{b,\omega}}{\partial b}\right| + |\omega_s|\left|\frac{\partial Q_{b,\omega}}{\partial \omega}\right| \le (\omega + |b|)(|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}}) \lesssim \delta(\kappa).$$

Using the Sobolev bounds (3-24) and the strategy for $f_{1,1}$, we have

$$|f_4| \lesssim \delta(\kappa)\left(\int (\omega|\varepsilon|^q + |\varepsilon|^5 + \varepsilon^2)\varphi_{i,B}'\right) \lesssim \delta(\kappa)\left(\int \varepsilon_{yy}^2\psi_B' + \int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'\right). \tag{3-39}$$

Control of f_5. From (2-34) we know that

$$|\omega_s| = m\omega\left|\frac{\lambda_s}{\lambda}\right| \lesssim \delta(\kappa).$$

Thus, by the Sobolev bounds (3-24) and the strategy for $f_{1,1}$, we have

$$|f_5| \lesssim \delta(\kappa)\int (\omega|\varepsilon|^{q+1} + \varepsilon^2)\varphi_{i,B}' \lesssim \delta(\kappa)\left(\int \varepsilon_{yy}^2\psi_B' + \int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'\right). \tag{3-40}$$

Combining (3-36), (3-38), (3-39) and (3-40), we conclude the proof of (3-17), and hence the proof of Proposition 3.1. □

3B. ***Dynamical control of the tail on the right.*** In order to close the bootstrap bound (H3), we need the dynamical control of the L^2 tail on the right introduced in [Martel, Merle and Raphaël 2014]. More precisely, we choose a smooth function

$$\varphi_{10}(y)=\begin{cases}0 & \text{for } y<0,\\ y^{10} & \text{for } y>1,\end{cases}\qquad \varphi_{10}'\geq 0.$$

Then we have:

Lemma 3.6 (dynamical control of the tail on the right [Martel, Merle and Raphaël 2014]). *Under the assumption of Proposition 3.1, it holds that*

$$\frac{1}{\lambda^{10}}\frac{d}{ds}\left(\lambda^{10}\int\varphi_{10}\varepsilon^2\right)\lesssim_B \mathcal{N}_{1,\mathrm{loc}}+b^2. \tag{3-41}$$

Proof. The proof of Lemma 3.6 is exactly the same as [Martel, Merle and Raphaël 2014, Lemma 3.7].

More precisely, from (3-22), we have

$$\frac{1}{2}\frac{d}{ds}\int\varphi_{10}\varepsilon^2=\int\varphi_{10}\varepsilon\biggl[\frac{\lambda_s}{\lambda}\Lambda\varepsilon+(-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon))_y+\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_{b,\omega}\\ +\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon)_y-b_s\frac{\partial Q_{b,\omega}}{\partial b}-\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}+\Psi_{b,\omega}\biggr],$$

where

$$\Delta_{b,\omega}(\varepsilon)=(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5-\omega(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}+\omega Q_{b,\omega}|Q_{b,\omega}|^{q-1}.$$

We integrate the linear term by parts using the fact that $y\varphi_{10}'=10\varphi_{10}$ for $y\geq 1$, and $\varphi_{10}'''\ll\varphi_{10}'$ for y large, to obtain

$$\int\varphi_{10}\varepsilon\left[\frac{\lambda_s}{\lambda}\Lambda\varepsilon+(-\varepsilon_{yy}+)_y\right]=-\frac{1}{2}\frac{\lambda_s}{\lambda}\int y\varphi_{10}'\varepsilon^2-\frac{3}{2}\int\varepsilon_y^2\varphi_{10}'-\frac{1}{2}\int\varepsilon^2\varphi_{10}'+\frac{1}{2}\int\varepsilon^2\varphi_{10}'''\\ \leq -5\frac{\lambda_s}{\lambda}\int\varphi_{10}\varepsilon^2-\frac{3}{2}\int\varepsilon_y^2\varphi_{10}'-\frac{1}{2}\int\varepsilon^2\varphi_{10}'+C\mathcal{N}_{1,\mathrm{loc}}.$$

Next, from (2-15), (2-34), and (2-35), it is easy to obtain

$$\left|\int\varphi_{10}\varepsilon\left[\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_{b,\omega}+\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon)_y-b_s\frac{\partial Q_{b,\omega}}{\partial b}-\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}+\Psi_{b,\omega}\right]\right|\lesssim b^2+\mathcal{N}_{1,\mathrm{loc}}.$$

While for the nonlinear term, we integrate by parts to remove all derivatives on ε to obtain

$$\left|\int\varphi_{10}\varepsilon[\Delta_{b,\omega}(\varepsilon)]_y\right|\lesssim\int\varphi_{10}\varepsilon^2e^{-\frac{|y|}{2}}(\|\varepsilon\|_{L^\infty(y>0)})+\int\varepsilon^6\varphi_{10}'+\omega\int|\varepsilon|^{q+1}\varphi_{10}'\lesssim\delta(\kappa)\int\varepsilon^2\varphi_{10}',$$

where we use the fact that $|Q_{b,\omega}|+|Q_{b,\omega}'|\lesssim e^{-|y|/2}$ for $y>0$ and

$$\|\varepsilon\|_{L^\infty(y>0)}\lesssim\mathcal{N}_1\ll 1.$$

Hence, we have

$$\frac{d}{ds}\int\varphi_{10}\varepsilon^2+10\frac{\lambda_s}{\lambda}\int\varphi_{10}\varepsilon^2\lesssim b^2+\mathcal{N}_{1,\mathrm{loc}},$$

which, together with Gronwall's inequality, implies (3-41) immediately. □

4. Rigidity of the dynamics in $\mathcal{A}_{\alpha_0}$ and proof of Theorem 1.3

We will classify the behavior of any solution with initial data in $\mathcal{A}_{\alpha_0}$, which directly implies Theorem 1.3. To begin, we define

$$t^* = \sup\{0 < t < +\infty \mid \text{for all } t' \in [0, t],\ u(t') \in \mathcal{T}_{\alpha^*,\gamma}\}. \tag{4-1}$$

Assume $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll 1$; then the condition on the initial data, i.e., $u_0 \in \mathcal{A}_{\alpha_0}$, implies $t^* > 0$.

Next, by Lemma 2.6, $u(t)$ admits the following geometrical decomposition on $[0, t^*]$:

$$u(t, x) = \frac{1}{\lambda^{1/2}(t)}[Q_{b(t),\omega(t)} + \varepsilon(t)]\left(\frac{x - x(t)}{\lambda(t)}\right).$$

The condition $u_0 \in \mathcal{A}_{\alpha_0}$ implies

$$\omega(0) + \|\varepsilon(0)\|_{H^1} + \omega(0)\|\varepsilon_y(0)\|^m_{L^2} + |b(0)| + |1 - \lambda(0)| \lesssim \delta(\alpha_0), \tag{4-2}$$

$$\int_{y>0} y^{10}\varepsilon^2(0)\, dy \le 2. \tag{4-3}$$

Using Hölder's inequality, we have

$$\mathcal{N}_2(0) \lesssim \delta(\alpha_0). \tag{4-4}$$

Then let us fix a $0 < \kappa \ll 1$ as in Propositions 2.9 and 3.1, and define

$$t^{**} = \sup\{0 < t < t^* \mid \text{(H1), (H2) and (H3) hold for all } t' \in [0, t]\}. \tag{4-5}$$

Note that from (4-2)–(4-4), we have $t^{**} > 0$. Let $s^* = s(t^*)$ and $s^{**} = s(t^{**})$.

4A. ***Consequence of the monotonicity formula.*** We derive some crucial estimates from the monotonicity formula introduced in Section 3.

Lemma 4.1. *We have the following*:

(1) *Almost monotonicity of the localized Sobolev norm*: *There exists a universal constant $K_0 > 1$ such that*, *for $i = 1, 2$ and $0 \le s_1 < s_2 \le s^{**}$*,

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2}\int (\varepsilon_y^2(s, y) + \varepsilon^2(s, y))\varphi'_{i,B}(y)\, dy\, ds \le K_0\Big[\mathcal{N}_i(s_1) + \sup_{s\in[s_1,s_2]} |b(s)|^3 + \sup_{s\in[s_1,s_2]} \omega^3(s)\Big], \tag{4-6}$$

$$\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \frac{1}{\lambda^2(s)}\left[\left(\int (\varepsilon_y^2 + \varepsilon^2)(s)\varphi'_{i,B}\right) + b^2(s)(|b(s)| + \omega(s))\right] ds \le K_0\left(\frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \left[\frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)}\right]\right). \tag{4-7}$$

(2) *Control of b and ω*: *For all $0 \le s_1 < s_2 \le s^{**}$*,

$$\omega(s_2) + \int_{s_1}^{s_2} b^2(s)\, ds \lesssim \mathcal{N}_1(s_1) + \omega(s_1) + \sup_{s\in[s_1,s_2]} |b(s)|. \tag{4-8}$$

(3) *Control of* b/λ^2: *Let* $c_1 = (m/(m+2))G'(0) > 0$, *where* G *is the* C^2 *function introduced in* (2-40). *Then there exists a universal constant* $K_1 > 1$ *such that, for all* $0 \le s_1 < s_2 \le s^{**}$,

$$\left|\frac{b(s_2)+c_1\omega(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)+c_1\omega(s_1)}{\lambda^2(s_1)}\right| \le K_1\left(\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right). \quad (4\text{-}9)$$

(4) *Refined control of* λ: *Let* $\lambda_0(s) = \lambda(s)(1-J_1(s))^2$. *Then there exists a universal constant* $K_2 > 1$ *such that, for all* $s \in [0, s^{**}]$,

$$\left|\frac{(\lambda_0)_s}{\lambda_0} + b\right| \le K_2[\mathcal{N}_1 + (|b|+\omega)(\mathcal{N}_2^{\frac{1}{2}} + |b|)]. \quad (4\text{-}10)$$

Proof. Proof of (4-6) *and* (4-8). From (2-50), we have

$$\frac{d}{ds}G(\omega) + b^2 \le -b_s + C\mathcal{N}_{1,\mathrm{loc}}.$$

Integrating from s_1 to s_2, we have

$$G(\omega(s_2)) + \int_{s_1}^{s_2} b^2 \lesssim \int_{s_1}^{s_2} \mathcal{N}_{1,\mathrm{loc}} + G(\omega(s_1)) + |b(s_2)-b(s_1)| \lesssim \int_{s_1}^{s_2} \mathcal{N}_{1,\mathrm{loc}} + G(\omega(s_1)) + \sup_{s\in[s_1,s_2]} |b(s)|.$$

Since $G(\omega) \sim \omega$, we obtain (4-8).

Next, from the monotonicity formulas (3-11) and (3-13) we obtain

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2}\int (\varepsilon_y^2(s,y) + \varepsilon^2(s,y))\varphi'_{i,B}(y)\,dy\,ds \lesssim \mathcal{N}_i(s_1) + \Big[\sup_{s\in[s_1,s_2]} b^2(s) + \sup_{s\in[s_1,s_2]} \omega^2(s)\Big]\int_{s_1}^{s_2} b^2. \quad (4\text{-}11)$$

Combining (4-8) and (4-11), we obtain (4-6).

Proof of (4-7). First, from (2-50) and (2-35), we have

$$\begin{aligned} 2\int_{s_1}^{s_2} \frac{|b|^3}{\lambda^2} &\le \int_{s_1}^{s_2}\Big[-\frac{|b|b_s - \omega_s G'(\omega)|b| + C\mathcal{N}_{1,\mathrm{loc}} + \delta(\kappa)|b|^3}{\lambda^2}\Big] \\ &\le -\frac{1}{2}\frac{b|b|}{\lambda^2}\Big|_{s_1}^{s_2} + O\Big(\int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\mathrm{loc}} + \omega b^2}{\lambda^2}\Big) + \delta(\kappa)\int_{s_1}^{s_2} \frac{|b|^3}{\lambda^2}. \end{aligned} \quad (4\text{-}12)$$

Recall that $\omega = \gamma/\lambda^m$. Then from (2-34) we have

$$\begin{aligned} \int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} &= -\int_{s_1}^{s_2} \frac{\lambda_s}{\lambda}\frac{\omega b}{\lambda^2} + \int_{s_1}^{s_2} \frac{\omega b}{\lambda^2}\Big(\frac{\lambda_s}{\lambda} + b\Big) \\ &\le \frac{1}{m+2}\int_{s_1}^{s_2} \Big(\frac{\omega}{\lambda^2}\Big)_s b + \delta(\kappa)\int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} + O\Big(\int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\Big) \\ &= \frac{1}{m+2}\int_{s_1}^{s_2} \frac{\omega}{\lambda^2}(-b_s) + \delta(\kappa)\int_{s_1}^{s_2} \frac{\omega b^2}{\lambda^2} \\ &\qquad + O\Big(\int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} + \frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\Big). \end{aligned} \quad (4\text{-}13)$$

From (2-50) and (2-35), we have

$$\int_{s_1}^{s_2}\frac{\omega}{\lambda^2}(-b_s)\le\int_{s_1}^{s_2}\frac{\omega}{\lambda^2}\bigg[\Big(2+\frac{m}{10}\Big)b^2+\omega_s G'(\omega)+C(m)\mathcal{N}_{1,\mathrm{loc}}\bigg]$$
$$\le\Big(2+\frac{m}{10}\Big)\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2}+\int_{s_1}^{s_2}\frac{\omega_s\omega G'(\omega)}{(\gamma/\omega)^{2/m}}$$
$$+O\bigg(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}+\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\bigg). \tag{4-14}$$

From (2-34) and (2-35) again, we have

$$\bigg|\int_{s_1}^{s_2}\frac{\omega_s\omega G'(\omega)}{(\gamma/\omega)^{2/m}}\bigg|=\frac{|M(\omega(s_2))-M(\omega(s_1))|}{\gamma^{2/m}}\lesssim\frac{\omega^2(s_1)}{\lambda^2(s_1)}+\frac{\omega^2(s_2)}{\lambda^2(s_2)}, \tag{4-15}$$

where

$$M(\omega)=\int_0^\omega x^{1+\frac{2}{m}}G'(x)\,dx\sim\omega^{2+\frac{2}{m}}.$$

Therefore, combining (4-13)–(4-15), we have

$$\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2}\le\bigg(\frac{2+\frac{1}{10}m}{m+2}+\delta(\kappa)\bigg)\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2}$$
$$+O\bigg(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}+\bigg[\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\bigg]\bigg). \tag{4-16}$$

Taking $\kappa>0$ small enough, from (4-12) and (4-16) we have

$$\int_{s_1}^{s_2}\frac{b^2(\omega+|b|)}{\lambda^2}\lesssim\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}+\bigg[\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\bigg]. \tag{4-17}$$

Now, integrating the monotonicity formula (3-12), we have

$$\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)}+\int_{s_1}^{s_2}\frac{1}{\lambda^2(s)}\bigg[\bigg(\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\bigg)\bigg]\,ds\lesssim\frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)}+\delta(\kappa)\int_{s_1}^{s_2}\frac{b^2(s)(\omega(s)+|b(s)|)}{\lambda^2(s)}\,ds,$$

which implies (4-7) immediately.

Proof of (4-9). The proof of (4-9) is based on integrating the equation of b/λ^2, i.e., (2-42). More precisely, from (2-34), (2-42) and the fact that $|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$ (recall that J given by (2-41) is a well-localized L^2 scalar product), we have

$$\bigg|\Big(\frac{b}{\lambda^2}e^J\Big)_s+\frac{\omega_s G'(\omega)}{\lambda^2}e^J\bigg|=\bigg|\Big(\frac{b}{\lambda^2}\Big)_s+\frac{b}{\lambda^2}J_s+\frac{\omega_s G'(\omega)}{\lambda^2}\bigg|e^J$$
$$\lesssim\bigg|\frac{\lambda_s}{\lambda}\frac{b}{\lambda^2}J\bigg|+O\bigg(\frac{1}{\lambda^2}\bigg(\int\varepsilon^2e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\bigg)\bigg)$$
$$\lesssim\frac{b^2}{\lambda^2}|J|+O\bigg(\frac{1}{\lambda^2}\bigg(\int\varepsilon^2e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\bigg)\bigg)$$
$$\lesssim O\bigg(\frac{1}{\lambda^2}(\mathcal{N}_{1,\mathrm{loc}}+(\omega+|b|)b^2)\bigg).$$

We integrate this estimate in time using (4-7) to get

$$\left|\left[\frac{b}{\lambda^2}e^J\right]_{s_1}^{s_2}+\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}e^J\right|\lesssim\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\left[\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right]. \tag{4-18}$$

Note that $|e^J-1|\le 2|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$. Together with (4-7), we have

$$\begin{aligned}\left[\frac{b}{\lambda^2}e^J\right]_{s_1}^{s_2}&=\frac{b}{\lambda^2}\bigg|_{s_1}^{s_2}+\left|\left[\frac{b}{\lambda^2}\mathcal{N}_{1,\mathrm{loc}}^{\frac{1}{2}}\right]_{s_1}^{s_2}\right|\\&=\frac{b}{\lambda^2}\bigg|_{s_1}^{s_2}+O\left(\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right).\end{aligned} \tag{4-19}$$

Next, from (2-35), (4-7) and $|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$, we have

$$\begin{aligned}\left|\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}(e^J-1)\right|&\lesssim\int_{s_1}^{s^2}\frac{(b^2+\omega^2)b^2+\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\\&\lesssim\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}.\end{aligned} \tag{4-20}$$

Finally, recall $\omega=\gamma/\lambda^m$, so we have

$$\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}=\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{(\gamma/\omega)^{2/m}}=\frac{\Sigma(\omega)}{\lambda^2}\bigg|_{s_1}^{s_2},$$

where

$$\Sigma(\omega):=\frac{1}{\omega^{2/m}}\int_0^\omega x^{\frac{2}{m}}G'(x)\,dx.$$

Recall that G is the C^2 function introduced in (2-40). We then have $\Sigma\in C^2$ and $c_1=\Sigma'(0)=(m/(m+2))G'(0)>0$. Hence, we have

$$\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}=\frac{c_1\omega}{\lambda^2}\bigg|_{s_1}^{s_2}+O\left(\frac{\omega^2(s_1)}{\lambda^2(s_1)}+\frac{\omega^2(s_2)}{\lambda^2(s_2)}\right). \tag{4-21}$$

Combining (4-18)–(4-21), we conclude the proof of (4-9).

Proof of (4-10). From (3-14), we have

$$\left|\frac{\lambda}{\lambda_0}-1\right|\lesssim|J_1|\lesssim\mathcal{N}_2^{\frac{1}{2}}\lesssim\delta(\kappa);$$

thus we obtain from (2-38)

$$\begin{aligned}\left|\frac{(\lambda_0)_s}{\lambda_0}+b\right|&=\left|\frac{1}{1-J_1}\left[(1-J_1)\frac{\lambda_s}{\lambda}+b-2(J_1)_s\right]-\frac{J_1}{1-J_1}b\right|\\&\lesssim\int\varepsilon^2e^{-\frac{|y|}{10}}+(|b|+\omega)(\mathcal{N}_2^{\frac{1}{2}}+|b|).\end{aligned}$$

This concludes the proof of (4-10), and hence the proof of Lemma 4.1. ☐

4B. ***Rigidity dynamics in $\mathcal{A}_{\alpha_0}$.*** In this part, we will give a specific classification for the asymptotic behavior of solution with initial data in $\mathcal{A}_{\alpha_0}$.

We first introduce the separation time t_1^*:

$$t_1^* = \begin{cases} 0 & \text{if } |b(0)+c_1\omega(0)| \geq C^*(\mathcal{N}_1(0)+b^2(0)+\omega^2(0)), \\ \sup\{0<t<t^* \mid \text{for all } t' \in [0,t],\ |b(t')+c_1\omega(t')| \leq C^*(\mathcal{N}_1(t')+b^2(t')+\omega^2(t'))\} & \text{else,} \end{cases} \tag{4-22}$$

where[10]

$$C^* = 100(K_1 + K_0K_2) > 0. \tag{4-23}$$

Then we have:

Proposition 4.2 (rigidity dynamics). *There exist universal constants $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll 1$ and $C^* > 1$ such that the following hold. Let $u_0 \in \mathcal{A}_{\alpha_0}$, and $u(t)$ be the corresponding solution to* (gKdV$_\gamma$). *Then we have*:

(1) *The following trichotomy holds*:

<u>Blow down</u>: *If $t_1^* = t^*$, then $t_1^* = t^* = T = +\infty$, with*

$$|b(t)| + \mathcal{N}_2(t) \to 0 \quad \text{as } t \to +\infty, \tag{4-24}$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}} \quad \text{as } t \to +\infty. \tag{4-25}$$

<u>Exit</u>: *If $t_1^* < t^*$ with*

$$b(t_1^*) + c_1\omega(t_1^*) \leq -C^*(\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)),$$

then $t^* < T = +\infty$. In particular,

$$\inf_{\lambda_0>0,\, \lambda_0^{-m}\gamma<\omega^*,\, x_0\in\mathbb{R}} \left\| u(t^*) - \frac{1}{\lambda_0^{1/2}} Q_{\lambda_0^{-m}\gamma}\left(\frac{x-x_0}{\lambda_0}\right) \right\|_{L^2} = \alpha^*. \tag{4-26}$$

Moreover, we have

$$b(t^*) \leq -C(\alpha^*) < 0, \quad \lambda(t^*) \geq \frac{C(\alpha^*)}{\delta(\alpha_0)} \gg 1. \tag{4-27}$$

<u>Soliton</u>: *If $t_1^* < t^*$ with*

$$b(t_1^*) + c_1\omega(t_1^*) \geq C^*(\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)),$$

then $t^* = T = +\infty$. Moreover, we have

$$\mathcal{N}_2(t) + |b(t)| \to 0 \quad \text{as } t \to +\infty, \tag{4-28}$$

$$\lambda(t) = \lambda_\infty(1+o(1)), \quad x(t) = \frac{t}{\lambda_\infty^2}(1+o(1)) \quad \text{as } t \to +\infty, \tag{4-29}$$

for some $\lambda_\infty \in (0, +\infty)$.

(2) *All of the three scenarios introduced in* (1) *are known to occur. Moreover, the initial data sets which lead to the soliton and exit cases are open in $\mathcal{A}_{\alpha_0}$ (under the topology of $H^1 \cap L^2(y_+^{10}\, dy)$).*

[10] Recall that K_0, K_1, K_2 and c_1 were introduced in Lemma 4.1.

Remark 4.3. It is easy to see Proposition 4.2 implies Theorem 1.3 immediately.

Remark 4.4. The constant C^* chosen here is not sharp. We can replace it by some slightly different ones.

Proof of Proposition 4.2. The basic idea of the proof is to show that the assumptions (H1)–(H3) introduced in Proposition 3.1 hold for all $t \in [0, t^*)$ (i.e., as long as the solution is close to the soliton manifold). And then together with the estimates obtained in Lemma 4.1, we can show that the error term ε does not perturb the ODE system, and hence the parameters (b, λ, x) have the same asymptotic behavior as the formal system (1-11), which concludes the proof of Theorem 1.3.

In the blow-down and exit cases, this is done by improving the estimates in (H1)–(H3) on $[0, t^{**}]$ (recall that t^{**} is the largest time t such that (H1)–(H3) hold on $[0, t]$), and then a standard bootstrap argument shows that $t^{**} = t^*$, i.e., (H1)–(H3) hold on $[0, t^*)$, while in the soliton case it seems hard to improve all the estimates on $[0, t^{**}]$. But, fortunately, we can use a similar bootstrap argument to show that some modified assumptions (H1)′, (H2)′, (H3)′ hold on $[0, t^*]$, which is also enough to obtain the asymptotic behavior of the parameters.

I. The blow-down case. Assume that $t_1^* = t^*$; i.e., for all $t \in [0, t^*]$,

$$|b(t) + c_1\omega(t)| \le C^*(\mathcal{N}_1(t) + b^2(t) + \omega^2(t)). \tag{4-30}$$

Step 1: Closing the bootstrap. We claim that $t^{**} = t^*$; i.e., the bootstrap assumptions (H1), (H2) and (H3) hold on $[0, t^*]$.

Indeed, we claim that for all $s \in [0, s^{**})$,

$$\omega(s) + |b(s)| + \|\varepsilon(s)\|_{L^2} + \mathcal{N}_2(s) \lesssim \delta(\alpha_0), \tag{4-31}$$

$$\lambda(s) \ge \tfrac{4}{5}, \tag{4-32}$$

$$\int_{y>0} y^{10}\varepsilon^2(s)\, dy \le 5. \tag{4-33}$$

Then choosing α^*, α_0, γ such that $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll \kappa$, we can see that (4-31)–(4-33) imply $t^{**} = t^*$ immediately.

First, from (4-30) we have, for all $s \in [0, s^{**})$,

$$b(s) \le 4C^*\mathcal{N}_1(s) - |b(s)|, \tag{4-34}$$

$$|b(s)| \lesssim \mathcal{N}_1(s) + \omega(s), \tag{4-35}$$

$$\omega(s) \lesssim \mathcal{N}_1(s) + |b(s)|. \tag{4-36}$$

Then we apply (4-34) and (4-36) to (4-10) to obtain

$$\begin{aligned}\frac{(\lambda_0)_s}{\lambda_0} &\ge -b - \mathcal{N}_1 - C(\omega + |b|)(\mathcal{N}_2^{\frac{1}{2}} + |b|)\\ &\ge -5C^*\mathcal{N}_1 + |b| - \delta(\kappa)|b| \gtrsim -\mathcal{N}_1.\end{aligned}$$

Integrating this from s_1 to s_2 for some $0 \le s_1 < s_2 \le s^{**}$, and using (4-6), we have

$$\lambda(s_2) \ge \tfrac{9}{10}\lambda(s_1). \tag{4-37}$$

In particular, we know from (4-2) that for all $s \in [0, s^{**})$

$$\lambda(s) \ge \tfrac{9}{10}\lambda(0) \ge \tfrac{4}{5}. \tag{4-38}$$

By our choice of γ, we have

$$\omega(s) = \frac{\gamma}{\lambda^m(s)} \le 2^m\gamma \lesssim \delta(\alpha_0). \tag{4-39}$$

Next, from (4-4), (4-6) and (4-35), we have for all $s \in [0, s^{**})$

$$\mathcal{N}_2(s) \lesssim \mathcal{N}_2(0) + \sup_{s' \in [0,s]} \mathcal{N}_2^3(s') + \sup_{s' \in [0,s]} \omega^3(s'),$$

which together with (4-35) implies

$$|b(s)| + \mathcal{N}_2(s) \lesssim \delta(\alpha_0)$$

for all $s \in [0, s^{**})$. Then from (2-32) and the condition on the initial data, we obtain

$$\|\varepsilon(s)\|_{L^2} \lesssim \delta(\alpha_0). \tag{4-40}$$

From (2-33) and the condition on the initial data, we have

$$\frac{\|\varepsilon_y(s)\|_{L^2}^2}{\lambda^2(s)} \lesssim \delta(\alpha_0) + \frac{\|\varepsilon_y(s)\|_{L^2}^{m+2}}{\lambda^{m+2}(s)}.$$

Since $\|\varepsilon_y(0)\|_{L^2} \lesssim \delta(\alpha_0)$, $\lambda(0) \sim 1$, from a standard bootstrap argument we have

$$\frac{\|\varepsilon_y(s)\|_{L^2}^2}{\lambda^2(s)} \lesssim \delta(\alpha_0).$$

Thus, we have

$$\omega(s)\|\varepsilon_y(s)\|_{L^2}^m \lesssim \gamma \frac{\|\varepsilon_y(s)\|_{L^2}^m}{\lambda^m(s)} \lesssim \delta(\alpha_0). \tag{4-41}$$

Finally, let us integrate (3-41) from 0 to s, using (4-3), (4-6), (4-8), (4-37) and (4-38) to obtain

$$\begin{aligned}\int \varphi_{10}\varepsilon^2(s)\,dy &\le \frac{\lambda^{10}(0)}{\lambda^{10}(s)} \int \varphi_{10}\varepsilon^2(0)\,dy + C\int_0^s \frac{\lambda^{10}(s')}{\lambda^{10}(s)}(\mathcal{N}_{1,\mathrm{loc}}(s') + b^2(s'))\,ds' \\ &\le 3 + C\int_0^s (\mathcal{N}_{1,\mathrm{loc}}(s') + b^2(s'))\,ds' \le 3 + \delta(\kappa) < 5.\end{aligned}$$

We therefore conclude the proof of (4-31)–(4-33), and obtain $t^{**} = t^*$. Since $0 < \alpha_0 \ll \alpha^*$, the estimate (4-31) implies $t^{**} = t^* = T = +\infty$.

Step 2: Proof of (4-24) and (4-25). We first claim that $\lambda(t) \to +\infty$ as $t \to +\infty$. Let

$$S = \int_0^{+\infty} \frac{1}{\lambda^3(\tau)}\,d\tau \in (0, +\infty].$$

From (2-35), (4-6), (4-8) and (4-36) we have

$$\int_0^{+\infty} |\omega_t|\,dt = \int_0^S |\omega_s|\,ds \lesssim \int_0^S (\mathcal{N}_{2,\mathrm{loc}}(s) + b^2(s))\,ds < +\infty,$$

$$\int_0^{+\infty} \frac{\gamma^2}{\lambda^{3+2m}(t)}\,dt = \int_0^S \omega^2(s)\,ds \lesssim \int_0^S (\mathcal{N}_{2,\mathrm{loc}}(s) + b^2(s))\,ds < +\infty.$$

This leads to $\lambda(t) \to +\infty$ as $t \to +\infty$, or equivalently $\lim_{t\to+\infty} \omega(t) = 0$.

Next, we claim that $S = +\infty$. Otherwise, $b(s), \omega(s) \in L^1([0, S))$. Applying this to (4-10), we obtain

$$\frac{(\lambda_0)_s}{\lambda_0} \in L^1([0, S)).$$

But since $\lambda_0(s) \to +\infty$ as $s \to S$, we have

$$\left|\int_0^{S-\delta_0} \frac{(\lambda_0)_s}{\lambda_0}(s')\,ds'\right| = \left|\log\left(\frac{\lambda_0(S-\delta_0)}{\lambda_0(0)}\right)\right| \to +\infty$$

as $\delta_0 \to 0$, which leads to a contradiction.

Now we can prove (4-24) and (4-25). To do this, we claim that, for all $s \in [0, +\infty)$,

$$\lambda^m(s)\mathcal{N}_2(s) + \int_0^s \lambda^m(s')(\varepsilon^2(s') + \varepsilon_y^2(s'))\varphi'_{2,B}\,ds' \lesssim 1. \tag{4-42}$$

From (3-11) we have

$$\frac{1}{\lambda^m}\frac{d}{ds}(\lambda^m \mathcal{F}_{2,1}) \le -\mu \int (\varepsilon^2 + \varepsilon_y^2)\varphi'_{2,B} + O(b^4 + \omega^2 b^2) - m\frac{\lambda_s}{\lambda}\mathcal{F}_{2,1}. \tag{4-43}$$

From (2-34), (3-13), (3-21) and (4-38), we have

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}\mathcal{F}_{2,1}\right| &\lesssim (|b| + \mathcal{N}_{1,\mathrm{loc}}^{\frac{1}{2}})\left[\left(1 + \frac{1}{\lambda^{10/9}(s)}\right)\mathcal{N}_{2,\mathrm{loc}}^{\frac{8}{9}} + \int \varepsilon_y^2 \psi_B\right]\\ &\lesssim b^2 + \delta(\kappa)\int (\varepsilon^2 + \varepsilon_y^2)\varphi'_{2,B}.\end{aligned}$$

Substituting this into (4-43) and integrating from 0 to s, using (4-35) and (4-36), we have,

$$\begin{aligned}\lambda^m(s)\mathcal{N}_2(s) + \int_0^s \lambda^m(s')(\varepsilon^2(s') + \varepsilon_y^2(s'))\varphi'_{2,B}\,ds' &\lesssim \int_0^s \lambda^m(s')\omega^4(s')\,ds' + \delta(\kappa)\int_0^s \lambda^m(s')\mathcal{N}_1(s')\,ds'\\ &\lesssim \gamma\int_0^s \omega^3(s')\,ds' + \delta(\kappa)\int_0^s \lambda^m(s')\mathcal{N}_1(s')\,ds'\\ &\lesssim \gamma\int_0^s b^2(s')\,ds' + \delta(\kappa)\int_0^s \lambda^m(s')\mathcal{N}_1(s')\,ds'.\end{aligned}$$

Together with (4-8), we obtain (4-42).

Since $\lambda(s) \to +\infty$ as $s \to +\infty$, we have

$$\mathcal{N}_2(s) \lesssim \lambda^{-m}(s) \to 0 \quad \text{as } s \to +\infty. \tag{4-44}$$

Now, using (4-10), (4-30) and (4-35), we have

$$\left|-\frac{(\lambda_0)_s}{\lambda_0}+c_1\omega\right| \lesssim \left|\frac{(\lambda_0)_s}{\lambda_0}+b\right| + |b+c_1\omega|$$
$$\lesssim \mathcal{N}_1 + b^2 + \omega^2 + (|b|+\omega)(\mathcal{N}_2^{\frac{1}{2}}+|b|) \lesssim \mathcal{N}_1 + \delta(\kappa)\omega.$$

Multiplying the above inequality by λ_0^m and integrating from 0 to s, we obtain

$$-C\int_0^s \lambda_0^m \mathcal{N}_1 + \tfrac{1}{2}c_1\gamma s \le \int_0^s (\lambda_0)_s \lambda_0^{m-1} \le C\int_0^s \lambda_0^m \mathcal{N}_1 + 2c_1\gamma s.$$

From (4-42) and $|1-\lambda/\lambda_0| \lesssim \delta(\kappa)$, we obtain

$$\lambda^m(s) \sim s \quad \text{as } s \to +\infty.$$

We then have

$$t(s) = \int_0^s \lambda^3(s')\,ds' \sim s^{\frac{m+3}{m}} = s^{\frac{q+1}{q-5}} \quad \text{as } s \to +\infty,$$

which implies

$$\lambda(t) \sim t^{\frac{2}{q+1}} \quad \text{as } t \to +\infty.$$

Next, from (4-30) and (4-35), we have

$$b(t) \to 0 \quad \text{as } t \to +\infty.$$

Finally, integrating (2-34), we obtain

$$x(t) \sim t^{\frac{q-3}{q+1}} \quad \text{as } t \to +\infty,$$

which concludes the proof of (4-24) and (4-25).

II. The exit case. Assume $t_1^* < t^*$ with

$$b(t_1^*) + c_1\omega(t_1^*) \le -C^*(\mathcal{N}_1(t_1^*) + b^2(t_1^*) + \omega^2(t_1^*)). \tag{4-45}$$

Step 1: Closing the bootstrap. First of all, following the same procedure as in the blow-down case, we have, for all $s \in [0, s_1^*]$,

$$\omega(s) + |b(s)| + \|\varepsilon(s)\|_{L^2} + \omega(s)\|\varepsilon_y(s)\|_{L^2}^m + \mathcal{N}_2(s) \lesssim \delta(\alpha_0), \tag{4-46}$$

$$\lambda(s) \ge \tfrac{4}{5}, \tag{4-47}$$

$$\int_{y>0} y^{10}\varepsilon^2(s)\,dy \le 5. \tag{4-48}$$

In particular, we have $t_1^* < t^{**} \le t^*$. Now, we claim $t^{**} = t^* < T = +\infty$.

To prove this, we use a standard bootstrap argument by improving (H1), (H2) and (H3) on $[t_1^*, t^{**}]$. Let

$$\ell^* = \frac{b(t_1^*) + c_1\omega(t_1^*)}{\lambda^2(t_1^*)} < 0.$$

It is easy to see that $|\ell^*| \lesssim \delta(\alpha_0)$. Now we observe from (4-9) that, for all $s \in [s_1^*, s^{**})$,

$$2\ell^* - C^* \frac{b^2(s) + \omega^2(s)}{\lambda^2(s)} \le \frac{b(s) + c_1\omega(s)}{\lambda^2(s)} \le \frac{\ell^*}{2} + C^* \frac{b^2(s) + \omega^2(s)}{\lambda^2(s)},$$

which implies

$$-b(s) \gtrsim \omega(s) > 0, \tag{4-49}$$

$$3\ell^* - C\frac{\omega(s)}{\lambda^2(s)} \le \frac{b(s)}{\lambda^2(s)} \le \frac{\ell^*}{3} < 0. \tag{4-50}$$

We then observe from (4-10) and (4-49) that

$$\frac{(\lambda_0)_s}{\lambda_0} \gtrsim -\mathcal{N}_{1,\mathrm{loc}},$$

which after integration, yields the almost monotonicity:

$$\text{for all } s_1^* \le s_1 < s_2 \le s^{**}, \quad \lambda(s_2) \ge \tfrac{9}{10}\lambda(s_1) \ge \tfrac{1}{2}. \tag{4-51}$$

So we obtain for all $s \in [s_1^*, s^{**})$,

$$\omega(s) + \frac{\omega(s)}{\lambda^2(s)} \lesssim \gamma \lesssim \delta(\alpha_0).$$

Together with (4-7) and (4-50), we have, for all $s \in [s_1^*, s^{**})$,

$$\frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0),$$

which improves (H2) if we choose $\alpha_0 \ll \kappa$. Next, using the same strategy as in the blow-down case, we have, for all $s \in [s_1^*, s^{**})$,

$$\int \varphi_{10}\varepsilon^2(s)\,dy \le 7.$$

Then, (H3) is improved. It now only remains to improve (H1). Since for all $t \in [t_1^*, t^*)$ we have $u(t) \in \mathcal{T}_{\alpha^*,\gamma}$, following the argument in Lemma 2.6, for all $t \in [0, t^*)$ we have $|b(t)| \lesssim \delta(\alpha^*)$. By (2-32), (4-6), and (4-49), we have, for all $s \in [s_1^*, s^{**})$,

$$\omega(s) + \|\varepsilon(s)\|_{L^2} + \mathcal{N}_2(s) \lesssim \delta(\alpha^*).$$

Now, following from the same argument as for (4-41), we have

$$\omega(s)\|\varepsilon_y(s)\|_{L^2}^m \lesssim \delta(\alpha_0).$$

Then (H1) is improved, due to our choice of the universal constant, i.e., $\alpha^* \ll \kappa$.

In conclusion, we have proved $t^{**} = t^*$.

Step 2: Proof of (4-26) and (4-27). We first claim that the exit case occurs in finite time $t^* < +\infty$. Dividing (4-10) by λ_0^2, and using (4-49) to estimate on $[t_1^*, t^*)$,

$$-\frac{\ell^*}{3} - C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} \le (\lambda_0)_t \le -3\ell^* + C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}.$$

Integrating from t_1^* to t, we get

$$\frac{|\ell^*|(t-t_1^*)}{3} - C_1\int_{t_1^*}^{t}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} \le \lambda_0(t)-\lambda_0(t_1^*) \le 3|\ell^*|(t-t_1^*) + C_2\int_{t_1^*}^{t}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}.$$

From (4-51), we have

$$\int_{t_1^*}^{t}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} = \int_{s_1^*}^{s}\lambda\mathcal{N}_{1,\mathrm{loc}} \lesssim \lambda(s)\int_{s_1^*}^{s}\mathcal{N}_{1,\mathrm{loc}} \lesssim \delta(\kappa)\lambda(t).$$

Thus, for all $t\in[t_1^*,t^*)$,

$$\tfrac{1}{4}(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*)) \le \lambda(t) \le 4(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*)).$$

Next, from (4-49), we have for all $t\in[t_1^*,t^*)$,

$$-100|\ell^*|(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*))^2 \le b(t) \le -\frac{|\ell^*|}{100}(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*))^2.$$

If $t^*=T=+\infty$, then the above estimate leads to $b(t)\to-\infty$ as $t\to+\infty$, which contradicts the fact that $|b(t)|\lesssim\delta(\alpha^*)$ for all $t\in[t_1^*,t^*)$. Thus, we have $t^*<T=+\infty$.

Finally, since $0<t^*<+\infty$, by the definition of t^*, we must have $-b(t^*)\ge C(\alpha^*)>0$. While from (4-49), we have

$$\lambda^2(t^*) \ge \frac{1}{2}\frac{|b(t^*)|}{|\ell^*|} \gtrsim \frac{C(\alpha^*)}{\delta(\alpha_0)} \gg 1,$$

which concludes the proof of (4-26) and (4-27).

III. The soliton case. Assume $t_1^*<t^*$ with

$$b(t_1^*)+c_1\omega(t_1^*) \ge C^*(\mathcal{N}_1(t_1^*)+b^2(t_1^*)+\omega^2(t_1^*)) > 0. \tag{4-52}$$

Step 1: Estimates on the rescaled solution. Similar to the exit case, we have, for all $s\in[0,s_1^*]$,

$$\omega(s)+|b(s)|+\|\varepsilon(s)\|_{L^2}+\omega(s)\|\varepsilon_y(s)\|_{L^2}^m+\mathcal{N}_2(s)\lesssim\delta(\alpha_0), \tag{4-53}$$

$$\lambda(s)\ge\tfrac{4}{5}, \tag{4-54}$$

$$\int_{y>0} y^{10}\varepsilon^2(s)\,dy \le 5. \tag{4-55}$$

But here we can't directly prove that $t^{**}=t^*$ as we did in the exit case. The main difficulty is that we lack some control on the upper bound of $\lambda(t_1^*)$, which makes it hard to improve the bootstrap assumptions (H2) and (H3). However, we will see that the bootstrap assumptions (H2) and (H3) are related to the scaling symmetry of the problem. If we use the pseudoscaling rule (1-1) on $[t_1^*,t^*)$ to rescale $\lambda(t_1^*)$ to 1, then we can get the desired result. Roughly speaking, on $[t_1^*,t^*]$, the bootstrap assumptions (H2) and (H3) should be replaced by some other suitable assumptions (H2)′ and (H3)′.

More precisely, we introduce the following change of coordinates. For all $t \in [t_1^*, t^*)$, let

$$\bar{t} = \frac{t - t_1^*}{\lambda^3(t_1^*)}, \quad \bar{x} = \frac{x - x(t_1^*)}{\lambda(t_1^*)}, \quad \bar{\gamma} = \frac{\gamma}{\lambda^m(t_1^*)}, \quad \bar{t}^* = \frac{t^* - t_1^*}{\lambda^3(t_1^*)}, \tag{4-56}$$

$$\bar{u}(\bar{t}, \bar{x}) = \lambda^{\frac{1}{2}} u\big(\lambda^3(t_1^*)\bar{t} + t_1^*, \lambda(t_1^*)\bar{x} + x(t_1^*)\big). \tag{4-57}$$

Then, from the pseudoscaling rule (1-1), $\bar{u}(\bar{t}, \bar{x})$ is a solution to the Cauchy problem

$$\begin{cases} \partial_{\bar{t}}\bar{u} + (\bar{u}_{\bar{x}\bar{x}} + \bar{u}^5 - \bar{\gamma}\bar{u}|\bar{u}|^{q-1})_{\bar{x}} = 0, & (\bar{t}, \bar{x}) \in [0, \bar{t}^*) \times \mathbb{R}, \\ \bar{u}(0, \bar{x}) = Q_{b(t_1^*), \omega(t_1^*)}(\bar{x}) + \varepsilon(t_1^*, \bar{x}) \in H^1(\mathbb{R}). \end{cases} \tag{4-58}$$

Moreover, for all $\bar{t} \in [0, \bar{t}^*)$ we define

$$\bar{\varepsilon}(\bar{t}, y) = \varepsilon(\lambda^3(t_1^*)\bar{t} + t_1^*, y), \quad \bar{\lambda}(\bar{t}) = \frac{\lambda(\lambda^3(t_1^*)\bar{t} + t_1^*)}{\lambda(t_1^*)}, \quad \bar{\omega}(\bar{t}) = \frac{\bar{\gamma}}{\bar{\lambda}^m(\bar{t})}, \tag{4-59}$$

$$\bar{b}(\bar{t}) = b(\lambda^3(t_1^*)\bar{t} + t_1^*), \quad \bar{x}(\bar{t}) = \frac{x(\lambda^3(t_1^*)\bar{t} + t_1^*) - x(t_1^*)}{\lambda(t_1^*)}. \tag{4-60}$$

Then, from (2-21), it is easy to check that

$$\bar{u}(\bar{t}, \bar{x}) = \frac{1}{\bar{\lambda}^{1/2}(\bar{t})}[Q_{\bar{b}(\bar{t}), \bar{\omega}(\bar{t})} + \bar{\varepsilon}(\bar{t})]\left(\frac{\bar{x} - \bar{x}(\bar{t})}{\bar{\lambda}(\bar{t})}\right),$$

with

$$(\bar{\varepsilon}(\bar{s}), Q_{\bar{\omega}(\bar{s})}) = (\bar{\varepsilon}(\bar{s}), \Lambda Q_{\bar{\omega}(\bar{s})}) = (\bar{\varepsilon}(\bar{s}), \bar{y}\Lambda Q_{\bar{\omega}(\bar{s})}) = 0,$$

where $(\bar{s}, \bar{y})$ are the scaling-invariant variables

$$\bar{s} = \int_0^{\bar{t}} \frac{1}{\bar{\lambda}^3(\tau)}\, d\tau, \quad \bar{y} = \frac{\bar{x} - \bar{x}(\bar{t})}{\bar{\lambda}(\bar{t})}.$$

We then introduce the weighted Sobolev norms

$$\begin{aligned} \overline{\mathcal{N}}_i(\bar{s}) &= \int \big(\bar{\varepsilon}_{\bar{y}}^2(\bar{s}, \bar{y})\psi_B(\bar{y}) + \bar{\varepsilon}^2(\bar{s}, \bar{y})\varphi_{i,B}(\bar{y})\big)\, d\bar{y}, \\ \overline{\mathcal{N}}_{i,\mathrm{loc}}(\bar{s}) &= \int \bar{\varepsilon}^2(\bar{s}, \bar{y})\varphi_{i,B}'(\bar{y})\, d\bar{y}, \end{aligned} \tag{4-61}$$

where $\varphi_{i,B}$ and ψ_B are the weight functions introduced in Section 3.

From now on, for all $\bar{t} \in [0, \bar{t}^*)$, we let $t = \lambda^3(t_1^*)\bar{t} + t_1^*$. In this setting, we have $\bar{s}(\bar{t}) = s(t) - s_1^*$. Since the pseudoscaling rule (1-1) is L^2 invariant, we have

$$\bar{u}(\bar{t}) \in \mathcal{T}_{\alpha^*, \bar{\gamma}} \iff u(t) \in \mathcal{T}_{\alpha^*, \gamma},$$

which yields

$$\bar{t}^* = \sup\{0 < \bar{t} < +\infty \mid \text{for all } t' \in [0, \bar{t}],\ \bar{u}(t') \in \mathcal{T}_{\alpha^*, \bar{\gamma}}\}.$$

Next, let $\kappa > 0$ be the universal constant introduced in Proposition 2.9, Proposition 3.1 and Lemma 4.1. We then define the following bootstrap assumptions for the rescaled solution $\bar{u}(\bar{t}, \bar{x})$. For all $\bar{s} \in [0, \bar{s}(\bar{t}))$:

(H1)′ Scaling-invariant bound:

$$\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s}) + \|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \le \kappa. \tag{4-62}$$

(H2)′ Bound related to H^1 scaling:

$$\frac{\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \le \kappa. \tag{4-63}$$

(H3)′ L^2 weighted bound on the right:

$$\int_{\bar{y}>0} \bar{y}^{10}\bar{\varepsilon}^2(\bar{s},\bar{y})\,d\bar{y} \le 50\left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right). \tag{4-64}$$

We define $\bar{t}^{**}$ as

$$\bar{t}^{**} = \sup\{0 < \bar{t} < \bar{t}^{*} \mid \text{(H1)}', \text{(H2)}' \text{ and (H3)}' \text{ hold for all } t' \in [0,\bar{t}]\}. \tag{4-65}$$

Our goal here is to prove that $\bar{t}^{**} = \bar{t}^{*} = +\infty$, which gives us the desired asymptotic behaviors.[11] Let $\bar{s}^{*} = \bar{s}(\bar{t}^{*})$, $\bar{s}^{**} = \bar{s}(\bar{t}^{**})$. Since

$$\bar{\lambda}(0) = 1, \quad \bar{x}(0) = 0, \quad \bar{b}(0) = b(t_1^*), \quad \bar{\omega}(0) = \omega(t_1^*), \quad \bar{\varepsilon}(0,\bar{y}) = \varepsilon(t_1^*,\bar{y}), \quad \bar{\gamma} \lesssim \gamma, \tag{4-66}$$

we know from (4-53)–(4-55), that $\bar{s}^{**} > 0$.

On the other hand, on $[0,\bar{s}^{**})$, all conditions of Propositions 2.9 and 3.1 and Lemmas 3.6 and 4.1 are satisfied for $\bar{u}(\bar{t},\bar{x})$. Repeating the same procedure, we have:

Lemma 4.5 (estimates for the rescaled solution). *For all $\bar{s} \in [0,\bar{s}^{**})$ or equivalently $s \in [s_1^*, s_1^* + \bar{s}^{**})$, all estimates of Propositions 2.9 and 3.1 and Lemmas 3.6 and 4.1 hold with*

$$(t, x, u, \gamma, \lambda(t), b(t), x(t), \omega(t), \varepsilon(t), s, y)$$

replaced by

$$(\bar{t}, \bar{x}, \bar{u}, \bar{\gamma}, \bar{\lambda}(\bar{t}), \bar{b}(\bar{t}), \bar{x}(\bar{t}), \bar{\omega}(\bar{t}), \bar{\varepsilon}(\bar{t}), \bar{s}, \bar{y}).$$

Remark 4.6. For simplicity, we skip the statement of these similar estimates for $\bar{u}$. We also refer to the equation number of the corresponding inequality for $u(t)$, when we need to use these estimates for $\bar{u}(\bar{t})$.

Step 2: Closing the bootstrap. In this part, we will close the bootstrap argument to show that $\bar{t}^{**} = \bar{t}^{*} = +\infty$. This is done through the following steps:

(1) We prove that for $\bar{t}$ large enough, we have $\bar{\omega}(\bar{t}) \gg |\bar{b}(\bar{t})|$, which coincides with the formal ODE system (1-11) in the soliton region, where we have $\omega(t)$ converges to a positive constant, while $b(t)$ converges to 0 as $t \to +\infty$. Indeed, if $|\bar{b}(\bar{t})| \gtrsim \bar{\omega}(\bar{t})$ holds for all $\bar{t} \in [0,\bar{t}^{**}]$, we will obtain finite time blow-up if $\bar{b}(0) > 0$ or exit behavior if $\bar{b}(0) < 0$. Both of them lead to a contradiction.

[11] Since $\lambda(t_1^*) \gtrsim 1$, we know that (H1) is equivalent to (H1)′ and (H2) is weaker than (H2)′, while (H3) is stronger than (H3)′. It is hard to determine whether $t^{**} = \lambda^3(t_1^*)\bar{t}^{**} + t_1^*$ holds.

(2) The hardest part of the analysis is to prove that the scaling parameter $\bar\lambda$ is bounded from both above and below for all $\bar t \in [0, \bar t^{**}]$. This is done by proving that[12]

$$\left(\frac{1}{\bar\lambda^2}\right)_s + C\bar\gamma\left(\frac{1}{\bar\lambda^2}\right)^{1+\frac{m}{2}} \sim \bar\ell^* > 0.$$

(3) The estimates of the rest of the terms can be done by arguments similar to those in the blow-down and exit regions.

Now we turn to the proof of $\bar t^{**} = \bar t^* = +\infty$. We first define

$$\bar t_2^* = \begin{cases} 0 & \text{if } |\bar b(0)| \le \frac{1}{100} c_1 \bar\omega(0), \\ \sup\{0 < \bar t < \bar t^* \mid \text{for all } t' \in [0, \bar t],\ |\bar b(t')| \ge \frac{1}{100} c_1 \bar\omega(t')\} & \text{else.} \end{cases}$$

Our first observation is that $\bar t_2^* < \bar t^*$. Otherwise, since $\bar t_2^* = \bar t^* \ge \bar t^{**} > 0$, we have, for all $\bar t \in [0, \bar t^{**})$, $\bar b(\bar t) \ne 0$.

If $\bar b(0) > 0$, we claim that $\bar t^{**} = \bar t_2^* = \bar t^* = +\infty$. To prove this, we need to improve (H1)′, (H2)′ and (H3)′ on $[0, \bar t^{**}]$. Indeed, from the definition of $\bar t_2^*$, we have

$$0 < \bar\omega(\bar t) \lesssim \bar b(\bar t) \tag{4-67}$$

for all $\bar t \in [0, \bar t^{**})$. Applying this to (4-10), we have

$$\frac{(\bar\lambda_0)_{\bar s}}{\bar\lambda_0} \le -\bar b + O(\bar{\mathcal N}_{2,\mathrm{loc}}) + \delta(\kappa)|\bar b|.$$

Integrating this from 0 to $\bar t$ using (4-6) and the fact that $\bar\lambda(0) = 1$, we obtain the almost monotonicity

$$\text{for all } 0 \le \bar s_1 < \bar s_2 \le \bar s^{**}, \quad \bar\lambda(\bar s_2) \le \tfrac{10}{9}\bar\lambda(\bar s_1) \le \tfrac{5}{4}. \tag{4-68}$$

On the other hand, we learn from (4-9), (4-52) and (4-66), that for all $\bar s \in [0, \bar s^{**})$

$$\frac{99}{100}\bar\ell^* - K_1\frac{\bar b^2(\bar s) + \bar\omega^2(\bar s)}{\bar\lambda^2(\bar s)} \le \frac{\bar b(\bar s) + c_1\bar\omega(\bar s)}{\bar\lambda^2(\bar s)} \le \frac{101}{100}\bar\ell^* + K_1\frac{\bar b^2(\bar s) + \bar\omega^2(\bar s)}{\bar\lambda^2(\bar s)}, \tag{4-69}$$

where

$$0 < \bar\ell^* = \frac{\bar b(0) + c_1\bar\omega(0)}{\bar\lambda^2(0)} = b(t_1^*) + c_1\omega(t_1^*) \lesssim \delta(\alpha_0).$$

Together with (4-67), we have for all $\bar s \in [0, \bar s^{**})$

$$\frac{\bar b(\bar s)}{\bar\lambda^2(\bar s)} \sim \bar\ell^* \lesssim \delta(\alpha_0), \quad \frac{\bar\omega(\bar s)}{\bar\lambda^2(\bar s)} \lesssim \bar\ell^* \lesssim \delta(\alpha_0). \tag{4-70}$$

Then from (4-68), (4-6) and (4-7), we have for all $\bar s \in [0, \bar s^{**})$

$$\frac{\bar{\mathcal N}_2(\bar s)}{\bar\lambda^2(\bar s)} \lesssim \delta(\alpha_0), \quad \bar{\mathcal N}_2(\bar s) + \bar\omega(\bar s) + |\bar b(\bar s)| \lesssim \bar\lambda^2(\bar s)\bar\ell^* + \delta(\alpha_0) \le \delta(\alpha_0). \tag{4-71}$$

[12] See (4-88) and (4-90) for details.

Then, from (2-32), (4-53) and fact

$$\bar{u}(0,\bar{x}) = Q_{b(t_1^*),\omega(t_1^*)}(\bar{x}) + \varepsilon(t_1^*,\bar{x}),$$

we know that

$$\begin{aligned}\|\bar{\varepsilon}(\bar{s})\|_{L^2} &\lesssim \delta(\alpha_0) + \left|\int \bar{u}^2(0) - \int Q^2\right|^{\frac{1}{2}} \\ &\lesssim \delta(\alpha_0) + \|\varepsilon(t_1^*)\|_{L^2} + |b(t_1^*)|^{\frac{1}{2}} + \omega^{\frac{1}{2}}(t_1^*) \lesssim \delta(\alpha_0).\end{aligned} \tag{4-72}$$

Now, from (2-33) and (4-71), we have

$$\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m = \bar{\gamma}\frac{\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m}{\bar{\lambda}^m(\bar{s})} \lesssim \delta(\alpha_0) + \left(\bar{\gamma}\frac{\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m}{\bar{\lambda}^m(\bar{s})}\right)^{\frac{m+2}{2}} + |\bar{\gamma}^{\frac{2}{m}}\bar{E}(\bar{u}(0))|^{\frac{m}{2}},$$

where $\bar{E}(\bar{u}(0))$ is the energy of the Cauchy problem (4-58), i.e.,

$$\bar{E}(\bar{u}(0)) = \frac{1}{2}\int \bar{u}_{\bar{x}}^2(0) - \frac{1}{6}\int \bar{u}^6(0) + \frac{\bar{\gamma}}{q+1}\int |\bar{u}(0)|^{q+1}.$$

Since

$$\bar{u}(0,\bar{x}) = \lambda^{\frac{1}{2}}(t_1^*)u(t_1^*,\lambda(t_1^*)\bar{x} + x(t_1^*)),$$

from the energy conservation law of (gKdV$_\gamma$) and the condition on the initial data, we have

$$|\bar{\gamma}^{\frac{2}{m}}\bar{E}(\bar{u}(0))| = \left|\gamma^{\frac{2}{m}}\frac{\bar{E}(\bar{u}(0))}{\lambda^2(t_1^*)}\right| = |\gamma^{\frac{2}{m}}E(u(t_1^*))| = |\gamma^{\frac{2}{m}}E_0| \lesssim \delta(\alpha_0).$$

Thus, for all $\bar{s} \in [0,\bar{s}^{**})$, we have

$$\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0) + (\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m)^{1+\frac{m}{2}}.$$

From (4-53) and (4-66), we have

$$\bar{\omega}(0)\|\bar{\varepsilon}_{\bar{y}}(0)\|_{L^2}^m = \omega(s_1^*)\|\varepsilon_y(s_1^*)\|_{L^2}^m \lesssim \delta(\alpha_0).$$

Then a standard bootstrap argument leads to

$$\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0) \tag{4-73}$$

for all $\bar{s} \in [0,\bar{s}^{**})$.

Finally, integrating (3-41), using (4-6) and (4-68) we obtain

$$\begin{aligned}\int \varphi_{10}(\bar{y})\bar{\varepsilon}^2(\bar{s},\bar{y})\,d\bar{y} &\le \frac{\bar{\lambda}^{10}(0)}{\bar{\lambda}^{10}(\bar{s})}\int \varphi_{10}(\bar{y})\bar{\varepsilon}^2(0,\bar{y})\,d\bar{y} + \frac{C}{\bar{\lambda}^{10}(\bar{s})}\int_0^{\bar{s}} \bar{\lambda}^{10}(\bar{\mathcal{N}}_{1,\mathrm{loc}} + \bar{b}^2) \\ &\le \frac{1}{\bar{\lambda}^{10}(\bar{s})}\left[5 + C\bar{\lambda}^{10}(0)\int_0^{\bar{s}}(\bar{\mathcal{N}}_{1,\mathrm{loc}} + \bar{b}^2)\right] \le \frac{5+\delta(\kappa)}{\bar{\lambda}^{10}(\bar{s})}.\end{aligned} \tag{4-74}$$

Combining (4-70)–(4-74), we conclude that $\bar{t}^{**} = \bar{t}^*$. Since all H^1 solutions of (4-58) are global in time, we must have $\bar{t}^{**} = \bar{t}^* = +\infty$, provided that $\alpha_0 \ll \alpha^*$. Now we substitute (4-70) into (4-10) to obtain

$$\frac{\bar{\ell}^*}{3} - C\frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2} \le -(\bar{\lambda}_0)_{\bar{t}} \le 3\bar{\ell}^* + C\frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2}.$$

Integrating in time, we have for all $\bar{t} \in [0, +\infty)$

$$0 < \bar{\lambda}_0(\bar{t}) \le \bar{\lambda}(0) - \frac{\bar{\ell}^*\bar{t}}{3} + C\int_0^{\bar{t}} \frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2}.$$

From (4-68) and (4-6) we have

$$\int_0^{\bar{t}} \frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2} = \int_0^{\bar{s}} \bar{\lambda}(\tau)\overline{\mathcal{N}}_{1,\mathrm{loc}}(\tau)\,d\tau \lesssim \int_0^{\bar{s}} \overline{\mathcal{N}}_{1,\mathrm{loc}}(\tau)\,d\tau \lesssim \delta(\kappa),$$

which implies that the solution blows up in finite time. This is a contradiction.

Now we consider the other case $\bar{b}(0) < 0$. We claim again that $\bar{t}_2^* = \bar{t}^{**} = \bar{t}^* = +\infty$. It is also done by improving the three bootstrap assumptions. First, we know from (4-9), (4-52) and (4-66) that (4-69) still holds in this case. And the definition of $\bar{t}_2^*$ implies

$$0 < \bar{\ell}^* \lesssim -\frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \sim \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}. \tag{4-75}$$

Then we apply the fact that $0 < \bar{\omega} \lesssim -\bar{b}$ to (4-10) to obtain

$$\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0} \ge -\tfrac{1}{2}\bar{b} - O(\overline{\mathcal{N}}_{2,\mathrm{loc}}).$$

Integrating in time we have

$$\text{for all } 0 \le \bar{s}_1 < \bar{s}_2 \le \bar{s}^{**}, \quad \bar{\lambda}(\bar{s}_2) \ge \tfrac{9}{10}\bar{\lambda}(\bar{s}_1) \ge \tfrac{4}{5}, \tag{4-76}$$

which yields for all $\bar{s} \in [0, \bar{s}^{**})$

$$\bar{\omega}(\bar{s}) + \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \bar{\gamma} \lesssim \delta(\alpha_0). \tag{4-77}$$

From (4-75), (4-6) and (4-7), we get

$$\overline{\mathcal{N}}_2(\bar{s}) + |\bar{b}(\bar{s})| + \frac{\overline{\mathcal{N}}_2(\bar{s}) + |\bar{b}(\bar{s})|}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0). \tag{4-78}$$

Using the same argument as we did for (4-72)–(4-74), we have

$$\|\bar{\varepsilon}(\bar{s})\|_{L^2} \lesssim \delta(\alpha_0), \quad \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0), \quad \int \varphi_{10}\bar{\varepsilon}^2(\bar{s})\,d\bar{y} \le 7. \tag{4-79}$$

Combining (4-77)–(4-79), we conclude that $\bar{t}^{**} = \bar{t}^* = +\infty$. But from (4-75), we have

$$-\bar{b} \sim \bar{\omega}(\bar{s}) \gtrsim \bar{\gamma}^{\frac{2}{m+2}}(\bar{\ell}^*)^{\frac{m}{m+2}} > 0. \tag{4-80}$$

On the other hand, from (4-8), we have

$$\int_0^{\bar{s}^{**}} \bar{b}^2(s')\,ds' \lesssim 1.$$

The above two estimates imply

$$\bar{s}^{**} = \int_0^{+\infty} \frac{1}{\bar{\lambda}^3(\tau)}\,d\tau < +\infty,$$

which leads to $\bar{\lambda}(\bar{t}_n) \to +\infty$ as $n \to +\infty$ for some sequence $\bar{t}_n \to +\infty$ or equivalently $\lim_{n\to+\infty} \bar{\omega}(\bar{t}_n) = 0$. This contradicts (4-80).

In conclusion, we have proved that $\bar{t}_2^* < \bar{t}^*$ with

$$|\bar{b}(\bar{t}_2^*)| \le \tfrac{1}{100} c_1 \bar{\omega}(\bar{t}_2^*).$$

Let $\bar{s}_2^* = \bar{s}(\bar{t}_2^*)$. Repeating the same procedure as before, we have for all $\bar{s} \in [0, \bar{s}_2^*]$

$$\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m + \bar{\mathcal{N}}_2(\bar{s}) \lesssim \delta(\alpha_0), \tag{4-81}$$

$$\frac{\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0), \tag{4-82}$$

$$\int_{\bar{y}>0} \bar{y}^{10} \bar{\varepsilon}^2(\bar{s})\,d\bar{y} \le 7\left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right). \tag{4-83}$$

In particular, we have $\bar{t}_2^* < \bar{t}^{**} \le \bar{t}^*$. Similarly, we need to improve the three bootstrap assumptions on $[\bar{t}_2^*, \bar{t}^{**})$ to obtain $\bar{t}^{**} = \bar{t}^* = +\infty$.

First, it is easy to see that (4-69) holds on $[\bar{s}_2^*, \bar{s}^{**})$. So the definition of $\bar{s}_2^*$ yields[13]

$$\frac{19}{20}\bar{\ell}^* \le \frac{c_1 \bar{\omega}(\bar{s}_2^*)}{\bar{\lambda}^2(\bar{s}_2^*)} \le \frac{21}{20}\bar{\ell}^*, \tag{4-84}$$

which implies

$$\frac{9}{10}\left(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\right)^{\frac{2}{m+2}} \le \frac{1}{\bar{\lambda}^2(\bar{s}_2^*)} \le \frac{11}{10}\left(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\right)^{\frac{2}{m+2}}. \tag{4-85}$$

Next, we let

$$C_1 = \frac{99}{100}c_1 < c_1, \quad C_2 = \frac{101}{100}c_1 > c_1.$$

Then, we learn from (4-69) that, for all $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$,

$$\begin{aligned}\frac{99}{100}\bar{\ell}^* &\le \frac{\bar{b}(\bar{s}) + C_2\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} - \frac{c_1}{100}\frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} + O\left(\frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right) \\ &\le \frac{\bar{b}(\bar{s}) + C_2\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} - \frac{c_1}{100}\frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} + \delta(\kappa)\left(\left|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right| + \left|\frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right|\right),\end{aligned}$$

[13] Recall that $c_1 = G'(0) > 0$, where G is the C^2 function introduced in (2-40).

which implies[14]

$$\frac{49}{50}\bar{\ell}^* \le \frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}, \tag{4-86}$$

where

$$\bar{\omega}_0(\bar{s}) = \frac{\bar{\gamma}}{\bar{\lambda}_0^m(\bar{s})}.$$

Substituting (4-10) into (4-86), using (4-7) and the fact that[15]

$$\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} > 0,$$

we have

$$\begin{aligned}\frac{49}{50}\bar{\ell}^* &\le \frac{101}{100}\left(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right) - \frac{1}{100}\left(\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right) - \frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\ &\qquad + \frac{101K_2}{100}\frac{\bar{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \delta(\kappa)\left(\left|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right| + \left|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|\right) \\ &\le \frac{101}{100}\left(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right) - \frac{1}{100}\left(\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right) - \frac{c_1}{300}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\ &\qquad + \frac{101K_0K_2}{100}\frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}_0^2(0)} + \delta(\kappa)\left(\left|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right| + \left|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|\right) \\ &\le \frac{101}{100}\left(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right) + \frac{51K_0K_2}{50}\frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}^2(0)}.\end{aligned} \tag{4-87}$$

From (4-52) and (4-66), we have

$$\bar{\ell}^* = \frac{\bar{b}(0) + c_1\bar{\omega}(0)}{\bar{\lambda}^2(0)} \ge 100(K_1 + K_0K_2)\frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}^2(0)}.$$

So (4-87) implies that for all $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$,

$$\frac{1}{2}\left(\frac{1}{\bar{\lambda}_0^2}\right)_{\bar{s}} + C_2\bar{\gamma}\left(\frac{1}{\bar{\lambda}_0^2}\right)^{1+\frac{m}{2}} \ge \frac{9}{10}\bar{\ell}^*. \tag{4-88}$$

Similar to (4-86), we have

$$\frac{51}{50}\bar{\ell}^* \ge \frac{\bar{b}(\bar{s}) + C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \delta(\kappa)\left|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|, \tag{4-89}$$

[14]Here we use the fact that $|1 - (\bar{\lambda}/\bar{\lambda}_0)| \lesssim |\bar{J}_1| \lesssim \delta(\kappa)$.

[15]This is a direct consequence of (4-86).

which leads to

$$\frac{51}{50}\bar{\ell}^* \geq \frac{99}{100}\biggl(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3}+\frac{C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr)+\frac{1}{100}\biggl(\frac{\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr)+\frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{99K_2}{100}\frac{\bar{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}-\delta(\kappa)\biggl(\biggl|\frac{\bar{b}(\bar{s})+C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr|+\biggl|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr|\biggr),$$

and

$$\frac{51}{50}\bar{\ell}^* \geq \frac{101}{100}\biggl(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3}+\frac{C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr)-\frac{1}{100}\biggl(\frac{\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr)+\frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{101K_2}{100}\frac{\bar{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}-\delta(\kappa)\biggl(\biggl|\frac{\bar{b}(\bar{s})+C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr|+\biggl|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\biggr|\biggr).$$

Using the same strategy as (4-87), and discussing the sign of $(\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s}))/\bar{\lambda}_0^2(\bar{s})$, we have

$$\frac{1}{2}\biggl(\frac{1}{\bar{\lambda}_0^2}\biggr)_{\bar{s}}+C_1\bar{\gamma}\biggl(\frac{1}{\bar{\lambda}_0^2}\biggr)^{1+\frac{m}{2}} \leq \frac{11}{10}\bar{\ell}^*. \tag{4-90}$$

Then we need following basic lemma:

Lemma 4.7. *Let* $F\colon [0,x_0)\to(0,+\infty)$ *be a* C^1 *function. Let* $\nu>0,\ L>0$ *be two positive constants. Then we have*:

(1) *If for all* $x\in[0,x_0)$

$$F_x+F^{1+\nu}\geq L,$$

then for all $x\in[0,x_0)$,

$$F(x)\geq \min(F(0),L^{\frac{1}{1+\nu}}).$$

(2) *If for all* $x\in[0,x_0)$

$$F_x+F^{1+\nu}\leq L,$$

then for all $x\in[0,x_0)$,

$$F(x)\leq \max(F(0),L^{\frac{1}{1+\nu}}).$$

It is easy to prove Lemma 4.7 by standard ODE theory. Now we apply Lemma 4.7 to (4-88) and (4-90) on $[\bar{s}_2^*,\bar{s}^{**})$, using (4-85) to obtain

$$\frac{90}{101}\biggl(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\biggr)^{\frac{2}{m+2}} \leq \frac{1}{\bar{\lambda}^2(\bar{s})} \leq \frac{10}{9}\biggl(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\biggr)^{\frac{2}{m+2}} \tag{4-91}$$

for all $\bar{s}\in[\bar{s}_2^*,\bar{s}^{**})$. This also implies that, for all $\bar{s}\in[\bar{s}_2^*,\bar{s}^{**})$,

$$\bar{\omega}(\bar{s})\sim\bar{\gamma}^{\frac{2}{m+2}}(\bar{\ell}^*)^{\frac{m}{m+2}}\lesssim\delta(\alpha_0),\qquad \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\sim\bar{\ell}^*\lesssim\delta(\alpha_0). \tag{4-92}$$

From (4-86) and (4-89), we have

$$\frac{\bar b(\bar s)+C_2\bar\omega_0(\bar s)}{\bar\lambda_0^2(\bar s)}\ge\frac{49}{50}\bar\ell^*,\quad \frac{\bar b(\bar s)+C_1\bar\omega_0(\bar s)}{\bar\lambda_0^2(\bar s)}\le 2\bar\ell^*;$$

together with (4-92), we have

$$\left|\frac{\bar b(\bar s)}{\bar\lambda^2(\bar s)}\right|\lesssim\bar\ell^*\lesssim\delta(\alpha_0),\quad |\bar b(\bar s)|\lesssim\bar\gamma^{\frac{2}{m+2}}(\bar\ell^*)^{\frac{m}{m+2}}\lesssim\delta(\alpha_0).\tag{4-93}$$

Again, from the mass conservation law (2-32), the energy conservation law (2-33) and the almost monotonicity (4-6), (4-7), we have for all $\bar s\in[\bar s_2^*,\bar s^{**})$

$$\|\bar\varepsilon(\bar s)\|_{L^2}+\bar\omega(\bar s)\|\bar\varepsilon_{\bar y}(\bar s)\|_{L^2}^m+\bar{\mathcal N}_2(\bar s)+\frac{\bar{\mathcal N}_2(\bar s)}{\bar\lambda^2(\bar s)}\lesssim\delta(\alpha_0).\tag{4-94}$$

Finally, we learn from (4-91) that, for all $\bar s_2^*\le\bar s_1<\bar s_2\le\bar s^{**}$,

$$\frac14<\left(\frac{81}{101}\right)^5\le\left(\frac{\bar\lambda(\bar s_1)}{\bar\lambda(\bar s_2)}\right)^{10}\le\left(\frac{101}{81}\right)^5<4.$$

Then for all $\bar s\in[\bar s_2^*,\bar s^{**})$, we integrate (3-41) from $\bar s_2^*$ to $\bar s$ to obtain

$$\begin{aligned}\int\varphi_{10}(\bar y)\bar\varepsilon^2(\bar s,\bar y)\,d\bar y&\le\frac{\bar\lambda^{10}(\bar s_2^*)}{\bar\lambda^{10}(\bar s)}\int\varphi_{10}(\bar y)\bar\varepsilon^2(\bar s_2^*,\bar y)\,d\bar y+\frac{C}{\bar\lambda^{10}(\bar s)}\int_{\bar s_2}^{\bar s}\bar\lambda^{10}(\bar{\mathcal N}_{1,\mathrm{loc}}+\bar b^2)\\&\le\frac{\bar\lambda^{10}(\bar s_2^*)}{\bar\lambda^{10}(\bar s)}\times 7\left(1+\frac{1}{\bar\lambda^{10}(\bar s_2^*)}\right)+4C\int_{\bar s_2}^{\bar s}(\bar{\mathcal N}_{1,\mathrm{loc}}+\bar b^2)\\&\le 28\left(1+\frac{1}{\bar\lambda^{10}(\bar s)}\right)+\delta(\kappa)<30\left(1+\frac{1}{\bar\lambda^{10}(\bar s)}\right).\end{aligned}\tag{4-95}$$

Combining (4-92)–(4-95), we have improved (H1)′, (H2)′ and (H3)′; hence $\bar t^{**}=\bar t^*=+\infty$. This also implies $t^*=+\infty$.

Step 3: Proof of (4-28) and (4-29). Now it is sufficient to prove

$$|\bar b(\bar t)|+\bar{\mathcal N}_2(\bar t)\to 0,\quad \bar\lambda(\bar t)\to\bar\lambda_\infty\in(0,+\infty)$$

as $\bar t\to+\infty$. First of all, from (4-91), we know that

$$\bar s^{**}=\bar s^*=\int_0^{+\infty}\frac{1}{\bar\lambda^3(\tau)}\,d\tau=+\infty.$$

Then we claim that $\bar b_{\bar s}\bar b\in L^1((0,+\infty))$. Indeed, from (2-50), we have

$$|\bar b\bar b_{\bar s}+\bar\omega_{\bar s}G'(\bar\omega)\bar b|\lesssim\bar b^2+\int\bar\varepsilon^2e^{-\frac{|\bar y|}{10}}\in L^1((0,+\infty)).$$

From (2-34), we have

$$\bar{\omega}_{\bar{s}} G'(\bar{\omega})\bar{b} = m\bar{\omega} G'(\bar{\omega})\bar{b}^2 + O\left(\bar{\omega}\left|\bar{b}\left(\frac{\bar{\lambda}_{\bar{s}}}{\bar{\lambda}} + \bar{b}\right)\right|\right) = O\left(\bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}}\right).$$

The above two estimates imply

$$\int_0^{+\infty} |\bar{b}_{\bar{s}}\bar{b}(s')|\,ds' = \int_0^{+\infty} \tfrac{1}{2}|(\bar{b}^2)_{\bar{s}}| < +\infty.$$

Together with

$$\int_0^{+\infty} \bar{b}^2(\bar{s})\,d\bar{s} < +\infty,$$

we conclude that $\bar{b}(\bar{t}) \to 0$ as $\bar{t} \to +\infty$. Next, we use (2-50) again to obtain

$$|\bar{b}_{\bar{s}} + \bar{\omega}_{\bar{s}} G'(\bar{\omega})| \lesssim \bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}} \in L^1((0,+\infty)).$$

Thus, we have

$$\int_0^{+\infty} |(\bar{b} + G(\bar{\omega}))_{\bar{s}}(s')|\,ds' < +\infty.$$

We then know that $b(\bar{t}) + G(\bar{\omega}(\bar{t}))$ has a limit as $\bar{t} \to +\infty$. Since $\lim_{\bar{t}\to+\infty} \bar{b}(\bar{t}) = 0$, we obtain that $G(\bar{\omega}(\bar{t}))$ has a limit as $\bar{t} \to +\infty$. On the other hand, we have $G'(0) > 0$, $\bar{\omega}(\bar{t}) \ll 1$, so there exists a constant $\bar{\omega}_\infty > 0$ such that

$$\lim_{\bar{t}\to+\infty} \bar{\omega}(\bar{t}) = \bar{\omega}_\infty \sim \bar{\gamma}^{\frac{2}{m+2}} (\bar{\ell}^*)^{\frac{m}{m+2}},$$

or equivalently

$$\lim_{\bar{t}\to+\infty} \bar{\lambda}(\bar{t}) = \bar{\lambda}_\infty \sim \left(\frac{c_1\bar{\gamma}}{\bar{\ell}^*}\right)^{\frac{1}{m+2}}.$$

Let

$$\ell^* = \frac{b(t_1^*) + c_1\omega(t_1^*)}{\lambda^2(t_1^*)} > 0.$$

Recall that

$$\bar{\gamma} = \frac{\gamma}{\lambda^m(t_1^*)}, \quad \bar{\ell}^* = b(t_1^*) + c_1\omega(t_1^*), \quad \bar{\lambda}(\bar{t}) = \frac{\lambda(\lambda^3(t_1^*)\bar{t} + t_1^*)}{\lambda(t_1^*)}.$$

We obtain

$$\lim_{t\to+\infty} \lambda(t) = \lambda_\infty \sim \left(\frac{c_1\gamma}{\ell^*}\right)^{\frac{1}{m+2}}. \tag{4-96}$$

Next, the inequality (4-6) implies the existence of a sequence $\bar{s}_n$ such that

$$\overline{\mathcal{N}}_1(\bar{s}_n) \lesssim \int (\bar{\varepsilon}^2(\bar{s}_n) + \bar{\varepsilon}_{\bar{y}}^2(\bar{s}_n))\varphi'_{2,B} \to 0 \quad \text{as } n \to +\infty,$$

where $\lim_{n\to+\infty} \bar{s}_n = +\infty$. Using the monotonicity (4-11), we have

$$\overline{\mathcal{N}}_1(\bar{s}) \to 0 \quad \text{as } \bar{s} \to +\infty.$$

Together with (3-21) and (4-91), we obtain

$$\overline{\mathcal{N}}_2(\bar{t}) \to 0 \quad \text{as } \bar{t} \to +\infty,$$

which implies

$$\mathcal{N}_2(t) \to 0 \quad \text{as } t \to +\infty.$$

Finally, from (2-34), we have

$$\lambda^2(t)x_t(t) \sim 1 \quad \text{as } t \to +\infty,$$

which after integration implies

$$x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as } t \to +\infty.$$

We then conclude the proof of (4-28) and (4-29), and hence the proof of the first part of Proposition 4.2.

IV. Nonemptiness and stability. Now we give the proof of the second part of Proposition 4.2.

First, we show that the soliton and exit regimes are stable under small perturbation in $\mathcal{A}_{\alpha_0}$. From (2-25), we know that the parameters depend continuously on the initial data, which implies that the exit and soliton cases are both open in $\mathcal{A}_{\alpha_0}$, since the separation condition is an open condition of initial data in $\mathcal{A}_{\alpha_0}$.

Indeed, for all $u_0 \in \mathcal{A}_{\alpha_0}$, if the corresponding solution $u(t)$ to (gKdV$_\gamma$) belongs to the soliton regime, we let t_1^* be the separation time introduced in Proposition 4.2. For all $\tilde{u}_0 \in \mathcal{A}_{\alpha_0}$, close enough to u_0, we let $\tilde{u}(t)$ be the corresponding solution to (gKdV$_\gamma$), and $\tilde{b}(t)$, $\tilde{x}(t)$, $\tilde{\lambda}(t)$, $\tilde{\varepsilon}(t)$ be the corresponding geometrical parameters and error term. Then from local theory, we have $\sup_{t\in[0,t_1^*]} \|u(t)-\tilde{u}(t)\|_{H^1} \ll 1$, which together with (2-25), leads to

$$\tilde{b}(t_1^*) + c_1\tilde{\omega}(t_1^*) \geq \tfrac{999}{1000} C^*(\widetilde{\mathcal{N}}_1(t_1^*) + \tilde{b}^2(t_1^*) + \tilde{\omega}^2(t_1^*)).$$

So $\tilde{u}(t)$ must belong to the soliton regime. This implies the openness of soliton regime. The openness of the exit regime follows from the same argument.

Next, we claim that there exists initial data in $\mathcal{A}_{\alpha_0}$ such that the corresponding solution to (gKdV$_\gamma$) belongs to the soliton and exit regimes respectively. First, it is easy to check that the traveling wave solution

$$u(t,x) = Q_\gamma(x-t)$$

belongs to the soliton regime. On the other hand, from (2-43), we can see, in both the soliton and blow-down cases, we have

$$\|u_0\|_{L^2} \geq \|Q\|_{L^2}.$$

Hence, for initial data $u_0 \in \mathcal{A}_{\alpha_0}$ with[16] $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the corresponding solution must belong to the exit regime.

Finally, since the sets of initial data which lead to the soliton and exit regimes are both open and nonempty in $\mathcal{A}_{\alpha_0}$, together with the fact that $\mathcal{A}_{\alpha_0}$ is connected, we conclude that there exists $u_0 \in \mathcal{A}_{\alpha_0}$ such that the corresponding solution to (gKdV$_\gamma$) belongs to the blow-down regime. □

[16] Since we assume that $\gamma \ll \alpha_0$, such u_0 exists in $\mathcal{A}_{\alpha_0}$.

5. Proof of Theorem 1.4

In this part we will use the local Cauchy theory of generalized KdV equations developed in [Kenig, Ponce and Vega 1993] to prove Theorem 1.4.

5A. *H^1 perturbation theory.* First of all, let us introduce the following:

Lemma 5.1 [Kenig, Ponce and Vega 1993]. *The following linear estimates hold*:

(1) *For all $u_0 \in H^1$*,

$$\left\| \frac{\partial}{\partial x} W(t) u_0 \right\|_{L_x^\infty L_t^2(\mathbb{R})} + \|W(t)u_0\|_{L_x^5 L_t^{10}(\mathbb{R})} \lesssim \|u_0\|_{L^2}, \tag{5-1}$$

$$\|D_x^{\alpha_q} D_t^{\beta_q} W(t) u_0\|_{L_x^p L_t^r(I)} \lesssim \|D_x^{s_q} u_0\|_{L^2}, \tag{5-2}$$

where $q > 5$ is the power of the defocusing nonlinear term of (gKdV$_\gamma$), *and*

$$W(t)f = e^{-t\partial_x^3} f, \qquad s_q = \frac{1}{2} - \frac{2}{(q-1)},$$
$$\alpha_q = \frac{1}{10} - \frac{2}{5(q-1)}, \quad \beta_q = \frac{3}{10} - \frac{6}{5(q-1)},$$
$$\frac{1}{p} = \frac{2}{5(q-1)} + \frac{1}{10}, \quad \frac{1}{r} = \frac{3}{10} - \frac{4}{5(q-1)}.$$

(2) *For all well-localized g, we have*

$$\sup_{t\in I} \left\| \frac{\partial}{\partial x} \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^2} \lesssim \|g\|_{L_x^1 L_t^2(I)}, \tag{5-3}$$

$$\left\| \frac{\partial^2}{\partial x^2} \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^\infty L_t^2(I)} \lesssim \|g\|_{L_x^1 L_t^2(I)}, \tag{5-4}$$

$$\left\| \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^5 L_t^{10}(I)} \lesssim \|g\|_{L_x^{5/4} L_t^{10/9}(I)}, \tag{5-5}$$

$$\left\| D_x^{\alpha_q} D_t^{\beta_q} \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^p L_t^r(I)} \lesssim \|g\|_{L_x^{p'} L_t^{r'}(I)}, \tag{5-6}$$

$$\|g\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}} \lesssim \|D_x^{\alpha_q} D_t^{\beta_q} g\|_{L_x^p L_t^r}, \tag{5-7}$$

where

$$1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'}.$$

Proof. See Theorem 3.5, Corollary 3.8, Lemma 3.14, Lemma 3.15 and Corollary 3.16 in [Kenig, Ponce and Vega 1993] for the proofs of (1) and (2). □

Now we define the norms

$$\eta_I^1(w) = \|w\|_{L_x^5 L_t^{10}(I)}, \quad \eta_I^2(w) = \|w_x\|_{L_x^\infty L_t^2(I)}, \quad \eta_I^3(w) = \|D_x^{\alpha_q} D_t^{\beta_q} w\|_{L_x^p L_t^r(I)},$$

$$\Omega_I(w) = \max_{j=1,2} [\eta_I^j(w) + \eta_I^j(w_x)] + \eta_I^3(w),$$

$$\Delta_I(h) = \|h\|_{L_x^1 L_t^2(I)} + \|h_x\|_{L_x^{5/4} L_t^{10/9}(I)} + \|h_x\|_{L_x^1 L_t^2(I)} + \|h_{xx}\|_{L_x^{5/4} L_t^{10/9}(I)} + \|h_x\|_{L_x^{p'} L_t^{r'}(I)}$$

for all interval $I \subset \mathbb{R}$.

Then we have the following:

Proposition 5.2 (modified long-time H^1 perturbation theory). *Let I be an interval containing 0 and $\tilde{u}$ be an H^1 solution to*

$$\begin{cases} \partial_t \tilde{u} + (\partial_{xx}\tilde{u} + \tilde{u}^5 - \gamma \tilde{u}|\tilde{u}|^{q-1})_x = e_x, & (t,x) \in I \times \mathbb{R}, \\ \tilde{u}(0,x) = \tilde{u}_0 \in H^1. \end{cases} \tag{5-8}$$

Suppose we have

$$\sup_{t\in I} \|\tilde{u}(t)\|_{H^1} + \Omega_I(\tilde{u}) \le M$$

for some $M > 0$ independent of γ. Let $u_0 \in H^1$ be such that

$$\|u_0 - \tilde{u}_0\|_{H^1} + \|e\|_{L_x^1 L_t^2(I)} + \|e_x\|_{L_x^{5/4} L_t^{10/9}(I)} + \|e_x\|_{L_x^1 L_t^2(I)} + \|e_{xx}\|_{L_x^{5/4} L_t^{10/9}(I)} + \|e_x\|_{L_x^{p'} L_t^{r'}(I)} \le \varepsilon$$

for some small $0 < \varepsilon < \varepsilon_0(M)$. Then the solution of (gKdV$_\gamma$) *with initial data u_0 satisfies*

$$\sup_{t\in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(u - \tilde{u}) \le C(M)\varepsilon. \tag{5-9}$$

Remark 5.3. The perturbation theory still holds true if we replace H^1 by H^s, with $s \ge \frac{1}{2} - 2/(q-1) > 0$.

Proof of Proposition 5.2. Without loss of generality, we assume that $I = [0, T_0]$ for some $T_0 > 0$.

We first claim the following:

Lemma 5.4 (short-time perturbation theory). *Under the same notation as Proposition 5.2, if we assume in addition that $\Omega_I(\tilde{u}) \le \varepsilon_0$ for some small $0 < \varepsilon_0 = \varepsilon_1(M) \ll 1$, then there exists a constant $C_0(M)$ which depends only on M such that if $0 < \varepsilon < \varepsilon_0 = \varepsilon_1(M)$, then*

$$\sup_{t\in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(u - \tilde{u}) \le C_0(M)\varepsilon. \tag{5-10}$$

We leave the proof of Lemma 5.4 for Appendix B.

Now we turn to the proof of Proposition 5.2. Let $\varepsilon_0 = \varepsilon_1(2M) > 0$ as in Lemma 5.4. We then choose $0 = t_0 < t_1 < \cdots < t_N = T_0$ (recall that we assume $I = [0, T_0]$) such that for all $j = 1, \dots, N$,

$$\Omega_{[t_{j-1}, t_j]}(\tilde{u}) \le \varepsilon_0.$$

From a standard argument, we know that $N = N(M, \varepsilon_0) = N(M) > 0$. We use Lemma 5.4 on each interval $[t_{j-1}, t_j]$ to obtain

$$\sup_{t\in[t_{j-1},t_j]} \|u(t) - \tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1},t_j]}(\tilde{u}) \le C_0(M) \max(\varepsilon, \|u(t_{j-1}) - \tilde{u}(t_{j-1})\|_{H^1}).$$

Arguing by induction, using $\|u(0)-\tilde{u}(0)\|_{H^1} \le \varepsilon$, we have for all $j=1,\dots,N$,

$$\sup_{t\in[t_{j-1},t_j]} \|u(t)-\tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1},t_j]}(\tilde{u}) \le C(j,M)\varepsilon.$$

Summarizing these estimates, we have

$$\begin{aligned}\sup_{t\in I} \|u-\tilde{u}\|_{H^1} + \Omega_I(\tilde{u}) &\le \sum_{j=1}^{N} \sup_{t\in[t_{j-1},t_j]} \|u(t)-\tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1},t_j]}(\tilde{u}) \\ &\le \sum_{j=1}^{N} C(j,M)\varepsilon = C(M)\varepsilon,\end{aligned}$$

which concludes the proof of Proposition 5.2. □

5B. ***End of the proof of Theorem 1.4.*** Now for $0<\gamma \ll \alpha_0 \ll \alpha^* \ll 1$, we choose a $u_0 \in \mathcal{A}_{\alpha_0/2} \subset \mathcal{A}_{\alpha_0}$ such that the corresponding solution $u(t)$ to (gKdV) belongs to the blow-up regime with blow-up time $T<+\infty$. Let $u_\gamma(t)$ be the corresponding solution to (gKdV$_\gamma$). From [Martel, Merle and Raphaël 2014, Section 4.4], we know that there exists a $0<T_1^*<T<+\infty$, geometrical parameters $(\lambda(t), b(t), x(t))$ and an error term $\varepsilon(t)$ such that the following geometrical decomposition holds on $[0,T_1^*]$:

$$u(t,x) = \frac{1}{\lambda(t)^{1/2}}[Q_{b(t)}+\varepsilon(t)]\left(\frac{x-x(t)}{\lambda(t)}\right), \tag{5-11}$$

with

$$(\varepsilon, Q) = (\varepsilon, \Lambda Q) = (\varepsilon, y\Lambda Q) = 0. \tag{5-12}$$

Moreover, we have for all $t \in [0,T_1^*]$

$$\mathcal{N}_2(t) + \|\varepsilon(t)\|_{L^2} + |b(t)| + |1-\lambda(t)| \lesssim \delta(\alpha_0), \tag{5-13}$$

$$\int_{y>0} y^{10}\varepsilon^2(t,y)\,dy \le 5, \tag{5-14}$$

$$b(T_1^*) \ge 2C^*\mathcal{N}_1(T_1^*), \tag{5-15}$$

where C^* is the universal constant[17] introduced in Section 4B. One may easily check that C^* defined by (4-23) is independent of γ.

Next, we claim that there exists a constant $C(u_0,q)>1$ which depends only on u_0 and q such that

$$\sup_{t\in[0,T_1^*]} \|u(t)\|_{H^1} + \Omega_{[0,T_1^*]}(u) + \Delta_{[0,T_1^*]}(u|u|^{q-1}) \le C(u_0,q) < +\infty. \tag{5-16}$$

Indeed, from [Kenig, Ponce and Vega 1993, Corollary 2.11] (taking $s=1$), we have

$$\eta^1_{[0,T_1^*]}(u) + \eta^1_{[0,T_1^*]}(u_x) + \eta^2_{[0,T_1^*]}(u) + \eta^2_{[0,T_1^*]}(u_x) \le C(u_0,q) < +\infty.$$

[17] The constant C^* chosen here might be different from the one in [Martel, Merle and Raphaël 2014, (4.23)]. But we can always replace C^* (both constants in this paper and in [Martel, Merle and Raphaël 2014]) by some larger universal constant.

Then, from Duhamel's principle, we have

$$u(t)=W(t)u_0+\int_0^t (W(t-t')\partial_x(u^5))\,dt'.$$

Together with (5-2), (5-6) and the Gagliardo–Nirenberg inequality introduced in [Bahouri, Chemin and Danchin 2011, Theorem 2.44], we have

$$\begin{aligned}\eta^3_{[0,T_1^*]}(u) &\lesssim \|u_xu^4\|_{L_x^{p'}L_t^{r'}}+\|u_0\|_{H^1}\lesssim \|u\|^4_{L_x^5L_t^{10}}\|u_x\|_{L_x^{p_0}L_t^{r_0}}+\|u_0\|_{H^1}\\ &\lesssim \|u\|^4_{L_x^5L_t^{10}}\|D_x^{s_q}u\|^{1-s_q}_{L_x^5L_t^{10}}\|D_x^{s_q}u_x\|^{s_q}_{L_x^\infty L_t^2}+\|u_0\|_{H^1}\\ &\lesssim \|u\|^4_{L_x^5L_t^{10}}(\|u\|^{1-s_q}_{L_x^5L_t^{10}}\|u_x\|^{s_q}_{L_x^5L_t^{10}})^{1-s_q}(\|u_x\|^{1-s_q}_{L_x^\infty L_t^2}\|u_{xx}\|^{s_q}_{L_x^\infty L_t^2})^{s_q}+\|u_0\|_{H^1}\\ &\lesssim \big(\eta^1_{[0,T_1^*]}(u)+\eta^1_{[0,T_1^*]}(u_x)+\eta^2_{[0,T_1^*]}(u)+\eta^2_{[0,T_1^*]}(u_x)\big)^5+\|u_0\|_{H^1},\end{aligned}$$

where

$$\frac{1}{p_0}=\frac{1}{10}-\frac{2}{5(q-1)},\qquad \frac{1}{r_0}=\frac{3}{10}+\frac{4}{5(q-1)}.$$

This implies $\Omega_{[0,T_1^*]}(u)\le C(u_0,q)<+\infty$.

Next, using the arguments in [Kenig, Ponce and Vega 1993, Section 6], we obtain

$$\Delta_{[0,T_1^*]}(u|u|^{q-1})\lesssim(\Omega_{[0,T_1^*]}(u))^q\le C(u_0,q),$$

which yields (5-16).

Then we apply Proposition 5.2 to $u(t)$ and $u_\gamma(t)$, with $e=\gamma u|u|^{q-1}$. Note that from (5-16), we have

$$\Delta_{[0,T_1^*]}(e)<\gamma C(u_0,q)\le\gamma^{\frac12}\ll\varepsilon_0(C(u_0,q)),$$

provided that $0<\gamma<\gamma(u_0,\alpha_0,\alpha^*,q)\ll 1$. Then Proposition 5.2 implies that, for all $t\in[0,T_1^*]$, we have

$$\|u(t)-u_\gamma(t)\|_{H^1}\lesssim\gamma^{\frac12}. \tag{5-17}$$

Combining with (5-11)–(5-14), we know that $u_\gamma(t)\in\mathcal{T}_{\alpha_0,\gamma}$ for all $t\in[0,T_1^*]$. This allows us to apply Lemma 2.6 to $u_\gamma(t)$ on $[0,T_1^*]$; i.e., there exist geometrical parameters $(b_\gamma(t),\lambda_\gamma(t),x_\gamma(t))$ and an error term $\varepsilon_\gamma(t)$, such that

$$u_\gamma(t,x)=\frac{1}{\lambda_\gamma(t)^{1/2}}[Q_{b_\gamma(t),\omega_\gamma(t)}+\varepsilon_\gamma(t)]\left(\frac{x-x_\gamma(t)}{\lambda_\gamma(t)}\right),$$

with

$$\omega_\gamma(t)=\frac{\gamma}{\lambda_\gamma^m(t)}.$$

Moreover, the orthogonality conditions (2-22) hold.

Now, from Lemma 2.6 and (5-17) we obtain that, for all $t\in[0,T_1^*]$,

$$\left|1-\frac{\lambda(t)}{\lambda_\gamma(t)}\right|+|b_\gamma(t)-b(t)|+|x_\gamma(t)-x(t)|+\|\varepsilon_\gamma(t)-\varepsilon(t)\|_{H^1}\lesssim\delta(\gamma). \tag{5-18}$$

Together with (5-13)–(5-15), we have the following:

(1) For all $t \in [0, T_1^*]$, (4-53)–(4-55) hold for $u_\gamma(t)$.

(2) At the time $t = T_1^*$,

$$b_\gamma(T_1^*) + c_1\omega_\gamma(T_1^*) \geq C^*(\mathcal{N}_{1,\gamma}(T_1^*) + b_\gamma^2(T_1^*) + \omega_\gamma^2(T_1^*)),$$

where

$$\mathcal{N}_{i,\gamma}(t) = \int (\varepsilon_\gamma)_y^2 \psi_B + \varepsilon_\gamma^2 \varphi_{i,B}.$$

By the argument in Section 4, we know that $u_\gamma(t)$ belongs to the soliton regime introduced in Theorem 1.3. Moreover, we also obtain (1-8) from (4-96). This concludes the proof of the first part of Theorem 1.4.

The second part of Theorem 1.4 follows from exactly the same procedure. Thus, we complete the proof of Theorem 1.4.

Appendix A. Proof of the geometrical decomposition

We will give the proof of Lemma 2.6. We first introduce the following notation: for all suitable, $(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v)$

$$F_1(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (Q_{\tilde{\omega}}, \varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}), \tag{A-1}$$

$$F_2(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (\Lambda Q_{\tilde{\omega}}, \varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}), \tag{A-2}$$

$$F_3(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (y\Lambda Q_{\tilde{\omega}}, \varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}), \tag{A-3}$$

where

$$\varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}(y) = \tilde{\lambda}^{\frac{1}{2}} v(\tilde{\lambda} y + \tilde{x}) - Q_{\tilde{b},\tilde{\omega}}(y).$$

We mention here that we don't assume

$$\tilde{\omega} = \frac{\gamma}{\tilde{\lambda}^m}.$$

At $(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (1, 0, 0, 0, Q)$, we have

$$\left(\frac{\partial F_1}{\partial \tilde{\lambda}}, \frac{\partial F_1}{\partial \tilde{x}}, \frac{\partial F_1}{\partial \tilde{b}}\right) = \big((\Lambda Q, Q), (Q', Q), (P, Q)\big),$$

$$\left(\frac{\partial F_2}{\partial \tilde{\lambda}}, \frac{\partial F_2}{\partial \tilde{x}}, \frac{\partial F_2}{\partial \tilde{b}}\right) = \big((\Lambda Q, \Lambda Q), (Q', \Lambda Q), (P, \Lambda Q)\big),$$

$$\left(\frac{\partial F_3}{\partial \tilde{\lambda}}, \frac{\partial F_3}{\partial \tilde{x}}, \frac{\partial F_3}{\partial \tilde{b}}\right) = \big((\Lambda Q, y\Lambda Q), (Q', y\Lambda Q), (P, y\Lambda Q)\big).$$

Since

$$(\Lambda Q, Q) = (Q', Q) = (Q', \Lambda Q) = (\Lambda Q, y\Lambda Q) = 0,$$

$$(P, Q) \neq 0, \quad (\Lambda Q, \Lambda Q) \neq 0, \quad (Q', y\Lambda Q) \neq 0,$$

it is easy to see that the above Jacobian is not degenerate. Hence, from implicit function theory, we have: there exist unique continuous maps

$$(\tilde{\lambda}_0, \tilde{x}_0, \tilde{b}_0) : (\tilde{\omega}, v) \mapsto (1-\delta, 1+\delta) \times (-\delta, \delta) \times (-\delta, \delta), \quad \delta > 0, \tag{A-4}$$

such that for all $\tilde{\omega} \ll 1$, $\|v - Q\|_{H^1} \ll 1$, it holds that

$$F_j\big(\tilde{\lambda}_0(\tilde{\omega}, v), \tilde{x}_0(\tilde{\omega}, v), \tilde{b}_0(\tilde{\omega}, v), \tilde{\omega}, v\big) = 0, \quad j = 1, 2, 3. \tag{A-5}$$

The uniqueness also implies that, for all $\tilde{\omega} \ll 1$, we have

$$\tilde{\lambda}_0(\tilde{\omega}, \mathcal{Q}_{\tilde{\omega}}) \equiv 1. \tag{A-6}$$

Next we fix a time $t \in [0, t_0)$ as in Lemma 2.6. For a solution $u(t)$ to (gKdV$_\gamma$) with

$$u(t, x) = \frac{1}{\lambda_1^{1/2}(t)} [\mathcal{Q}_{\omega_1(t)} + \varepsilon_1(t)] \left(\frac{x - x_1(t)}{\lambda_1(t)} \right),$$

and

$$\omega_1(t) = \frac{\gamma}{\lambda_1^m(t)} \ll 1,$$

we let

$$v(t, \cdot) = \lambda_1^{\frac{1}{2}}(t) u(t, \lambda_1(t) \cdot + x_1(t)) = \mathcal{Q}_{\omega_1(t)}(\cdot) + \varepsilon_1(t, \cdot).$$

Then we have $\|v(t, \cdot) - Q(\cdot)\|_{H^1} \ll 1$.

We claim that there exists a $\underline{\lambda}(t) > 0$ such that

$$\lambda_1(t) \tilde{\lambda}_0 \left(\frac{\gamma}{\underline{\lambda}^m(t)}, v(t) \right) = \underline{\lambda}(t), \quad \frac{\gamma}{\underline{\lambda}^m(t)} \ll 1. \tag{A-7}$$

This is easily verified by implicit function theory. We let

$$M(\underline{\lambda}, v) = \underline{\lambda} - \lambda_1(t) \tilde{\lambda}_0 \left(\frac{\gamma}{\underline{\lambda}^m}, v \right).$$

Then we have

$$M(\lambda_1(t), \mathcal{Q}_{\omega_1(t)}) = 0,$$

$$\frac{\partial M}{\partial \underline{\lambda}} \bigg|_{(\underline{\lambda}, v) = (\lambda_1(t), \mathcal{Q}_{\omega_1(t)})} = 1 + m \omega_1(t) \frac{\partial \tilde{\lambda}_0}{\partial \tilde{\omega}} (\omega_1(t), \mathcal{Q}_{\omega_1(t)}) > 0,$$

which implies (A-7) immediately.

Applying (A-4)–(A-7) to $v(t)$, we have

$$F_j\big(\tilde{\lambda}_0(\omega(t), v(t)), \tilde{x}_0(\omega(t), v(t)), \tilde{b}_0(\omega(t), v(t)), \omega(t), v(t)\big) = 0, \quad j = 1, 2, 3, \tag{A-8}$$

$$\lambda_1(t) \tilde{\lambda}_0(\omega(t), v(t)) = \underline{\lambda}(t), \tag{A-9}$$

where

$$\omega(t) = \frac{\gamma}{\underline{\lambda}^m(t)}.$$

Now, we let

$$\lambda(t)=\underline{\lambda}(t),\quad b(t)=\tilde{b}_0(\omega(t),v(t)),\quad x(t)=x_1(t)+\lambda_1(t)\tilde{x}_0(\omega(t),v(t)), \tag{A-10}$$

$$\omega(t)=\frac{\gamma}{\lambda^m(t)},\quad \varepsilon(t,y)=\lambda^{\frac{1}{2}}(t)u(t,\lambda(t)\,\cdot\,+x(t))-Q_{b(t),\omega(t)}. \tag{A-11}$$

We claim that this $(\lambda(t),x(t),b(t))$ satisfies the orthogonality conditions (2-22). Indeed, from (A-7)–(A-9), we have

$$\begin{aligned}
0&=F_1\big(\tilde{\lambda}_0(\omega(t),v(t)),\tilde{x}_0(\omega(t),v(t)),\tilde{b}_0(\omega(t),v(t)),\omega(t),v(t)\big)\\
&=\Big(\mathcal{Q}_{\omega(t)}(\cdot),\ \tilde{\lambda}_0^{\frac{1}{2}}(\omega(t),v(t))v\big(t,\tilde{\lambda}_0(\omega(t),v(t))\cdot+\tilde{x}_0(\omega(t),v(t))\big)-Q_{b(t),\omega(t)}(\cdot)\Big)\\
&=\Big(\mathcal{Q}_{\omega(t)}(\cdot),\ [\lambda_1(t)\tilde{\lambda}_0(\omega(t),v(t))]^{\frac{1}{2}}\\
&\qquad\qquad\times u\big(t,\lambda_1(t)[\tilde{\lambda}_0(\omega(t),v(t))\cdot+\tilde{x}_0(\omega(t),v(t))]+x_1(t)\big)-Q_{b(t),\omega(t)}(\cdot)\Big)\\
&=\big(\mathcal{Q}_{\omega(t)}(\cdot),\ \lambda^{\frac{1}{2}}(t)u(t,\lambda(t)\cdot+x(t))-Q_{b(t),\omega(t)}(\cdot)\big)\\
&=(\mathcal{Q}_{\omega(t)},\varepsilon(t)).
\end{aligned}$$

The other two orthogonality conditions can be verified similarly.

Finally, since the maps

$$(\tilde{\lambda}_0,\tilde{x}_0,\tilde{b}_0):\ (\tilde{\omega},v)\mapsto(1-\delta,1+\delta)\times(-\delta,\delta)\times(-\delta,\delta)$$

are continuous, the remaining part of Lemma 2.6 follows immediately.

Appendix B. Proof of Lemma 5.4

We give the proof of the modified short-time perturbation theory, i.e., Lemma 5.4.

First, we let $v(t,x)=u(t,x)-\tilde{u}(t,x),\ S(t)=\Omega_{[0,t]}(v)$. We claim the following estimate holds true for all $t\in I$:

$$S(t)\lesssim_M \varepsilon+S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde{u})^4+\Omega_I(\tilde{u})^{q-1}). \tag{B-1}$$

Since $S(0)=0$ and $\Omega_I(\tilde{u})\le\varepsilon_0$, we know that Lemma 5.4 follows from a standard bootstrap argument. Now it only remains to prove (B-1).

First, by Duhamel's principle, we have

$$\begin{aligned}
v(t)&=W(t)(\tilde{u}_0-u_0)+\int_0^t\big(W(t-t')\partial_x[\tilde{u}^5-\gamma\tilde{u}|\tilde{u}|^{q-1}-(\tilde{u}+v)^5+\gamma(\tilde{u}+v)|\tilde{u}+v|^{q-1}-e]\big)\,dt'\\
&=v_L(t)+v_N(t).
\end{aligned}$$

For the linear part v_L, from Lemma 5.1, we have

$$\Omega_{[0,t]}(v_L)+\sup_{t'\in[0,t]}\|v_L\|_{H^1}\lesssim\|\tilde{u}_0-u_0\|_{H^1}\lesssim\varepsilon. \tag{B-2}$$

Now, for the nonlinear part v_N, we use Lemma 5.1 to estimate

$$\eta^1_{[0,t]}(v_N) \lesssim \|e_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \|(v+\tilde{u})^4(v+\tilde{u})_x - \tilde{u}^4\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \| |v+\tilde{u}|^{q-1}(v+\tilde{u})_x - |\tilde{u}|^{q-1}\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])}.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\|(v+\tilde{u})^4(v+\tilde{u})_x - \tilde{u}^4\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} \\
&\quad\lesssim \|((v+\tilde{u})^4-\tilde{u}^4)\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}} + \|(v+\tilde{u})^4 v_x\|_{L_x^{5/4}L_t^{10/9}} \\
&\quad\lesssim (\|\tilde{u}\|^3_{L_x^5L_t^{10}} + \|v\|^3_{L_x^5L_t^{10}})\|v\|_{L_x^5L_t^{10}}\|\tilde{u}_x\|_{L_x^\infty L_t^2} + \|v\|^4_{L_x^5L_t^{10}}(\|v_x\|_{L_x^\infty L_t^2} + \|\tilde{u}_x\|_{L_x^\infty L_t^2}) \\
&\quad\lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\| |v+\tilde{u}|^{q-1}(v+\tilde{u})_x - |\tilde{u}|^{q-1}\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} \\
&\quad\lesssim \|(|v+\tilde{u}|^{q-1} - |\tilde{u}|^{q-1})\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}} + \| |v+\tilde{u}|^{q-1}v_x\|_{L_x^{5/4}L_t^{10/9}} \\
&\quad\lesssim (\|\tilde{u}\|^{q-2}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}} + \|v\|^{q-2}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\|v\|_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}\|\tilde{u}_x\|_{L_x^\infty L_t^2} \\
&\qquad\qquad + \|v\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}(\|v_x\|_{L_x^\infty L_t^2} + \|\tilde{u}_x\|_{L_x^\infty L_t^2}) \\
&\quad\lesssim (\|D_x^{\alpha_q}D_t^{\beta_q}\tilde{u}\|^{q-2}_{L_x^pL_t^r} + \|D_x^{\alpha_q}D_t^{\beta_q}v\|^{q-2}_{L_x^pL_t^r})\|D_x^{\alpha_q}D_t^{\beta_q}v\|_{L_x^pL_t^r}\|\tilde{u}_x\|_{L_x^\infty L_t^2} \\
&\qquad\qquad + \|D_x^{\alpha_q}D_t^{\beta_q}v\|^{q-1}_{L_x^pL_t^r}(\|v_x\|_{L_x^\infty L_t^2} + \|\tilde{u}_x\|_{L_x^\infty L_t^2}) \\
&\quad\lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),
\end{aligned}$$

where we used (5-7) for the last two inequalities. The above two estimates imply

$$\eta^1_{[0,t]}(v_N) \lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon. \tag{B-3}$$

Similarly, we have

$$\eta^1_{[0,t]}(\partial_x v_N) \lesssim \|e_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \|((v+\tilde{u})^5 - \tilde{u}^5)_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \|((v+\tilde{u})|v+\tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1})_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])}.$$

By Hölder's inequality again, we have

$$\begin{aligned}
&\|((v+\tilde{u})^5 - \tilde{u}^5)_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])} \\
&\quad\lesssim \|(v+\tilde{u})^4 v_{xx}\|_{L_x^{5/4}L_t^{10/9}} + \|(v+\tilde{u})^3(v_x + 2\tilde{u}_x)v_x\|_{L_x^{5/4}L_t^{10/9}} \\
&\quad\lesssim (\|\tilde{u}\|^4_{L_x^5L_t^{10}} + \|v\|^4_{L_x^5L_t^{10}})\|v_{xx}\|_{L_x^\infty L_t^2} + \|v\|^3_{L_x^5L_t^{10}}\|v_x\|_{L_x^\infty L_t^2}(\|v_x\|_{L_x^5L_t^{10}} + \|\tilde{u}_x\|_{L_x^5L_t^{10}}) \\
&\quad\lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\|((v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1})_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])}\\
&\quad\lesssim \||v+\tilde u|^{q-1}v_{xx}\|_{L_x^{5/4}L_t^{10/9}}+\||v+\tilde u|^{q-2}(v_x+2\tilde u_x)v_x\|_{L_x^{5/4}L_t^{10/9}}\\
&\quad\lesssim (\|\tilde u\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}+\|v\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\|v_{xx}\|_{L_x^\infty L_t^2}\\
&\qquad\qquad+\|v\|^{q-2}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}\|v_x\|_{L_x^\infty L_t^2}(\|v_x+2\tilde u_x\|_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\\
&\quad\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1}).
\end{aligned}$$

Collecting these estimates, we have

$$\eta^1_{[0,t]}(\partial_x v_N)\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1})+\varepsilon. \tag{B-4}$$

Next, using a similar strategy, we have

$$\begin{aligned}
\eta^2_{[0,t]}(v_N)&\lesssim \|e\|_{L_x^1L_t^2([0,t])}+\|(v+\tilde u)^5-\tilde u^5\|_{L_x^1L_t^2([0,t])}+\|(v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1}\|_{L_x^1L_t^2([0,t])}\\
&\lesssim \varepsilon+(\|\tilde u\|^4_{L_x^5L_t^{10}}+\|v\|^4_{L_x^5L_t^{10}})\|v\|_{L_x^5L_t^{10}}\\
&\qquad+(\|v\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}+\|\tilde u\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\|v\|_{L_x^5L_t^{10}}\\
&\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1})+\varepsilon,
\end{aligned}\tag{B-5}$$

and

$$\begin{aligned}
\eta^2_{[0,t]}(\partial_x v_N)&\lesssim \|e_x\|_{L_x^1L_t^2([0,t])}+\|((v+\tilde u)^5-\tilde u^5)_x\|_{L_x^1L_t^2([0,t])}\\
&\qquad+\|((v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1})_x\|_{L_x^1L_t^2([0,t])}\\
&\lesssim \varepsilon+\|((v+\tilde u)^4-\tilde u^4)\tilde u_x\|_{L_x^1L_t^2([0,t])}+\|(v+\tilde u)^4v_x\|_{L_x^1L_t^2([0,t])}\\
&\qquad+\|(|v+\tilde u|^{q-1}-|\tilde u|^{q-1})\tilde u_x\|_{L_x^1L_t^2([0,t])}+\||v+\tilde u|^{q-1}v_x\|_{L_x^1L_t^2([0,t])}\\
&\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1})+\varepsilon.
\end{aligned}\tag{B-6}$$

Finally, we need to estimate $\eta^3_{[0,t]}(v_N)$. From Lemma 5.1, we have

$$\begin{aligned}
\eta^3_{[0,t]}(v_N)&\lesssim \|e_x\|_{L_x^{p'}L_t^{r'}([0,t])}+\|((v+\tilde u)^5-\tilde u^5)_x\|_{L_x^{p'}L_t^{r'}([0,t])}\\
&\qquad+\|((v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1})_x\|_{L_x^{p'}L_t^{r'}([0,t])}\\
&\lesssim \varepsilon+\|((v+\tilde u)^4-\tilde u^4)\tilde u_x\|_{L_x^{p'}L_t^{r'}}+\|(v+\tilde u)^4v_x\|_{L_x^{p'}L_t^{r'}}\\
&\qquad+\|(|v+\tilde u|^{q-1}-|\tilde u|^{q-1})\tilde u_x\|_{L_x^{p'}L_t^{r'}}+\||v+\tilde u|^{q-1}v_x\|_{L_x^{p'}L_t^{r'}}.
\end{aligned}$$

By similar technique to that used for (B-6), we have

$$\begin{aligned}
&\|((v+\tilde u)^4-\tilde u^4)\tilde u_x\|_{L_x^{p'}L_t^{r'}}+\|(v+\tilde u)^4v_x\|_{L_x^{p'}L_t^{r'}}\\
&\qquad\lesssim \|(v+\tilde u)^4-\tilde u^4\|_{L_x^{5/4}L_t^{5/2}}\|\tilde u_x\|_{L_x^{p_0}L_t^{r_0}}+\|(v+\tilde u)^4\|_{L_x^{5/4}L_t^{5/2}}\|v_x\|_{L_x^{p_0}L_t^{r_0}}\\
&\qquad\lesssim \|v_x\|_{L_x^{p_0}L_t^{r_0}}(S(t)^4+\Omega_{[0,t]}(\tilde u)^4)+\|\tilde u_x\|_{L_x^{p_0}L_t^{r_0}}S(t)(S(t)^3+\Omega_{[0,t]}(\tilde u)^3),
\end{aligned}$$

and

$$\|(|v+\tilde{u}|^{q-1}-|\tilde{u}|^{q-1})\tilde{u}_x\|_{L_x^{p'}L_t^{r'}}+\||v+\tilde{u}|^{q-1}v_x\|_{L_x^{p'}L_t^{r'}}$$
$$\lesssim \|v_x\|_{L_x^{p_0}L_t^{r_0}}(S(t)^{q-1}+\Omega_{[0,t]}(\tilde{u})^{q-1})+\|\tilde{u}_x\|_{L_x^{p_0}L_t^{r_0}}S(t)(S(t)^{q-2}+\Omega_{[0,t]}(\tilde{u})^{q-2}),$$

where

$$\frac{1}{p_0}=\frac{1}{10}-\frac{2}{5(q-1)},\quad \frac{1}{r_0}=\frac{3}{10}+\frac{4}{5(q-1)}.$$

By the Gagliardo–Nirenberg inequality introduced in [Bahouri, Chemin and Danchin 2011, Theorem 2.44], we have

$$\|v_x\|_{L_x^{p_0}L_t^{r_0}}\lesssim \|D_x^{s_q}v\|_{L_x^5L_t^{10}}^{1-s_q}\|D_x^{s_q}v_x\|_{L_x^\infty L_t^2}^{s_q}$$
$$\lesssim (\|v\|_{L_x^5L_t^{10}}^{1-s_q}\|v_x\|_{L_x^5L_t^{10}}^{s_q})^{1-s_q}(\|v_x\|_{L_x^\infty L_t^2}^{1-s_q}\|v_{xx}\|_{L_x^\infty L_t^2}^{s_q})^{s_q}\lesssim S(t).$$

Similarly, we have

$$\|\tilde{u}_x\|_{L_x^{p_0}L_t^{r_0}}\lesssim \Omega_{[0,t]}(\tilde{u});$$

hence

$$\eta_{[0,t]}^3(v_N)\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde{u})^4+\Omega_I(\tilde{u})^{q-1})+\varepsilon. \tag{B-7}$$

Combining (B-2)–(B-7), we conclude the proof of (B-1), and hence the proof of Lemma 5.4.

Acknowledgement

The author would like to thank his supervisors F. Merle and T. Duyckaerts for having suggested this problem to him and giving a lot of guidance.

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Received 26 Nov 2016. Revised 31 Aug 2017. Accepted 19 Apr 2018.

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ANALYSIS AND PDE
Vol. 12, No. 1, 2019
dx.doi.org/10.2140/apde.2019.12.113

ON THE STABILITY OF TYPE II BLOWUP FOR THE 1-COROTATIONAL ENERGY-SUPERCRITICAL HARMONIC HEAT FLOW

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We consider the energy-supercritical harmonic heat flow from $\mathbb{R}^d$ into the d-sphere $\mathbb{S}^d$ with $d \geq 7$. Under an additional assumption of 1-corotational symmetry, the problem reduces to the one-dimensional semilinear heat equation

$$\partial_t u = \partial_r^2 u + \frac{(d-1)}{r}\partial_r u - \frac{(d-1)}{2r^2}\sin(2u).$$

We construct for this equation a family of $\mathcal{C}^\infty$ solutions which blow up in finite time via concentration of the universal profile

$$u(r,t) \sim Q\left(\frac{r}{\lambda(t)}\right),$$

where Q is the stationary solution of the equation and the speed is given by the quantized rates

$$\lambda(t) \sim c_u (T-t)^{\frac{\ell}{\gamma}}, \quad \ell \in \mathbb{N}^*,\ 2\ell > \gamma = \gamma(d) \in (1,2].$$

The construction relies on two arguments: the reduction of the problem to a finite-dimensional one thanks to a robust universal energy method and modulation techniques developed by Merle, Raphaël and Rodnianski (*Camb. J. Math.* **3**:4 (2015), 439–617) for the energy supercritical nonlinear Schrödinger equation and by Raphaël and Schweyer (*Anal. PDE* **7**:8 (2014), 1713–1805) for the energy critical harmonic heat flow. Then we proceed by contradiction to solve the finite-dimensional problem and conclude using the Brouwer fixed-point theorem. Moreover, our constructed solutions are in fact $(\ell-1)$-codimension stable under perturbations of the initial data. As a consequence, the case $\ell = 1$ corresponds to a stable type II blowup regime.

1. Introduction

We consider the harmonic map heat flow which is defined as the negative gradient flow of the Dirichlet energy of maps between manifolds. Indeed, if Φ is a map from $\mathbb{R}^d \times [0,T)$ to a compact Riemannian

Ibrahim is partially supported by NSERC Discovery Grant # 371637-2014.
MSC2010: primary 35B40, 35K50; secondary 35K55, 35K57.
Keywords: harmonic heat flow, blowup, stability, differential geometry.

manifold $\mathcal{M} \subset \mathbb{R}^n$, with second fundamental form Υ, then Φ solves

$$\begin{cases} \partial_t \Phi - \Delta\Phi = \Upsilon(\Phi)(\nabla\Phi, \nabla\Phi), \\ \Phi(t=0) = \Phi_0. \end{cases} \tag{1-1}$$

We assume that the target manifold is the d-sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. Then, (1-1) becomes

$$\begin{cases} \partial_t \Phi - \Delta\Phi = |\nabla\Phi|^2\Phi, \\ \Phi(t=0) = \Phi_0. \end{cases} \tag{1-2}$$

We will study the problem (1-2) under an additional assumption of 1-corotational symmetry, namely that a solution of (1-2) takes the form

$$\Phi(x,t) = \begin{pmatrix} \cos(u(|x|,t)) \\ (x/|x|)\sin(u(|x|,t)) \end{pmatrix}. \tag{1-3}$$

Under this ansatz, the problem (1-2) reduces to the one-dimensional semilinear heat equation

$$\begin{cases} \partial_t u = \partial_r^2 u + \dfrac{(d-1)}{r}\partial_r u - \dfrac{(d-1)}{2r^2}\sin(2u), \\ u(t=0) = u_0, \end{cases} \tag{1-4}$$

where $u(t) : r \in \mathbb{R}_+ \to u(r,t) \in [0,\pi]$. The set of solutions to (1-4) is invariant by the scaling symmetry

$$u_\lambda(r,t) = u\left(\frac{r}{\lambda}, \frac{t}{\lambda^2}\right) \quad \text{for all } \lambda > 0.$$

The energy associated to (1-4) is given by

$$\mathcal{E}[u](t) = \int_0^{+\infty} \left(|\partial_r u|^2 + \frac{(d-1)}{r^2}\sin^2(u)\right) r^{d-1}\,dr, \tag{1-5}$$

which satisfies

$$\mathcal{E}[u_\lambda] = \lambda^{d-2}\mathcal{E}[u].$$

The criticality of the problem is reflected by the fact that the energy (1-5) is left invariant by the scaling property when $d = 2$; hence, the case $d \geq 3$ corresponds to the energy-supercritical case.

The problem (1-4) is locally well-posed for data which are close in L^∞ to a uniformly continuous map, see [Koch and Lamm 2012], or in BMO, see [Wang 2011]. Actually, Eells and Sampson [1964] introduced the harmonic map heat flow as a process to deform any smooth map Φ_0 into a harmonic map via (1-2). They also proved that the solution exists globally if the sectional curvature of the target manifold is negative. There exist other assumptions for the global existence; for example, assuming the image of the initial data u_0 is contained in a ball of radius $\pi/(2\sqrt{\kappa})$, where κ is an upper bound on the sectional curvature of the target manifold $\mathcal{M}$; see [Jost 1981; Lin and Wang 2008]. Without these assumptions, the solution $u(r,t)$ may develop singularities in some finite time; see, for example, [Coron and Ghidaglia 1989; Chen and Ding 1990] for $d \geq 3$, and [Chang, Ding and Ye 1992] for $d = 2$. In this case, we say that $u(r,t)$ blows up in a finite time $T < +\infty$ in the sense that

$$\lim_{t\to T} \|\nabla u(t)\|_{L^\infty} = +\infty.$$

Here we call T the blowup time of $u(x,t)$. The blowup has been divided by Struwe [1996] into two types:

$$u \text{ blows up with type I if } \limsup_{t\to T}(T-t)^{\frac{1}{2}}\|\nabla u(t)\|_{L^\infty} < +\infty,$$
$$u \text{ blows up with type II if } \limsup_{t\to T}(T-t)^{\frac{1}{2}}\|\nabla u(t)\|_{L^\infty} = +\infty.$$

Struwe [1988] showed that the type I singularities are asymptotically self-similar; that is, their profile is given by a smooth shrinking function

$$u(r,t) = \phi\left(\frac{r}{\sqrt{T-t}}\right) \quad \text{for all } t \in [0,T),$$

where ϕ solves the equation

$$\phi'' + \left(\frac{d-1}{y} + \frac{y}{2}\right)\phi' - \frac{d-1}{2y^2}\sin(2\phi) = 0. \tag{1-6}$$

Thus, the study of type I blowup reduces to the study of nonconstant solutions of (1-6).

When $3 \le d \le 6$, by using a shooting method, Fan [1999] proved that there exists an infinite sequence of globally regular solutions ϕ_n of (1-6) which are called “shrinkers” (corresponding to the existence of type I blowup solutions of (1-4)), where the integer index n denotes the number of intersections of the function ϕ_n with $\frac{\pi}{2}$. More detailed quantitative properties of such solutions were studied in [Biernat and Bizoń 2011], where the authors conjectured that ϕ_1 is linear stable and provided numerical evidence supporting that ϕ_1 corresponds to a generic profile of type I blowup. Very recently, Biernat, Donninger and Schörkhuber [2016] proved the existence of a stable self-similar blowup solution for $d = 3$. Since (1-2) is not time-reversible, there exists another family of self-similar solutions called “expanders”, which were introduced in [Germain and Rupflin 2011]. These expanders have been recently proved to be nonlinearly stable in [Germain, Ghoul and Miura 2017]. To our knowledge, the question on the existence of type II blowup solutions for (1-4) remains open for $3 \le d \le 6$.

When $d \ge 7$, Bizoń and Wasserman [2015] proved that (1-4) has no self-similar shrinking solutions. According to [Struwe 1988], this result implies that in dimensions $d \ge 7$, all singularities for (1-4) must be of type II (see also [Biernat 2015] for a recent analysis of such singularities). Recently, Biernat and Seki [2016], via the matched asymptotic method developed in [Herrero and Velázquez 1994], constructed for (1-4) a countable family of type II blowup solutions, each characterized by a different blowup rate:

$$\lambda(t) \sim (T-t)^{\frac{\ell}{\gamma}} \quad \text{as } t \to T, \tag{1-7}$$

where $\ell \in \mathbb{N}^*$ such that $2\ell > \gamma$ and $\gamma = \gamma(d)$ is given by

$$\gamma(d) = \tfrac{1}{2}(d-2-\tilde{\gamma}) \in (1,2] \quad \text{for } d \ge 7, \tag{1-8}$$

where $\tilde{\gamma} = \sqrt{d^2-8d+8}$. The blowup rate (1-7) is in fact driven by the asymptotic behavior of a stationary solution of (1-4), say Q, which is the unique (up to scaling) solution of the equation

$$Q'' + \frac{(d-1)}{r}Q' - \frac{(d-1)}{2r^2}\sin(2Q) = 0, \quad Q(0) = 0, \;\; Q'(0) = 1, \tag{1-9}$$

and admits the behavior for r large

$$Q(r) = \frac{\pi}{2} - \frac{a_0}{r^\gamma} + \mathcal{O}\left(\frac{1}{r^{2+\gamma}}\right) \quad \text{for some } a_0 = a_0(d) > 0, \tag{1-10}$$

(see the Appendix in [Biernat 2015] for a proof of the existence of Q). Note that the case $2\ell = \gamma$ only happens in dimension $d = 7$. In this case, Biernat [2015] used the method of [Herrero and Velázquez 1994] and formally derived the blowup rate

$$\lambda(t) \sim \frac{(T-t)^{\frac{1}{2}}}{|\log(T-t)|} \quad \text{as } t \to T. \tag{1-11}$$

He also provided numerical evidence supporting that the case $\ell = 1$ in (1-7) or (1-11) corresponds to a generic blowup solution.

In the energy-critical case, i.e., $d = 2$, van den Berg, Hulshof and King [2003], through a formal analysis based on the matched asymptotic technique of [Herrero and Velázquez 1994], predicted that there are type II blowup solutions to (1-4) of the form

$$u(r,t) \sim Q\left(\frac{r}{\lambda(t)}\right),$$

where

$$Q(r) = 2\tan^{-1}(r) \tag{1-12}$$

is the unique (up to scaling) solution of (1-9), and the blowup speed is governed by the quantized rates:

$$\lambda(t) \sim \frac{(T-t)^\ell}{|\log(T-t)|^{\frac{2\ell}{2\ell-1}}} \qquad \text{for } \ell \in \mathbb{N}^*.$$

This result was later confirmed by Raphaël and Schweyer [2014b]. Note that the case $\ell = 1$ was treated in [Raphaël and Schweyer 2013] and corresponds to a stable blowup. In particular, in those papers, they adapted the strategy developed in [Raphaël and Rodnianski 2012; Merle, Raphaël and Rodnianski 2011] for the study of wave and Schrödinger maps to construct for (1-4) type II blowup solutions. Their method relies on a two-step procedure:

- Construction of a suitable approximate blowup profile through iterated resolutions of elliptic equations. The *tail computation* allows us to formally derive the blowup speed. As a matter of fact, the asymptotic behavior at infinity of the stationary solution (1-12) is an essential algebraic fact for their analysis, which in fact drives the derivation of the blowup law and the possibility of a blowup solution with Q profile.
- Implementation of a robust universal energy method to control the solution in the blowup regime through the derivation of suitable "Lyapunov" functionals involving critical Sobolev norms adapted to the linearized flow near the ground state, which relies on neither spectral estimates nor the maximum principle and may be easily applied to various settings.

In this work, by considering $d \geq 7$, we ask whether we can carry out the analysis of [Raphaël and Schweyer 2014b] for the energy-critical case $d = 2$ to the construction of blowup solutions for (1-4) in the case $d \geq 7$. It happens that the asymptotic behavior (1-10) is perfectly suitable to replace the explicit profile (1-12) for an implementation of the strategy of [Raphaël and Schweyer 2014b]. The following theorem is the main result of this paper.

Theorem 1.1 (existence of type II blowup solutions to (1-4) with prescribed behavior). *Let $d \geq 7$ and γ be defined as in* (1-8), *we fix an integer*

$$\ell \in \mathbb{N}^* \quad \textit{such that} \quad 2\ell > \gamma,$$

and an arbitrary Sobolev exponent

$$\mathfrak{s} \in \mathbb{N}, \qquad \mathfrak{s} = \mathfrak{s}(\ell) \to +\infty \quad \textit{as } \ell \to +\infty.$$

Then there exists a smooth corotational radially symmetric initial data u_0 such that the corresponding solution to (1-4) *is of the form*

$$u(r,t) = Q\left(\frac{r}{\lambda(t)}\right) + q\left(\frac{r}{\lambda(t)}, t\right), \tag{1-13}$$

where

$$\lambda(t) = c(u_0)(T-t)^{\frac{\ell}{\gamma}}(1 + o_{t\to T}(1)), \quad c(u_0) > 0, \tag{1-14}$$

and

$$\lim_{t\to T} \|\nabla^\sigma q(t)\|_{L^2} = 0 \quad \textit{for all } \sigma \in \left(\tfrac{d}{2} + 3, \mathfrak{s}\right]. \tag{1-15}$$

Moreover, the case $\ell = 1$ corresponds to a stable blowup regime.

Remark 1.2. Since $\gamma = 2$ for $d = 7$ and $\gamma \in (1,2)$ for $d \geq 8$, the condition $2\ell > \gamma$ means that $\ell \geq 2$ for $d = 7$ and $\ell \geq 1$ for $d \geq 8$. Note that the condition $2\ell > \gamma$ allows us to avoid the presence of logarithmic corrections in the construction of the approximate profile. In other words, the case $2\ell = \gamma$ (equivalent to $\ell = 1$ and $d = 7$) would involve an additional logarithmic gain related to the growth of the approximate profile at infinity, which turns out to be essential for the derivation of the speed (1-11). Although our analysis could be naturally extended to this case ($\ell = 1$ and $d = 7$) with some complicated computations, we hope to treat this case in a separate work.

Remark 1.3. The quantization of the blowup rate (1-14) is the same as the one obtained in [Biernat and Seki 2016]. Note that in that paper, they only claim the existence result of a type II blowup solution with the rate (1-14) and say nothing about the dynamical description of the solution. On the contrary, our result shows that the constructed solution blows up in finite time by concentration of a stationary state in the supercritical regime. Moreover, our constructed solution is in fact ($\ell-1$)-codimension stable in the sense that we will precise shortly.

Remark 1.4. Fix $\ell \in \mathbb{N}^*$ such that $2\ell > \gamma$, an integer $L \gg \ell$ and $\mathfrak{s} \sim L \gg 1$. Then our initial data is of the form

$$u_0 = Q_{b(0)} + \varepsilon_0, \tag{1-16}$$

where Q_b is a deformation of the ground state Q and $b = (b_1, \dots, b_L)$ corresponds to possible unstable directions of the flow in the $\dot{H}^{\mathfrak{s}}$ topology in a suitable neighborhood of Q. We will show that for all $\varepsilon_0 \in \dot{H}^\sigma \cap \dot{H}^\mathfrak{s}$ $\left(\text{for some } \sigma = \sigma(d) > \frac{d}{2}\right)$ small enough, for all $(b_1(0), b_{\ell+1}(0), \dots, b_L(0))$ small enough, there exists a choice of unstable directions $(b_2(0), \dots, b_\ell(0))$ such that the solution of (1-4) with the data (1-16) satisfies the conclusion of Theorem 1.1. This implies that our constructed solution is $(\ell-1)$-codimension stable. In other words, the case $\ell = 1$ corresponds to a stable type II blowup regime, which is in agreement with numerical evidence given in [Biernat 2015].

Remark 1.5. The harmonic heat flow shares many features with the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u \quad \text{in } \mathbb{R}^d. \tag{1-17}$$

Two important critical exponents appear when considering the dynamics of (1-17):

$$p_S = \frac{d+2}{d-2} \quad \text{and} \quad p_{JL} = \begin{cases} +\infty & \text{for } d \leq 10, \\ 1 + \dfrac{4}{d-4-2\sqrt{d-1}} & \text{for } d \geq 11 \end{cases}$$

correspond to the cases $d = 2$ and $d = 7$ in the study of (1-4) respectively.

When $1 < p \leq p_S$, Giga and Kohn [1987] and Giga, Matsui and Sasayama [2004] showed that all blowup solutions are of type I. Here the type I blowup means that

$$\limsup_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty} < +\infty;$$

otherwise we say the blowup solution is of type II.

When $p = p_S$, Filippas, Herrero and Velázquez [2000] formally constructed for (1-17) type II blowup solutions in dimensions $3 \leq d \leq 6$; however, they could not do the same in dimensions $d \geq 7$. This formal result is partly confirmed by Schweyer [2012] in dimension $d = 4$. Interestingly, Collot, Merle and Raphaël [2017] showed that type II blowup is ruled out in dimension $d \geq 7$ near the solitary wave.

When $p_S < p < p_{JL}$, Matano and Merle [2004], see also [Mizoguchi 2004], proved that only type I blowup occurs in the radial setting.

When $p > p_{JL}$, Herrero and Velázquez [1994] formally derived the existence of type II blowup solutions with the quantized rates

$$\|u(t)\|_{L^\infty} \sim (T-t)^{\frac{2\ell}{(p-1)\alpha(d,p)}}, \quad \ell \in \mathbb{N}, \; 2\ell > \alpha.$$

The formal result was clarified in [Matano and Merle 2009; Mizoguchi 2007; Collot 2017]. The collection of these works yields a complete classification of the type II blowup scenario for the radially symmetric energy-supercritical case.

In comparison to the case of the semilinear heat equation (1-17), it might be possible to prove that all blowup solutions to (1-4) are of type I in dimensions $3 \leq d \leq 6$. However, due to the lack of monotonicity of the nonlinear term, the analysis of the harmonic heat flow (1-4) is much more difficult than the case of the semilinear heat equation (1-17) treated in [Matano and Merle 2004].

Let us briefly explain the main steps of the proof of Theorem 1.1, which follows the method of [Raphaël and Schweyer 2014b] treated for the critical case $d = 2$. This kind of method has been successfully applied for various nonlinear evolution equations, in particular in the dispersive setting for the nonlinear Schrödinger equation both in the mass-critical [Merle and Raphael 2005a; 2005b; 2004; 2003] and mass-supercritical [Merle, Raphaël and Rodnianski 2015] cases, the mass-critical gKdV equation [Martel, Merle and Raphaël 2015a; 2015b; 2014], the energy-critical [Duyckaerts, Kenig and Merle 2013; Hillairet and Raphaël 2012] and energy-supercritical [Collot 2018] wave equation, the two-dimensional critical geometric equations, the wave maps [Raphaël and Rodnianski 2012], the Schrödinger maps [Merle, Raphaël and Rodnianski 2013] and the harmonic heat flow [Raphaël and Schweyer 2013; 2014b], the semilinear heat equation (1-17) in the energy-critical [Schweyer 2012] and energy-supercritical [Collot 2017] cases, and the two-dimensional Keller–Segel model [Raphaël and Schweyer 2014a; Ghoul and Masmoudi 2016]. In all these works, the method relies on two arguments:

- Reduction of an infinite-dimensional problem to a finite-dimensional one, through the derivation of suitable Lyapunov functionals and the robust energy method as mentioned in the two-step procedure above.
- The control of the finite-dimensional problem thanks to a topological argument based on index theory.

Note that this kind of topological argument has proved to be successful also for the construction of type I blowup solutions for the semilinear heat equation (1-17) in [Bricmont and Kupiainen 1994; Merle and Zaag 1997; Nguyen and Zaag 2017] (see also [Nguyen and Zaag 2016] for the case of logarithmic perturbations, [Bressan 1990; 1992; Ghoul, Nguyen and Zaag 2017] for the exponential source and [Nouaili and Zaag 2015] for the complex-valued case), the Ginzburg–Landau equation in [Masmoudi and Zaag 2008] (see also [Zaag 1998] for an earlier work), a nonvariational parabolic system in [Ghoul, Nguyen and Zaag 2018] and the semilinear wave equation in [Côte and Zaag 2013].

For the reader's convenience and for a better explanation, we first introduce notation used throughout this paper.

Notation. For each $d \geq 7$, we define

$$\begin{cases} \hbar = \lfloor \frac{1}{2}\left(\frac{d}{2} - \gamma\right) \rfloor \in \mathbb{N}, \\ \delta = \frac{1}{2}\left(\frac{d}{2} - \gamma\right) - \hbar, \quad \delta \in (0, 1), \end{cases} \tag{1-18}$$

where $\lfloor x \rfloor \in \mathbb{Z}$ stands for the integer part of x, which is defined by $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Note that $\delta \neq 0$. Indeed, if $\delta = 0$, then there is $m \in \mathbb{N}$ such that $2\gamma = d - 4m \in \mathbb{N}$. This only happens when $\gamma = 2$ or $\gamma = \frac{3}{2}$ because $\gamma \in (1, 2]$. The case $\gamma = 2$ gives $d = 7$ and $m = \frac{3}{4} \notin \mathbb{N}$. The case $\gamma = \frac{3}{2}$ gives $d = \frac{17}{2} \notin \mathbb{N}$.

Given a large integer $L \gg 1$, we set

$$\Bbbk = L + \hbar + 1. \tag{1-19}$$

Given $b_1 > 0$ and $\lambda > 0$, we define

$$B_0 = \frac{1}{\sqrt{b_1}}, \qquad B_1 = B_0^{1+\eta}, \quad 0 < \eta \ll 1, \tag{1-20}$$

and

$$f_\lambda(r) = f(y) \quad \text{with } y = \frac{r}{\lambda}.$$

Let $\chi \in \mathcal{C}_0^\infty([0,+\infty))$ be a positive nonincreasing cutoff function with $\mathrm{supp}(\chi) \subset [0,2]$ and $\chi \equiv 1$ on $[0,1]$. For all $M > 0$, we define

$$\chi_M(y) = \chi\left(\frac{y}{M}\right). \tag{1-21}$$

We also introduce the differential operator

$$\Lambda f = y\partial_y f$$

and the Schrödinger operator

$$\mathscr{L} = -\partial_{yy} - \frac{(d-1)}{y}\partial_y + \frac{Z}{y^2}, \quad \text{with } Z(y) = (d-1)\cos(2Q(y)). \tag{1-22}$$

Strategy of the proof. We now summary the main ideas of the proof of Theorem 1.1, which follows the route map in [Raphaël and Schweyer 2014b; Merle, Raphaël and Rodnianski 2015]:

(i) *Renormalized flow and iterated resonances*. Following the scaling invariance of (1-4), let us make the change of variables

$$w(y,s) = u(r,t), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)},$$

which leads to the renormalized flow

$$\partial_s w = \partial_y^2 w + \frac{(d-1)}{y}\partial_y w - b_1\Lambda w - \frac{(d-1)}{2y^2}\sin(2w), \quad b_1 = -\frac{\lambda_s}{\lambda}. \tag{1-23}$$

Assuming that the leading part of the solution $w(y,s)$ is given by the ground state profile Q admitting the asymptotic behavior (1-10), the remaining part is governed by the Schrödinger operator $\mathscr{L}$ defined by (1-22). The linear operator $\mathscr{L}$ admits the factorization (see Lemma 2.2 below)

$$\mathscr{L} = \mathscr{A}^*\mathscr{A}, \quad \mathscr{A}f = -\Lambda Q\partial_y\left(\frac{f}{\Lambda Q}\right), \quad \mathscr{A}^*f = \frac{1}{y^{d-1}\Lambda Q}\partial_y(y^{d-1}\Lambda Q f), \tag{1-24}$$

which directly implies

$$\mathscr{L}(\Lambda Q) = 0,$$

where from a direct computation,

$$\Lambda Q \sim \frac{c_0}{y^\gamma} \quad \text{as } y \to +\infty, \text{ with } \gamma \text{ defined in (1-8).}$$

More generally, we can compute the kernel of the powers of $\mathscr{L}$ through the iterative scheme

$$\mathscr{L}T_{k+1} = -T_k, \quad T_0 = \Lambda Q, \tag{1-25}$$

which displays a nontrivial tail at infinity (see Lemma 2.9 below),

$$T_k(y) \sim c_k y^{2k-\gamma} \quad \text{for } y \gg 1. \tag{1-26}$$

(ii) *Tail dynamics*. Following the approach in [Raphaël and Schweyer 2014b], we look for a slowly modulated approximate solution to (1-23) of the form

$$w(y,s) = Q_{b(s)}(y),$$

where

$$b = (b_1, \dots, b_L), \quad Q_{b(s)}(y) = Q(y) + \sum_{i=1}^{L} b_i T_i(y) + \sum_{i=2}^{L+2} S_i(y) \tag{1-27}$$

with a priori bounds

$$b_i \sim b_1^i, \quad |S_i(y)| \lesssim b_1^i y^{2(i-1)-\gamma},$$

so that S_i is in some sense homogeneous of degree i in b_1, and behaves better than T_i at infinity. The construction of S_i with the above a priori bounds is possible for a specific choice of the universal dynamical system which drives the modes $(b_i)_{1\le i\le L}$. This is so-called the *tail computation*. Let us illustrate the procedure of the *tail computation*. We plug the decomposition (1-27) into (1-23) and choose the law for $(b_i)_{1\le i\le L}$ which cancels the leading-order terms at infinity:

- At the order $\mathcal{O}(b_1)$: We cannot adjust the law of b_1 for the first term[1] and obtain from (1-23),

$$b_1(\mathcal{L}T_1 + \Lambda Q) = 0.$$

- At the order $\mathcal{O}(b_1^2, b_2)$: We obtain

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 \mathscr{L} T_2 + \mathscr{L} S_2 = b_1^2 \,\mathrm{NL}_1(T_1, Q),$$

where $\mathrm{NL}_1(T_1, Q)$ corresponds to nonlinear interaction terms. Note from (1-26) and (1-25), we have

$$\Lambda T_1 \sim (2-\gamma) T_1 \quad \text{for } y \gg 1, \qquad \mathscr{L} T_2 = -T_1,$$

and thus,

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 \mathscr{L} T_2 \sim [(b_1)_s + (2-\gamma) b_1^2 - b_2] T_1.$$

Hence the leading-order growth for y large is canceled by the choice

$$(b_1)_s + (2-\gamma) b_1^2 - b_2 = 0.$$

We then solve for

$$\mathscr{L} S_2 = -b_1^2(\Lambda T_1 - (2-\gamma) T_1) + b_1^2 \,\mathrm{NL}_1(T_1, Q),$$

and check the improved decay

$$|S_2(y)| \lesssim b_1^2 y^{2-\gamma} \quad \text{for } y \gg 1.$$

- At the order $\mathcal{O}(b_1^{k+1}, b_{k+1})$: We obtain an elliptic equation of the form

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} \mathscr{L} T_{k+1} + \mathscr{L} S_{k+1} = b_1^{k+1} \,\mathrm{NL}_k(T_1, \dots, T_k, Q).$$

[1] If $(b_1)_s = -c_1 b_1$, then $-\lambda_s/\lambda \sim b_1 \sim e^{-c_1 s}$; hence after an integration in time, $|\log \lambda| \lesssim 1$ and there is no blowup.

From (1-26) and (1-25), we have

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} \mathscr{L} T_{k+1} \sim [(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}] T_k,$$

which leads to the choice

$$(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = 0$$

for the cancellation of the leading-order growth at infinity. We then solve for the remaining S_{k+1}-term and check that $|S_{k+1}(y)| \lesssim b_1^{k+1} y^{2k-\gamma}$ for y large. We refer to Proposition 2.11 for all the details of the *tail computation*.

(iii) *The universal system of ODEs*. The above procedure leads to the following universal system of ODEs after L iterations:

$$\begin{cases} (b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \; b_{L+1} = 0, \\ -\dfrac{\lambda_s}{\lambda} = b_1, \quad \dfrac{ds}{dt} = \dfrac{1}{\lambda^2}. \end{cases} \tag{1-28}$$

Unlike the critical case treated in [Raphaël and Schweyer 2014b], there is no further logarithmic correction to take into account. The set of solutions to (1-28) (see Lemma 2.13 below) is explicitly given by

$$\begin{cases} b_k^e(s) = \dfrac{c_k}{s^k}, \quad 1 \le k \le L, \\ c_1 = \dfrac{\ell}{2\ell-\gamma}, \quad \ell \in \mathbb{N}^*, \; 2\ell > \gamma, \\ c_{k+1} = -\dfrac{\gamma(\ell-k)}{2\ell-\gamma} c_k, \quad 1 \le k \le \ell-1, \; \ell \ge 2 \\ c_j = 0, \quad j \ge \ell+1, \\ \lambda(s) \sim s^{-\frac{\ell}{2\ell-\gamma}}. \end{cases} \tag{1-29}$$

In the original time variable t, this implies that $\lambda(t)$ goes to zero in finite time T with the asymptotic

$$\lambda(t) \sim (T-t)^{\frac{\ell}{\gamma}}.$$

Moreover, the linearized flow of (1-28) near the solution (1-29) is explicit and displays $\ell - 1$ unstable directions (see Lemma 2.14 below). This implies that the case $\ell = 1$ corresponds to a stable type II blowup regime.

(iv) *Decomposition of the flow and modulation equations*. Let the approximate solution Q_b be given by (1-27), which by construction generates an approximate solution to the renormalized flow (1-23),

$$\Psi_b = \partial_s Q_b - \Delta Q_b + b \Lambda Q_b + \frac{(d-1)}{2y^2} \sin(2Q_b) = \mathrm{Mod}(t) + O(b_1^{2L+2}),$$

where the modulation equation term is roughly of the form

$$\mathrm{Mod}(t) = \sum_{i=1}^{L} [(b_i)_s + (2i-\gamma) b_1 b_i - b_{i+1}] T_i.$$

We localize Q_b in the zone $y \le B_1$ to avoid the irrelevant growing tails for $y \gg 1/\sqrt{b_1}$. We then take initial data of the form

$$u_0(y) = Q_{b(0)}(y) + q_0(y),$$

where q_0 is small in some suitable sense and $b(0)$ is chosen to be close to the exact solution (1-29). By a standard modulation argument, we introduce the decomposition of the flow

$$u(r,t) = w(y,s) = (Q_{b(s)} + q)(y,s) = (Q_{b(t)} + v)\left(\frac{r}{\lambda(t)}, t\right), \tag{1-30}$$

where $L+1$ modulation parameters $(b(t), \lambda(t))$ are chosen in order to manufacture the orthogonality conditions

$$\langle q, \mathscr{L}^i \Phi_M \rangle = 0, \quad 0 \le i \le L, \tag{1-31}$$

where Φ_M, see (3-4), is some fixed direction depending on some large constant M, generating an approximation of the kernel of the powers of $\mathscr{L}$. This orthogonal decomposition (1-30), which follows from the implicit function theorem, allows us to compute the modulation equations governing the parameters $(b(t), \lambda(t))$ (see Lemmas 4.2 and 4.3 below),

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{i=1}^{L} |(b_i)_s + (2i-\gamma) b_1 b_i - b_{i+1}| \lesssim \|q\|_{\text{loc}} + b_1^{L+1+\nu(\delta,\eta)}, \tag{1-32}$$

where $\|q\|_{\text{loc}}$ measures a spatially localized norm of the radiation q and $\nu(\delta,\eta) > 0$.

(v) *Control of Sobolev norms*. According to (1-32), we need to show that local norms of q are under control and do not perturb the dynamical system (1-28). This is achieved via high-order mixed energy estimates which provide controls of the Sobolev norms adapted to the linear flow and based on the powers of the linear operator $\mathscr{L}$. In particular, we have the following coercivity of the high energy under the orthogonality conditions (1-31) (see Lemma A.5):

$$\mathscr{E}_{2\Bbbk}(s) = \int |\mathscr{L}^{\Bbbk} q|^2 \gtrsim \int |\nabla^{2\Bbbk} q|^2 + \int \frac{|q|^2}{1+y^{4\Bbbk}},$$

where $\Bbbk$ is given by (1-19). Here the factorization (1-24) will help to simplify the proof. As in [Raphaël and Rodnianski 2012; Raphaël and Schweyer 2014b; Merle, Raphaël and Rodnianski 2015], the control of $\mathscr{E}_{2\Bbbk}$ is done through the use of the linearized equation in the original variables (r,t); i.e., we work with v in (1-30) and not q. The energy estimate is of the form (see Proposition 4.4)

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}\right\} \lesssim \frac{b_1^{2L+1+2\nu(\delta,\eta)}}{\lambda^{4\Bbbk-d}}, \quad \nu(\delta,\eta) > 0, \tag{1-33}$$

where the right-hand side is controlled by the size of the error Ψ_b in the construction of the approximate profile Q_b above. An integration of (1-33) in time by using initial smallness assumptions, $b_1 \sim b_1^e$ and $\lambda(s) \sim b_1^{\ell/(2\ell-\gamma)}$ yields the estimate

$$\int |\nabla^{2\Bbbk} q|^2 + \int \frac{|q|^2}{1+y^{4\Bbbk}} \lesssim \mathscr{E}_{2\Bbbk}(s) \lesssim b_1^{2L+2\nu(\delta,\eta)},$$

which is good enough to control the local norms of q and close the modulation equations (1-32).

Note that we also need to control lower energies $\mathscr{E}_{2m}$ for $\hbar + 2 \le m \le \Bbbk - 1$ because the control of the high energy $\mathscr{E}_{2\Bbbk}$ alone is not enough to control a nonlinear term appearing in the linearized equation around Q_b. In particular, we exhibit a Lyapunov functional with the dynamical estimate

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2m}}{\lambda^{4m-d}}\right\} \lesssim \frac{b_1^{2(m-\hbar)-1+2\nu'(\delta,\eta)}}{\lambda^{4m-d}}, \quad \nu'(\delta,\eta) > 0.$$

Then, an integration in time yields

$$\mathscr{E}_{2m}(s) \lesssim \begin{cases} b_1^{\frac{\ell}{2\ell-\gamma}(4m-d)} & \text{for } \hbar+2 \le m \le \ell+\hbar, \\ b_1^{2(m-\hbar-1)+2\nu'(\delta,\eta)} & \text{for } \hbar+\ell+1 \le m \le \Bbbk-1, \end{cases}$$

which is enough to control the nonlinear term. Let us remark that the condition $m \ge \hbar + 2$ ensures $4m - d > 0$ so that $\mathscr{E}_{2m}$ is always controlled. By the coercivity of $\mathscr{E}_{2m}$, this means that we are only able to control the Sobolev norms $\|\nabla^{2\sigma} q\|_{L^2}^2$ for $\sigma \ge \hbar + 2$, resulting in the asymptotic (1-15).

The above scheme designs a bootstrap regime (see Definition 3.2 for a precise definition) which traps the blowup solution with speed (1-14). According to Lemmas 2.13 and 2.14, such a regime displays $\ell - 1$ unstable modes $(b_2, \dots, b_\ell)$ which we can control through a topological argument based on the Brouwer fixed-point theorem (see the proof of Proposition 3.5), and the proof of Theorem 1.1 follows.

The paper is organized as follows. In Section 2, we give the construction of the approximate solution Q_b of (1-4) and derive estimates on the generated error term Ψ_b (Proposition 2.11), as well as its localization (Proposition 2.12). We also give in this section some elementary facts on the study of the system (1-28) (Lemmas 2.13 and 2.14). Section 3 is devoted to the proof of Theorem 1.1, assuming a main technical result (Proposition 3.6). In particular, we give the proof of the existence of the solution trapped in some shrinking set to zero (Proposition 3.5) such that the constructed solution satisfies the conclusion of Theorem 1.1. Readers not interested in technical details may stop there. In Section 4, we give the proof of Proposition 3.6 which gives the reduction of the problem to a finite-dimensional one, and this is the heart of our analysis.

2. Construction of an approximate profile

This section is devoted to the construction of a suitable approximate solution to (1-4) by using the same approach developed in [Raphaël and Rodnianski 2012]. Similar approaches can also be found in [Raphaël and Schweyer 2013; 2014a; Hillairet and Raphaël 2012; Schweyer 2012; Merle, Raphaël and Rodnianski 2015]. The key to this construction is the fact that the linearized operator $\mathscr{L}$ around Q is completely explicit in the radial setting thanks to the explicit formulas of the kernel elements.

Following the scaling invariance of (1-4), we introduce the change of variables

$$w(y,s) = u(r,t), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \tag{2-1}$$

which leads to the renormalized flow

$$\partial_s w = \partial_y^2 w + \frac{(d-1)}{y}\partial_y w + \frac{\lambda_s}{\lambda}\Lambda w - \frac{(d-1)}{2y^2}\sin(2w), \tag{2-2}$$

where $\lambda_s = d\lambda/ds$. Noticing that in the setting (2-1) we have

$$\partial_r u(r,t) = \frac{1}{\lambda(t)} \partial_y w(y,s)$$

and since we deal with the finite-time blowup of the problem (1-4), we naturally impose the condition

$$\lambda(t) \to 0 \quad \text{as } t \to T$$

for some $T \in (0, +\infty)$. Hence, $\partial_r u(r,t)$ blows up in finite time T.

Let us assume that the leading part of the solution of (2-2) is given by the harmonic map Q, which is a unique solution (up to scaling) of the equation

$$Q'' + \frac{(d-1)}{y} Q' - \frac{(d-1)}{2y^2} \sin(2Q) = 0, \quad Q(0) = 0, \quad Q'(0) = 1. \tag{2-3}$$

We aim to construct an approximate solution of (2-2) close to Q. The natural way is to linearize (2-2) around Q, which generates the Schrödinger operator defined by (1-22). Let us now recall the main properties of $\mathscr{L}$ in the following subsection.

2A. ***Structure of the linearized Hamiltonian.*** We recall the main properties of the linearized Hamiltonian close to Q, which is the heart of both construction of the approximate profile and the derivation of the coercivity properties serving for the high Sobolev energy estimates. Let us start by recalling the following result from [Biernat 2015], which gives the asymptotic behavior of the harmonic map Q:

Lemma 2.1 (development of the harmonic map Q). *Let $d \geq 7$. There exists a unique solution Q to* (2-3) *which admits the following asymptotic behavior. For any $k \in \mathcal{N}^*$:*

(i) (*asymptotic behavior of Q*)

$$Q(y) = \begin{cases} y + \sum_{i=1}^{k} c_i y^{2i+1} + \mathcal{O}(y^{2k+3}) & \text{as } y \to 0, \\ \frac{\pi}{2} - \frac{a_0}{y^\gamma}\left[1 + \mathcal{O}\left(\frac{1}{y^2}\right) + \mathcal{O}\left(\frac{1}{y^{\tilde{\gamma}}}\right)\right] & \text{as } y \to +\infty, \end{cases} \tag{2-4}$$

where γ is defined in (1-8), $\tilde{\gamma} = \sqrt{d^2 - 8d + 8}$ *and* $a_0 = a_0(d) > 0$.

(ii) (*degeneracy*)

$$\Lambda Q > 0, \quad \Lambda Q(y) = \begin{cases} y + \sum_{i=1}^{k} c_i' y^{2i+1} + \mathcal{O}(y^{2k+3}) & \text{as } y \to 0, \\ \frac{a_0 \gamma}{y^\gamma}\left[1 + \mathcal{O}\left(\frac{1}{y^2}\right) + \mathcal{O}\left(\frac{1}{y^{\tilde{\gamma}}}\right)\right] & \text{as } y \to +\infty. \end{cases} \tag{2-5}$$

Proof. The proof of (2-4) is done through the introduction of the variables $x = \log y$ and $v(x) = 2Q(y) - \pi$ and consists of the phase portrait analysis of the autonomous equation

$$v''(x) + (d-2)v'(x) + (d-2)\sin(v(x)) = 0.$$

All details of the proof can be found in [Biernat 2015, pages 184–185]. The proof of (2-5) directly follows from the expansion (2-4). □

The linearized operator $\mathscr{L}$ displays a remarkable structure given by the following lemma:

Lemma 2.2 (factorization of $\mathscr{L}$). *Let $d \geq 7$ and define the first-order operators*

$$\mathscr{A}w = -\partial_y w + \frac{V}{y}w = -\Lambda Q\,\partial_y\left(\frac{w}{\Lambda Q}\right), \tag{2-6}$$

$$\mathscr{A}^*w = \frac{1}{y^{d-1}}\partial_y(y^{d-1}w) + \frac{V}{y}w = \frac{1}{y^{d-1}\Lambda Q}\partial_y(y^{d-1}\Lambda Q w), \tag{2-7}$$

where

$$V(y) := \Lambda \log(\Lambda Q) = \begin{cases} 1 + \mathcal{O}(y^2) & \text{as } y \to 0,\\ -\gamma + \mathcal{O}\left(\frac{1}{y^2}\right) + \mathcal{O}\left(\frac{1}{y^{\tilde{\gamma}}}\right) & \text{as } y \to +\infty. \end{cases} \tag{2-8}$$

We have

$$\mathscr{L} = \mathscr{A}^*\mathscr{A}, \quad \widetilde{\mathscr{L}} = \mathscr{A}\mathscr{A}^*, \tag{2-9}$$

where $\widetilde{\mathscr{L}}$ stands for the conjugate Hamiltonian.

Remark 2.3. The adjoint operator $\mathscr{A}^*$ is defined with respect to the Lebesgue measure

$$\int_0^{+\infty} (\mathscr{A}u)wy^{d-1}\,dy = \int_0^{+\infty} u(\mathscr{A}^*w)y^{d-1}\,dy.$$

Remark 2.4. We have

$$\mathscr{L}(\Lambda w) = \Lambda(\mathscr{L}w) + 2\mathscr{L}w - \frac{\Lambda Z}{y^2}w. \tag{2-10}$$

Since $\mathscr{L}(\Lambda Q) = 0$, one can express the definition of Z through the potential V as

$$Z(y) = V^2 + \Lambda V + (d-2)V. \tag{2-11}$$

Let $\widetilde{Z}$ be defined by

$$\widetilde{\mathscr{L}} = -\partial_{yy} - \frac{d-1}{y}\partial_y + \frac{\widetilde{Z}}{y^2}. \tag{2-12}$$

Then, a direct computation yields

$$\widetilde{Z}(y) = (V+1)^2 + (d-2)(V+1) - \Lambda V. \tag{2-13}$$

From (2-6) and (2-7), we see that the kernels of $\mathscr{A}$ and $\mathscr{A}^*$ are explicit:

$$\mathscr{A}w = 0 \quad \text{if and only if} \quad w \in \operatorname{Span}(\Lambda Q),$$

$$\mathscr{A}^*w = 0 \quad \text{if and only if} \quad w \in \operatorname{Span}\left(\frac{1}{y^{d-1}\Lambda Q}\right).$$

Hence, the elements of the kernel of $\mathscr{L}$ are given by

$$\mathscr{L}w = 0 \quad \text{if and only if} \quad w \in \operatorname{Span}(\Lambda Q, \Gamma), \tag{2-14}$$

where Γ can be found from the Wronskian relation

$$\Gamma'\Lambda Q - \Gamma(\Lambda Q)' = \frac{1}{y^{d-1}}, \tag{2-15}$$

that is,

$$\Gamma(y) = \Lambda Q(y)\int_1^y \frac{d\xi}{\xi^{d-1}(\Lambda Q(\xi))^2},$$

which admits the asymptotic behavior

$$\Gamma(y) = \begin{cases} \dfrac{1}{dy^{d-1}} + \mathcal{O}(y) & \text{as } y \to 0, \\ \dfrac{1}{a_0\gamma(d-2-2\gamma)y^{d-2-\gamma}} + \mathcal{O}\Big(\dfrac{1}{y^{d-\gamma}}\Big) & \text{as } y \to +\infty. \end{cases} \tag{2-16}$$

From (2-14), we may invert $\mathscr{L}$ as follows:

$$\mathscr{L}^{-1} f = -\Gamma(y)\int_0^y f(x)\Lambda Q(x)x^{d-1}\,dx + \Lambda Q(y)\int_0^y f(x)\Gamma(x)x^{d-1}\,dx. \tag{2-17}$$

The factorization of $\mathscr{L}$ allows us to compute $\mathscr{L}^{-1}$ in an elementary two-step process that will help us to avoid tracking the cancellation in the formula (2-17) induced by the Wronskian relation when estimating the growth of $\mathscr{L}^{-1} f$. In particular, we have the following:

Lemma 2.5 (inversion of $\mathscr{L}$). *Let f be a $\mathcal{C}^\infty$ radially symmetric function and $w = \mathscr{L}^{-1} f$ be given by* (2-17). *Then*

$$\mathscr{L}w = f, \quad \mathscr{A}w = \frac{1}{y^{d-1}\Lambda Q}\int_0^y f(x)\Lambda Q(x)x^{d-1}\,dx, \quad w = -\Lambda Q\int_0^y \frac{\mathscr{A}w(x)}{\Lambda Q(x)}\,dx. \tag{2-18}$$

Proof. From the relation (2-15), we compute

$$\mathscr{A}\Gamma = -\frac{1}{y^{d-1}\Lambda Q}.$$

Applying $\mathscr{A}$ to (2-17) and using the cancellation $\mathscr{A}(\Lambda Q) = 0$, we obtain

$$\mathscr{A}w = \frac{1}{y^{d-1}\Lambda Q}\int_0^y f(x)\Lambda Q(x)x^{d-1}\,dx.$$

From the definition (2-6) of $\mathscr{A}$, we write

$$w = -\Lambda Q\int_0^y \frac{\mathscr{A}w}{\Lambda Q}\,dx. \qquad \square$$

2B. *Admissible functions.* We define a class of admissible functions which display a suitable behavior both at the origin and infinity.

Definition 2.6 (admissible function). Fix $\gamma > 0$, we say that a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is admissible of degree $(p_1, p_2) \in \mathbb{N}\times\mathbb{Z}$ if:

(i) f admits a Taylor expansion to all orders around the origin,

$$f(y)=\sum_{k=p_1}^{p} c_k y^{2k+1}+\mathcal{O}(y^{2p+3});$$

(ii) f and its derivatives admit the bounds, for $y\geq 1$,

$$\text{for all } k\in\mathbb{N},\quad |\partial_y^k f(y)|\lesssim y^{2p_2-\gamma-k}.$$

Remark 2.7. By (2-5), ΛQ is admissible of degree $(0,0)$.

Note that $\mathscr{L}$ naturally acts on the class of admissible functions in the following way:

Lemma 2.8 (action of $\mathscr{L}$ and $\mathscr{L}^{-1}$ on admissible functions). *Let f be an admissible function of degree $(p_1,p_2)\in\mathbb{N}\times\mathbb{Z}$. Then*:

(i) *Λf is admissible of degree (p_1,p_2).*

(ii) *$\mathscr{L}f$ is admissible of degree* $(\max\{0,p_1-1\},p_2-1)$.

(iii) *$\mathscr{L}^{-1}f$ is admissible of degree (p_1+1,p_2+1).*

Proof. (i)–(ii) This is simply a consequence of Definition 2.6.

(iii) We aim to prove that if f is admissible of degree (p_1,p_2), then $w=\mathscr{L}^{-1}f$ is admissible of degree (p_1+1,p_2+1). To do so, we use Lemma 2.5 to estimate

- for $y\ll 1$,

$$\mathscr{A}w=\frac{1}{y^{d-1}\Lambda Q}\int_0^y f\Lambda Q x^{d-1}\,dx=\mathcal{O}\left(\frac{1}{y^d}\int_0^y x^{2p_1+1+d}\,dx\right)=\mathcal{O}(y^{2p_1+2}),$$

$$w=-\Lambda Q\int_0^y\frac{\mathscr{A}w}{\Lambda Q}\,dx=\mathcal{O}\left(y\int_0^y x^{2p_1+1}\,dx\right)=\mathcal{O}(y^{2(p_1+1)+1}),$$

- for $y\geq 1$,

$$\mathscr{A}w=\mathcal{O}\left(\frac{1}{y^{d-1-\gamma}}\int_0^y x^{2p_2-2\gamma+d-1}\,dx\right)=\mathcal{O}(y^{2p_2+1-\gamma}),$$

$$w=\mathcal{O}\left(\frac{1}{y^\gamma}\int_0^y x^{2p_2+1}\right)=\mathcal{O}(y^{2(p_2+1)-\gamma}).$$

From the last formula in (2-18) and (2-8), we estimate

$$\partial_y w=-\partial_y\Lambda Q\int_0^y\frac{\mathscr{A}w}{\Lambda Q}\,dx-\mathscr{A}w=-\frac{\partial_y\Lambda Q}{\Lambda Q}w-\mathscr{A}w=\mathcal{O}(y^{2(p_2+1)-\gamma-1}).$$

Using $\mathscr{L}w=f$, we get

$$\partial_{yy}w=\mathcal{O}\left(\frac{|\partial_y w|}{y}+\frac{|w|}{y^2}+|f|\right)=\mathcal{O}(y^{2(p_2+1)-\gamma-2}).$$

By taking radial derivatives of $\mathscr{L}w=f$, we obtain by induction

$$|\partial_y^k w|\lesssim y^{2(p_2+1)-\gamma-k},\quad k\in\mathbb{N},\ \ y\geq 1.$$

□

The following lemma is a consequence of Lemma 2.8:

Lemma 2.9 (generators of the kernel of $\mathscr{L}^k$). *Consider the sequence of profiles*

$$T_k = (-1)^k \mathscr{L}^{-k} \Lambda Q, \quad k \in \mathbb{N}. \tag{2-19}$$

Then:

(i) T_k *is admissible of degree* (k, k) *for* $k \in \mathbb{N}$.

(ii) $\Lambda T_k - (2k - \gamma) T_k$ *is admissible of degree* $(k, k-1)$ *for* $k \in \mathbb{N}^*$.

Proof. (i) We note from (2-5) that ΛQ is admissible of degree $(0, 0)$. By induction and part (iii) of Lemma 2.8, the conclusion then follows.

(ii) We proceed by induction. For $k = 1$, we explicitly compute $T_1 = -\mathscr{L}^{-1} \Lambda Q$ by using Lemma 2.5 and the expansion (2-5) to get

$$\text{for all } m \in \mathbb{N}, \quad \partial_y^m T_1(y) = e_{1,m} y^{2-\gamma-m} + \mathcal{O}(y^{-\gamma-m}) \quad \text{as } y \to +\infty.$$

By induction, one can easily check that $\partial_y^m \Lambda f = \Lambda \partial_y^m f + m \partial_y^m f$ for $m \in \mathbb{N}^*$. Hence,

$$\partial_y^m [\Lambda T_1 - (2-\gamma) T_1] = \Lambda \partial_y^m T_1 - (2 - \gamma - m) \partial_y^m T_1 = \mathcal{O}(y^{-\gamma-m}) \quad \text{as } y \to +\infty.$$

Since T_1 and ΛT_1 are admissible of degree $(1, 1)$, we deduce that $\Lambda T_1 - (2-\gamma) T_1$ is admissible of degree $(1, 0)$.

We now assume the claim for $k \geq 1$, namely that $\Lambda T_k - (2k - \gamma) T_k$ is admissible of degree $(k, k-1)$. Let us prove that $\Lambda T_{k+1} - (2(k+1) - \gamma) T_{k+1}$ is admissible of degree $(k+1, k)$. We use formula (2-10) and definition (2-19) to write

$$\begin{aligned} \mathscr{L}(\Lambda T_{k+1} - (2k + 2 - \gamma) T_{k+1}) &= \Lambda \mathscr{L} T_{k+1} - (2k - \gamma) \mathscr{L} T_{k+1} - \frac{\Lambda Z}{y^2} T_{k+1} \\ &= \Lambda T_k - (2k - \gamma) T_k - \frac{\Lambda Z}{y^2} T_{k+1}. \end{aligned} \tag{2-20}$$

From part (i), we know that T_{k+1} is admissible of degree $(k+1, k+1)$. From (2-11) and (2-8), one can check that $(\Lambda Z / y^2) T_{k+1}$ admits the asymptotic

$$\frac{\Lambda Z}{y^2} T_{k+1} = \mathcal{O}(y^{2k+1}) \quad \text{as } y \to 0,$$

and

$$\partial_y^j \left(\frac{\Lambda Z}{y^2} T_{k+1} \right) = \mathcal{O}(y^{2(k+1) - j - \gamma - 3}) \ll y^{2(k-1) + j - \gamma} \quad \text{as } y \to +\infty.$$

Together with the induction hypothesis, we deduce that the right-hand side of (2-20) is admissible of degree $(k, k-1)$. The conclusion then follows by using part (iii) of Lemma 2.8. □

We end this subsection by introducing a simple notion of homogeneous admissible function.

Definition 2.10 (homogeneous admissible function). Let $L \gg 1$ be an integer and $m = (m_1, \dots, m_L) \in \mathbb{N}^L$. We say that a function $f(b, y)$ with $b = (b_1, \dots, b_L)$ is homogeneous of degree $(p_1, p_2, p_3) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ if it is a finite linear combination of monomials

$$\tilde{f}(y) \prod_{k=1}^{L} b_k^{m_k},$$

with $\tilde{f}(y)$ admissible of degree (p_1, p_2) in the sense of Definition 2.6 and

$$(m_1, \dots, m_L) \in \mathbb{N}^L, \quad \sum_{k=1}^{L} k m_k = p_3.$$

We set

$$\deg(f) := (p_1, p_2, p_3).$$

2C. *Slowly modulated blowup profile.* We use the explicit structure of the linearized operator $\mathscr{L}$ to construct an approximate blowup profile. In particular, we claim the following:

Proposition 2.11 (construction of the approximate profile). *Let $d \geq 7$ and $L \gg 1$ be an integer. Let $M > 0$ be a large enough universal constant. Then there exists a small enough universal constant $b^*(M, L) > 0$ such that the following holds true. Consider a $\mathcal{C}^1$ map*

$$b = (b_1, \dots, b_L) : [s_0, s_1] \mapsto (-b^*, b^*)^L,$$

with a priori bounds in $[s_0, s_1]$,

$$0 < b_1 < b^*, \qquad |b_k| \lesssim b_1^k, \quad 2 \leq k \leq L. \tag{2-21}$$

Then there exist homogeneous profiles

$$S_1 = 0, \qquad S_k = S_k(b, y), \quad 2 \leq k \leq L + 2,$$

such that

$$Q_{b(s)}(y) = Q(y) + \sum_{k=1}^{L} b_k(s) T_k(y) + \sum_{k=2}^{L+2} S_k(b, y) \equiv Q(y) + \Theta_{b(s)}(y) \tag{2-22}$$

generates an approximate solution to the renormalized flow (2-2)

$$\partial_s Q_b - \partial_{yy} Q_b - \frac{(d-1)}{y} \partial_y Q_b + b_1 \Lambda Q_b + \frac{(d-1)}{2y^2} \sin(2Q_b) = \Psi_b + \mathrm{Mod}(t), \tag{2-23}$$

with the following properties:

(*i*) (*modulation equation*)

$$\mathrm{Mod}(t) = \sum_{k=1}^{L} [(b_k)_s + (2k - \gamma) b_1 b_k - b_{k+1}] \left[T_k + \sum_{j=k+1}^{L+2} \frac{\partial S_j}{\partial b_k} \right], \tag{2-24}$$

where we use the convention $b_j = 0$ for $j \geq L + 1$.

(*ii*) (*estimate on the profiles*) *The profiles* $(S_k)_{2\le k\le L+2}$ *are homogeneous with*

$$\deg(S_k) = (k, k-1, k) \quad \textit{for } 2 \le k \le L+2,$$
$$\frac{\partial S_k}{\partial b_m} = 0 \quad \textit{for } 2 \le k \le m \le L.$$

(*iii*) (*estimate on the error* Ψ_b) *For all* $0 \le m \le L$, *we have*:

- (*global weight bound*)

$$\int_{y\le 2B_1} |\mathscr{L}^{\hbar+m+1}\Psi_b|^2 + \int_{y\le 2B_1} \frac{|\Psi_b|^2}{1+y^{4(\hbar+m+1)}} \lesssim b_1^{2m+4+2(1-\delta)-C_L\eta}, \tag{2-25}$$

where B_1, $\hbar$, δ *are defined in* (1-20) *and* (1-18).

- (*improved local bound*)

$$\textit{For all } M \ge 1, \quad \int_{y\le 2M} |\mathscr{L}^{\hbar+m+1}\Psi_b|^2 \lesssim M^C b_1^{2L+6}. \tag{2-26}$$

Proof. We aim to construct the profiles $(S_k)_{2\le k\le L+2}$ such that $\Psi_b(y)$ defined from (2-23) has the *least possible growth* as $y \to +\infty$. The key to this construction is the fact that the structure of the linearized operator $\mathscr{L}$ defined in (1-22) is completely explicit in the radial sector thanks to the explicit formulas of the elements of the kernel. This procedure will lead to the leading-order modulation equation

$$(b_k)_s = -(2k-\gamma)b_1 b_k + b_{k+1} \quad \text{for } 1 \le k \le L, \tag{2-27}$$

which actually cancels the worst growth of S_k as $y \to +\infty$.

- Expansion of Ψ_b. From (2-23) and (2-3), we write

$$\begin{aligned}
\partial_s Q_b - \partial_{yy} Q_b - \frac{(d-1)}{y}\partial_y Q_b + b_1 \Lambda Q_b + \frac{(d-1)}{2y^2}\sin(2Q_b) \\
= b_1 \Lambda Q + \partial_s \Theta_b - \partial_{yy}\Theta_b - \frac{(d-1)}{y}\partial_y \Theta_y + \frac{(d-1)}{y^2}\cos(2Q)\Theta_b + b_1 \Lambda \Theta_b \\
+ \frac{(d-1)}{2y^2}[\sin(2Q+2\Theta_b) - \sin(2Q) - 2\cos(2Q)\Theta_b] \\
:= A_1 + A_2.
\end{aligned}$$

Using the expression (2-22) of Θ_b and the definition (2-19) of T_k (note that $\mathscr{L}T_k = -T_{k-1}$ with the convention $T_0 = \Lambda Q$), we write

$$\begin{aligned}
A_1 &= b_1 \Lambda Q + \sum_{k=1}^{L} [(b_k)_s T_k + b_k \mathscr{L} T_k + b_1 b_k \Lambda T_k] + \sum_{k=2}^{L+2} [\partial_s S_k + \mathscr{L} S_k + b_1 \Lambda S_k] \\
&= \sum_{k=1}^{L} [(b_k)_s T_k - b_{k+1} T_k + b_1 b_k \Lambda T_k] + \sum_{k=2}^{L+2} [\partial_s S_k + \mathscr{L} S_k + b_1 \Lambda S_k]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{L} [(b_k)_s - b_{k+1} + (2k-\gamma) b_1 b_k] T_k \\
&\quad + \sum_{k=1}^{L} [\mathscr{L} S_{k+1} + \partial_s S_k + b_1 b_k [\Lambda T_k - (2k-\gamma) T_k] + b_1 \Lambda S_k] \\
&\qquad + [\mathscr{L} S_{L+2} + \partial_s S_{L+1} + b_1 \Lambda S_{L+1}] + [\partial_s S_{L+2} + b_1 \Lambda S_{L+2}].
\end{aligned}
$$

We now write

$$
\partial_s S_k = \sum_{j=1}^{L} (b_j)_s \frac{\partial S_k}{\partial b_j} = \sum_{j=1}^{L} [(b_j)_s + (2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j} - \sum_{j=1}^{L} [(2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}.
$$

Hence,

$$
A_1 = \mathrm{Mod}(t) + \sum_{k=1}^{L+1} [\mathscr{L} S_{k+1} + E_k] + E_{L+2},
$$

where for $k = 1, \dots, L$,

$$
E_k = b_1 b_k [\Lambda T_k - (2k-\gamma) T_k] + b_1 \Lambda S_k - \sum_{j=1}^{k-1} [(2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}, \tag{2-28}
$$

and for $k = L+1, L+2$,

$$
E_k = b_1 \Lambda S_k - \sum_{j=1}^{L} [(2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}. \tag{2-29}
$$

For the expansion of the nonlinear term A_2, let us set

$$
f(x) = \sin(2x)
$$

and use a Taylor expansion to write (see page 1740 in [Raphaël and Schweyer 2014b] for a similar computation)

$$
A_2 = \frac{(d-1)}{2y^2} \left[\sum_{i=2}^{L+2} \frac{f^{(i)}(Q)}{i!} \Theta_b^i + R_2 \right] = \frac{(d-1)}{2y^2} \left[\sum_{i=2}^{L+2} P_i + R_1 + R_2 \right],
$$

where

$$
P_i = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j, |J|_2 = i} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \tag{2-30}
$$

$$
R_1 = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j, |J|_2 \geq L+3} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \tag{2-31}
$$

$$
R_2 = \frac{\Theta_b^{L+3}}{(L+2)!} \int_0^1 (1-\tau)^{L+2} f^{(L+3)}(Q + \tau \Theta_b)\, d\tau, \tag{2-32}
$$

with $J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}) \in \mathbb{N}^{2L+1}$ and

$$|J|_1 = \sum_{k=1}^{L} i_k + \sum_{k=2}^{L+2} j_k, \quad |J|_2 = \sum_{k=1}^{L} k i_k + \sum_{k=2}^{L+2} k j_k. \tag{2-33}$$

In conclusion, we have

$$\Psi_b = \sum_{k=1}^{L+1} \left[\mathscr{L} S_{k+1} + E_k + \frac{(d-1)}{2y^2} P_{k+1} \right] + E_{L+2} + \frac{(d-1)}{2y^2}(R_1 + R_2). \tag{2-34}$$

• Construction of S_k. From the expression of Ψ_b given in (2-34), we construct iteratively the sequences of profiles $(S_k)_{1 \le k \le L+2}$ through the scheme

$$\begin{cases} S_1 = 0, \\ S_k = -\mathscr{L}^{-1} F_k, \quad 2 \le k \le L+2, \end{cases} \tag{2-35}$$

where

$$F_k = E_{k-1} + \frac{(d-1)}{2y^2} P_k \quad \text{for } 2 \le k \le L+2.$$

We claim by induction on k that F_k is homogeneous with

$$\deg(F_k) = (k-1, k-2, k) \quad \text{for } 2 \le k \le L+2, \tag{2-36}$$

and

$$\frac{\partial F_k}{\partial b_m} = 0 \quad \text{for } 2 \le k \le m \le L+2. \tag{2-37}$$

From item (iii) of Lemma 2.8 and (2-36), we deduce that S_k is homogeneous with

$$\deg(S_k) = (k, k-1, k) \quad \text{for } 2 \le k \le L+2,$$

and from (2-37), we get

$$\frac{\partial S_k}{\partial b_m} = 0 \quad \text{for } 2 \le k \le m \le L+2,$$

which is the conclusion of item (ii).

Let us now give the proof of (2-36) and (2-37). We proceed by induction.

Case $k = 2$: We compute explicitly from (2-28) and (2-30),

$$F_2 = E_1 + \frac{(d-1)}{2y^2} P_2 = b_1^2 \left[\Lambda T_1 - (2-\gamma) T_1 + \frac{(d-1) f''(Q)}{2y^2} T_1^2 \right],$$

which directly follows (2-37). From Lemma 2.9, we know that T_1 and $\Lambda T_1 - (2-\gamma)T_1$ are admissible of degrees $(1, 1)$ and $(1, 0)$ respectively. Using (2-4), one can check the bound

$$\text{for all } m, j \in \mathbb{N}^2, \quad \left| \partial_y^m \left(\frac{f^{(j)}(Q)}{y^2} \right) \right| \lesssim y^{-\gamma-2-m} \quad \text{as } y \to +\infty. \tag{2-38}$$

Since T_1 is admissible of degree $(1, 1)$, we have

$$\text{for all } m \in \mathbb{N}, \quad |\partial_y^m (T_1^2)| \lesssim y^{4-2\gamma-m} \quad \text{as } y \to +\infty.$$

By the Leibniz rule and the fact that $2\gamma - 2 > 0$, we get

$$\text{for all } m, j \in \mathbb{N}^2, \quad \left|\partial_y^m \left(\frac{f^{(j)}(Q)}{y^2} T_1^2\right)\right| \lesssim y^{-\gamma-m-(2\gamma-2)} \lesssim y^{-\gamma-m}.$$

We also have the expansion near the origin,

$$\frac{f^{(j)}(Q)}{y^2} T_1^2 = \sum_{i=2}^{k} c_i y^{2i+1} + \mathcal{O}(y^{2k+3}), \quad k \geq 1.$$

Hence, $(f''(Q)/y^2)T_1^2$ is admissible of degree $(2, 0)$, which concludes the proof of (2-36) for $k = 2$.

Case $k \to k+1$: Estimate (2-37) holds by direct inspection. Let us now assume that S_k is homogeneous of degree $(k, k-1, k)$ and prove that S_{k+1} is homogeneous of degree $(k+1, k, k+1)$. In particular, the claim immediately follows from part (iii) of Lemma 2.8 once we show that F_{k+1} is homogeneous with

$$\deg(F_{k+1}) = \deg\left(E_k + \frac{P_{k+1}}{y^2}\right) = (k, k-1, k+1). \tag{2-39}$$

From part (ii) of Lemma 2.9 and the a priori assumption (2-21), we see that $b_1 b_k(\Lambda T_k - (2k-\gamma)T_k)$ is homogeneous of degree $(k, k-1, k+1)$. From part (i) of Lemma 2.8 and the induction hypothesis, $b_1 \Lambda S_k$ is also homogeneous of degree $(k, k-1, k+1)$. By definition, $b_1(\partial S_k/\partial b_1)$ is homogeneous and has the same degree as S_k. Thus,

$$\left((2j-\gamma)b_1 - \frac{b_2}{b_1}\right)\left(b_1 \frac{\partial S_k}{\partial b_1}\right)$$

is homogeneous of degree $(k, k-1, k+1)$. From definitions (2-28) and (2-29), we derive

$$\deg(E_k) = (k, k-1, k+1), \quad k \geq 1.$$

It remains to control the term P_{k+1}/y^2. From the definition (2-30), we see that P_{k+1}/y^2 is a linear combination of monomials of the form

$$M_J(y) = \frac{f^{(j)}(Q)}{y^2} \prod_{m=1}^{L} b_m^{i_m} T_m^{i_m} \prod_{m=2}^{L+2} S_m^{j_m},$$

with

$$J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}), \qquad |J|_1 = j, \quad |J|_2 = k+1, \quad 2 \leq j \leq k+1.$$

Recall from part (i) of Lemma 2.9 the bound

$$\text{for all } n \in \mathbb{N}, \quad |\partial_y^n T_m| \lesssim y^{2m-\gamma-n} \quad \text{as } y \to +\infty,$$

and from the induction hypothesis and the a priori bound (2-21),

$$\text{for all } n \in \mathbb{N}, \quad |\partial_y^n S_m| \lesssim b_1^m y^{2(m-1)-\gamma-n} \quad \text{as } y \to +\infty.$$

Together with the bound (2-38), we obtain the following bound at infinity:

$$|M_J| \lesssim b_1^{|J|_2} y^{2|J|_2-\gamma-|J|_1\gamma-2-2\sum_{m=2}^{L+2} j_m} \lesssim b_1^{k+1} y^{2(k-1)-\gamma}.$$

The control of $\partial_y^n M_J$ follows by the Leibniz rule and the above estimates. One can also check that M_J is of order $y^{2|J|_2+|J|_1-1}$ near the origin. This concludes the proof of (2-39) as well as part (ii) of Proposition 2.11.

• Estimate on Ψ_b. From (2-34) and (2-35), the expression of Ψ_b is now reduced to

$$\Psi_b = E_{L+2} + \frac{(d-1)}{y^2}(R_1 + R_2),$$

where E_{L+2}, R_1 and R_2 are given by (2-29), (2-31) and (2-32).

We start by estimating the E_{L+2} term defined by (2-29). Since S_{L+2} is homogeneous of degree $(L+2, L+1, L+2)$, so are ΛS_{L+2} and $b_1(\partial S_{L+2}/\partial b_1)$. It follows that E_{L+2} is homogeneous of degree $(L+2, L+1, L+3)$. Using part (ii) of Lemma 2.8 and the relation $d-2\gamma-4\hbar = 4\delta$, see (1-18), we estimate for all $0 \le m \le L$

$$\begin{aligned}\int_{y\le 2B_1} |\mathscr{L}^{\hbar+m+1} E_{L+2}|^2 &\lesssim b_1^{2L+6} \int_{y\le 2B_1} |y^{2(L+1)-\gamma-2(\hbar+m+1)}|^2 y^{d-1}\, dy \\ &\lesssim b_1^{2L+6} \int_{y\le 2B_1} y^{4(L-m+\delta)-1}\, dy \\ &\lesssim b_1^{(2L+6)-2(L-m+\delta)(1+\eta)} \lesssim b_1^{2m+4+2(1-\delta)-C_L\eta},\end{aligned}$$

where $\eta = \eta(L)$, $0 < \eta \ll 1$.

We now turn to the control of the term R_1/y^2, which is a linear combination of terms of the form, see (2-31),

$$\widetilde{M}_J = \frac{f^{(j)}(Q)}{y^2} \prod_{n=1}^{L} b_n^{i_n} T_n^{i_n} \prod_{n=2}^{L+2} S_k^{j_n},$$

with

$$J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}), \qquad |J|_1 = j, \quad |J|_2 \ge L+3, \quad 2 \le j \le L+2.$$

Using the admissibility of T_n and the homogeneity of S_n, we get the bounds

$$|\widetilde{M}_J| \lesssim b_1^{L+3} y^{2|J|_2+j-1} \lesssim b_1^{L+3} y^{2L+6} \quad \text{as } y \to 0,$$

and

$$|\widetilde{M}_J| \lesssim b_1^{|J|_2} y^{2|J|_2-j\gamma-2-\gamma} \quad \text{as } y \to +\infty,$$

where we used the facts that $j \ge 2$ and $2 - j\gamma < 0$, and similarly for higher derivatives by the Leibniz rule. Thus, we obtain the round estimate for all $0 \le m \le L$,

$$\begin{aligned}\int_{y\le 2B_1} \left|\mathscr{L}^{\hbar+m+1}\left(\frac{R_1}{y^2}\right)\right|^2 &\lesssim b_1^{2|J|_2} \int_{y\le 2B_1} |y^{2|J|_2-j\gamma-\gamma-2-2(\hbar+m+1)}|^2 y^{d-1}\, dy \\ &\lesssim b_1^{2m+4+2(1-\delta)-C_L\eta}.\end{aligned}$$

The term R_2/y^2 is estimated exactly as the term R_1/y^2 using the definition (2-32). Similarly, the control of $\int_{y\le 2B_1}|\Psi_b|^2/(1+y^{4(\hbar+m+1)})$ is obtained along the exact same lines as above. This concludes the proof of (2-25). The local estimate (2-26) directly follows from the homogeneity of S_k and the admissibility of T_k. □

We now proceed to a simple localization of the profile Q_b to avoid the growth of tails in the region $y \ge 2B_1 \gg B_0$. More precisely, we claim the following:

Proposition 2.12 (estimates on the localized profile). *Under the assumptions of Proposition 2.11, we assume in addition the a priori bound*

$$|(b_1)_s| \lesssim b_1^2. \tag{2-40}$$

Consider the localized profile

$$\widetilde{Q}_{b(s)}(y) = Q(y) + \sum_{k=1}^{L} b_k \widetilde{T}_k + \sum_{k=2}^{L+2} \widetilde{S}_k \quad \textit{with } \widetilde{T}_k = \chi_{B_1} T_k,\ \ \widetilde{S}_k = \chi_{B_1} S_k, \tag{2-41}$$

where B_1 and χ_{B_1} are defined as in (1-20) *and* (1-21). *Then*

$$\partial_s \widetilde{Q}_b - \partial_{yy}\widetilde{Q}_b - \frac{(d-1)}{y}\partial_y \widetilde{Q}_b + b_1\Lambda\widetilde{Q}_b + \frac{(d-1)}{2y^2}\sin(2\widetilde{Q}_b) = \widetilde{\Psi}_b + \chi_{B_1}\operatorname{Mod}(t), \tag{2-42}$$

where $\widetilde{\Psi}_b$ satisfies the bounds:

(i) (*large Sobolev bound*) *For all* $0 \le m \le L-1$,

$$\int |\mathscr{L}^{\hbar+m+1}\widetilde{\Psi}_b|^2 + \int \frac{|\mathscr{A}\mathscr{L}^{\hbar+m}\widetilde{\Psi}_b|^2}{1+y^2} + \int \frac{|\mathscr{L}^{\hbar+m}\widetilde{\Psi}_b|^2}{1+y^4} + \int \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(\hbar+m+1)}} \lesssim b_1^{2m+2+2(1-\delta)-C_L\eta}, \tag{2-43}$$

and

$$\int |\mathscr{L}^{\hbar+L+1}\widetilde{\Psi}_b|^2 + \int \frac{|\mathscr{A}\mathscr{L}^{\hbar+L}\widetilde{\Psi}_b|^2}{1+y^2} + \int \frac{|\mathscr{L}^{\hbar+L}\widetilde{\Psi}_b|^2}{1+y^4} + \int \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(\hbar+L+1)}} \lesssim b_1^{2L+2+2(1-\delta)(1+\eta)}, \tag{2-44}$$

where $\hbar$ and δ are defined by (1-18).

(ii) (*very local bound*) *For all $M \le \frac{1}{2}B_1$ and $0 \le m \le L$,*

$$\int_{y\le 2M} |\mathscr{L}^{\hbar+m+1}\widetilde{\Psi}_b|^2 \lesssim M^C b_1^{2L+6}. \tag{2-45}$$

(iii) (*refined local bound near B_0*) *For all* $0 \le m \le L$,

$$\int_{y\le 2B_0} |\mathscr{L}^{\hbar+m+1}\widetilde{\Psi}_b|^2 + \int_{y\le 2B_0} \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(\hbar+m+1)}} \lesssim b_1^{2m+4+2(1-\delta)-C_L\eta}. \tag{2-46}$$

Proof. By a direct computation, we have

$$\begin{aligned}
&\partial_s \widetilde{Q}_b - \partial_{yy}\widetilde{Q}_b - \frac{(d-1)}{y}\partial_y \widetilde{Q}_b + b_1\Lambda\widetilde{Q}_b + \frac{(d-1)}{2y^2}\sin(2\widetilde{Q}_b)\\
&\qquad = \chi_{B_1}\Big[\partial_s Q_b - \partial_{yy}Q_b - \frac{(d-1)}{y}\partial_y Q_b + b_1\Lambda Q_b + \frac{(d-1)}{2y^2}\sin(2Q_b)\Big]\\
&\qquad\quad + \Theta_b\Big[\partial_s\chi_{B_1} - \Big(\partial_{yy}\chi_{B_1} + \frac{d-1}{y}\partial_y\chi_{B_1}\Big) + b_1\Lambda\chi_{B_1}\Big] - 2\partial_y\chi_{B_1}\partial_y\Theta_b + b_1(1-\chi_{B_1})\Lambda Q\\
&\qquad\quad + \frac{(d-1)}{2y^2}[\sin(2\widetilde{Q}_b) - \sin(2Q) - \chi_{B_1}(\sin(2Q_b) - \sin(2Q))].
\end{aligned}$$

According to (2-23) and (2-42), we write

$$\widetilde{\Psi}_b = \chi_{B_1}\Psi_b + \widehat{\Psi}_b,$$

where

$$\begin{aligned}
\widehat{\Psi}_b = \underbrace{b_1(1-\chi_{B_1})\Lambda Q}_{\widehat{\Psi}_b^{(1)}} + \underbrace{\frac{(d-1)}{2y^2}[\sin(2\widetilde{Q}_b) - \sin(2Q) - \chi_{B_1}(\sin(2Q_b) - \sin(2Q))]}_{\widehat{\Psi}_b^{(2)}}\\
+ \underbrace{\Theta_b\Big[\partial_s\chi_{B_1} - \Big(\partial_{yy}\chi_{B_1} + \frac{d-1}{y}\partial_y\chi_{B_1}\Big) + b_1\Lambda\chi_{B_1}\Big] - 2\partial_y\chi_{B_1}\partial_y\Theta_b}_{\widehat{\Psi}_b^{(3)}}.
\end{aligned}$$

The contribution of the term $\chi_{b_1}\Psi_b$ to the bounds (2-43), (2-44), (2-45) and (2-46) follows in exactly the same way as in the proof of (2-25) and (2-26). We are therefore left to estimate the term $\widehat{\Psi}_b$. All the terms in the expression of $\widehat{\Psi}_b$ are localized in $B_1 \le y \le 2B_1$, except for the first one whose support is a subset of $\{y \ge B_1\}$. Hence, the estimates (2-45) and (2-46) directly follow from (2-26) and (2-25).

Let us now find the contribution of $\widehat{\Psi}_b$ to the bounds (2-43) and (2-44). We estimate

$$\text{for all } n \in \mathbb{N},\quad \Big|\frac{d^n}{dy^n}(1-\chi_{B_1})\Lambda Q\Big| \lesssim \frac{1}{y^{\gamma+n}}\mathbf{1}_{y\ge B_1};$$

hence, using the relation $d - 2\gamma - 4\hbar = 4\delta$, see (1-18), and the definition (1-20) of B_1, we estimate for all $0 \le m \le L$,

$$\int |\mathscr{L}^{\hbar+m+1}\widehat{\Psi}_b^{(1)}|^2 \lesssim b_1^2 \int_{y\ge B_1} \frac{y^{d-1}}{y^{4(\hbar+m+1)+2\gamma}} \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)+2m\eta}.$$

For the nonlinear term $\widehat{\Psi}_b^{(2)}$, we note from the admissibility of T_k and the homogeneity of S_k that the T_k-terms dominate for $y \ge B_1$ in Θ_b. Thus, for $y \ge B_1$,

$$\text{for all } n \in \mathbb{N},\quad |\partial_y^n\Theta_b| \lesssim \sum_{k=1}^{L} b_1^k y^{2k-\gamma-n}\mathbf{1}_{y\ge B_1}. \tag{2-47}$$

Using (2-47) and noting that $\widehat{\Psi}_b^{(2)}$ is localized in $B_1 \le y \le 2B_1$, we obtain the round bound

$$|\partial_y^n \widehat{\Psi}_b^{(2)}| \lesssim \sum_{k=1}^{L} b_1^k y^{2(k-1)-\gamma-n} \mathbf{1}_{B_1 \le y \le 2B_1} \lesssim \frac{b_1}{y^{\gamma+n}} \sum_{k=1}^{L} b_1^{-(k-1)\eta} \mathbf{1}_{B_1 \le y \le 2B_1}.$$

We then estimate for $0 \le m \le L$,

$$\int |\mathscr{L}^{\hbar+m+1} \widehat{\Psi}_b^{(2)}| \lesssim b_1^2 \sum_{k=1}^{L} b_1^{-2(k-1)\eta} \int_{B_1 \le y \le 2B_1} \frac{y^{d-1}}{y^{4(\hbar+m+1)+2\gamma}} \, dy \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)} \sum_{k=1}^{L} b_1^{(2m-2k+2)\eta}.$$

To control $\widehat{\Psi}_b^{(3)}$, we first note from the definition (1-21) and the assumption (2-40) that

$$|\partial_s \chi_{B_1}| \lesssim \frac{(b_1)_s}{b_1} \frac{y}{B_1} \mathbf{1}_{B_1 \le y \le 2B_1} \lesssim b_1 \mathbf{1}_{B_1 \le y \le 2B_1}.$$

Using (2-47), we estimate for $0 \le m \le L$,

$$\int |\mathscr{L}^{\hbar+m+1} \widehat{\Psi}_b^{(3)}| \lesssim \sum_{k=1}^{L} b_1^2 b_1^{2k} \int_{B_1 \le y \le 2B_1} \frac{y^{d-1}}{y^{4(\hbar+m+1)+2\gamma-4k+2}} \, dy \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)} \sum_{k=1}^{L} b_1^{(2m-2k)\eta}.$$

Gathering all the bounds yields

$$\int |\mathscr{L}^{\hbar+m+1} \widehat{\Psi}_b|^2 \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)} \sum_{k=1}^{L} b_1^{(2m-2k)\eta} \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)+2\eta(m-L)}.$$

The control of

$$\int \frac{|\mathscr{A} \mathscr{L}^{\hbar+m} \widetilde{\Psi}_b|^2}{1+y^2}, \quad \int \frac{|\mathscr{L}^{\hbar+m} \widetilde{\Psi}_b|^2}{1+y^4}, \quad \text{and} \quad \int \frac{|\widehat{\Psi}_b|^2}{1+y^{4(\hbar+m+1)}}$$

is obtained along the exact same lines as above. This concludes the proof of (2-43) and (2-44), as well as Proposition 2.12. □

2D. ***Study of the dynamical system for $b = (b_1, \dots, b_L)$.*** The construction of the Q_b profile formally leads to the finite-dimensional dynamical system for $b = (b_1, \dots, b_L)$ by setting to zero the inhomogeneous $\mathrm{Mod}(t)$ term given in (2-24):

$$(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \ b_{L+1} = 0. \tag{2-48}$$

Unlike the critical case ($d = 2$) treated in [Raphaël and Schweyer 2014b], there is no further logarithmic correction to be taken into account in the system (2-48). In particular, the system (2-48) admits explicit solutions and the linearized operator near these solutions is explicit.

Lemma 2.13 (solution to the system (2-48)). *Let*

$$\tfrac{1}{2}\gamma < \ell \ll L, \quad \ell \in \mathbb{N}^*,$$

and consider the sequence

$$\begin{cases} c_1 = \dfrac{\ell}{2\ell-\gamma}, \\ c_{k+1} = -\dfrac{\gamma(\ell-k)}{2\ell-\gamma} c_k, & 1 \le k \le \ell-1, \\ c_{k+1} = 0, & k \ge \ell. \end{cases} \tag{2-49}$$

Then the explicit choice

$$b_k^e(s) = \frac{c_k}{s^k}, \quad s > 0, \ 1 \le k \le L, \tag{2-50}$$

is a solution to (2-48).

The proof of Lemma 2.13 directly follows from an explicit computation which is left to the reader. We claim that the linearized flow of (2-48) near the solution (2-50) is explicit and displays $\ell - 1$ unstable directions. Note that the stability is considered in the sense that

$$\sup_s s^k |b_k(s)| \le C_k, \quad 1 \le k \le L.$$

In particular, we have the following result which was proved in [Merle, Raphaël and Rodnianski 2015]:

Lemma 2.14 (linearization of (2-48) around (2-50)). *Let*

$$b_k(s) = b_k^e(s) + \frac{\mathcal{U}_k(s)}{s^k}, \quad 1 \le k \le \ell, \tag{2-51}$$

and note that $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_\ell)$. *Then, for* $1 \le k \le \ell - 1$,

$$(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = \frac{1}{s^{k+1}} [s(\mathcal{U}_k)_s - (A_\ell \mathcal{U})_k + \mathcal{O}(|\mathcal{U}|^2)], \tag{2-52}$$

$$(b_\ell)_s + (2\ell-\gamma) b_1 b_\ell = \frac{1}{s^{k+1}} [s(\mathcal{U}_\ell)_s - (A_\ell \mathcal{U})_\ell + \mathcal{O}(|\mathcal{U}|^2)], \tag{2-53}$$

where

$$A_\ell = (a_{i,j})_{1 \le i,j \le \ell} \quad \textit{with} \quad \begin{cases} a_{1,1} = \dfrac{\gamma(\ell-1)}{2\ell-\gamma} - (2-\gamma)c_1, \\ a_{i,i} = \dfrac{\gamma(\ell-i)}{2\ell-\gamma}, & 2 \le i \le \ell, \\ a_{i,i+1} = 1, & 1 \le i \le \ell-1, \\ a_{1,i} = -(2i-\gamma)c_i, & 2 \le i \le \ell, \\ a_{i,j} = 0 & \text{otherwise.} \end{cases}$$

Moreover, A_ℓ *is diagonalizable*:

$$A_\ell = P_\ell^{-1} D_\ell P_\ell, \quad D_\ell = \operatorname{diag}\left\{-1, \frac{2\gamma}{2\ell-\gamma}, \frac{3\gamma}{2\ell-\gamma}, \dots, \frac{\ell\gamma}{2\ell-\gamma}\right\}. \tag{2-54}$$

Proof. Since we have an analogous system to the one in [Merle, Raphaël and Rodnianski 2015] and the proof is essentially the same as written there, we kindly refer the reader to Lemma 3.7 in that paper for all details of the proof. □

3. Proof of Theorem 1.1 assuming technical results

This section is devoted to the proof of Theorem 1.1. We proceed in three subsections:

- In the first subsection, we give an equivalent formulation of the linearization of the problem in the setting (1-30).
- In the second subsection, we prepare the initial data and define the shrinking set $\mathcal{S}_K$ (see Definition 3.2) such that the solution trapped in this set satisfies the conclusion of Theorem 1.1.
- In the third subsection, we give all arguments of the proof of the existence of solutions trapped in $\mathcal{S}_K$ (Proposition 3.5) assuming an important technical result (Proposition 3.6) whose proof is left to the next section. Then we conclude the proof of Theorem 1.1.

3A. ***Linearization of the problem.*** Let $L \gg 1$ be an integer and $s_0 \gg 1$. We introduce the renormalized variables

$$y = \frac{r}{\lambda(t)}, \quad s = s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \tag{3-1}$$

and the decomposition

$$u(r,t) = w(y,s) = (\tilde{Q}_{b(s)} + q)(y,s) = (\tilde{Q}_{b(t)} + q)\left(\frac{r}{\lambda(t)}, t\right), \tag{3-2}$$

where $\tilde{Q}_b$ is constructed in Proposition 2.12 and the modulation parameters

$$\lambda(t) > 0, \quad b(t) = (b_1(t), \dots, b_L(t))$$

are determined from the $L+1$ orthogonality conditions

$$\langle q, \mathscr{L}^k \Phi_M \rangle = 0, \quad 0 \le k \le L, \tag{3-3}$$

where Φ_M is a fixed direction depending on some large constant M defined by

$$\Phi_M = \sum_{k=0}^{L} c_{k,M} \mathscr{L}^k (\chi_M \Lambda Q), \tag{3-4}$$

with

$$c_{0,M} = 1, \qquad c_{k,M} = (-1)^{k+1} \frac{\sum_{j=0}^{k-1} c_{j,M} \langle \chi_M \mathscr{L}^j (\chi_M \Lambda Q), T_k \rangle}{\langle \chi_M \Lambda Q, \Lambda Q \rangle}, \quad 1 \le k \le L. \tag{3-5}$$

Here, Φ_M is built to ensure the nondegeneracy

$$\langle \Phi_M, \Lambda Q\rangle = \langle \chi_M \Lambda Q, \Lambda Q\rangle \gtrsim M^{d-2\gamma} \tag{3-6}$$

and the cancellation

$$\langle \Phi_M, T_k\rangle = \sum_{j=0}^{k-1} c_{j,M} \langle \mathscr{L}^j(\chi_M \Lambda Q), T_k\rangle + c_{k,M}(-1)^k \langle \chi_M \Lambda Q, \Lambda Q\rangle = 0. \tag{3-7}$$

In particular, we have

$$\langle \mathscr{L}^i T_k, \Phi_M\rangle = (-1)^k \langle \chi_M \Lambda Q, \Lambda Q\rangle \delta_{i,k}, \quad 0 \le i, k \le L. \tag{3-8}$$

From (2-2), we see that q satisfies the equation

$$\partial_s q - \frac{\lambda_s}{\lambda}\Lambda q + \mathscr{L} q = -\widetilde{\Psi}_b - \widehat{\mathrm{Mod}} + \mathcal{H}(q) - \mathcal{N}(q) \equiv \mathcal{F}, \tag{3-9}$$

where

$$\widehat{\mathrm{Mod}} = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda \widetilde{Q}_b - \chi_{B_1} \,\mathrm{Mod}, \tag{3-10}$$

$\mathcal{H}$ is the linear part given by

$$\mathcal{H}(q) = \frac{(d-1)}{y^2}[\cos(2Q) - \cos(2\widetilde{Q}_b)]q, \tag{3-11}$$

and $\mathcal{N}$ is the purely nonlinear term

$$\mathcal{N}(q) = \frac{(d-1)}{2y^2}[\sin(2\widetilde{Q}_b + 2q) - \sin(2\widetilde{Q}_b) - 2q\cos(2\widetilde{Q}_b)]. \tag{3-12}$$

We also need to write (3-9) in the original variables. To do so, consider the rescaled linearized operator

$$\mathscr{L}_\lambda = -\partial_{rr} - \frac{(d-1)}{r}\partial_r + \frac{Z_\lambda}{r^2} \tag{3-13}$$

and the renormalized function

$$v(r,t) = q(y,s), \quad \partial_t v = \frac{1}{\lambda^2(t)}\left(\partial_s q - \frac{\lambda_s}{\lambda}\Lambda q\right)_\lambda.$$

Then from (3-9), v satisfies

$$\partial_t v + \mathscr{L}_\lambda v = \frac{1}{\lambda^2}\mathcal{F}_\lambda, \quad \mathcal{F}_\lambda(r,t) = \mathcal{F}(y,s). \tag{3-14}$$

Note that

$$\mathscr{L}_\lambda = \frac{1}{\lambda^2}\mathscr{L}.$$

3B. ***Preparation of the initial data.*** We now describe the set of initial data u_0 of the problem (1-4), as well as the initial data for (b,λ) leading to the blowup scenario of Theorem 1.1. Assume that $u_0 \in H^\infty(\mathbb{R}^d)$ satisfies

$$\|u_0 - Q\|_{\dot{H}^s} \ll 1 \quad \text{for } \tfrac{d}{2} \le s \le \Bbbk. \tag{3-15}$$

By continuity of the flow and a standard argument, the smallness assumption (3-15) is propagated on a small time interval $[0, t_1)$. Thus, the decomposition (3-2),

$$u(r,t) = (\widetilde{Q}_{b(t)} + q)\left(\frac{r}{\lambda(t)}, t\right), \quad \lambda(t) > 0, \;\; b = (b_1, \dots, b_L), \tag{3-16}$$

can be uniquely defined on the interval $t \in [0, t_1]$.

The existence of the decomposition (3-16) is a standard consequence of the implicit function theorem and the explicit relations

$$\frac{\partial}{\partial \lambda}(\widetilde{Q}_{b(t)})_\lambda, \frac{\partial}{\partial b_1}(\widetilde{Q}_{b(t)})_\lambda, \dots, \frac{\partial}{\partial b_L}(\widetilde{Q}_{b(t)})_\lambda \bigg|_{\lambda=1, b=0} = (\Lambda Q, T_1, \dots, T_L),$$

which implies the nondegeneracy of the Jacobian

$$\left| \left\langle \frac{\partial}{\partial(\lambda, b_j)}(\widetilde{Q}_{b(t)})_\lambda, \mathscr{L}^i \Phi_M \right\rangle_{1 \le j \le L,\, 0 \le i \le L} \right|_{\lambda=1, b=0} = |\langle \chi_M \Lambda Q, \Lambda Q \rangle|^{L+1} \neq 0.$$

In fact, the decomposition (3-16) exists as long as $t < T$ and q remains small in the energy topology. We now set up the bootstrap for the control of the parameters (b, λ) and the radiation q. We will measure the regularity of the map through the following coercive norms of q:

$$\mathscr{E}_{2k} = \int |\mathscr{L}^k q|^2 \ge C(M) \sum_{m=0}^{k-1} \int \frac{|\mathscr{L}^m q|^2}{1 + y^{4(k-m)}} \quad \text{for } \hbar + 1 \le k \le \Bbbk. \tag{3-17}$$

Our construction is built on a careful choice of the initial data for the modulation parameter b and the radiation q at time $s = s_0$. In particular, we will choose it in the following way:

Definition 3.1 (choice of the initial data). Take η and δ as in (1-20) and (1-18). Let consider the variable

$$\mathcal{V} = P_\ell \mathcal{U}, \tag{3-18}$$

where $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_\ell)$ is introduced in the linearization (2-51), namely

$$\mathcal{U}_k = s^k b_k - c_k, \quad \text{with } c_k \text{ given by (2-49)},$$

and P_ℓ refers to the diagonalization (2-54) of A_ℓ.

Let $s_0 \ge 1$. We assume

- (smallness of the initial perturbation for the b_k-unstable modes)

$$|s_0^{\frac{\eta}{2}(1-\delta)} \mathcal{V}_k(s_0)| < 1 \quad \text{for } 2 \le k \le \ell, \tag{3-19}$$

- (smallness of the initial perturbation for the b_k-stable modes)

$$|s_0^{\frac{\eta}{2}(1-\delta)} \mathcal{V}_1(s_0)| < 1, \qquad |b_k(s_0)| < s_0^{-\frac{5\ell(2k-\gamma)}{2\ell-\gamma}} \quad \text{for } \ell + 1 \le k \le L, \tag{3-20}$$

- (smallness of the data)
$$\sum_{k=\hbar+2}^{\Bbbk} \mathscr{E}_{2k}(s_0) < s_0^{-\frac{10L\ell}{2\ell-\gamma}}, \tag{3-21}$$
- (normalization) up to a fixed rescaling, we may always assume
$$\lambda(s_0) = 1. \tag{3-22}$$

In particular, the initial data described in Definition 3.1 belongs to the following set which shrinks to zero as $s \to +\infty$:

Definition 3.2 (definition of the shrinking set). Take η and δ as in (1-20) and (1-18). For all $K \geq 1$ and $s \geq 1$, we define $\mathcal{S}_K(s)$ as the set of all $(b_1(s), \dots, b_L(s), q(s))$ such that

$$\begin{aligned}
&|\mathcal{V}_k(s)| \leq 10 s^{-\frac{\eta}{2}(1-\delta)} && \text{for } 1 \leq k \leq \ell, \\
&|b_k(s)| \leq s^{-k} && \text{for } \ell+1 \leq k \leq L, \\
&\mathscr{E}_{2\Bbbk}(s) \leq K s^{-(2L+2(1-\delta)(1+\eta))}, && \\
&\mathscr{E}_{2m}(s) \leq \begin{cases} K s^{-\frac{\ell}{2\ell-\gamma}(4m-d)} \\ s^{-2(m-\hbar-1)-2(1-\delta)+K\eta} \end{cases} && \begin{aligned}&\text{for } \hbar+2 \leq m \leq \ell+\hbar, \\ &\text{for } \ell+\hbar+1 \leq m \leq \Bbbk-1.\end{aligned}
\end{aligned}$$

Remark 3.3. From (2-51), the bounds given in Definition 3.2 imply that for η small enough,
$$b_1(s) \sim \frac{c_1}{s}, \quad |b_k(s)| \lesssim |b_1(s)|^k.$$
Hence, the choice of the initial data $(b(s_0), q(s_0))$ belongs in $\mathcal{S}_K(s_0)$ if s_0 is large enough.

Remark 3.4. The introduction of the high Sobolev norm $\mathscr{E}_{2\Bbbk}$ is reflected in the relation
$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{k=1}^{L} |(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}| \lesssim C(M)\sqrt{\mathscr{E}_{2\Bbbk}} + \text{l.o.t.}, \tag{3-23}$$
which is computed thanks to the $L+1$ orthogonality conditions (3-3) (see Lemmas 4.2 and 4.3 below).

3C. *Existence of solutions trapped in $\mathcal{S}_K(s)$ and conclusion of Theorem 1.1.*

We claim the following proposition:

Proposition 3.5 (existence of solutions trapped in $\mathcal{S}_K(s)$). *There exists $K_1 \geq 1$ such that for $K \geq K_1$, there exists $s_{0,1}(K)$ such that for all $s_0 \geq s_{0,1}$, there exists initial data for the unstable modes*
$$(\mathcal{V}_2(s_0), \dots, \mathcal{V}_\ell(s_0)) \in [-s_0^{-\frac{\eta}{2}(1-\delta)}, s_0^{-\frac{\eta}{2}(1-\delta)}]^{\ell-1}$$
such that the corresponding solution $(b(s), q(s))$ is in $\mathcal{S}_K(s)$ for all $s \geq s_0$.

Let us briefly give the proof of Proposition 3.5. Let us consider $K \geq 1$ and $s_0 \geq 1$ and $(b(s_0), q(s_0))$ as in Definition 3.1. We introduce the exit time
$$s_* = s_*(b(s_0), q(s_0)) = \sup\{s \geq s_0 \text{ such that } (b(s), q(s)) \in \mathcal{S}_K(s)\},$$

and assume that for any choice of

$$(\mathcal{V}_2(s_0),\dots,\mathcal{V}_\ell(s_0)) \in [-s_0^{-\frac{\eta}{2}(1-\delta)}, s_0^{-\frac{\eta}{2}(1-\delta)}]^{\ell-1},$$

the exit time satisfies $s_* < +\infty$ and look for a contradiction. By the definition of $\mathcal{S}_K(s_*)$, at least one of the inequalities in that definition is an equality. Owing the following proposition, this can happen only for the components $(\mathcal{V}_2(s_*),\dots,\mathcal{V}_\ell(s_*))$. Precisely, we have the following result which is the heart of our analysis:

Proposition 3.6 (control of $(b(s), q(s))$ in $\mathcal{S}_K(s)$ by $(\mathcal{V}_2(s),\dots,\mathcal{V}_\ell(s))$). *There exists $K_2 \ge 1$ such that for each $K \ge K_2$, there exists $s_{0,2}(K) \ge 1$ such that for all $s_0 \ge s_{0,2}(k)$, the following holds: Given the initial data at $s = s_0$ as in Definition 3.1, if $(b(s), q(s)) \in \mathcal{S}_K(s)$ for all $s \in [s_0, s_1]$, with $(b(s_1), q(s_1)) \in \partial\mathcal{S}_K(s_1)$ for some $s_1 \ge s_0$, then:*

(i) (*reduction to a finite-dimensional problem*)

$$(\mathcal{V}_2(s_1),\dots,\mathcal{V}_\ell(s_1)) \in \partial\left[-\frac{K}{s_1^{\frac{\eta}{2}(1-\delta)}}, \frac{K}{s_1^{\frac{\eta}{2}(1-\delta)}}\right]^{\ell-1}.$$

(ii) (*transverse crossing*)

$$\frac{d}{ds}\left(\sum_{i=2}^{\ell} |s^{\frac{\eta}{2}(1-\delta)}\mathcal{V}_i(s)|^2\right)\Bigg|_{s=s_1} > 0.$$

Let us assume Proposition 3.6 and continue the proof of Proposition 3.5. From part (i) of Proposition 3.6, we see that

$$(\mathcal{V}_2(s_*),\dots,\mathcal{V}_\ell(s_*)) \in \partial\left[-\frac{K}{s_*^{\frac{\eta}{2}(1-\delta)}}, \frac{K}{s_*^{\frac{\eta}{2}(1-\delta)}}\right]^{\ell-1},$$

and the mapping

$$\Upsilon : [-1,1]^{\ell-1} \to \partial([-1,1]^{\ell-1}),$$
$$s_0^{\frac{\eta}{2}(1-\delta)}(\mathcal{V}_2(s_0),\dots,\mathcal{V}_\ell(s_0)) \mapsto \frac{s_*^{\frac{\eta}{2}(1-\delta)}}{K}(\mathcal{V}_2(s_*),\dots,\mathcal{V}_\ell(s_*)),$$

is well-defined. Applying the transverse-crossing property given in part (ii) of Proposition 3.6, we see that $(b(s), q(s))$ leaves $\mathcal{S}_K(s)$ at $s = s_0$; hence, $s_* = s_0$. This is a contradiction since Υ is the identity map on the boundary sphere and it cannot be a continuous retraction of the unit ball. This concludes the proof of Proposition 3.5, assuming that Proposition 3.6 holds.

• Conclusion of Theorem 1.1 assuming Proposition 3.6. From Proposition 3.5, we know that there exists initial data $(b(s_0), q(s_0))$ such that

$$(b(s), q(s)) \in \mathcal{S}_K(s) \quad \text{for all } s \ge s_0.$$

From (4-57), (4-58), we have

$$-\lambda\lambda_t = c(u_0)\lambda^{\frac{2\ell-\gamma}{\ell}}[1 + o(1)],$$

which yields

$$-\lambda^{1-\frac{2\ell-\gamma}{\ell}}\lambda_t = c(u_0)(1+o(1)).$$

We easily conclude that λ vanishes in finite time $T = T(u_0) < +\infty$ with the following behavior near the blowup time:

$$\lambda(t) = c(u_0)(1+o(1))(T-t)^{\frac{\ell}{\gamma}},$$

which is the conclusion of item (i) of Theorem 1.1.

For the control of the Sobolev norms, we observe from (B-3) and Definition 3.2 that

$$\text{for all } \hbar + 2 \le m \le \Bbbk, \quad \int |\partial_y^{2m} q|^2 \lesssim \mathscr{E}_{2m} \to 0 \quad \text{as } s \to +\infty.$$

From the relation $d = 4\hbar + 4\delta + 2\gamma$, we deduce that

$$\text{for all } \sigma \in \left[\tfrac{d}{2} + 3, 2\Bbbk\right], \quad \int |\nabla^\sigma q|^2 \to 0 \quad \text{as } s \to +\infty,$$

which yields (ii) of Theorem 1.1.

4. Reduction of the problem to a finite-dimensional one

We now prove Proposition 3.6, which is the heart of our analysis. We proceed in three separate subsections:

- In the first subsection, we derive the laws for the parameters (b, λ) thanks to the orthogonality condition (3-3) and the coercivity of the powers of $\mathscr{L}$.
- In the second subsection, we prove the main monotonicity tools for the control of the infinite-dimensional part of the solution. In particular, we derive a suitable Lyapunov functional for the $\mathscr{E}_{2\Bbbk}$ energy, as well as the monotonicity formula for the lower Sobolev energy.
- In the third subsection, we conclude the proof of Proposition 3.6 thanks to the identities obtained in the first two parts.

4A. ***Modulation equations.*** We derive here the modulation equations for (b, λ). The derivation is mainly based on the orthogonality (3-3) and the coercivity of the powers of $\mathscr{L}$. Let us start with elementary estimates relating to the fixed direction Φ_M.

Lemma 4.1 (estimate for Φ_M). *Given Φ_M as defined in* (3-4), *we have*

$$|c_{k,M}| \lesssim M^{2k} \quad \textit{for all } 1 \le k \le L,$$

$$\int |\Phi_M|^2 \lesssim M^{d-2\gamma}, \quad \int |\mathscr{L}\Phi_M|^2 \lesssim M^{d-2\gamma-4}.$$

Proof. Arguing by induction, we assume that

$$|c_{j,M}| \lesssim M^{2j}, \quad 1 \le j \le k.$$

Using the fact that $\mathscr{L}^j T_i$ is admissible of degree $(\max\{0, i-j\}, i-j)$, we estimate from the definition (3-5),

$$|c_{k+1,M}| \lesssim \frac{1}{M^{d-2\gamma}} \sum_{j=0}^{k} M^{2j} \int |\chi_M \Lambda Q \mathscr{L}^j(T_{k+1})|$$
$$\lesssim \frac{1}{M^{d-2\gamma}} \sum_{j=0}^{k} M^{2j} \int_{y\le M} \frac{y^{d-1}}{y^\gamma} y^{2(k+1-j)-\gamma}\,dy \lesssim M^{2(k+1)}.$$

Using the estimate for $c_{k,M}$ yields

$$\int |\Phi_M|^2 \lesssim \int |\chi_M \Lambda Q|^2 + \sum_{j=1}^{L} |c_{j,M}|^2 \int |\mathscr{L}^j(\chi_M \Lambda Q)|^2 \lesssim M^{d-2\gamma-4},$$

and

$$\int |\mathscr{L}\Phi_M|^2 \lesssim \sum_{j=0}^{L} |c_{j,M}|^2 \int |\mathscr{L}^{j+1}(\chi_M \Lambda Q)|^2 \lesssim M^{d-2\gamma}. \qquad \square$$

From the orthogonality conditions (3-3) and (3-9), we claim the following:

Lemma 4.2 (modulation equations). *Take $\hbar$, δ and η as defined in* (1-18) *and* (1-20). *For $K \ge 1$, we assume that there is $s_0(K) \gg 1$ such that $(b(s), q(s)) \in \mathcal{S}_K(s)$ for $s \in [s_0, s_1]$ for some $s_1 \ge s_0$. Then, the following hold for $s \in [s_0, s_1]$:*

$$\sum_{k=1}^{L-1} |(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}| + \left| b_1 + \frac{\lambda_s}{\lambda} \right| \lesssim b_1^{L+1+(1-\delta)(1+\eta)}, \tag{4-1}$$

and

$$|(b_L)_s + (2L-\gamma) b_1 b_L| \lesssim \frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)}. \tag{4-2}$$

Proof. We start with the law for b_L. Let

$$D(t) = \left| b_1 + \frac{\lambda_s}{\lambda} \right| + \sum_{k=1}^{L} |(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}|,$$

where we recall that $b_k \equiv 0$ if $k \ge L+1$.

Now, we take the inner product of (3-9) with $\mathscr{L}^L \Phi_M$ and use the orthogonality (3-3) to write

$$\langle \widehat{\mathrm{Mod}}(t), \mathscr{L}^L \Phi_M \rangle = -\langle \mathscr{L}^L \widetilde{\Psi}_b, \Phi_M \rangle - \langle \mathscr{L}^{L+1} q, \Phi_M \rangle - \left\langle -\frac{\lambda_s}{\lambda} \Lambda q - \mathcal{L}(q) + \mathcal{N}(q), \mathscr{L}^L \Phi_M \right\rangle. \tag{4-3}$$

From the definition (3-4), we see that Φ_M is localized in $y \le 2M$. From (3-10) and (2-24), we compute by using the identity (3-8),

$$\langle \widehat{\mathrm{Mod}}(t), \mathscr{L}^L \Phi_M \rangle = (-1)^L \langle \Lambda Q, \Phi_M \rangle [(b_L)_s + (2L-\gamma) b_1 b_L] + \mathcal{O}(M^C b_1 D(t)).$$

The error term is estimated by using (2-26) with $m = L - \hbar - 1$ and Lemma 4.1:

$$|\langle \mathscr{L}^L \widetilde{\Psi}_b, \Phi_M \rangle| \le \left(\int_{y \le 2M} |\mathscr{L}^L \widetilde{\Psi}_b|^2 \right)^{\frac{1}{2}} \left(\int_{y \le 2M} |\Phi_M|^2 \right)^{\frac{1}{2}} \lesssim M^C b_1^{L+3} \lesssim b_1^{L+1+(1-\delta)(1+\eta)}.$$

For the linear term, we apply Lemma A.5 with $k = \Bbbk - 1$:

$$\mathscr{E}_{2\Bbbk}(q) \gtrsim \int \frac{|\mathscr{L}^{L+1} q|^2}{y^4(1 + y^{4(\hbar-1)})} \gtrsim \int \frac{|\mathscr{L}^{L+1} q|^2}{1 + y^{4\hbar}}.$$

Hence, the Cauchy–Schwarz inequality yields

$$|\langle \mathscr{L}^{L+1} q, \Phi_M \rangle| \lesssim M^{2\hbar} \left(\int \frac{|\mathscr{L}^{L+1} q|^2}{1 + y^{4\hbar}} \right)^{\frac{1}{2}} \left(\int |\Phi_M|^2 \right)^{\frac{1}{2}} \lesssim M^{2\hbar + \frac{d}{2} - \gamma} \sqrt{\mathscr{E}_{2\Bbbk}}.$$

The remaining terms are easily estimated by using the following bound coming from Lemma A.5 and Lemma A.4:

$$\mathscr{E}_{2\Bbbk}(q) \gtrsim \int \frac{|\mathscr{L} q|^2}{y^4(1 + y^{4(\Bbbk-2)})} \gtrsim \int \frac{|\partial_y q|^2}{y^4(1 + y^{4(\Bbbk-2)+2})} + \int \frac{q^2}{y^6(1 + y^{4(\Bbbk-2)+2})}. \tag{4-4}$$

This implies

$$\left| \left\langle -\frac{\lambda_s}{\lambda} \Lambda q + \mathcal{L}(q) + \mathcal{N}(q), \mathscr{L}^L \Phi_M \right\rangle \right| \lesssim M^C b_1 (\sqrt{\mathscr{E}_{2\Bbbk}} + D(t)).$$

Putting all the above estimates into (4-3) and using (3-6) together with the relation (1-18), we arrive at

$$|(b_L)_s + (2L - \gamma) b_1 b_L| \lesssim \frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)} + M^C b_1 D(t). \tag{4-5}$$

For the modulation equations for b_k with $1 \le k \le L - 1$, we take the inner product of (3-9) with $\mathscr{L}^k \Phi_M$ and use the orthogonality (3-3) to write for $1 \le k \le L - 1$,

$$\langle \widehat{\mathrm{Mod}}(t), \mathscr{L}^k \Phi_M \rangle = -\langle \mathscr{L}^k \widetilde{\Psi}_b, \Phi_M \rangle - \left\langle -\frac{\lambda_s}{\lambda} \Lambda q - \mathcal{L}(q) + \mathcal{N}(q), \mathscr{L}^k \Phi_M \right\rangle.$$

Proceeding as for b_L, we end up with

$$|(b_k)_s + (2k - \gamma) b_1 b_k - b_{k+1}| \lesssim b_1^{L+1+(1-\delta)(1+\eta)} + M^C b_1 (\sqrt{\mathscr{E}_{2\Bbbk}} + D(t)). \tag{4-6}$$

Similarly, we have by taking the inner product of (3-9) with Φ_M,

$$\left| \frac{\lambda_s}{\lambda} + b_1 \right| \lesssim b_1^{L+1+(1-\delta)(1+\eta)} + M^C b_1 (\sqrt{\mathscr{E}_{2\Bbbk}} + D(t)). \tag{4-7}$$

From (4-5), (4-6) and (4-7), we obtain the round bound

$$D(t) \lesssim M^C \sqrt{\mathscr{E}_{2\Bbbk}} + b_1^{L+1+(1-\delta)(1+\eta)}.$$

The conclusion then follows by substituting this bound into (4-5), (4-6) and (4-7). □

From the bound for $\mathscr{E}_{\Bbbk}$ given in Definition 3.2 and the modulation equation (4-2), we only have the pointwise bound

$$|(b_L)_s + (2L-\gamma)b_1 b_L| \lesssim b_1^{L+(1-\delta)(1+\eta)},$$

which is not good enough to close the expected one

$$|(b_L)_s + (2L-\gamma)b_1 b_L| \ll b_1^{L+1}.$$

We claim that the main linear term can be removed up to an oscillation in time leading to the improved modulation equation for b_L as follows:

Lemma 4.3 (improved modulation equation for b_L). *Under the assumption of Lemma 4.2, the following bound holds for all* $s \in [s_0, s_1]$:

$$\left|(b_L)_s + (2L-\gamma)b_1 b_L + \frac{d}{ds}\left\{\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right\}\right| \lesssim \frac{1}{B_0^{2\delta}}\left[C(M)\sqrt{\mathcal{E}_{2\Bbbk}} + b_1^{L+1+(1-\delta)-C_L\eta}\right]. \tag{4-8}$$

Proof. We commute (3-9) with $\mathscr{L}^L$ and take the inner product with $\chi_{B_0}\Lambda Q$ to get

$$\begin{aligned}\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle &\left\{\frac{d}{ds}\left[\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right] - \langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle \frac{d}{ds}\left[\frac{1}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right]\right\} \\ &= \langle \mathscr{L}^L q, \Lambda Q \partial_s(\chi_{B_0})\rangle - \langle \mathscr{L}^{L+1} q, \chi_{B_0}\Lambda Q\rangle + \frac{\lambda_s}{\lambda}\langle \mathscr{L}^L \Lambda q, \chi_{B_0}\Lambda Q\rangle \\ &\quad - \langle \mathscr{L}^L \widetilde{\Psi}_b, \chi_{B_0}\Lambda Q\rangle - \langle \mathscr{L}^L \widehat{\mathrm{Mod}}(t), \chi_{B_0}\Lambda Q\rangle + \langle \mathscr{L}^L(\mathcal{L}(q) - \mathcal{N}(q)), \chi_{B_0}\Lambda Q\rangle. \end{aligned} \tag{4-9}$$

We recall from (2-5) that

$$B_0^{d-2\gamma} \lesssim |\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle| \lesssim B_0^{d-2\gamma}. \tag{4-10}$$

Let us estimate the second term in the left-hand side of (4-9). We use Cauchy–Schwarz and Lemma A.5 to estimate

$$|\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle| \lesssim B_0^{2\hbar+2}\|\chi_{B_0}\Lambda Q\|_{L^2}\left(\int \frac{|\mathscr{L}^L q|^2}{1+y^{4\hbar+4}}\right)^{\frac{1}{2}} \lesssim B_0^{\frac{d}{2}-\gamma+2\hbar+2}\sqrt{\mathscr{E}_{2\Bbbk}}. \tag{4-11}$$

We write

$$\begin{aligned}\left|\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle \frac{d}{ds}\left[\frac{1}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right]\right| &\lesssim \frac{|\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle|}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle^2}\left|\frac{(b_1)_s}{b_1}\right| \int_{B_0 \le y \le 2B_0} |\Lambda Q|^2 \\ &\lesssim b_1 \frac{B_0^{\frac{d}{2}-\gamma+2\hbar+2}\sqrt{\mathscr{E}_{2\Bbbk}}}{B_0^{2d-4\gamma}} B_0^{d-2\gamma} \lesssim \frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{B_0^{2\delta}},\end{aligned}$$

where we used the relation (1-18).

For the first three terms in the right-hand side of (4-9), we use Cauchy–Schwarz, Lemma A.5 and the fact that $\mathscr{L}(\Lambda Q)=0$ to find that

$$\begin{aligned}|\langle \mathscr{L}^L q, \Lambda Q \partial_s(\chi_{B_0})\rangle| &\lesssim \left|\frac{(b_1)_s}{b_1}\right| \left(\int_{B_0\le y\le 2B_0} (1+y^{4\hbar+4})|\Lambda Q|^2\right)^{\frac12}\left(\int \frac{|\mathscr{L}^L q|^2}{1+y^{4\hbar+4}}\right)^{\frac12}\\ &\lesssim b_1 B_0^{\frac{d}{2}-\gamma+2\hbar+2}\sqrt{\mathscr{E}_{2\Bbbk}} \lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}},\\ |\langle \mathscr{L}^{L+1} q, \chi_{B_0}\Lambda Q\rangle| &\lesssim \left(\int (1+y^{4\hbar})|\chi_{B_0}\Lambda Q|^2\right)^{\frac12}\left(\int \frac{|\mathscr{L}^{L+1}q|^2}{1+y^{4\hbar}}\right)^{\frac12} \lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}\langle \mathscr{L}^L \Lambda q, \chi_{B_0}\Lambda Q\rangle\right| &\lesssim b_1 \left(\int (1+y^{4(L+\hbar)+2})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac12}\left(\int \frac{|\partial_y q|^2}{1+y^{4(L+\hbar)+2}}\right)^{\frac12}\\ &\lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}}.\end{aligned}$$

The error term is estimated by using (2-46):

$$\begin{aligned}|\langle \mathscr{L}^L \widetilde{\Psi}_b, \chi_{B_0}\Lambda Q\rangle| &\lesssim \left(\int (1+y^{4(L+\hbar+1)})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac12}\left(\int \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(L+\hbar+1)}}\right)^{\frac12}\\ &\lesssim B_0^{\frac{d}{2}-\gamma+2\hbar+2} b_1^{L+2+(1-\delta)-C_L\eta}.\end{aligned}$$

The last term in the right-hand side of (4-9) is estimated in the same way:

$$\begin{aligned}|\langle \mathscr{L}^L(\mathcal{L}(q)-\mathcal{N}(q)), \chi_{B_0}\Lambda Q\rangle| &\lesssim \int |\mathcal{L}(q)\mathscr{L}^L(\chi_{B_0}\Lambda Q)| + \int |\mathcal{N}(q)\mathscr{L}^L(\chi_{B_0}\Lambda Q)|\\ &\lesssim \left(\int \frac{|\mathcal{L}(q)|^2}{1+y^{4\Bbbk-4}}\right)^{\frac12}\left(\int (1+y^{4\Bbbk-4})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac12}\\ &\qquad + \left(\int \frac{|\mathcal{N}(q)|^2}{1+y^{4\Bbbk}}\right)^{\frac12}\left(\int (1+y^{4\Bbbk})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac12}\\ &\lesssim B_0^{\frac{d}{2}-1-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}} + b_1 B_0^2 B_0^{\frac{d}{2}-1-\gamma+2\hbar}\sqrt{\mathscr{E}_{\Bbbk}}\\ &\lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}}.\end{aligned}$$

For the remaining term, we recall that $\mathscr{L}(\Lambda Q)=0$, $\mathscr{L}^L T_k = 0$ for $1\le k\le L-1$, and $\mathscr{L}^L T_L = (-1)^L \Lambda Q$, from which

$$\mathscr{L}^L(T_k\chi_{B_1}) = -\mathscr{L}^L(T_k(1-\chi_{B_1})), \quad 1\le k\le L-1.$$

From (3-10), (2-24) and the fact that $\chi_{B_0}(1-\chi_{B_1})=0$, we write

$$\begin{aligned}&\left|\langle \mathscr{L}^L \widehat{\mathrm{Mod}}(t), \chi_{B_0}\Lambda Q\rangle - (-1)^L\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle[(b_L)_s + (2L-\gamma)b_1 b_L]\right|\\ &\quad\lesssim \sum_{k=1}^{L} |(b_k)_s + (2k-\gamma)b_1 b_L - b_{k+1}| \left|\left\langle \sum_{j=k+1}^{L+2}\frac{\partial \widetilde{S}_j}{\partial b_k}, \mathscr{L}^L(\chi_{B_0}\Lambda Q)\right\rangle\right| + \left|\frac{\lambda_s}{\lambda}+b_1\right| |\langle \Lambda\widetilde{\Theta}_b, \mathscr{L}^L(\chi_{B_0}\Lambda Q)\rangle|.\end{aligned}$$

Recalling that T_k is admissible of degree (k,k) and S_k is homogeneous of degree $(k,k-1,k)$, we derive the round bounds for $y\sim B_0$:

$$|\Lambda\Theta_b|\lesssim b_1y^{2-\gamma},\qquad \sum_{j=k+1}^{L+2}\left|\frac{\partial S_j}{\partial b_k}\right|\le\sum_{j=k+1}^{L+2}b_1^{j-k}y^{2(j-1)-\gamma}\lesssim b_1y^{2k-\gamma}.$$

Thus, from Lemma 4.2, we derive the bound

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}+b_1\right||\langle\Lambda\widetilde{\Theta}_b,\mathscr{L}^L(\chi_{B_0}\Lambda Q)\rangle|+\sum_{k=1}^{L}|(b_k)_s+(2k-\gamma)b_1b_L-b_{k+1}|\left|\left\langle\sum_{j=k+1}^{L+2}\frac{\partial\widetilde{S}_j}{\partial b_k},\mathscr{L}^L(\chi_{B_0}\Lambda Q)\right\rangle\right|\\ \lesssim(C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)})b_1\int_{B_0\le y\le 2B_0}\frac{y^{2L-\gamma}y^{d-1}}{y^{2L+\gamma}}\,dy\\ \lesssim(C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)})b_1B_0^{d-2\gamma}.\end{aligned}$$

The equation (4-8) follows by gathering all the above estimates into (4-9), dividing both sides of (4-9) by $(-1)^L\langle\Lambda Q,\chi_{B_0}\Lambda Q\rangle$ and using the relation (1-18). □

4B. ***Monotonicity.*** We derive in this subsection the main monotonicity formula for $\mathscr{E}_{2k}$ for $\hbar+1\le k\le\Bbbk$. We claim the following which is the heart of this paper:

Proposition 4.4 (Lyapounov monotonicity for the high Sobolev norm). *We have*

$$\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1^{\eta(1-\delta)})]\right\}\le\frac{b_1}{\lambda^{4\Bbbk-d+2}}\left[\frac{\mathscr{E}_{2\Bbbk}}{M^{2\delta}}+b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{2L+2(1-\delta)(1+\eta)}\right],\tag{4-12}$$

and for $\hbar+2\le m\le\Bbbk-1$,

$$\frac{d}{dt}\left\{\frac{\mathscr{E}_{2m}}{\lambda^{4m-d}}[1+\mathcal{O}(b_1)]\right\}\le\frac{b_1}{\lambda^{4m-d+2}}[b_1^{m-\hbar-1+(1-\delta)-C\eta}\sqrt{\mathscr{E}_{2m}}+b_1^{2(m-\hbar-1)+2(1-\delta)-C\eta}].\tag{4-13}$$

Proof. The proof uses some ideas developed in [Raphaël and Schweyer 2014b; Merle, Raphaël and Rodnianski 2015]. Because the proof of (4-13) follows exactly the same lines as for (4-12), we only deal with the proof of (4-12). Let us start the proof of (4-12).

Step 1: suitable derivatives and energy identity. For $k\in\mathbb{N}$, we define the suitable derivatives of q and v as follows:

$$q_{2k}=\mathscr{L}^kq,\quad q_{2k+1}=\mathscr{A}\mathscr{L}^kq,\quad v_{2k}=\mathscr{L}_\lambda^kv,\quad v_{2k+1}=\mathscr{A}_\lambda\mathscr{L}_\lambda^kv,\tag{4-14}$$

where $q=q(y,s)$ and $v=v(r,t)$ satisfy (3-9) and (3-14) respectively, the linearized operator $\mathscr{L}$ and $\mathscr{L}_\lambda$ are defined by (1-22) and (3-13), $\mathscr{A}$ and $\mathscr{A}^*$ are the first-order operators defined by (2-6) and (2-7), and

$$\mathscr{A}_\lambda f=-\partial_rf+\frac{V_\lambda}{r}f,\quad \mathscr{A}_\lambda^*f=\frac{1}{r^{d-1}}\partial_r(r^{d-1}f)+\frac{V_\lambda}{r}f,$$

with $V=\Lambda\log\Lambda Q$ admitting the asymptotic behaviors as in (2-8).

With the notation (4-14), we note that

$$q_{2k+1}=\mathscr{A}q_{2k},\quad q_{2k+2}=\mathscr{A}^*q_{2k+1},\quad v_{2k+1}=\mathscr{A}_\lambda v_{2k},\quad v_{2k+2}=\mathscr{A}_\lambda^*v_{2k+1}.$$

Recall from Lemma 2.2, we have the factorization

$$\mathscr{L} = \mathscr{A}^*\mathscr{A}, \quad \widetilde{\mathscr{L}} = \mathscr{A}\mathscr{A}^*, \quad \mathscr{L}_\lambda = \mathscr{A}_\lambda^*\mathscr{A}_\lambda, \quad \widetilde{\mathscr{L}}_\lambda = \mathscr{A}_\lambda\mathscr{A}_\lambda^*,$$

where

$$\widetilde{\mathscr{L}} = -\partial_{yy} - \frac{d-1}{y}\partial_y + \frac{\widetilde{Z}}{y^2}, \tag{4-15}$$

$$\widetilde{\mathscr{L}}_\lambda = -\partial_{rr} - \frac{d-1}{r}\partial_r + \frac{\widetilde{Z}_\lambda}{r^2}, \tag{4-16}$$

with $\widetilde{Z}$ expressed in terms of V as in (2-13).

We commute (3-14) with $\mathscr{L}_\lambda^{\Bbbk-1}$ and use the notation (4-14) to derive

$$\partial_t v_{2\Bbbk-2} + \mathscr{L}_\lambda v_{2\Bbbk-2} = [\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right). \tag{4-17}$$

Now commuting this equation with $\mathscr{A}_\lambda$ yields

$$\partial_t v_{2\Bbbk-1} + \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} = \frac{\partial_t V_\lambda}{r} v_{2\Bbbk-2} + \mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right). \tag{4-18}$$

Since $\mathscr{L}_\lambda = (1/\lambda^2)\mathscr{L}$, we then have

$$\mathscr{L}_\lambda^k v = \frac{1}{\lambda^{2k}}\mathscr{L}^k q;$$

hence,

$$\int |\mathscr{L}_\lambda^k v|^2 = \frac{1}{\lambda^{4k-d}}\int |\mathscr{L}^k q|^2.$$

Using the definition (4-16) of $\widetilde{\mathscr{L}}_\lambda$ and an integration by parts, we write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\lambda^{4\Bbbk-d}}\mathscr{E}_{2\Bbbk}\right) &= \frac{1}{2}\frac{d}{dt}\int |\mathscr{L}_\lambda^\Bbbk v|^2 = \frac{1}{2}\frac{d}{dt}\int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} v_{2\Bbbk-1} \\
&= \int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\partial_t v_{2\Bbbk-1} + \frac{1}{2}\int \frac{\partial_t(\widetilde{Z}_\lambda)}{r^2} v_{2\Bbbk-1}^2 \\
&= \int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\partial_t v_{2\Bbbk-1} + b_1\int \frac{(\Lambda\widetilde{Z})_\lambda}{2\lambda^2 r^2} v_{2\Bbbk-1}^2 - \left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda\widetilde{Z})_\lambda}{2\lambda^2 r^2} v_{2\Bbbk-1}^2.
\end{aligned}$$

Using the definition (2-7) of $\mathscr{A}^*$ and an integration by parts together with the definition (2-13) of $\widetilde{Z}$, we write

$$\begin{aligned}
\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\mathscr{A}_\lambda^* v_{2\Bbbk-1} &= \frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda V}{y} q_{2\Bbbk-1}\mathscr{A}^* q_{2\Bbbk-1} \\
&= \frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda V(2V+d) - \Lambda^2 V}{2y^2} q_{2\Bbbk-1}^2 \\
&= \frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda\widetilde{Z}}{2y^2} q_{2\Bbbk-1}^2 = \int \frac{b_1(\Lambda\widetilde{Z})_\lambda}{2\lambda^2 r^2} v_{2\Bbbk-1}^2.
\end{aligned}$$

From (4-17), we write

$$\frac{d}{dt}\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}v_{2\Bbbk-2} = \int \frac{d}{dt}\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\right) v_{2\Bbbk-1}v_{2\Bbbk-2} + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\partial_t v_{2\Bbbk-1} + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\left[-\mathscr{A}_\lambda^* v_{2\Bbbk-1} + [\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right].$$

Gathering all the above identities and using (4-18) yields the energy identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left\{\left(\frac{1}{\lambda^{4\Bbbk-d}}\mathscr{E}_{2\Bbbk}\right) + 2\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}v_{2\Bbbk-2}\right\}\\ &= -\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 - \left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda \widetilde{Z})_\lambda}{2\lambda^2 r^2} v_{2\Bbbk-1}^2 - \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\\ &\quad + \int \frac{d}{dt}\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\right) v_{2\Bbbk-1}v_{2\Bbbk-2} + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\left[[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]\\ &\quad + \int \left(\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\right)\left[\frac{\partial_t V_\lambda}{r} v_{2\Bbbk-2} + \mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{A}_\lambda \mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right]. \end{aligned}\tag{4-19}$$

We now estimate all terms in (4-19). The proof uses the coercivity estimate given in Lemma A.5. In particular, we shall apply Lemma A.5 with $k = \Bbbk - 1$ to get the estimate

$$\mathscr{E}_{2\Bbbk} \gtrsim \int \frac{|q_{2\Bbbk-1}|^2}{y^2} + \sum_{m=0}^{\Bbbk-1}\int \frac{|q_{2m}|^2}{y^4(1+y^{4(\Bbbk-1-m)})} + \sum_{m=0}^{\Bbbk-2}\int \frac{|q_{2m+1}|^2}{y^6(1+y^{4(\Bbbk-2-m)})}. \tag{4-20}$$

<u>Step 2</u>: control of the lower-order quadratic terms. Let us start with the second term in the left-hand side of (4-19). From (2-8) and (2-13), we have the round bound

$$|\Lambda \widetilde{Z}(y)| + |\Lambda V(y)| \lesssim \frac{y^2}{1+y^4} \quad \text{for all } y \in [0, +\infty). \tag{4-21}$$

Making a change of variables and using the Cauchy–Schwarz inequality together with (4-20), we estimate

$$\begin{aligned}\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}v_{2\Bbbk-2}\right| &= \left|\frac{b_1}{\lambda^{4\Bbbk-d}}\int \frac{\Lambda V}{y} q_{2\Bbbk-1}q_{2\Bbbk-2}\right|\\ &\lesssim \frac{b_1}{\lambda^{4\Bbbk-d}}\left(\int \frac{|q_{2\Bbbk-1}|^2}{y^2}\right)^{\frac{1}{2}}\left(\int \frac{|q_{2\Bbbk-2}|^2}{1+y^4}\right)^{\frac{1}{2}} \lesssim \frac{b_1}{\lambda^{4\Bbbk-d}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

Using (4-21), (4-1) and (4-20), we estimate

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda \widetilde{Z})_\lambda}{\lambda^2 r} v_{2\Bbbk-1}^2\right| &= \left|\left(\frac{\lambda_s}{\lambda} + b_1\right)\frac{1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda \widetilde{Z}}{y^2} q_{2\Bbbk-1}^2\right|\\ &\lesssim \frac{b_1^{L+1+(1-\delta)(1+\eta)}}{\lambda^{4\Bbbk-d+2}}\int \frac{q_{2\Bbbk-1}^2}{y^2} \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

For the third term in the right-hand side of (4-19), we write

$$\begin{aligned}\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\right| &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + 4\int \left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\right)^2 v_{2\Bbbk-2}^2 \\ &= \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{4b_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{|\Lambda V|^2}{y^2} q_{2\Bbbk-2}^2 \\ &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

A direct computation yields the round bound

$$\left|\frac{d}{dt}\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2}\right)\right| \lesssim \frac{b_1^2}{\lambda^4}(|\Lambda V| + |\Lambda^2 V|).$$

Thus, we use (4-21), the Cauchy–Schwarz inequality and (4-20) to estimate

$$\begin{aligned}\left|\int \frac{d}{dt}\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\right) v_{2\Bbbk-1}v_{2\Bbbk-2}\right| &\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{|\Lambda V| + |\Lambda^2 V|}{y}|q_{2\Bbbk-1}q_{2\Bbbk-2}| \\ &\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\left(\int \frac{q_{2\Bbbk-1}^2}{y^2}\right)^{\frac{1}{2}}\left(\int \frac{q_{2\Bbbk-2}^2}{1+y^4}\right)^{\frac{1}{2}} \\ &\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\int \left(\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}v_{2\Bbbk-2}\right)\frac{\partial_t V_\lambda}{r}v_{2\Bbbk-2}\right| &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{|\Lambda V|^2}{y^2}q_{2\Bbbk-2}^2 \\ &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}\end{aligned}$$

and

$$\begin{aligned}&\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v\right| + \left|\int \left(\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}v_{2\Bbbk-2}\right)\mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v\right| \\ &\qquad \le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + C\left(\frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk} + \int \frac{|[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2}{\lambda^2(1+y^2)} + \int |\mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2\right).\end{aligned}$$

We claim the bound

$$\int \frac{|[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2}{\lambda^2(1+y^2)} + \int |\mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}, \tag{4-22}$$

whose proof is left to Appendix C.

The collection of all the above estimates to (4-19) yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\} \le -\frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk} \\
+\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) \\
+\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) \\
+\int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right).
\end{aligned} \tag{4-23}
$$

<u>Step 3</u>: further use of dissipation. We aim to estimate all terms in the right-hand side of (4-23). From (4-21), (4-20) and the Cauchy–Schwarz inequality, we write

$$
\begin{aligned}
\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right| &= \left|\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda V}{y} q_{2\Bbbk-1}\mathscr{L}^{\Bbbk-1}\mathcal{F}\right| \\
&\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\int \frac{q_{2\Bbbk-1}^2}{y^2}\right)^{\frac{1}{2}}\left(\int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^4}\right)^{\frac{1}{2}} \\
&\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\sqrt{\mathscr{E}_{2\Bbbk}}\left(\int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^4}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right| &= \left|\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda V}{y} q_{2\Bbbk-2}\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}\right| \\
&\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\int \frac{q_{2\Bbbk-2}^2}{1+y^4}\right)^{\frac{1}{2}}\left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^2}\right)^{\frac{1}{2}} \\
&\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\sqrt{\mathscr{E}_{2\Bbbk}}\left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

For the last term in (4-23), let us introduce the function

$$
\xi_L = \frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\widetilde{T}_L \tag{4-24}
$$

and the decomposition

$$
\mathcal{F} = \partial_s \xi_L + \mathcal{F}_0 + \mathcal{F}_1, \quad \mathcal{F}_0 = -\widetilde{\Psi}_b - \widehat{\mathrm{Mod}} - \partial_s\xi_L, \quad \mathcal{F}_1 = \mathcal{H}(q) - \mathcal{N}(q), \tag{4-25}
$$

where $\widetilde{\Psi}_b$ is as referred to in (2-42), and $\widehat{\mathrm{Mod}}$, $\mathcal{H}(q)$ and $\mathcal{N}(q)$ are as defined in (3-10) (3-11) and (3-12) respectively. Actually, we introduced the decomposition (4-25) and ξ_L to take advantage of the improved bound obtained in Lemma 4.3. We now write

$$\begin{aligned}\int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\mathscr{A}_\lambda \mathscr{L}_\lambda^{\Bbbk-1}\Big(\frac{1}{\lambda^2}\mathcal{F}_\lambda\Big)&\\ &= \frac{1}{\lambda^{4\Bbbk-d+2}}\Big(\int \mathscr{A}^* q_{2\Bbbk-1}\mathscr{L}^{\Bbbk}(\partial_s\xi_L) + \int \mathscr{A}^* q_{2\Bbbk-1}\mathscr{L}^{\Bbbk}\mathcal{F}_0 + \int \widetilde{\mathscr{L}} q_{2\Bbbk-1}\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\Big)\\ &\le \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q\mathscr{L}^{\Bbbk}(\partial_s\xi_L) + \frac{C}{\lambda^{4\Bbbk-d+2}}\Big(\int |\mathscr{L}^{\Bbbk} q|^2\Big)^{\frac12}\Big(\int |\mathscr{L}^{\Bbbk}\mathcal{F}_0|\Big)^{\frac12}\\ &\qquad + \frac18\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{C}{\lambda^{4\Bbbk-d+2}}\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1|^2\\ &= \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q\mathscr{L}^{\Bbbk}(\partial_s\xi_L) + \frac18\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2\\ &\qquad + \frac{C}{\lambda^{4\Bbbk-d+2}}\big(\sqrt{\mathscr{E}_{2\Bbbk}}\|\mathscr{L}^{\Bbbk}\mathcal{F}_0\|_{L^2} + \|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\|_{L^2}^2\big).\end{aligned}$$

Injecting all these bounds into (4-23) yields

$$\begin{aligned}\frac12\frac{d}{dt}\Big\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\Big\} &\le -\frac18\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk} + \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q\mathscr{L}^{\Bbbk}(\partial_s\xi_L)\\ &\quad + \frac{b_1}{\lambda^{4\Bbbk-d+2}}\sqrt{\mathscr{E}_{2\Bbbk}}\Big[\Big(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^2}\Big)^{\frac12} + \Big(\int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^4}\Big)^{\frac12}\Big]\\ &\quad + \frac{C}{\lambda^{4\Bbbk-d+2}}\big(\sqrt{\mathscr{E}_{2\Bbbk}}\|\mathscr{L}^{\Bbbk}\mathcal{F}_0\|_{L^2} + \|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\|_{L^2}^2\big).\end{aligned} \tag{4-26}$$

Step 4: estimates for $\widetilde{\Psi}_b$ term. Recall from (2-44) that we already have the following estimate for $\widetilde{\Psi}_b$:

$$\|\mathscr{L}^{\Bbbk}\widetilde{\Psi}_b\|_{L^2} + \Big(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\widetilde{\Psi}_b|^2}{1+y^2}\Big)^{\frac12} + \Big(\int \frac{|\mathscr{L}^{\Bbbk-1}\widetilde{\Psi}_b|^2}{1+y^4}\Big)^{\frac12} \lesssim b_1^{L+1+(1-\delta)(1+\eta)}. \tag{4-27}$$

Step 5: estimates for $\widehat{\mathrm{Mod}}$ term. We claim the following:

$$\Big(\int \frac{|\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2}{1+y^4}\Big)^{\frac12} + \Big(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2}{1+y^2}\Big)^{\frac12} \lesssim b_1^{(1-\delta)(1+\eta)}\Big(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)}\Big), \tag{4-28}$$

$$\Big(\int |\mathscr{L}^{\Bbbk}\widetilde{\mathrm{Mod}}|^2\Big)^{\frac12} \lesssim b_1\Big(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{\eta(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}} + b_1^{L+1+(1-\delta)(1+\eta)}\Big), \tag{4-29}$$

where

$$\widetilde{\mathrm{Mod}} = \widehat{\mathrm{Mod}} + \partial_s\xi_L.$$

Let us prove (4-28). We only deal with the first term since the second term is estimated similarly. We recall from (3-10) the definition of $\widehat{\mathrm{Mod}}$:

$$\widehat{\mathrm{Mod}} = -\Big(\frac{\lambda_s}{\lambda} + b_1\Big)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-\gamma)b_1b_i - b_{i+1}]\Big(\widetilde{T}_i + \sum_{j=i+1}\frac{\partial S_j}{\partial b_i}\chi_{B_1}\Big),$$

where $\widetilde{Q}_b$ is defined as in (2-41) and we know from Lemma 2.9 that T_i is admissible of degree (i, i) and from Proposition 2.11 that S_j is homogeneous of degree $(j, j-1, j)$.

Since $|b_j| \lesssim b_1^j$ and $\mathscr{L}\Lambda Q = 0$, we use Lemma 2.8 to estimate

$$\begin{aligned}\int \frac{|\mathscr{L}^{\Bbbk-1}\Lambda \widetilde{Q}_b|^2}{1+y^4} &\lesssim \sum_{i=1}^{L} b_i^2 \int \frac{|\mathscr{L}^{\Bbbk-1}\Lambda \widetilde{T}_i|^2}{1+y^4} + \sum_{i=2}^{L+2}\int \frac{|\mathscr{L}^{\Bbbk-1}\Lambda \widetilde{S}_i|^2}{1+y^4} \\ &\lesssim \sum_{i=1}^{L} b_1^{2i} \int_{y\le 2B_1} \frac{y^{d-1}\,dy}{1+y^{4(\Bbbk-i)+2\gamma}} + \sum_{i=2}^{L+1} b_1^{2i}\int_{y\le 2B_1} \frac{y^{d-1}\,dy}{1+y^{4(\Bbbk-i+1)+2\gamma}} + b_1^{2L+4}\int_{y\le 2B_1}\frac{y^{d-1}\,dy}{1+y^{4\hbar+2\gamma}} \\ &\lesssim b_1^2,\end{aligned}$$

where we used the algebra $4(\Bbbk - L) + 2\gamma - d + 1 = 5 - 4\delta > 1$.

Using the cancellation $\mathscr{L}^{\Bbbk} T_i = 0$ for $1 \le i \le L$ and the admissibility of T_i, we estimate

$$\sum_{i=1}^{L}\int \frac{|\mathscr{L}^{\Bbbk-1}(\chi_{B_1}T_i)|^2}{1+y^4} \lesssim \sum_{i=1}^{L}\int_{B_1\le y\le 2B_1} y^{4(i-\Bbbk)-2\gamma+d-1}\,dy \lesssim b_1^{2(1-\delta)(1+\eta)}.$$

Using the homogeneity of S_j, we estimate for $1 \le i \le L$,

$$\sum_{j=i+1}^{L+2}\int \frac{1}{1+y^4}\left|\mathscr{L}^{\Bbbk-1}\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right)\right|^2 \lesssim \sum_{j=i+1}^{L+2} b_1^{2(j-i)}\int_{B_1\le y\le 2B_1} y^{4(j-1-\Bbbk)-2\gamma}y^{d-1}\,dy \lesssim b_1^2,$$

provided that $\eta \le \frac{1}{\delta} - 1$.

The collection of the above bounds together with (4-1) and (4-2) yields

$$\left(\int \frac{|\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2}{1+y^4}\right)^{\frac{1}{2}} \lesssim b_1^{(1-\delta)(1+\eta)}\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)}\right).$$

The same estimate holds for $\left(\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2/(1+y^2)\right)^{1/2}$ by following the same lines as above. This concludes the proof of (4-28).

We now prove (4-29). Let us write

$$\begin{aligned}\widetilde{\mathrm{Mod}} = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda \widetilde{Q}_b &+ \sum_{i=1}^{L-1}[(b_i)_s + (2i-\gamma)b_1 b_i - b_{i+1}]\widetilde{T}_i \\ &+ \sum_{i=1}^{L}[(b_i)_s + (2i-\gamma)b_1b_i - b_{i+1}]\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i} \\ &+ \left[(b_L)_s + (2i-\gamma)b_1b_L + \frac{d}{ds}\left\{\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right\}\right]\widetilde{T}_L + \frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\partial_s \widetilde{T}_L.\end{aligned}$$

Proceeding as in the proof of (4-28) yields the estimate

$$\int |\mathscr{L}^{\Bbbk}\Lambda\widetilde{Q}_b|^2+\sum_{i=1}^{L-1}\int |\mathscr{L}^{\Bbbk}\widetilde{T}_i|^2+\sum_{i=1}^{L}\sum_{j=i+1}^{L+2}\int\left|\mathscr{L}^{\Bbbk}\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right)\right|^2\lesssim b_1^2,$$

and

$$\int |\mathscr{L}^{\Bbbk}\widetilde{T}_L|^2\lesssim b_1^{2(1-\delta)(1+\eta)}. \tag{4-30}$$

From (4-10) and (4-11), we have the bound

$$\left|\frac{\langle \mathscr{L}^L q,\chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q,\chi_{B_0}\Lambda Q\rangle}\right|\lesssim B_0^{2(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}=b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}. \tag{4-31}$$

We also have

$$\int |\mathscr{L}^{\Bbbk}(\partial_s\chi_{B_1}T_L)|^2\lesssim b_1^2\int_{B_1\le y\le 2B_1}\frac{y^{d-1}\,dy}{y^{4(\Bbbk-L)+2\gamma}}\lesssim b_1^2 b_1^{2(1-\delta)(1+\eta)}.$$

The collection of the above bounds together with Lemmas 4.2 and 4.3 yields

$$\begin{aligned}\left(\int |\mathscr{L}^{\Bbbk}\widetilde{\mathrm{Mod}}|^2\right)^{\frac12}&\lesssim b_1\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}}+b_1^{L+1+(1-\delta)(1+\eta)}\right)\\&\quad+b_1^{(1-\delta)(1+\eta)}b_1^{\delta}(C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)})\\&\quad+b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}b_1 b_1^{(1-\delta)(1+\eta)}\\&\lesssim b_1\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}}+b_1^{\eta(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)}\right),\end{aligned}$$

which is the conclusion of (4-29).

Injecting the estimates (4-27), (4-28) and (4-29) into (4-26), we arrive at

$$\begin{aligned}\frac12\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\}&\le-\frac18\int|\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2+\frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\frac{\mathscr{E}_{2\Bbbk}}{M^{2\delta}}+b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}+b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}\right)\\&\quad+\frac{b_1\sqrt{\mathscr{E}_{2\Bbbk}}}{\lambda^{4\Bbbk-d+2}}\left[\left(\int\frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1|^2}{1+y^2}\right)^{\frac12}+\left(\int\frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}_1|^2}{1+y^4}\right)^{\frac12}\right]\\&\quad+\frac{1}{\lambda^{4\Bbbk-d+2}}\|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\|_{L^2}^2+\frac{1}{\lambda^{4\Bbbk-d+2}}\int\mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}(\partial_s\xi_L).\end{aligned} \tag{4-32}$$

<u>Step 6</u>: estimates for the linear small term $\mathcal{H}(q)$. We claim

$$\int|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2+\int\frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2}{1+y^2}+\frac{|\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2}{1+y^4}\lesssim b_1^2\mathscr{E}_{2\Bbbk}. \tag{4-33}$$

We only deal with the estimate for the first term because the last two terms are estimated similarly. Let us rewrite from (3-11) the definition of $\mathcal{H}(q)$,

$$\mathcal{H}(q)=\Phi q\quad\text{with }\Phi=\frac{(d-1)}{y^2}[\cos(2Q)-\cos(2Q+2\widetilde{\Theta}_b)],$$

where

$$\widetilde{\Theta}_b = \sum_{i=1}^{L} b_i \widetilde{T}_i + \sum_{i=2}^{L+2} \widetilde{S}_i(b, y).$$

From the asymptotic behavior of Q given in (2-4), the admissibility of T_i and the homogeneity of S_i, we deduce that Φ is a regular function both at the origin and at infinity. We then apply the Leibniz rule (C-2) with $k = \Bbbk - 1$ and $\phi = \Phi$ to write

$$\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q) = \sum_{m=0}^{\Bbbk-1} [q_{2m+1}\Phi_{2\Bbbk-1,2m+1} + q_{2m}\Phi_{2\Bbbk-1,2m}],$$

where $\Phi_{2\Bbbk-1,i}$ with $0 \le i \le 2\Bbbk - 1$ are defined by the recurrence relation given in Lemma C.1. In particular, we have the estimate

$$|\Phi_{k,i}| \lesssim \frac{b_1}{1 + y^{\gamma+(k-i)}} \lesssim \frac{b_1}{1 + y^{1+k-i}} \quad \text{for all } k \ge 1,\ \ 0 \le i \le k.$$

Hence, we estimate from (4-20),

$$\begin{aligned}
\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2 &\lesssim \sum_{m=0}^{\Bbbk-1} \left[\int |q_{2m+1}\Phi_{2\Bbbk-1,2m+1}|^2 + \int |q_{2m}\Phi_{2\Bbbk-1,2m}|^2 \right] \\
&\lesssim b_1^2 \sum_{m=0}^{\Bbbk-1} \left[\int \frac{|q_{2m+1}|^2}{1 + y^{2+2(2\Bbbk-1-2m-1)}} + \int \frac{|q_{2m}|^2}{1 + y^{2+2(2\Bbbk-1-2m)}} \right] \\
&\lesssim b_1^2 \sum_{m=0}^{\Bbbk-1} \left[\int \frac{|q_{2m+1}|^2}{1 + y^{2+4(\Bbbk-1-m)}} + \int \frac{|q_{2m}|^2}{1 + y^{4+4(\Bbbk-1-m)}} \right] \lesssim b_1^2 \mathscr{E}_{2\Bbbk}.
\end{aligned}$$

This concludes the proof of (4-33).

<u>Step 7</u>: estimates for the nonlinear term $\mathcal{N}(q)$. This is the most delicate point in the proof of (4-12). We claim the following:

$$\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2 \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}, \tag{4-34}$$

$$\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2}{1 + y^2} + \int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2}{1 + y^4} \lesssim b_1^{2L+2+2(1-\delta)(1+\eta)}, \tag{4-35}$$

provided that η and $1/L$ are small enough. We only deal with the proof of (4-34) since the same proof holds for (4-35).

Control for $y < 1$. Let us rewrite from (3-12) the definition of $\mathcal{N}(q)$:

$$\mathcal{N}(q) = \frac{q^2}{y}\Phi \quad \text{with } \Phi = \left[-\frac{(d-1)}{y} \int_0^1 (1-\tau)\sin(2\widetilde{Q}_b + 2\tau q)\, d\tau \right].$$

From (B-2) and the admissibility of T_i, we write

$$\frac{q^2}{y} = \frac{1}{y}\left(\sum_{i=0}^{\mathbb{k}} c_i T_i(y) + r_q(y)\right)^2 = \sum_{i=0}^{\mathbb{k}-1} \tilde{c}_i y^{2i+1} + \tilde{r}_q \quad \text{for } y < 1, \tag{4-36}$$

where

$$|\tilde{c}_i| \lesssim \mathscr{E}_{2\mathbb{k}}, \qquad |\partial_y^j \tilde{r}_q(y)| \lesssim y^{2\mathbb{k}-\frac{d}{2}-j} |\ln y|^{\mathbb{k}} \mathscr{E}_{2\mathbb{k}}, \quad 0 \le j \le 2\mathbb{k}-1, \ \ y < 1.$$

Let $\tau \in [0, 1]$ and

$$v_\tau = \widetilde{Q}_b + \tau q.$$

We obtain from Proposition 2.11 and the expansion (B-2),

$$v_\tau = \sum_{i=0}^{\mathbb{k}-1} \hat{c}_i y^{2i+1} + \hat{r}_q,$$

with

$$|\hat{c}_i| \lesssim 1, \qquad |\partial_y^j \hat{r}_q| \lesssim y^{2\mathbb{k}-\frac{d}{2}-j} |\ln y|^{\mathbb{k}}, \quad 0 \le j \le 2\mathbb{k}-1, \ \ y < 1.$$

Together with the Taylor expansion of $\sin(x)$ at $x = 0$, we write

$$\Phi(q) = \sum_{i=0}^{\mathbb{k}-1} \bar{c}_i y^{2i} + \bar{r}_q, \tag{4-37}$$

with

$$|\bar{c}_i| \lesssim 1, \qquad |\partial_y^j \bar{r}_q| \lesssim y^{2\mathbb{k}-\frac{d}{2}-1-j} |\ln y|^{\mathbb{k}}, \quad 0 \le j \le 2\mathbb{k}-1, \ \ y < 1.$$

From (4-36) and (4-37), we have the expansion of $\mathcal{N}$ near the origin,

$$\mathcal{N}(q) = \sum_{i=0}^{\mathbb{k}-1} \hat{\hat{c}}_i y^{2i+1} + \hat{\hat{r}}_q,$$

with

$$|\hat{\hat{c}}_i| \lesssim \mathscr{E}_{2\mathbb{k}}, \qquad |\partial_y^j \hat{\hat{r}}_q| \lesssim y^{2\mathbb{k}-\frac{d}{2}-j} |\ln y|^{\mathbb{k}} \mathscr{E}_{2\mathbb{k}}, \quad 0 \le j \le 2\mathbb{k}-1, \ \ y < 1.$$

From the definitions of $\mathscr{A}$ and $\mathscr{A}^*$, see (2-6) and (2-7), one can check that for $y < 1$,

$$|\mathscr{A}\mathscr{L}^{\mathbb{k}-1} \hat{\hat{r}}_q| \lesssim \sum_{i=0}^{2\mathbb{k}-1} \frac{\partial_y^i \hat{\hat{r}}_q}{y^{2\mathbb{k}-1-i}} \lesssim \mathscr{E}_{2\mathbb{k}} \sum_{i=0}^{2\mathbb{k}-1} \frac{y^{2\mathbb{k}-\frac{d}{2}-i} |\ln y|^{\mathbb{k}}}{y^{2\mathbb{k}-1-i}} \lesssim y^{-\frac{d}{2}+1} |\ln y|^{\mathbb{k}} \mathscr{E}_{2\mathbb{k}}.$$

Note from the asymptotic behavior (2-8) of V that $\mathscr{A}(y) = \mathcal{O}(y^2)$ for $y < 1$, which implies

$$\left|\mathscr{A}\mathscr{L}^{\mathbb{k}-1}\left(\sum_{i=0}^{\mathbb{k}-1} \hat{\hat{c}}_i y^{2i+1}\right)\right| \lesssim \sum_{i=0}^{\mathbb{k}-1} |\hat{\hat{c}}_i| y^2 \lesssim y^2 \mathscr{E}_{2\mathbb{k}}.$$

We then conclude

$$\int_{y<1} |\mathscr{A}\mathscr{L}^{\mathbb{k}-1} \mathcal{N}(q)|^2 \lesssim \mathscr{E}_{2\mathbb{k}}^2 \int_{y<1} y |\ln y|^{2\mathbb{k}} \, dy \lesssim \mathscr{E}_{2\mathbb{k}}^2 \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}.$$

Control for $y \geq 1$. Let us rewrite the definition of $\mathcal{N}(q)$:

$$\mathcal{N}(q) = Z^2\psi, \quad Z = \frac{q}{y}, \quad \psi = -(d-1)\int_0^1 (1-\tau)\sin(2\widetilde{Q}_b + 2\tau q)\,d\tau. \tag{4-38}$$

Note from the definitions of $\mathscr{A}$ and $\mathscr{A}^*$ that

$$\text{for all } k \in \mathbb{N}, \quad |\mathscr{A}\mathscr{L}^k f| \lesssim \sum_{i=0}^{2k+1} \frac{|\partial_y^i f|}{y^{2k+1-i}},$$

from which and the Leibniz rule, we write

$$\begin{aligned}\int_{y\geq 1} |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2 &\lesssim \sum_{k=0}^{2\Bbbk-1}\int_{y\geq 1}\frac{|\partial_y^k\mathcal{N}(q)|^2}{y^{4\Bbbk-2k-2}}\\ &\lesssim \sum_{k=0}^{2\Bbbk-1}\sum_{i=0}^{k}\int_{y\geq 1}\frac{|\partial_y^i Z^2|^2|\partial_y^{k-i}\psi|^2}{y^{4\Bbbk-2k-2}}\\ &\lesssim \sum_{k=0}^{2\Bbbk-1}\sum_{i=0}^{k}\sum_{m=0}^{i}\int_{y\geq 1}\frac{|\partial_y^m Z|^2|\partial_y^{i-m}Z|^2|\partial_y^{k-i}\psi|^2}{y^{4\Bbbk-2k-2}}.\end{aligned}$$

We aim to use the pointwise estimate (B-5) to prove that for $0 \leq k \leq 2\Bbbk-1$, $0 \leq i \leq k$ and $0 \leq m \leq i$,

$$A_{k,i,m} := \int_{y\geq 1}\frac{|\partial_y^m Z|^2|\partial_y^{i-m}Z|^2|\partial_y^{k-i}\psi|^2}{y^{4\Bbbk-2k-2}} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}, \tag{4-39}$$

which concludes the proof of (4-34).

To prove (4-39), we distinguish three cases:

Case I: $k=0$. Since $0 \leq m \leq i \leq k$, we have $k=i=m=0$. Although this is the simplest case, it gives us a basic idea to handle the other cases. From (4-38), it is obvious that $|\psi|$ is uniformly bounded. We write

$$A_{0,0,0} = \int_{y\geq 1}\frac{|q|^4|\psi|^2}{y^{4\Bbbk+2}}y^{d-1}\,dy \lesssim \int_{1\leq y\leq B_0}\frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy + \int_{y\geq B_0}\frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy.$$

Using (B-5), Definition 3.2, $b_1 \sim \frac{1}{s}$ and the fact that $d = 4\hbar + 2\gamma + 4\delta$, see (1-18), we estimate

$$\begin{aligned}\int_{1\leq y\leq B_0}\frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy &\lesssim \left\|\frac{y^{d-2}|q|^2}{y^{2(2\Bbbk-1)}}\right\|_{L^\infty(y>1)}\left\|\frac{y^{d-2}|q|^2}{y^{2(2\ell+2\hbar+3)}}\right\|_{L^\infty(y>1)}\int_{1\leq y\leq B_0} y^{4\ell+5-4\delta-2\gamma}\,dy\\ &\lesssim \mathscr{E}_{2\Bbbk}\mathscr{E}_{2(\ell+\hbar+2)}B_0^{4\ell+6-4\delta-2\gamma}\\ &\lesssim Kb_1^{2L+2(1-\delta)(1+\eta)}b_1^{2(\ell+1)+2(1-\delta)-K\eta}b_1^{-2\ell-3+2\delta+\gamma}\\ &\lesssim Kb_1^{2L+2(1-\delta)(1+\eta)}b_1^{1+\gamma-K\eta} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}.\end{aligned}$$

For the integral on the domain $y \geq B_0$, let us write

$$\begin{aligned}\int_{y\geq B_0} \frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy &\lesssim \left\| \frac{y^{d-2}|q|^2}{y^{2(2\Bbbk-2\ell-1)}} \right\|_{L^\infty(y>1)} \left\| \frac{y^{d-2}|q|^2}{y^{2(2\ell+2\hbar+1)}} \right\|_{L^\infty(y>1)} \int_{y\geq B_0} \frac{dy}{y^{4\delta+2\gamma-1}} \\ &\lesssim \mathscr{E}_{2(\Bbbk-\ell)}\mathscr{E}_{2(\ell+\hbar+1)} B_0^{2-4\delta-2\gamma} \\ &\lesssim b_1^{2(\Bbbk-\ell-\hbar-1)+2(1-\delta)-K\eta} b_1^{2\ell+2(1-\delta)-K\eta} b_1^{2\delta+\gamma-1} \\ &\lesssim b_1^{2L+2(1-\delta)(1+\eta)} b_1^{1+\gamma-(K+2(1-\delta))\eta} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}.\end{aligned}$$

This concludes the proof of (4-39) when $k = i = m = 0$.

<u>Case II</u>: $k \geq 1$ and $k = i$. We first use the Leibniz rule to write

$$\text{for all } l \in \mathbb{N}, \quad |\partial_y^l Z|^2 \lesssim \sum_{j=0}^{l} \frac{|\partial_y^j q|^2}{y^{2+2l-2j}}, \tag{4-40}$$

from which,

$$A_{k,k,m} \lesssim \sum_{j=0}^{m}\sum_{l=0}^{k-m} \int_{y\geq 1} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2}} y^{d-1}\,dy.$$

We claim that for all $(j,l) \in \mathbb{N}^2$ and $1 \leq j+l \leq 2\Bbbk-1$,

$$B_{j,l,0} := \int_{y\geq 1} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2}} y^{d-1}\,dy \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{(\gamma-1)}{2}}, \tag{4-41}$$

which immediately follows from (4-39) for the case when $k = i$.

To prove (4-41), we proceed as for the case $k = 0$ by splitting the integral in two parts as follows:

$$\begin{aligned}&B_{j,l,0} \\ &= \int_{1\leq y\leq B_0} \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar+6}} y^{7-4\delta-2\gamma}\,dy + \int_{y\geq B_0} \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar}} \frac{dy}{y^{4\delta+2\gamma-1}} \\ &\lesssim \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar+6}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-4} + \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-1} \\ &= \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{2J_1-2j+2J_2-2l}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-4} + \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{2J_3-2j+2J_4-2l}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-1} \\ &:= B_{j,l,0,J_1,J_2} b_1^{2\delta+\gamma-4} + B_{j,l,0,J_3,J_4} b_1^{2\delta+\gamma-1},\end{aligned}$$

where $J_n (n = 1,2,3,4)$ satisfy

$$J_1 + J_2 = 2\Bbbk + 2\hbar + 3, \quad J_3 + J_4 = 2\Bbbk + 2\hbar.$$

We now estimate $B_{j,l,0,J_1,J_2}$.

• If l is even, we take

$$J_2 = \begin{cases} l+2 & \text{if } l \le 2\Bbbk - 4, \\ l & \text{if } l = 2\Bbbk - 2. \end{cases}$$

This gives

$$2\hbar + 4 \le J_2 \le 2\Bbbk - 2, \quad 2\hbar + 5 \le J_1 = 2\Bbbk + 2\hbar + 3 - J_2 \le 2\Bbbk - 1.$$

Using (B-5), we have the estimate

$$B_{j,l,0,J_1,J_2} \lesssim \left\| \frac{y^{d-2}|\partial_y^j q|^2}{y^{2J_1-2j}} \right\|_{L^\infty(y\ge 1)} \left\| \frac{y^{d-2}|\partial_y^l q|^2}{y^{2J_2-2l}} \right\|_{L^\infty(y\ge 1)} \lesssim \mathscr{E}_{J_1+1}\sqrt{\mathscr{E}_{J_2}\mathscr{E}_{J_2+2}}.$$

• If l is odd, we simply take $J_2 = l+1$, which gives

$$2\hbar + 4 \le J_2 \le 2\Bbbk - 2, \quad 2\hbar + 5 \le J_1 \le 2\Bbbk - 1.$$

Hence,

$$B_{j,l,0,J_1,J_2} \lesssim \mathscr{E}_{J_1+1}\sqrt{\mathscr{E}_{J_2}\mathscr{E}_{J_2+2}}.$$

Recall from Definition 3.2 that for all even integers m in the range $2\hbar + 4 \le m \le 2\Bbbk$,

$$\mathscr{E}_m \le \begin{cases} b_1^{\frac{\ell}{2\ell-\gamma}(2m-d)} & \text{for } 2\hbar+4 \le m \le 2\hbar + 2\ell, \\ b_1^{m-2\hbar-2+2(1-\delta)-K\eta} & \text{for } 2\hbar + 2\ell + 2 \le m \le 2\Bbbk. \end{cases} \tag{4-42}$$

• If $J_1 + 1 \ge 2\hbar + 2\ell + 2$ and $J_2 \ge 2\hbar + 2\ell + 2$, then

$$B_{j,l,0,J_1,J_2} \lesssim b_1^{J_1+J_2-4\hbar-2+4(1-\delta)-2K\eta} \lesssim b_1^{2L+2+4(1-\delta)-K\eta}.$$

• If $J_1+1 \le 2\hbar+2\ell$, then $J_2 = 2\Bbbk + 2\hbar + 3 - J_1 \ge 2\Bbbk - 2\ell + 4 \ge 2\hbar + 2\ell + 2$ because $\Bbbk \gg \ell$. This implies

$$B_{j,l,0,J_1,J_2} \lesssim b_1^{\frac{\ell}{2\ell-\gamma}(2J_1+2-d)+J_2+1-2(\hbar+1)+2(1-\delta)-K\eta} \lesssim b_1^{2L+2+4(1-\delta)-K\eta}.$$

Hence, we obtain

$$B_{j,l,0,J_1,J_2} \lesssim b_1^{2L+2+4(1-\delta)-K\eta} \quad \text{for } J_1 + J_2 = 2\Bbbk + 2\hbar + 3.$$

Similarly, one can prove that

$$B_{j,l,0,J_3,J_4} \lesssim b_1^{2L-1+4(1-\delta)-K\eta} \quad \text{for } J_3 + J_4 = 2\Bbbk + 2\hbar.$$

Therefore,

$$\begin{aligned} B_{j,l,0} &\lesssim b_1^{2L+2+4(1-\delta)-K\eta} b_1^{2\delta+\gamma-4} + b_1^{2L-1+4(1-\delta)-K\eta} b_1^{2\delta+\gamma-1} \\ &\lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+(\gamma-1)-(K+2-2\delta)\eta} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{(\gamma-1)}{2}} \end{aligned}$$

for $\eta \le (\gamma-1)/(2(K+2-2\delta))$. This concludes the proof of (4-41) as well as (4-39) when $k = i$.

Case III: $k \ge 1$ and $k - i \ge 1$. Let us write from (4-39) and (4-40),

$$A_{k,m,i} \lesssim \sum_{j=0}^{m}\sum_{l=0}^{i-m} \int_{y\ge 1} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2}} \frac{|\partial_y^{k-i}\psi|^2}{y^{-2(k-i)}}. \tag{4-43}$$

At this stage, we need to make precise the decay of $|\partial_y^n\psi|$ to archive the bound (4-39). To do so, let us recall that T_i is admissible of degree (i,i) (see Lemma 2.9) and S_i is homogeneous of degree $(i,i-1,i)$ (see Proposition 2.11). Together with (2-4), we estimate

$$\text{for all } j\ge 1,\quad |\partial_y^j\widetilde{Q}_b|\lesssim \frac{1}{y^{\gamma+j}}+\sum_{l=1}^{2L+2}\frac{b_1^l y^{2l}}{y^{\gamma+j}}\mathbf{1}_{\{y\le 2B_1\}}\lesssim \frac{b_1^{-(2L+2)\eta}}{y^{\gamma+j}}.$$

Let $\tau\in[0,1]$ and $v_\tau=\widetilde{Q}_b+\tau q$. We use the Faà di Bruno formula to write

$$\begin{aligned}\text{for all } n\in\mathbb{N},\quad |\partial_y^n\psi|^2 &\lesssim \int_0^1\sum_{m^*=n}|\partial_{v_\tau}^{m_1+\cdots+m_n}\sin(v_\tau)|^2\prod_{i=1}^n|\partial_y^i\widetilde{Q}_b+\partial_y^i q|^{2m_i}\,d\tau\\ &\lesssim \sum_{m^*=n}\prod_{i=1}^n\Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}}+|\partial_y^i q|^2\Big)^{m_i},\quad m^*=\sum_{i=1}^n i m_i.\end{aligned}$$

For $1\le y\le B_0$, we use (B-5) to estimate

$$|\partial_y^i q|^2=y^{4\Bbbk-2i-2}\Big|\frac{\partial_y^i q}{y^{2\Bbbk-i-1}}\Big|^2\le B_0^{4\Bbbk-2i-d}\mathscr{E}_{2\Bbbk}\le b_1^{-C(K)\eta+i+\gamma}\le \frac{b_1^{-C(K)\eta}}{y^{2\gamma+2i}},$$

from which, we have

$$|\partial_y^n\psi|^2\lesssim\sum_{m^*=n}\prod_{i=1}^n\Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}}+\frac{b_1^{-C(K)\eta}}{y^{2\gamma+2i}}\Big)^{m_i}\lesssim\frac{b_1^{-C(K,L)\eta}}{y^{2\gamma+2n}}\quad\text{for all } 1\le y\le B_0. \tag{4-44}$$

For $y\ge B_0$, we use again (B-5) to write for all $1\le n\le 2\Bbbk-1$,

$$\begin{aligned}|\partial_y^n\psi|^2 &\lesssim \sum_{m^*=n}\prod_{i=1}^{2\hbar+2\ell+1}\Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}}+y^{4\hbar+4\ell+2-2i}\Big|\frac{\partial_y^i q}{y^{2\hbar+2\ell+1-i}}\Big|^2\Big)^{m_i}\prod_{i=2\hbar+2\ell+1}^n\Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}}+|\partial_y^i q|^2\Big)^{m_i}\\ &\lesssim \sum_{m^*=n}\prod_{i=1}^{2\hbar+2\ell+1}(b_1^{-C(L)\eta+\gamma+i}+b_1^{-K\eta+i+\gamma}b_1^{2\ell+2(1-\delta)}y^{4\ell+4(1-\delta)})^{m_i}\\ &\qquad\times\prod_{i=2\hbar+2\ell+1}^n(b_1^{-C(L)\eta+\gamma+i}+b_1^{-K\eta+\gamma+i})^{m_i}\\ &\lesssim b_1^{-C(L,K)\eta+n+\gamma\sum_{i=1}^n m_i}(b_1y^2)^{(2\ell+2(1-\delta))\sum_{i=1}^{2\hbar+2\ell+1}m_i}\quad\text{for all } y\ge B_0.\end{aligned} \tag{4-45}$$

Injecting (4-44) and (4-45) into (4-43), we arrive at

$$A_{k,i,m}\lesssim b_1^{-C\eta}\sum_{j=0}^m\sum_{l=0}^{i-m}\Big(\int_{1\le y\le B_0}\frac{|\partial_y^j q|^2|\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2+2\gamma}}+b_1^\alpha\int_{y\ge B_0}\frac{|\partial_y^j q|^2|\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2-2\alpha}}\Big),$$

where $\alpha=k-i+(2\ell+2(1-\delta))\sum_{i=1}^{2\hbar+2\ell+1}m_i$. Arguing as for the proof of (4-41), we end up with

$$A_{k,i,m}\lesssim b_1^{-C\eta}(b_1^{2L+1+\gamma+2(1-\delta)(1-\eta)+\frac{(\gamma-1)}{2}}+b_1^{2L+1+2(1-\delta)(1-\eta)+\frac{(\gamma-1)}{2}})\lesssim b_1^{2L+1+2(1-\delta)(1-\eta)}$$

for η small enough. This finishes the proof of (4-39) as well as (4-34). Since the proof of (4-35) follows exactly the same lines as the proof of (4-34), we omit it.

Inserting (4-33), (4-34) and (4-35) into (4-32) and recalling from Definition 3.2 that

$$\mathscr{E}_{2\Bbbk} \le K b_1^{2L+2(1-\delta)(1+\eta)},$$

we arrive at

$$\frac{1}{2}\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\} \lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\frac{\mathscr{E}_{2\Bbbk}}{M^{2\delta}}+b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{2L+2(1-\delta)(1+\eta)}\right)+\frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}(\partial_s\xi_L). \quad (4\text{-}46)$$

<u>Step 8</u>: time oscillations. In this step, we want to find the contribution of the last term in (4-46) to the estimate (4-12). Let us write

$$\begin{aligned}\frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}(\partial_s\xi_L) &= \frac{d}{ds}\left\{\frac{1}{\lambda^{4\Bbbk-d+2}}\left[\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}\xi_L-\frac{1}{2}\int|\mathscr{L}^{\Bbbk}\xi_L|^2\right]\right\}\\ &\quad+\frac{4\Bbbk-d+2}{\lambda^{4\Bbbk-d+2}}\frac{\lambda_s}{\lambda}\left[\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}\xi_L+\frac{1}{2}\int|\mathscr{L}^{\Bbbk}\xi_L|^2\right]\\ &\quad-\frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk}(\partial_s q-\partial_s\xi_L)\mathscr{L}^{\Bbbk}\xi_L. \end{aligned}\quad (4\text{-}47)$$

From (4-30) and (4-31), we have

$$\int |\mathscr{L}^{\Bbbk}\xi_L|^2 \lesssim b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk}. \quad (4\text{-}48)$$

This implies

$$\begin{aligned}\left|\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}\xi_L\right| &\lesssim \left(\int|\mathscr{L}^{\Bbbk}q|^2\right)^{\frac{1}{2}}\left(\int|\mathscr{L}^{\Bbbk}\xi_L|^2\right)^{\frac{1}{2}}\\ &\lesssim \sqrt{\mathscr{E}_{2\Bbbk}}\,b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}\,b_1^{(1-\delta)(1+\eta)} = b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

Since $dt/ds=\lambda^2$, we then write

$$\frac{d}{ds}\left\{\frac{1}{\lambda^{4\Bbbk-d+2}}\left[\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}\xi_L-\frac{1}{2}\int|\mathscr{L}^{\Bbbk}\xi_L|^2\right]\right\} = \frac{d}{dt}\left(\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}\mathcal{O}(b_1^{\eta(1-\delta)})\right). \quad (4\text{-}49)$$

Noting from (4-1) that $|\lambda_s/\lambda|\lesssim b_1$, this gives

$$\left|\frac{\lambda_s}{\lambda}\left[\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}\xi_L+\frac{1}{2}\int|\mathscr{L}^{\Bbbk}\xi_L|^2\right]\right| \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}. \quad (4\text{-}50)$$

For the last term in (4-47), we use (3-9) and the decomposition (4-25) to write

$$\begin{aligned}&\int \mathscr{L}^{\Bbbk}(\partial_s q-\partial_s\xi_L)\mathscr{L}^{\Bbbk}\xi_L\\ &\quad=\left[-\int \mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk+1}\xi_L+\frac{\lambda_s}{\lambda}\int \Lambda q\mathscr{L}^{2\Bbbk}\xi_L\right]+\int \mathscr{L}^{\Bbbk}[-\widetilde{\Psi}_b-\widetilde{\mathrm{Mod}}+\mathcal{H}(q)+\mathcal{N}(q)]\mathscr{L}^{\Bbbk}\xi_L.\end{aligned}\quad (4\text{-}51)$$

Using (4-31), the admissibility of T_L and the fact that $\mathscr{L}^k T_i = 0$ if $i < k$, we estimate

$$\begin{aligned}\int |\mathscr{L}^{\Bbbk+1}\xi_L|^2 &\lesssim \left|\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right|^2 \int |\mathscr{L}^{\Bbbk+1}[(1-\chi_{B_1})T_L]|^2 \\ &\lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}\int_{y\geq B_1} y^{2(2L-\gamma-2(\Bbbk+1))}y^{d-1}\,dy \\ &\lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk} b_1^{(4-2\delta)(1+\eta)} \lesssim b_1^2 b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk},\end{aligned}$$

from which we obtain

$$\left|\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk+1}\xi_L\right| \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}.$$

Similarly, we have the estimate

$$\int (1+y^{4\Bbbk})|\mathscr{L}^{2\Bbbk}\xi_L|^2 \lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}\int_{y\geq B_1} y^{4\Bbbk}y^{2(2L-\gamma-4\Bbbk)}y^{d-1}\,dy \lesssim b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk};$$

hence, using (4-4) and (4-1), we get

$$\left|\frac{\lambda_s}{\lambda}\int \Lambda q \mathscr{L}^{2\Bbbk}\xi_L\right| \lesssim b_1\left(\int \frac{|\partial_y q|^2}{1+y^{4\Bbbk-2}}\right)^{\frac{1}{2}}\left(\int (1+y^{4\Bbbk})|\mathscr{L}^{2\Bbbk}\xi_L|^2\right)^{\frac{1}{2}} \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}.$$

From (4-48), (4-27) and (4-29), we have

$$\begin{aligned}\left|\int \mathscr{L}^{\Bbbk}(\widetilde{\Psi}_b + \widetilde{\mathrm{Mod}})\mathscr{L}^{\Bbbk}\xi_L\right| &\lesssim \left(\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right)^{\frac{1}{2}}\left(\int |\mathscr{L}^{\Bbbk}(\widetilde{\Psi}_b + \widetilde{\mathrm{Mod}})|^2\right)^{\frac{1}{2}} \\ &\lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk} + b_1 b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}.\end{aligned}$$

In the same manner, we have the estimate

$$\int (1+y^4)|\mathscr{L}^{\Bbbk+1}\xi_L|^2 \lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}\int_{y\geq B_1} y^4 y^{2(2L-\gamma-2(\Bbbk+1))}y^{d-1}\,dy \lesssim b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk},$$

from which, together with (4-33) and (4-35), we get the bound

$$\begin{aligned}\left|\int \mathscr{L}^{\Bbbk-1}(\mathcal{H}(q)+\mathcal{N}(q))\mathscr{L}^{\Bbbk+1}\xi_L\right| &\lesssim \left(\int \frac{|\mathscr{L}^{\Bbbk-1}(\mathcal{H}(q)+\mathcal{N}(q))|^2}{1+y^4}\right)^{\frac{1}{2}}\left(\int (1+y^4)|\mathscr{L}^{\Bbbk+1}\xi_L|^2\right)^{\frac{1}{2}} \\ &\lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk} + b_1 b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}.\end{aligned}$$

Collecting these final bounds into (4-51) yields

$$\left|\int \mathscr{L}^{\Bbbk}(\partial_s q - \partial_s \xi_L)\mathscr{L}^{\Bbbk}\xi_L\right| \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk} + b_1 b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}. \tag{4-52}$$

Substituting (4-47), (4-49), (4-50) and (4-52) into (4-46) concludes the proof of (4-12) as well as Proposition 4.4. □

4C. ***Conclusion of Proposition 3.6.*** We give the proof of Proposition 3.6 in this subsection in order to complete the proof of Theorem 1.1. Note that this section corresponds to Section 6.1 of [Merle, Raphaël and Rodnianski 2015]. Here we follow exactly the same lines as in that paper and no new ideas are needed. We divide the proof into two parts:

Part 1: reduction to a finite-dimensional problem. Assume that for a given $K > 0$ large and an initial time $s_0 \geq 1$ large, we have $(b(s), q(s)) \in \mathcal{S}_K(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$. By using (4-1), (4-8), (4-12) and (4-13), we derive new bounds on $\mathcal{V}_1(s)$, $b_k(s)$ for $\ell + 1 \leq k \leq L$ and $\mathscr{E}_{2(\hbar+m)}$ for $1 \leq m \leq L+1$, which are better than those defining $\mathcal{S}_K(s)$ (see Definition 3.2). It then remains to control $(\mathcal{V}_2(s), \dots, \mathcal{V}_\ell(s))$. This means that the problem is reduced to the control of a finite-dimensional function $(\mathcal{V}_2(s), \dots, \mathcal{V}_\ell(s))$, and then we get the conclusion (i) of Proposition 3.6.

Part 2: transverse crossing. We aim to prove that if $(\mathcal{V}_2(s), \dots, \mathcal{V}_\ell(s))$ touches

$$\partial \widehat{\mathcal{S}}_K(s) := \partial\left(-\frac{K}{s^{\frac{\eta}{2}}(1-\delta)}, \frac{K}{s^{\frac{\eta}{2}}(1-\delta)}\right)^{\ell-1}$$

at $s = s_1$, it actually leaves $\partial \widehat{\mathcal{S}}_K(s)$ at $s = s_1$ for $s_1 \geq s_0$, provided that s_0 is large enough. We then get the conclusion (ii) of Proposition 3.6.

Part 1: reduction to a finite-dimensional problem. We give the proof of item (i) of Proposition 3.6 in this part. Given $K > 0$, $s_0 \geq 1$ and the initial data at $s = s_0$ as in Definition 3.1, we assume for all $s \in [s_0, s_1]$, $(b(s), q(s)) \in \mathcal{S}_K(s)$ for some $s_1 \geq s_0$. We claim that for all $s \in [s_0, s_1]$,

$$|\mathcal{V}_1(s)| \leq s^{-\frac{\eta}{2}(1-\delta)}, \tag{4-53}$$

$$|b_k(s)| \lesssim s^{-(k+\eta(1-\delta))} \qquad \text{for } \ell + 1 \leq k \leq L, \tag{4-54}$$

$$\mathscr{E}_{2m} \leq \begin{cases} \frac{1}{2} K s^{-\frac{\ell(4m-d)}{2\ell-\gamma}} & \text{for } \hbar + 2 \leq m \leq \ell + \hbar, \\ \frac{1}{2} s^{-2(m-\hbar-1)-2(1-\delta)+K\eta} & \text{for } \ell + \hbar + 1 \leq m \leq \Bbbk - 1, \end{cases} \tag{4-55}$$

$$\mathscr{E}_{2\Bbbk} \leq \tfrac{1}{2} K s^{-(2L+2(1-\delta)(1+\eta))}, \tag{4-56}$$

Once these estimates are proved, it immediately follows from Definition 3.2 of $\mathcal{S}_K$ that if $(b(s_1), q(s_1)) \in \partial \mathcal{S}_K(s_1)$, then $(\mathcal{V}_2, \dots, \mathcal{V}_\ell))(s_1)$ must be in $\partial \widehat{\mathcal{S}}_K(s_1)$, which concludes the proof of Proposition 3.6(i).

Before going to the proof of (4-53)–(4-56), let us compute explicitly the scaling parameter λ. To do so, let us note from (2-51) and the a priori bound on $\mathcal{U}_1$ given in Definition 3.2

$$b_1(s) = \frac{c_1}{s} + \frac{\mathcal{U}_1}{s} = \frac{\ell}{(2\ell - \gamma)s} + \mathcal{O}\left(\frac{1}{s^{1+c\eta}}\right).$$

Using (4-1) yields

$$-\frac{\lambda_s}{\lambda} = \frac{\ell}{(2\ell - \gamma)s} + \mathcal{O}\left(\frac{1}{s^{1+c\eta}}\right), \tag{4-57}$$

from which we write

$$\left|\frac{d}{ds}\{\log(s^{\frac{\ell}{2\ell-\gamma}} \lambda(s))\}\right| \lesssim \frac{1}{s^{1+c\eta}}.$$

We now integrate by using the initial data value $\lambda(s_0) = 1$ to get

$$\lambda(s) = \left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\gamma}}[1+\mathcal{O}(s^{-c\eta})] \quad \text{for } s_0 \gg 1. \tag{4-58}$$

This implies

$$s_0^{-\frac{\ell}{2\ell-\gamma}} \lesssim \frac{s^{-\frac{\ell}{2\ell-\gamma}}}{\lambda(s)} \lesssim s_0^{-\frac{\ell}{2\ell-\gamma}}. \tag{4-59}$$

Improved control of $\mathscr{E}_{2\Bbbk}$: We aim to use (4-12) to derive the improved bound (4-56). To do so, we inject the bound of $\mathscr{E}_{2\Bbbk}$ given in Definition 3.2 into the monotonicity formula (4-12) and integrate in time by using $\lambda(s_0) = 1$: For all $s \in [s_0, s_1)$,

$$\mathscr{E}_{2\Bbbk}(s) \le C\lambda(s)^{4\Bbbk-d}\left[\mathscr{E}_{2\Bbbk}(s_0) + \left(\frac{K}{M^{2\delta}} + \sqrt{K} + 1\right)\int_{s_0}^{s} \frac{\tau^{-(2L+1+2(1-\delta)(1+\eta))}}{\lambda(\tau)^{4\Bbbk-d}}\,d\tau\right].$$

Using (4-59), we estimate

$$\begin{aligned}\lambda(s)^{4\Bbbk-d}\int_{s_0}^{s} \frac{\tau^{-(2L+1+2(1-\delta)(1+\eta))}}{\lambda(\tau)^{4\Bbbk-d}}\,d\tau &\lesssim s^{-\frac{\ell(4\Bbbk-d)}{2\ell-\gamma}}\int_{s_0}^{s} \tau^{\frac{\ell(4\Bbbk-d)}{2\ell-\gamma}-(2L+1+2(1-\delta)(1+\eta))}\,d\tau \\ &\lesssim s^{-(2L+2(1-\delta)(1+\eta))}.\end{aligned}$$

Here we used the fact that the integral is divergent because

$$\frac{\ell(4\Bbbk-d)}{2\ell-\gamma} - [2L+1+2(1-\delta)(1+\eta)] = \frac{2\gamma L}{2\ell-\gamma} + \mathcal{O}_{L\to+\infty}(1) \gg -1.$$

Using again (4-59) and the initial bound (3-21), we estimate

$$\lambda(s)^{4\Bbbk-d}\mathscr{E}_{2\Bbbk}(s_0) \le \left(\frac{s_0}{s}\right)^{\frac{\ell(4\Bbbk-d)}{2\ell-\gamma}} s_0^{-\frac{10L\ell}{2\ell-\gamma}} \lesssim s^{-(2L+2(1-\delta)(1+\eta))}$$

for L large enough. Therefore, we obtain

$$\mathscr{E}_{2\Bbbk}(s) \le C\left(\frac{K}{M^{2\delta}} + \sqrt{K} + 1\right)s^{-(2L+2(1-\delta)(1+\eta))} \le \frac{K}{2}s^{-(2L+2(1-\delta)(1+\eta))}$$

for $K = K(M)$ large enough. This concludes the proof of (4-56).

Improved control of $\mathscr{E}_{2m}$: We can improve the control of $\mathscr{E}_{2m}$ by using the monotonicity formula (4-13). We distinguish two cases:

<u>Case 1</u>: $\hbar + 2 \le m \le \ell + \hbar$. From the bound of $\mathscr{E}_{2m}$ given in Definition 3.2 and $b_1(s) \sim \frac{1}{s}$, we integrate (4-13) in time s by using $\lambda(s_0) = 1$ to find that

$$\begin{aligned}\mathscr{E}_{2m}(s) \le C\lambda(s)^{4m-d}\Bigg[\mathscr{E}_{2m}(s_0) &+ \sqrt{K}\int_{s_0}^{s} \frac{\tau^{-\frac{\ell}{2\ell-\gamma}(2m-\frac{d}{2})-(m-\hbar+1-\delta-C\eta)}}{\lambda(\tau)^{4m-d}}\,d\tau \\ &+ \int_{s_0}^{s} \frac{\tau^{-(2m-2\hbar-1+2(1-\delta)-C\eta)}}{\lambda(\tau)^{4m-d}}\,d\tau\Bigg].\end{aligned}$$

Using the initial bound (3-21) and (4-59), we estimate

$$C\lambda(s)^{4m-d}\mathscr{E}_{2m}(s_0) \lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}$$

for s_0 large.

Using (4-59) and the identity

$$\begin{aligned}\frac{\ell}{2\ell-\gamma}\left(2m-\frac{d}{2}\right)-(m-\hbar+1-\delta-C\eta) &= -\frac{\gamma}{2}-1+C\eta+\frac{\gamma}{2\ell-\gamma}\left(m-\hbar-\delta-\frac{\gamma}{2}\right)\\ &\le -1-\frac{\gamma\delta}{2\ell-\gamma}+C\eta < -1,\end{aligned}$$

we estimate

$$\begin{aligned}\lambda(s)^{4m-d}\int_{s_0}^{s}\frac{\tau^{-\frac{\ell}{2\ell-\gamma}(2m-\frac{d}{2})-(m-\hbar+1-\delta-C\eta)}}{\lambda(\tau)^{4m-d}}\,d\tau &\lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}\int_{s_0}^{s}\tau^{\frac{\ell}{2\ell-\gamma}(2m-\frac{d}{2})-(m-\hbar+1-\delta-C\eta)}\,d\tau\\ &\lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}\int_{s_0}^{s}\frac{d\tau}{\tau^{1+\varepsilon}} \lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}.\end{aligned}$$

Similarly, thanks to the identity

$$\begin{aligned}&\frac{\ell}{2\ell-\gamma}(4m-d)-(2m-2\hbar-1+2(1-\delta)-C\eta)\\ &\qquad = -\gamma-1+C\eta+\frac{\gamma}{2\ell-\gamma}(2m-2\hbar-2\delta-\gamma) \le -1-\frac{2\gamma\delta}{2\ell-\gamma}+C\eta < -1,\end{aligned}$$

we obtain

$$\lambda(s)^{4m-d}\int_{s_0}^{s}\frac{\tau^{-(2m-2\hbar-1+2(1-\delta)-C\eta)}}{\lambda(\tau)^{4m-d}}\,d\tau \lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}.$$

Therefore, we deduce that

$$\mathscr{E}_{2m}(s) \le C(1+\sqrt{K})s^{-\frac{\ell}{2\ell-\gamma}(4m-d)} \le \frac{K}{2}s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}$$

for K large, which yields the improved bound (4-55) for $\hbar+2 \le m \le \ell+\hbar$.

Case 2: $\ell+\hbar+1 \le m \le \Bbbk-1$. Proceeding as in the previous case, we arrive at

$$\mathscr{E}_{2m}(s) \le C\lambda(s)^{4m-d}\left[\mathscr{E}_{2m}(s_0)+\int_{s_0}^{s}\frac{\tau^{-[2m-2\hbar-1+2(1-\delta)-(C+\frac{K}{2})\eta]}}{\lambda(\tau)^{4m-d}}\,d\tau\right].$$

From the identity

$$\begin{aligned}\frac{\ell}{2\ell-\gamma}(4m-d)-\left(2m-2\hbar-1+2(1-\delta)-\left(C+\frac{K}{2}\right)\eta\right) &= -\gamma-1+\left(C+\frac{K}{2}\right)\eta+\frac{\gamma}{2\ell-\gamma}(2m-2\hbar-2\delta-\gamma)\\ &\ge -1+\frac{2\gamma(1-\delta)}{2\ell-\gamma}+\left(C+\frac{K}{2}\right)\eta > -1, \qquad (4\text{-}60)\end{aligned}$$

together with (4-59), we estimate

$$\lambda(s)^{4m-d}\int_{s_0}^{s}\frac{\tau^{-[2m-2\hbar-1+2(1-\delta)-(C+\frac{K}{2})\eta]}}{\lambda(\tau)^{4m-d}}\,d\tau \lesssim s^{-\frac{\ell(4m-d)}{2\ell-\gamma}}\int_{s_0}^{s}\tau^{\frac{\ell(4m-d)}{2\ell-\gamma}-[2m-2\hbar-1+2(1-\delta)-(C+\frac{K}{2})\eta]}\,d\tau \lesssim s^{-[2(m-\hbar-1)+2(1-\delta)-(C+\frac{K}{2})\eta]} \le \tfrac{1}{4}s^{-[2(m-\hbar-1)+2(1-\delta)-K\eta]}.$$

Using (4-60), (4-59) and the initial bound (3-21), we derive

$$C\lambda(s)^{4m-d}\mathscr{E}_{2m}(s_0) \lesssim s^{-\frac{\ell(4m-d)}{2\ell-\gamma}} \lesssim s^{-[2(m-\hbar-1)+2(1-\delta)-(C+\frac{K}{2})\eta]} \le \tfrac{1}{4}s^{-[2(m-\hbar-1)+2(1-\delta)-K\eta]}.$$

This concludes the proof of (4-55).

Control of the stable modes, b_k's. We now close the control of the stable modes $(b_{\ell+1},\dots,b_L)$; in particular, we prove (4-54). We first treat the case when $k=L$. Let

$$\tilde{b}_L = b_L + \frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}.$$

Then from (4-31) and (4-56),

$$|\tilde{b}_L - b_L| \lesssim b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}} \lesssim b_1^{L+\eta(1-\delta)},$$

and hence from the improved modulation equation (4-8),

$$|(\tilde{b}_L)_s + (2L-\gamma)b_1\tilde{b}_L| \lesssim b_1|\tilde{b}_L - b_L| + \frac{1}{B_0^{2\delta}}[C(M)\sqrt{\mathscr{E}_{2\Bbbk}} + b_1^{L+(1-\delta)}] \lesssim b_1^{L+1+\eta(1-\delta)}.$$

This implies

$$\left|\frac{d}{ds}\left\{\frac{\tilde{b}_L}{\lambda^{2L-\gamma}}\right\}\right| \lesssim \frac{b_1^{L+1+\eta(1-\delta)}}{\lambda^{2L-\gamma}}.$$

Integrating this identity in time from s_0 and recalling that $\lambda(s_0)=1$ yields

$$\tilde{b}_L(s) \lesssim C\lambda(s)^{2L-\gamma}\left(\tilde{b}_L(s_0) + \int_{s_0}^{s}\frac{b_1(\tau)^{L+1+\eta(1-\delta)}}{\lambda(\tau)^{2L-\gamma}}\,d\tau\right).$$

Using (4-31), $b_1(s)\sim \frac{1}{s}$, the initial bounds (3-20) and (3-21) together with (4-59), we estimate

$$\lambda(s)^{2L-\gamma}\tilde{b}_L(s_0) \lesssim \left(\frac{s_0}{s}\right)^{\frac{\ell(2L-\gamma)}{2\ell-\gamma}}\left(s_0^{-\frac{5\ell(2L-\gamma)}{2\ell-\gamma}} + s_0^{\eta(1-\delta)}s_0^{-\frac{5L\ell}{2\ell-\gamma}}\right) \lesssim s^{-L-\eta(1-\delta)}$$

and

$$\lambda(s)^{2L-\gamma}\int_{s_0}^{s}\frac{b_1(\tau)^{L+1+\eta(1-\delta)}}{\lambda(\tau)^{2L-\gamma}}\,d\tau \lesssim s^{-\frac{\ell(2L-\gamma)}{2\ell-\gamma}}\int_{s_0}^{s}\tau^{\frac{\ell(2L-\gamma)}{2\ell-\gamma}-L-1-\eta(1-\delta)}\,d\tau \lesssim s^{-L-\eta(1-\delta)}.$$

Therefore,

$$b_L(s) \lesssim |\tilde{b}_L(s)| + |\tilde{b}_L(s) - b_L(s)| \lesssim s^{-L-\eta(1-\delta)},$$

which concludes the proof of (4-54) for $k=L$. Now we will propagate this improvement that we found for the bound of b_L to all b_k for all $\ell+1\le k\le L-1$. To do so we do a descending induction where the initialization is for $k=L$. Assume the bound

$$|b_k|\lesssim b_1^{k+\eta(1-\delta)}$$

for $k+1$ and let's prove it for k. Indeed, from (4-1) and the induction bound, we have

$$\left|(b_k)_s-(2k-\gamma)\frac{\lambda_s}{\lambda}b_k\right|\lesssim b_1^{L+1}+|b_{k+1}|\lesssim b_1^{k+1+\eta(1-\delta)},$$

which implies

$$\left|\frac{d}{ds}\left\{\frac{b_k}{\lambda^{2k-\gamma}}\right\}\right|\lesssim\frac{b_1^{k+1+\eta(1-\delta)}}{\lambda^{2k-\gamma}}.$$

Integrating this identity in time as for the case $k=L$, we end up with

$$b_k(s)\lesssim C\lambda(s)^{2k-\gamma}\left(b_k(s_0)+\int_{s_0}^{s}\frac{b_1(\tau)^{k+1+\eta(1-\delta)}}{\lambda(\tau)^{2k-\gamma}}\,d\tau\right)\lesssim s^{-k-\eta(1-\delta)},$$

where we used the initial bound (3-20), (4-59) and $k\ge\ell+1$. This concludes the proof of (4-54).

Control of the stable mode $\mathcal{V}_1$. We recall from (2-51) and (3-18) that

$$b_k=b_k^e+\frac{\mathcal{U}_k}{s^k},\quad 1\le k\le\ell,\quad \mathcal{V}=P_\ell\,\mathcal{U},$$

where P_ℓ diagonalizes the matrix A_ℓ with spectrum (2-54). From (2-52), and (4-1), we estimate for $1\le k\le\ell-1$,

$$|s(\mathcal{U}_k)_s-(A_\ell\mathcal{U})_k|\lesssim s^{k+1}|(b_k)_s+(2k-\gamma)b_1b_k-b_{k+1}|+|\mathcal{U}|^2\lesssim s^{-L+k}+|\mathcal{U}|^2.$$

From (2-53), (4-1) and the improved bound (4-54), we have

$$|s(\mathcal{U}_\ell)_s-(A_\ell\mathcal{U})_\ell|\lesssim s^{\ell+1}\big(|(b_k)_s+(2k-\gamma)b_1b_\ell-b_{\ell+1}|+|b_{\ell+1}|\big)+|\mathcal{U}|^2\lesssim s^{-\eta(1-\delta)}+|\mathcal{U}|^2.$$

Using the diagonalization (2-54), we obtain

$$s\mathcal{V}_s=D_\ell\mathcal{V}+\mathcal{O}(s^{-\eta(1-\delta)}).\tag{4-61}$$

Using (2-54) again yields the control of the stable mode $\mathcal{V}_1$:

$$|(s\mathcal{V}_1)_s|\lesssim s^{-\eta(1-\delta)}.$$

Thus from the initial bound (3-20),

$$|s^{\eta(1-\delta)}\mathcal{V}_1(s)|\le\left(\frac{s_0}{s}\right)^{1-\eta(1-\delta)}s_0^{\eta(1-\eta)}\mathcal{V}_1(s_0)+1\lesssim s_0^{\eta(1-\delta)},$$

which yields (4-53) for $s_0\ge s_0(\eta)$ large enough.

Part 2: transverse crossing. We give the proof of item (ii) of Proposition 3.6 in this part. We compute from (4-61) and (2-54) at the exit time $s = s_1$

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left(\sum_{k=2}^{\ell}|s^{\frac{\eta}{2}(1-\delta)}\mathcal{V}_k(s)|^2\right)\Big|_{s=s_1} &= \left(s^{\eta(1-\delta)-1}\sum_{k=2}^{\ell}\left[\frac{\eta}{2}(1-\delta)\mathcal{V}_k^2(s)+s\mathcal{V}_k(\mathcal{V}_k)_s\right]\right)\Big|_{s=s_1}\\ &= \left(s^{\eta(1-\delta)-1}\left[\sum_{k=2}^{\ell}\left[\frac{k\gamma}{2k-\gamma}+\frac{\eta}{2}(1-\delta)\right]\mathcal{V}_k^2(s)+\mathcal{O}\left(\frac{1}{s^{\frac{3}{2}\eta(1-\delta)}}\right)\right]\right)\Big|_{s=s_1}\\ &\geq \frac{1}{s_1}\left[c(d,\ell)\sum_{k=2}^{\ell}|s_1^{\frac{\eta}{2}(1-\delta)}\mathcal{V}_k(s_1)|^2+\mathcal{O}\left(\frac{1}{s_1^{\frac{\eta}{2}(1-\delta)}}\right)\right]\\ &\geq \frac{1}{s_1}\left[c(d,\ell)+\mathcal{O}\left(\frac{1}{s_1^{\frac{\eta}{2}(1-\delta)}}\right)\right] > 0,\end{aligned}$$

where we used item (i) of Proposition 3.6 in the last step. This completes the proof of Proposition 3.6.

Appendix A: Coercivity of the adapted norms

We give in this section the coercivity estimates for the operator $\mathscr{L}$ as well as the iterates of $\mathscr{L}$ under some suitable orthogonality condition. We first recall the standard Hardy-type inequalities for the class of radially symmetric functions,

$$\mathcal{D}_{\text{rad}} = \{f \in \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ with radial symmetry}\}.$$

For simplicity, we write

$$\int f := \int_0^{+\infty} f(y)y^{d-1}\,dy$$

and

$$D^k = \begin{cases}\Delta^m & \text{if } k = 2m,\\ \partial_y\Delta^m & \text{if } k = 2m+1.\end{cases}$$

We have the following:

Lemma A.1 (Hardy-type inequalities). *Let $d \geq 7$ and $f \in \mathcal{D}_{\text{rad}}$. Then*:

(i) (*Hardy near the origin*)

$$\int_0^1 \frac{|\partial_y f|^2}{y^{2i}} \geq \frac{(d-2-2i)^2}{4}\int_0^1 \frac{f^2}{y^{2+2i}} - C(d)f^2(1), \quad i = 0, 1, 2.$$

(ii) (*Hardy away from the origin for the noncritical exponent*) *Let $\alpha > 0$, $\alpha \neq \frac{1}{2}(d-2)$. Then*

$$\int_1^{+\infty} \frac{|\partial_y f|^2}{y^{2\alpha}} \geq \left(\frac{d-(2\alpha+2)}{2}\right)^2\int_1^{+\infty} \frac{f^2}{y^{2+2\alpha}} - C(\alpha,d)f^2(1).$$

(iii) (*Hardy away from the origin for the critical exponent*) *Let* $\alpha = \frac{1}{2}(d-2)$. *Then*

$$\int_1^{+\infty} \frac{|\partial_y f|^2}{y^{2\alpha}} \geq \frac{1}{4}\int_1^{+\infty} \frac{f^2}{y^{2+2\alpha}(1+\log y)^2} - C(d) f^2(1).$$

(iv) (*general weighted Hardy*) *For any* $\mu > 0$, $k \geq 2$ *an integer and* $1 \leq j \leq k-1$,

$$\int \frac{|D^j f|^2}{1+y^{\mu+2(k-j)}} \lesssim_{j,\mu} \int \frac{|D^k f|^2}{1+y^{\mu}} + \int \frac{f^2}{1+y^{\mu+2k}}.$$

Proof. The proof can be found in [Merle, Raphaël and Rodnianski 2015, Lemma B.1]. □

From the Hardy-type inequalities, we derive the following coercivity of $\mathscr{A}^*$:

Lemma A.2 (weight coercivity of $\mathscr{A}^*$). *Let* $\alpha \geq 0$. *There exists* $c_\alpha > 0$ *such that for all* $f \in \mathcal{D}_{\mathrm{rad}}$,

$$\int \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} \geq c_\alpha \biggl(\int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2\alpha})} + \int \frac{f^2}{y^{2i+2}(1+y^{2\alpha})} \biggr), \quad i = 0,1,2. \tag{A-1}$$

Proof. We proceed in two steps:

Step 1: subcoercive estimate for $\mathscr{A}^*$. We first prove the following subcoercive bound for $\mathscr{A}^*$: for $i = 0,1,2$ and $\alpha \geq 0$,

$$\int \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} \gtrsim \int \frac{f^2}{y^{2i+2}(1+y^{2\alpha})} + \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2\alpha})} - f^2(1) - \int \frac{f^2}{1+y^{2i+2\alpha+4}}. \tag{A-2}$$

From the definition (2-7) of $\mathscr{A}^*$ and the asymptotic of V given in (2-8), we use an integration by parts to estimate near the origin

$$\begin{aligned}
\int_{y\leq 1} \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} &\gtrsim \int_{y\leq 1} \frac{1}{y^{2i}} \left| \partial_y f + \frac{d}{y} f + \mathcal{O}(|yf|) \right|^2 \\
&\gtrsim \int_{y\leq 1} \frac{|\partial_y f|^2}{y^{2i}} + d\int_{y\leq 1} \frac{\partial_y (f^2)}{y^{2i+1}} + d^2 \int_{y\leq 1} \frac{f^2}{y^{2i+2}} + \mathcal{O}\biggl(\int_{y\leq 1} \frac{f^2}{y^{2i-2}} \biggr) \\
&\gtrsim \int_{y\leq 1} \frac{|\partial_y f|^2}{y^{2i}} + (2+2i)d \int_{y\leq 1} \frac{f^2}{y^{2i+2}} + df^2(1) + \mathcal{O}\biggl(\int_{y\leq 1} \frac{f^2}{y^{2i-2}} \biggr) \\
&\gtrsim \int_{y\leq 1} \biggl(\frac{|\partial_y f|^2}{y^{2i}} + \frac{f^2}{y^{2i+2}} \biggr) - \int_{y\leq 1} y^2 f^2.
\end{aligned}$$

Away from the origin, we use (2-8) to estimate

$$\int_{y\geq 1} \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} \gtrsim \int_{y\geq 1} \frac{1}{y^{2i+2\alpha}} \biggl(\partial_y f + \frac{d-1-\gamma}{y} f \biggr)^2 - \int_{y\geq 1} \frac{f^2}{y^{2i+2\alpha+4}}.$$

We make the change of variable $g = y^{d-1-\gamma} f$ and use the Hardy inequality given in part (ii) of Lemma A.1 to write

$$\int_{y\geq 1} \frac{|\partial_y (y^{d-1-\gamma} f)|^2}{y^{2i+2\alpha+2(d-1-\gamma)}}\, dy = \int_{y\geq 1} \frac{|\partial_y g|^2}{y^{2i+2\alpha+2(d-1-\gamma)}}\, dy \gtrsim \int_{y\geq 1} \frac{g^2}{y^{2i+2\alpha+2(d-1-\gamma)+2}}\, dy - g^2(1)$$
$$\gtrsim \int_{y\geq 1} \frac{f^2}{y^{2i+2\alpha+2}} - f^2(1).$$

Gathering the above bounds together with the trivial bound from (2-8),

$$\int_{y\geq 1} \frac{|\partial_y f|^2}{y^{2i+2\alpha}} \lesssim \int_{y\geq 1} \frac{|\mathscr{A}^* f|^2}{y^{2i+2\alpha}} + \int_{y\geq 1} \frac{f^2}{y^{2i+2\alpha+2}}$$

yields the subcoercivity (A-2).

Step 2: coercivity of $\mathscr{A}^*$. We now argue by contradiction to show the coercivity of $\mathscr{A}^*$. Assume that (A-1) does not hold. Up to a renormalization, we consider the sequence $f_n \in \mathcal{D}_{\text{rad}}$ with

$$\int \frac{f_n^2}{y^{2i+2}(1+y^{2\alpha})} + \int \frac{|\partial_y f_n|^2}{y^{2i}(1+y^{2\alpha})} = 1 \quad \text{and} \quad \int \frac{|\mathscr{A}^* f_n|^2}{y^{2i}(1+y^{2\alpha})} \leq \frac{1}{n}. \tag{A-3}$$

This implies by (A-2),

$$f_n^2(1) + \int \frac{f_n^2}{1+y^{2i+2\alpha+4}} \gtrsim 1. \tag{A-4}$$

From (A-3), the sequence f_n is bounded in H^1_{loc}. Hence, from a standard diagonal extraction argument, there exists $f_\infty \in H^1_{\text{loc}}$ such that up to a subsequence,

$$f_n \rightharpoonup f_\infty \quad \text{in } H^1_{\text{loc}},$$

and from the local compactness of one-dimensional Sobolev embeddings

$$f_n \to f_\infty \quad \text{in } L^2_{\text{loc}}, \qquad f_n(1) \to f_\infty(1).$$

This implies by (A-3) and (A-4),

$$f_\infty^2(1) + \int \frac{f_\infty^2}{1+y^{2i+2\alpha+4}} \gtrsim 1 \quad \text{and} \quad \int \frac{f_\infty^2}{y^{2i+2}(1+y^{2\alpha})} \lesssim 1, \tag{A-5}$$

which means that $f_\infty \neq 0$. On the other hand, from (A-3) and the lower semicontinuity of norms for the weak topology, we have

$$\mathscr{A}^* f_\infty = 0.$$

Hence,

$$f_\infty = \frac{\beta}{y^{d-1}\Lambda Q} \quad \text{for some } \beta \neq 0.$$

Since $\Lambda Q \sim y$ near the origin, we have

$$\int_{y\leq 1} \frac{f_\infty^2}{y^{2i+2}} \gtrsim \int_{y\leq 1} \frac{y^{d-1}}{y^{2d+2i+2}}\, dy = \int_{y\leq 1} \frac{dy}{y^{d+2i+3}} = +\infty,$$

which contradicts the a priori regularity of f_∞ given in (A-5). □

We also need the following subcoercivity of $\mathscr{A}$.

Lemma A.3 (weight coercivity of $\mathscr{A}$). *Let $p \geq 0$ and $i = 0, 1, 2$ such that $|2p + 2i - (d - 2 - 2\gamma)| \neq 0$, where $\gamma \in (1, 2]$ is defined by* (1-8). *We have*

$$\int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} \gtrsim \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2p})} + \int \frac{f^2}{y^{2i+2}(1+y^{2p})} - \left[f^2(1) + \int \frac{f^2}{1+y^{2i+2p+4}} \right]. \quad \text{(A-6)}$$

Assume in addition that

$$\langle f, \Phi_M \rangle = 0 \quad \text{if } 2i + 2p > d - 2\gamma - 2,$$

where Φ_M is defined in (3-4). *Then we have*

$$\int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} \gtrsim \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2p})} + \int \frac{f^2}{y^{2i+2}(1+y^{2p})}. \quad \text{(A-7)}$$

Proof. The proof is very similar to the proof of Lemma A.2. We proceed into two steps. The first step is to derive the subcoercive estimate (A-6). In the second step, we use a compactness argument to show the coercivity of $\mathscr{A}$ under a suitable condition.

Step 1: subcoercive estimate for $\mathscr{A}$. From the definition (2-6) of $\mathscr{A}$ and the asymptotic of V given in (2-8), we estimate near the origin

$$\begin{aligned}
\int_{y \leq 1} \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} &\gtrsim \int_{y \geq 1} \frac{1}{y^{2i}} \left| -\partial_y f + \frac{f}{y} + \mathcal{O}(|y f|) \right|^2 \\
&\gtrsim \int_{y \leq 1} \frac{|\partial_y f|^2}{y^{2i}} + \int_{y \leq 1} \frac{f^2}{y^{2i+2}} - \int_{y \leq 1} \frac{\partial_y (f^2)}{y^{2i+1}} - \int_{y \leq 1} \frac{f^2}{y^{2i-2}} \\
&\gtrsim \int_{y \leq 1} \frac{|\partial_y f|^2}{y^{2i}} + (d - 2i - 1) \int_{y \leq 1} \frac{f^2}{y^{2i+2}} - f^2(1) - \int_{y \leq 1} \frac{f^2}{y^{2i-2}} \\
&\gtrsim \int_{y \leq 1} \frac{|\partial_y f|^2}{y^{2i}} + \int_{y \leq 1} \frac{f^2}{y^{2i+2}} - f^2(1) - \int_{y \leq 1} y^2 f^2.
\end{aligned}$$

Away from the origin, we estimate from (2-8)

$$\int_{y \geq 1} \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} \gtrsim \int_{y \geq 1} \frac{1}{y^{2i+2p}} \left(\partial_y f + \frac{\gamma}{y} f \right)^2 - \int_{y \geq 1} \frac{f^2}{y^{2i+2p+4}}.$$

We make the change of variable $g = y^\gamma f$. From the assumption $|2i + 2p - (d - 2 - 2\gamma)| \neq 0$, we use the Hardy inequality given in part (ii) of Lemma A.1 to write

$$\int_{y \geq 1} \frac{|\partial_y (y^\gamma f)|^2}{y^{2i+2p+2\gamma}} = \int_{y \geq 1} \frac{|\partial_y g|^2}{y^{2i+2p+2\gamma}} \gtrsim \int_{y \geq 1} \frac{g^2}{y^{2i+2p+2+2\gamma}} - g^2(1) \gtrsim \int_{y \geq 1} \frac{f^2}{y^{2i+2p+2}} - f^2(1).$$

Note also that we have the trivial bound from (2-8),

$$\int_{y \geq 1} \frac{|\mathscr{A} f|^2}{y^{2i+2p}} + \int_{y \geq 1} \frac{f^2}{y^{2i+2p+2}} \gtrsim \int_{y \geq 1} \frac{|\partial_y f|^2}{y^{2i+2p}}.$$

The collection of the above bounds yields the subcoercivity (A-6).

Step 2: coercivity of $\mathscr{A}$. Arguing as the proof of (A-1), we end up with the existence of $f_\infty \neq 0$ such that

$$\int \frac{f_\infty^2}{y^{2i+2}(1+y^{2p})} \lesssim 1 \quad \text{and} \quad \mathscr{A} f_\infty = 0.$$

Hence, from the definition (2-6) of $\mathscr{A}$, we have

$$f_\infty = \beta \Lambda Q \quad \text{for some } \beta \neq 0.$$

If $2i+2p > d-2\gamma-2$, we use the orthogonality condition to deduce that

$$0 = \langle f_\infty, \Phi_M \rangle = \beta \langle \Lambda Q, \chi_M \Lambda Q \rangle.$$

Thus, $\beta = 0$. If $2i+2p \leq d-2\gamma-2$, we use the fact that $\Lambda Q \sim 1/y^\gamma$ as $y \to +\infty$ to estimate

$$\int_{y\geq 1} \frac{|\Lambda Q|^2 y^{d-1}\, dy}{y^{2i+2}(1+y^{2p})} \gtrsim \int_{y\geq 1} y^{d-1-2\gamma-2i-2p-2}\, dy \gtrsim \int_{y \geq 1} y^{-1}\, dy = +\infty,$$

which contradicts with the regularity of f_∞. □

From the coercivities of $\mathscr{A}$ and $\mathscr{A}^*$, we claim the following coercivity for $\mathscr{L}$:

Lemma A.4 (weighted coercivity of $\mathscr{L}$ under a suitable orthogonality condition). *Let $k \in \mathbb{N}$, $i = 0, 1, 2$, and $M = M(k)$ large enough. Then there exists $c_{M,k} > 0$ such that for all $f \in \mathcal{D}_{\text{rad}}$ satisfying the orthogonality*

$$\langle f, \Phi_M \rangle = 0 \quad \text{if } 2i+2k > d-2\gamma-4,$$

where Φ_M is defined by (3-4) *and $\hbar$ is given in* (1-18), *we have*

$$\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} \geq c_{M,k} \int \left(\frac{|\partial_{yy} f|^2}{y^{2i}(1+y^{2k})} + \frac{|\partial_y f|^2}{y^{2i}(1+y^{2k+2})} + \frac{|f|^2}{y^{2i+2}(1+y^{2k+2})} \right), \tag{A-8}$$

and

$$\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} \geq c_{M,k} \int \left(\frac{|\mathscr{A} f|^2}{y^{2i+2}(1+y^{2k})} + \int \frac{|f|^2}{y^{2i}(1+y^{2k+4})} \right). \tag{A-9}$$

Proof. We proceed in two steps:

Step 1: subcoercivity of $\mathscr{L}$. We apply Lemma A.2 to $\mathscr{A} f$ with $\alpha = k$ and note that

$$\partial_y(\mathscr{A} f) = \mathscr{A}(\partial_y f) + \partial_y \left(\frac{V}{y} \right) f,$$

to write

$$\begin{aligned}
\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} &\gtrsim \int \frac{|\mathscr{A} f|^2}{y^{2i+2}(1+y^{2k})} + \int \frac{|\partial_y(\mathscr{A} f)|^2}{y^{2i}(1+y^{2k})} && \text{(A-10)}\\
&\gtrsim \int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{|\partial_y(\mathscr{A} f)|^2}{y^{2i}(1+y^{2k})} \\
&\gtrsim \int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{|\mathscr{A}(\partial_y f)|^2}{y^{2i}(1+y^{2k})} - \int \frac{|f|^2}{y^{2i+2}(1+y^{2k})}. && \text{(A-11)}
\end{aligned}$$

Applying Lemma A.3 to f with $p=k+1$ and noting that the condition $|2(k+1)+2i-(d-2-2\gamma)|\neq 0$ is always satisfied (if not, we have $d=4+2\sqrt{(k+1+i)^2+2}\notin\mathbb{N}$), we have

$$\int \frac{|\mathscr{A}f|^2}{y^{2i}(1+y^{2k+2})} \gtrsim \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{f^2}{y^{2i+2}(1+y^{2k+2})} - \left[f^2(1) + \int \frac{f^2}{1+y^{2k+2i+6}} \right].$$

We apply again Lemma A.3 to $\partial_y f$ with $p=k$ to estimate

$$\int \frac{|\mathscr{A}(\partial_y f)|^2}{y^{2i}(1+y^{2k})} \gtrsim \int \frac{|\partial_{yy} f|^2}{y^{2i}(1+y^{2k})} + \int \frac{|\partial_y f|^2}{y^{2i+2}(1+y^{2k})} - \left[|\partial_y f(1)|^2 + \int \frac{|\partial_y f|^2}{1+y^{2k+2i+4}} \right].$$

Injecting these bounds into (A-11) yields the subcoercive estimate for $\mathscr{L}$,

$$\int \frac{|\mathscr{L}f|^2}{y^{2i}(1+y^{2k})} \gtrsim \int \frac{|\partial_{yy} f|^2}{y^{2i}(1+y^{2k})} + \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{f^2}{y^{2i+2}(1+y^{2k+2})} - \left[f^2(1) + |f_y(1)|^2 + \int \frac{|f_y|^2}{1+y^{2k+2i+4}} + \int \frac{f^2}{1+y^{2k+2i+6}} \right]. \quad \text{(A-12)}$$

<u>Step 2</u>: coercivity of $\mathscr{L}$. We argue by contradiction. Assume that (A-8) does not hold. Up to a renormalization, there exists a sequence of functions $f_n \in \mathcal{D}_{\text{rad}}$ such that

$$\int \frac{|\mathscr{L}f_n|^2}{y^{2i}(1+y^{2k})} \leq \frac{1}{n}, \quad \int \frac{|\partial_{yy} f_n|^2}{y^{2i}(1+y^{2k})} + \int \frac{|\partial_y f_n|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{|f_n|^2}{y^{2i+2}(1+y^{2k+2})} = 1. \quad \text{(A-13)}$$

This implies by (A-12),

$$f_n^2(1) + |\partial_y f_n(1)|^2 + \int \frac{|\partial_y f_n|^2}{1+y^{2k+2i+4}} + \int \frac{f_n^2}{y^2(1+y^{2k+2i+6})} \gtrsim 1. \quad \text{(A-14)}$$

From (A-13), the sequence f_n is bounded in H^2_{loc}. Hence, from a standard diagonal extraction argument, there exists $f_\infty \in H^2_{\text{loc}}$ such that up to a subsequence,

$$f_n \rightharpoonup f_\infty \quad \text{in } H^2_{\text{loc}},$$

and from the local compactness of one-dimensional Sobolev embeddings

$$f_n \to f_\infty \quad \text{in } H^1_{\text{loc}},$$

and

$$f_n(1) \to f_\infty(1), \quad \partial_y f_n(1) \to \partial_y f_\infty(1).$$

This implies by (A-13) and (A-14),

$$f_\infty^2(1) + |\partial_y f_\infty(1)|^2 + \int \frac{|\partial_y f_\infty|^2}{1+y^{2k+2i+4}} + \int \frac{f_\infty^2}{y^2(1+y^{2k+2i+6})} \gtrsim 1,$$

which means that $f_\infty \neq 0$. On the other hand, from (A-13) and the lower semicontinuity of norms for the weak topology, we deduce that f_∞ is a nontrivial function in the kernel of $\mathscr{L}$, namely that

$$\mathscr{L} f_\infty = 0,$$

which implies

$$f_\infty = \mu\Gamma + \beta\Lambda Q,$$

where μ and β two real numbers.

From (A-13) and the lower semicontinuity, we have

$$\int \frac{f_\infty^2}{y^{2i+2}(1+y^{2k+2})} < +\infty.$$

Recall from (2-16) that $\Gamma \sim 1/y^{d-1}$ as $y \to 0$. This yields the estimate

$$\int_{y\le 1} \frac{\Gamma^2}{y^{2i+2}(1+y^{2k+2})} \gtrsim \int_{y\le 1} \frac{dy}{y^{2i+2+d-1}} = +\infty;$$

hence, $\mu = 0$.

From (2-5), we have $\Lambda Q \sim 1/y^\gamma$ as $y \to +\infty$. If $2i+2k \le d-2\gamma-4$, we have

$$\int_{y\ge 1} \frac{|\Lambda Q|^2 y^{d-1}\,dy}{y^{2i+2}(1+y^{2k+2})} \gtrsim \int_{y\ge 1} y^{d-1-2i-2k-4-2\gamma}\,dy \gtrsim \int_{y\ge 1} y^{-1}\,dy = +\infty;$$

hence, $\beta = 0$. If $2i+2k > d-2\gamma-4$, we use the orthogonality condition to deduce

$$0 = \langle f_\infty, \Phi_M\rangle = \beta\langle \Lambda Q, \chi_M \Lambda Q\rangle,$$

which yields $\beta = 0$; hence $f_\infty = 0$. The contradiction then follows and the coercivity (A-8) is proved. The estimate (A-9) simply follows from (A-8) and (A-10). ☐

We are now in a position to prove the coercivity of $\mathscr{L}^k$ under a suitable orthogonality condition. We claim the following:

Lemma A.5 (coercivity of the iterate of $\mathscr{L}$). *Let $k \in \mathbb{N}$ and $M = M(k)$ large enough. Then there exists $c_{M,k} > 0$ such that for all $f \in \mathcal{D}_{\mathrm{rad}}$ satisfying the orthogonality condition*

$$\langle f, \mathscr{L}^m \Phi_M\rangle = 0, \quad 0 \le m \le k - \hbar,$$

where $\hbar$ is defined as in (1-18), *we have*

$$\begin{aligned}\mathscr{E}_{2k+2}(f) &= \int |\mathscr{L}^{k+1} f|^2 \\ &\ge c_{M,k}\left\{\int \frac{|\mathscr{A}(\mathscr{L}^k f)|^2}{y^2} + \sum_{m=0}^{k}\int \frac{|\mathscr{L}^m f|^2}{y^4(1+y^{4(k-m)})} + \sum_{m=0}^{k-1} \frac{|\mathscr{A}(\mathscr{L}^m f)|^2}{y^6(1+y^{4(k-m-1)})}\right\}. \end{aligned} \tag{A-15}$$

Proof. We argue by induction on k. For $k = 0$, we apply Lemma A.2 to $\mathscr{A} f$ with $i = 0$ and $\alpha = 0$, then Lemma A.3 to f with $i = 1$ and $p = 0$ to write

$$\mathscr{E}_2(f) = \int |\mathscr{L} f|^2 \gtrsim \int \frac{|\mathscr{A} f|^2}{y^2} \gtrsim \int \frac{|\mathscr{A} f|^2}{y^2} + \int \frac{f^2}{y^4}.$$

Note that we had to use the orthogonality condition $\langle f, \Phi_M\rangle$ when $\hbar = 0$. In fact, the case $\hbar = 0$ only happens when $d = 7$. In this case, the condition $2 > d - 2\gamma - 2$ is fulfilled when applying Lemma A.2 with $i = 1$ and $p = 0$.

We now assume the claim for $k \geq 0$ and prove it for $k+1$. We have the orthogonality condition

$$\langle f, \mathscr{L}^m \Phi_M \rangle = 0, \quad 0 \leq m \leq k+1-\hbar.$$

Let $g = \mathscr{L} f$, then we have

$$\langle g, \mathscr{L}^m \Phi_M \rangle = 0, \quad 0 \leq m \leq k-\hbar.$$

By induction hypothesis, we write

$$\begin{aligned}\int |\mathscr{L}^{k+2} f|^2 &= \int |\mathscr{L}^{k+1} g|^2 \\ &\gtrsim \int \frac{|\mathscr{A}(\mathscr{L}^k g)|^2}{y^2} + \sum_{m=0}^{k} \int \frac{|\mathscr{L}^m g|^2}{y^4(1+y^{4(k-m)})} + \sum_{m=0}^{k-1} \frac{|\mathscr{A}(\mathscr{L}^m g)|^2}{y^6(1+y^{4(k-m-1)})} \\ &= \int \frac{|\mathscr{A}(\mathscr{L}^{k+1} f)|^2}{y^2} + \sum_{m=1}^{k+1} \int \frac{|\mathscr{L}^m f|^2}{y^4(1+y^{4(k+1-m)})} + \sum_{m=1}^{k} \frac{|\mathscr{A}(\mathscr{L}^m f)|^2}{y^6(1+y^{4(k-m)})}.\end{aligned}$$

Note that we have the orthogonality condition $\langle f, \Phi_M \rangle = 0$ when $k \geq \hbar - 1$. The case $k \leq \hbar - 2$ implies

$$4 + 4k \leq 4 + 4\left(\frac{d}{4} - \frac{\gamma}{2} - \delta\right) - 8 \leq d - 2\gamma - 4.$$

Hence, we use the coercivity bound (A-9) to derive

$$\int \frac{|\mathscr{L} f|^2}{y^4(1+y^{4k})} \gtrsim \int \frac{|\mathscr{A} f|^2}{y^6(1+y^{4k})} + \int \frac{f^2}{y^4(1+y^{4k+4})},$$

which concludes the proof of Lemma A.5. ☐

Appendix B: Interpolation bounds

We derive in this section interpolation bounds on q which are the consequence of the coercivity property given in Lemma A.5. We have the following:

Lemma B.1 (interpolation bounds). (i) *Weighted bounds for q_i: for $1 \leq m \leq \Bbbk$,*

$$\int |q_{2m}|^2 + \sum_{i=0}^{2k-1} \int \frac{|q_i|^2}{y^2(1+y^{4m-2i-2})} \leq C(M)\mathscr{E}_{2m}. \tag{B-1}$$

(ii) *Development near the origin*:

$$q = \sum_{i=1}^{\Bbbk} c_i T_{\Bbbk-i} + r_q, \tag{B-2}$$

with bounds

$$|c_i| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}},$$
$$|\partial_y^j r_q| \lesssim y^{2\Bbbk-\frac{d}{2}-j} |\ln(y)|^{\Bbbk} \sqrt{\mathscr{E}_{2\Bbbk}}, \quad 0 \leq j \leq 2\Bbbk-1, \ \ y < 1.$$

(iii) *Bounds near the origin for* q_i *and* $\partial_y^i q$: *for* $y \le \frac{1}{2}$,

$$|q_{2i}| + |\partial_y^{2i} q| \lesssim y^{-\frac{d}{2}+2} |\ln y|^{\Bbbk} \sqrt{\mathscr{E}_{2\Bbbk}} \quad \text{for } 0 \le i \le \Bbbk - 1,$$
$$|q_{2i-1}| + |\partial_y^{2i-1} q| \lesssim y^{-\frac{d}{2}+1} |\ln y|^{\Bbbk} \sqrt{\mathscr{E}_{2\Bbbk}} \quad \text{for } 1 \le i \le \Bbbk.$$

(iv) *Weighted bounds for* $\partial_y^i q$: *for* $1 \le m \le \Bbbk$,

$$\sum_{i=0}^{2m} \int \frac{|\partial_y^i q|^2}{1 + y^{4m-2i}} \lesssim \mathscr{E}_{2m}. \tag{B-3}$$

Moreover, let $(i, j) \in \mathbb{N} \times \mathbb{N}^*$ *with* $2 \le i + j \le 2\Bbbk$. *Then*

$$\int \frac{|\partial_y^i q|^2}{1 + y^{2j}} \lesssim \begin{cases} \mathscr{E}_{2m} & \text{for } i + j = 2m, \ 1 \le m \le \Bbbk, \\ \sqrt{\mathscr{E}_{2m}} \sqrt{\mathscr{E}_{2(m+1)}} & \text{for } i + j = 2m + 1, \ 1 \le m \le \Bbbk - 1. \end{cases} \tag{B-4}$$

(v) *Pointwise bound far away*: *Let* $(i, j) \in \mathbb{N} \times \mathbb{N}$ *with* $1 \le i + j \le 2\Bbbk - 1$. *We have for* $y \ge 1$,

$$\left| \frac{\partial_y^i q}{y^j} \right|^2 \lesssim \frac{1}{y^{d-2}} \begin{cases} \mathscr{E}_{2m} & \text{for } i + j + 1 = 2m, \ 1 \le m \le \Bbbk, \\ \sqrt{\mathscr{E}_{2m}} \sqrt{\mathscr{E}_{2(m+1)}} & \text{for } i + j = 2m, \ 1 \le m \le \Bbbk - 1. \end{cases} \tag{B-5}$$

Proof. (i) The estimate (B-1) directly follows from Lemma A.5.

(ii) For $1 \le m \le \Bbbk$, we claim that $q_{2\Bbbk-2m}$ admits the Taylor expansion at the origin

$$q_{2\Bbbk-2m} = \sum_{i=1}^{m} c_{i,m} T_{m-i} + r_{2m}, \tag{B-6}$$

with the bounds

$$|c_{i,m}| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}},$$
$$|\partial_y^j r_{2m}| \lesssim y^{2m-\frac{d}{2}-j} |\ln(y)|^m \sqrt{\mathscr{E}_{2\Bbbk}}, \quad 0 \le j \le 2m - 1, \ \ y < 1,$$

The expansion (B-2) then follows from (B-6) with $m = \Bbbk$.

We proceed by induction in m for the proof of (B-6). For $m = 1$, we write from the definition (2-7) of $\mathscr{A}^*$,

$$r_1(y) = q_{2\Bbbk-1}(y) = \frac{1}{y^{d-1} \Lambda Q} \int_0^y q_{2\Bbbk} \Lambda Q x^{d-1}\, dx + \frac{d_1}{y^{d-1} \Lambda Q}.$$

Note from (B-1) that $\int |q_{2\Bbbk-1}|^2 / y^2 \lesssim \mathscr{E}_{2\Bbbk}$ and from (2-5) that $\Lambda Q \sim y$ as $y \to 0$; we deduce that $d_1 = 0$. Using the Cauchy–Schwarz inequality, we derive the pointwise estimate

$$|r_1(y)| \le \frac{1}{y^d} \left(\int_0^y |q_{2\Bbbk}|^2 x^{d-1}\, dx \right)^{\frac{1}{2}} \left(\int_0^y x^2 x^{d-1}\, dx \right)^{\frac{1}{2}} \lesssim y^{-\frac{d}{2}+1} \sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

We remark that there exists $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2\Bbbk-1}(a)|^2 \lesssim \int_{y \le 1} |q_{2\Bbbk-1}|^2 \lesssim \mathscr{E}_{2\Bbbk}.$$

We then define

$$r_2(y) = -\Lambda Q \int_a^y \frac{r_1}{\Lambda Q}\,dx,$$

and obtain from the pointwise estimate of r_1,

$$|r_2(y)| \lesssim y y^{-\frac{d}{2}+1}\sqrt{\mathscr{E}_{2\Bbbk}} \int_a^y \frac{dx}{x} \lesssim y^{-\frac{d}{2}+2}|\ln(x)|\sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

By construction and the definition (2-6) of $\mathscr{A}$, we have

$$\mathscr{A} r_2 = r_1 = q_{2\Bbbk-1}, \quad \mathscr{L} r_2 = \mathscr{A}^* q_{2\Bbbk-1} = q_{2\Bbbk} = \mathscr{L} q_{2\Bbbk-2}.$$

Recall that $\operatorname{Span}(\mathscr{L}) = \{\Lambda Q, \Gamma\}$, where Γ admits the singular behavior (2-16). From (B-1), we have $\int |q_{2\Bbbk-2}|^2/y^4 \lesssim \mathscr{E}_{2\Bbbk} < +\infty$. This implies that there exists $c_2 \in \mathbb{R}$ such that

$$q_{2\Bbbk-2} = c_2 \Lambda Q + r_2.$$

Moreover, there exists $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2\Bbbk-2}(a)|^2 \lesssim \int_{|y|\le 1} |q_{2\Bbbk-2}|^2 \lesssim \mathscr{E}_{2\Bbbk},$$

which implies

$$|c_2| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}}, \quad |q_{2\Bbbk-2}| \lesssim y^{-\frac{d}{2}+2}|\ln(y)|\sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

Since $\mathscr{A} r_2 = r_1$, we then write from the definition (2-6) of $\mathscr{A}$,

$$|\partial_y r_2| \lesssim |r_1| + \left|\frac{r_2}{y}\right| \lesssim y^{-\frac{d}{2}+2}|\ln(y)|\sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

This concludes the proof of (B-6) for $m = 1$.

We now assume that (B-6) holds for $m \ge 1$ and prove it for $m+1$. The term r_{2m} is built as follows:

$$r_{2m-1} = \frac{1}{y^{d-1}\Lambda Q}\int_0^y r_{2m-2}\Lambda Q x^{d-1}\,dx, \quad r_{2m} = -\Lambda Q \int_a^y \frac{r_{2m-1}}{\Lambda Q}\,dx, \quad a \in \left(\tfrac{1}{2}, 1\right).$$

We now use the induction hypothesis to estimate

$$\begin{aligned} |r_{2m+1}| &= \left|\frac{1}{y^{d-1}\Lambda Q}\int_0^y r_{2m}\Lambda Q x^{d-1}\,dx\right| \\ &\lesssim \frac{1}{y^d}\sqrt{\mathscr{E}_{2\Bbbk}}\int_0^y x^{2m+\frac{d}{2}}|\ln(x)|^m\,dx \\ &\lesssim y^{2m-\frac{d}{2}}\sqrt{\mathscr{E}_{2\Bbbk}}\int_0^y |\ln(x)|^m\,dx \\ &\lesssim y^{2m-\frac{d}{2}+1}|\ln(y)|^m\sqrt{\mathscr{E}_{2\Bbbk}}. \end{aligned}$$

Here we used the identity

$$I_m = \int_0^y [\ln(x)]^m\,dx \lesssim y|\ln(y)|^m, \quad m \ge 1, \quad y < 1.$$

Indeed, we have $I_1 = \int_0^y \ln(x)\,dx = y\ln(y) - y \lesssim y|\ln(y)|$ for $y < 1$. Assuming the claim for $m \geq 1$, we use an integration by parts to estimate for $m+1$

$$I_{m+1} = \int_0^y [\ln(x)]^m (x\ln(x) - x)'\,dx$$
$$= y[\ln(y)]^{m+1} - y[\ln(y)]^m - m(I_m - I_{m-1}) \lesssim y|\ln(y)|^{m+1}.$$

Using an integration by parts yields

$$\int_a^y \frac{[\ln(x)]^m}{x}\,dx = \frac{[\ln(y)]^{m+1} - [\ln(a)]^{m+1}}{m+1}.$$

Hence, we have the estimate

$$|r_{2m+2}| = \left|\Lambda Q \int_a^y \frac{r_{2m+1}}{\Lambda Q}\,dx\right| \lesssim y^{2m-\frac{d}{2}+2}\sqrt{\mathscr{E}_{2\Bbbk}} \int_a^y \frac{|\ln(x)|^m}{x}\,dx$$
$$\lesssim y^{2m-\frac{d}{2}+2}|\ln(y)|^{m+1}\sqrt{\mathscr{E}_{2\Bbbk}}.$$

By construction, we have

$$\mathscr{A} r_{2m+2} = r_{2m+1}, \quad \mathscr{L} r_{2m+2} = r_{2m}.$$

From the induction hypothesis and the definition (2-19) of T_k, we write

$$\mathscr{L} q_{2\Bbbk-2(m+1)} = q_{2\Bbbk-2m} = \sum_{i=1}^m c_{i,m} T_{m-i} + r_{2m} = \sum_{i=1}^m c_{i,m} \mathscr{L} T_{m+1-i} + \mathscr{L} r_{2m+2}.$$

The singularity (2-16) of Γ at the origin and the bound $\int_{y\leq 1} |q_{2\Bbbk-2(m+1)}|^2/y^4 \lesssim \mathscr{E}_{2\Bbbk}$ allows us to deduce

$$q_{2\Bbbk-2(m+1)} = \sum_{i=1}^m c_{i,m} T_{m+1-i} + c_{2m+2}\Lambda Q + r_{2m+2}.$$

From (B-1), we see that there exists $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2\Bbbk-2(m+1)}(a)|^2 \lesssim \int_{y\leq 1} |q_{2\Bbbk-2(m+1)}|^2 \lesssim \mathscr{E}_{2\Bbbk}.$$

Together with the induction hypothesis $|c_{i,m}| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}}$ and the pointwise estimate on r_{2m+2}, we get the bound $|c_{2m+2}| \leq \sqrt{\mathscr{E}_{2\Bbbk}}$.

A brute force computation using the definitions of $\mathscr{A}$ and $\mathscr{A}^*$ and the asymptotic behavior (2-8) ensure that for any function f,

$$\partial_y^j f = \sum_{i=0}^j P_{i,j} f_i, \quad |P_{i,j}| \lesssim \frac{1}{y^{j-i}}, \tag{B-7}$$

and we estimate

$$|\partial_y^j r_{2m+2}| \lesssim \sum_{i=0}^j \frac{|r_{2m+2-i}|}{y^{j-i}} \lesssim \sqrt{\mathscr{E}_{2\Bbbk}} \sum_{i=0}^j \frac{y^{2m+2-i-\frac{d}{2}}|\ln(y)|^{m+1}}{y^{j-i}} \lesssim y^{2m+2-\frac{d}{2}-j}|\ln(y)|^{m+1}\sqrt{\mathscr{E}_{2\Bbbk}}.$$

This concludes the proof of (B-6) as well as (B-2).

(iii) The proof of (iii) directly follows from (B-6).

(iv) We have from (B-7),

$$|\partial_y^k q| \lesssim \sum_{j=0}^{k} \frac{|q_j|}{y^{k-j}},$$

and thus, using (B-1) and the pointwise bounds given in part (iii) yields

$$\begin{aligned}\sum_{i=0}^{2m}\int \frac{|\partial_y^i q|^2}{1+y^{4m-2i}} &\lesssim \mathscr{E}_{2m} + \sum_{i=0}^{2m-1}\int_{y<1} |\partial_y^i q|^2 + \sum_{i=0}^{2m-1}\int_{y>1} \frac{|\partial_y^i q|^2}{y^{4m-2i}}\\ &\lesssim \mathscr{E}_{2m} + \mathscr{E}_{2\Bbbk}\int_{y<1} y|\ln y|^{\Bbbk}\,dy + \sum_{i=0}^{2m-1}\sum_{j=0}^{i}\int_{y>1}\frac{|q_j|^2}{y^{4m-2j}} \lesssim \mathscr{E}_{2m},\end{aligned}$$

which concludes the proof of (B-3).

The estimate (B-4) simply follows from (B-3). Indeed, if $i+j=2m$ with $1\le m\le \Bbbk$, we have

$$\int \frac{|\partial_y^i q|^2}{1+y^{2j}} = \int \frac{|\partial_y^i q|^2}{1+y^{4m-2i}} \lesssim \mathscr{E}_{2m}.$$

If $i+j=2m+1$ with $1\le m\le \Bbbk-1$, we write

$$\begin{aligned}\int \frac{|\partial_y^i q|^2}{1+y^{2j}} = \int \frac{|\partial_y^i q|^2}{1+y^{4m-2i+2}} &\lesssim \left(\int \frac{|\partial_y^i q|^2}{1+y^{4m-2i}}\right)^{\frac12}\left(\int \frac{|\partial_y^i q|^2}{1+y^{4m-2i+4}}\right)^{\frac12}\\ &\lesssim \sqrt{\mathscr{E}_{2m}}\sqrt{\mathscr{E}_{2(m+1)}}.\end{aligned}$$

(v) Let $i, j\ge 0$ with $1\le i+j\le 2\Bbbk-1$. Then $2\le i+j+1\le 2\Bbbk$ and we conclude from (B-4) that for $y\ge 1$,

$$\begin{aligned}\left|\frac{\partial_y^i q}{y^j}\right|^2 \lesssim \left|\int_y^{+\infty} \partial_x\left(\frac{(\partial_x^i q)^2}{x^{2j}}\right)dx\right| &\lesssim \frac{1}{y^{d-2}}\left\{\int_y^{+\infty}\frac{|\partial_x^i q|^2}{x^{2j+2}} + \int_y^{+\infty}\frac{|\partial_x^{i+1} q|^2}{x^{2j}}\right\}\\ &\lesssim \frac{1}{y^{d-2}}\begin{cases}\mathscr{E}_{2m} & \text{for } i+j+1=2m,\ 1\le m\le \Bbbk,\\ \sqrt{\mathscr{E}_{2m}}\sqrt{\mathscr{E}_{2(m+1)}} & \text{for } i+j+1=2m+1,\ 1\le m\le \Bbbk-1.\end{cases}\end{aligned}$$

□

Appendix C: Proof of (4-22)

We give here the proof of (4-22). Before going to the proof, we need the following Leibniz rule for $\mathscr{L}^k$.

Lemma C.1 (Leibniz rule for $\mathscr{L}^k$). *Let ϕ be a smooth function and $k\in\mathbb{N}$, we have*

$$\mathscr{L}^{k+1}(\phi f) = \sum_{m=0}^{k+1} f_{2m}\phi_{2k+2,2m} + \sum_{m=0}^{k} f_{2m+1}\phi_{2k+2,2m+1}, \tag{C-1}$$

$$\mathscr{A}\mathscr{L}^{k}(\phi f) = \sum_{m=0}^{k} f_{2m+1}\phi_{2k+1,2m+1} + \sum_{m=0}^{k} f_{2m}\phi_{2k+1,2m}, \tag{C-2}$$

where for $k=0$,

$$\phi_{1,0}=-\partial_y\phi,\quad \phi_{1,1}=\phi,$$

$$\phi_{2,0}=-\partial_y^2\phi-\frac{d-1+2V}{y}\partial_y\phi,\quad \phi_{2,1}=2\partial_y\phi,\quad \phi_{2,2}=\phi,$$

and for $k\geq 1$,

$$\begin{aligned}
\phi_{2k+1,0}&=-\partial_y\phi_{2k,0},\\
\phi_{2k+1,2i}&=-\partial_y\phi_{2k,2i}-\phi_{2k,2i-1},\quad 1\leq i\leq k,\\
\phi_{2k+1,2i+1}&=\phi_{2k,2i}+\frac{d-1+2V}{y}\phi_{2k,2i+1}-\partial_y\phi_{2k,2i+1},\quad 0\leq i\leq k-1,\\
\phi_{2k+1,2k+1}&=\phi_{2k,2k}=\phi,\\
\phi_{2k+2,0}&=\partial_y\phi_{2k+1,0}+\frac{d-1+2V}{y}\phi_{2k+1,0},\\
\phi_{2k+2,2i}&=\phi_{2k+1,2i-1}+\partial_y\phi_{2k+1,2i}+\frac{d-1+2V}{y}\phi_{2k+1,2i},\quad 1\leq i\leq k,\\
\phi_{2k+2,2i+1}&=-\phi_{2k+1,2i}+\partial_y\phi_{2k+1,2i+1},\quad 0\leq i\leq k,\\
\phi_{2k+2,2k+2}&=\phi_{2k+1,2k+1}=\phi.
\end{aligned}$$

Proof. We use the relations

$$\mathscr{A}(\phi f)=\phi\mathscr{A}f-\partial_y\phi f,\quad \mathscr{A}^*(\phi f)=\phi\mathscr{A}^*f+\partial_y\phi f,$$
$$\mathscr{A}f+\mathscr{A}^*f=\frac{d-1+2V}{y}f$$

to compute

$$\begin{aligned}
\mathscr{A}(\phi f)&=f_1\phi+f(-\partial_y\phi),\\
\mathscr{L}(\phi f)&=\mathscr{A}^*\mathscr{A}(\phi f)=f_2\phi+f_1(2\partial_y\phi)+f\left(-\partial_y^2\phi-\frac{d-1+2V}{y}\partial_y\phi\right),
\end{aligned}$$

which is the conclusions of (C-1) and (C-2) for $k=0$.

Assume that (C-1) and (C-2) hold for $k\in\mathbb{N}$; let us compute for $k\to k+1$. Using (C-1), we write

$$\begin{aligned}
\mathscr{A}\mathscr{L}^{k+1}(\phi f)&=\sum_{m=0}^{k+1}\mathscr{A}[f_{2m}\phi_{2k+2,2m}]+\sum_{m=0}^{k}\left[-\mathscr{A}^*+\frac{d-1+2V}{y}\right]f_{2m+1}\phi_{2k+2,2m+1}\\
&=\sum_{m=0}^{k+1}\{f_{2m+1}\phi_{2k+2,2m}+f_{2m}(-\partial_y\phi_{2k+2,2m})\}\\
&\quad+\sum_{m=0}^{k}\bigg\{f_{2m+2}(-\phi_{2k+2,2m+1})+f_{2m+1}(-\partial_y\phi_{2k+2,2m+1})\\
&\qquad\qquad+f_{2m+1}\left(\frac{d-1+2V}{y}\phi_{2k+2,2m+1}\right)\bigg\}
\end{aligned}$$

$$= \sum_{m=0}^{k} f_{2m+1}\Big(\phi_{2k+2,2m} - \partial_y \phi_{2k+2,2m+1} + \frac{d-1+2V}{y}\phi_{2k+2,2m+1}\Big)$$
$$+ \sum_{m=1}^{k} f_{2m}(-\partial_y \phi_{2k+2,2m} - \phi_{2k+2,2m+1}) + f_{2k+3}\phi_{2k+2,2k+2} + f(-\partial_y \phi_{2k+2,0}),$$

which yields the recurrence relation for $\phi_{2k+3,j}$ with $0 \le j \le 2k+3$.

Similarly, we write $\mathscr{L}^{k+2}(\phi f) = \mathscr{A}^*[\mathscr{A}\mathscr{L}^{k+1}(\phi f)]$ and use the formula (C-2) with $k+1$ to obtain the recurrence relation for $\phi_{2k+4,j}$ with $0 \le j \le 2k+4$. □

Let us now give the proof of (4-22). By induction and the definition (3-13), we have

$$[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v = \sum_{m=0}^{\Bbbk-2} \mathscr{L}_\lambda^m([\partial_t, \mathscr{L}_\lambda]\mathscr{L}_\lambda^{\Bbbk-2-m} v) = \sum_{m=0}^{\Bbbk-2} \mathscr{L}_\lambda^m\Big(\frac{\partial_t Z_\lambda}{r^2}\mathscr{L}_\lambda^{\Bbbk-2-m} v\Big).$$

Noting that

$$\frac{\partial_t Z_\lambda}{r^2} = \frac{b_1 \Lambda Z}{\lambda^4 y^2},$$

we make a change of variables to obtain

$$\int \frac{1}{\lambda^2(1+y^2)} |[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 = \frac{b_1^2}{\lambda^{4\Bbbk-d+2}} \int \frac{1}{1+y^2}\Big|\sum_{m=0}^{\Bbbk-2} \mathscr{L}^m\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2-m} q\Big)\Big|^2$$
$$\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}} \sum_{m=0}^{\Bbbk-2} \int \frac{1}{1+y^2}\Big|\mathscr{L}^m\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2-m} q\Big)\Big|^2.$$

For $m = 0$, we use (4-21) and (4-20) to estimate

$$\int \frac{1}{1+y^2}\Big|\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2} q\Big)\Big|^2 \lesssim \int \frac{|q_{2\Bbbk-4}^2|}{1+y^{10}} \lesssim \mathscr{E}_{2\Bbbk}.$$

For $m = 1, \dots, \Bbbk-2$, we apply (C-1) with

$$\phi = \frac{\Lambda Z}{y^2} = \frac{(d-1)\Lambda \cos(2Q)}{y^2}$$

and note from (2-4) that

$$|\phi_{k,i}| \lesssim \frac{1}{1+y^{2\gamma+2+(2k-i)}} \lesssim \frac{1}{1+y^{4+(2k-i)}}, \quad k \in \mathbb{N}^*, \ 0 \le i \le 2k,$$

which yields

$$\int \frac{1}{1+y^2}\Big|\mathscr{L}^m\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2-m} q\Big)\Big|^2 \lesssim \sum_{i=0}^{2m} \int \frac{q_{2\Bbbk-4-2m-i}^2}{(1+y^{10+(4m-2i)})} \lesssim \mathscr{E}_{2\Bbbk}.$$

Thus,

$$\int \frac{1}{\lambda^2(1+y^2)} |[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}} \mathscr{E}_{2\Bbbk}.$$

Similarly, we use (C-2) to get the estimate

$$\int |\mathscr{A}[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.$$

This concludes the proof of (4-22).

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Received 23 Sep 2017. Accepted 9 Apr 2018.

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mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 12, No. 1, 2019
dx.doi.org/10.2140/apde.2019.12.189

msp

ON PROPAGATION OF HIGHER SPACE REGULARITY FOR NONLINEAR VLASOV EQUATIONS

DANIEL HAN-KWAN

This work is concerned with the broad question of propagation of regularity for smooth solutions to nonlinear Vlasov equations. For a class of equations (that includes Vlasov–Poisson and relativistic Vlasov–Maxwell systems), we prove that higher regularity in space is propagated, locally in time, into higher regularity for the moments in velocity of the solution. This in turn can be translated into some anisotropic Sobolev higher regularity for the solution itself, which can be interpreted as a kind of weak propagation of space regularity. To this end, we adapt the methods introduced by D. Han-Kwan and F. Rousset (*Ann. Sci. École Norm. Sup.* **49**:6 (2016) 1445–1495) in the context of the quasineutral limit of the Vlasov–Poisson system.

1. Introduction

This paper is concerned with the broad question of propagation of regularity for smooth solutions to Vlasov equations of the general form

$$\partial_t f + a(v)\cdot\nabla_x f + F(t,x,v)\cdot\nabla_v f = 0, \tag{1-1}$$

set in the phase space $\mathbb{T}^d\times\mathbb{R}^d$ (with $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ endowed with normalized Lebesgue measure), where $F:\mathbb{R}^+\times\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}^d$ is a force field satisfying $\nabla_v\cdot F=0$ and $a:\mathbb{R}^d\to\mathbb{R}^d$ is an advection field satisfying suitable assumptions, $a(v)=v$ being the main example to be considered. The (scalar) function $f(t,x,v)$ may be understood as the distribution function of a family of particles, which can be, depending

MSC2010: 35Q83.
Keywords: kinetic transport equations, kinetic averaging lemmas.

on the physical context, e.g., electrons, ions in plasma physics, or stars in galactic dynamics. The choice of the periodic torus $\mathbb{T}^d$ is made for simplicity.

The two precise examples of equations we specifically have in mind are the Vlasov equations arising from a coupling with Poisson or Maxwell equations, in which case the resulting coupled system is called the Vlasov–Poisson or the relativistic Vlasov–Maxwell system (we will discuss as well several other models).

• The *Vlasov–Poisson* system — either the repulsive or the attractive version, the sign of the interaction here does not matter here — is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f \pm E \cdot \nabla_v f = 0, \\ E(t,x) = -\nabla_x \phi(t,x), \\ -\Delta_x \phi = \int_{\mathbb{R}^d} f \, dv - \int_{\mathbb{T}^d \times \mathbb{R}^d} f \, dv \, dx, \\ f|_{t=0} = f_0. \end{cases} \tag{1-2}$$

In the repulsive version (that is, with the sign $+$ in the Vlasov equation), this system describes the dynamics of charged particles in a nonrelativistic plasma, with a self-induced electric field.

In the attractive version (that is, with the sign $-$ in the Vlasov equation), it describes the dynamics of stars or planets with gravitational interaction.

• The *relativistic Vlasov–Maxwell* system, in dimension $d = 3$, is given by

$$\begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + F \cdot \nabla_v f = 0, \\ \hat{v} := \dfrac{v}{\sqrt{1 + |v|^2/c^2}}, \quad F(t,x,v) := E(t,x) + \dfrac{1}{c} \hat{v} \times B(t,x), \\ \dfrac{1}{c} \partial_t B + \nabla_x \times E = 0, \quad \nabla_x \cdot E = \int_{\mathbb{R}^3} f \, dv - \int_{\mathbb{T}^3 \times \mathbb{R}^3} f \, dv \, dx, \\ -\dfrac{1}{c} \partial_t E + \nabla_x \times B = \dfrac{1}{c} \int_{\mathbb{R}^3} \hat{v} f \, dv, \quad \nabla_x \cdot B = 0, \\ f|_{t=0} = f_0, \quad (E,B)|_{t=0} = (E_0, B_0), \end{cases} \tag{1-3}$$

in which the parameter c is the speed of light. There are also related versions of (1-3) in lower dimensions. This system describes the dynamics of charged particles in a relativistic plasma, with a self-induced electromagnetic field. We recall that the (repulsive) Vlasov–Poisson system can be derived from (1-3) in the nonrelativistic regime, that is to say, in the limit $c \to \infty$, as studied in [Asano and Ukai 1986; Degond 1986; Schaeffer 1986].

In this paper, we will consider weighted Sobolev norms and associated weighted Sobolev spaces (based on L^2), defined, for $k \in \mathbb{N},\ r \in \mathbb{R}$, as

$$\| f \|_{\mathcal{H}^k_r} := \Big(\sum_{|\alpha| + |\beta| \le k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (1 + |v|^2)^r \, |\partial_x^\alpha \partial_v^\beta f|^2 \, dv \, dx \Big)^{\frac{1}{2}}, \tag{1-4}$$

where for $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \{1, \dots, d\}^n$, we write

$$|\alpha| = n, \quad |\beta| = n,$$

and

$$\partial_x^\alpha := \partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_n}}, \quad \partial_v^\beta := \partial_{v_{\beta_1}} \cdots \partial_{v_{\beta_n}}.$$

As usual the notation H^s will stand for the standard Sobolev spaces, without weight.

It will be also useful to introduce the weighted $W^{k,\infty}$ space, whose norm is defined, for $k \in \mathbb{N}$, $r \in \mathbb{R}$, by

$$\|f\|_{\mathcal{W}_r^{k,\infty}} := \sum_{|\alpha|+|\beta| \le k} \|(1+|v|^2)^{\frac{r}{2}}\, \partial_x^\alpha \partial_v^\beta f\|_{L^\infty_{x,v}}. \tag{1-5}$$

For the Vlasov–Poisson or Vlasov–Maxwell couplings, given an initial condition f_0 satisfying

$$f_0 \in \mathcal{H}_r^n$$

for $n, r > 0$ large enough (and with a smooth enough initial force $F(0)$), it is standard that there exists a unique local solution $f(t) \in C(0, T; \mathcal{H}_r^n)$. Under fairly general assumptions on the advection field a and the force F, the same result can also be shown for (1-1), as we will soon see.

Let us now present the precise problem we tackle in this work. Assuming some higher *space* regularity such as

$$\partial_x^{n+1} f_0 \in \mathcal{H}_r^0 \quad (\text{or } \ \partial_x^p f_0 \in \mathcal{H}_r^0 \ \text{ for } p \ge n+1), \tag{1-6}$$

the question we ask is the following: is there also propagation of any higher regularity for the solution $f(t)$? A first remark to be made is that there is no hope of proving that this sole additional assumption implies that the solution $f(t)$ also satisfies $\partial_x^{n+1} f(t) \in \mathcal{H}_r^0$, even for small values of t. Indeed, regularity in x and v is intricately intertwined for solutions of the Vlasov equation, as can be seen from the representation of the solution using the method of characteristics.

For $s, t \ge 0$ and $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, we define as usual the characteristic curves $(X(s, t, x, v), V(s, t, x, v))$ as the solutions to the system of ODEs

$$\begin{cases} \dfrac{d}{ds} X(s,t,x,v) = a(V(s,t,x,v)), & X(t,t,x,v) = x, \\ \dfrac{d}{ds} V(s,t,x,v) = F(s, X(s,t,x,v), V(s,t,x,v)), & V(t,t,x,v) = v. \end{cases} \tag{1-7}$$

The existence and uniqueness of such curves are consequences of the Cauchy–Lipschitz theorem (assuming we deal with smooth forces). The method of characteristics asserts that one can represent the solution of (1-1) as

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)). \tag{1-8}$$

Therefore we see (except maybe in trivial cases such as $F \equiv 0$) that derivatives in x of $f(t)$ involve derivatives in x and in v of f_0, so that regularity in x only of f_0 cannot in general be propagated for $f(t)$. However, given some smooth test function $\psi(v)$ (the case $\psi = 1$ is already interesting), we can also wonder about the higher regularity of the moment $m_\psi(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\psi(v)\, dv$. Such moments, which can be interpreted as hydrodynamic quantities, are important objects in kinetic theory. We have the representation formula

$$m_\psi(t,x) = \int_{\mathbb{R}^d} f_0(X(0,t,x,v), V(0,t,x,v))\, \psi(v)\, dv.$$

We note that for t small enough, the map $v \mapsto V(s,t,x,v)$ is a diffeomorphism for all $s \in [0,t]$. Indeed for $s = t$ this map is the identity and integrating with respect to s the equation satisfied by $V(s,t,x,v)$, we note that for t small enough and $s \in [0,t]$, the map $v \mapsto V(s,t,x,v)$ is a small perturbation of the identity, hence our claim that it is a diffeomorphism. In particular the map $v \mapsto V(0,t,x,v)$ is a diffeomorphism and we denote by $V^{-1}(t,x,v)$ its inverse. Using this diffeomorphism as a change of variables (in v) we get, for t small enough,

$$m_\psi(t,x) = \int_{\mathbb{R}^d} f_0(X(0,t,x,V^{-1}(t,x,v)), v)\, \psi(V^{-1}(t,x,v))\, |\det D_v V(0,t,x,v)|^{-1}\, dv.$$

Thanks to this formula, at least formally, the Leibniz rule ensures that derivatives in x of the moment m_ψ only involve derivatives in x of f_0. Recalling the extra higher regularity (1-6), it seems maybe natural to expect that the moment m_ψ belongs to the Sobolev space H^{n+1} in x. In the case where F is a *fixed* external force, assumed to be very smooth, say C^∞ with respect to all variables, since t is fixed, the fact that $m_\psi(t,\cdot)$ belongs to H^{n+1}_x follows indeed from the Leibniz formula, using the fact the characteristic curves (X,V) inherit the C^∞ regularity of F.

However, this argument seems to break down in the case where F depends on the solution $f(t)$ itself, as the regularity of F is then tightly linked to that of f. Let us discuss for instance the Poisson case — the Maxwell case is actually worse in the sense that in the Vlasov–Poisson coupling, F gains, loosely speaking, one derivative in x compared to f. As already mentioned, the local Cauchy theory yields $f(t) \in C(0,T;\mathcal{H}^n_r)$, and we have $F \in C(0,T;H^{n+1}_x)$. Note then that when applying $n+1$ derivatives in x on m_ψ, one needs to apply $n+1$ derivatives in x on $|\det D_v V(0,t,x,v)|^{-1}$, which amounts to applying in total $n+2$ derivatives to $V(0,t,x,v)$. However, by (1-7), we observe that (X,V) inherits the same order of regularity as F, and therefore it does not seem licit to take as many as $n+2$ derivatives.

The goal of this work is to show that despite this apparent shortcoming, it is indeed possible to show for a fairly wide class of nonlinear Vlasov equations (including the Vlasov–Poisson and Vlasov–Maxwell system) a result of propagation of regularity in x for the moments, assuming higher-order space regularity for the initial condition. This in turn can be translated into some anisotropic Sobolev higher regularity for the solution itself, which can be interpreted as a kind of weak propagation of space regularity.

It turns out that the *lagrangian approach*, that is to say, the approach that we have just underlined, based on representation formulas using characteristics, is not adapted to answer this question. Instead we shall rely on an *eulerian approach*, which is based to a larger extent on the PDE itself, inspired by the recent work of the author in collaboration with F. Rousset on the quasineutral limit of the Vlasov–Poisson system [Han-Kwan and Rousset 2016; ≥ 2019]. The quasineutral limit is a singular limit which loosely consists in a penalization of the laplacian in the Poisson equation. The small parameter is the scaled Debye length, which appears to be very small in several usual plasma settings. The limit leads to *singular*

Vlasov equations, which display a loss of regularity of the force field compared to that of the distribution function. As a consequence, these equations are in general ill-posed in the sense of Hadamard; see [Bardos and Nouri 2012; Han-Kwan and Nguyen 2016]. This problem might therefore look quite different from the one considered here; the similarity comes from the fact that the justification of the quasineutral limit ultimately loosely comes down to the proof of a *uniform*[1] propagation of one order of higher regularity for moments of solutions of the Vlasov–Poisson equation. Note though that the analysis of [Han-Kwan and Rousset 2016; ≥ 2019] requires the introduction of pointwise Penrose stability conditions, and also relies on pseudodifferential tools, which will not be the case in this paper. As a matter of fact, the singular Vlasov equations which can be formally derived in the quasineutral limit will not enter the class of Vlasov equations we will deal with in this work, precisely because of the aforementioned loss of derivative.

The methodology of [Han-Kwan and Rousset 2016] was also used in the context of large time estimates for data close to stable equilibria for the Vlasov–Maxwell system in the nonrelativistic regime, in a recent work in collaboration with T. Nguyen and F. Rousset [Han-Kwan et al. 2017].

As a matter of fact, the approach can be considered as *semilagrangian*, in the sense that at some point we still rely on characteristics as in the lagrangian approach but at the level of the PDEs that arise after applying derivatives on the Vlasov equation, whereas in the lagrangian approach, derivatives are taken after using the representation of the solution by characteristics.

2. Main results

2A. ***The abstract framework.*** Let us now describe precisely the class of Vlasov equations we deal with. We consider in this work the abstract equation

$$\partial_t f + a(v)\cdot\nabla_x f + F\cdot\nabla_v f = 0, \tag{2-1}$$

with the following structural assumptions. Among all these assumptions, we highlight that the force depends on the distribution function itself, but only through some of its moments in velocity.

• *Assumptions on the advection field.* The map $a : \mathbb{R}^d \to \mathbb{R}^d$ is a one-to-one C^∞ function such that

$$|a(v)| \le C(1+|v|) \quad \text{for all } v\in\mathbb{R}^d, \tag{2-2}$$

$$\|\partial_v^\alpha a\|_{L^\infty} \le C_\alpha \quad \text{for all } |\alpha|\neq 0, \tag{2-3}$$

and its inverse a^{-1} (defined on $a(\mathbb{R}^d)$) satisfies, for some $\lambda>0$,

$$|\partial_v^\alpha a^{-1}(w)| \le C_\alpha(1+|a^{-1}(w)|)^{1+\lambda|\alpha|} \quad \text{for all } w\in a(\mathbb{R}^d), \ \text{ for all } \alpha. \tag{2-4}$$

• *Assumptions on the force field.* The vector field F is divergence-free in v (i.e., satisfies $\nabla_v\cdot F=0$) and we have the following decomposition for some $\ell\in\mathbb{N}^*$:

$$F(t,x,v) = \sum_{j=1}^{\ell} A_j(v)\, F^j(t,x). \tag{2-5}$$

[1] With respect to the scaled Debye length.

We assume that for all $j \in \{1, \dots, \ell\}$, A_j is a C^∞ scalar function satisfying

$$\|\partial_v^\alpha A_j\|_{L^\infty} \le C_\alpha \quad \text{for all } \alpha. \tag{2-6}$$

Furthermore, there exist C^∞ functions $\psi_1(v), \dots, \psi_r(v)$ with at most polynomial growth, i.e., there is $r_0 > 0$ such that

$$\|\psi_i(v)\|_{\mathcal{W}^{k,\infty}_{-r_0}} \le C_{i,k} \quad \text{for all } k \in \mathbb{N} \tag{2-7}$$

such that, defining

$$m_{\psi_i}(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, \psi_i(v)\, dv$$

for all $j = 1, \dots, \ell$, the vector field F^j is *uniquely determined by these moments and the initial conditions*, through a map

$$\big((m_{\psi_i})_{i=1,\dots,r}, (F^j(0))_{j=1,\dots,\ell}\big) \mapsto F^j, \tag{2-8}$$

and for all large enough $n > 1 + d$, and all $t > 0$, we have

$$\|F^j\|_{L^2(0,t;H^n_x)} \le \Gamma_n^{(j)}\Big(t, \|m_{\psi_1}\|_{L^2(0,t;H^n_x)}, \dots, \|m_{\psi_r}\|_{L^2(0,t;H^n_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\Big), \tag{2-9}$$

$$\|F^j\|_{L^\infty(0,t;H^n_x)} \le \Gamma_n^{(j)'}\Big(t, \|m_{\psi_1}\|_{L^\infty(0,t;H^n_x)}, \dots, \|m_{\psi_r}\|_{L^\infty(0,t;H^n_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\Big), \tag{2-10}$$

where $\Gamma_n^{(j)}$, $\Gamma_n^{(j)'}$ are polynomial functions that are nonincreasing with respect to each of their arguments (the others being fixed nonnegative numbers).

Finally, the force field satisfies the following stability property. Let f and g be two solutions of (2-1), and denote by $F[f]$ and $F[g]$ their associated force fields. Assume that the initial conditions $(F^j(0))_{j=1,\dots,\ell}$ are the same. Then, we have for all $j = 1, \dots, \ell$,

$$\|F^j[f] - F^j[g]\|_{L^2(0,t;H^n_x)} \le \Gamma_n^{(j)\sharp}\Big(t, \Big\|\int (f-g)\psi_i(v)\,dv\Big\|_{L^2(0,t;H^n_x)}, \dots, \Big\|\int (f-g)\psi_r(v)\,dv\Big\|_{L^2(0,t;H^n_x)}\Big), \tag{2-11}$$

where $\Gamma_n^{(j)\sharp}$ is a polynomial function that is nonincreasing with respect to each of its arguments and such that $\Gamma_n^{(j)'}(0,\cdot) = 0$.

We shall explain later why both Vlasov–Poisson and relativistic Vlasov–Maxwell systems enter the abstract framework.

2B. ***Statement of the main results.*** The regularity and integrability indices that will be useful to handle such equations will depend on the dimension d, the maximal growth of the moments that intervene in the definition of F, which is r_0, and the parameter of growth of the inverse of a, which is λ; let us set

$$N := \tfrac{3}{2}d + 4, \quad R := \max\big(\tfrac{1}{2}d + 2(1+\lambda)(1+d) + r_0\big). \tag{2-12}$$

We use in the following statement the notation $\lfloor \cdot \rfloor$ for the floor function.

The main result proved in this paper is the following theorem.

Theorem 2.1. *Let $n \geq N$ and $r > R$. Let $n' > n$ be an integer such that $n > \lfloor \frac{1}{2} n' \rfloor + 1$. Assume that $f_0 \in \mathcal{H}^n_r$ and $F^j(0) \in H^{n'}_x$ for all $j \in \{1, \dots, \ell\}$. Assume furthermore that the initial data f_0 satisfies the following higher anisotropic regularity*:

$$\partial_x^{2(n-\lfloor \frac{1}{2} n' \rfloor + k)} \partial_x^\alpha \partial_v^\beta f_0 \in \mathcal{H}^0_r \quad \text{for all } |\alpha| + |\beta| = n' - n - k, \ \text{for all } k \in \{1, \dots, 2\lfloor \tfrac{1}{2} n' \rfloor - n\}. \quad (2\text{-}13)$$

Then there is $T > 0$ such that the following holds. There exists a unique solution $(f(t), F(t))$ with initial data $(f_0, F(0))$ to (2-1) *such that $f(t) \in C(0, T; \mathcal{H}^n_r)$.*

Moreover, for all test functions $\psi \in L^\infty(0, T; \mathcal{W}^{n',\infty}_{-r_0})$, we have

$$\left\| \int f \psi \, dv \right\|_{L^2(0,T;H^{n'}_x)} \leq \Lambda_\psi(T, M), \quad (2\text{-}14)$$

where Λ_ψ is a polynomial function and

$$M = \|f_0\|_{\mathcal{H}^n_r} + \sum_{j=1}^{\ell} \|F^j(0)\|_{H^{n'}_x} + \sum_{k=1}^{2\lfloor \frac{1}{2} n' \rfloor - n} \sum_{|\alpha|+|\beta|=n'-n-k} \|\partial_x^{2(n-\lfloor \frac{n'}{2} \rfloor + k)} \partial_x^\alpha \partial_v^\beta f_0\|_{\mathcal{H}^0_r}.$$

Thanks to (2-9), we immediately deduce from (2-14) that the force field satisfies as well the higher regularity

$$F^j \in L^2(0, T; H^{n'}_x).$$

Another consequence concerns the flow $(X, V) = (X(t, 0, x, v), V(t, 0, x, v))$ as defined in (1-7), for which we also obtain a higher regularity property.

Corollary 2.2. *For some $T' \leq T$, we have*

$$\partial^\gamma_{x,v}(X - x - tv, V - v) \in L^\infty(0, T'; L^\infty_v L^2_x) \quad \text{for all } |\gamma| \leq n'.$$

Remark 2.3. Some remarks about Theorem 2.1 are in order:

- In the case where $n = 2m - 1$ and $n' = n + 1 = 2m$, the assumption (2-13) is simply given by $\partial_x^{n+1} f_0 \in \mathcal{H}^0_r$ and we obtain the $L^2_t H^{n+1}_x$ smoothness of the moments: in other words this gives an answer to the question raised in the beginning of the Introduction. Note though that the regularity result we prove is not pointwise in t.

- Observe that it is required that the higher regularity index n' is not too large compared to n (i.e., $n > \lfloor \frac{1}{2} n' \rfloor + 1$); such a restriction is somehow reminiscent of a similar one appearing in the celebrated result of Bony [1981, Théorème 6.1] concerning the propagation of Sobolev microlocal regularity at characteristic points for general nonlinear PDEs. We remark however that the class of PDEs considered in this work does not enter the framework of [Bony 1981], in particular because of the "nonlocality" in velocity. We refer to Section 10 for some remarks and (counter-)examples in this direction.

- As a matter of fact, our result can be somehow interpreted as a kinetic (and nonlocal) analogue of Bony's aforementioned theorem.

• If it is ensured that the solution $(f(t), F(t))$ to (2-1) is *global*, (e.g., for the Vlasov–Poisson system in dimension $d \le 3$, see [Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991; Batt and Rein 1991; Horst 1993]), we do not know if the higher propagation of regularity for the moments is global.

• Let us mention that in a somewhat different direction, a vector field method was devised in [Smulevici 2016] (see also [Fajman et al. 2017]) in order to prove time decay of moments for Vlasov equations set in unbounded spaces.

In the case where the force is one derivative smoother than the distribution function f itself (that is to say, when estimates (2-9) hold with $n-1$ instead of n in the right-hand side), the statement of Theorem 2.1 may be strengthened, insofar as one may ask only for derivatives in x in the regularity assumption (2-13). We refer to such a case as the transport/elliptic case, which includes in particular the Vlasov–Poisson system; see Theorem 9.1 in Section 9.

As already mentioned in the Introduction, the higher regularity for moments as obtained in Theorem 2.1 actually yields regularity for the solution itself (see [Gérard 1990] for a microlocal version of this fact, in the context of averaging lemmas) in anisotropic Sobolev spaces (as defined in [Hörmander 1976, Chapter II, Section 2.5]), which we first introduce.

Definition 2.4. Let $m, n \in \mathbb{R}$. The anisotropic Sobolev space $H^{m,n}_{x,v}$ is defined as

$$H^{m,n}_{x,v} := \{ g \in \mathcal{S}'(\mathbb{T}^d \times \mathbb{R}^d) : (1+|k|^2)^{\frac{m}{2}} (1+|\eta|^2)^{\frac{n}{2}} \hat{g}(k,\eta) \in L^2(\mathbb{Z}^d \times \mathbb{R}^d) \},$$

where $\hat{g}$ stands for the Fourier transform[2] of g. We also define

$$H^{m,-\infty}_{x,v} := \bigcup_{p \in \mathbb{R}} H^{m,p}_{x,v}.$$

Corollary 2.5. *Consider the same assumptions and notation as in Theorem 2.1. We have*

$$f(t,x,v) \in L^2(0,T; H^{n',-\infty}_{x,v}).$$

Corollary 2.5 is a direct consequence of some estimates obtained in the proof of Theorem 2.1; we will provide a proof of this fact in Section 7. It is actually possible to give an estimate of a value of $p < 0$ such that $f \in L^2(0,T; H^{n',p}_{x,v})$.

2C. ***Overview of the proof.*** We discuss in this section the ingredients, inspired by [Han-Kwan and Rousset 2016], leading to the higher propagation of regularity for the moments (the local well-posedness theory is fairly standard; see Section 3). We shall discuss here the case $n = 2k-1$ and $n' = n+1 = 2k$. To ease readability, we assume here that the dimension is $d = 1$ (in higher dimensions, the algebra is more involved but the basic principle is the same).

Taking derivatives. Since we intend to propagate regularity in space, the first step consists in understanding how to appropriately apply derivatives in x to the Vlasov equation (2-1).

[2] Where $\hat{g}(k,\eta) = 1/(2\pi)^d \int_{\mathbb{T}^d \times \mathbb{R}^d} g(x,v) e^{-ix\cdot k} e^{-iv\cdot\eta}\, dx\, dv$, although the convention that is chosen for the writing of the Fourier transform does not matter here.

We note that applying the operator ∂_x^α does not seem relevant, as it does not commute well with the operator $F\partial_v$: as a result it is not possible to obtain a closed equation bearing on $\partial_x^\alpha f$ without appealing to $\partial_x^\beta \partial_v^\gamma f$ for $\gamma \neq 0$, and therefore such an approach would require a control of derivatives in v which we do not have at initial time (this is of course reminiscent of the mixing in x and v that we have evoked in the Introduction).

The idea is to look for more appropriate differential operators, with nonconstant coefficients, satisfying the following three key properties:

- At initial time, they involve only derivatives in x.
- They enjoy good commutation properties with the transport operator, so that it is eventually possible to obtain closed systems involving these differential operators alone.
- They allow a good control of the Sobolev norm of the moments.

It turns out that second-order differential operators in x and v, with coefficients depending on the solution itself, will be appropriate. More precisely, we consider the operator

$$L := \partial_x^2 + \varphi(t,x)\partial_x\partial_v + \psi(t,x)\partial_v^2,$$

whose coefficients φ and ψ will depend on the force field F. Setting $\mathcal{T} := \partial_t + a(v)\partial_x + F\partial_v$ as the transport operator, we ask that the coefficients φ, ψ solve a semilinear system of the form

$$\begin{cases} \mathcal{T}\phi = 2\partial_x F + G_1(\phi, \psi, \partial_{x,v}F), \\ \mathcal{T}\psi = G_2(\phi, \psi, \partial_{x,v}F), \\ \varphi|_{t=0} = 0, \quad \psi|_{t=0} = 0, \end{cases}$$

where G_1, G_2 are polynomial functions of degree greater than or equal to 2; this corresponds to zero-order coupling terms. Note in particular that by definition, $L = \partial_x^2$ at time $t = 0$. The semilinear system is precisely chosen in order to cancel bad terms in the commutation between L and $\mathcal{T}$, so that for any function g,

$$L\mathcal{T}(g) = \mathcal{T}L(g) + (LF)\,\partial_v g + (La)\,\partial_x g + (\partial_v a)\varphi Lg.$$

Applying this identity to the solution f of the Vlasov equation (1-1), this yields

$$\mathcal{T}L(f) = -(LF)\,\partial_v f - (La)\,\partial_x f - (\partial_v a)\varphi Lf.$$

This formula will play a key role in the analysis. The main term (in terms of regularity issues) is $-\partial_x^2 F\partial_v f$, since the others involve either more regular quantities (we recall indeed that F and a are assumed to be smooth with respect to v), or the quantity Lf, which paves the way for a closed system involving only compositions of L applied to f. As a consequence, the operators obtained as compositions of L appear to be relevant for applying higher-order derivatives in x, since by construction:

- They require only a control of space regularity at initial time.

• Denoting by L_k the composition of k operators L, one can obtain that $L_k f$ satisfies an equation of the form

$$\mathcal{T}(L_k f) = A(L_k f) - (\partial_x^{2k} F)\, \partial_v f + G((\partial_{x,v}^{\alpha} f)_{|\alpha| \le 2k-1}), \tag{2-15}$$

where A, G are bounded linear operators. We note that this equation involves derivatives in v of the solution, but only of order $2k-1 = n$, which we control thanks to the local well-posedness theory. This can therefore be seen as a closed equation for $L_k f$.

• One can show that for any smooth test function ψ,

$$\int_{\mathbb{R}} (L_k f)\, \psi(t,x,v)\, dv = \int_{\mathbb{R}} (\partial_x^{2k} f)\, \psi(t,x,v)\, dv + \text{“controlled terms”}.$$

In the controlled terms, the overload of derivatives in v falling on f is transferred to ψ by an integration-by-parts argument.

All in all, this eventually shows that the L_k are indeed well-suited to study the regularity of moments. This step is fully developed in Section 4. There are two separate difficulties in order to complete this task: obtaining the right algebra as discussed here, and proving Sobolev estimates for all the involved objects.

(In the case where $n' > n+1$, we need to set up an induction argument, and this leads the study of successive systems of coupled kinetic transport equations, which build on the general equation (2-15).)

Propagation of regularity on moments. We then turn to the study of moments of the solutions to (2-15). This step is partly inspired from (and thus related to) the treatment of linear Landau damping in [Mouhot and Villani 2011].

We first use the method of characteristics to invert the operator $\mathcal{T} - A$. It is convenient at this stage to use changes of variables in velocity (introduced and studied in Section 5) in order to straighten characteristics and eventually, roughly speaking, come down from $\mathcal{T}$ to the free transport operator $\partial_t + a(v) \cdot \nabla_x$. To this end, it turns out to be efficient to introduce the change of variables $v \mapsto \Phi$ where Φ solves the Burgers' equation

$$\partial_t \Phi + a(\Phi) \cdot \nabla_x \Phi = F(\,\cdot\,, \Phi), \quad \Phi|_{t=0} = v,$$

where we can prove that Φ remains close to v in small time (in terms of Sobolev norms). The problem comes down to the understanding of the contribution of the term $-(\partial_x^{2k} F)\, \partial_v f$, and eventually roughly reduces to the study of an equation of the type

$$H_1(t,x) = \int_0^t \int_{\mathbb{R}} (\partial_x H_2)(s, x-(t-s)a(v))\, U(t,s,x,v)\, dv\, ds + \text{“controlled terms”},$$

where we know only that H_2 is controlled in $L^2(0,T; L_x^2)$ and U is smooth, and we seek a bound of H_1 in $L^2(0,T; L_x^2)$ (such an estimate corresponds to a control on the moments of $L_k f$). The integral in time is due to the use of Duhamel's formula, and the integral in v to the fact that we study moments in v. We observe that the operator in the right-hand side seems to feature a loss of derivative in x. However, we use a smoothing effect to overcome this apparent loss, which was proved in [Han-Kwan and Rousset

2016]. The outcome is the estimate

$$\left\| \int_0^t \int_{\mathbb{R}^d} (\nabla_x H_2)(s, x-(t-s)a(v))\, U(t,s,x,v)\, dv\, ds \right\|_{L^2(0,t;L^2_x)} \lesssim \|H_2\|_{L^2(0,t;L^2_x)} \sup_{0\le t,s\le T} \|U(t,s,\cdot)\|,$$

where $\|\cdot\|$ stands for a high-order weighted Sobolev norm (in x and v) which we will make precise later. As noted in [Han-Kwan and Rousset 2016], this is reminiscent of (but different from) classical *kinetic averaging lemmas*, as it loosely speaking involves the gain of one full derivative; we refer to Section 6 for a thorough discussion.

2D. ***Content of the end of the paper.*** The paper is then organised as follows: the proofs of Corollaries 2.2 and 2.5 are provided at the end of Section 7. In Section 8, we check the general assumptions for the Vlasov–Poisson and relativistic Vlasov–Maxwell equations, and discuss some extensions as well. As already mentioned, Section 9 is devoted to the particular case of the transport/elliptic case, for which Theorem 2.1 can be improved. We end the paper with the study of two examples that we cook up in order to discuss the regularity assumptions of Theorem 2.1.

We will prove Theorem 2.1 when n is odd, of the form $n = 2m-1$, and the higher regularity index n' is even of the form $n' = 2(m+p)$. The other cases follow by the same arguments. The requirement on n and n' is $m > p+2$. The assumption (2-13) in this case is given by

$$\partial_x^{2(m-p+k)} \partial_x^\alpha \partial_v^\beta f_0 \in \mathcal{H}^0_r \quad \text{for all } |\alpha|+|\beta| = 2p-k, \ \text{ for all } k = 0,\dots,2p. \tag{2-16}$$

3. Local well-posedness

We prove in this section a basic local Sobolev well-posedness result for (2-1). We start by recalling useful product estimates in weighted Sobolev spaces, taken from [Han-Kwan and Rousset 2016].

Lemma 3.1. *Let s be a nonnegative integer. Consider a smooth nonnegative function $\chi = \chi(v)$ that satisfies $|\partial^\alpha \chi| \le C_\alpha \chi$ for every multi-index α such that $|\alpha| \le s$:*

- *Consider two functions $f = f(x,v)$, $g = g(x,v)$; then we have for $k \ge \frac{1}{2}s$,*

$$\|\chi f g\|_{H^s_{x,v}} \lesssim \|f\|_{W^{k,\infty}_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|g\|_{W^{k,\infty}_{x,v}} \|\chi f\|_{H^s_{x,v}}. \tag{3-1}$$

- *Consider a function $E = E(x)$ and a function $F(x,v)$; then we have for any $s_0 > d$,*

$$\|\chi E F\|_{H^s_{x,v}} \lesssim \|E\|_{H^{s_0}_x} \|\chi F\|_{H^s_{x,v}} + \|E\|_{H^s_x} \|\chi F\|_{H^s_{x,v}}. \tag{3-2}$$

- *Consider a vector field $E = E(x)$, a function $A(v)$, and a function $f = f(x,v)$; then we have for any $s_0 > 1+d$ and for any multi-indices α, β such that $|\alpha|+|\beta| = s \ge 1$,*

$$\|\chi[\partial_x^\alpha \partial_v^\beta, A(v)E(x)\cdot\nabla_v] f\|_{L^2_{x,v}} \lesssim \|A\|_{W^{s,\infty}_v} (\|E\|_{H^{s_0}_x} \|\chi f\|_{H^s_{x,v}} + \|E\|_{H^s_x} \|\chi f\|_{H^s_{x,v}}). \tag{3-3}$$

- *Consider two functions $f = f(x,v)$, $g = g(x,v)$; then we have for multi-indices α, β with $|\alpha|+|\beta| \le s$,*

$$\|\partial^\alpha_{x,v} f\, \partial^\beta_{x,v} g\|_{L^2} \lesssim \left\| \frac{1}{\chi} f \right\|_{L^\infty_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|\chi g\|_{L^\infty_{x,v}} \left\| \frac{1}{\chi} f \right\|_{H^s_{x,v}}. \tag{3-4}$$

Proposition 3.2. *Let $n > d+1$ and $r > r_0 + \frac{1}{2}d$. Assume that $f_0 \in \mathcal{H}^n_r$ and $F^j(0) \in H^n_x$. Then there exists $T > 0$ such that there is a unique solution $(f(t), F(t))$ with initial data $(f_0, F(0))$ to* (2-1) *such that $f(t) \in C(0,T;\mathcal{H}^n_r)$ and $F^j(t) \in L^\infty(0,T;H^n_x)$.*

Proof of Proposition 3.2. The existence part follows from a standard iterative construction. We define recursively a sequence of distribution functions $(f_{(k)})_{k\in\mathbb{N}}$, denoting by $F_{(k)}$ the force field associated to $f_{(k)}$ and the initial condition $(F^j(0))$. Let us define

$$R_0 := \|f_0\|_{\mathcal{H}^n_r} + \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}.$$

We set $f_{(0)} := f_0$ and assume that $f_{(k)}$ is already constructed (with associated force field $F_{(k)}$), and is such that for some $T_k > 0$, we have $f_{(k)} \in C(0,T_k;\mathcal{H}^n_r)$, and

$$\|f_{(k)}\|_{L^\infty(0,T_k;\mathcal{H}^n_r)} \le 2R_0. \tag{3-5}$$

We define $f_{(k+1)}$ as the unique solution on $[0,T_k)$ to the equation

$$\partial_t f_{(k+1)} + a(v)\cdot\nabla_x f_{(k+1)} + F_{(k)}\cdot\nabla_v f_{(k+1)} = 0, \quad f_{(k+1)}|_{t=0} = f_0, \tag{3-6}$$

obtained by the method of characteristics.

Applying the operator $\partial^\alpha_x\partial^\beta_v$ to (3-6) for $|\alpha|+|\beta| \le n$ yields

$$(\partial_t + a(v)\cdot\nabla_x + F_{(k)}\cdot\nabla_v)(\partial^\alpha_x\partial^\beta_v f_{(k+1)}) + [\partial^\alpha_x\partial^\beta_v, a(v)\cdot\nabla_x + F_{(k)}\cdot\nabla_v]f_{(k+1)} = 0.$$

We then take the L^2 scalar product with $(1+|v|^2)^r\,\partial^\alpha_x\partial^\beta_v f_{(k+1)}$ and sum for all $|\alpha|+|\beta| \le n$. By using (2-3), we have

$$\sum_{|\alpha|+|\beta|\le n} \int \big|[\partial^\alpha_x\partial^\beta_v, a(v)\cdot\nabla_x]f_{(k+1)}\,\partial^\alpha_x\partial^\beta_v f_{(k+1)}\big|(1+|v|^2)^r\,dv\,dx \le \|f_{(k+1)}\|^2_{\mathcal{H}^n_r}.$$

Thanks to (2-6) and estimate (3-3) in Lemma 3.1 with $s = n$, $\chi(v) = (1+|v|^2)^{\frac{1}{2}r}$ and $s_0 = n$ (recall that $n > d+1$), we have for all $j \in \{1,\dots,\ell\}$,

$$\big\|\chi[\partial^\alpha_x\partial^\beta_v, A_j(v)F^j_{(k)}(x)\cdot\nabla_v]f_{(k+1)}\big\|_{L^2_{x,v}} \lesssim \|F^j_{(k)}\|_{H^n_x}\|f_{(k+1)}\|_{\mathcal{H}^n_r}.$$

Therefore by Cauchy–Schwarz, we get

$$\sum_{|\alpha|+|\beta|\le n} \int \big|[\partial^\alpha_x\partial^\beta_v, F_{(k)}\cdot\nabla_v]f_{(k+1)}\,\partial^\alpha_x\partial^\beta_v f_{(k+1)}\big|(1+|v|^2)^r\,dv\,dx \lesssim \|F^j_{(k)}\|_{H^n_x}\|f_{(k+1)}\|^2_{\mathcal{H}^n_r}.$$

Recalling that $\nabla_v\cdot F = 0$, we deduce that for all $t \in (0,T_k)$,

$$\frac{d}{dt}\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r} \lesssim \Big(1 + \sum_{j=1}^{\ell}\|F^j_{(k)}\|_{H^n_x}\Big)\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r}$$

so that

$$\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r} \lesssim \|f_0\|_{\mathcal{H}^n_r}\exp\Big[C\int_0^t\Big(1 + \sum_{j=1}^{\ell}\|F^j_{(k)}(s)\|_{H^n_x}\Big)\,ds\Big]. \tag{3-7}$$

We set

$$m_{\psi_i,(k)}(t,x) = \int_{\mathbb{R}^d} f_{(k)}(t,x,v)\,\psi_i(v)\,dv,$$

and by Cauchy–Schwarz and (2-7), we get, for $r' > \frac{1}{2}d$ such that $r \geq r_0 + r'$, which is possible thanks to the assumption $r > r_0 + \frac{1}{2}d$, that

$$\begin{aligned}
\|m_{\psi_i,(k)}\|_{L^2(0,t;H^n)} &= \sum_{|\alpha|\leq n} \left\| \left(\int_{\mathbb{R}^d} \partial_x^\alpha f_{(k)} \psi_i \, dv \right)^2 \right\|^{\frac{1}{2}}_{L^2(0,t;L^1_x)} \\
&\lesssim \sum_{|\alpha|\leq n} \left\| \left(\int_{\mathbb{R}^d} |\partial_x^\alpha f_{(k)}|^2 (1+|v|^2)^{r_0+r'} \, dv \right) \left(\int_{\mathbb{R}^d} \frac{|\psi_i|^2\, dv}{(1+|v|^2)^{r_0+r'}} \right) \right\|^{\frac{1}{2}}_{L^2(0,t;L^1_x)} \\
&\lesssim \sum_{|\alpha|\leq n} \left\| \left(\int_{\mathbb{R}^d} |\partial_x^\alpha f_{(k)}|^2 (1+|v|^2)^{r_0+r'} \, dv \right) \left(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^{r'}} \right) \right\|^{\frac{1}{2}}_{L^2(0,t;L^1_x)} \\
&\lesssim \|f_{(k)}\|_{L^2(0,t;\mathcal{H}^n_r)}.
\end{aligned}$$

Therefore, by (2-9), denoting by $C > 0$ a generic constant that does not depend on t or k, we obtain

$$\begin{aligned}
&\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r} \\
&\quad\lesssim \|f_0\|_{\mathcal{H}^n_r} \exp\left[Ct + C\sqrt{t} \sum_{j=1}^{\ell} \|F^j_{(k)}\|_{L^2(0,t;H^n_x)} \right] \\
&\quad\lesssim \|f_0\|_{\mathcal{H}^n_r} \exp\left[Ct + C\sqrt{t} \sum_{j=1}^{\ell} \Gamma^{(j)}_n \left(t, (\sqrt{t}\|m_{\psi_i,(k)}\|_{L^\infty(0,t;H^n_x)})_{i=1,\dots,r}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x} \right) \right] \\
&\quad\lesssim \|f_0\|_{\mathcal{H}^n_r} \exp\left[Ct + C\sqrt{t} \sum_{j=1}^{\ell} \Gamma^{(j)}_n \left(t, \sqrt{t}\|f_{(k)}\|_{L^\infty(0,t;\mathcal{H}^n_r)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x} \right) \right].
\end{aligned}$$

We now observe that if we choose $T > 0$ small enough so that

$$R_0 \exp\left[CT + C\sqrt{T} \sum_{j=1}^{\ell} \Gamma^{(j)}_n (T, 2\sqrt{T} R_0, R_0) \right] < 2R_0, \tag{3-8}$$

and $T_k \geq T$, then,

$$\|f_{(k+1)}(t)\|_{L^\infty(0,T;\mathcal{H}^n_r)} \leq 2R_0. \tag{3-9}$$

Therefore, by induction, we obtain that for all $k \in \mathbb{N}$, we have $f_{(k)} \in C(0,T;\mathcal{H}^n_r)$, and

$$\|f_{(k)}\|_{L^\infty(0,T;\mathcal{H}^n_r)} \leq 2R_0. \tag{3-10}$$

For $k \in \mathbb{N} \setminus \{0\}$, we set $h_k := f_{(k+1)} - f_{(k)}$, which satisfies the equation

$$\partial_t h_k + a(v)\cdot\nabla_x h_k + F[f_k]\cdot\nabla_v h_k + (F[f_{(k)}] - F[f_{(k-1)}])\cdot\nabla_v f_k = 0. \tag{3-11}$$

By weighted L^2 estimates, proceeding as before, we get

$$\frac{d}{dt}\|h_k(t)\|^2_{\mathcal{H}^0_r} \lesssim \Big(1+\sum_{j=1}^{\ell}\|F^j[f_{(k)}]\|_{H^n_x}\Big)\|h_k(t)\|^2_{\mathcal{H}^0_r} + \|f_{(k)}\|_{\mathcal{H}^n_r}\sum_{j=1}^{\ell}\|F^j[f_{(k)}]-F^j[f_{(k-1)}]\|_{L^2_x}\|h_k(t)\|_{\mathcal{H}^0_r}.$$

Let $t\in(0,T)$. Integrating in time, applying Cauchy–Schwarz and using the stability property (2-11) and the uniform estimates (3-10) for $(f_{(k)})$, we obtain

$$\begin{aligned}
\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)}
&\lesssim \int_0^t \Big(1+\sum_{j=1}^{\ell}\|F^j[f_{(k)}]\|_{H^n_x}\Big)\|h_k(s)\|_{\mathcal{H}^0_r}\,ds + \sum_{j=1}^{\ell}\int_0^t \|F^j[f_{(k)}]-F^j[f_{(k-1)}]\|_{L^2_x}\,ds\\
&\lesssim \sqrt{t}\Big[(\sqrt{t}+\|F^j[f_{(k)}]\|_{L^2(0,t;H^n_x)})\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)} + \sum_{j=1}^{\ell}\|F^j[f_{(k)}]-F^j[f_{(k-1)}]\|_{L^2(0,t;L^2_x)}\Big]\\
&\lesssim \sqrt{t}\Big[\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)} + \sqrt{t}\sum_{i=1}^{r}\Gamma^{(j)\sharp}_n\Big(t,\Big(\sqrt{t}\Big\|\int(f_{(k)}-f_{(k-1)})\psi_i(v)\,dv\Big\|_{L^\infty(0,t;L^2_x)}\Big)_{i=1,\dots,r}\Big)\Big]\\
&\lesssim \sqrt{t}\Big[\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)} + \sqrt{t}\sum_{i=1}^{r}\Gamma^{(j)\sharp}_n(t,\sqrt{t}\|h_{k-1}\|_{L^\infty(0,t;\mathcal{H}^0_r)})\Big].
\end{aligned}$$

We can thus pick a small enough time $T'>0$, independently of k such that for all $k\in\mathbb{N}\setminus\{0\}$,

$$\|f_{(k+1)}-f_{(k)}\|_{L^\infty(0,T';\mathcal{H}^0_r)} \le \tfrac{1}{2}\|f_{(k)}-f_{(k-1)}\|_{L^\infty(0,T';\mathcal{H}^0_r)}.$$

We can therefore pass to the limit in (3-6) and find that the limit $(f,F[f])$ satisfies (in the sense of distributions)

$$\partial_t f + a(v)\cdot\nabla_x f + F[f]\cdot\nabla_v f = 0, \tag{3-12}$$

with the initial conditions $(f_0, F^j(0))$. We deduce from (3-12) that $f\in C^0(0,T';\mathcal{H}^n_r)$ and $\partial_t f\in L^2(0,T';\mathcal{H}^{n-1}_{r-1})$. Also, thanks to (2-10), we deduce $F^j\in L^\infty(0,T';H^n_x)$. That the equation is satisfied in a classical way follows from the smoothness of $(f,F[f])$]. Uniqueness is also a consequence of the contraction estimate. □

The main matter is now to obtain the higher regularity statement for the moments. To this end, we will focus only on the task of obtaining a priori estimates for *smooth* solutions of (2-1); setting

$$M := \|f_0\|_{\mathcal{H}^{2m-1}_r} + \sum_{k=0}^{2p}\sum_{|\alpha|+|\beta|=2p-k}\|\partial_x^{2(m-p+k)}\partial_x^\alpha\partial_v^\beta f_0\|_{\mathcal{H}^0_r} + \sum_{j=1}^{\ell}\|F^j(0)\|_{H_x^{2(m+p)}}, \tag{3-13}$$

we look for some time $T_0 > 0$ depending only on M such that given a smooth test function $\psi \in L^\infty(0, T_0; \mathcal{W}^{2(m+p),\infty}_{-r_0})$, the following estimate holds:

$$\left\| \int f\psi(v)\,dv \right\|_{L^2(0,T_0;H_x^{2(m+p)})} \le C_\psi \Lambda(T_0, M), \tag{3-14}$$

where Λ is a polynomial function which is nondecreasing with respect to each of its arguments, once the others are fixed nonnegative numbers. In what follows, the function Λ may change from line to line but will always refer to such a function.

Once a priori estimates such as (3-14) as are obtained, we apply them to the sequence of solutions built in the iteration scheme proving the existence of solutions in the proof of Proposition 3.2. Passing to the limit yields the higher regularity for the moments of the solution $f(t)$.

4. Differential operators

In this section, we introduce and study the second-order differential operators (with coefficients depending on t and x) that we use in order to apply derivatives in x on the Vlasov equation (2-1).

The basic operators are defined in (4-3) and the definition of the coefficients is provided in Lemma 4.1. By definition these operators involve only derivatives in x at initial time. The key algebraic result reflecting the good commutation properties of these operators with the transport operator is stated in Lemma 4.2.

The composition of these operators is then studied:

- In Lemma 4.4, it is shown that they are indeed well suited to study the regularity of moments, as after integration in v, they act like derivations in x only (plus remainders that we can control). The proof is quite technical as one needs to be careful of the limited available smoothness on the coefficients of the differential operators. Note that in the statement, one does assume some (limited) higher-order smoothness for the moments: this is in prevision of a forthcoming induction argument.

- In Lemmas 4.5 and 4.6, the equations satisfied by the functions obtained after composition of these operators is established. This is where the key algebraic Lemma 4.2 appears to be crucial. Whereas the formal computation is straightforward, here again, the proof appears to be quite technical in order to justify that remainders are indeed well controlled. One also needs to be careful in order to get some Sobolev regularity for the coefficients involved in the equations.

- As the systems of equations in Lemmas 4.5 and 4.6 are not closed, this invites one to study the system satisfied by a larger set of appropriate functions; this is the purpose of Lemmas 4.7 and 4.8 (whose proof is similar to that of Lemmas 4.5 and 4.6).

4A. ***Second-order operators.*** As in the Introduction, we set $\mathcal{T} := \partial_t + a(v)\cdot\nabla_x + F\cdot\nabla_v$ as the transport operator to ease readability.

Lemma 4.1. *Let $n > d+1$. Assume that $(F^j) \in L^2(0, T'; H^n_x)$ with norm bounded by $\Lambda(T', M)$. There is $T \in (0, T')$ such that there exists a unique smooth solution $(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})_{i,j,k,l\in\{1,\dots,d\}}$ on $[0, T]$ of the* *system*

$$\begin{cases} \mathcal{T}\varphi^{i,j}_{k,l} = \sum_{k'} \partial_{v_{k'}} a(v)_k \, \psi^{i,j}_{k',l} + \sum_{k'} \partial_{v_{k'}} a(v)_k \, \psi^{i,j}_{l,k'} - \sum_{k',l',m} \partial_{v_{l'}} a(v)_m \, \varphi^{i,j}_{k',l'} \varphi^{k',m}_{k,l} \\ \qquad\qquad + \delta_{k,j} \, \partial_{x_i} F_l + \delta_{k,i} \, \partial_{x_j} F_l + \sum_{l'} \varphi^{i,j}_{k,l'} \, \partial_{v_{l'}} F_l, \\ \mathcal{T}\psi^{i,j}_{k,l} = - \sum_{k',l',m} \partial_{v_{l'}} a(v)_m \, \varphi^{i,j}_{k',l'} \psi^{k',m}_{k,l} + \varphi^{i,j}_{k,l} \, \partial_{x_k} F_k + \sum_{k'} \psi^{i,j}_{k',l} \partial_{v_{k'}} F_k + \sum_{l'} \psi^{i,j}_{k,l'} \, \partial_{v_{l'}} F_l, \\ \varphi^{i,j}_{k,l}|_{t=0} = \psi^{i,j}_{k,l}|_{t=0} = 0, \end{cases} \tag{4-1}$$

where δ denotes the Kronecker function and $a(v)_k$ and F_k stand for the k-th coordinates of $a(v)$ and F. Moreover we have the following estimates:

$$\begin{aligned} &\sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{W^{p,\infty}_{x,v}} \lesssim \Lambda(T, M) \quad \text{for all } p < n - 1 - \tfrac{1}{2}d, \\ &\sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{\mathcal{H}^{n-1}_{-\tilde r}} \lesssim \Lambda(T, M) \quad \text{for all } \tilde r > \tfrac{1}{2}d. \end{aligned} \tag{4-2}$$

We will not reproduce the proof of Lemma 4.1, since it follows, mutatis mutandis, that of Lemma 4.2 of [Han-Kwan and Rousset 2016]: System (4-1) is solved as a semilinear system of coupled kinetic transport equations. Note that we use the assumptions (2-3) on a and (2-6) on A to control the contribution of the additional linear and semilinear terms that appear compared to Lemma 4.2 of [Han-Kwan and Rousset 2016].

We introduce now the second-order operators

$$L_{i,j} := \partial^2_{x_i,x_j} + \sum_{1\le k,l\le d} (\varphi^{i,j}_{k,l} \, \partial_{x_k} \partial_{v_l} + \psi^{i,j}_{k,l} \, \partial^2_{v_k,v_l}) \quad \text{for all } i, j \in \{1, \dots, d\}. \tag{4-3}$$

We observe that by uniqueness of the solution of (4-1) and a symmetry argument, $L_{i,j} = L_{j,i}$.

One of the interests of the operators $L_{i,j}$ comes from the following lemma.

Lemma 4.2. *For all smooth functions f, we have the formula*

$$\begin{aligned} L_{i,j}\mathcal{T}(f) = \mathcal{T}L_{i,j}(f) + \Big(\partial^2_{x_i,x_j} F + \sum_{k,l} \varphi^{i,j}_{k,l} \, \partial_{x_k} \partial_{v_l} F + \psi^{i,j}_{k,l} \, \partial^2_{v_k,v_l} F\Big) \cdot \nabla_v f \\ + \sum_{k,l} \psi^{i,j}_{k,l} \, \partial^2_{v_k,v_l} a(v) \cdot \nabla_x f + \sum_{k,l,m} \partial_{v_l} a(v)_m \, \varphi^{i,j}_{k,l} L_{k,m} f. \end{aligned} \tag{4-4}$$

Remark 4.3. Formula (4-4) can also be written in a more synthetic form:

$$L_{i,j}\mathcal{T}(f) = \mathcal{T}L_{i,j}(f) + (L_{i,j} F) \cdot \nabla_v f + (L_{i,j} a) \cdot \nabla_x f + \sum_{k,l,m} \partial_{v_l} a(v)_m \, \varphi^{i,j}_{k,l} L_{k,m} f.$$

Proof of Lemma 4.2. We have by direct computations

$$
\begin{aligned}
\partial^2_{x_i x_j}(\mathcal{T} f) &= \mathcal{T}(\partial^2_{x_i,x_j} f) + \partial^2_{x_i,x_j} F \cdot \nabla_v f + \partial_{x_i} F \cdot \nabla_v \partial_{x_j} f + \partial_{x_j} F \cdot \nabla_v \partial_{x_i} f,\\
\varphi^{i,j}_{k,l}\, \partial_{x_k} \partial_{v_l} (\mathcal{T} f) &= \mathcal{T}(\varphi^{i,j}_{k,l}\, \partial_{x_k}\partial_{v_l} f) - \mathcal{T}(\varphi^{i,j}_{k,l}) \partial_{x_k}\partial_{v_l} f\\
&\quad + \varphi^{i,j}_{k,l}\big(\partial_{v_l} a(v)\cdot \nabla_x \partial_{x_k} f + \partial_{x_k} F\cdot \nabla_v \partial_{v_l} f + \partial_{v_l} F \cdot \nabla_v \partial_{x_k} f + \partial_{x_k}\partial_{v_l} F\cdot \nabla_v f\big),\\
\psi^{i,j}_{k,l}\, \partial^2_{v_k,v_l}(\mathcal{T} f) &= \mathcal{T}(\psi^{i,j}_{k,l}\, \partial^2_{v_k,v_l} f) - \mathcal{T}(\psi^{i,j}_{k,l}) \partial^2_{v_k,v_l} f\\
&\quad + \psi^{i,j}_{k,l}\big(\partial_{v_l} a(v)\cdot \nabla_x \partial_{v_k} f + \partial_{v_k} a(v) \cdot \nabla_x \partial_{v_l} f + \partial^2_{v_k,v_l} a(v)\cdot \nabla_v f\\
&\qquad + \partial_{v_k} F\cdot \nabla_v \partial_{v_l} f + \partial_{v_l} F\cdot \nabla_v \partial_{v_k} f + \partial^2_{v_k,v_l} F\cdot \nabla_v f\big).
\end{aligned}
$$

We can rewrite

$$
\begin{aligned}
\varphi^{i,j}_{k,l}\, \partial_{v_l} a(v)\cdot \nabla_x \partial_{x_k} f &= \varphi^{i,j}_{k,l} \sum_m \partial_{v_l} a(v)_m\, \partial_{x_m}\partial_{x_k} f\\
&= \varphi^{i,j}_{k,l} \sum_m \partial_{v_l} a(v)_m \Big(L_{k,m} f - \sum_{k',l'} (\varphi^{k,m}_{k',l'}\, \partial_{x_{k'}}\partial_{v_{l'}} + \psi^{k,m}_{k',l'}\, \partial^2_{v_{k'},v_{l'}}) f\Big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
L_{i,j}\mathcal{T}(f) = \mathcal{T} L_{i,j}(f) &+ \partial_{x_i x_j} F\cdot \nabla_v f\\
&+ \sum_{k,l} \big(\varphi^{i,j}_{k,l}\, \partial_{x_k}\partial_{v_l} F\cdot\nabla_v f + \psi^{i,j}_{k,l}\, \partial^2_{v_k,v_l} F\cdot \nabla_v f + \psi^{i,j}_{k,l}\, \partial^2_{v_k,v_l} a(v)\cdot \nabla_x f\big)\\
&+ \sum_{k,l,m} \partial_{v_l} a(v)_m \varphi^{i,j}_{k,l} L_{k,m} f\\
&+ \sum_{k,l} \partial_{x_k}\partial_{v_l} f \Big[-\mathcal{T}\varphi^{i,j}_{k,l} + \sum_{k'} \partial_{v_{k'}} a(v)_k \psi^{i,j}_{k',l} + \sum_{k'} \partial_{v_{k'}} a(v)_k \psi^{i,j}_{l,k'}\\
&\qquad - \sum_{k',l',m} \partial_{v_{l'}} a(v)_m \varphi^{i,j}_{k',l'} \varphi^{k',m}_{k,l} + \delta_{k,j}\, \partial_{x_i} F_l + \delta_{k,i}\, \partial_{x_j} F_l + \sum_{l'} \varphi^{i,j}_{k,l'}\, \partial_{v_{l'}} F_l\Big]\\
&+ \sum_{k,l} \partial^2_{v_k,v_l} f \Big[-\mathcal{T}\psi^{i,j}_{k,l} - \sum_{k',l',m} \partial_{v_{l'}} a(v)_m \varphi^{i,j}_{k',l'} \psi^{k',m}_{k,l} + \varphi^{i,j}_{k,l}\, \partial_{x_k} F_k\\
&\qquad + \sum_{k'} \psi^{i,j}_{k',l}\, \partial_{v_{k'}} F_k + \sum_{l'} \psi^{i,j}_{k,l'}\, \partial_{v_{l'}} F_l\Big].
\end{aligned}
$$

We therefore deduce (4-4), because $(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})$ solves (4-1). □

4B. ***Composition of the second-order operators.*** Relying on Lemma 4.2, we shall use the $L_{i,j}$ operators in order to apply derivatives to the solution f of the Vlasov equation (2-1).

Set for $I, J \in \{1, \dots, d\}^k$,

$$L^{I,J} := L_{i_1,j_1} \cdots L_{i_k,j_k}. \tag{4-5}$$

Let us also introduce the following useful notation. Given $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$, we set

$$\alpha(I, J) := (i_1, j_1, \dots, i_k, j_k), \tag{4-6}$$

and

$$\partial_x^{\alpha(I,J)} = \partial_{x_{i_1}} \partial_{x_{j_1}} \cdots \partial_{x_{i_k}} \partial_{x_{j_k}}. \tag{4-7}$$

Note that by construction,

$$L^{I,J}|_{t=0} = \partial_x^{\alpha(I,J)}.$$

In what follows, f will systematically stand for the solution of (2-1), starting from f_0 satisfying the assumptions of Theorem 2.1.

4C. *Moments in v.* We study in this section the moments in v of the $L^{I,J} f$. Until the end of the section, the times $T > 0$ will be such that the solution to (2-1) satisfies

$$\|f\|_{L^\infty(0,T;\mathcal{H}_r^{2m-1})} \le 2R_0,$$

thanks to Proposition 3.2.

Lemma 4.4. • *Let $k = 0, \dots, p$ and $I, J \in \{1, \dots, d\}^{m+k}$. Assume that the force field satisfies $F^j \in L^2(0, T; H_x^{2(m+k)-1})$ with norm bounded by $\Lambda(T, M)$. Assume that for all $n = 2m, \dots, 2(m+k)-1$, for all $\varphi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+2+n-2m,\infty})$ such that $\|\varphi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+2+n-2m,\infty})} \le \Lambda(T, M)$, and all $|\alpha| = n$, we have*

$$\left\| \int_{\mathbb{R}^d} (\partial_x^\alpha f)\, \varphi(t, x, v)\, dv \right\|_{L^2(0,T;L_x^2)} \le \Lambda(T, M). \tag{4-8}$$

Let $\psi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+2+2k,\infty})$ satisfy $\|\psi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+2+2k,\infty})} \le \Lambda(T, M)$. We have

$$\int_{\mathbb{R}^d} L^{I,J} f\, \psi(t, x, v)\, dv = \int_{\mathbb{R}^d} \partial_x^{\alpha(I,J)} f\, \psi(t, x, v)\, dv + \mathfrak{R}_{I,J,\psi}, \tag{4-9}$$

where $\mathfrak{R}_{I,J,\psi}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2(0,T;L_x^2)} \le \Lambda(T, M). \tag{4-10}$$

• *Let $k = 0, \dots, p-1$ and $I, J \in \{1, \dots, d\}^{m+k}$. Assume the force field satisfies $F^j \in L^2(0, T; H_x^{2(m+k)})$ with norm bounded by $\Lambda(T, M)$. Assume that for all $n = 2m, \dots, 2(m+k)$, for all $|\alpha| = n$, and for all $\varphi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+2+n-2m,\infty})$ such that $\|\varphi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+2+n-2m,\infty})} \le \Lambda(T, M)$, we have*

$$\left\| \int_{\mathbb{R}^d} (\partial_x^\alpha f)\, \varphi(t, x, v)\, dv \right\|_{L^2(0,T;L_x^2)} \le \Lambda(T, M). \tag{4-11}$$

Let $\psi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+3+2k,\infty})$ satisfy $\|\psi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+3+2k,\infty})} \le \Lambda(T, M)$. Let $\partial = \partial_{x_i}$ or ∂_{v_i} for some $i \in \{1, \dots, d\}$. We have

$$\int_{\mathbb{R}^d} \partial L^{I,J} f\, \psi(t, x, v)\, dv = \int_{\mathbb{R}^d} \partial_x^{\alpha(I,J)} \partial f\, \psi(t, x, v)\, dv + \mathfrak{R}_{I,J,\psi}, \tag{4-12}$$

where $\mathfrak{R}_{I,J,\psi}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2(0,T;L_x^2)} \le \Lambda(T, M). \tag{4-13}$$

This result will allow us to set up an induction argument: indeed, with the assumption (4-8) (resp. (4-11)) that corresponds to regularity of the moments up to order $2(m+k)-1$ (resp. $2(m+k)$), this will imply that controlling the moments of the $(L^{I,J} f)$ gives information on the regularity of the moments up to order $2(m+k)$ (resp. $2(m+k)+1$).

Proof of Lemma 4.4. Let us focus only on the first item (the proof of the second one is completely similar). Let $\psi \in L^\infty(0,T;\mathcal{W}^{d+2+2k,\infty}_{-r_0})$. The beginning of the proof closely follows that of Lemma 4.3 of [Han-Kwan and Rousset 2016]. At first, we can expand $f_{I,J} = L^{I,J} f$ in a more tractable form. Let us set for readability

$$U := (\varphi^{i_\alpha, j_\beta}_{k',l}, \psi^{i_\alpha, j_\beta}_{k',l})_{1\le k',l\le d,\, 1\le \alpha,\beta\le m+k}.$$

Then, by induction, we obtain

$$\begin{aligned} f_{I,J} &= \partial_x^{\alpha(I,J)} f + \sum_{s=0}^{2(m+k)-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} P^{k_0}_{s,e,\alpha}(U)\, P^{k_1}_{s,e,\alpha}(\partial U) \cdots P^{k_s}_{s,e,\alpha}(\partial^s U)\, \partial_v^e \partial^\alpha f \\ &=: \partial_x^{\alpha(I,J)} f + \sum_{s=0}^{2(m+k)-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} \mathcal{R}^{k_0,\dots,k_s}_{s,e,\alpha}, \end{aligned} \tag{4-14}$$

where the sum is taken on indices such that

$$\begin{gathered} |e| = 1, \quad |\alpha| = 2(m+k)-1-s, \\ k_0 + k_1 + \cdots + k_s \le m+k, \quad k_0 \ge 1, \quad k_1 + 2k_2 + \cdots + s k_s = s, \end{gathered} \tag{4-15}$$

and for all $0 \le p \le s$, we have $P^{k_p}_{s,e,\alpha}(X)$ is a polynomial of degree smaller than k_p (we denote by $\partial^k U$ the vector made of all the partial derivatives of length k of all components of U). We can set

$$\mathfrak{R}_{I,J,\psi} = \int_{\mathbb{R}^d} \psi(\cdot, v) \sum_{s=0}^{2(m+k)-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} \mathcal{R}^{k_0,\dots,k_s}_{s,e,\alpha}\, dv,$$

so that we have to estimate $\int_{\mathbb{R}^d} \psi \mathcal{R}^{k_0,\dots,k_s}_{s,e,\alpha}\, dv$. All the following estimates are uniform in time for $t \in [0,T]$ and we shall dismiss the time parameter to ease readability.

We begin by estimating the terms for which $s \ge 2k+1$. Note that for all these terms the total number of derivatives applied to f is at most $2m-1$.

- When $s < 2(m+k) - \frac{1}{2}d - 2$, we can use estimate (4-2) in Lemma 4.1 to obtain

$$\| P^{k_0}_{s,e,\alpha}(U)\, P^{k_1}_{s,e,\alpha}(\partial U) \cdots P^{k_s}_{s,e,\alpha}(\partial^s U) \|_{L^\infty_{x,v}} \le \Lambda(T, M),$$

and hence using that

$$\sup_v |(1+|v|^2)^{-\frac{1}{2}r_0} \psi(\cdot, v)| \le \Lambda(T, M),$$

we obtain by Cauchy–Schwarz that since $r > r_0 + r'$ for some $r' > \frac{1}{2}d$, we have

$$\begin{aligned}\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} &\le \big\| \|(1+|v|^2)^{-\frac{1}{2}(r_0-r')}\psi\|_{L^2_v} \|(1+|v|^2)^{\frac{1}{2}(r_0+r')}\,\partial_v^e\partial^\alpha f\|_{L^2_v}\big\| \\ &\le \Lambda(T,M)\left(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^{r'}}\right)^{\frac{1}{2}} \|f\|_{\mathcal{H}^{2m-1}_r} \\ &\le \Lambda(T,M).\end{aligned}$$

• Let us now consider $s \ge 2(m+k)-2-\frac{1}{2}d$. We start with the case where in the sequence $(k_1,\dots,k_s)$, the largest index l such that $k_l \ne 0$ and $k_p = 0$ for every $p > l$ is such that $l > \frac{1}{2}s$. In this case, since $lk_l \le s$ has to hold, we necessarily have $k_l = 1$. Moreover, for the indices $p < l$ such that $k_p \ne 0$, we must have $p \le pk_p < \frac{1}{2}s$. Thus, we can use estimate (4-2) in Lemma 4.1 to bound $\|\partial^p U\|_{L^\infty_{x,v}}$ provided $\frac{1}{2}s \le 2(m+k)-\frac{1}{2}d-2$. Since $s \le 2(m+k)-2$, this is satisfied thanks to the assumption that $2m > 2+d$. We thus obtain

$$\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \le \Lambda(T,M) \left\| \int \psi\, \partial^l U\, \partial_v^e\partial^\alpha f\, dv \right\|_{L^2_x}.$$

Next, we can use the fact that

$$\begin{aligned}\left\| \int \psi\, \partial^l U\, \partial_v^e\partial^\alpha f \right\|_{L^2_x} &\lesssim \Lambda(T,M) \big\| \|(1+|v|^2)^{-\frac{1}{2}r}\,\partial^l U\|_{L^2_v} \|(1+|v|^2)^{\frac{1}{2}(r_0+r)}\,\partial_v^e\partial^\alpha f\|_{L^2_v}\big\|_{L^2_x} \\ &\lesssim \Lambda(T,M)\|U\|_{\mathcal{H}^{2m-2}_{-r}} \sup_x \|(1+|v|^2)^{\frac{1}{2}r}\,\partial_v^e\partial^\alpha f\|_{L^2_v}.\end{aligned}$$

By the Sobolev embedding in x, we have

$$\sup_x \|(1+|v|^2)^{\frac{1}{2}r}\,\partial_v^e\partial^\alpha f\|_{L^2_v} \lesssim \|f\|_{\mathcal{H}^{2m-1}_r}$$

as soon as $2m-1 > 1+|\alpha|+\frac{1}{2}d = 1+2(m+k)-1-s+\frac{1}{2}d$, which is equivalent to $s > 1+2k+\frac{1}{2}d$. Since we are in the case where $s \ge 2(m+k)-2-\frac{1}{2}d$, the condition is matched, thanks to the assumption $2m > 3+d$. Consequently, by using estimate (4-2) in Lemma 4.1, we obtain again that

$$\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \lesssim \Lambda(T,M).$$

Finally, it remains to handle the case where $k_l = 0$ for every $l > \frac{1}{2}s$. As above, we necessarily have $\frac{1}{2}s < 2(m+k)-\frac{1}{2}d-2$ and hence by using again estimate (4-2) in Lemma 4.1, we find

$$\|\partial^l U\|_{L^\infty_{x,v}} \le \Lambda(T,M), \quad l \le \tfrac{1}{2}s.$$

We deduce

$$\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \le \Lambda(T,M)\|f\|_{\mathcal{H}^{2m-1}_r} \le \Lambda(T,M).$$

It remains to treat the cases corresponding to $s \le 2k$; that is to say, $\mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}$ contains the maximal number of derivatives applied to f. This means that $|\alpha| = 2m-1,\dots,2(m+k)-1$ so that at least

$2m$ derivatives of f are involved. We define for readability the associated coefficient

$$\Gamma := \psi\, P^{k_0}_{s,e,\alpha}(U)\, P^{k_1}_{s,e,\alpha}(\partial U) \cdots P^{k_s}_{s,e,\alpha}(\partial^s U),$$

and we have to study the L^2_x norm of $\int \Gamma\, \partial^e_v \partial^\alpha f\, dv$.

First, assume that $|\alpha| \le 2(m+k)-2$ (which corresponds to $s \ge 1$). We note that for all $s' = 0, \dots, 2(m+k)-1-|\alpha|$, we have by Lemma 4.1 that

$$\|\partial^{s'} U\|_{W^{k,\infty}_{x,v}} \le \Lambda(T,M) \quad \text{for all } k < 2(m+k)-2-\tfrac{1}{2}d - s'.$$

Since $s' \le 2(m+k)-1-|\alpha|$, we have $2(m+k)-2-\frac{1}{2}d-s' \ge |\alpha| - \frac{1}{2}d - 1 > d+2+|\alpha|+1-2m$ because $2m > \frac{3}{2}d+4$. Therefore

$$\|\partial^{s'} U\|_{W^{d+2+|\alpha|+1-2m,\infty}_{x,v}} \le \Lambda(T,M),$$
$$\|\Gamma\|_{\mathcal{W}^{d+2+|\alpha|+1-2m,\infty}_{-r_0}} \le \Lambda(T,M).$$

We can thus use the assumption (4-8) to obtain the bound

$$\left\| \int \Gamma\, \partial^e_v \partial^\alpha f\, dv \right\|_{L^2_x} \le \Lambda(T,M). \tag{4-16}$$

Assume finally that $|\alpha| = 2(m+k)-1$ (which corresponds to $s=0$); that is to say, $2(m+k)$ derivatives of f are involved. We can write, by integration by parts in v (relying on the fast decay of f and its derivatives at infinity)

$$\int_{\mathbb{R}^d} \Gamma\, \partial^e_v \partial^\alpha f\, dv = -\int_{\mathbb{R}^d} \partial^e_v \Gamma\, \partial^\alpha f\, dv.$$

We have

$$\|\partial^e_v \Gamma\|_{\mathcal{W}^{d+1+2k,\infty}_{-r_0}} \le \Lambda(T,M),$$

and we can use again (4-8) to obtain

$$\left\| \int \partial^e_v \Gamma\, \partial^\alpha f\, dv \right\|_{L^2_x} \le \Lambda(T,M).$$

In summary, we have proved

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2_x} \le \Lambda(T,M). \qquad \square$$

4D. *The equation satisfied by $L^{I,J} f$.* Using the algebraic identities of Lemma 4.2, we obtain:

Lemma 4.5. *For all $k = 0, \dots, p$, the following holds. Assume that $(F^j) \in L^2(0,T; H^{2(m+k)-1}_x)$ with norm bounded by $\Lambda(T,M)$. For all $I, J \in \{1, \dots, d\}^{m+k}$, we have*

$$\mathcal{T}(L^{I,J} f) + \partial^{\alpha(I,J)}_x F \cdot \nabla_v f = \sum_{r=m-k}^{m+k} \sum_{K,L \in \{1,\dots,d\}^r} \sum_{|\alpha|+|\beta| = m+k-r} \gamma^{I,J}_{K,L,\alpha,\beta} L^{K,L} \partial^\alpha_x \partial^\beta_v f + R_{I,J}, \tag{4-17}$$

where

- $\gamma^{I,J}_{K,L,\alpha,\beta}$ *are coefficients satisfying*

$$\|\gamma^{I,J}_{K,L,\alpha,\beta}\|_{L^2(0,T; W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M), \tag{4-18}$$

- $R_{I,J}$ *is a remainder satisfying*

$$\|R_{I,J}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} \lesssim \Lambda(T,M) \quad \text{for all } \tilde r \le r - \tfrac{1}{2}d.$$

A version of this lemma was proved in [Han-Kwan and Rousset 2016] in the case $k=0$.

Lemma 4.5 will be useful in the induction argument to treat the case of even integers. For odd integers, we have the following result.

Lemma 4.6. *For all $k = 0,\dots,p-1$, the following holds. Assume that $(F^j) \in L^2(0,T;H_x^{2(m+k)})$ with norm bounded by $\Lambda(T,M)$. For all $I,J \in \{1,\dots,d\}^{m+k}$, and $i=1,\dots,d$, we have*

$$\mathcal{T}(L^{I,J}\,\partial_{x_i} f) + \partial_{x_i}\partial_x^{\alpha(I,J)} F\cdot\nabla_v f = \sum_{r=m-k-1}^{m+k}\ \sum_{K,L\in\{1,\dots,d\}^r}\ \sum_{|\alpha|+|\beta|=m+k+1-r} \gamma^{x_i,I,J}_{K,L,\alpha,\beta}\, L^{K,L}\,\partial_x^\alpha\partial_v^\beta f + R_{x_i,I,J}, \quad (4\text{-}19)$$

$$\mathcal{T}(L^{I,J}\,\partial_{v_i} f) + \partial_{v_i}\partial_x^{\alpha(I,J)} F\cdot\nabla_v f = \sum_{r=m-k-1}^{m+k}\ \sum_{K,L\in\{1,\dots,d\}^r}\ \sum_{|\alpha|+|\beta|=m+k+1-r} \gamma^{v_i,I,J}_{K,L,\alpha,\beta} L^{K,L}\,\partial_x^\alpha\partial_v^\beta f + R_{v_i,I,J}, \quad (4\text{-}20)$$

where

- $\gamma^{x_i I,J}_{K,L,\alpha,\beta}, \gamma^{v_i I,J}_{K,L,\alpha,\beta}$ *are coefficients satisfying*

$$\|\gamma^{x_i,I,J}_{K,L,\alpha,\beta}, \gamma^{v_i,I,J}_{K,L,\alpha,\beta}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M), \quad (4\text{-}21)$$

- $R_{x_i,I,J}$, $R_{v_i,I,J}$ *are remainders satisfying*

$$\|R_{x_i,I,J}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} + \|R_{v_i,I,J}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} \lesssim \Lambda(T,M) \quad \text{for all } \tilde r \le r - \tfrac{1}{2}d.$$

4E. ***The equation satisfied by $L^{I,J}\,\partial_x^\alpha\partial_v^\beta f$.*** Lemma 4.5 invites us to search for a closed equation on $L^{I,J}\,\partial_x^\alpha\partial_v^\beta f$ for $k\in\{0,\dots,p\}$, $r\in\{m-k,\dots,m+k\}$, $I,J\in\{1,\dots,d\}^r$ and all $|\alpha|+|\beta| = m+k-r$ (and similarly for Lemma 4.6). This is the purpose of the next two lemmas.

Lemma 4.7. *Let $k=0,\dots,p$. Let $r=m-k,\dots,m+k$. Assume that $(F^j)\in L^2(0,T;H_x^{2(m+k)-1})$ with norm bounded by $\Lambda(T,M)$. For all $I,J\in\{1,\dots,d\}^r$ and all $|\alpha|+|\beta| = m+k-r$, we have*

$$\mathcal{T}(L^{I,J}\,\partial_x^\alpha\partial_v^\beta f) + \partial_x^\alpha\partial_v^\beta\partial_x^{\alpha(I,J)} F\cdot\nabla_v f = \sum_{r'=m-k}^{r}\ \sum_{K,L\in\{1,\dots,d\}^{r'}}\ \sum_{|\alpha'|+|\beta'|=m+k-r'} \gamma^{I,J,\alpha,\beta}_{K,L,\alpha',\beta'}\, L^{K,L}\,\partial_x^{\alpha'}\partial_v^{\beta'} f + R_{I,J,\alpha,\beta}, \quad (4\text{-}22)$$

where

- $\gamma^{I,J,\alpha,\beta}_{K,L,\alpha',\beta'}$ *are coefficients satisfying*

$$\|\gamma^{x_i,I,J}_{K,L,\alpha,\beta}, \gamma^{v_i,I,J}_{K,L,\alpha,\beta}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M), \quad (4\text{-}23)$$

- $R_{I,J,\alpha,\beta}$ *is a remainder satisfying*

$$\|R_{I,J,\alpha,\beta}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} \lesssim \Lambda(T,M) \quad \text{for all } \tilde r \le r - \tfrac{1}{2}d.$$

Lemma 4.8. *Let $k = 0, \dots, p-1$. Let $r = m-k-1, \dots, m+k$. Assume that $(F^j) \in L^2(0, T; H_x^{2(m+k)})$ with norm bounded by $\Lambda(T, M)$. For all $I, J \in \{1, \dots, d\}^r$, and all $|\alpha| + |\beta| = m + k + 1 - r$, we have*

$$\mathcal{T}(L^{I,J}\, \partial_x^\alpha \partial_v^\beta f) + \partial_x^\alpha \partial_v^\beta \partial_x^{\alpha(I,J)} F \cdot \nabla_v f = \sum_{r'=m-k-1}^{r} \sum_{K,L \in \{1,\dots,d\}^{r'}} \sum_{|\alpha'|+|\beta'|=m+k+1-r'} \gamma_{K,L,\alpha',\beta'}^{I,J,\alpha,\beta} L^{K,L}\, \partial_x^{\alpha'} \partial_v^{\beta'} f + R_{I,J,\alpha,\beta}, \tag{4-24}$$

where

- $\gamma_{K,L,\alpha',\beta'}^{I,J,\alpha,\beta}$ *are coefficients satisfying*

$$\|\gamma_{K,L,\alpha,\beta}^{x_i,I,J}, \gamma_{K,L,\alpha,\beta}^{v_i,I,J}\|_{L^2(0,T;W_{x,v}^{d+2,\infty})} \lesssim \Lambda(T, M), \tag{4-25}$$

- $R_{I,J,\alpha,\beta}$ *is a remainder satisfying*

$$\|R_{I,J,\alpha,\beta}\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^0)} \lesssim \Lambda(T, M) \quad \textit{for all } \tilde r \le r - \tfrac{1}{2}d.$$

We observe that as wanted, Lemmas 4.7 and 4.8 provide *closed* systems of equations.

To conclude this section, we shall give the proofs of Lemmas 4.5 and 4.7 (the proofs of the remaining Lemmas 4.6 and 4.8 being very similar).

4F. *Proofs of Lemmas 4.5 and 4.7.*

Proof of Lemma 4.5. Let $\tilde r < r - \frac{1}{2}d$. Since $r > d$, we can assume, without loss of generality, that $\tilde r > \frac{1}{2}d$. We can write, by an induction argument relying on Lemma 4.2, that

$$\mathcal{T}(L^{I,J} f) = F_{I,J},$$

with the source term $F_{I,J}$ given by $F_{I,J} = -\sum_{i=1}^4 F_i$, where

$$F_1 = \sum_{\ell=1}^{m+k-1} L_{i_1,j_1} \cdots L_{i_{m+k-\ell},j_{m+k-\ell}} \times \big((\partial^2_{x_{i_{m+k-\ell+1}},x_{j_{m+k-\ell+1}}} F) \cdot \nabla_v L_{i_{m+k-\ell+2},j_{m+k-\ell+2}} \cdots L_{i_{m+k},j_{m+k}} f\big), \tag{4-26}$$

$$F_2 = \sum_{\ell=1}^{m+k-1} L_{i_1,j_1} \cdots L_{i_{m+k-\ell},j_{m+k-\ell}} \times\Bigg(\Bigg[\sum_{k,l} \varphi_{k,l}^{i_{m+k-\ell+1},j_{m+k-\ell+1}}\, \partial_{x_{i_{m+k-\ell+1}}} \partial_{v_{j_{m+k-\ell+1}}} F + \psi_{k,l}^{i_{m+k-\ell+1},j_{m+k-\ell+1}} \partial^2_{v_{i_{m+k-\ell+1}},v_{j_{m+k-\ell+1}}} F\Bigg] \cdot \nabla_v L_{i_{m+k-\ell+2},j_{m+k-\ell+2}} \cdots L_{i_{m+k},j_{m+k}} f\Bigg), \tag{4-27}$$

$$F_3 = \sum_{\ell=1}^{m+k-1} L_{i_1,j_1} \cdots L_{i_{m+k-\ell},j_{m+k-\ell}} \times\Bigg(\Bigg[\sum_{k,l} \psi_{k,l}^{i_{m+k-\ell+1},j_{m+k-\ell+1}}\, \partial^2_{v_{i_{m+k-\ell+1}},v_{j_{m+k-\ell+1}}} a\Bigg] \cdot \nabla_x L_{i_{m+k-\ell+2},j_{m+k-\ell+2}} \cdots L_{i_{m+k},j_{m+k}} f\Bigg), \tag{4-28}$$

$$F_4 = \sum_{\ell=1}^{m+k-1} L_{i_1,j_1} \cdots L_{i_{m+k-\ell},j_{m+k-\ell}} \times \sum_{k',l',m'} \partial_{v_{l'}} a(v)_{m'} \, \varphi_{k',l'}^{i_{m+k-\ell+1},j_{m+k-\ell+1}} L_{k',m'} L_{i_{m+k-\ell+2},j_{m+k-\ell+2}} \cdots L_{i_{m+k},j_{m+k}} f. \tag{4-29}$$

We shall focus on the contribution of F_1. We have to estimate terms of the form

$$F_{1,\ell} = L^{m+k-\ell} G_\ell, \quad G_\ell = \partial^2 E \cdot \nabla_v L^{\ell-1}, \tag{4-30}$$

where we use the notation L^n for the composition of n operators of type $L_{i,j}$ (the exact combination of the operators involved in the composition does not matter here). Note that as in (4-14), we can write L^n in the form

$$L^n = \partial_x^{\alpha_n} + \sum_{s=0}^{2n-2} \sum_{e,\alpha,k_0,\ldots,k_s} P_{s,e,\alpha}^{k_0}(U)\, P_{s,e,\alpha}^{k_1}(\partial U) \cdots P_{s,e,\alpha}^{k_s}(\partial^s U)\, \partial_v^e \partial^\alpha, \tag{4-31}$$

where for all $0 \le p \le s$, we have $P_{s,e,\alpha}^{k_p}(X)$ is a polynomial of degree smaller than k_p, the multi-index α_n has length $2n$ and the sum is taken on indices such that

$$|e| = 1, \quad |\alpha| = 2n-1-s, \quad k_0 + k_1 + \cdots + k_s \le n, \quad k_0 \ge 1, \quad k_1 + 2k_2 + \cdots + sk_s = s. \tag{4-32}$$

Let us first establish a general estimate, adapted from [Han-Kwan and Rousset 2016]. We set for any function $G(x,v)$,

$$J_p(G)(x,v) = \sum_{s,\beta,K\in E} J_{p,s,\beta,K}(G), \tag{4-33}$$

where $K = (k_0,\ldots,k_s)$ and

$$J_{p,s,\beta,K}(G)(x,v) = P_{s,\beta}^{k_0}(U)\, P_{s,\beta}^{k_1}(\partial U) \cdots P_{s,\beta}^{k_s}(\partial^s U)\, \partial^\beta G, \tag{4-34}$$

where for all $0 \le r \le s$, we have $P_{s,\beta}^{k_r}(X)$ is a polynomial of degree smaller than k_r and the sum is taken over indices belonging to the set E defined by

$$|\beta| = p - s, \quad k_0 + k_1 + \cdots + k_s \le \tfrac{1}{2}p, \quad k_1 + 2k_2 + \cdots + sk_s = s, \quad 0 \le s \le p-2. \tag{4-35}$$

Lemma 4.9. *For $2(m+k)-1 \ge p \ge 1$, $2m > d+3$, $\tilde r > \frac{1}{2}d$ and s, p, K satisfying* (4-35), *we have the estimate*

$$\|J_p(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M)\Big(\|G\|_{\mathcal{H}^p_{\tilde r}} + \sum_{\substack{l \ge 2(m+k)-\frac{1}{2}d-2 \\ l+|\alpha|\le p,\, |\alpha|\ge 2}} \|\partial^l U\, \partial^\alpha G\|_{\mathcal{H}^0_{\tilde r}}\Big). \tag{4-36}$$

Proof of Lemma 4.9. For the terms in the sum such that $s < 2(m+k) - \frac{1}{2}d - 2$, we can use estimate (4-2) in Lemma 4.1 to obtain

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M)\|G\|_{\mathcal{H}^p_{\tilde r}}.$$

When $s \ge 2(m+k) - \frac{1}{2}d - 2$, we first consider the terms for which in the sequence $(k_1,\ldots,k_s)$ the largest index l for which $k_l \ne 0$ is such that $l < 2(m+k) - \frac{1}{2}d - 2$. Then again thanks to estimate (4-2) in Lemma 4.1, we obtain

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M)\|G\|_{\mathcal{H}^p_{\tilde r}}.$$

When $l \ge 2(m+k)-\frac{1}{2}d-2$, we first observe that we necessarily have $k_l=1$. Indeed if $k_l \ge 2$, because of (4-35), we must have $l \le \frac{1}{2}s$. This is possible only if $2(m+k)-\frac{1}{2}d-2 \le \frac{1}{2}p-2 \le \frac{1}{2}(2(m+k)-3)$, which corresponds to $m+k \le \frac{1}{2}d+1$, and hence this is impossible. Consequently $k_l=1$. Moreover, we note that for the other indices $\tilde l$ for which $k_{\tilde l} \ne 0$, because of (4-35), we must have $\tilde l k_{\tilde l} \le s - lk_l$, so that

$$\tilde l \le s-l \le s-2(m+k)+\tfrac{1}{2}d+2 \le \tfrac{1}{2}d-1,$$

and we observe that $\frac{1}{2}d-1 < 2m-\frac{1}{2}d-2$. Consequently, by another use of estimate (4-2) in Lemma 4.1, we obtain

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M) \sum_{\substack{l\ge 2(m+k)-\frac{1}{2}d-2\\ l+|\alpha|\le p,\,|\alpha|\ge 2}} \|\partial^l U\, \partial^\alpha G\|_{\mathcal{H}^0_{\tilde r}}.$$

The fact that $|\alpha| \ge 2$ comes from (4-35). □

We shall now estimate $F_{1,\ell}$. Looking at the expansion of $L^{m+k-\ell}$ given by (4-31), we have to estimate terms of the form $J_p(G_\ell)$ with $p \le 2(m+k-\ell)$. Using (4-31), we decompose G_ℓ as

$$G_\ell = \partial^2 F\cdot\nabla_v L^{\ell-1} f = \partial^2 F\cdot\nabla_v(H_{\ell,+}+H_{\ell,-}) =: G_{\ell,+}+G_{\ell,-},$$

where:

- In $H_{\ell+}$, we gather all terms of the form (4-34), with $G=f$, such that $2k+1+|\beta| \ge 2\ell$. These terms may contribute to terms with at least $2m$ derivatives of f.
- On the other hand in $H_{\ell,-}$, the terms that arise correspond to $2k+1+|\beta| < 2\ell$, which involve at most $2m-1$ derivatives of f.

We first focus on the contribution of $G_{\ell,-}$; we define

$$F_{1,\ell,-} := L^{m+k-\ell} G_{\ell,-}.$$

Let us start with the case $\ell \ge \frac{1}{2}(m+k)$. We can use the decomposition (4-31), which means that we have to estimate terms of the form $J_p(G_{\ell,-})$ with $p \le 2(m+k-\ell) \le 2(m+k)-1$, and apply Lemma 4.9 to get

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M)\Bigg(\|G_{\ell,-}\|_{L^2(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})} + \sum_{\substack{l\ge 2(m+k)-\frac{1}{2}d-2\\ l+|\alpha|\le 2(m+k-\ell),\,|\alpha|\ge 2}} \|\partial^l U\,\partial^\alpha G_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})}\Bigg). \quad (4\text{-}37)$$

We observe that in the right-hand side of (4-37), we have $l \le 2(m+k-\ell)-2 \le m+k-2$; consequently, since $2m-1 > d-1$, we have $l < 2(m+k)-\frac{1}{2}d-2$, and hence we can estimate $\|\partial^l U\|_{L^\infty}$ by using estimate (4-2) in Lemma 4.1. This yields

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M)\|G_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}, \quad \ell \ge \tfrac{1}{2}(m+k).$$

Then we use estimate (3-2) in Lemma 3.1 with $s = 2(m+k-\ell)$ and $s_0 = d+1$, and the definition of $G_{\ell,-}$ to estimate $\|G_{\ell,-}\|_{\mathcal{H}^{2(m+k-\ell)}_{\tilde r}}$. Since $d+2 < 2m-1$ and $2(m+k-\ell)+2 \le 2(m+k)-1$ (because $\ell \ge \frac{1}{2}(m+k) \ge 2$), we obtain

$$\begin{aligned}\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} &\le \Lambda(T,M)\big(\sup_j \|F^j\|_{L^2(0,T;H^{d+1})}\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}\\ &\qquad\qquad + \sup_j \|F^j\|_{L^2(0,T;H^{2(m+k-\ell)+2})}\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}\big)\\ &\le \Lambda(T,M)\sup_j \|F^j\|_{L^2(0,T;H^{2(m+k)-1})}\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}. \end{aligned}\tag{4-38}$$

By using the regularity assumption on F^j, this yields

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M)\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}.$$

To estimate the above right-hand side, we need to estimate $\partial^\gamma_{x,v} H_{\ell,-}$ with $|\gamma| \le 2(m+k-\ell)+1$. Recalling the definition of $H_{\ell,-}$, by taking derivatives using the expression (4-31), we see that we have to estimate terms under the form $J_p(f)$ with $p \le 2m-1$. Using Lemma 4.9 one more time, we thus obtain

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M)\Big(\|f\|_{L^\infty(0,T;\mathcal{H}^{2m-1}_{\tilde r})} + \sum_{\substack{l \ge 2(m+k)-\frac{1}{2}d-2\\ l+|\alpha| \le 2m-1,\, |\alpha|\ge 2}} \|\partial^l U\,\partial^\alpha f\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})}\Big).$$

To estimate the right-hand side, we argue as follows. Let $r' > \frac{1}{2}d$ be such that $\tilde r + r' \le r$. Since $|\alpha| \ge 2$ and $|\alpha|-2+l \le 2m-3$, we can use estimate (3-4) in Lemma 3.1 (taking $\chi(v) = (1+|v|^2)^{\frac{1}{2}r'}$) to obtain

$$\|\partial^l U (1+|v|^2)^{\frac{1}{2}\tilde r}\,\partial^\alpha f\|_{L^2_{x,v}} \lesssim \|U\|_{\mathcal{H}^{2m-3}_{-r'}}\|(1+|v|^2)^r\,\partial^2 f\|_{L^\infty} + \|U\|_{L^\infty}\|f\|_{\mathcal{H}^{2m-1}_r}. \tag{4-39}$$

By using again estimate (4-2) in Lemma 4.1 and the Sobolev embedding, we finally obtain

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M)\|f\|_{L^\infty(0,T;\mathcal{H}^{2m-1}_r)} \le \Lambda(T,M), \quad \ell \ge \tfrac{1}{2}(m+k). \tag{4-40}$$

It remains to handle the case $\ell \le \frac{1}{2}(m+k)$. Note that necessarily, for these cases to be meaningful, we must have $2k+1 < 2\ell$. Assume first $\ell \ge 2$. We obtain again (4-37). We first need to estimate $\|\partial^2 F \cdot \nabla_v H_{\ell,-}\|_{L^2(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}$. We thus have to study

$$\|\partial^\beta \partial^2 F \cdot \nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})},$$

with $|\beta|+|\gamma| \le 2(m+k-\ell)$. Since $\ell \ge 2$, we have $|\beta|+2 \le 2(m+k-1)$. If $|\beta|+2 < 2(m+k)-1-\frac{1}{2}d$, then we get, by the Sobolev embedding, the bound

$$\begin{aligned}\|\partial^\beta \partial^2 F \cdot \nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} &\le \sup_j \|\partial^\beta \partial^2 F^j\|_{L^2(0,T;L^\infty_x)}\|\nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})}\\ &\le \sup_j \|F^j\|_{L^2(0,T;H^{2(m+k)-1}_x)}\|f\|_{L^2(0,T;\mathcal{H}^{2m-1}_{\tilde r})}\\ &\le \Lambda(T,M),\end{aligned}$$

recalling the definition of $H_{\ell,-}$. If $|\beta| \geq 2(m+k)-3-\frac{1}{2}d$, then $|\gamma| \leq 2(m+k-\ell)-2(m+k)+3+\frac{1}{2}d$ and thus the term $\nabla_v \partial^\gamma H_{\ell,-}$ involves at most $\frac{1}{2}d+2$ derivatives. Since $2m-1 > \frac{3}{2}d+2$, we have

$$\begin{aligned}\|\partial^\beta \partial^2 F \cdot \nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde{r}})} &\leq \sup_j \|\partial^\beta \partial^2 F^j\|_{L^2(0,T;L^2_x)} \|H_{\ell,-}\|_{L^2(0,T;\mathcal{W}^{d/2+2,\infty}_{\tilde{r}})} \\ &\leq \sup_j \|F^j\|_{L^2(0,T;H^{2(m+k)-1}_x} \|f\|_{L^2(0,T;\mathcal{H}^{2m-1}_r)} \\ &\leq \Lambda(T,M).\end{aligned}$$

We also have to estimate terms in (4-37) of the form

$$\|\partial^l U\, \partial^\beta \partial^2 F\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}},$$

with $l \geq 2(m+k)-\frac{1}{2}d-2$ and $l+|\beta|+|\gamma| \leq 2(m+k-\ell)$. Note that this implies $|\beta| \leq 2(m+k-\ell)-l \leq \frac{1}{2}d+2-2\ell \leq \frac{1}{2}d$ since we have $\ell \geq 1$. In particular this yields $|\beta|+2 < 2m-1-\frac{1}{2}d$ since $2m > 3+\frac{1}{2}d$, and thus by using the Sobolev embedding and (2-9), we obtain

$$\begin{aligned}\|\partial^l U\, \partial^\beta \partial^2 F\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}} &\lesssim \sup_j \|F^j\|_{H^{2m-1}_x} \|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}} \\ &\lesssim (\|f\|_{\mathcal{H}^{2m-1}_r} + \sup_j \|F^j(0)\|_{H^{2m-1}_x}) \|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}} \\ &\leq \Lambda(T,M) \|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}}.\end{aligned}$$

Consequently, it remains to estimate $\|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}}$ for $l \geq 2(m+k)-\frac{1}{2}d-2$ and $l+|\gamma| \leq 2(m+k-\ell)$. By using again (4-31) and the definition of $H_{\ell,-}$, we can expand $\partial^\gamma \nabla_v H_{\ell,-}$ as terms of the form $J_p(f)$, with $p \leq 2(\ell-k)+|\gamma|-1$. Since we have $2(\ell-k)+|\gamma|-1 \leq 1+\frac{1}{2}d < 2(m+k)-\frac{1}{2}d-2$, we can use estimate (4-2) in Lemma 4.1 again to estimate all the terms in the expression of $J_p(f)$ involving U and its derivatives in L^∞. This yields

$$\|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}} \leq \Lambda(T,M) \sum_{\tilde{\gamma}} \|\partial^l U\, \partial^{\tilde{\gamma}} f\|_{\mathcal{H}^0_{\tilde{r}}},$$

with $|\tilde{\gamma}| \leq |\gamma|+2(\ell-k)-1$. Consequently, arguing as for (4-39), we obtain

$$\|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde{r}}} \leq \Lambda(T,M)\big(\|U\|_{L^\infty} \|f\|_{\mathcal{H}^{2m-1}_r} + \|(1+|v|^2)^r f\|_{L^\infty_{x,v}} \|U\|_{\mathcal{H}^{2m-1}_{-r'}}\big),$$

where we recall $r' > \frac{1}{2}d$ and we conclude finally by invoking estimate (4-2) in Lemma 4.1 and the Sobolev embedding that

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde{r}})} \leq \Lambda(T,M), \quad 2 \leq \ell \leq \tfrac{1}{2}(m+k). \tag{4-41}$$

For the case $\ell = 1$ to be meaningful, k must be equal to 0. We set aside the term $\partial_x^{\alpha(I,J)} F \cdot \nabla_v f$, which appears in the formula (4-17), and we thus have to study the term

$$L_{i_1,j_1} \cdots L_{i_{m-1},j_{m-1}} (\partial^2_{x_{i_m},x_{j_m}} F \cdot \nabla_v f) - \partial_x^{\alpha(I,J)} F \cdot \nabla_v f.$$

We argue exactly as before to obtain a bound in $L^2(0,T;\mathcal{H}^0_{\tilde r})$ by $\Lambda(T,M)$ (note indeed that at most $2m-1$ derivatives of f and F are involved). Gathering all pieces together, we have thus proven that

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M). \tag{4-42}$$

Let us now treat the contribution of $G_{\ell,+}$, which will give rise to terms involving $2m$ up to $2(m+k)$ derivatives of f. Let $j\in\{0,\dots,2k\}$. Let us describe the form of the terms involving derivatives of order $2m+j$ of f. We note that $2m+j-1\ge 2m-1>m+p-1\ge m+k-1$. This means that such terms are necessarily of the form

$$\big(\partial_x^{\alpha^0}\partial_v^{\beta^0} L_{i_1,j_1}\cdots L_{i_k,j_k}\,\partial_x^{\alpha^k}\partial_v^{\beta^k}\cdots L_{i_{m+j-k},j_{m+j-k}}\,\partial_x^{\alpha^{m+j-k}}\partial_v^{\beta^{m+j-k}}\big)f, \tag{4-43}$$

with

$$\sum_{k=0}^{m+j-k} |\alpha^k|+|\beta^k| = 2k-j, \qquad \sum_{k=0}^{m+j-k} |\beta^k|\neq 0.$$

In order to have exactly $2m+j$ derivatives of f, this expression can be rewritten as $L^{K,L}\,\partial_x^\alpha\partial_v^\beta f$, where $|K|=|L|=m+j-k$ and $|\alpha|+|\beta|=2k-j$, $|\beta|\ge 1$. Indeed if derivatives fall on a coefficient of one of the L_{i_k,j_k}, then there are less than $2m+j$ derivatives of f.

We denote by $\gamma^{I,J,1}_{K,L,\alpha,\beta}$ the coefficient associated to such terms. Note that for $|\gamma|\le 2k-j-1$, we have $\partial_x^\gamma\partial^2 F^i\in L^2(0,T;H_x^{2m+j-2})$. Since we have $2m>\frac{3}{2}d+4$, we can bound this term in $L^2(0,T;W_x^{d+2,\infty})$ by the Sobolev embedding. Likewise, for $|\gamma|\le 2k-j-1$, since $2m+j-1-\frac{1}{2}d> d+2$ we have $\partial^\gamma_{x,v}U\in L^\infty(0,T;W^{d+2,\infty}_{x,v})$. All in all, we deduce

$$\|\gamma^{I,J,1}_{K,L,\alpha,\beta}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})}\le \Lambda(T,M).$$

It remains to treat the other terms that all involve at most $2m-1$ derivatives on f. If $k\ge 1$, we set aside the term $\partial_x^{\alpha(I,J)}F\cdot\nabla_v f$ in (4-17), which corresponds to the case $\ell=1$ treated above (relevant when $k=0$).

The other terms can be considered as remainders that are uniformly bounded in $L^2(0,T;\mathcal{H}^0_{\tilde r})$, since at most $2m-1$ derivatives are involved on f and at most $2(m+k)-1$ derivatives are involved on F; these terms can be treated exactly as we treated the remainders in $G_{\ell,-}$.

The treatment of F_2, F_3, F_4 gives rise to similar terms and we omit it. ☐

Proof of Lemma 4.7. The proof is similar to the previous one. We shall only explain why the terms involving at least $2m$ derivatives of f are indeed of the form appearing in (4-22).

Let $k=0,\dots,p-1$, and $r=m+j$ for $j=-k-1,\dots,k$. We look for the terms involving $2m+l$ derivatives of f for $l=0,\dots,k+1+j$. Among the operators in $L^{I,J}$ there are exactly $2m+l-(m+k+1-r)=2m+j+l-k-1$ derivatives to be applied on f. Since $m>p\ge k+1$, we have $2m+j+l-k-1>m+j$. This means that these derivatives must be of the form $L^{K,L}\,\partial_x^\gamma\partial_v^\delta$, with $|K|=|L|=m+l-k-1$ and $|\gamma|+|\delta|=j-l+k+1$ (up to commutations between the differential operators as in (4-43), which is treated as in the previous proof). In the end, the terms involving $2m+l$ derivatives of f are thus necessarily of the form $L^{K,L}\,\partial_x^\gamma\partial_v^\delta f$, with

$$|K|=|L|=m+l-k-1, \quad |\gamma|+|\delta|=2k+2-l,$$

as appearing in (4-22). ☐

Remark 4.10. An inspection of the proof reveals that the uniform regularity of the coefficients in (4-18), (4-21), (4-23), (4-25) can be improved to $L^2(0,T;W^{p,\infty}_{x,v})$ for all $p < 2m-2-\frac{1}{2}d$.

5. Burgers' equation and the semilagrangian approach

In this section, we explain the procedure to straighten the transport operator $\mathcal{T}$, which allows, loosely speaking, to come down to the operator $\partial_t + a(v)\cdot\nabla_v$. This relies on several changes of variables in velocity that we introduce now.

Let $\Phi(t,x,v)$ satisfy the Burgers' equation

$$\begin{cases}\partial_t\Phi + a(\Phi)\cdot\nabla_x\Phi = F(t,x,\Phi),\\ \Phi(0,x,v) = v.\end{cases} \tag{5-1}$$

The interest in introducing the vector field Φ comes from the following algebraic identity.

Lemma 5.1. *Given a smooth function g satisfying $\mathcal{T}g = R$, the function*

$$G(t,x,v) := g(t,x,\Phi(t,x,v))$$

solves the equation

$$\partial_t G + a(\Phi(t,x,v))\cdot\nabla_x G = R(t,x,\Phi(t,x,v)). \tag{5-2}$$

Proof of Lemma 5.1. We compute

$$\partial_t G = (\partial_t g)(t,x,\Phi(t,x,v)) + \partial_t\Phi\cdot(\nabla_v g)(t,x,\Phi(t,x,v)),$$
$$a(\Phi)\cdot\nabla_x G = a(\Phi)\cdot(\nabla_x g)(t,x,\Phi(t,x,v)) + [a(\Phi)\cdot\nabla_x\Phi]\cdot(\nabla_v g)(t,x,\Phi(t,x,v)).$$

Since $\mathcal{T}g = R$, we have

$$(\partial_t g)(t,x,\Phi(t,x,v)) + a(\Phi)\cdot(\nabla_x g)(t,x,\Phi(t,x,v)) = -F(t,x,\Phi)\cdot(\nabla_v g)(t,x,\Phi(t,x,v)) + R(t,x,\Phi(t,x,v)).$$

From (5-1), we deduce (5-2) as claimed. □

In other words, the change of variables in velocity $v\mapsto\Phi(t,x,v)$ allows us to straighten the vector field $\mathcal{T}$.

We now provide a lemma concerning the existence, uniqueness and regularity of solutions of (5-1).

Lemma 5.2. *Assume that for all $j = 1,\dots,\ell$, we have $F^j \in L^2(0,T';H^n_x)$ with norm bounded by $\Lambda(T',M)$. There is $T\in(0,T']$ such that the following holds. There exists a unique solution $\Phi(t,x,v)\in C^0(0,T;W^{k,\infty}_{x,v})$ of* (5-1) *and we have the following estimates*:

$$\sup_{[0,T]}\sup_v\sum_{|\alpha|\le n}\|\partial^\alpha_{x,v}(\Phi-v)\|_{L^2_{x,v}} + \sup_{[0,T]}\|\Phi-v\|_{W^{k,\infty}_{x,v}} \lesssim \Lambda(T,M), \tag{5-3}$$

$$\sup_{[0,T]}\sup_v\sum_{|\alpha|\le n}\|\partial^\alpha_{x,v}(a(\Phi)-a(v))\|_{L^2_{x,v}} + \sup_{[0,T]}\|a(\Phi)-a(v)\|_{W^{k,\infty}_{x,v}} \lesssim \Lambda(T,M) \tag{5-4}$$

for all $k < n-\frac{1}{2}d$.

We shall not provide the proof of Lemma 5.2 as it follows closely the proof of Lemma 4.6 in [Han-Kwan and Rousset 2016]. Here the source is semilinear, whereas there it is linear; however, the proof is essentially the same (see also [Han-Kwan et al. 2017] for a similar issue).

We now introduce the characteristics associated to Φ, defined as the solution to

$$\begin{cases} \partial_t \mathrm{X}(t,s,x,v) = a(\Phi)(t, \mathrm{X}(t,s,x,v), v), \\ \mathrm{X}(s,s,x,v) = x, \end{cases} \tag{5-5}$$

and study the deviation of X from the (modified) free transport flow.[3]

Lemma 5.3. *Assume that for all $j = 1, \dots, \ell$, we have $F^j \in L^\infty(0, T'; H^n_x)$ with norm bounded by $\Lambda(T', M)$. There is $T \in (0, T']$ such that the following holds. For every $0 \le s, t \le T$, we can write*

$$\mathrm{X}(t,s,x,v) = x + (t-s)(a(v) + \widetilde{\mathrm{X}}(t,s,x,v)), \tag{5-6}$$

with $\widetilde{\mathrm{X}}$ that satisfies the estimate

$$\sup_{t,s\in[0,T]} \sup_v \sum_{|\alpha|\le n} \|\partial^\alpha_{x,v} \widetilde{\mathrm{X}}(t,s,x,v)\|_{L^2_x} + \sup_{t,s\in[0,T]} \|\widetilde{\mathrm{X}}(t,s,x,v)\|_{W^{k,\infty}_{x,v}} \lesssim \Lambda(T,M) \tag{5-7}$$

for all $k < n - \frac{1}{2}d$. Moreover, the map $x \mapsto x + (t-s)\widetilde{\mathrm{X}}(t,s,x,v)$ is a diffeomorphism, and there exists $\Psi(t,s,x,v)$ such that the identity

$$\mathrm{X}(t,s,x,\Psi(t,s,x,v)) = x + (t-s)a(v)$$

holds. Finally, we have the estimate

$$\sup_{t,s\in[0,T]} \left[\sup_v \sum_{|\alpha|\le n} \|\partial^\alpha_{x,v}(\Psi(t,s,x,v) - v)\|_{L^2_x} + \|\Psi(t,s,x,v) - v\|_{W^{k,\infty}_{x,v}} \right] \lesssim \Lambda(T,M) \tag{5-8}$$

for all $k < n - \frac{1}{2}d$.

Again, we will not reproduce the proof of Lemma 5.3 as it follows closely that of Lemma 5.1 in [Han-Kwan and Rousset 2016].

In what follows, the procedure will consist in applying derivatives on (2-1) using multiple combinations of the operators $L^{I,J}$ that were introduced and studied in the previous section. This yields systems of Vlasov equations with sources, such as (4-22) in Lemma 4.7. It is on these equations that we will apply the change of variables $v \mapsto \Phi(t,x,v)$ in order to straighten the transport operator $\mathcal{T}$. We then integrate along characteristics, which is why the $\mathrm{X}(t,s,x,v)$ are useful, and average in velocity to obtain equations bearing on moments. In these equations, it will be crucial to apply various changes of variables based on the $\widetilde{\mathrm{X}}$ and Ψ introduced in Lemma 5.3.

This is what we refer to as the *semilagrangian* approach.

[3] Note that the X introduced here is not the same as the X previously defined in (1-7).

6. Averaging operators

For $i \in \{1, \dots, d\}$ and a smooth function $U(t, s, x, v)$, we define the following integral operator $K_U^{(i)}$ acting on scalar functions $H(t, x)$:

$$K_U^{(i)}(H)(t,x) = \int_0^t \int_{\mathbb{R}^d} (\partial_{x_i} H)(s, x-(t-s)a(v))\, U(t,s,x,v)\, dv\, ds. \tag{6-1}$$

The integral operator K can be seen as a modified version of the operator $\mathsf{K}_{\mathsf{U}}^{(i)}$

$$\mathsf{K}_{\mathsf{U}}^{(i)}(H)(t,x) = \int_0^t \int_{\mathbb{R}^3} (\partial_{x_i} H)(s, x-(t-s)v)\, \mathsf{U}(t,s,x,v)\, dv\, ds$$

that was studied in [Han-Kwan and Rousset 2016].

6A. *The smoothing estimate.* We note that the operators $K_U^{(i)}$ and $\mathsf{K}_{\mathsf{U}}^{(i)}$ seem to feature a loss of derivative in x. It was nevertheless proved in [Han-Kwan and Rousset 2016, Proposition 5.1 and Remark 5.1] that for the operators $\mathsf{K}_{\mathsf{U}}^{(i)}$, this loss is only apparent, provided that U is sufficiently smooth in x, v and decaying in v: this is the content of the following theorem.

Theorem 6.1 [Han-Kwan and Rousset 2016]. *Let $k > 1 + d$ and $\sigma > \frac{1}{2}d$. For all $H \in L^2(0, T; L^2_x)$, and for all $i \in \{1, \dots, d\}$, we have*

$$\|\mathsf{K}_{\mathsf{U}}^{(i)}(H)\|_{L^2(0,T;L^2_x)} \lesssim \sup_{0\le s,t\le T} \|\mathsf{U}(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \|H\|_{L^2(0,T;L^2_x)}. \tag{6-2}$$

Based on this result, we deduce the following smoothing estimate[4] for the operators $K_U^{(i)}$.

Proposition 6.2. *Let $k > 1 + d$ and $\sigma > \frac{1}{2}d$. For all $H \in L^2(0, T; L^2_x)$, and for all $i \in \{1, \dots, d\}$, we have*

$$\|K_U^{(i)}(H)\|_{L^2(0,T;L^2_x)} \lesssim \sup_{0\le s,t\le T} \|U(t,s,\cdot)\|_{\mathcal{H}^k_{r_k}} \|H\|_{L^2(0,T;L^2_x)}, \tag{6-3}$$

with $r_k = \sigma + (1+\lambda)(d+k)$.

Proof of Proposition 6.2. To ease readability we set $\partial_x = \partial_{x_i}$ and we forget about the subscript i. We come down from the modified to the straight operator by using the change of variable $w := a(v)$. We get

$$\begin{aligned} K_U(H)(t,x) &= \int_0^t \int_{a(\mathbb{R}^d)} (\partial_x H)(s, x-(t-s)w)\, U(t,s,x,a^{-1}(w)) |\det Da(a^{-1}(w))|^{-1}\, dw\, ds \\ &= \mathsf{K}_{\mathsf{U}}(H)(t,x), \end{aligned}$$

with

$$\mathsf{U}(t,s,x,w) := U(t,s,x,a^{-1}(w)) |\det Da(a^{-1}(w))|^{-1} 1_{a(\mathbb{R}^d)}.$$

Let $k > 1 + d$ and $\sigma > \frac{1}{2}d$. By Theorem 6.1, we get

$$\|K_U(H)\|_{L^2([0,T];L^2_x)} = \|\mathsf{K}_{\mathsf{U}}(H)\|_{L^2([0,T];L^2_x)} \lesssim \sup_{0\le s,t\le T} \|\mathsf{U}(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \|H\|_{L^2([0,T];L^2_x)}.$$

[4] A close version of this result is also stated in [Han-Kwan et al. 2017] for the special case $a(v) = \hat{v}$.

By the assumption on a, we have

$$|\partial_w^\alpha a^{-1}(w)| \lesssim (1+|a^{-1}(w)|)^{1+\lambda|\alpha|}.$$

In particular, we deduce

$$|\det Da(a^{-1}(w))|^{-1} \lesssim (1+|a^{-1}(w)|)^{d(1+\lambda)}.$$

As a consequence, we have, by the Faà di Bruno formula, and using the reverse change of variable $v = a^{-1}(w)$ and (2-3), that

$$\|\mathsf{U}(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \lesssim \|U(t,s,\cdot)\|_{\mathcal{H}^k_{\sigma+(d+k)(1+\lambda)}},$$

and hence the claimed estimate. □

6B. ***Intermission: a comparison to averaging lemmas.*** We end this section with a comparison of the smoothing estimate we have just shown, in the simple case where $a(v) = v$, which corresponds to Theorem 6.1, with kinetic averaging lemmas. Averaging lemmas were introduced in [Golse et al. 1985; 1988; Agoshkov 1984] and now generically stand for various smoothing effects in average for kinetic transport-type equations.[5] They proved over the years to be fundamental in several contexts of kinetic theory, as they provide compactness and regularity. There exist many versions of these, involving several different assumptions on the functional spaces, on the number of derivatives in v or in x in the source etc.; see, e.g., [DiPerna et al. 1991; Perthame and Souganidis 1998; Golse and Saint-Raymond 2002; Bouchut 2002; Jabin and Vega 2004; Jabin 2009; Arsénio and Saint-Raymond 2011; Arsénio and Masmoudi 2014] for more recent advances. The closest (to Theorem 6.1) analogue of averaging lemmas is arguably the following result.

Theorem 6.3 [Perthame and Souganidis 1998]. *Let $1 < p < \infty$. Let $f, g = (g_j)_{j=1,\dots,d} \in L^p_{t,x,v}$ satisfy the transport equation*

$$\partial_t f + v\cdot\nabla_x f = \sum_{j=1}^d \partial_{x_j}\partial_v^k g_j, \tag{6-4}$$

where k is an arbitrary multi-index. Let $\varphi(v)$ be a C^∞ compactly supported function and set

$$\rho_\varphi(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\varphi(v)\,dv.$$

Then we have, for all $\alpha \in \left[0, \min\left(\frac{1}{p}, \frac{1}{p'}\right)\right)$,

$$\|\rho_\varphi\|_{L^p_{t,x}} \le C_{d,p,\alpha,\varphi}\|f\|_{L^p_{t,x,v}}^{1-\frac{\alpha}{|k|+1}}\|g\|_{L^p_{t,x,v}}^{\frac{\alpha}{|k|+1}}. \tag{6-5}$$

Let us focus especially on the case $p = 2$, $|k| = 0$, in which case (6-5) actually also holds for $\alpha = \frac{1}{2}$. Theorem 6.1 can also be understood as a kind of averaging lemma for the moments in v of the kinetic

[5] Actually this can be embedded in a more general framework; see in particular [Gérard 1990; Gérard and Golse 1992; Gérard et al. 1996].

equation (6-4), in the special case where the source has the form

$$\sum_{j=1}^{d} \partial_{x_j} H_j(t,x)\, \partial_v^k \mathcal{U}_j(t,x,v), \tag{6-6}$$

where $\mathcal{U}_j$ is smooth in x and v, and the initial condition is $f|_{t=0} = 0$. Let $\varphi(t,x,v)$ be a smooth and decaying test function. Then by the method of characteristics,

$$f(t,x,v) = \int_0^t \sum_{j=1}^{d} \partial_{x_j} H_j(s, x-(t-s)v)\, \partial_v^k \mathcal{U}_j(s, x-(t-s)v, v)\, ds,$$

and thus

$$\rho_\varphi(t,x) = \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^{d} \partial_{x_j} H_j(s, x-(t-s)v)\, \partial_v^k \mathcal{U}_j(s, x-(t-s)v, v)\, \varphi(t,x,v)\, ds = \sum_{j=1}^{d} \mathsf{K}^{(j)}_{U_j}(H_j)(t,x),$$

setting $U_j(s,t,x,v) = \partial_v^k \mathcal{U}_j(s, x-(t-s)v, v)\, \varphi(t,x,v)$. The regularity assumption of Theorem 6.1 can be written as

$$\sup_{0\le s,t\le T} \|U_j(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} < \infty$$

for $k > 1+d$, $\sigma > \frac{1}{2}d$, and the consequence is

$$\|\rho_\psi\|_{L^2(0,t;L^2_x)} \lesssim \sup_{0\le s,t\le T} \sum_j \|U_j(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \|H_j\|_{L^2(0,t;L^2_x)}. \tag{6-7}$$

This estimate is not a consequence of Theorem 6.3. Indeed, note that it does not involve the L^2 norm of the solution f: somehow, this can be roughly seen as a version of Theorem 6.3 allowing $\alpha = 1$, whereas Theorem 6.3 only allows $\alpha \le \frac{1}{2}$, at the expense of asking for the structure assumption (6-6) on the source g and of considering a norm for the source that is more demanding than the L^2 norm of estimate (6-5).

Observe also that Theorem 6.1 does not require the test function in v to be decaying at infinity, as long as for all j, we have $\mathcal{U}_j$ in (6-6) is itself decaying sufficiently fast at infinity.

7. Proofs of Theorem 2.1 and Corollaries 2.2 and 2.5

We finally set up an induction argument, which relies on the machinery developed in the previous sections, and will ultimately lead to the proof of Theorem 2.1. We summarize the procedure below:

- By induction, we assume smoothness on the moments until order $n'-1$. We can first apply Lemma 4.1 to obtain the same smoothness for the coefficients of the operators $L_{i,j}$.
- We apply Lemma 4.7 or 4.8 in order to get the system of equations satisfied by $(L^{K,L}\partial_x^\alpha \partial_v^\beta f)$, which is of the abstract form

$$\mathcal{T}(\mathfrak{F}) + \mathfrak{A}\mathfrak{F} = \mathcal{B},$$

where $\mathfrak{A}$ is a matrix whose coefficients we control and $\mathcal{B}$ is the rest we need to control. Loosely speaking, $\mathcal{B}$ consists either of remainders we can control thanks to the induction assumption, and terms of the form $-\partial_x^{\alpha(K,L)} F \cdot \nabla_v f$ for $K, L \in \{1,\dots,d\}^{m+k}$, whose contribution is the main matter.

• We then invert the operator $\mathcal{T}+\mathfrak{A}$ in order to solve the equation. At this stage, after integration in velocity (remember that we are interested in the regularity of moments), we use the changes of variables introduced in Lemmas 5.1, 5.2 and 5.3.

• What is rather straightforward then is the study of the contribution of the initial data and of the remainder terms in $\mathcal{B}$. As already said, the contribution of the terms $-\partial_x^{\alpha(K,L)} F\cdot\nabla_v f$ is more serious and involves the study of integrals of the form

$$\int_0^t \int_{\mathbb{R}^d} (\partial_{x_i}\partial_x^{\alpha(K,L)} F)(s, x-(t-s)a(v))\, U(t,s,x,v)\, dv\, ds,$$

which seem to feature a loss of derivative in x. We recognize the integral operators introduced and studied in Section 6. This is where the smoothing estimate of Proposition 6.2 proves to be crucial.

7A. ***End of the proof of Theorem 2.1.*** For $\boldsymbol{n}\geq 2m-1$, let $\mathcal{P}(\boldsymbol{n})$ be the following statement:

There is $T>0$ such that for all test functions

$$\psi(t,x,v)\in L^\infty(0,T;\mathcal{W}^{d+2+n-2m,\infty}_{-r_0}),$$

setting for all $|\alpha|=n$,

$$m_{\psi,\alpha}(t,x)=\int_{\mathbb{R}^d}\partial_x^\alpha f(t,x,v)\,\psi(t,x,v)\,dv,$$

there exists Λ for which

$$\sum_{|\alpha|=n}\|m_{\psi,\alpha}\|_{L^2(0,T;L^2_x)}\lesssim\Lambda(T,M). \tag{7-1}$$

By Proposition 3.2, it is clear that $\mathcal{P}(2m-1)$ is verified.

Let $\boldsymbol{n}\in\{2m,\dots,2(m+p)\}$. Let us assume that $\boldsymbol{n}$ is even, of the form $2(m+k)$. We shall not proceed with the case where $\boldsymbol{n}$ is odd, as it follows by completely similar arguments. Assume that $\mathcal{P}(2m),\dots,\mathcal{P}(\boldsymbol{n}-1)$ are satisfied and let $T>0$ be a time on which the estimates (7-1) (for $2m,\dots,\boldsymbol{n}-1$) are satisfied. We shall prove that $\mathcal{P}(\boldsymbol{n})$ is also verified. Once this is done, we deduce by induction that $\mathcal{P}(2m),\dots,\mathcal{P}(2(m+p))$ are satisfied; we then deduce the required estimates (3-14).

Thanks to the property $\mathcal{P}(\boldsymbol{n}-1)$ applied to the $(\psi_j)_{j=1,\dots,r}$, and (2-9), we first have

$$\sum_{j=1}^{\ell}\|F^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\leq\Lambda(T,M). \tag{7-2}$$

We can therefore apply Lemma 4.1 and obtain a possible smaller time, still denoted by T, and operators $L_{i,j}$ with coefficients $(\varphi^{i,j}_{k,l},\psi^{i,j}_{k,l})_{i,j,k,l\in\{1,\dots,d\}}$ belonging to $L^\infty(0,T;\mathcal{H}^{2(m+k)-2}_{-\tilde r})$ for all $\tilde r>\frac12 d$, with uniform regularity

$$\|(\varphi^{i,j}_{k,l},\psi^{i,j}_{k,l})_{i,j,k,l}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k)-2}_{-\tilde r})}\leq\Lambda(T,M).$$

Let us consider the vector (the precise ordering does not matter)

$$\mathfrak{F}=(L^{K,L}\partial_x^\alpha\partial_v^\beta f)_{r\in\{m-k,\dots,m+k\},\,K,L\in\{1,\dots,d\}^r,\,|\alpha|+|\beta|=m+k-r}. \tag{7-3}$$

By Lemma 4.7, it follows that $\mathfrak{F}$ satisfies the system

$$\mathcal{T}(\mathfrak{F}) + \mathfrak{A}\mathfrak{F} = \mathcal{B} + \mathfrak{R}, \tag{7-4}$$

where $\mathfrak{A}(t,x,v)$ is a matrix with coefficients in $L^2(0,T;W^{d+2,\infty}_{x,v})$, satisfying

$$\|\mathfrak{A}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M). \tag{7-5}$$

(The term $\mathfrak{A}\mathfrak{F}$ encodes the contribution of the leading-order terms in the triple sum of the right-hand side of (4-22).) On the other hand, $\mathfrak{R}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}\|_{L^2(0,T;\mathcal{H}^0_{\tilde{r}})} \lesssim \Lambda(T,M) \tag{7-6}$$

for all $\tilde{r} < r - \frac{1}{2}d$ and $\mathcal{B}$ is defined as follows: all its components are equal to 0 except those corresponding to the components associated to some $K, L \in \{1,\dots,d\}^{m+k}$, in which case it is equal to

$$-\partial_x^{\alpha(K,L)} F \cdot \nabla_v f.$$

The next step consists in using the change of variables $v \mapsto \Phi(t,x,v)$, where Φ solves (5-1), in order to straighten the vector field $\mathcal{T}$; see Lemma 5.1. To this end, we use Lemma 5.2 (reduce again $T>0$ if necessary) and use the notation $\circ\,\Phi$ to denote the composition in v with Φ. Setting $\mathcal{F} = \mathfrak{F}\circ\Phi$, we obtain

$$(\partial_t + a(\Phi)\cdot\nabla_x)\mathcal{F} + (\mathfrak{A}\circ\Phi)\mathcal{F} = \mathcal{B}\circ\Phi + \mathfrak{R}\circ\Phi. \tag{7-7}$$

Let $\mathcal{A}(s,t,x,v)$ be the operator, whose existence is ensured by the Cauchy–Lipschitz theorem, as the solution of the following *linear* ODE

$$\partial_s\mathcal{A}(s,t,x,v) = \mathfrak{A}(s,x,\Phi(s,x,v))\,A(s,t,x,v), \quad A(t,t,x,v) = \mathrm{Id}.$$

Thanks to (7-5), we also have

$$\|\mathcal{A}(\,\cdot\,,t,\cdot\,)\|_{L^\infty(0,T;W^{d+2,\infty}_{x,v})} + \|\partial_s\mathcal{A}(\,\cdot\,,t,\cdot\,)\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M). \tag{7-8}$$

By the method of characteristics we get

$$\begin{aligned}\mathcal{F}(t,x,v) &= \mathcal{A}(t,0,x,v)\,\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)\\ &\quad+\int_0^t \mathcal{A}(t,s,x,v)\,\mathcal{B}\circ\Phi(s,\mathrm{X}(s,t,x,v),v)\,ds + \int_0^t \mathcal{A}(t,s,x,v)\,\mathfrak{R}\circ\Phi(s,\mathrm{X}(s,t,x,v),v)\,ds.\end{aligned} \tag{7-9}$$

Suppose $\psi(t,x,v) \in L^\infty(0,T;\mathcal{W}^{d+2+2k,\infty}_{-r_0})$. Then we multiply the representation formula (7-9) by $\psi(t,x,\Phi(t,x,v))|\det D_v\Phi(t,x,v)|$ and integrate in v to obtain

$$\int_{\mathbb{R}^d} \mathcal{F}(t,x,v)\,\psi(t,x,\Phi(t,x,v))|\det D_v\Phi(t,x,v)|\,dv = I_0 + I_1 + I_2, \tag{7-10}$$

with

$$
\begin{aligned}
I_0 &= \int_{\mathbb{R}^d} \mathcal{A}(t,0,x,v)\,\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)\,\psi\circ\Phi|\det D_v\Phi(t,x,v)|\,dv,\\
I_1 &= \int_0^t\int_{\mathbb{R}^d} \mathcal{A}(t,s,x,v)\,(\mathfrak{R}\circ\Phi)(s,\mathrm{X}(s,t,x,v),v)\,\psi\circ\Phi|\det D_v\Phi(t,x,v)|\,dv\,ds, \qquad (7\text{-}11)\\
I_2 &= \int_0^t\int_{\mathbb{R}^d} \mathcal{A}(t,s,x,v)\,(\mathcal{B}\circ\Phi)(s,\mathrm{X}(s,t,x,v),v)\,\psi\circ\Phi|\det D_v\Phi(t,x,v)|\,dv\,ds.
\end{aligned}
$$

By the change of variables $v\mapsto\Phi(t,x,v)$, we have

$$
\int_{\mathbb{R}^d}\mathcal{F}(t,x,v)\,\psi(t,x,\Phi(t,x,v))|\det D_v\Phi(t,x,v)|\,dv = \int_{\mathbb{R}^d}\mathfrak{F}(t,x,v)\,\psi(t,x,v)\,dv.
$$

Let us first study this term. Since $\mathcal{P}(2m),\dots,\mathcal{P}(2(m+k)-1)$ are satisfied, we can apply Lemma 4.4 (the assumption (4-8) is indeed verified), which yields, see (4-9) and (4-10), that for all $I,J\in\{1,\dots,d\}^{m+k}$,

$$
\int_{\mathbb{R}^d} L^{I,J} f\,\psi(t,x,v)\,dv = \int_{\mathbb{R}^d}\partial_x^{\alpha(I,J)} f\,\psi(t,x,v)\,dv + \mathfrak{R}_{I,J,\psi},
$$

where $\mathfrak{R}_{I,J,\psi}$ is a remainder satisfying the estimate

$$
\|\mathfrak{R}_{I,J,\psi}\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M).
$$

Consequently, recalling the definition of $\mathfrak{F}$ in (7-3), if we are able to obtain the bound

$$
\|I_0\|_{L^2(0,T;L^2_x)} + \|I_1\|_{L^2(0,T;L^2_x)} + \|I_2\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M),
$$

then we deduce the bound

$$
\sum_{I,J}\left\|\int_{\mathbb{R}^d}\partial_x^{\alpha(I,J)} f\psi\,dv\right\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M);
$$

that is, we obtain the sought bound (7-1) at rank n.

7A1. *Study of* I_0. Let us begin by treating the contribution of the initial data, which corresponds to the term I_0. First by using estimate (5-3) in Lemma 5.2, the L^∞ bound for $\mathcal{A}$ in (7-8), and the estimate

$$
\|(1+|v|^2)^{-\frac12 r_0}\psi\|_{L^\infty_{x,v}} \lesssim 1, \qquad (7\text{-}12)
$$

we have for all $x\in\mathbb{T}^d$,

$$
\left|\int_{\mathbb{R}^d}\mathcal{A}(t,0,x,v)\,\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)(1+|v|^2)^{\frac12 r_0}|\det D_v\Phi(t,x,v)|\,dv\right| \le \Lambda(T,M)\int |\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)|(1+|v|^2)^{\frac12 r_0}\,dv.
$$

Therefore, we get that

$$
\|I_0\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M)\left\|\int_{\mathbb{R}^d}\|\mathcal{F}(0,\mathrm{X}(0,t,\cdot,v),v)\|_{L^2_x}(1+|v|^2)^{\frac12 r_0}\,dv\right\|_{L^2(0,T)}.
$$

By using the change of variable $y = \mathrm{X}(0,t,x,v) + ta(v) = x - t\widetilde{\mathrm{X}}(0,t,x,v)$ and Lemma 5.3, we obtain

$$\|\mathcal{F}(0,\mathrm{X}(0,t,\cdot,v),v)\|_{L^2_x} \le \Lambda(T,M)\|\mathcal{F}(0,\cdot - ta(v),v)\|_{L^2_x} \le \Lambda(T,M)\|\mathcal{F}(0,\cdot,v)\|_{L^2_x}$$

and hence, we deduce that since $r > r_0 + \frac{1}{2}d$, for some $r' > \frac{1}{2}d$, it holds that

$$\|I_0\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M)\left(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^{r'}}\right)^{\frac{1}{2}} \|\mathcal{F}(0)\|_{\mathcal{H}^0_r}.$$

By using the fact that at $t=0$ we have $\Phi(0,x,v) = v$ and $L^{(K,L)}|_{t=0} = \partial_x^{\alpha(K,L)}$, we end up with

$$\|\mathcal{F}(0)\|_{\mathcal{H}^0_r} = \|\mathfrak{F}(0)\|_{\mathcal{H}^0_r} \le \Lambda(M) \sum_{j=m-k}^{m+k} \sum_{|\alpha|+|\beta|=m+k-j} \|\partial_x^{2j}\partial_x^\alpha\partial_v^\beta f_0\|_{\mathcal{H}^0_r},$$

and hence we finally obtain

$$\|I_0\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M).$$

7A2. *Study of* I_1. We treat the other remainder term I_1 in a similar fashion. Indeed, using again estimate (5-3) in Lemma 5.2, (7-8) and (7-12), we first get

$$\|I_1\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M)\left\| \int_0^t \int_{\mathbb{R}^d} \|\mathfrak{R}(s,\mathrm{X}(s,t,\cdot,v),\Phi(s,\mathrm{X}(s,t,\cdot,v),v))\|_{L^2_x}(1+|v|^2)^{\frac{1}{2}r_0}\,dv\,ds\right\|_{L^2(0,T)}.$$

Thanks to the change of variable $x \mapsto \mathrm{X}(s,t,x,v)$ and to the estimates of Lemma 5.3, it follows that

$$\begin{aligned}\|I_1\|_{L^2(0,T;L^2_x)} &\le \Lambda(T,M)\left\|\int_0^t\int_{\mathbb{R}^d} \|\mathfrak{R}(s,\cdot,\Phi(s,\cdot,v))\|_{L^2_x}(1+|v|^2)^{\frac{1}{2}r_0}\,dv\,ds\right\|_{L^2(0,T)} \\ &\le \Lambda(T,M)\left\|\int_0^t \|(\mathfrak{R}\circ\Phi)(s)\|_{\mathcal{H}^0_{\tilde r}}\,ds\right\|_{L^2(0,T)} \\ &\le \Lambda(T,M)\,T\,\|\mathfrak{R}\circ\Phi\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})},\end{aligned}$$

by choosing $\tilde r > r_0 + \frac{1}{2}d$, which is possible since $r > r_0 + d$. Using again the change of variables $v \mapsto \Phi(t,x,v)$, Lemma 5.2 and the estimate (7-6), we thus obtain

$$\|I_1\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M).$$

7A3. *Study of* I_2. The main matter thus concerns the contribution of the term I_2, which features an apparent loss of derivative in x. This is however not the case, thanks to Proposition 6.2. Let $K, L \in \{1,\dots,d\}^{m+k}$. Writing $\partial_x^{\alpha(K,L)} = \partial_x\partial_x^{\alpha'}$ with $|\alpha'| = |\alpha(K,L)| - 1$, we are led to study terms of the form (here F_i^j stands for the i-th coordinate of F^j)

$$\sum_{j=1}^{\ell}\int_0^t\int_{\mathbb{R}^d}(\partial_x\partial_x^{\alpha'}F_i^j)(s,\mathrm{X}(s,t,x,v))\,\psi(t,\mathrm{X}(s,t,x,v),\Phi(s,\mathrm{X}(s,t,x,v),v))$$
$$\times\mathcal{A}_{K,L}^{I,J}\big(t,s,,\mathrm{X}(s,t,x,v),\Phi(s,\mathrm{X}(s,t,x,v),v)\big)\,A_j(\Phi(s,\mathrm{X}(s,t,x,v),v))$$
$$\times\partial_{v_i}f\big(s,\mathrm{X}(s,t,x,v),\Phi(s,\mathrm{X}(s,t,x,v),v)\big)|\det D_v\Phi(t,x,v)|\,dv\,ds,$$

where $\|\mathcal{A}_{K,L}^{I,J}\|_{L^\infty(0,T;W^{d+2,\infty}_{x,v})}\le\Lambda(T,M)$.

We use the change of variables $v=\Psi(s,t,x,w)$ to rewrite this expression as $\sum_{j=1}^{\ell}K_{U_j}(\partial_x^{\alpha'}F_i^j)$, with

$$\begin{aligned}U_j(s,t,x,v)=\;&A_j\big(\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\&\times\mathcal{A}_{K,L}^{I,J}\big(t,s,x-(t-s)\,a(v),\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\&\times\psi\big(t,x-(t-s)\,a(v),\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\&\times\partial_{v_i}f\big(s,x-(t-s)\,a(v),\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\&\times|\det D_v\Phi(t,x,\Psi(s,t,x,v))|\,|\det D_v\Psi(s,t,x,v)|,\end{aligned}\tag{7-13}$$

where we recall the operators K were introduced in Section 6. In order to apply Proposition 6.2, we have to estimate, s,t being fixed, U_j in $\mathcal{H}^{2+d}_{r'}$, with $r'>\frac{1}{2}d+2(1+\lambda)(1+d)$ and $r\ge r'+r_0$, which is possible since $r>R$ as defined in (2-12). First, by (2-3), (2-6), (7-8), (5-3) in Lemma 5.2 and estimate (5-8) in Lemma 5.3, we can uniformly bound in L^∞ all terms involving A_j, Φ, Ψ and their derivatives (since only at most $2+d$ derivatives can be involved). For ψ, we use

$$\|(1+|v|^2)^{-\frac{1}{2}r_0}\partial^\alpha\psi\|_{L^\infty_{x,v}}\lesssim 1\quad\text{for all }|\alpha|\le d+2.$$

We are therefore led to estimate integrals of the form

$$I=\left|\int_{\mathbb{T}^d\times\mathbb{R}^d}|g(x-(t-s)\,a(v),\Phi(s,x-(t-s)v,\Psi(s,t,x,v)))|^2(1+|v|^2)^{r_0+r'}\,dv\,dx\right|,$$

where $g=\partial^\alpha f$, $|\alpha|\le d+3$. To this end, we can use the change of variables $v\mapsto w=\Psi(s,t,x,v)$ and rely on estimate (5-7) in Lemma 5.3 to obtain the bound

$$I\le\Lambda(T,M)\int_{\mathbb{T}^d\times\mathbb{R}^d}|g(\mathrm{X}(s,t,x,w),\Phi(s,\mathrm{X}(s,t,x,w),w))|^2(1+|w|^2)^{r_0+r'}\,dx\,dw.$$

Next, arguing as for I_1, we can use successively the change of variable $x\mapsto y=\mathrm{X}(s,t,x,w)$ with the estimates of Lemma 5.3, and the change of variable $w\mapsto u=\Phi(s,y,w)$ with estimate (5-3) in Lemma 5.2, to finally obtain

$$I\le\Lambda(T,M)\|g\|^2_{\mathcal{H}^0_r}\le\Lambda(T,M)\|f\|^2_{\mathcal{H}^{2m-1}_r},$$

since $2m-1\ge d+3$ and $r>R$. As a result we obtain the bound

$$\sup_{s,t}\|U_j\|_{\mathcal{H}^{2+d}_{r'}}\le\Lambda(T,M)\|f\|^2_{\mathcal{H}^{2m-1}_r}\le\Lambda(T,M).\tag{7-14}$$

We can therefore apply Proposition 6.2 to get the bound

$$\begin{aligned}\|K_{U_j,i}(F_i^j)\|_{L^2(0,T;L_x^2)} &\lesssim \sup_{s,t}\|U_j\|_{\mathcal{H}^{2+d}_{r'}}\|F_i^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\\ &\le \Lambda(T,M)\|F_i^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\\ &\le \Lambda(T,M), \end{aligned}\tag{7-15}$$

thanks to estimate (7-2). We deduce

$$\|I_2\|_{L^2(0,T;L_x^2)} \le \Lambda(T,M)$$

and gathering all pieces together, we therefore obtain (7-1) at rank $\boldsymbol{n}$, and the induction argument is complete. Theorem 2.1 follows.

7B. ***Proof of Corollary 2.2.*** In order to prove the higher-order regularity for the characteristics, we proceed as in [Han-Kwan and Rousset 2016, Lemma 5.1].

By Theorem 2.1 and the assumption (2-9), we have for all $j = 1, \dots, \ell$,

$$F^j \in L^2(0,T;H_x^{n'})$$

and thus by Sobolev embedding, we deduce that for $k < n' - \frac{1}{2}d$,

$$F^j \in L^2(0,T;W_x^{k,\infty}). \tag{7-16}$$

We set

$$Z := (Y,W) := (X - tv - x, V - v).$$

Let us first prove that $Z \in L^\infty(0,T;W^{k,\infty}_{x,v})$ for $k < n' - \frac{1}{2}d$. Note that by the definition of (X,V), we know Z satisfies the equation

$$Z = \left(\int_0^t (Y+v)\,ds,\ \int_0^t \sum_{j=1}^{\ell} A_j(W+v)\,F^j(Y+x+tv)\,ds\right).$$

By (2-6) and (7-16), we obtain by induction (on the number of applied derivatives) that for $t \le T$,

$$\sup_{|\alpha|<n'-\frac{1}{2}d}\ \sup_{[0,t]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_{x,v}} \lesssim \int_0^t \lambda(s)\big(1+\sup_{|\alpha|<n'-\frac{1}{2}d}\ \sup_{[0,s]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_{x,v}}\big)\,ds,$$

where λ is a nonnegative function belonging to $L^2(0,T)$, with norm bounded by $\Lambda(T,M)$. We deduce our claim thanks to the Gronwall inequality, which yields

$$\sup_{|\alpha|<n'-\frac{1}{2}d}\ \sup_{[0,t]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_{x,v}} \le \sqrt{t}\,\Lambda(T,M). \tag{7-17}$$

We deduce in particular from this estimate that for $T' \in (0,T]$ small enough, for all $v \in \mathbb{R}^d$, the map $x \mapsto X(T',0,x,v)$ is a C^1 diffeomorphism.

Next, let us turn to the $L_t^\infty L_v^\infty L_x^2$ estimate. We set

$$\mathcal{N}(t) := \sup_{|\alpha|\le n'}\ \sup_{[0,t]}\|\partial^\alpha_{x,v}Z\|_{L_v^\infty L_x^2}.$$

By an application of the Faà di Bruno formula, we obtain

$$\mathcal{N}(t) \lesssim \sum_{j=1}^{\ell} \int_0^t \sum_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}}\, ds,$$

with

$$J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} := \big\| |(D_v^{k_1} A_j)\circ V(s) (D_x^{k_2} F^j)\circ X(s)| \, |\partial^{\beta_1}_{x,v}(X,V)| \cdots |\partial^{\beta_{k_1+k_2}}_{x,v}(X,V)| \big\|_{L^\infty_v L^2_x},$$

and where the sum is taken only on indices such that $k_1+k_2 =: k \le |\alpha| \le n'$, $\beta_1+\dots+\beta_k = |\alpha|$ with for every j, $|\beta_j| \ge 1$ and $|\beta_1| \le |\beta_2| \le \dots \le |\beta_k|$.

Let us observe that in the sum, if $k_1+k_2 = k \ge 2$, we necessarily have $|\beta_{k-1}| < n' - \frac{1}{2}d$. Indeed, otherwise, we would have $|\beta_1|+\dots+|\beta_k| \ge 2n'-d$ and thus $n' \ge 2n'-d$, which means $n' \le d$. This is impossible by assumption on n'. Next:

- If $k_2 < n' - \frac{1}{2}d$ and $k_1+k_2 = k \ge 2$, we obtain thanks to the above observation and (7-17) that for $i = 1, \dots, k-1$,

$$\|\partial^{\beta_i}_{x,v}(X,V)\|_{L^\infty_{x,v}} \lesssim 1 + T + \|\partial^{\beta_i}_{x,v}(Z)\|_{L^\infty_{x,v}} \lesssim \Lambda(T,M). \tag{7-18}$$

Moreover, using (2-6), (7-16) we get

$$\begin{aligned} J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} &\le \|D^{k_1} A_j\|_{L^\infty_{x,v}} \|D^{k_2} F^j\|_{L^\infty_{x,v}} \Big\| \prod_{i=1}^{k-1} \partial^{\beta_i}_{x,v}(X,V) \Big\|_{L^\infty_{x,v}} \|\partial^{\beta_k}_{x,v}(X,V)\|_{L^\infty_v L^2_x} \\ &\le \Lambda(T,M) \|D^{k_2} F^j\|_{L^\infty_{x,v}} (1+\mathcal{N}(s)). \end{aligned}$$

If $k = 1$, the above estimate is clearly also valid.

- If $k_2 \ge n' - \frac{1}{2}d$, we observe that for every i, we have $|\beta_i| \le |\beta_k| \le n' - (k-1) < \frac{1}{2}d$. In particular $|\beta_i| < n' - \frac{1}{2}d$ by assumption on n' and we have that (7-18) holds for all $i = 1, \dots, k$. This yields

$$\begin{aligned} J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} &\lesssim \|(D_v^{k_1} A_j)\circ V\|_{L^\infty_v L^2_x} \|(D_x^{k_2} F^j)\circ X\|_{L^\infty_v L^2_x} \Lambda(T,M) \\ &\lesssim \|D^{k_1}_{x,v} A_j\|_{L^\infty_v} \|(D_x^{k_2} F^j)\circ X\|_{L^\infty_v L^2_x} \Lambda(T,M) \\ &\lesssim \Lambda(T,M) \|(D_x^{k_2} F^j)\|_{L^2_x}. \end{aligned}$$

To get the last estimate, we restrict to $T' \le T$ small enough so that we can use the change of variable $y = X(t,0,x,v)$ when computing the L^2_x norm of $(D^{k_2}_{x,v} F^j)\circ X$.

By combining the above estimates, we obtain that for $t \le T'$,

$$\mathcal{N}(t) \le \sqrt{t}\,\Lambda(T,M) + \int_0^t \Lambda(T,M) \sup_j \|F^j(s)\|_{H^{n'}_x} \mathcal{N}(s)\, ds.$$

By using again (7-16) and the Gronwall inequality, we thus obtain that for $t \le T'$,

$$\mathcal{N}(t) \lesssim \sqrt{t}\,\Lambda(T,M),$$

which concludes the proof of Corollary 2.2.

7C. *Proof of Corollary 2.5.* The idea, as in [Gérard 1990, Proposition 5.2], consists in applying Theorem 2.1 with the test function

$$\psi_\eta(v) = e^{-v\cdot\eta} \in W^{n',\infty}_{x,v},$$

where $\eta \in \mathbb{R}^d$ has to be seen as the Fourier variable in velocity. A close inspection of the proofs reveals that the conclusion of Theorem 2.1 can be refined into

$$\text{for all } \eta \in \mathbb{R}^d, \quad \left\| \int f \psi_\eta \, dv \right\|_{L^2(0,T_0;H^{n'}_x)} \le \Lambda(T_0, M, \|\psi_\eta\|_{W^{n',\infty}_v}), \tag{7-19}$$

where Λ is a polynomial function. Moreover, $\|\psi_\eta\|_{W^{n',\infty}_v} \lesssim \Lambda'(|\eta|)$, where Λ' is also a polynomial function (of degree n'). Since

$$\frac{1}{(2\pi)^{\frac{1}{2}d}} \int f \psi_\eta \, dv = \mathcal{F}_v f(t, x, \eta),$$

we deduce from (7-19) that for some $p > 0$ taken large enough,

$$\| \hat{f}(t, k, \eta)(1 + |k|^2)^{\frac{1}{2}n'} (1 + |\eta|^2)^{-\frac{1}{2}p} \|_{L^2(0,T_0;L^2(\mathbb{Z}^d \times \mathbb{R}^d))} < \infty,$$

which means that $f \in L^2(0, T_0; H^{n',-p}_{x,v})$.

8. Application to classical models from physics

The goal of this section is to briefly explain why both Vlasov–Poisson and relativistic Vlasov–Maxwell systems enter the abstract framework, and thus why Theorem 2.1 (and its corollaries) apply to these classical models.

8A. *Vlasov–Poisson.* The Cauchy problem for the Vlasov–Poisson system (1-2) was studied (among many other references)

- for (global) weak solutions in [Arsen'ev 1975],
- for local strong solutions in [Ukai and Okabe 1978], and for global strong solutions in [Bardos and Degond 1985; Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991; Glassey 1996; Batt and Rein 1991; Horst 1993].

Let us check the following structural assumptions for (1-2).

• *Assumptions on the advection field.* In this model, $a(v) = v$, so that all required assumptions on a are straightforward properties. One can take $\lambda = 0$ in (2-4).

• *Assumptions on the force field.* For the force field F, we can write $\ell = 1$, $A_1 = 1$ and $F^1 = -\nabla_x \phi$, where ϕ is computed thanks to the moment of order 0 of f only; that is, $\psi_1 = 1$ (thus $r_0 = 0$) and

$$m_{\psi_1} = \int_{\mathbb{R}^d} f \, dv.$$

The assumption (2-9) follows straightforwardly from the Poisson equation, as for all $n \in \mathbb{N}$, it holds that

$$\text{for all } t \geq 0, \quad \|F^1(t)\|_{H^n_x} \lesssim \|m_{\psi_1}(t)\|_{H^{n-1}_x}.$$

We however do not need the smoothing effect due to the Poisson equation. It follows directly that both estimates (2-9) and (2-10) hold. The stability estimate (2-11) holds because of the same estimate, by linearity of the Poisson equation. It turns out that using the smoothing estimate, we can obtain a stronger version of Theorem 2.1: we embed this situation in what we refer to as transport/elliptic systems, and refer to Theorem 9.1 in Section 9.

Note also that the Vlasov–Poisson system with dynamics constrained on geodesics introduced in the context of stellar dynamics in [Diacu et al. 2016] enters the abstract framework as well (and in this model there is no smoothing of the force field).

8B. ***Relativistic Vlasov–Maxwell.*** The Cauchy problem for the relativistic Vlasov–Maxwell system (1-3) was studied (among many other references)

- for (global) weak solutions in [DiPerna and Lions 1989],
- for (local) strong solutions in [Wollman 1984; 1987; Degond 1986; Asano 1986; Glassey and Strauss 1986; 1987; Glassey 1996; Schaeffer 2004; Bouchut et al. 2003; Klainerman and Staffilani 2002; Pallard 2015; Luk and Strain 2016].

Let us check the following structural assumptions for (1-3).

• *Assumptions on the advection field.* In this model, $a(v) = \hat{v}$, and one can check by a straightforward induction that

$$\|\partial_v^\alpha \hat{v}\|_{L^\infty_v} \leq C_\alpha \quad \text{for all } \alpha.$$

We have $a(\mathbb{R}^d) = B(0, c)$ and the explicit formula

$$\text{for all } w \in B(0, c), \quad a^{-1}(w) = \frac{w}{\sqrt{1 - |w|^2/c^2}}.$$

It follows that one can take $\lambda = 2$ in (2-4).

• *Assumptions on the force field.* For the force field F, we observe that we can take $\ell = 4$ and write

$$A_1 = 1, \quad F^1 = E, \tag{8-1}$$

and setting $B = (B_1, B_2, B_3)$ in an orthonormal basis (e_1, e_2, e_3),

$$\begin{aligned} A_2 &= \hat{v}_1, \quad F^2 = B_2 e_3 - B_3 e_1, \\ A_3 &= \hat{v}_2, \quad F^3 = B_3 e_1 - B_1 e_3, \\ A_4 &= \hat{v}_3, \quad F^4 = B_1 e_2 - B_2 e_1. \end{aligned} \tag{8-2}$$

The electromagnetic field (E, B) is computed only from initial data (E_0, B_0) and the moments of order 0 and 1, which correspond to $\psi_1 = 1, \psi_2 = \hat{v}$ (so that $r_0 = 0$) and

$$m_{\psi_1} = \int_{\mathbb{R}^d} f \, dv, \quad m_{\psi_2} = \int_{\mathbb{R}^d} f \hat{v} \, dv.$$

The assumption (2-9) follows from classical energy estimates for Maxwell equations: we have for all $n \in \mathbb{N}$ and all $t \geq 0$,

$$\|(E, B)\|_{L^2(0,t;H^n_x)} \leq C_n t^{\frac{3}{2}} \sum_{i=1}^{2} \|m_{\psi_i}\|_{L^2(0,t;H^n_x)} + \|(E, B)(0)\|_{L^2(0,t;H^n_x)};$$

see, e.g., [Han-Kwan et al. 2017, Lemma 3.2]. The estimate (2-10) is proved similarly. The stability estimate (2-11) holds because of the same energy estimate, by linearity of the Maxwell equations.

8C. ***Remarks.*** Some remarks about possible generalizations of the abstract framework are in order:

- It is possible to add a smooth force, of C^k regularity with k large enough, and still adapt the results of Theorem 2.1, without significantly modifying the analysis. This allows one for instance to consider Vlasov–Poisson systems with a smooth external magnetic field.

- The so-called relativistic gravitational Vlasov–Poisson system (which may be relevant for galactic dynamics) enters the abstract framework as well, by a combination of the estimates of Section 8A and 8B (see, e.g., [Glassey and Schaeffer 1985; Hadžić and Rein 2007; Kiessling and Tahvildar-Zadeh 2008; Lemou et al. 2008] for some references about this system).

- The divergence-free (in v) condition for F is not an absolute requirement for the analysis. It may be dropped, but would sometimes necessitate introducing more complicated formulas. In particular, it is likely that fluid/kinetic systems for sprays such as Vlasov–Stokes or Vlasov–Navier–Stokes in dimension $d = 2$ enter this framework (or a slightly modified version of it) as well. We refer, e.g., to [Jabin 2000; Boudin et al. 2009; 2015; Desvillettes 2010] for some references about these equations. See also [Baranger and Desvillettes 2006; Moussa and Sueur 2013] for other fluid/kinetic systems.

- Note that the so-called nonrelativistic Vlasov–Maxwell system (that is system (1-3) with v replacing all occurrences of $\hat{v}$) does not enter the abstract framework. Indeed the assumption (2-6) is not satisfied. However, we claim that (2-6) is crucial only for having a good local well-posedness theory in $\mathcal{H}^n_r$ spaces. This means that without (2-6), we can still obtain a result similar to that of Theorem 2.1, except that we have to *assume* the existence of a solution of (2-1) with the required regularity. For the nonrelativistic Vlasov–Maxwell system, such solutions do exist, following [Asano 1986], which requires the introduction of Sobolev spaces with loss of integrability in velocity.

9. The case of transport/elliptic-type Vlasov equations

9A. ***An improvement of Theorem 2.1.*** Let us assume in this section that the following strengthened version of (2-9) is satisfied:

$$\|F^j\|_{L^2(0,t;H^n_x)} \leq \Gamma^{(j)}_n\left(t, \|m_{\psi_1}\|_{L^2(0,t;H^{n-1}_x)}, \dots, \|m_{\psi_r}\|_{L^2(0,t;H^{n-1}_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\right). \tag{9-1}$$

In other words, the force is smoothed out and gains one derivative compared to the distribution function. We refer to such a situation as the transport/elliptic-type case. This includes in particular the Vlasov–Poisson system. We then have the following version of Theorem 2.1. This is an improved version in the sense that the higher regularity we ask for is only regularity in x and not at all in v (compare (9-2) below to (2-13) in Theorem 2.1).

Theorem 9.1. *Let $n \geq N$ and $r > R$. Let $n' > n$ be an integer such that $n > \lfloor \frac{1}{2}n' \rfloor + d + 1$. Assume that $f_0 \in \mathcal{H}^n_r$ and $F^j(0) \in H^{n'}_x$ for all $j \in \{1, \dots, \ell\}$. Assume furthermore that the initial data f_0 satisfies the following higher space regularity*:

$$\partial^\alpha_x f_0 \in \mathcal{H}^0_r \quad \text{for all } |\alpha| = n'. \tag{9-2}$$

Then there is $T > 0$ such that the following holds. There exists a unique solution $(f(t), F(t))$ with initial data $(f_0, F(0))$ to (2-1) *such that $f(t) \in C(0, T; \mathcal{H}^n_r)$.*

Moreover, for all test functions $\psi \in L^\infty(0, T; \mathcal{W}^{n',\infty}_{-r_0})$, we have

$$\int f \psi \, dv \in L^2(0, T; H^{n'}_x). \tag{9-3}$$

As in Corollary 2.5, we may deduce as well under the assumptions of Theorem 9.1 that

$$f \in L^2(0, T; H^{n',-\infty}_{x,v}). \tag{9-4}$$

Proof of Theorem 9.1. The beginning of the proof is the same as for Theorem 2.1 (of which we keep the notations). Let us set in this context

$$M := \|f_0\|_{\mathcal{H}^{2m-1}_r} + \sum_{k=0}^{2p} \sum_{|\alpha|=2m+k} \|\partial^\alpha_x f_0\|_{\mathcal{H}^0_r} + \sum_{j=1}^{\ell} \|F^j(0)\|_{H^{2(m+p)}_x}. \tag{9-5}$$

We proceed with the same induction argument, treating all terms similarly except for[6] the treatment of the term I_0, for which the following is an improvement of Section 7. The idea will be to use integration by parts in v to trade derivatives in v against derivatives in x, allowing us to obtain estimates depending on (9-5) (compared to (3-13) for Theorem 2.1).

First note using the smoothing estimate (9-1) that we improve (7-2) to

$$\sum_{j=1}^{\ell} \|F^j\|_{L^2(0,T; H^{2(m+k)}_x)} \leq \Lambda(T, M). \tag{9-6}$$

We can use this improved estimate with Remark 4.10 to deduce that the coefficients of $\mathfrak{A}$, as appearing in (7-4), satisfy the improved form of (7-5)

$$\|\mathfrak{A}\|_{L^2(0,T; W^{p,\infty}_{x,v})} \lesssim \Lambda(T, M) \quad \text{for all } p < 2m - \tfrac{1}{2}d - 1. \tag{9-7}$$

Therefore we deduce the improved form of (7-8):

$$\|\mathcal{A}(\cdot, t, \cdot)\|_{L^\infty(0,T; W^{p,\infty}_{x,v})} \lesssim \Lambda(T, M) \quad \text{for all } p < 2m - \tfrac{1}{2}d - 1. \tag{9-8}$$

[6] We also remark that in order to treat the term I_2, we do not absolutely need to use Proposition 6.2; we can indeed rely on the smoothing estimate (9-6) on the force instead and argue as we did for I_1. This observation will be useful later in order to treat other Vlasov models.

The treatment of I_0 then leads to the study of terms of the general form

$$J = \int_{\mathbb{R}^d} (\partial_x^\alpha \partial_v^\beta \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\, m(t,x,v)\, dv,$$

where, for $j = m-k, \dots, m+k$, we have $|\alpha| + |\beta| = m+k+j$, $|\alpha| \geq 2j$, and

$$\|m\|_{L^\infty(0,T;\mathcal{H}^N_{-r'-r_0})} \leq \Lambda(T,M)$$

for all $N < 2m - \frac{1}{2}d - 1$ and all $r' > \frac{1}{2}d$. If $|\beta| = 0$, there is nothing special to do, as only derivatives in x are involved, so let us assume that $|\beta| \geq 1$. We write $\partial_v^\beta = \partial_v^{\beta'} \partial_v$. We have

$$J = \int_{\mathbb{R}^d} \partial_v [(\partial_x^\alpha \partial_v^{\beta'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, m(t,x,v)\, dv \\ - \int_{\mathbb{R}^d} (\partial_x^\alpha \partial_v^{\beta'})(\partial_v \mathrm{X}(0,t,x,v) \cdot \nabla_x \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\, m(t,x,v)\, dv,$$

and thus by integration by parts in v, we get

$$J = -\int_{\mathbb{R}^d} [(\partial_x^\alpha \partial_v^{\beta'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, \partial_v m(t,x,v)\, dv \\ - \int_{\mathbb{R}^d} (\partial_x^\alpha \partial_v^{\beta'})(\partial_v \mathrm{X}(0,t,x,v) \cdot \nabla_x \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\, m(t,x,v)\, dv.$$

We therefore observe that this procedure allows us to trade derivatives in v for derivatives in x.

Assume now that one can write, for some $l \in \{1, \dots, |\beta|\}$,

$$J = \sum_{|\beta'| \leq l} \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \partial_v^{\beta'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, m_{\alpha',\beta'}(t,x,v)\, dv + R_l,$$

where

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^{N_l}_{-r'-r_0})} \leq \Lambda(T,M) \tag{9-9}$$

for all $N_l < 2m - \frac{1}{2}d - 1 - |\beta| + l$ and all $r' > \frac{1}{2}d$, and R_l is a remainder satisfying

$$\|R_l\|_{L^2(0,T;L^2_x)} \leq \Lambda(T,M).$$

Let us show that this property holds as well for at rank $l-1$. Following the same integration-by-parts argument as above, we may write

$$J - R_l = J_1 + J_2 + J_3,$$

where

$$J_1 = \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, m_{\alpha',0}(t,x,v)\, dv,$$

$$J_2 = -\sum_{|\beta'| \leq l} \sum_{\beta' = (\beta'', j)} \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \partial_v^{\beta''} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, \partial_{v_j} m_{\alpha',\beta'}(t,x,v)\, dv,$$

$$J_3 = -\sum_{|\beta'| \leq l} \sum_{\beta' = (\beta'', j)} \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} (\partial_x^{\alpha'} \partial_v^{\beta''})(\partial_{v_j} \mathrm{X}(0,t,x,v) \cdot \nabla_x \mathcal{F})(0, \mathrm{X}(0,t,x,v), v) \\ \times m_{\alpha',\beta'}(t,x,v)\, dv.$$

The terms J_1 and J_2 have good forms already. For J_3, by using the Leibniz rule, we observe that we need to study terms of the form

$$\bar{J} = \int_{\mathbb{R}^d} \partial^{\gamma}_{x,v} \mathrm{X}(0,t,x,v)\, \partial^{\eta_1}_x \partial^{\eta_2}_v \mathcal{F}(0, \mathrm{X}(0,t,x,v), v)\, m_{\alpha',\beta'}(t,x,v)\, dv,$$

with $|\eta_2| \le |\beta''| = l-1,\ 1 \le |\eta_1| \le |\alpha'|+1$, and $|\gamma| = |\alpha'| + |\beta''| - |\eta_1| - |\eta_2| + 2$.

Assume first that $|\eta_1| + |\eta_2| \le 2m-1$. If $|\eta_1| + |\eta_2| < 2m-1-d$, then by the Sobolev embedding we have the bound

$$\|(1+|v|^2)^{\frac{1}{2}r} (\partial^{\eta_1}_x \partial^{\eta_2}_v \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\|_{L^\infty_{x,v}} \le \|f_0\|_{\mathcal{H}^{2m-1}_r} \le \Lambda(M).$$

Since $0 < |\gamma| \le 2(m+k)$, we use (9-6) and Lemma 5.2 to get

$$\|\partial^{\gamma} \mathrm{X}\|_{L^\infty(0,T;L^\infty_v L^2_x)} \le \Lambda(T,M).$$

(This is where the elliptic estimate (9-1) is crucially used.) Furthermore, since $2m - \frac{1}{2}d - 1 - 2p > d$, we have the bound

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{W}^{0,\infty}_{-r'-r_0})} \le \Lambda(T,M)$$

for $r' > \frac{1}{2}d$ such that $r > r' + r_0 + d$. Therefore such terms satisfy the bound

$$\|\bar{J}\|_{L^2(0,T;L^2_{x,v})} \le \Lambda(T,M),$$

and thus can be put into the remainder R_{l-1}. If $|\eta_1| + |\eta_2| \ge 2m-1-d$, then $|\gamma| \le 2k+d+1$. Since $2m-d-1 > \frac{1}{2}d$, we can use $\|\partial^{\gamma}\mathrm{X}\|_{L^\infty(0,T;L^\infty_{x,v})} \le \Lambda(T,M)$ and again, arguing as in the treatment of I_0 in the proof of Theorem 2.1, such terms are remainders.

Otherwise $|\eta_1| + |\eta_2| \ge 2m$. Then we have $|\gamma| \le 2k$ and thus $2(m+k) - |\gamma| \ge 2m$. We set in this case $m_{\eta_1,\eta_2} := \partial^{\gamma}_{x,v} \mathrm{X}\, m_{\alpha',\beta'}$. In order to show that $\partial^{\gamma}_{x,v}\mathrm{X}\, m_{\alpha',\beta'}$ has the required regularity, we are led to study terms of the form

$$\widetilde{J} = \|\partial^{a}_{x,v} \partial^{\gamma}_{x,v} \mathrm{X}\, \partial^{b}_{x,v} m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^0_{-r'-r_0})}, \quad |a| + |b| = N_{l-1},$$

for all $N_l < 2m - \frac{1}{2}d - 1 - |\beta| + l - 1$ and all $r' > \frac{1}{2}d$. Assume first that $|a| < 2m - \frac{1}{2}d$; then we have $|a| + |\gamma| < 2(m+k) - \frac{1}{2}d$ and we use estimate (5-7) in Lemma 5.2 to get $\|\partial^{a}_{x,v}\partial^{\gamma}_{x,v}\mathrm{X}\|_{L^\infty(0,T;L^\infty_{x,v})} \le \Lambda(T,M)$, and apply (9-9) to obtain the bound

$$\|\partial^{b}_{x,v} m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^0_{-r'-r_0})} \le \Lambda(T,M).$$

Otherwise, $|a| \ge 2m - \frac{1}{2}d$. Since we have $2(m+k) - |\gamma| \ge N_{l-1}$ for all $N_{l-1} < 2m - 2 - |\beta| + l$, we can use estimate (5-7) in Lemma 5.2 to get

$$\sum_{|a| \le N_{l-1}} \|\partial^{a}_{x,v} \partial^{\gamma}_{x,v} \mathrm{X}\|_{L^\infty(0,T;L^\infty_v L^2_x)} \le \Lambda(T,M).$$

Since $|b| = N_{l-1} - |a| \leq N_{l-1} - 2m + \frac{1}{2}d$, we have $N_l - |b| \geq 2m + 1 - \frac{1}{2}d > d$. As a result, by (9-9) and the Sobolev embedding we get

$$\|\partial_{x,v}^b m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{W}^{0,\infty}_{-r'-r_0})} \leq \Lambda(T, M).$$

In all cases, we have obtained

$$\widetilde{J} \leq \Lambda(T, M).$$

Therefore the corresponding terms of J_3 can be written in the form

$$\int_{\mathbb{R}^d} (\partial_x^{\eta_1} \partial_v^{\eta_2} \mathcal{F})(0, \mathrm{X}(0, t, x, v), v)\, m_{\eta_1,\eta_2}(t, x, v)\, dv,$$

with

$$\|m_{\eta_1,\eta_2}\|_{L^\infty(0,T;\mathcal{H}^{N_{l-1}}_{-r'-r_0})} \leq \Lambda(T, M)$$

for all $N_{l-1} < 2m - 1 - |\beta| + (l-1)$ and $r' > \frac{1}{2}d$.

We conclude by induction that we can write at rank $l = 0$

$$J = \sum_{|\alpha'| \leq m+k+j} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \mathcal{F})(0, \mathrm{X}(0, t, x, v), v)]\, m_{\alpha',0}(t, x, v)\, dv + R,$$

with

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^{N}_{-r'-r_0})} \leq \Lambda(T, M)$$

for all $N < 2m - 1 - |\beta|$ and $r' > \frac{1}{2}d$, and $\|R\|_{L^2(0,T;L^2_x)} \leq \Lambda(T, M)$ is a remainder.

We then note that $2m - 2 - 2k > d$, so that

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{W}^{0,\infty}_{-r'-r_0})} \leq \Lambda(T, M).$$

Arguing as in the previous treatment of I_0 in the proof of Theorem 2.1, we finally conclude that

$$\|I_0\|_{L^2(0,T;L^2_x)} \leq \Lambda(T, M) \sum_{j=m-k}^{m+k} \sum_{|\alpha|=m+k-j} \|\partial_x^\alpha f_0\|_{\mathcal{H}^0_r}. \tag{9-10}$$

This allows us to conclude the proof. □

As already noted in the proof of Theorem 9.1, we actually do not need to use Proposition 6.2 to treat the term I_2 in view of Theorem 9.1; we can indeed rely on the smoothing estimate (9-1) on the force instead. Furthermore, one can obtain L^∞_t estimates instead of the L^2_t theory that we have developed. This observation implies the following fact: replacing (9-1) by the slightly weaker estimate (in the sense that it is implied by (9-1))

$$\|F^j\|_{L^2(0,t;H^n_x)} \leq \Gamma_n^{(j)}\Bigg(t, \|m_{\psi_1}\|_{L^\infty(0,t;H^{n-1}_x)}, \ldots, \|m_{\psi_r}\|_{L^\infty(0,t;H^{n-1}_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\Bigg), \tag{9-11}$$

together with an associated stability estimate replacing (2-11) with L^∞_t norms instead of L^2_t for the moments on the right-hand side, Theorem 9.1 still holds. It suffices to estimate all terms (that is to say, the moments, $I_0, I_1, I_2, \ldots$) in $L^\infty(0,T;L^2_x)$ instead of $L^2(0,T;L^2_x)$ as previously done. This remark is useful in particular to treat the so-called Vlasov–Darwin model from plasma physics, which we introduce in the following section.

9B. ***Vlasov–Darwin.*** The Vlasov–Darwin system is another model that allows one to describe the dynamics of charged particles in a plasma, which lies between Vlasov–Poisson and relativistic Vlasov Maxwell systems. Like Vlasov–Poisson, it can be derived from the Vlasov–Maxwell system in the nonrelativistic regime, that is to say, in the limit $c \to \infty$. The difference is that the Vlasov–Darwin system happens to be a higher-order approximation than the Vlasov–Poisson, see [Bauer and Kunze 2005]; in particular it retains self-induced magnetic effects that have disappeared completely in the Vlasov–Poisson dynamics. It is given by

$$\begin{cases} \partial_t f + \hat{v}\cdot\nabla_x f + \Big(E + \frac{1}{c}\hat{v}\times B\Big)\cdot\nabla_v f = 0, \\ E = -\nabla_x\phi - \frac{1}{c}\partial_t A, \quad B = \nabla_x \times A, \\ -\nabla_x\phi = \int_{\mathbb{R}^3} f\,dv - \int_{\mathbb{T}^3\times\mathbb{R}^3} f\,dv\,dx, \\ -\Delta_x A = \frac{1}{c}\mathbb{P}\int_{\mathbb{R}^3}\hat{v}f\,dv, \quad \nabla_x\cdot A = 0, \end{cases} \tag{9-12}$$

where $c > 0$ is the speed of light and $\mathbb{P}$ denotes the Leray projection. The Cauchy problem for the Vlasov–Darwin system (1-3) was studied (among many other references)

- for (global) weak solutions in [Pallard 2006],
- for strong solutions in [Pallard 2006; Seehafer 2008; Sospedra-Alfonso et al. 2012].

To embed this system into the abstract framework, we need to make the additional assumption that all initial conditions f_0 that are considered are a.e. nonnegative. By a standard property of the Vlasov equation, any associated solution $f(t)$ is also a.e. nonnegative.

• *Assumptions on the advection field.* In this model $a(v) = \hat{v}$, which is already treated for the relativistic Vlasov–Maxwell case.

• *Assumptions on the force field.* We have the decomposition (8-1)–(8-2) as well. Let us set

$$E = E_L + E_T, \quad E_L = \nabla_x\phi, \quad E_T = -\frac{1}{c}\partial_t A$$

and introduce

$$\psi_1 = 1, \quad \psi_2 = \hat{v}, \quad \psi_3 = \frac{\hat{v}\otimes\hat{v}}{\sqrt{1+|v|^2/c^2}}, \quad \psi_3 = \mathrm{Id} - m_{\psi_3}$$

(so that $r_0 = 0$) and

$$m_{\psi_i} = \int_{\mathbb{R}^d} \psi_i f\,dv,$$

where m_{ψ_3} and m_{ψ_4} are symmetric matrices. Since E_L and E_T derive from potentials solving a Poisson equation, we have

$$\text{for all } t \geq 0, \quad \|(E_L, B)(t)\|_{H_x^n} \lesssim \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{H_x^{n-1}},$$

and thus

$$\|(E_L, B)\|_{L^\infty(0,t;H_x^n)} \lesssim \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{L^\infty(0,t;H_x^{n-1})}.$$

For E_T, this is a little more subtle; this is where we need that $f(t) \geq 0$ a.e. As in [Pallard 2006, Lemma 2.10], we obtain that E_T satisfies the inhomogeneous elliptic equation

$$-\Delta E_T + \frac{1}{c} m_{\psi_4} E_T = -\frac{1}{c}(m_{\psi_4} E_L - m_{\psi_2} \times B - \nabla_x : m_{\psi_3}). \tag{9-13}$$

We fix the time $t \geq 0$, which is a parameter here (we take the L_t^∞ norm in the end). Let $n > d$. By [Pallard 2006, Lemma 2.10], which relies on the fact that m_{ψ_4} is actually a *semidefinite* symmetric matrix, it follows that (9-13) has a unique solution E_T in H_x^1, with the bound

$$\begin{aligned}
\|E_T\|_{H_x^1} &\lesssim \|m_{\psi_4} E_L\|_{H_x^{-1}} + \|m_{\psi_2} \times B\|_{H_x^{-1}} + \|\nabla : m_{\psi_3}\|_{H_x^{-1}} \\
&\lesssim \|m_{\psi_4} E_L\|_{L_x^2} + \|m_{\psi_2} \times B\|_{L_x^2} + \|m_{\psi_3}\|_{L_x^2} \\
&\lesssim \|m_{\psi_4}\|_{H_x^n} \|E_L\|_{H_x^n} + \|m_{\psi_2}\|_{H_x^n} \|B\|_{H_x^n} + \|m_{\psi_3}\|_{H_x^n} \\
&\lesssim \left(1 + \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{H_x^n}\right)(\|m_{\psi_4}\|_{H_x^n} + \|m_{\psi_2}\|_{H_x^n} + \|m_{\psi_3}\|_{H_x^n}).
\end{aligned}$$

Then assume by induction that we have a bound of the form

$$\text{for all } k = 1, \dots, N, \quad \|E_T\|_{H_x^k} \lesssim \Gamma_k(\|m_{\psi_1}\|_{H_x^n}, \dots, \|m_{\psi_4}\|_{H_x^n}) \tag{9-14}$$

for $N \leq n$, where Γ_k is a polynomial function. Assume first that $N < n - \frac{1}{2}d$. Let $|\alpha| = N$. We note that $\partial_x^\alpha E_T$ satisfies

$$-\Delta \partial_x^\alpha E_T + \frac{1}{c} m_{\psi_4} \partial_x^\alpha E_T = -\frac{1}{c} \partial_x^\alpha (m_{\psi_4} E_L - m_{\psi_2} \times B - \nabla_x : m_{\psi_3}) - [\partial_x^\alpha, m_{\psi_4}] E_T.$$

We have by standard tame Sobolev estimates

$$\begin{aligned}
&\left\| \partial_x^\alpha (m_{\psi_4} E_L - m_{\psi_2} \times B - \nabla_x : m_{\psi_3}) \right\|_{H_x^{-1}} \\
&\qquad \lesssim \left(1 + \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{H_x^n}\right)(\|m_{\psi_4}\|_{H_x^n} + \|m_{\psi_2}\|_{H_x^n} + \|m_{\psi_3}\|_{H_x^n}).
\end{aligned} \tag{9-15}$$

Since $N < n - \frac{1}{2}d$, we can use the Sobolev embedding to obtain

$$\begin{aligned}
\|[\partial_x^\alpha, m_{\psi_4}] E_T\|_{H_x^{-1}} &\lesssim \|m_{\psi_4}\|_{W_x^{N,\infty}} \|E_T\|_{H_x^N} \\
&\lesssim \|m_{\psi_4}\|_{H_x^n} \Gamma_k(\|m_{\psi_1}\|_{H_x^n}, \dots, \|m_{\psi_4}\|_{H_x^n}).
\end{aligned}$$

We apply again the H^1_x estimate of [Pallard 2006, Lemma 2.10] to obtain a bound of the form

$$\|E_T\|_{H_x^{N+1}} \lesssim \Gamma_{N+1}(\|m_{\psi_1}\|_{H_x^n}, \dots, \|m_{\psi_4}\|_{H_x^n}).$$

We deduce by induction that for all $N < n - \frac{1}{2}d$,

$$\|E_T\|_{H_x^{N+1}} \lesssim \Gamma_{N+1}(\|m_{\psi_1}\|_{H_x^n}, \dots, \|m_{\psi_4}\|_{H_x^n}).$$

In particular, since $n > d$, we deduce

$$\|E_T\|_{L_x^\infty} \lesssim \Gamma(\|m_{\psi_1}\|_{H_x^n}, \dots, \|m_{\psi_4}\|_{H_x^n}). \tag{9-16}$$

Now assume we have (9-14) for some $N \le n$. We have the tame Sobolev estimate

$$\begin{aligned}\|[\partial_x^\alpha, m_{\psi_4}]E_T\|_{H_x^{-1}} &\lesssim \|m_{\psi_4}\|_{H^n}(\|E_T\|_{H_x^N} + \|E_T\|_{L_x^\infty})\\ &\lesssim \|m_{\psi_4}\|_{H_x^n}\Gamma_N(\|m_{\psi_1}\|_{H_x^n}, \dots, \|m_{\psi_4}\|_{H_x^n}),\end{aligned}$$

by (9-14) at rank N and (9-16). Thus using the H^1_x estimate of [Pallard 2006, Lemma 2.10], we obtain (9-14) at rank $N+1$. By induction, we conclude that

$$\|E_T\|_{L^\infty(0,T;H_x^{n+1})} \lesssim \Gamma_{n+1}(\|m_{\psi_1}\|_{L^\infty(0,T;H_x^n)}, \dots, \|m_{\psi_4}\|_{L^\infty(0,T;H_x^n)}),$$

which is an estimate of the requested form (9-11). A stability estimate of the same form also holds because of similar considerations.

10. On the regularity assumptions of Theorem 2.1

The goal of this short last section is to discuss the type of regularity assumptions which could be conceivable for proving propagation of higher regularity.

Example 1. Consider the free transport equation

$$\partial_t f + v\,\partial_x f = 0, \tag{10-1}$$

set in $\mathbb{R} \times \mathbb{R}$ to simplify the discussion. Let $\varphi(v)$ be a C^∞ function, with compact support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and such that $\int_{\mathbb{R}} \varphi\, dv = 0$. Let g be the piecewise continuous function defined by $g(x) = 1$ for $x \in [-1, 1]$ and 0 elsewhere. Observe that in the sense of distributions, we have $g'(x) = \delta_{x=-1} - \delta_{x=-1}$, where δ stands for the Dirac measure. We consider the initial condition

$$f|_{t=0} = g(x)\,\varphi(v) \in L^2_{x,v},$$

and the solution to (10-1) can be written as

$$f(t,x,v) = g(x - tv)\,\varphi(v).$$

It follows by explicit computations that $\rho(t,x) := \int_{\mathbb{R}} f\, dv$ satisfies

$$\begin{aligned}\partial_x \rho(t,x) &= \varphi\left(\frac{x+1}{t}\right) - \varphi\left(\frac{x-1}{t}\right),\\ \partial_x^k \rho(t,x) &= \frac{1}{t^{k-1}}\left(\varphi^{(k-1)}\left(\frac{x+1}{t}\right) - \varphi^{(k-1)}\left(\frac{x-1}{t}\right)\right) \quad \text{for all } k \in \mathbb{N}^*.\end{aligned}$$

We have for $t < 4$,

$$\|\partial_x^k \rho(t)\|_{L_x^2}^2 = \frac{1}{t^{2(k-1)}}\left(\left\|\varphi^{(k-1)}\left(\frac{x+1}{t}\right)\right\|_{L_x^2}^2 + \left\|\varphi^{(k-1)}\left(\frac{x-1}{t}\right)\right\|_{L_x^2}^2\right),$$

since φ is compactly supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and thus

$$\|\partial_x^k \rho(t)\|_{L_x^2}^2 = \frac{2}{t^{2(k-1)-1}}\|\varphi^{(k-1)}\|_{L_x^2}^2.$$

We deduce that for any $T > 0$, we have $\rho \notin L^2(0, T; H_x^2)$. However, $\rho(0, x) = 0 \in H_x^k$ for all $k \in \mathbb{N}$.

This example shows that regularity of moments at initial time may not be propagated, and more precise information such as (2-13) is somehow required to obtain higher regularity for moments.

Example 2. Consider the equation

$$\partial_t f + v\,\partial_x f + F(t, x)\,\partial_v f = 0 \tag{10-2}$$

on $\mathbb{T} \times \mathbb{R}$, with

$$F(t, x) = \int_{\mathbb{R}} \psi(v) f(t, x, v)\,dv,$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ with compact support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. It is clear that (10-2) enters the abstract framework of this work.

We consider the initial condition

$$f|_{t=0} = f_0^{(1)} + f_0^{(2)},$$

where $f_0^{(2)}$ is a smooth nonnegative function, with support in $\mathbb{T}\times\left[-\frac{1}{2}, \frac{1}{2}\right]$ and $f_0^{(1)}$ is a smooth nonnegative function, with support in $\mathbb{T} \times [1, 2]$.

Consider $f^{(1)}$ the solution of (10-2) associated to the initial condition $f_0^{(1)}$, and assume that it is defined on an interval $[0, T]$ for $T > 0$ small enough. Now define $f^{(2)}$ as the solution on $[0, T]$ of the *linear* kinetic transport equation

$$\partial_t f + v\,\partial_x f + \left(\int_{\mathbb{R}} \psi(v) f^{(1)}\,dv\right)\partial_v f = 0,$$

with initial condition $f_0^{(2)}$.

Because of the form of the force F, notably because ψ is localised in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we observe that up to reducing $T > 0$, the solution f on $[0, T]$ of (10-2) can be written as

$$f = f^{(1)} + f^{(2)},$$

since $T > 0$ can be chosen small enough so that the support in velocity of $f^{(2)}(t)$ is disjoint from that of ψ, and thus

$$\int_{\mathbb{R}} \psi(v)\, f^{(2)}(t)\,dv = 0.$$

Now let $k \in \mathbb{N}$ and assume that there is $(x_0, v_0) \in \mathbb{T} \times (1, 2)$ such that $f|_{t=0}(x_0, v_0)$ is not zero and is locally H^k around this point. Because of the assumptions on the supports, this is equivalent to asking that $f_0^{(2)}(x_0, v_0)$ is not zero and is locally H^k around this point. However, we can choose (independently of $f_0^{(2)}$) $f_0^{(1)}$ so that $\int_{\mathbb{R}} \psi(v) f^{(1)}\, dv$ is not H^k, in such a way that $f^{(2)}(t)$ (and thus $f(t)$) is not locally H^k around points of the form $(X(0,t,x_0,v_0), V(0,t,x_0,v_0))$, where (X,V) denotes the characteristics associated to F, as defined in (1-7).

This example shows that local regularity may not be propagated (along characteristics), contrary to what happens for the class of PDEs considered in [Bony 1981]. This is due to the "nonlocality" in velocity. Therefore a global regularity assumption is required in order to obtain propagation of higher regularity.

This example can (also) be slightly modified, in order to prove that a local version of (2-13) cannot either be propagated into higher local regularity of moments; see the next (and last) example.

Example 3. Consider the equation

$$\partial_t f + v\,\partial_x f + F\big(t, x + \tfrac{1}{4}\big)\,\partial_v f = 0 \tag{10-3}$$

on $\mathbb{T} \times \mathbb{R}$ (here we identify $\mathbb{T}$ with $[0,1)$ with periodic boundary conditions). Let us consider as in the previous example

$$F(t,x) = \int_{\mathbb{R}} \psi(v)\, f(t,x,v)\, dv.$$

We consider the initial condition

$$f|_{t=0} = f_0^{(1)} + f_0^{(2)},$$

where $f_0^{(1)}$ is a nonnegative function, with compact support in $\big[0, \tfrac{1}{8}\big] \times \mathbb{R}$, and $f_0^{(2)}$ is a nonnegative function, with compact support in $\big[\tfrac{1}{4}, \tfrac{3}{8}\big] \times \mathbb{R}$.

Observe that because of the shift in the argument of the force, by looking at the supports in x, the solution $f^{(2)}$ associated to the initial condition $f_0^{(2)}$ is equal to $f_0^{(2)}(t, x - tv, v)$ on $[0,T]$ for $T > 0$ small enough. Moreover, we have

$$\left(\int_{\mathbb{R}} \psi(v) f^{(2)}\big(t, x + \tfrac{1}{4}, v\big)\, dv\right) \partial_v f^{(2)} = 0.$$

Now define $f^{(1)}$ as the solution on $[0,T]$ of the *linear* kinetic transport equation

$$\partial_t f + v\,\partial_x f + \left(\int_{\mathbb{R}} \psi(v)\, f_0^{(2)}\big(x + \tfrac{1}{4} - tv, v\big)\, dv\right) \partial_v f = 0,$$

with initial condition $f_0^{(2)}$.

We observe that up to reducing $T > 0$, the solution f on $[0,T]$ of (10-2) can be written as

$$f = f^{(1)} + f^{(2)}.$$

Indeed, by looking at the supports in x, we can impose $T > 0$ small enough so that

$$\left(\int_{\mathbb{R}} \psi(v)\, f^{(1)}\big(t, x + \tfrac{1}{4}, v\big)\, dv\right) \partial_v f^{(2)} = 0,$$
$$\left(\int_{\mathbb{R}} \psi(v)\, f^{(1)}\big(t, x + \tfrac{1}{4}, v\big)\, dv\right) \partial_v f^{(1)} = 0.$$

Now let $k \in \mathbb{N}$ and assume that there is $x_0 \in \big(0, \tfrac{1}{8}\big)$ such that $\int_{\mathbb{R}} f|_{t=0}(x_0, v)\, dv \neq 0$ and $f|_{t=0}$ is locally H_x^k around this point. This is equivalent to asking that $\int_{\mathbb{R}} f_0^{(1)}(x_0, v)\, dv \neq 0$ and $f_0^{(1)}$ is locally H_x^k

around this point. This corresponds to a local analogue of (2-13). However, we can choose (independently of $f_0^{(1)}$) $f_0^{(2)}$ so that $\int_{\mathbb{R}} \psi(v) f_0^{(2)}(x - tv, v)\, dv$ is not locally H^k, in such a way that the moments in velocity of $f^{(1)}(t)$ (and thus of $f(t)$) are not locally H_x^k around points of the form $X(0, t, x_0, v_0)$, for some $v_0 \in \mathbb{R}$.

Acknowledgements

The author would like to thank Frédéric Rousset, as this work wouldn't have been possible without [Han-Kwan and Rousset 2016], and Jérémie Szeftel for a stimulating discussion about the second-order operators introduced in that work and used in this one as well.

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Received 12 Oct 2017. Revised 19 Mar 2018. Accepted 19 Apr 2018.

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ANALYSIS AND PDE
Vol. 12, No. 1, 2019
dx.doi.org/10.2140/apde.2019.12.245

ON A BOUNDARY VALUE PROBLEM FOR CONICALLY DEFORMED THIN ELASTIC SHEETS

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We consider a thin elastic sheet in the shape of a disk that is clamped at its boundary such that the displacement and the deformation gradient coincide with a conical deformation with no stretching there. These are the boundary conditions of a so-called "d-cone". We define the free elastic energy as a variation of the von Kármán energy, which penalizes bending energy in L^p with $p \in \left(2, \frac{8}{3}\right)$ (instead of, as usual, $p = 2$). We prove ansatz-free upper and lower bounds for the elastic energy that scale like $h^{p/(p-1)}$, where h is the thickness of the sheet.

1. Introduction

Strong deformations of thin elastic sheets under the influence of some external force have been a topic of considerable interest in the physics and engineering community over the last decades. These "postbuckling" phenomena are relevant on many length scales, e.g., for structural failure, for the design of protective structures, or in atomic-force microscopy of virus capsids and bacteria. In the physics literature, one finds numerous contributions that discuss the focusing of elastic energy in ridges and conical vertices; see [Cerda et al. 1999; Venkataramani 2004; Lobkovsky and Witten 1997]. The overview article [Witten 2007] contains a comprehensive review of the activities in that area of physics. However, quoting the seminal work [Lobkovsky et al. 1995], the "understanding of the strongly buckled state remains primitive", and this fact has not changed fundamentally since the publication of that article more than 20 years ago.

In the mathematical literature on thin elastic sheets, there have been two major topics: On the one hand, there are the derivations of lower-dimensional models starting from three-dimensional finite elasticity [Ciarlet 1997; Le Dret and Raoult 1995; Friesecke et al. 2002; 2006]. On the other hand, there has been quite some effort to investigate the qualitative properties of plate models by determining the scaling behavior of the free elastic energy with respect to the small parameters in the model (such as the thickness of the sheet). Such scaling laws have been derived, e.g., in [Bella and Kohn 2014; Ben Belgacem et al. 2002; Bourne et al. 2017; Kohn and Nguyen 2013]. Building on the results from [Venkataramani 2004], it has been proved in [Conti and Maggi 2008] that the free energy per unit thickness of the so-called "single fold" scales like $h^{5/3}$, where h is the thickness of the sheet. This is also the conjectured scaling behavior for the confinement problem, which consists in determining the minimum of elastic energy necessary to fit a thin elastic sheet into a container whose size is smaller than the diameter of the sheet. The energy focusing in conical vertices has been investigated in [Brandman et al. 2013; Müller and Olbermann 2014],

MSC2010: 49Q10, 74K20.
Keywords: thin elastic sheets, d-cone.

where the following has been proved: Consider a thin elastic sheet in the shape of a disc, and fix it at the boundary and at the center such that it agrees with a (nonflat) conical configuration there. Then the elastic energy scales like $h^2\log(1/h)$. On a technical level, [Conti and Maggi 2008; Brandman et al. 2013; Müller and Olbermann 2014] consider an energy functional of the form

$$I_h(y)=\int_\Omega |Dy^T Dy-\mathrm{Id}_{2\times 2}|^2+h^2|D^2y|^2\,\mathrm{d}x, \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is the undeformed sheet, $y:\Omega\to\mathbb{R}^3$ is the deformation, and $\mathrm{Id}_{2\times 2}$ is the 2-by-2 identity matrix. The first term is the (nonconvex) membrane energy, and the second is the bending energy. If one manages to derive scaling laws for this two-dimensional model, then as a consequence, it is often the case that analogous results for three-dimensional elasticity are not difficult to derive as a corollary by the results from [Friesecke et al. 2002]; see for example [Conti and Maggi 2008; Brandman et al. 2013]. Of course, the character of the variational problem heavily depends on the chosen boundary conditions.

While the mentioned articles have contributed a lot to the mathematical understanding of folds and vertices in thin sheets, they do not consider situations where the constraints prevent the sheet from adopting an isometric immersion with respect to the reference metric as its configuration, but do not prevent it from adopting a *short* map as its configuration. (We recall that a map $y:\Omega\to\mathbb{R}^3$ is short if every path $\gamma\subset\Omega$ is mapped to a shorter path $y(\gamma)\subset\mathbb{R}^3$.) Such a situation is characteristic of postbuckling, and in particular, the confinement problem.

The reason why short maps are problematic can be found in the famous Nash–Kuiper theorem [Nash 1954; Kuiper 1955a; 1955b]: if one is given a short map $y_0\in C^1(\Omega;\mathbb{R}^3)$ and $\varepsilon>0$, then there exists an isometric immersion $y\in C^1(\Omega;\mathbb{R}^3)$ with $\|y-y_0\|_{C^0}<\varepsilon$. This is relevant in the present context, since the difference between the induced metric and the flat reference metric is the leading-order term in the energy (1). Thus, if short maps are permissible, then there exists a vast amount of configurations with vanishing or very small membrane energy. One needs a principle that is capable of showing that all these maps are associated with a large amount of bending energy. As has recently been shown in [Lewicka and Pakzad 2017], this problem is not only encountered when dealing with the geometrically fully nonlinear plate model (1). It is also present in the von Kármán model, which we are going to treat here. In fact, the proof in [Lewicka and Pakzad 2017] is based on a suitable adaptation of the Nash–Kuiper argument to the von Kármán model.

Possibly the simplest example of a variational problem where isometric immersions are prohibited by the boundary conditions, but short maps are not, is given by a modification of the "conically constrained" sheets from [Brandman et al. 2013; Müller and Olbermann 2014]. The modification consists in considering clamped boundary conditions (for displacements and deformation gradients), and dropping the constraint on the deformation at the center of the sheet. This completely changes the character of the problem, and the method of proof from [Brandman et al. 2013; Müller and Olbermann 2014] breaks down.

This is the variational problem we will consider here, and we will prove an energy scaling law for it; see Theorem 2.1 below. There is one caveat: we penalize the bending energy in L^p with $p\in\left(2,\frac{8}{3}\right)$, see (6), instead of, as would be dictated by a heuristic derivation of the von Kármán model from three-dimensional elasticity, $p=2$. For a discussion of this modification, see Remark 2.2.

Our method of proof builds on the observations we made in [Olbermann 2016; 2017], where we proved scaling laws for an elastic sheet with a single disclination. The guiding principle is that the (linearized) Gauss curvature is controlled by both the membrane and the bending energy, in different function spaces. The boundary conditions can be used to show that the Gauss curvature is bounded from below in a certain space "in between" in the sense of interpolation. In the recent paper [Olbermann 2018], we showed that for the setting of [Olbermann 2016; 2017], it is not necessary to use interpolation, and lower bounds for the bending energy can be obtained by using the control over the membrane energy alone. The present setting with a flat reference metric however defines an interpolation-type problem for the Gauss curvature, and we hope that this approach can also yield results for similar variational problems.

This paper is structured as follows: In Section 2, we state our main result, Theorem 2.1. In Section 3, we collect some facts from the literature, concerning the Brouwer degree, Sobolev and Triebel–Lizorkin spaces, and interpolation theory. The proof of Theorem 2.1 is contained in Section 4.

Notation. We write $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $S^1 = \partial B_1$. When dealing with functions on S^1, we will identify S^1 with the one-dimensional torus $\mathbb{R}/(2\pi\mathbb{Z})$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write $\hat{x} = x/|x|$ and $x^\perp = (-x_2, x_1)$. In Section 2 below, we introduce a function $\beta \in W^{2,p}(S^1)$ that can be considered as fixed for the rest of the paper. The symbol "C" is used as follows: A statement such as "$f \le Cg$" is shorthand for "there exists a constant $C > 0$ that only depends on β such that $f \le Cg$". The value of C may change within the same line. For $f \le Cg$, we also write $f \lesssim g$. The symmetrized gradient of a function $u : U \to \mathbb{R}^2$ with $U \subseteq \mathbb{R}^2$ is denoted by $\operatorname{sym} Du = \frac{1}{2}(Du + Du^T)$.

2. Setting and statement of the main theorem

Let $\beta \in W^{2,p}(S^1)$ with

$$\int_{S^1} (\beta^2(t) - \beta'^2(t))\,\mathrm{d}t = 0 \quad \text{and} \quad \int_{S^1} |\beta + \beta''|\,\mathrm{d}t > 0. \tag{2}$$

Using the identification of $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ with the torus $\mathbb{R}/(2\pi\mathbb{Z})$, we define

$$\gamma(t) := -\frac{\beta^2(t)}{2} \qquad \zeta(t) := \frac{1}{2}\int_0^t \beta^2(s) - \beta'^2(s)\,\mathrm{d}s, \tag{3}$$

and we define $u_\beta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$u_\beta(x) \cdot \hat{x} := |x|\gamma(\hat{x}), \quad u_\beta(x) \cdot \hat{x}^\perp := |x|\zeta(\hat{x}), \tag{4}$$

Furthermore, we set

$$v_\beta(x) = |x|\beta(\hat{x}). \tag{5}$$

Note that the deformation defined by u_β, v_β is an isometric immersion in the von Kármán sense; i.e.,

$$\operatorname{sym} Du_\beta + \tfrac{1}{2} Dv_\beta \otimes Dv_\beta = 0,$$

but $D^2 v_\beta \notin L^p$ for $p \ge 2$. The set of allowed configurations is given by

$$\mathcal{A}_{\beta,p} := \{(u, v) \in W^{1,2}(B_1) \times W^{2,p}(B_1) : v = v_\beta,\ Dv = Dv_\beta \text{ and } u = u_\beta \text{ on } S^1\}.$$

The energy is given by a sum of membrane and bending energy,

$$I_{h,p}(u,v)=\left\|\operatorname{sym} Du+\tfrac{1}{2}Dv\otimes Dv\right\|^2_{L^2(B_1)}+h^2\|D^2v\|^2_{L^p(B_1)}. \tag{6}$$

In the statement of our main theorem, the dual exponent p' is defined as usual by $1/p+1/p'=1$. We are going to prove:

Theorem 2.1. *Let $p\in\left(2,\frac{8}{3}\right)$. Then there exists a constant $C=C(\beta,p)>0$ such that*

$$C^{-1}h^{p'}\leq\inf_{y\in\mathcal{A}_{\beta,p}} I_{h,p}(y)\leq Ch^{p'}.$$

Remark 2.2. (i) The arguments of the energy functional $(u,v):B_1\to\mathbb{R}^3$ can be thought of as the displacements of a deformation $x\mapsto x+\varepsilon^2u(x)+\varepsilon v(x)e_3$, where ε is another small parameter (with $h\ll\varepsilon$). The membrane energy geometrically corresponds to the deviation of the induced metric tensor from the flat Euclidean metric: the induced metric is given by $(\mathrm{Id}_{2\times 2}+\varepsilon^2Du+\varepsilon e_3\otimes Dv)^T(\mathrm{Id}_{2\times 2}+\varepsilon^2Du+\varepsilon e_3\otimes Dv)$, and the membrane term $\operatorname{sym} Du+\frac{1}{2}Dv\otimes Dv$ is the leading-order term of the difference to the flat reference metric. We say that $\det D^2v$ is the "linearized Gauss curvature" since we have that the Gauss curvature is given by $K=\varepsilon^2\det D^2v+o(\varepsilon^2)$. Rigorously, the von Kármán energy, (6) with $p=2$, has only been justified as a limit of three-dimensional finite elasticity for small deformations [Ciarlet 1980; Friesecke et al. 2006]. Nevertheless, it has a long and successful history of describing phenomena including moderate deformations.

(ii) The conditions on the boundary values in (2) are the von Kármán version of the requirement that the associated conical deformation defined by u_β, v_β has no membrane energy and is not contained in a plane.

(iii) The restriction to the range $p\in\left(2,\frac{8}{3}\right)$ is due to our method of proof, which is an application of the Gagliardo–Nirenberg inequality to the linearized Gauss curvature $\det D^2v$. This interpolation inequality is only valid for that range. The standard von Kármán model is linear in the material response, and hence it penalizes the bending energy in L^2. In this case, one expects an energy scaling law of the form $I_{h,2}\sim h^2\log(1/h)$, as is the case when the center of the sheet is fixed; see [Brandman et al. 2013; Müller and Olbermann 2014]. In order to obtain lower bounds for this case, one would have to show "additional regularity", in the sense that one would need to control higher L^p norms of D^2v by the L^2 norm. One might hope that such estimates are possible, e.g., for minimizers of the functional. However, we do not know if this is possible.

(iv) We do not know if our method of proof can be adapted to prove an analogous result for the geometrically fully nonlinear plate model that is given by the energy $\tilde{I}_{h,p}:W^{2,p}(B_1;\mathbb{R}^3)\to\mathbb{R}$,

$$\tilde{I}_{h,p}(y)=\int_{B_1}|Dy^TDy-\mathrm{Id}_{2\times 2}|^2\,\mathrm{d}x+h^2\|D^2y\|^2_{L^p(B_1)}.$$

The reason is that it seems much more complicated to obtain a good test function in $W^{1,p}$ for $\sum_i\det D^2y_i$ (which is the appropriate linearization of Gauss curvature in that setting) that would yield a lower bound for this quantity in the Sobolev space $W^{-1,p'}$. In the von Kármán case, we can simply use the identity

$$(\operatorname{div}\psi)|_{Dv(x)}\det D^2v(x)=\operatorname{div}\big(\psi(Dv(x))\operatorname{cof} D^2v(x)\big)$$

and compute a lower bound for this quantity by Gauss' theorem, using the boundary values of Dv. In the case of $y \in W^{2,p}(B_1;\mathbb{R}^3)$, we cannot argue similarly component by component: only the *sum* $\sum_i \det D^2 y_i$ is controlled by the energy. The task is to find a test function that (a) allows us to use Gauss' theorem and the boundary values to obtain a lower bound of order 1, and (b) is controlled in $W^{1,p}$ by the bending energy. We have not found a way to do so.

3. Preliminaries

The Brouwer degree. At the heart of our proof of the lower bound for the energy is an interpolation estimate for the linearized Gauss curvature. This quantity can be thought of as a pull-back of the volume form on $\mathbb{R}^2$ under the map $Dv : B_1 \to \mathbb{R}^2$. This is where the Brouwer degree becomes relevant, since integrals over the linearized Gauss curvature "downstairs" (on B_1) can be expressed as integrals "upstairs" (on $\mathbb{R}^2$) over the Brouwer degree of Dv.

For a bounded set $U \subset \mathbb{R}^n$, $f \in C^\infty(\overline{U};\mathbb{R}^n)$ and $y \in \mathbb{R}^n \setminus f(\partial U)$, the Brouwer degree $\deg(f, U, y)$ may be defined as follows: Let $A_{y,f}$ denote the connected component of $\mathbb{R}^n \setminus f(\partial U)$ that contains y, and let μ be a smooth n-form on $\mathbb{R}^n$ with support in $A_{y,f}$ such that $\int_{\mathbb{R}^n} \mu = 1$. Then we set

$$\deg(f, U, y) = \int_U f^\# \mu,$$

where $f^\#$ denotes the pull-back under f. By approximation with smooth functions, $\deg(f, U, y)$ may be defined for every $f \in C^0(\overline{U};\mathbb{R}^n)$ and $y \in \mathbb{R}^n \setminus f(\partial U)$. If $f \in W^{1,\infty}(\overline{U};\mathbb{R}^n)$ and μ is an n-form with regularity $W^{1,\infty}$, it follows straightforwardly from the definition that

$$\int_{\mathbb{R}^n} \deg(f, U, \cdot)\mu = \int_U f^\# \mu.$$

If $\mu = \varphi\, \mathrm{d}z$, where $\mathrm{d}z$ is the canonical volume form on $\mathbb{R}^n$, this can be written as

$$\int_{\mathbb{R}^n} \varphi(z) \deg(f, U, z)\, \mathrm{d}z = \int_U \varphi(f(x)) \det Df(x)\, \mathrm{d}x. \tag{7}$$

If $f \in C^1(U;\mathbb{R}^n)$, U has Lipschitz boundary and μ is a smooth $(n-1)$-form on $\mathbb{R}^n$, then we have

$$\int_{\mathbb{R}^n} \deg(f, U, \cdot)\, \mathrm{d}\mu = \int_U f^\#(\mathrm{d}\mu) = \int_{\partial U} f^\# \mu. \tag{8}$$

It can be shown that $y \mapsto \deg(f, U, y)$ is constant on the connected components of $\mathbb{R}^n \setminus f(\partial U)$. Finally, we are going to use the fact that $\deg(f, U, y)$ only depends on $f|_{\partial U}$. Thus for every continuous function $\tilde{f} : \partial U \to \mathbb{R}^N$, and $y \notin \tilde{f}(\partial U)$, we may define

$$\deg^\partial(\tilde{f}, \partial U, y) = \deg(f, U, y),$$

where f is any continuous extension of $\tilde{f}$ to $\overline{U}$. For more details (in particular for the proofs of the statements made here), see [Fonseca and Gangbo 1995].

Function spaces. Our main estimate for the Gauss curvature is a version of the Gagliardo–Nirenberg inequality for the spaces $W^{-m,p}$ with $m \in \mathbb{N}$ and $p \in (1,\infty)$. To define these spaces, let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $u \in L^1(\Omega)$ with compact support in Ω, we set

$$\|u\|_{W_0^{m,p}(\Omega)} := \left(\int_\Omega |D^m u|^p \,\mathrm{d}x\right)^{1/p}.$$

This defines a norm on the space $W_0^{m,p}(\Omega)$ which is defined as the set of those $u \in L^1(\Omega)$ that are compactly supported in Ω and satisfy $\|u\|_{W_0^{m,p}(\Omega)} < \infty$. The dual space of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$, where p' satisfies $1/p + 1/p' = 1$. The norm on $W^{-m,p'}(\Omega)$ is given by

$$\|f\|_{W^{-m,p'}(\Omega)} = \sup\{\langle f,\varphi\rangle : \varphi \in W_0^{m,p}(\Omega),\ \|\varphi\|_{W_0^{m,p}(\Omega)} \le 1\}.$$

Additionally, we define the space $W^{m,p}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{W^{m,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |D^m u|^p \,\mathrm{d}x\right)^{1/p}.$$

The Gagliardo–Nirenberg inequality that we want to prove is an interpolation inequality for the spaces $W^{-m,p'}(\Omega)$. In fact, the interpolation can be carried out in the spaces $W^{m,p}$ (with $m \ge 0$ and the understanding $W^{0,p} \equiv L^p$). These will be derived by appealing to results from the literature, where one finds a well-developed interpolation theory for the Triebel–Lizorkin spaces $F^s_{p,q}$, which contains the appropriate interpolation between Lebesgue and Sobolev spaces as a special case.

Let $\mathcal{D}'(\mathbb{R}^n)$ denote the space of temperate distributions on $\mathbb{R}^n$, and let $\mathcal{F} : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ denote the Fourier transform. We briefly recall the Littlewood–Paley decomposition of temperate distributions: Let $\eta_0 \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \le \eta_0 \le 1$, $\eta_0(x) = 1$ for $|x| \le 1$, $\eta_0(x) = 0$ for $|x| \ge 2$. Set $\eta_j(x) = \eta_0(2^{-j}x) - \eta_0(2^{-j+1}x)$ for $j \ge 1$.

Definition 3.1 [Triebel 1983, Chapter 2.3.1]. For $-\infty < s < \infty$, $0 < p, q < \infty$, let

$$F^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : \|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \|\|2^{sj}\mathcal{F}^{-1}\eta_j\mathcal{F}f\|_{l^q}\|_{L^p(\mathbb{R}^n)} < \infty\}.$$

The following special cases of the Triebel–Lizorkin spaces will be relevant for us (see [Triebel 1983, Sections 2.2.2 and 2.3.5]):

$$\begin{aligned} L^p(\mathbb{R}^n) &= F^0_{p,2}(\mathbb{R}^n), \\ W^{k,p}(\mathbb{R}^n) &= F^k_{p,2}(\mathbb{R}^n) \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{9}$$

Apart from their interpolation properties, the following embedding theorem will play a role in our proof:

Theorem 3.2 [Triebel 1983, Theorem 2.7.1]. *Suppose $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 < \infty$ and $0 < q_0, q_1 < \infty$ such that*

$$s_1 - \frac{n}{p_1} = s_0 - \frac{n}{p_0}.$$

Then we have the continuous embedding

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^n) \subset F^{s_1}_{p_1,q_1}(\mathbb{R}^n).$$

Real interpolation. We recall some basic facts concerning the real interpolation method. Let X_0, X_1 be Banach spaces such that there exists a topological vector space Z with continuous embeddings $X_0, X_1 \subset Z$. In such a situation, let $t > 0$ and $x \in X_0 + X_1$. We define

$$K(t,x) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0,\ x_1 \in X_1,\ x_0 + x_1 = x\}.$$

Let $0 \le \theta \le 1$ and $p \ge 1$. The real interpolation space $(X_0, X_1)_{\theta,p}$ is defined as

$$(X_0, X_1)_{\theta,p} = \{x \in X_0 + X_1 : \Phi_{\theta,p}(x) < \infty\},$$

where

$$\Phi_{\theta,p}(x) = \begin{cases} \left(\int_0^\infty |t^{-\theta} K(t,x)|^p \,\frac{\mathrm{d}t}{t}\right)^{1/p} & \text{if } p < \infty, \\ \sup_{t>0} |t^{-\theta} K(t,x)| & \text{else.} \end{cases}$$

The interpolation space $(X_0, X_1)_{\theta,p}$ is a normed space with the norm $\Phi_{\theta,p}(x)$. For every $p < \infty$, we have the continuous embedding

$$(X_0, X_1)_{\theta,p} \subset (X_0, X_1)_{\theta,\infty}. \tag{10}$$

For a proof, see, e.g., Chapter 1.3 of [Triebel 1978]. Concerning real interpolation of Triebel–Lizorkin spaces, we have the following theorem:

Theorem 3.3 [Triebel 1978, Theorem 1 in Chapter 2.4.2]. *Let* $-\infty < s_0, s_1 < \infty,\ 1 < p_0, p_1, q_0, q_1 < \infty$, $0 < \theta < 1$ *and*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then we have

$$(F^{s_0}_{p_0,q_0}(\mathbb{R}^n), F^{s_1}_{p_1,q_1}(\mathbb{R}^n))_{\theta,p} = F^s_{p,p}(\mathbb{R}^n).$$

4. Proof of Theorem 2.1

A sketch of the proof of Theorem 2.1 goes as follows: As usual, the upper bound is provided by a conical construction that is smoothed on a ball around the origin with the appropriate length scale; see Lemma 4.1. At the heart of the lower bound, we have an interpolation inequality for the linearized Gauss curvature $\det D^2 v$. The Gagliardo–Nirenberg inequality [Nirenberg 1959] yields

$$\|\det D^2 v\|_{W^{-1,p'}(B_1)} \lesssim \|\det D^2 v\|^{1-\alpha}_{W^{-2,2}(B_1)} \|\det D^2 v\|^{\alpha}_{L^{p/2}(B_1)} \quad \text{(formally)}, \tag{11}$$

with $\alpha \in \left[\frac{1}{2}, 1\right]$ determined by

$$\frac{2}{p'} - 1 = \left(\frac{2}{p/2} - 2\right)\alpha + 1 - \alpha,$$

i.e.,

$$\alpha = \frac{2}{3p-4}.$$

In (11), the left-hand side can be bounded from below using the boundary conditions and an argument involving the mapping degree. Namely, for an appropriately chosen test function $\varphi \in C_c^\infty(\mathbb{R}^2)$, we have

$$\int_{B_1} \varphi \circ Dv(x) \det D^2 v \,\mathrm{d}x = \int_{\mathbb{R}^2} \varphi(z) \deg(Dv, B_1, z) \,\mathrm{d}z = O(1).$$

For the details see Lemma 4.2.

The exponents in (11) are chosen such that the terms on the right-hand side can be estimated by the energy,

$$\begin{aligned} \|\det D^2 v\|_{W^{-2,2}} &\lesssim \left\|\operatorname{sym} Du + \tfrac{1}{2} Dv \otimes Dv\right\|_{L^2} \lesssim I_{h,p}(u,v)^{1/2}, \\ \|\det D^2 v\|_{L^{p/2}} &\lesssim \|D^2 v\|_{L^p}^2 \lesssim h^{-2} I_{h,p}(u,v). \end{aligned} \tag{12}$$

With these estimates, we obtain the desired lower bound.

Basically, all that remains is to prove the aforementioned lemmas, and justify (11). We could not find a proof of the Gagliardo–Nirenberg inequality for "negative orders of differentiation" in the literature. We believe that it holds true, and that a proof could be given using the machinery from [Triebel 1978]. However, in our case a shorter route exists, using the fact that $v : B_1 \to \mathbb{R}$ has a natural extension to $\mathbb{R}^2$ with vanishing membrane energy on $\mathbb{R}^2 \setminus B_1$, and existing results on interpolation of Sobolev and Triebel–Lizorkin spaces on $\mathbb{R}^n$; see again [Triebel 1978].

Now we start with the proof.

Lemma 4.1. *We have*

$$\inf_{y \in \mathcal{A}_{\beta,p}} I_{h,p}(y) < C h^{p'}.$$

Proof. Recall the definition of u_β, v_β from (3)–(5). Let $\eta \in C^\infty([0,\infty))$ with $\eta(t) = 0$ for $t < \frac{1}{2}$, $\eta(t) = 1$ for $t \geq 1$. We set

$$v_{\beta,h}(x) = \eta\left(\frac{|x|}{h^{p'/2}}\right) v_\beta(x).$$

Now we have

$$\left|\operatorname{sym} Du_\beta(x) + \tfrac{1}{2} Dv_{\beta,h}(x) \otimes Dv_{\beta,h}(x)\right| = \begin{cases} 0 & \text{if } |x| \geq h^{p'/2}, \\ O(1) & \text{else.} \end{cases}$$

Furthermore, we have

$$\int_{B_1} |D^2 v_{\beta,h}|^p = \int_{B_1 \setminus B_{h^{p'/2}}} \mathrm{d}x \left|\frac{\beta''(\hat{x}) + \beta(\hat{x})}{|x|}\right|^p + \int_{B_{h^{p'/2}}} O(h^{-p(p'/2)}) \,\mathrm{d}x \lesssim h^{(2-p)p'/2}.$$

This implies

$$I_{h,p}(u_\beta, v_{\beta,h}) = \int_{B_1} \left|\operatorname{sym} Du_\beta(x) + \tfrac{1}{2} Dv_{\beta,h}(x) \otimes Dv_{\beta,h}(x)\right|^2 \mathrm{d}x + h^2 \left(\int_{B_1} |D^2 v_{\beta,h}|^p \,\mathrm{d}x\right)^{2/p} \lesssim h^{p'}.$$

□

Lemma 4.2. *Assume that $\beta \in W^{2,p}(S^1)$ with*

$$\int \beta^2(t) - \beta'^2(t) \,\mathrm{d}t = 0 \quad \textit{and} \quad \int |\beta + \beta''| \,\mathrm{d}t \neq 0,$$

and let v_β *be defined by* (5)*. Then there exists* $\varphi_\beta \in C_c^\infty(\mathbb{R}^2)$ *such that* $\operatorname{supp}\varphi_\beta \cap Dv_\beta(S^1) = \varnothing$ *and*

$$\int_{\mathbb{R}^2} \varphi_\beta(z) \deg^\partial(Dv_\beta, S^1, z)\,\mathrm{d}z > 0.$$

Proof. Step 1: reduction to the smooth case. We claim that we may assume $\beta \in C^\infty(S^1)$. Indeed, for every $\varepsilon > 0$ we may choose $\tilde\beta \in C^\infty(S^1;\mathbb{R}^2)$ such that

$$\|\beta - \tilde\beta\|_{W^{2,p}} < \varepsilon \quad \text{and} \quad \int |\tilde\beta + \tilde\beta''|\,\mathrm{d}t \neq 0.$$

Additionally, we may choose $\tilde\beta$ such that

$$\int_{S^1} (\tilde\beta^2 - \tilde\beta'^2)\,\mathrm{d}t = 0.$$

We have

$$Dv_\beta = \beta(\hat x)\hat x + \beta'(\hat x)\hat x^\perp, \quad Dv_{\tilde\beta} = \tilde\beta(\hat x)\hat x + \tilde\beta'(\hat x)\hat x^\perp.$$

By the continuous embedding $W^{2,p} \to C^1$, we have that $\|Dv_\beta - Dv_{\tilde\beta}\|_{C^0(S^1)}$ and hence we can also make $\|\deg^\partial(Dv_\beta, S^1, \cdot) - \deg^\partial(Dv_{\tilde\beta}, S^1, \cdot)\|_{L^1(\mathbb{R}^2)}$ arbitrarily small by a suitable choice of ε. If we manage to show $\deg^\partial(Dv_{\tilde\beta}, S^1, \cdot) \neq 0$ in $L^1(\mathbb{R}^2)$, then we have also proved the claim of the lemma. Hence, from now on we prove the claim of the lemma for $\beta \in C^\infty(S^1)$.

Step 2: taking the derivative of "deg". For $t \in S^1$, let $e_t = (\cos t, \sin t)$. Let $\gamma : S^1 \to \mathbb{R}^2$ be defined by

$$\gamma(t) = \beta(t)e_t + \beta'(t)e_t^\perp.$$

It is enough to show that $\deg^\partial(Dv_\beta, S^1, \cdot) = \deg^\partial(\gamma, S^1, \cdot)$ is nonzero in $L^1(\mathbb{R}^2)$. By (8), we have for any smooth one-form $\omega = \omega_1\,\mathrm{d}x_1 + \omega_2\,\mathrm{d}x_2$ on $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \deg^\partial(\gamma, S^1, \cdot)\,\mathrm{d}\omega = \int_{S^1} \gamma^\#\omega.$$

If we show that the right-hand side is nonzero for some choice of ω, we are done. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$x \mapsto \sum_{t\in\gamma^{-1}(x)} \gamma'(t) = \sum_{t\in\gamma^{-1}(x)} (\beta(t) + \beta''(t))e_t^\perp. \tag{13}$$

Then we have

$$(\gamma^\#\omega)(t) = (\omega_1(\gamma(t)), \omega_2(\gamma(t))) \cdot f(\gamma(t))\,\mathrm{d}t,$$

and we see that it suffices to show that $f \neq 0$ on a set of positive $\mathcal{H}^1$ measure to prove the claim of the lemma.

Step 3: proof of the lemma by contradiction. We assume that $f = 0$ $\mathcal{H}^1$-almost everywhere and show that this leads to a contradiction. Since $\gamma'(t) = (\beta(t) + \beta''(t))e_t^\perp$, we have that $\gamma'(t) = 0$ if and only if $\beta(t) + \beta''(t) = 0$. Let U be an open interval such that $\gamma' \neq 0$ on U and $\gamma : U \to \gamma(U)$ is a diffeomorphism.

Our aim is now to show that up to $\mathcal{H}^1$ null sets, we have

$$\gamma^{-1}(\gamma(U)) \setminus U = U + \pi,$$

where we are using the identification of S^1 with $\mathbb{R}/(2\pi\mathbb{Z})$.

Since $f = 0$ $\mathcal{H}^1$-almost everywhere on $\gamma(U)$ and by the explicit form (13) of f, there exists $E_1 \subset S^1$ with $\mathcal{H}^1(E_1) = 0$ such that

$$\gamma(U \setminus E_1) \subseteq \gamma(S^1 \setminus U). \tag{14}$$

Next let

$$E_2 := \{t \in S^1 : \gamma'(t) = 0\}, \quad A := \gamma(E_2).$$

By Sard's lemma, we have $\mathcal{H}^1(A) = 0$. Furthermore, let

$$E_3 := \gamma^{-1}(\gamma(U \setminus E_1) \setminus A) \setminus U,$$

and let $E_4 \subseteq E_3$ be the set of points that are not of density 1, i.e.,

$$E_4 := \left\{ x \in E_3 : \liminf_{\varepsilon \to 0} \frac{\mathcal{H}^1((x-\varepsilon, x+\varepsilon) \cap E_3)}{2\varepsilon} < 1 \right\}.$$

It is a well-known fact from measure theory that $\mathcal{H}^1(E_4) = 0$. Let $E_5 := \gamma^{-1}(\gamma(E_4)) \cap U$. Then also $\mathcal{H}^1(E_5) = 0$.

Now let $p \in U \setminus (E_1 \cup E_5)$. Then $\gamma(p) \notin A \cup \gamma(E_4)$, and by (14), we have

$$\gamma(p) \in \gamma(S^1 \setminus U) \setminus (A \cup \gamma(E_5)).$$

Hence there exists $p' \in E_3 \setminus E_4$ with $\gamma(p') = \gamma(p)$. We may choose a sequence p'_k, $k \in \mathbb{N}$, with $p'_k \in E_3$, $\gamma(p'_k) \neq \gamma(p')$, and $p'_k \to p'$. Since $\gamma|_U$ is a diffeomorphism, we may set

$$p_k := \gamma|_U^{-1}(\gamma(p'_k))$$

and obtain a sequence $p_k \to p$ in U, with $\gamma(p_k) \neq \gamma(p)$. Now we have for every k,

$$\frac{\gamma(p_k) - \gamma(p)}{|\gamma(p_k) - \gamma(p)|} = \frac{\gamma(p'_k) - \gamma(p')}{|\gamma(p'_k) - \gamma(p')|}.$$

Passing to a suitable subsequence and taking the limit $k \to \infty$ in that equation, we obtain that the vectors $\gamma'(p)$ and $\gamma'(p')$ are parallel. Since

$$\gamma'(t) = (\beta(t) + \beta''(t))e_t^\perp$$

and $p \neq p'$, we must have $e_p = -e_{p'}$, and hence (using the identification of S^1 with $\mathbb{R}/(2\pi\mathbb{Z})$)

$$p' = p + \pi.$$

Summarizing, we have shown that for $\mathcal{H}^1$-almost every $p \in U$, we have $p + \pi \in \gamma^{-1}(\gamma(U)) \setminus U$. We also may conclude that for every $p' \in E_3 \setminus E_4$, it holds that $p' + \pi \in U$. Hence, as desired, we have

$$\gamma^{-1}(\gamma(U)) \setminus U = U + \pi \quad \text{up to } \mathcal{H}^1 \text{ null sets.}$$

Since for every $x \in S^1 \setminus E_2$ there exists a neighborhood U of x with the properties we have assumed above, we obtain that for $\mathcal{H}^1$-almost every $t \in S^1 \setminus E_2$, we have $\gamma(t) = \gamma(t+\pi)$. Hence,

$$\beta(t)e_t + \beta'(t)e_t^\perp = -\beta(t+\pi)e_t - \beta'(t+\pi)e_t^\perp,$$

which implies

$$\beta(t+\pi) = -\beta(t), \quad \beta'(t+\pi) = -\beta'(t) \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in S^1 \setminus E_2. \tag{15}$$

We claim that we even have

$$\beta(t+\pi) = -\beta(t) \quad \text{for } t \in S^1. \tag{16}$$

Indeed, let $t \in S^1$. If $t \in \overline{S^1 \setminus E_2}$, then the claim follows from (15). If t is in the interior of E_2, then let $T \in \partial E_2$ such that $(t, T) \subset E_2$. Then we have that also $(t+\pi, T+\pi) \subset E_2$, and $\beta(T+\pi) = -\beta(T)$, $\beta'(T+\pi) = -\beta'(T)$. The values of $\beta(t), \beta(t+\pi)$ are then determined by the initial values of β, β' at the points $T, T+\pi$ and by the ODE $\beta + \beta'' = 0$. By the linearity and translation invariance of this initial value problem, we obtain $\beta(t+\pi) = -\beta(t)$ as desired. This proves the claim (16).

By (16), we have $\int_{S^1} \beta(t)\,\mathrm{d}t = 0$. By the Poincaré–Wirtinger inequality, we have that

$$\int_{S^1} (\beta^2 - \beta'^2)\,\mathrm{d}t \le 0,$$

with equality only if β is of the form $\beta(t) = C\sin(t+\alpha)$ for some $C, \alpha \in \mathbb{R}$. Equality must hold true by assumption, which yields

$$\beta + \beta'' = 0 \quad \text{on } S^1,$$

in contradiction to our assumptions. This proves the lemma. □

Lemma 4.3. *Let $p \in \left(2, \frac{8}{3}\right)$, and*

$$\theta = 1 - \frac{2}{3p-4}, \quad \frac{1}{q} = \frac{1-\theta}{p/(p-2)} + \frac{\theta}{2}.$$

Then we have

$$W^{1,p}(\mathbb{R}^2) \subset (L^{p/(p-2)}(\mathbb{R}^2), W^{2,2}(\mathbb{R}^2))_{\theta,q}.$$

Proof. By (9), we have $L^{p/(p-2)}(\mathbb{R}^2) = F^0_{p/(p-2),2}(\mathbb{R}^2)$ and $W^{2,2}(\mathbb{R}^2) = F^2_{2,2}(\mathbb{R}^2)$. By Theorem 3.3, we obtain

$$(L^{p/(p-2)}(\mathbb{R}^2), W^{2,2}(\mathbb{R}^2))_{\theta,q} = F^{2\theta}_{q,q}(\mathbb{R}^2).$$

Finally, by Theorem 3.2, we have

$$W^{1,p}(\mathbb{R}^2) = F^1_{p,2}(\mathbb{R}^2) \subset F^{2\theta}_{q,q}(\mathbb{R}^2).$$

Note that the assumption $s_1 < s_0$ in Theorem 3.2 is fulfilled by $1 > 2\theta$, which in turn is a consequence of $p \in \left(2, \frac{8}{3}\right)$. □

In the next lemma, we use the following notation: for $(u,v)\in\mathcal{A}_{\beta,p}$, we let $\bar v:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$\bar u(x)=\begin{cases}u(x) & \text{if } x\in B_1,\\ u_\beta(x) & \text{if } x\in\mathbb{R}^2\setminus B_1,\end{cases}\qquad \tilde v(x)=\begin{cases}v(x) & \text{if } x\in B_1,\\ v_\beta(x) & \text{if } x\in\mathbb{R}^2\setminus B_1,\end{cases}$$

where u_β, v_β are as defined in (3)–(5).

Lemma 4.4. *Let $(u,v)\in\mathcal{A}_{\beta,p}$. Then*

$$\|\det D^2\bar v\|_{W^{-2,2}(\mathbb{R}^2)}\lesssim \left\|\operatorname{sym} Du+\tfrac12 Dv\otimes Dv\right\|_{L^2(B_1)}.$$

Proof. We write down the Hessian determinant of $\bar v$ in its very weak form,

$$\det D^2\bar v=(\bar v_{,1}\bar v_{,2})_{,12}-\tfrac12(\bar v_{,1}^2)_{,22}-\tfrac12(\bar v_{,2}^2)_{,11}-=-\tfrac12\operatorname{curl}\operatorname{curl} D\bar v\otimes D\bar v.$$

Here, we have introduced $\operatorname{curl}(w_1,w_2)=w_{1,2}-w_{2,1}$. (In the formula above, curl is first applied in each row of the matrix $D\bar v\otimes D\bar v$, and then on the components of the resulting column vector.) Since we have $\operatorname{curl}\operatorname{curl}(Dw^T+Dw)=0$ for every $w\in W^{1,2}(B_1;\mathbb{R}^2)$, we obtain

$$\det D^2\bar v=-\operatorname{curl}\operatorname{curl}\bigl(\operatorname{sym} D\bar u+\tfrac12 D\bar v\otimes D\bar v\bigr).$$

We note that

$$\operatorname{sym} D\bar u+\tfrac12 D\bar v\otimes D\bar v=0\quad\text{on } \mathbb{R}^2\setminus B_1.$$

Hence for every $\varphi\in W^{2,2}(\mathbb{R}^2)$, we obtain by two integrations by parts, and the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^2}\det D^2\bar v\,\varphi\,\mathrm{d}x&=-\int_{\mathbb{R}^2}\bigl(\operatorname{sym} D\bar u+\tfrac12 D\bar v\otimes D\bar v\bigr):\operatorname{cof} D^2\varphi\,\mathrm{d}x\\ &\le \left\|\operatorname{sym} Du+\tfrac12 Dv\otimes Dv\right\|_{L^2(B_1)}\|\varphi\|_{W^{2,2}(\mathbb{R}^2)}.\end{aligned}$$

□

Proof of Theorem 2.1. The upper bound has been proved in Lemma 4.1. It remains to prove the lower bound.

For any $(u,v)\in\mathcal{A}_{\beta,p}$ we have $Dv|_{S^1}=Dv_\beta|_{S^1}$, and hence $\deg(Dv,B_1,\cdot)=\deg^\partial(Dv,S^1,\cdot)=\deg^\partial(Dv_\beta,S^1,\cdot)$. By Lemma 4.2, we may choose $\varphi\in C_c^\infty(\mathbb{R}^2)$ such that $\varphi\circ Dv\in W_0^{1,p}(B_1)\subset W^{1,p}(\mathbb{R}^2)$ and

$$\begin{aligned}0<C(\beta)&=\int_{\mathbb{R}^2}\varphi(z)\deg(Dv,B_1,z)\,\mathrm{d}z\\ &=\int_{B_1}\det D^2v(x)\varphi(Dv(x))\,\mathrm{d}x\\ &=\int_{\mathbb{R}^2}\det D^2\bar v(x)\varphi(D\bar v(x))\,\mathrm{d}x,\end{aligned}$$

where we have used the notation introduced above Lemma 4.4, and the fact that $\det D^2\bar v=0$ on $\mathbb{R}^2\setminus B_1$. By Lemma 4.3, $\psi:=\varphi\circ Dv\in(L^{p/(p-2)}(\mathbb{R}^2),W^{2,2}(\mathbb{R}^2))_{\theta,q}$. Hence by (10), there exist functions $\psi_0:\mathbb{R}^+\to L^{p/(p-2)}(\mathbb{R}^2)$ and $\psi_1:\mathbb{R}^+\to W^{2,2}(\mathbb{R}^2)$ such that $\psi_0(t)+\psi_1(t)=\psi$ for all $t\in\mathbb{R}^+$ and

$$t^{-\theta}\|\psi_0(t)\|_{L^{p/(p-2)}(\mathbb{R}^2)}+t^{1-\theta}\|\psi_1(t)\|_{W^{2,2}(\mathbb{R}^2)}\lesssim\|\psi\|_{W^{1,p}(\mathbb{R}^2)}.$$

Rearranging, we have for every $t > 0$ that

$$\begin{aligned}\|\psi_0(t)\|_{L^{p/(p-2)}(\mathbb{R}^2)} &\lesssim t^{\theta}\|\psi\|_{W^{1,p}(\mathbb{R}^2)},\\ \|\psi_1(t)\|_{W^{2,2}(\mathbb{R}^2)} &\lesssim t^{\theta-1}\|\psi\|_{W^{1,p}(\mathbb{R}^2)}.\end{aligned}$$

Now we fix the argument,

$$t := \frac{\|\det D^2\bar{v}\|_{W^{-2,2}}}{\|\det D^2\bar{v}\|_{L^{p/2}}},$$

and write $\psi_0 = \psi_0(t)$, $\psi_1 = \psi_1(t)$. Hence we may estimate

$$\begin{aligned}C(\beta) &= \int_{\mathbb{R}^2} \det D^2\bar{v}(x)\varphi(D\bar{v}(x))\,\mathrm{d}x\\ &\lesssim \|\det D^2\bar{v}\|_{L^{p/2}}\|\psi_0\|_{L^{p/(p-2)}} + \|\det D^2\bar{v}\|_{W^{-2,2}}\|\psi_1\|_{W^{2,2}}\\ &\lesssim \|\det D^2\bar{v}\|^{\theta}_{W^{-2,2}}\|\det D^2\bar{v}\|^{1-\theta}_{L^{p/2}}\|\psi\|_{W^{1,p}}\\ &\lesssim I_{h,p}^{\theta/2}(h^{-2}I_{h,p})^{1-\theta}(h^{-2}I_{h,p})^{1/2}\\ &\lesssim I_{h,p}^{(3-\theta)/2}h^{2\theta-3},\end{aligned}\tag{17}$$

where we have used Lemma 4.4 and the facts

$$|\det D^2v| \le |D^2v|^2, \qquad \det D^2\bar{v} = 0 \quad \text{on } \mathbb{R}^2\setminus B_1$$

to obtain the fourth line from the third. This implies

$$I_{h,p} \gtrsim h^{(6-4\theta)/(3-\theta)} = h^{p/(p-1)} = h^{p'},$$

which proves the theorem. □

Acknowledgment

This work has been supported by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) as part of project 350398276.

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Received 4 Jan 2018. Revised 5 Mar 2018. Accepted 10 Apr 2018.

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ANALYSIS & PDE

Volume 12 No. 1 2019

2157-5045(2019)12:1;1-P