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# BOUNDARY BEHAVIOR OF SOLUTIONS TO THE PARABOLIC $p$ -LAPLACE EQUATION

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We establish boundary estimates for nonnegative solutions to the  $p$ -parabolic equation in the degenerate range  $p > 2$ . Our main results include new parabolic intrinsic Harnack chains in cylindrical NTA domains together with sharp boundary decay estimates. If the underlying domain is  $C^{1,1}$ -regular, we establish a relatively complete theory of the boundary behavior, including boundary Harnack principles and Hölder continuity of the ratios of two solutions, as well as fine properties of associated boundary measures. There is an intrinsic waiting-time phenomenon present which plays a fundamental role throughout the paper. In particular, conditions on these waiting times rule out well-known examples of explicit solutions violating the boundary Harnack principle.

## 1. Introduction and results

This paper is devoted to a study of the boundary behavior of nonnegative solutions to the  $p$ -parabolic equation, in the degenerate range  $p > 2$ . We restrict the analysis to space-time cylinders  $\Omega_T = \Omega \times (0, T)$ ,  $T > 0$ , where  $\Omega \subset \mathbb{R}^n$  is a bounded domain, i.e., an open and connected set. Given  $p$ ,  $1 < p < \infty$ , fixed, recall that the  $p$ -parabolic equation is

$$\partial_t u - \Delta_p u := \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0. \quad (1-1)$$

In the special case  $p = 2$ , the  $p$ -parabolic equation coincides with the heat equation, and in this case we refer to [Kemper 1972; Salsa 1981], and also [Fabes et al. 1984; 1986; 1999; Fabes and Safonov 1997; Garofalo 1984; Nyström 1997] concerning the boundary behavior of nonnegative solutions. Key results established in these works, in the context of Lipschitz-cylinders  $\Omega_T$ , include Carleson-type estimates, the relation between the associated parabolic measure and the Green's function, the backward-in-time Harnack inequality, boundary Harnack principles (local and global) and Hölder continuity up to the boundary of quotients of nonnegative solutions vanishing on the lateral boundary.

On the contrary for  $p \neq 2$ ,  $1 < p < \infty$ , much less is known concerning these problems and we refer the reader to [Avelin 2016; Avelin et al. 2016; Kuusi et al. 2014] for accounts of the current literature. For a relatively complete picture in the case of nonlinear parabolic operators with linear growth, we refer to [Nyström et al. 2015]. However, it is also important to mention that there is interesting and related recent literature devoted to the asymptotic and pointwise behavior of solutions to nonlinear diffusion

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equations on bounded domains; see [Stan and Vázquez 2013], and also [Bonforte and Vázquez 2015] for the porous medium-type equations.

Considering nonnegative solutions to the  $p$ -parabolic equation, for  $p$  in the degenerate range  $p > 2$ , it is a priori not clear to what extent and in what sense the above-mentioned results can hold. Indeed, on the one hand we have to account for the lack of homogeneity of the  $p$ -parabolic equation, and on the other hand we have to account for the fact that in the degenerate regime the phenomenon of finite speed propagation is present. As a matter of fact, simple examples show that in this case there are, compared to the case  $p = 2$ , much more delicate waiting-time phenomena to take into account.

To discuss the aspects of the waiting time phenomena further, we here first briefly describe some by now classical results in the case  $p = 2$ ; see [Fabes et al. 1986; Salsa 1981]. Assume that  $\Omega$  is, say, a Lipschitz domain, that  $x_0 \in \partial\Omega$  and let  $A_{\pm} := (a_r(x_0), t_0 \pm r^2)$ , where  $a_r(x_0)$  is an interior point of  $\Omega$  with distance to the boundary comparable to  $r$ . Assume also that  $u$  and  $v$  are nonnegative caloric functions in  $\Omega_T$ , i.e., functions satisfying (1-1) with  $p = 2$  in  $\Omega_T$ , vanishing continuously on  $(\partial\Omega \cap B_r(x_0)) \times (t_0 - r^2, t_0 + r^2)$ , where  $B_r(x_0) \subset \mathbb{R}^n$  is the standard Euclidean ball of radius  $r$  and centered at  $x_0 \in \mathbb{R}^n$ . Then

$$c^{-1} \frac{u(A_-)}{v(A_+)} \leq \frac{u(x, t)}{v(x, t)} \leq c \frac{u(A_+)}{v(A_-)}, \quad (1-2)$$

for a universal constant  $c$ , whenever  $(x, t) \in (\Omega \cap B_{r/2}(x_0)) \times (t_0 - (r/2)^2, t_0 + (r/2)^2)$ . However, in general an estimate like (1-2) dramatically fails in the case  $p \neq 2$ . To see this, recall the following two classical solutions, see, e.g., [Avelin et al. 2016], in the case when  $\Omega := \mathbb{R}^{n-1} \times \{x_n > 0\}$ :

$$u(x, t) = c_p (T - t)^{-1/(p-2)} x_n^{p/(p-2)}, \quad v(x, t) = x_n. \quad (1-3)$$

In view of the examples in (1-3) it is not clear under what conditions the boundary Harnack principle in (1-2) could hold. Let us make a few observations. When defining  $u$  as in (1-3), we see that the larger we take  $T$ , the longer the solution  $u$  exists and the smaller its pointwise values become at a fixed time  $t < T$ . If we wish to show an estimate as in (1-2), we need to be able to rule out examples like  $u$  in (1-3); see also some other examples in [Avelin et al. 2016]. We do this by simply requiring, for  $(x, t) \in \Omega_T$  fixed, that

$$T - t > C_0 u(x, t)^{2-p} d(x, \partial\Omega)^p \quad (1-4)$$

for a large enough constant  $C_0$ . It is easily seen that the solution  $u$  in (1-3) does not satisfy (1-4) at any point  $(x, t) \in \Omega_T$  if we require  $C_0 \geq c_p^{p-2}$ .

In this context, and for the  $p$ -parabolic equation, it is here natural to make a link to the by-now classical method of *intrinsic scaling* due to [DiBenedetto 1993]. The intrinsic scalings define the canonical geometry in which weak solutions to the  $p$ -parabolic equation become homogenized in a sense to be made precise. Indeed, in this geometry we consider, instead of the standard parabolic cylinders, intrinsically time-scaled cylinders of the type

$$\begin{aligned} Q_r^{\lambda,+}(x, t) &:= B_r(x) \times (t, t + \lambda^{2-p} r^p), \\ Q_r^{\lambda,-}(x, t) &:= B_r(x) \times (t - \lambda^{2-p} r^p, t), \end{aligned} \quad \lambda := u(x, t).$$

These kinds of intrinsic cylinders appear naturally in the context of Harnack inequalities, oscillation reduction estimates, and decay estimates, and define the correct geometry in our setting.

The main goal of this paper is to study to what extent the theory developed in [Fabes et al. 1986; Salsa 1981] generalizes to the case  $p > 2$ , under suitable intrinsic conditions. We have already seen that it rules out the pathological examples like  $u$  in (1-3). In fact, we prove that (1-4) is a sufficient condition for developing a rather general theory concerning the boundary behavior of nonnegative solutions to the  $p$ -parabolic equation. For instance, (1-4) allows us to prove a counterpart of (1-2) valid for  $2 < p < \infty$ ; see Theorem 9.4 below.

**1A. Summary of results.** We will now give an informal summary of our results. The precise statements can be found in the body of the paper.

*Harnack chains.* Fundamental tools in the study of the boundary behavior of nonnegative solutions are the Harnack inequality and Harnack chains. Harnack chains allow one to relate the value of nonnegative solutions at different space-time points in the domain. For  $p = 2$  the Harnack inequality is homogeneous and, roughly, to control the values of the solution in a ball of size  $r$  requires a waiting time comparable to  $r^2$ . For  $p > 2$  we have to use an intrinsic version of the Harnack inequality [DiBenedetto 1993; DiBenedetto et al. 2012]. In particular, the intrinsic Harnack inequality states, see Theorem 3.1, that if we have a nonnegative solution  $u$  to the  $p$ -parabolic equation in  $\Omega_T$ , with  $(x, t) \in \Omega_T$ , and  $Q_{4r}^{u(x,t)/c_h, -}(x, t) \subset \Omega_T$ , then

$$u(x, t) \leq C_h \inf_{y \in B_r(x)} u\left(y, t + \left[\frac{c_h}{u(x, t)}\right]^{p-2} r^p\right),$$

provided that  $t + [c_h/u(x, t)]^{p-2} r^p < T$ . The intrinsic waiting time required in this Harnack inequality is consistent with the condition stated in (1-4). In Section 3 we develop a sequence of Harnack chain estimates and the goal of that section is twofold. First, we want to establish estimates applicable in cylindrical NTA domains; see Definition 2.1. Second, we want to establish a  $p$ -stable counterpart of the sharp Harnack chain estimate proved by Salsa [1981, Theorem C], which in the case  $p = 2$  can be written as

$$u(y, s) \leq u(x, t) \exp\left[C\left(\frac{|x-y|^2}{t-s} + \frac{t-s}{k} + 1\right)\right], \quad (1-5)$$

where  $k = \min\{1, s, d(x, \partial\Omega)^2, d(y, \partial\Omega)^2\}$ ,  $s < t$ , and  $(x, t), (y, s) \in \Omega_T$ . To do this we develop Harnack chains based on the weak Harnack inequality proved in [Kuusi 2008], see Theorem 3.2 below, valid for supersolutions to the  $p$ -parabolic equation. As truncations of our solutions are supersolutions to (1-1), we are able to control the waiting times more precisely by adjusting the levels at which the solutions are truncated. This is in sharp contrast to the Harnack chains developed in [Avelin 2016; Avelin et al. 2016] for which there is very little control over the waiting time. Our approach to Harnack chains has at least three advantages. First, it allows us to construct Harnack chains starting from the measure  $u(x, t_0) dx$  at the initial time  $t_0$ . Second, it allows us to develop a flexible Carleson estimate, see Section 4, generalizing the one in [Avelin et al. 2016] and which, in addition, remains valid in the context of time-independent NTA cylinders. We note that although the Carleson estimate proved in [Avelin 2016] is valid in the setting of time-independent NTA cylinders, a difference compared to the results in this paper is that the Carleson

estimate proved there is not  $p$ -stable as  $p \rightarrow 2$ . Third, we develop a version of (1-5) which is  $p$ -stable in the sense that we recover (1-5) as  $p \rightarrow 2$ . To establish this Gaussian-type behavior for  $p > 2$  is technically rather involved.

*Estimates of associated boundary measures.* In the study of the boundary behavior of quasilinear equations of  $p$ -Laplace type, certain Riesz measures supported on the boundary and associated to nonnegative solutions vanishing on a portion of the boundary are important; see [Lewis and Nyström 2007; 2010]. These measures are nonlinear generalizations of the harmonic measure relevant in the study of the Laplace equation and the Green's function. Corresponding measures can also be associated to solutions to the  $p$ -parabolic equation. Indeed, let  $u$  be a nonnegative solution in  $\Omega_T$ , assume that  $u$  is continuous on the closure of  $\Omega_T$ , and that  $u$  vanishes on  $\partial_p \Omega_T \cap Q$  with some open set  $Q$ . Extending  $u$  to be zero in  $Q \setminus \Omega_T$ , it is straightforward to see that  $u$  is a continuous weak subsolution to (1-1) in  $Q$ . Using this, one can conclude that there exists a unique locally finite positive Borel measure  $\mu$ , supported on  $S_T \cap Q$ , such that

$$\int_Q u \partial_t \phi \, dx \, dt - \int_Q |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \, dt = \int_Q \phi \, d\mu \quad (1-6)$$

whenever  $\phi \in C_0^\infty(Q)$ . In Section 5 we establish, in cylindrical NTA domains, both upper and lower bounds for the measure  $\mu$  in terms of  $u$ . If  $\Omega$  is smooth, then  $d\mu = |\nabla u|^{p-1} dH^{n-1} dt$ . Based on this, the lower bound established on  $\mu$  can be interpreted as a nondegeneracy estimate, close to the boundary, of the solution. Our proof of the lower bound for the measure  $\mu$  is a modification of the elliptic proof; see for example [Avelin and Nyström 2013; Kilpeläinen and Zhong 2003]. However, our proof is genuinely nonlinear, it applies to much more general operators of  $p$ -parabolic type, and the result seems to be new already in the case  $p = 2$ .

*A “complete theory” in  $C^{1,1}$ -domains.* We establish a “complete theory” concerning the boundary behavior of nonnegative solutions in  $\Omega_T$  in the case when  $\Omega$  is a  $C^{1,1}$ -domain. As comprehensive literature is missing, we in Sections 8 and 9 develop both a local, as well as a global, theory of boundary behavior in  $C^{1,1}$ -cylinders. In the global setting we are able, as in [Fabes et al. 1986] with corresponding estimates in the case  $p = 2$ , to give a rather complete picture. For nonnegative solutions vanishing on the lateral boundary, our results include a global boundary Harnack principle and Hölder continuity of ratios. On the other hand, in the local setting we prove a new intrinsic local boundary Harnack principle. In the context of  $C^{1,1}$ -cylinders we are also able to show that the boundary measure in (1-6) is mutually absolutely continuous with respect to the surface measure in a suitably chosen intrinsic geometry. The results in Sections 8 and 9 are obtained by combining Harnack chains and Carleson estimates with explicit barrier constructions from Section 6 and decay estimates from Section 7.

## 2. Notation and preliminaries

Points in  $\mathbb{R}^{n+1}$  are denoted by  $x = (x_1, \dots, x_n, t)$ . Given a set  $E \subset \mathbb{R}^n$ , let  $\bar{E}$ ,  $\partial E$ ,  $\text{diam } E$ ,  $E^c$ ,  $E^\circ$ , denote the closure, boundary, diameter, complement and interior of  $E$ , respectively. Let  $\cdot$  denote the standard inner product on  $\mathbb{R}^n$ , let  $|x| = (x \cdot x)^{1/2}$  be the Euclidean norm of  $x$ , and let  $dx$  be the Lebesgue

$n$ -measure on  $\mathbb{R}^n$ . Given  $x \in \mathbb{R}^n$  and  $r > 0$ , let  $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ . Given  $E, F \subset \mathbb{R}^n$ , let  $d(E, F)$  be the Euclidean distance from  $E$  to  $F$ . When  $E = \{y\}$ , we write  $d(y, F)$ . For simplicity, we define ‘‘sup’’ to be the essential supremum and ‘‘inf’’ to be the essential infimum. If  $O \subset \mathbb{R}^n$  is open and  $1 \leq q \leq \infty$ , then by  $W^{1,q}(O)$  we denote the space of equivalence classes of functions  $f$  with distributional gradient  $\nabla f = (f_{x_1}, \dots, f_{x_n})$ , and both  $f$  and  $\nabla f$  are  $q$ -th power integrable on  $O$ . Let

$$\|f\|_{W^{1,q}(O)} = \|f\|_{L^q(O)} + \|\nabla f\|_{L^q(O)}$$

be the norm in  $W^{1,q}(O)$ , where  $\|\cdot\|_{L^q(O)}$  denotes the usual Lebesgue  $q$ -norm in  $O$ . We define  $C_0^\infty(O)$  to be the set of infinitely differentiable functions with compact support in  $O$  and we let  $W_0^{1,q}(O)$  denote the closure of  $C_0^\infty(O)$  in the norm  $\|\cdot\|_{W^{1,q}(O)}$ . We define  $W_{\text{loc}}^{1,q}(O)$  in the standard way. By  $\nabla \cdot$  we denote the divergence operator. Given  $t_1 < t_2$  we denote by  $L^q(t_1, t_2, W^{1,q}(O))$  the space of functions such that for almost every  $t$ ,  $t_1 \leq t \leq t_2$ , the function  $x \rightarrow u(x, t)$  belongs to  $W^{1,q}(O)$  and

$$\|u\|_{L^q(t_1, t_2, W^{1,q}(O))} := \left( \int_{t_1}^{t_2} \int_O \left( |u(x, t)|^q + |\nabla u(x, t)|^q \right) dx dt \right)^{1/q} < \infty.$$

The spaces  $L^q(t_1, t_2, W_0^{1,q}(O))$  and  $L_{\text{loc}}^q(t_1, t_2, W_{\text{loc}}^{1,q}(O))$  are defined analogously. Finally, for  $I \subset \mathbb{R}$ , we define  $C(I; L^q(O))$  as the space of functions such that  $t \rightarrow \|u(t, \cdot)\|_{L^q(O)}$  is continuous whenever  $t \in I$ . We define  $C_{\text{loc}}(I; L_{\text{loc}}^q(O))$  analogously.

**2A. Weak solutions.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain, i.e., a connected open set. For  $t_1 < t_2$ , we let  $\Omega_{t_1, t_2} := \Omega \times (t_1, t_2)$ . Given  $p$ ,  $1 < p < \infty$ , we say that  $u$  is a weak solution to

$$\partial_t u - \Delta_p u = 0 \tag{2-1}$$

in  $\Omega_{t_1, t_2}$  if  $u \in L_{\text{loc}}^p(t_1, t_2, W_{\text{loc}}^{1,p}(\Omega))$  and

$$\int_{\Omega_{t_1, t_2}} (-u \partial_t \phi + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi) dx dt = 0 \tag{2-2}$$

whenever  $\phi \in C_0^\infty(\Omega_{t_1, t_2})$ . If  $u$  is a weak solution to (2-1) in the above sense, then we will often refer to  $u$  as being  $p$ -parabolic in  $\Omega_{t_1, t_2}$ . For  $p \in (2, \infty)$  we have by the parabolic regularity theorem, see [DiBenedetto 1993], that any  $p$ -parabolic function  $u$  has a locally Hölder continuous representative. In particular, in the following we will assume that  $p \in (2, \infty)$  and any solution  $u$  is continuous. If (2-2) holds with ‘‘=’’ replaced by ‘‘ $\geq$ ’’ (‘‘ $\leq$ ’’) for all  $\phi \in C_0^\infty(\Omega_{t_1, t_2})$ ,  $\phi \geq 0$ , then we will refer to  $u$  as a weak supersolution (subsolution).

**2B. Geometry.** We here state the geometrical notions used throughout the paper.

**Definition 2.1.** A bounded domain  $\Omega$  is called nontangentially accessible (NTA) if there exist  $M \geq 2$  and  $r_0$  such that the following are fulfilled:

(1) *Corkscrew condition:* for any  $w \in \partial\Omega$ ,  $0 < r < r_0$ , there exists a point  $a_r(w) \in \Omega$  such that

$$M^{-1}r < |a_r(w) - w| < r, \quad d(a_r(w), \partial\Omega) > M^{-1}r.$$

- (2)  $\mathbb{R}^n \setminus \Omega$  satisfies (1).
- (3) *Uniform condition:* if  $w \in \partial\Omega$ ,  $0 < r < r_0$ , and  $w_1, w_2 \in B_r(w) \cap \Omega$ , then there exists a rectifiable curve  $\gamma : [0, 1] \rightarrow \Omega$  with  $\gamma(0) = w_1$ ,  $\gamma(1) = w_2$ , such that
- (a)  $H^1(\gamma) \leq M|w_1 - w_2|$ ,
  - (b)  $\min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq M d(\gamma(t), \partial\Omega)$  for all  $t \in [0, 1]$ .

We choose this definition as it very useful when we explicitly construct the parabolic Harnack chains in [Section 3](#); see specifically [Theorem 3.5](#). The values  $M$  and  $r_0$  will be called the NTA constants of  $\Omega$ . For more on the notion of NTA domains we refer to [\[Jerison and Kenig 1982\]](#).

**Definition 2.2.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. We say that  $\Omega$  satisfies the ball condition with radius  $r_0 > 0$  if for each point  $y \in \partial\Omega$  there exist points  $x^+ \in \Omega$  and  $x^- \in \Omega^c$  such that  $B_{r_0}(x^+) \subset \Omega$ ,  $B_{r_0}(x^-) \subset \Omega^c$ ,  $\partial B_{r_0}(x^+) \cap \partial\Omega = \{y\} = \partial B_{r_0}(x^-) \cap \partial\Omega$ , and such that the points  $x^+(y)$ ,  $x^-(y)$ ,  $y$  are collinear for each  $y \in \partial\Omega$ .

**Remark 2.3.** It is easy to see that a domain satisfying the ball condition with radius  $r_0 > 0$  is an NTA domain with a constant  $M$  and  $r_0$ . In particular, we may canonically choose

$$a_r(x_0) := x_0 + \frac{r}{2} \frac{x^+ - x_0}{|x^+ - x_0|},$$

since the direction given by  $(x^+ - x_0)/|x^+ - x_0|$  is unique. The exterior corkscrew point is defined analogously.

**Remark 2.4.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Then  $\Omega$  is a  $C^{1,1}$  domain if and only if it satisfies the ball condition. For a proof of this fact, see for example [\[Aikawa et al. 2007, Lemma 2.2\]](#).

**2C. The continuous Dirichlet problem.** Assuming that  $\Omega$  is a bounded NTA domain one can prove, see [\[Björn et al. 2015; Kilpeläinen and Lindqvist 1996\]](#), that all points on the parabolic boundary

$$\partial_p \Omega_T = S_T \cup (\bar{\Omega} \times \{0\}), \quad S_T = \partial\Omega \times [0, T],$$

of the cylinder  $\Omega_T$  are regular for the Dirichlet problem for (2-1). In particular, for any  $f \in C(\partial_p \Omega_T)$ , there exists a unique Perron solution  $u = u_f^{\Omega_T} \in C(\bar{\Omega}_T)$  to the Dirichlet problem  $\partial_t u - \Delta_p u = 0$  in  $\Omega_T$  and  $u = f$  on  $\partial_p \Omega_T$ .

### 3. Harnack chains

In this section we prove a sequence of results concerning intrinsic Harnack chains. Forward-in-time chains describe the diffusion with an appropriate waiting time. On the other hand, backward-in-time chains say that if the solution has existed for a long enough time, the future values will control the values from the past as well. Throughout the section we let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and given  $T > 0$  we let  $\Omega_T = \Omega \times (0, T)$ .

**3A. Local Harnack inequalities.** We here collect two estimates from the literature. The following theorem can be found in [DiBenedetto 1993; DiBenedetto et al. 2008; 2012].

**Theorem 3.1.** *Let  $u$  be a nonnegative  $p$ -parabolic function in  $\Omega_T$ , let  $(x_0, t_0) \in \Omega_T$  and assume that  $u(x_0, t_0) > 0$ . There exist positive constants  $c_h$  and  $C_h$ , depending only on  $p, n$ , such that if  $B_{4r}(x_0) \subset \Omega$  and*

$$(t_0 - \theta(4r)^p, t_0 + \theta(4r)^p) \subset (0, T],$$

where  $\theta = (c_h/u(x_0, t_0))^{p-2}$ , then

$$u(x_0, t_0) \leq C_h \inf_{B_r(x_0)} u(\cdot, t_0 + \theta r^p).$$

The constants  $c_h$  and  $C_h$  are stable as  $p \rightarrow 2$  and deteriorate as  $p \rightarrow \infty$  in the sense that  $c_h(p), C_h(p) \rightarrow \infty$  as  $p \rightarrow \infty$ .

The next theorem is instead valid for nonnegative weak supersolutions. See [Kuusi 2008] for the proof.

**Theorem 3.2.** *Let  $u$  be a nonnegative weak supersolution in  $B_{4r}(x_0) \times (0, T)$ . There exist constants  $C_i \equiv C_i(p, n)$ ,  $i = 1, 2$ , such that*

$$\int_{B_r(x_0)} u(x, t_1) \, dx \leq \frac{1}{2} \left( \frac{C_1 r^p}{T - t_1} \right)^{1/(p-2)} + C_2 \inf_Q u$$

for almost every  $0 < t_1 < T$ , where  $Q := B_{2r}(x_0) \times (t_1 + T_1/2, t_1 + T_1)$ , and

$$T_1 = \min \left\{ T - t_1, C_1 r^p \left( \int_{B_r(x_0)} u(x, t_1) \, dx \right)^{2-p} \right\}.$$

In particular, if  $T_1 < T - t_1$ , then

$$\int_{B_r(x_0)} u(x, t_1) \, dx \leq 2C_2 \inf_Q u.$$

**3B. Forward Harnack chains.** We begin by describing a simple Harnack chain for weak supersolutions.

**Lemma 3.3** (weak forward Harnack chains). *Let  $\Omega \subset \mathbb{R}^n$  be a domain and let  $T > 0$ . Let  $x, y$  be two points in  $\Omega$  and assume that there exist a sequence of balls  $\{B_{4r}(x_j)\}_{j=0}^k$  such that  $x_0 = x$ ,  $x_k = y$ ,  $B_{4r}(x_j) \subset \Omega$  for all  $j = 0, \dots, k$  and  $x_{j+1} \in B_r(x_j)$ ,  $j = 0, \dots, k-1$ . Assume that  $u$  is a continuous nonnegative weak supersolution in  $\Omega_T$  with*

$$\bar{\Lambda} := \int_{B_r(x_0)} u(x, t_0) \, dx > 0.$$

There exist constants  $\bar{c}_i \equiv \bar{c}_i(p, n) > 1$ ,  $i \in \{1, 2\}$ , such that if

$$t_0 + \tau_k \bar{\Lambda}^{2-p} r^p < T, \quad \tau_k := \bar{c}_1 \sum_{j=0}^k \bar{c}_2^j (p-2),$$

then

$$\int_{B_r(x)} u(x, t_0) \, dx \leq \bar{c}_2^{k+1} \inf_{z \in B_{2r}(y)} u(z, t_0 + \tau_k \bar{\Lambda}^{2-p} r^p).$$

Furthermore, the constants  $\bar{c}_i$ ,  $i \in \{1, 2\}$ , are stable as  $p \rightarrow 2^+$ . In particular, when  $p = 2$ , we have  $\tau_k = \bar{c}_1(k+1)$  with  $\bar{c}_1 = \bar{c}_1(n)$ .



*Proof.* Using [Theorem 3.2](#) we first get

$$\frac{\bar{\Lambda}}{2C_2} \leq \inf_{z \in B_{2r}(x_0)} u(z, t_1), \quad t_1 := t_0 + C_1 \bar{\Lambda}^{2-p} r^p. \quad (3-1)$$

Define then

$$u_j := \min(u, \Lambda_j), \quad \Lambda_j := (2C_2)^{-j} \bar{\Lambda}, \quad t_{j+1} := t_j + C_1 \Lambda_j^{2-p} r^p$$

for  $j = \{1, \dots, k-1\}$ . Assume inductively that for  $t_{i+1} \leq T$  we have

$$B_r(x_i) \subset B_{2r}(x_{i-1})$$

and

$$u_i(z, t_i) = \Lambda_i \quad \text{for } z \in B_r(x_i)$$

hold for  $i \in \{0, \dots, j\}$ . For  $j = 1$  this is certainly the case as we see from (3-1). Since  $u_j$  is a nonnegative weak supersolution, [Theorem 3.2](#) gives us

$$\Lambda_{j+1} = \frac{\Lambda_j}{2C_2} \leq \inf_{z \in B_{2r}(x_j)} u_j(z, t_{j+1}),$$

and hence also

$$u_{j+1}(z, t_{j+1}) = \Lambda_{j+1} \quad \text{for } z \in B_r(x_{j+1}).$$

This proves the induction step. By the construction,

$$\inf_{z \in B_r(y)} u_k(z, t_k) = \Lambda_k$$

holds. Thus, applying [Theorem 3.2](#) one more time we get

$$\inf_{z \in B_{2r}(y)} u(z, \bar{t}) \geq (2C_2)^{-(k+1)} \bar{\Lambda},$$

with

$$\bar{t} := t_0 + C_1 \sum_{j=0}^k (2C_2)^{j(p-2)} \bar{\Lambda}^{2-p} r^p.$$

Setting  $\bar{c}_1 = C_1$  and  $\bar{c}_2 = 2C_2$  completes the proof of the lemma.  $\square$

For  $p$ -parabolic functions we have the following pointwise version of [Lemma 3.3](#).

**Proposition 3.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain and let  $T > 0$ . Let  $x, y$  be two points in  $\Omega$  and assume that there exists a sequence of balls  $\{B_{4r}(x_j)\}_{j=0}^k$  such that  $x_0 = x$ ,  $x_k = y$ ,  $B_{4r}(x_j) \subset \Omega$  for all  $j = 0, \dots, k$ , and  $x_{j+1} \in B_r(x_j)$ ,  $j = 0, \dots, k-1$ . Assume that  $u$  is a nonnegative  $p$ -parabolic function in  $\Omega_T$  and assume that  $u(x, t_0) > 0$ . There exist constants  $c \equiv c(p, n)$  and  $c_1 \equiv c_1(p, n, k) > 1$  such that if*

$$t_0 - \left( \frac{c_h}{u(x, t_0)} \right)^{p-2} (4r)^p > 0, \quad t_0 + c_1(k)u(x, t_0)^{2-p} r^p < T,$$

then

$$u(x, t_0) \leq c^k \inf_{z \in B_r(y)} u(z, t_0 + c_1(k)u(x, t_0)^{2-p} r^p).$$

Furthermore,  $c_1$  satisfies the estimate

$$\tilde{c}_1 k \leq c_1 \leq \tilde{c}_1 (k+1) c^{(k+1)(p-2)},$$

with  $\tilde{c}_1 \equiv \tilde{c}_1(p, n)$  and  $\tilde{c}_1(p, n) \rightarrow \tilde{c}_1(n)$  as  $p \rightarrow 2$ .

*Proof.* After applying [Theorem 3.1](#) once, the result follows from [Lemma 3.3](#).  $\square$

We next focus on cylindrical NTA domains. The first theorem, [Theorem 3.5](#), holds for weak supersolutions and shows how to bound the values of a solution at points close to the boundary using pointwise interior values. A remarkable fact of the proof is that the waiting time is explicitly defined and, as  $p \rightarrow 2$ , it gives a supersolution version of [[Salsa 1981](#), Theorem C], as alluded to in the [Introduction](#); see (1-5). The proof uses heavily the assumptions on NTA domains and iterations of [Lemma 3.3](#).

**Theorem 3.5.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ , let  $x_0 \in \partial\Omega$ ,  $T > 0$  and  $0 < r < r_0$ . Let  $x, y$  be two points in  $\Omega \cap B_r(x_0)$  such that*

$$\varrho := d(x, \partial\Omega) \leq r \quad \text{and} \quad d(y, \partial\Omega) \geq \frac{r}{4}.$$

*Assume that  $u$  is a nonnegative continuous weak supersolution in  $\Omega_T$ , and assume that*

$$\Lambda := \int_{B_{\varrho/4}(x)} u(z, t_0) \, dz > 0. \quad (3-2)$$

*Let  $\delta \in (0, 1]$ . Then there exist positive constants  $c_i \equiv c_i(M, p, n)$ ,  $i \in \{1, 2, 3\}$ , such that if  $t_0 + \tau < T$ , where*

$$\tau := \delta^{p-1} \left( c_2^{-1/\delta} \left( \frac{r}{\varrho} \right)^{-c_3/\delta} \Lambda \right)^{2-p} r^p,$$

*then*

$$\int_{B_{\varrho/4}(x)} u(z, t_0) \, dz \leq c_1^{1/\delta} \left( \frac{r}{\varrho} \right)^{c_3/\delta} \inf_{z \in B_{r/16}(y)} u(z, t_0 + \tau).$$

*Furthermore, the constants  $c_i$ ,  $i \in \{1, 2, 3\}$ , are stable as  $p \rightarrow 2^+$ .*

*Proof.* We split the proof into three steps.

*Step 1: parametrization of the curve connecting  $x$  and  $y$ .* According to the uniform condition (3) in [Definition 2.1](#), we can find a rectifiable curve  $\gamma$  connecting  $x$  and  $y$  such that  $\gamma(0) = x$ ,  $\gamma(1) = y$ , and

- (a)  $H^1(\gamma) \leq M |w_1 - w_2|$ ,
- (b)  $\min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq M d(\gamma(t), \partial\Omega)$  for all  $t \in [0, 1]$ .

We call a ball  $B \subset \Omega$  *admissible* if  $4B \subset \Omega$  and is thus eligible for the Harnack inequality. Our goal in this step is to construct a sequence of admissible balls covering the curve  $\gamma$ . In the following we may, without loss of generality, assume that  $H^1(\gamma([0, 1])) > 2^{-4}r$ . We define  $\hat{t}_1, \hat{t}_2 \in (0, 1)$  such that  $H^1(\gamma([0, \hat{t}_1])) = 2^{-5}r$  and  $H^1(\gamma([\hat{t}_2, 1])) = 2^{-5}r$ . The technical part will be in the interval  $(0, \hat{t}_1)$ . To

continue we choose  $k$  as the integer which satisfies  $2^{-k}r \in (\varrho/16, \varrho/8]$ . We define sequence of real numbers  $\{s_j\}$  through

$$H^1(\gamma([0, s_j])) = 2^{-k+j} H^1(\gamma([0, \hat{t}_1])) := 2^{-k-5+j}r, \quad s_0 = 0.$$

Then, for any  $s \in [s_j, s_{j+1})$ , (a) and (b) imply

$$d(\gamma(s), \partial\Omega) \geq \frac{2^{-k+j-5}r}{M}$$

and

$$H^1(\gamma([s_j, s_{j+1}])) = 2^{-k+j-5}r.$$

Thus, defining

$$\varrho_j := N^{-1}2^{-k+j-5}r, \quad N \in \mathbb{N}, \quad N \geq 2^9M,$$

we see that the piece  $\gamma([s_j, s_{j+1}])$  can be covered with  $N$  admissible balls of the type  $B^{i,j} := B_{\varrho_j}(y_{i,j})$  such that  $y_{i,j} \in \gamma([s_j, s_{j+1}])$  for  $i \in \{1, \dots, N\}$ ,  $y_{i,j-1} \in B^{i,j}$  and  $\gamma([s_j, s_{j+1}]) \subset \cup_i B^{i,j}$ . Finally, we observe that the middle piece of the curve  $\gamma([\hat{t}_1, \hat{t}_2])$ , due to the definitions of  $\hat{t}_1, \hat{t}_2$  together with (b) can be covered with  $MN$  admissible balls of size  $r/N$ . Moreover we can cover the end piece  $\gamma([\hat{t}_2, 1])$  with  $N$  admissible balls of size  $r/N$  since  $\gamma([\hat{t}_2, 1]) \subset B_{r/16}(y) \subset \Omega$ . At this point, we consider  $N \in \mathbb{N}$  to be a free parameter such that  $N \geq 2^9M$ .

*Step 2: iteration via Harnack estimates.* Let now  $\Lambda$  be as in (3-2). Theorem 3.2 implies that if

$$t_0 + C_1\Lambda^{2-p}\varrho_0^p < T,$$

then

$$\inf_{z \in B_{\varrho_0}(x_0)} u(z, t_1) \geq \frac{1}{2C_2}\Lambda, \quad t_1 := t_0 + C_1\Lambda^{2-p}\varrho_0^p.$$

Let

$$\Lambda_1 = \sigma\Lambda, \quad \sigma \in (0, (2C_2)^{-1}).$$

Defining thus  $u_1 := \min(u, \Lambda_1)$ , we obtain by Lemma 3.3 (see also its proof) that there exist constants  $\bar{c}_1 \equiv \bar{c}_1(p, n)$  and  $\bar{c}_2 \equiv \bar{c}_2(p, n)$ , such that if

$$t_2 := t_1 + \tau_N\Lambda_1^{2-p}\varrho_1^p < T, \quad \tau_N := \bar{c}_1 \sum_{j=0}^{N-1} \bar{c}_2^{j(p-2)} \in [\bar{c}_1N, \bar{c}_1N\bar{c}_2^{N(p-2)}), \quad (3-3)$$

then

$$\inf_{z \in B_{\varrho_2}(\gamma(s_2))} u(z, t_2) = \inf_{z \in B_{2\varrho_1}(\gamma(s_2))} u(z, t_2) \geq \frac{\Lambda_1}{\bar{c}_2^N} =: \Lambda_2.$$

Define

$$\Lambda_{j+1} := \bar{c}_2^{-(j+1)N} \Lambda_1, \quad u_j := \min(u, \Lambda_j), \quad j \in \mathbb{N},$$

let  $\hat{k} := k + M + 1$ , and let

$$t_{j+1} := \begin{cases} t_j + \tau_N\Lambda_j^{2-p}\varrho_j^p & \text{if } j \in \{1, \dots, k\}, \\ t_j + \tau_N\Lambda_j^{2-p}(r/N)^p & \text{if } j \in \{k+1, \dots, \hat{k}\}. \end{cases}$$

Iterating [Lemma 3.3](#) it follows by induction that

$$u(y, t_{\hat{k}+1}) \geq \Lambda_{\hat{k}+1}. \quad (3-4)$$

*Step 3: waiting time.* Let us now analyze the waiting time  $t_{\hat{k}+1}$ , which we want to show to be precisely  $t_0 + \tau$  by a suitable choice of  $\Lambda_1$ . We have

$$\begin{aligned} t_{\hat{k}+1} &= t_1 + C_1 \Lambda^{2-p} \varrho_0^p + \tau_N \left[ \sum_{j=1}^k \Lambda_j^{2-p} \varrho_j^p + \sum_{j=k+1}^{\hat{k}} \Lambda_j^{2-p} \varrho_j^p \right] \\ &= t_0 + C_1 \Lambda^{2-p} \varrho_0^p + N^{-p} \tau_N \Lambda_1^{2-p} r^p \left[ 2^{-kp} \sum_{j=0}^k (2^p \bar{c}_2^{(p-2)N})^j \right. \\ &\quad \left. + \bar{c}_2^{(p-2)kN} \sum_{j=0}^M (\bar{c}_2^{(p-2)N})^j \right] \end{aligned} \quad (3-5)$$

$$=: t_0 + \mathbf{T} \Lambda^{2-p} r^p. \quad (3-6)$$

We can write the sum in (3-5) as

$$2^{-kp} \sum_{j=0}^k (2^p \bar{c}_2^{(p-2)N})^j = 2^p \frac{\bar{c}_2^{(p-2)(k+1)N} - 2^{-(k+1)p}}{2^p \bar{c}_2^{(p-2)N} - 1},$$

while the sum in (3-6) can be estimated similarly to  $\tau_N$ , see (3-3):

$$\bar{c}_2^{(p-2)kN} \sum_{j=0}^M (\bar{c}_2^{(p-2)N})^j \in (M \bar{c}_2^{(p-2)kN}, M \bar{c}_2^{(p-2)(k+M)N}].$$

Hence, recalling the definitions of  $\tau_N$  and  $\mathbf{T}$ , we get after some straightforward estimation that

$$\frac{1}{2} \bar{c}_1 \sigma^{2-p} N^{1-p} \left( \frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 N} \leq \mathbf{T} \leq 2 \bar{c}_3 \sigma^{2-p} N^{1-p} \bar{c}_3^{(p-2)N} \left( \frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 N} \quad (3-7)$$

for new constants  $\bar{c}_3, \bar{c}_4$  depending only on  $p, n, M$ . We now choose  $N = \tilde{c}/\delta$  and let  $\tilde{c}$  be a degree of freedom. First note that choosing  $\sigma_1 = (2C_2)^{-1}$ , then choosing  $c_3 = \tilde{c}c_4$  and  $c_2 = [\bar{c}_3]^{\tilde{c}}$ , for a large enough  $\tilde{c} = \tilde{c}(p, n, M)$  we have

$$2 \bar{c}_3 (2C_2)^{p-2} \tilde{c}^{1-p} \bar{c}_3^{(p-2)\tilde{c}/\delta} \left( \frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 \tilde{c}/\delta} < c_2^{(p-2)/\delta} \left( \frac{r}{\varrho} \right)^{(p-2)c_3/\delta}. \quad (3-8)$$

Second we see that choosing  $\sigma_2 = c_5^{-1/\delta}$  for large enough  $c_5 = c_5(p, n, M)$  the following holds:

$$\frac{1}{2} \bar{c}_1 \sigma_2^{2-p} \tilde{c}^{1-p} \left( \frac{r}{\varrho} \right)^{(p-2)\bar{c}_4 \tilde{c}/\delta} > c_2^{(p-2)/\delta} \left( \frac{r}{\varrho} \right)^{(p-2)c_3/\delta}. \quad (3-9)$$

With (3-8) and (3-9) and (3-7) at hand we see that there is a choice of  $\sigma \in [\sigma_2, \sigma_1]$  such that

$$\mathbf{T} = \delta^{p-1} c_2^{(p-2)/\delta} \left( \frac{r}{\varrho} \right)^{(p-2)c_3/\delta},$$

and thus we have proved  $t_{\hat{k}+1} = t_0 + \mathbf{T} \Lambda^{2-p} r^p = t_0 + \tau$ . This together with (3-4) finishes the proof by taking suitably large  $c_1$  in the statement.  $\square$

For  $p$ -parabolic functions we have the following pointwise version of [Theorem 3.5](#).

**Theorem 3.6.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ , let  $x_0 \in \partial\Omega$ ,  $T > 0$  and  $0 < r < r_0$ . Let  $x, y$  be two points in  $\Omega \cap B_r(x_0)$  such that*

$$\varrho := d(x, \partial\Omega) \leq r \quad \text{and} \quad d(y, \partial\Omega) \geq \frac{r}{4}.$$

*Assume that  $u$  is a nonnegative  $p$ -parabolic function in  $\Omega_T$ , and assume that  $u(x, t_0)$  is positive. Let  $\delta \in (0, 1]$ . Then there exist constants  $c_i \equiv c_i(M, p, n)$ ,  $i \in \{1, 2, 3\}$ , such that if*

$$t_0 - (c_h/u(x, t_0))^{p-2}(\delta\varrho)^p > 0, \quad t_0 + \tau < T,$$

*with*

$$\tau := \delta^{p-1} \left( c_2^{-1/\delta} \left( \frac{r}{\varrho} \right)^{-c_3/\delta} u(x, t_0) \right)^{2-p} r^p,$$

*then*

$$u(x, t_0) \leq c_1^{1/\delta} \left( \frac{r}{\varrho} \right)^{c_3/\delta} \inf_{z \in B_{r/16}(y)} u(z, t_0 + \tau).$$

*Furthermore, the constants  $c_i$ ,  $i \in \{1, 2, 3\}$ , are stable as  $p \rightarrow 2^+$ .*

*Proof.* Applying [Theorem 3.1](#) once, we see that the theorem follows from [Theorem 3.5](#).  $\square$

**3C. Backward Harnack chains.** The philosophy of the forward Harnack chains in [Section 3B](#) is that the data at the starting point will start to diffuse according to the intrinsic Harnack inequality. The finite speed of diffusion forces the waiting time to blow up if we wish to spread our information in an infinite chain. In the backward Harnack chains that we develop in this section the philosophy is reversed. Instead of looking to the future we look to the past. This means that if the value of the solution at a point  $(y, s)$  is, say, 1, then we ask the question: how large can the values in the past be without violating the fact that the solution is 1 at  $(y, s)$ .

We start with the weak version of the backward Harnack chains, valid for weak supersolutions.

**Theorem 3.7.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ , let  $x_0 \in \partial\Omega$ ,  $T > 0$ , and  $0 < r < r_0$ . Let  $x, y$  be two points in  $\Omega \cap B_r(x_0)$  such that*

$$\varrho := d(x, \partial\Omega) \leq r \quad \text{and} \quad d(y, \partial\Omega) \geq \frac{r}{4}.$$

*Assume that  $u$  is a nonnegative continuous weak supersolution in  $\Omega_T$ , and assume that  $u(y, s)$  is positive. Let  $\delta \in (0, 1]$ . Then there exist positive constants  $C_i \equiv C_i(p, n)$  and  $c_i \equiv c_i(p, n, M)$ ,  $i \in \{4, 5\}$ , such that if  $s \in (\tau, T)$  and*

$$t \in [s - \tau, s - \delta^{p-1}\tau],$$

*with*

$$\tau := C_4 [C_5 u(y, s)]^{2-p} r^p$$

*then*

$$\int_{B_{\varrho/4}(x)} u(z, t) \, dz \leq c_4^{1/\delta} \left( \frac{r}{\varrho} \right)^{c_5/\delta} u(y, s). \quad (3-10)$$

*Furthermore, the constants  $c_i, C_i$ ,  $i \in \{4, 5\}$ , are stable as  $p \rightarrow 2^+$ .*

**Remark 3.8.** If we assume that  $u \in C([0, T]; L^2(\Omega))$  then we can replace  $s \in (\tau, T)$  with  $s \in [\tau, T)$ . That is, the chain can be taken all the way to the initial time, i.e.,  $t = 0$ .

*Proof.* After scaling, we may assume that  $u(y, s) = 1$ . Assume now the contrary to (3-10), i.e.,

$$\int_{B_{\varrho/4}(x)} u(z, t) \, dz > H \left( \frac{r}{\varrho} \right)^{c_5/\delta} \quad (3-11)$$

for constants  $c_5, H$  to be fixed. Then [Theorem 3.5](#) implies that with

$$\tilde{\tau} := \tilde{\delta}^{p-1} \left( c_2^{-1/\tilde{\delta}} \left( \frac{r}{\varrho} \right)^{-c_3/\tilde{\delta}} \int_{B_{\varrho/4}(x)} u(z, t) \, dz \right)^{2-p} r^p$$

we get

$$\int_{B_{\varrho/4}(x)} u(z, t) \, dz \leq c_1^{1/\tilde{\delta}} \left( \frac{r}{\varrho} \right)^{c_3/\tilde{\delta}} \inf_{z \in B_{r/16}(y)} u(z, t + \tilde{\tau}) \quad (3-12)$$

with constants  $c_i \equiv c_i(p, n, M)$ ,  $i \in \{1, 2, 3\}$ , and  $\tilde{\delta} \in (0, \delta]$ . Now we have an upper bound for  $\tilde{\tau}$  by means of (3-11) as follows:

$$\tilde{\tau} \leq \tilde{\delta}^{p-1} \left( c_2^{-1/\tilde{\delta}} \left( \frac{r}{\varrho} \right)^{-c_3/\tilde{\delta}} H \left( \frac{r}{\varrho} \right)^{c_5/\delta} \right)^{2-p} r^p = \tilde{\delta}^{p-1} (c_2^{-1/\tilde{\delta}} H)^{2-p} r^p \leq \delta^{p-1} \tau,$$

provided that

$$H \geq C_5 c_2^{1/\tilde{\delta}}, \quad \tilde{\delta} := \delta \min\{1, C_4\}^{1/(p-1)}, \quad c_5 := \frac{c_3}{\min\{1, C_4\}^{1/(p-1)}}. \quad (3-13)$$

Therefore we have

$$t + \tilde{\tau} \leq s.$$

Observe that both  $C_4$  and  $C_5$  are still to be fixed. Thus we need to carry the information from the time  $t + \tilde{\tau}$  up to  $s$ . To this end, connecting (3-11) and (3-12) with the choices in (3-13) leads to

$$H c_1^{-1/\tilde{\delta}} < \inf_{z \in B_{r/16}(y)} u(z, t + \tilde{\tau}). \quad (3-14)$$

Truncate  $u$  as

$$\tilde{u} = \min(4C_2, u),$$

and take

$$H := c_4^{1/\delta}, \quad c_4 := \max\{4C_2 c_1, C_5 c_2\}^{1/\min\{1, C_4\}^{1/(p-1)}}, \quad (3-15)$$

where  $C_2$  is as in [Theorem 3.2](#). Then  $\tilde{u}$  is a continuous weak supersolution, and we have by (3-13) and (3-14) that

$$\int_{B_{\tilde{r}}(y)} \tilde{u}(z, t + \tilde{\tau}) \, dz = 4C_2, \quad \tilde{r} \in \left(0, \frac{r}{16}\right].$$

Applying thus the forward-in-time weak Harnack estimate in [Theorem 3.2](#) gives

$$4C_2 \leq 2C_2 \inf_{z \in B_{2\tilde{r}}(y)} \tilde{u}(z, t + \tilde{\tau} + C_1(4C_2)^{2-p} \tilde{r}^p),$$

provided that

$$t + \tilde{\tau} + C_1(4C_2)^{2-p}\tilde{r}^p < T.$$

Choosing  $C_4 = 16^{-p}C_1$  and  $C_5 = 4C_2$ , we can always find  $\tilde{r} \leq r/16$  such that

$$t + \tilde{\tau} + C_1(4C_2)^{2-p}\tilde{r}^p = s,$$

and hence

$$2 \leq \inf_{z \in B_{2\tilde{r}}(y)} \tilde{u}(z, s).$$

This gives a contradiction since we assumed that  $u(y, s) = 1$ , and thus the proof is complete.  $\square$

The following theorem is the corresponding result for weak solutions, where we use pointwise information in the past instead of information in mean. The main difference this imposes on the assumptions in [Theorem 3.7](#) is that we have to require the solution to have lived for a certain amount of time, which is precisely the price we have to pay if we wish to control pointwise values in the past.

**Theorem 3.9.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ , let  $x_0 \in \partial\Omega$ ,  $T > 0$ , and  $0 < r < r_0$ . Let  $x, y$  be two points in  $\Omega \cap B_r(x_0)$  such that*

$$\varrho := d(x, \partial\Omega) \leq r \quad \text{and} \quad d(y, \partial\Omega) \geq \frac{r}{4}.$$

*Assume that  $u$  is a nonnegative  $p$ -parabolic function in  $\Omega_T$ , and assume that  $u(y, s)$  is positive. Let  $\delta \in (0, 1]$ . Then there exist positive constants  $C_i \equiv C_i(p, n)$  and  $c_i \equiv c_i(p, n, M)$ ,  $i \in \{4, 5\}$ , such that if  $s < T$  and*

$$\max \left\{ \left( \frac{c_4^{1/\delta}}{c_h} \left( \frac{r}{\varrho} \right)^{c_5/\delta} u(y, s) \right)^{2-p} (\delta\varrho)^p, s - \tau \right\} \leq t \leq s - \delta^{p-1}\tau, \quad (3-16)$$

with

$$\tau := C_4[C_5 u(y, s)]^{2-p} r^p,$$

then

$$u(x, t) \leq c_4^{1/\delta} \left( \frac{r}{\varrho} \right)^{c_5/\delta} u(y, s).$$

Furthermore, the constants  $c_i, C_i$ ,  $i \in \{4, 5\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* To prove the lemma we follow the same outline as the proof of [Theorem 3.7](#), but instead of assuming (3-11) we assume the contrary assumption

$$u(x, t) > H \left( \frac{r}{\varrho} \right)^{c_5/\delta}$$

for some constants  $c_5, H$  to be fixed. Applying [Theorem 3.6](#) instead of [Theorem 3.5](#) we get

$$u(x, t) \leq c_1^{1/\tilde{\delta}} \left( \frac{r}{\varrho} \right)^{c_3/\tilde{\delta}} \inf_{z \in B_{r/16}(y)} u(z, t + \tilde{\tau}).$$

Note that it is the usage of [Theorem 3.6](#) which requires (3-16). The proof now follows repeating the remaining part of the proof of [Theorem 3.7](#) essentially verbatim.  $\square$

#### 4. Carleson estimate

In this section we prove, using the improved Harnack chain estimate in [Theorem 3.9](#), a flexible Carleson estimate valid in cylindrical NTA domains. Versions of the Carleson estimate were originally proved, for equations of  $p$ -parabolic type, in [\[Avelin et al. 2016\]](#) in Lipschitz cylinders, and in [\[Avelin 2016\]](#) in certain time-dependent space-time domains. We begin with a standard oscillation decay lemma valid for weak subsolutions; see for example [\[DiBenedetto 1993\]](#).

**Lemma 4.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ . Let  $u$  be a nonnegative, continuous weak subsolution in  $\Omega_T$ . Let  $(x_0, t_0) \in S_T$ ,  $0 < r < r_0$ ,*

$$Q_r^\lambda(x_0, t_0) := (B_r(x_0) \cap \Omega) \times (t_0 - \lambda^{2-p} r^p, t_0) \subset \Omega_T,$$

and assume that  $u$  vanishes continuously on  $S_T \cap Q_r^\lambda(x_0, t_0)$  and that

$$\sup_{Q_r^\lambda(x_0, t_0)} u \leq \lambda.$$

Then there is a constant  $\sigma$  depending only on  $p, n, M$  such that  $Q_{\sigma r}^{\lambda/2}(x_0, t_0) \subset Q_r^\lambda(x_0, t_0)$  and

$$\sup_{Q_{\sigma r}^{\lambda/2}(x_0, t_0)} u \leq \frac{\lambda}{2}.$$

In particular, we have

$$\sup_{Q_{\sigma^j r}^{2^{-j}\lambda}(x_0, t_0)} u \leq 2^{-j}\lambda$$

for any  $j \in \mathbb{N}$ .

The following theorem is usually referred to as a Carleson estimate. We want to point out that, compared to [\[Avelin et al. 2016\]](#), not only does it hold for cylindrical NTA domains, but also the formulation is more flexible for applications. In particular, we are able to adjust the waiting time, the height of the cylinder, and the distance to the initial boundary. All these parameters influence the constant in the inequality and a Gaussian-type behavior is proved to be present.

**Theorem 4.2.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ . Let  $u$  be a nonnegative, weak solution in  $\Omega_T$ . Let  $(x, t) \in S_T$  and  $0 < r < r_0$ . Assume that  $u(a_r(x), t) > 0$  and let*

$$\tau = \frac{C_4}{4} [C_5 u(a_r(x), t)]^{2-p} r^p,$$

where  $C_4$  and  $C_5$ , both depending on  $p, n$ , are as in [Theorem 3.9](#). Assume that  $t > (\delta_1^{p-1} + \delta_2^{p-1} + 2\delta_3^{p-1})\tau$  for  $0 < \delta_1 \leq \delta_3 \leq 1$ ,  $\delta_2 \in (0, 1)$ , and that for a given  $\lambda \geq 0$ , the function  $(u - \lambda)_+$  vanishes continuously on  $S_T \cap B_r(x) \times (t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau, t - \delta_1^{p-1}\tau)$  from  $\Omega_T$ . Then there exist constants  $c_i \equiv c_i(M, p, n)$ ,  $i \in \{6, 7\}$ , such that

$$\sup_Q u \leq \left( \frac{c_6}{\delta_3} \right)^{c_7/\delta_1} u(a_r(x), t) + \lambda,$$

where  $Q := B_r(x) \times (t - (\delta_1^{p-2} + \delta_2^{p-1})\tau, t - \delta_1^{p-1}\tau)$ . Furthermore, the constants  $c_i$ ,  $i \in \{6, 7\}$ , are stable as  $p \rightarrow 2^+$ .



**Remark 4.3.** Note that in the case  $p = 2$ , [Theorem 4.2](#) for  $\lambda = 0$  is equivalent to the estimate above by linearity. However, for  $p > 2$  this result extends the ones in [[Avelin 2016](#); [Avelin et al. 2016](#)] if  $\lambda > 0$ .

*Proof.* By scaling the function  $u$  we can assume that  $u(a_r(x), t) = 1$ , and replacing  $\lambda$  with its scaled version. Consider the boxes

$$\tilde{Q}_\nu := (\Omega \cap B_r(x)) \times (t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau, t - \nu^{p-1}\tau), \quad \nu \in \{\delta_1, \delta_3\},$$

and define also

$$Q_r^\theta(x, t) := (\Omega \cap B_r(x)) \times (t - \theta^{2-p}r^p, t), \quad \theta > 0.$$

Observe that with the choice of  $\tau$ , we have by [Theorem 3.9](#), for any  $(y, s) \in \tilde{Q}_\nu$  with  $d(y, \partial\Omega) \leq r$  and  $\nu \in \{\delta_1, \delta_3\}$ , that

$$u(y, s) \leq c_8^{1/\nu} \left( \frac{r}{d(y, \partial\Omega)} \right)^{c_9/\nu} \quad (4-1)$$

holds with  $c_8, c_9$  depending only on  $p, n, M$  (apply [Theorem 3.9](#) with the choice  $\delta := \nu \min\{1, C_4/2\}$  in order to guarantee the conditions in (3-16)).

We proceed by induction via a contradiction assumption. Assume that  $P_0 = (x_0, t_0) \in Q$ , where  $Q$  is as in the statement, is such that  $u(P_0) > H + \lambda$  for some large  $H$  to be fixed. Assume then that we find inductively points  $P_j = (x_j, t_j) \in \tilde{Q}_{\hat{\delta}_j}$ , where  $\hat{\delta}_j = \delta_3$  if  $t_j \leq t_0 - \delta_3^{p-1}\tau/2$  and  $\hat{\delta}_j = \delta_1$  if  $t_j > t_0 - \delta_3^{p-1}\tau/2$  for any  $j \in \mathbb{N}$ . Set  $r_j := d(x_j, \partial\Omega)$  and let  $x'_j \in \partial\Omega$  be such that  $r_j = |x_j - x'_j|$ . Assume inductively that

$$u(P_j) > 2^j H + \lambda \quad \text{and} \quad t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau < t_j \leq t_{j-1} \leq t_j + (2^j H)^{2-p} (r_{j-1}/\sigma)^p \quad (4-2)$$

hold for all  $j \in \{1, \dots, k\}$ , where  $\sigma \equiv \sigma(p, n, M) \in (0, 1)$  is as in [Lemma 4.1](#). We then want to show that for large enough  $H$  this continues to hold for  $j = k + 1$  as well.

To show the induction step, observe that (4-1) and the induction (4-2) imply

$$2^j H + \lambda < u(P_j) \leq c_8^{1/\hat{\delta}_j} \left( \frac{r}{r_j} \right)^{c_9/\hat{\delta}_j} \implies r_j \leq (2^j H c_8^{-1/\hat{\delta}_j})^{-\hat{\delta}_j/c_9}.$$

Fixing

$$H := \left( \frac{4c_8^{p/c_9} c_9 \tau}{p\sigma^p \log 2 r^p \delta_3^p} \right)^{c_9/[p\delta_1]} =: \left( \frac{c_6}{\delta_3} \right)^{c_7/\delta_1},$$

we have after simple manipulations that

$$r_j^p \leq \sigma^p \frac{\delta_3^{p-1}\tau}{4} 2^{-j\hat{\delta}_j p/c_9} \frac{\delta_3 p \log 2}{c_9} \leq \sigma^p \frac{\delta_3^{p-1}\tau}{4} \frac{2^{-j\hat{\delta}_j p/c_9}}{\sum_{j=0}^{\infty} 2^{-j\delta_3 p/c_9}}. \quad (4-3)$$

In particular,

$$(2^{k+1} H)^{2-p} \left( \frac{r_k}{\sigma} \right)^p \leq \left( \frac{r_k}{\sigma} \right)^p \leq \frac{\delta_3^{p-1}\tau}{4}.$$

Hence we set  $K_1 := Q_{r_k}^{2^k H}(x'_k, t_k)$  and  $K_2 := Q_{r_k/\sigma}^{2^{k+1} H}(x'_k, t_k)$ , and we deduce, by the induction assumption and the estimate in the previous display, that  $K_2 \subset \Omega_T$ . Now, if

$$\sup_{K_2} (u - \lambda)_+ \leq 2^{k+1} H,$$

then, using that  $(u - \lambda)_+$  is a weak subsolution, [Lemma 4.1](#) would imply

$$\sup_{K_1} u \leq 2^k H + \lambda,$$

which is a contradiction since  $P_k \in \bar{K}_1$ . Thus there is  $P_{k+1} \in K_2$  such that

$$u(P_{k+1}) > 2^{k+1} H + \lambda.$$

By the definition of  $K_2$  and  $P_{k+1}$ , we must have

$$t_{k+1} \leq t_k \leq t_{k+1} + (2^{k+1} H)^{2-p} (r_k/\sigma)^p.$$

Therefore we are left to show that  $t_{k+1} \geq t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau$  in order to prove the induction step.

To this end, let now  $\hat{k} \leq k + 1$  be the largest integer such that  $t_0 - t_{\hat{k}} \leq \delta_3^{p-1} \tau / 2$ . We may without loss of generality assume that  $\hat{k} < k + 1$ , since otherwise  $t_{k+1} \geq t - (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau$ , because  $t_0 > t - (\delta_1^{p-1} + \delta_2^{p-1})\tau$ . Now [\(4-3\)](#) and [\(4-2\)](#), together with the fact that  $\hat{\delta}_j = \delta_3$  for  $j > \hat{k}$ , give

$$\begin{aligned} t_0 - t_{k+1} &= (t_0 - t_{\hat{k}}) + (t_{\hat{k}} - t_{k+1}) \\ &\leq \frac{\delta_3^{p-1} \tau}{2} + \sum_{j=\hat{k}}^k (t_j - t_{j+1}) \\ &\leq \frac{\delta_3^{p-1} \tau}{2} + (2^{\hat{k}+1} H)^{2-p} \left( \frac{r_{\hat{k}}}{\sigma} \right)^p + \sum_{j=\hat{k}+1}^k (2^{j+1} H)^{2-p} \left( \frac{r_j}{\sigma} \right)^p \\ &\leq \frac{\delta_3^{p-1} \tau}{2} + \frac{\delta_3^{p-1} \tau}{4} + \frac{\delta_3^{p-1} \tau}{4} \left[ \sum_{j=0}^{\infty} 2^{-j \delta_3 p / c_9} \right]^{-1} \sum_{j=\hat{k}+1}^k 2^{-j \delta_3 p / c_9} < \delta_3^{p-1} \tau. \end{aligned} \quad (4-4)$$

Therefore, since  $t_0 > t - (\delta_1^{p-1} + \delta_2^{p-1})\tau$ , we have

$$t - t_{k+1} = t - t_0 + t_0 - t_k < (\delta_1^{p-1} + \delta_2^{p-1} + \delta_3^{p-1})\tau,$$

which was to be proven. Hence we have concluded the proof of the induction step. As a consequence, we have constructed a sequence of points  $P_j = (x_j, t_j) \in \tilde{Q}_{\delta_1}$  such that  $d(x_j, \partial\Omega) \rightarrow 0$  and  $u(P_j) \rightarrow \infty$  as  $j \rightarrow \infty$ . This violates the assumed continuity of  $(u - \lambda)_+$  in the neighborhood of  $S_T \cap \tilde{Q}_{\delta_1}$ , giving the contradiction. Hence,

$$\sup_Q u \leq H + \lambda,$$

completing the proof of the theorem.  $\square$

### 5. Estimating the boundary-type Riesz measure

In this section we establish, in NTA cylinders, upper and lower bounds for the measure  $\mu$  defined in (1-6).

**5A. Upper estimate on  $\mu$ .** We will first provide an upper bound on the measure. The proof relies on the Carleson estimate in [Theorem 4.2](#) and the following standard Caccioppoli-type estimate; see [[DiBenedetto 1993](#)].

**Lemma 5.1.** *Let  $u$  be a nonnegative weak subsolution in  $\Omega_T$ , and  $\phi \in C_0^\infty(\Omega \times (t_1, T))$  with  $\phi \geq 0$ . Then*

$$\int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^p \phi^p \, dx \, dt \leq C \left( \int_{t_1}^{t_2} \int_{\Omega} u^p |\nabla \phi|^p \, dx \, dt + \int_{t_1}^{t_2} \int_{\Omega} u^2 (\phi_t)_+ \phi^{p-1} \, dx \, dt \right)$$

for  $C = C(p, n)$ .

**Theorem 5.2.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ . Let  $0 < r \leq r_0$  and let  $u$  be a weak nonnegative solution in  $\Omega_T$ . Fix a point  $x_0 \in \partial\Omega$  and define*

$$\tau = \frac{C_4}{16} [C_5 u(a_r(x_0), t_0)]^{2-p} r^p,$$

where  $C_4$  and  $C_5$ , both depending on  $p, n$ , are as in [Theorem 3.9](#). Let  $0 < \delta \leq \tilde{\delta} \leq 1$  and assume that  $t_0 > 5\tilde{\delta}^{p-1}\tau$  and that  $u$  vanishes continuously on  $S_T \cap (B_r(x) \times (t_0 - 4\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau))$  from  $\Omega_T$ . Then there is a constant  $C \equiv C(p, n)$  and  $c_8 \equiv c_8(p, n, M)$  such that

$$\frac{\mu(Q)}{r^n} \leq C \left( \frac{c_6}{\tilde{\delta}} \right)^{c_8/\delta} u(a_r(x_0), t_0),$$

where  $\mu$  is the measure from (1-6),

$$Q := B_{r/2}(x_0) \times (t_0 - 2\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau),$$

and  $c_6$  is from [Theorem 4.2](#). Furthermore, the constants  $C, c_8$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* After scaling, we may assume that  $u(a_r(x_0), t_0) = 1$ . Let

$$\hat{Q} = B_r(x_0) \times (t_0 - 3\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau),$$

and observe, by our assumptions, that [Theorem 4.2](#) implies

$$\sup_{\hat{Q}} u \leq \left( \frac{c_6}{\tilde{\delta}} \right)^{c_7/\delta} =: \Lambda. \tag{5-1}$$

As in the construction of the measure  $\mu$  in (1-6), we see that extending  $u$  to the entire cylinder  $\hat{Q}$  as zero, we obtain a weak subsolution in  $\hat{Q}$ . Take a cut-off function  $\phi \in C^\infty(\hat{Q})$  vanishing on  $\partial_p \hat{Q}$  such that  $0 \leq \phi \leq 1$ ,  $\phi$  is 1 on  $Q$ , and

$$|\nabla \phi| < \frac{4}{r} \quad \text{and} \quad (\phi_t)_+ < \frac{4}{[\tilde{\delta}^{p-1}\tau]}.$$

Then by (1-6), the definition of  $\phi$  and Hölder's inequality we get

$$\begin{aligned} \int_{\widehat{Q}} \phi^p \, d\mu &\leq \int_{\widehat{Q}} |\nabla u|^{p-1} |\nabla \phi| \phi^{p-1} \, dx \, dt + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \, dx \, dt \\ &\leq \frac{4}{r} \int_{\widehat{Q}} |\nabla u|^{p-1} \phi^{p-1} \, dx \, dt + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \, dx \, dt \\ &\leq \frac{4}{r} |\widehat{Q}|^{1/p} \left( \int_{\widehat{Q}} |\nabla u|^p \phi^p \, dx \, dt \right)^{(p-1)/p} + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \, dx \, dt. \end{aligned}$$

Now using Lemma 5.1 and (5-1) we see that

$$\begin{aligned} \mu(Q) &\leq C \frac{|\widehat{Q}|^{1/p}}{r} \left( \int_{\widehat{Q}} u^p |\nabla \phi|^p + u^2 (\phi_t)_+ \phi^{p-1} \, dx \, dt \right)^{(p-1)/p} + \int_{\widehat{Q}} u(\phi_t)_+ \phi^{p-1} \, dx \, dt \\ &\leq C \frac{|\widehat{Q}|^{1/p}}{r} \left( |\widehat{Q}| \left( \frac{\Lambda^p}{r^p} + \frac{\Lambda^2}{\widetilde{\delta}^{p-1} \tau} \right) \right)^{(p-1)/p} + C |\widehat{Q}| \frac{\Lambda}{\widetilde{\delta}^{p-1} \tau} \\ &\leq C \frac{|\widehat{Q}|}{\widetilde{\delta}^{p-1} r^p} \Lambda^{p-1}. \end{aligned}$$

After scaling back, this can be rewritten in the homogeneous form

$$\frac{\mu(Q_r)}{r^n} \leq C \left( \frac{c_6}{\widetilde{\delta}} \right)^{(p-1)c_7/\delta} u(a_r(x_0), t_0),$$

completing the proof with  $c_8 = (p-1)c_7$ . □

**5B. Lower estimate on  $\mu$ .** We next prove the lower bound for the measure  $\mu$ .

**Theorem 5.3.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0$ , and let  $u$  be a weak nonnegative solution in  $\Omega_T$ . Fix a point  $(x_0, t_0) \in \partial\Omega \times (0, T]$ , and define  $A_r^- = (a_{r/2}(x_0), t_0)$  for  $0 < r < r_0$ . There exist  $C, \tau_0, \tau_1$ , all depending only on  $p, n, M$ , such that if*

$$(t_0 - \tau_0 u(A_r^-)^{2-p} r^p, t_0 + (\tau_0 + \tau_1) u(A_r^-)^{2-p} r^p) \subset (0, T),$$

and if  $u$  vanishes continuously on  $S_T \cap (B_r(x) \times (t_0, t_0 + (\tau_0 + \tau_1) u(A_r^-)^{2-p} r^p))$  from  $\Omega_T$ , then

$$u(A_r^-) \leq C \frac{\mu(Q)}{r^n},$$

where  $\mu$  is the measure from (1-6) and

$$Q := B_r(x_0) \times (t_0 + \tau_0 u(A_r^-)^{2-p} r^p, t_0 + (\tau_0 + \tau_1) u(A_r^-)^{2-p} r^p).$$

Furthermore, the constants  $C, \tau_0, \tau_1$ , are stable as  $p \rightarrow 2^+$ .

To prove Theorem 5.3 we first consider the model problem in Lemma 5.4, and we prove that the measure associated to this model problem is bounded from below by a constant. Returning to Theorem 5.3, we then apply the intrinsic Harnack inequality to obtain a lower bound on the function such that by the comparison principle the solution  $v$  in Lemma 5.4 is below our solution  $u$ . The result then follows by the

fact that the corresponding measures are ordered according to [Lemma 5.5](#), a fact easily realized if the domain is smooth, as in this case the measure is just the modulus of the gradient to the power  $p - 1$ .

**Lemma 5.4.** *Let  $\Omega \subset \mathbb{R}^n$  be an NTA domain with constants  $M$  and  $r_0 = 2$ . There exist constants  $C, T_M$ , both depending on  $p, n, M$ , such that if  $v$  is a continuous solution to the problem*

$$\begin{cases} v_t - \Delta_p v = 0 & \text{in } (\Omega \cap B_2(0)) \times (0, T_0), \\ v = 0 & \text{on } \partial(\Omega \cap B_2(0)) \times [0, T_0), \\ v = \chi_{B_{1/(4M)}(a_1(0))} & \text{on } (\Omega \cap B_2(0)) \times \{0\}, \end{cases}$$

then

$$\mu_v(B_2(0) \times (0, T_0)) \geq \frac{1}{C}.$$

Furthermore, the constants  $C, T_M$  are stable as  $p \rightarrow 2^+$ .

*Proof.* To begin with, extend  $v$  as zero to the rest of  $Q = B_2(0) \times (0, T_0)$ , i.e., set  $v \equiv 0$  in  $(B_2(0) \setminus \Omega) \times (0, T_0)$ , and let  $\mu_v$  be the associated measure as in [\(1-6\)](#). Let then  $h$  be the solution to the problem

$$\begin{cases} h_t - \Delta_p h = 0 & \text{in } Q, \\ h = w & \text{on } \partial_p Q. \end{cases}$$

We observe that the supremum of  $h$  and  $v$ , which is 1, is attained at the bottom of the cylinder. Let us now recall the decay estimate in [Lemma 4.1](#), which implies that for  $Q_r^\lambda(0, t_0) := (\Omega \cap B_r(0)) \times (t_0 - \lambda^{2-p} r^p, t_0)$  we have

$$\sup_{Q_{\sigma^j}^{2^{-j}}(0, t_0)} v \leq 2^{-j} \tag{5-2}$$

for  $j \in \mathbb{N}$  provided that  $t_0 \in [1, T_0]$  and  $T_0 > 1$ . On the other hand, [Lemma 3.3](#) gives us

$$1 = \int_{B_{(4M)^{-1}}(a_1(0))} h(x, 0) \, dx \leq C \inf_{z \in B_{(2M)^{-1}}(0)} u(z, \tau),$$

with  $\tau$  and  $C$  depending on  $p, n, M$ . We then apply [Theorem 3.2](#) in order to get

$$1 \leq \widehat{C} \inf_{\widehat{Q}} h, \quad \widehat{Q} := B_{1/M}(0) \times \left( \frac{T_0}{2}, T_0 \right), \tag{5-3}$$

by properly choosing  $T_0$  by means of  $\tau$  and  $C$  to be larger than 2. We then choose large enough  $j^* \in \mathbb{N}$  so that

$$2^{-j^*} \leq \frac{1}{2\widehat{C}} \quad \text{and} \quad \sigma^{j^*} \leq \frac{1}{M}.$$

Then, sliding  $t_0$  along  $(1, T_0]$  in [\(5-2\)](#), we obtain by combining [\(5-2\)](#) and [\(5-3\)](#) that there is  $r_1 = r_1(p, n, M)$  such that

$$\inf_{\widetilde{Q}} (h - v) \geq \frac{1}{2\widehat{C}} =: \epsilon, \quad \widetilde{Q} := B_{r_1}(0) \times \left( \frac{T_0}{2}, T_0 \right),$$

where  $\epsilon \equiv \epsilon(p, n, M)$ .

Let us now define  $\phi = \min\{h - v, \epsilon\}$ , which is vanishing on  $\partial_p Q$  and is  $\epsilon$  on  $\tilde{Q}$ . Then from the weak formulation of  $h$  and  $v$  we get

$$\int_{Q \cap \{t < \tau\}} (h - v)_t \phi + (|\nabla h|^{p-2} \nabla h - |\nabla v|^{p-2} \nabla v) \cdot \nabla \phi \, dx \, dt = \int_{Q \cap \{t < \tau\}} \phi \, d\mu_w.$$

For the time term, we integrate to obtain

$$\int_{Q \cap \{t < \tau\}} (h - v)_t \phi \, dx \, dt = \int_{Q \cap \{t < \tau\}} \partial_t \left( \int_0^{h-v} \min(s, \epsilon) \, ds \right) \, dx \, dt \geq \frac{1}{2} \int_{B_2(0)} \phi^2(x, \tau) \, dx.$$

We also have the elementary inequality

$$(|\nabla h|^{p-2} \nabla h - |\nabla v|^{p-2} \nabla v) \cdot \nabla \phi \geq \frac{1}{C} |\nabla \phi|^p,$$

since  $p > 2$ . Hence by combining the last three displays we arrive at

$$\sup_{0 < t < T_0} \int_{B_1(0)} \phi^2(x, t) \, dx + \int_Q |\nabla \phi|^p \, dx \, dt \leq C \epsilon \mu_v(Q).$$

Using the parabolic Sobolev inequality [DiBenedetto 1993, Corollary I.3.1] we obtain

$$\begin{aligned} \epsilon^p |\tilde{Q}| &\leq \int_Q \phi^p \, dx \, dt \\ &\leq C \left( \sup_{0 < t < T_0} \int_{B_2(0)} \phi^p(x, t) \, dx + \int_Q |\nabla \phi|^p \, dx \, dt \right) \\ &\leq C \left( \epsilon^{p-2} \sup_{0 < t < T_0} \int_{B_2(0)} \phi^2(x, t) \, dx + \int_Q |\nabla \phi|^p \, dx \, dt \right). \end{aligned}$$

Hence we see that

$$1 \leq C \mu_v(Q)$$

with a constant  $C \equiv C(p, n, M)$  through the dependencies of  $\epsilon, r_1, T_0$ .  $\square$

The next lemma provides a comparison estimate for the measures. If two solutions are ordered, then the corresponding measures will be ordered as well.

**Lemma 5.5.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain. Let  $u$  and  $v$  be weak solutions in  $(\Omega \cap B_r(0)) \times (0, T)$  such that  $u \geq v \geq 0$  and both vanish continuously on the lateral boundary  $(\partial\Omega \cap B_r(0)) \times (0, T)$ . Then*

$$\mu_v \leq \mu_u \quad \text{in } B_r(0) \times (0, T)$$

*in the sense of measures.*

*Proof.* To show this, consider the test function  $\phi = \min(1, (u - v - \epsilon)_+ / \epsilon) \psi$ , where  $\psi$  is nonnegative and belongs to  $C_0^\infty(Q)$  with  $Q = B_r(0) \times (0, T)$ . Obviously  $\phi$  is supported in  $(\Omega \cap B_r(0)) \times (0, T)$ , because both  $u$  and  $v$  vanish continuously on the lateral boundary  $(\partial\Omega \cap B_r(0)) \times (0, T)$ . Since both  $u$  and  $v$  are weak solutions, we have, by extending them both by zero in  $(B_r(0) \setminus \Omega) \times (0, T)$ , that

$$\int_Q (u - v)_t \phi \, dx \, dt + \int_Q (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \phi \, dx \, dt = 0. \quad (5-4)$$

Let us first treat the time term. Integrating by parts we get

$$\begin{aligned}
\int_Q (u-v)_t \phi \, dx \, dt &= \int_Q \partial_t \left( \int_0^{u-v} \min \left( 1, \frac{(s-\epsilon)_+}{\epsilon} \right) ds \right) \psi \, dx \, dt \\
&= - \int_Q \left( \int_0^{u-v} \min \left( 1, \frac{(s-\epsilon)_+}{\epsilon} \right) ds \right) \partial_t \psi \, dx \, dt \\
&\rightarrow - \int_Q (u-v) \partial_t \psi \, dx \, dt
\end{aligned} \tag{5-5}$$

as  $\epsilon \rightarrow 0$ . To treat the elliptic term, we begin by noting that

$$\nabla \phi = \nabla \psi \min \left( 1, \frac{(u-v-\epsilon)_+}{\epsilon} \right) + \frac{1}{\epsilon} \psi \nabla (u-v) \chi_{U_\epsilon}, \tag{5-6}$$

where  $U_\epsilon := \{u-v > \epsilon\} \cap (\Omega \cap B_r(0)) \times (0, T)$ . The second term in (5-6) will give rise to a positive term in (5-4); hence we discard it and obtain the inequality

$$\begin{aligned}
&\int_Q (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \phi \, dx \, dt \\
&\geq \int_Q (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \psi \min \left( 1, \frac{(u-v-\epsilon)_+}{\epsilon} \right) \, dx \, dt \\
&\rightarrow \int_Q (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \psi \, dx \, dt
\end{aligned} \tag{5-7}$$

as  $\epsilon \rightarrow 0$  by dominated convergence. Combining the convergence in (5-5) and (5-7) with (5-4) we arrive at the inequality

$$- \int_Q (u-v) \partial_t \psi \, dx \, dt + \int_Q (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla \psi \, dx \, dt \leq 0 \tag{5-8}$$

after sending  $\epsilon \rightarrow 0$ . Since the nonnegative function  $\psi \in C_0^\infty(Q)$  is arbitrary, (5-8) finishes the proof after recalling the definitions of  $\mu_u$  and  $\mu_v$ .  $\square$

We now have all the technical tools to complete the proof of [Theorem 5.3](#).

*Proof of [Theorem 5.3](#).* Let  $u$  be as in [Theorem 5.3](#) with  $A_r^- := (a_{r/2}(x_0), t_0)$ . Applying the Harnack estimate in [Theorem 3.6](#) yields for a constant  $C = C(p, n, M)$

$$u(A_r^-) \leq \tilde{C} \inf_{y \in B_{r/(8M)}(a_{r/2}(x_0))} u(y, t_0 + \tau_0 u(A_r^-)^{2-p} r^p), \quad \tau_0 := \frac{C}{(2M)^p},$$

since

$$t_0 - \tau_0 u(A_r^-)^{2-p} r^p > 0.$$

Consider the scaled solution

$$\hat{u}(x, t) = \frac{1}{\lambda} u \left( x_0 + \frac{r}{2} x, t_0 + \tau_0 u(A_r^-)^{2-p} r^p + \lambda^{2-p} \left( \frac{r}{2} \right)^p t \right), \quad \lambda := \frac{u(A_r^-)}{\tilde{C}},$$

so that

$$\inf_{y \in B_{1/(4M)}(a_1(0))} \hat{u}(y, 0) \geq 1,$$

and  $\Omega$  is mapped to  $\hat{\Omega}$  and  $(x_0, t_0 + \tau_0 u(A_r^-)^{2-p} r^p)$  to  $(0, 0)$ . The comparison principle shows that the function  $v$  defined in [Lemma 5.4](#) satisfies  $v \leq \hat{u}$  in  $(\hat{\Omega} \cap B_2(0)) \times (0, T_0)$  provided that

$$t_0 + \tau_0 u(A_r^-)^{2-p} r^p + \lambda^{2-p} \left(\frac{r}{2}\right)^p T_0 < T.$$

Thus we choose  $\tau_1 := \tilde{C}^{p-2} 2^{-p} T_0$  in the statement. Applying [Lemmas 5.4](#) and [5.5](#)

$$\mu_{\hat{u}}(B_2(0) \times (0, T_0)) \geq \mu_v(B_2(0) \times (0, T_0)) \geq \frac{1}{C}.$$

Scaling back to  $u$  gives us the result. □

## 6. Construction of barriers

In this section we construct the barriers that will serve as the starting point for the estimates of the decay rate of the solutions. The upper barrier in [Lemma 6.4](#) is based on the function constructed in [\[DiBenedetto et al. 1991, Theorem 4.1\]](#). However the subsolution constructed in [Lemma 6.1](#) seems to be new and allows us to obtain  $p$ -stable estimates from below on the decay rate.

**Lemma 6.1.** *Let  $T = (n^{p-1} p)^{-1}$  and for  $a \in (0, 1)$ , let*

$$\varrho_0 := \min \left\{ \frac{ap}{n(p-2)} + 1, 2 \right\}^{(p-1)/p}.$$

Then the function  $h$ ,

$$h(x, t) = g(|x|, t) - g(\varrho_0, t), \quad g(r, t) = \left[ 1 - \frac{p-2}{p^{p/(p-1)}} \frac{r^{p/(p-1)} - 1}{t^{1/(p-1)}} \right]_+^{(p-1)/(p-2)},$$

is a classical subsolution in  $(B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$  satisfying the boundary conditions

$$\begin{cases} h = 0 & \text{on } \partial B_{\varrho_0}(0) \times (0, T), \\ h = 0 & \text{on } (B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times \{0\}, \\ h = 1 - g(\varrho_0, t) & \text{on } \partial B_1(0) \times (0, T). \end{cases} \quad (6-1)$$

Furthermore,  $x \mapsto h(x, t)$  is a radially decreasing function satisfying

$$\inf_{1 \leq |x| \leq \varrho_0} |\nabla h(x, T)| \geq n \exp\left(-\frac{n}{p}\right) (1-a)^{n/p},$$

and  $h(x, t) \equiv h_p(x, t)$  tends to

$$\exp\left(-\frac{|x|^2 - 1}{4t}\right) - \exp\left(-\frac{1}{4t}\right)$$

as  $p \rightarrow 2^+$ .



**Remark 6.2.** Note that the function in [Lemma 6.1](#) is not continuous up to the boundary at the corner  $\partial B_1(0) \times \{0\}$ . However, the limsup as we approach a point on this piece from the inside of  $D := (B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$  is 1 for  $h$ . This implies, see [\[Kilpeläinen and Lindqvist 1996, Lemma 4.4\]](#), that if we have a weak supersolution  $u$  in  $D$ , staying above the boundary conditions in (6-1) in liminf sense, and staying above 1 on the corner  $\partial B_1(0) \times \{0\}$  again in the liminf sense, then  $u$  will be above  $h$  in  $D$ .

*Proof.* Let  $h$  and  $\varrho_0$  be as stated. By construction, the boundary conditions for  $h$  are in force. To verify that  $h$  is a classical subsolution in  $(B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$ , we first compute

$$\nabla g(|x|, t) = -\frac{x}{|x|} \left[ \frac{1}{p} \frac{|x|}{t} g(|x|, t) \right]^{1/(p-1)},$$

$$|\nabla g(|x|, t)|^{p-2} \nabla g(|x|, t) = -\frac{1}{p} \frac{x}{t} g(|x|, t),$$

$$-\Delta_p g(|x|, t) = \left[ \frac{n}{pt} g(|x|, t)^{(p-2)/(p-1)} - p^{-p/(p-1)} \frac{|x|^{p/(p-1)}}{t^{p/(p-1)}} \right] g(|x|, t)^{1/(p-1)},$$

$$\partial_t g(r, t) = p^{-p/(p-1)} \frac{r^{p/(p-1)} - 1}{t^{p/(p-1)}} g(r, t)^{1/(p-1)}.$$

Observing that  $\partial_t h(x, t) \leq \partial_t g(|x|, t)$ , it is enough to verify  $g_t - \Delta_p g \geq 0$  in  $(B_{\varrho_0}(0) \setminus \overline{B_1(0)}) \times (0, T)$  for  $g > 0$ . Assuming  $g > 0$ , we see, since  $g(|x|, t) \leq 1$  for  $|x| > 1$ , that

$$\begin{aligned} \frac{(h_t - \Delta_p h)(x, t)}{g(|x|, t)^{1/(p-1)}} &\leq \left[ p^{-p/(p-1)} \frac{|x|^{p/(p-1)} - 1}{t^{p/(p-1)}} + \frac{n}{pt} g(|x|, t)^{(p-2)/(p-1)} - p^{-p/(p-1)} \frac{|x|^{p/(p-1)}}{t^{p/(p-1)}} \right] \\ &\leq \frac{1}{pt} \left[ n - \left( \frac{1}{pt} \right)^{1/(p-1)} \right]. \end{aligned}$$

Since  $0 < t < T = (n^{p-1} p)^{-1}$  we have  $h_t - \Delta_p h \leq 0$  for  $|x| > 1$  and  $t \in (0, T)$ . Note also that our choices of parameters are stable as  $p \rightarrow 2$ . Next, by yet another explicit calculation we obtain

$$\inf_{1 \leq |x| \leq \varrho_0} |\nabla h(x, T)| \geq \left( \frac{1}{pT} g(\varrho_0, T) \right)^{1/(p-1)} = n g(\varrho_0, T)^{1/(p-1)}.$$

To complete the proof we need to estimate  $g(\varrho_0, T)$  from below. To do this we note that

$$g(\varrho_0, T)^{1/(p-1)} = \left[ 1 - \frac{n(p-2)}{p} (\varrho_0^{p/(p-1)} - 1) \right]^{1/(p-2)},$$

and we consider two cases. First, if  $\varrho_0 = 2^{(p-1)/p}$ , then  $ap \geq n(p-2)$  and

$$g(\varrho_0, T)^{1/(p-1)} = (1-s)^{b/s}, \quad s = \frac{n(p-2)}{p}, \quad b = \frac{n}{p}.$$

Furthermore, for  $\eta \in [0, 1)$  we have

$$\begin{aligned} (1-\eta)^{b/\eta} &= \exp\left(-\frac{b}{\eta} \sum_{k=1}^{\infty} \frac{\eta^k}{k}\right) = \exp\left(-b \sum_{k=1}^{\infty} \frac{\eta^{k-1}}{k}\right) = \exp\left(-b - b \sum_{k=1}^{\infty} \frac{\eta^k}{k+1}\right) \\ &\geq \exp\left(-b - b \sum_{k=1}^{\infty} \frac{\eta^k}{k}\right) = e^{-b}(1-\eta)^b. \end{aligned} \quad (6-2)$$

Since  $ap \geq n(p-2)$  implies  $s \leq a < 1$ , we can apply (6-2) to get

$$g(\varrho_0, T)^{1/(p-1)} \geq \exp(-b)(1-s)^b \geq \exp(-b)(1-a)^b.$$

Second, if  $\varrho_0 < 2^{(p-1)/p}$ , then  $ap < n(p-2)$  and using (6-2) we get

$$g(\varrho_0, T) = (1-a)^{(p-1)/(p-2)} \geq (1-a)^{b/a} \geq \exp(-b)(1-a)^b.$$

Collecting the results of the two cases completes the proof of the lemma.  $\square$

**Remark 6.3.** Note that we could, as in [Stan and Vázquez 2013], instead of the function in Lemma 6.1 use the Barenblatt fundamental solution together with the barriers from [Bidaut-Véron 2009] to establish a version of Lemma 6.1. However, this would result in a construction which is not  $p$ -stable.

In the next lemma we construct a certain supersolution to be used in the subsequent arguments.

**Lemma 6.4.** *Let  $T, H > 0$  be given degrees of freedom. Let*

$$k \in (0, k_0], \quad k_0 := \min\left\{\frac{p-1}{n}, T^{1/(p-1)} H^{(p-2)/(p-1)}\right\}.$$

*There exists a classical supersolution  $\tilde{h}$  in*

$$N = \{(x, t) : 1 < |x| < 1+k, 0 < t < T\}$$

*such that*

$$\begin{cases} \tilde{h} \geq 0 & \text{in } \partial B_1(0) \times (0, T], \\ \tilde{h} \geq H & \text{on } (\overline{B_{1+k}(0)} \setminus B_1(0)) \times \{0\}, \\ \tilde{h} \geq H & \text{on } \partial B_{1+k}(0) \times [0, T], \end{cases}$$

*and such that*

$$\tilde{h}(x, T) \leq \frac{H \exp(2)}{k} (|x| - 1) \quad (6-3)$$

*whenever  $x \in B_{1+k}(0) \setminus B_1(0)$ .*

*Proof.* This type of construction was originally carried out in [DiBenedetto et al. 1991, Theorem 4.1] and we here include a proof for completeness. Let

$$v(x, t) = \exp\left(\frac{t-T}{T} - \frac{|x|-1}{k}\right),$$

and let

$$\tilde{h}(x, t) = \tilde{H}(1 - v(x, t)), \quad \tilde{H} = H \exp(2),$$

accordingly. Then

$$\partial_t \tilde{h}(x, t) = -\frac{\tilde{H}}{T} v(x, t) \quad \text{and} \quad \nabla \tilde{h}(x, t) = \frac{\tilde{H}x}{k|x|} v(x, t).$$

Observe also that  $\tilde{H}v \geq H$  for all  $(x, t) \in N$  and that  $\tilde{h}$  satisfies the boundary conditions. We now show that  $\tilde{h}$  is a classical supersolution in  $N$ . Indeed, by a straightforward calculation we see that

$$\begin{aligned} \tilde{h}_t - \Delta_p \tilde{h} &= -\frac{\tilde{H}}{T} v + \tilde{H}^{p-1} v^{p-1} \left( (p-1)k^{-p} - k^{1-p} \frac{n-1}{|x|} \right) \\ &\geq \frac{\tilde{H}}{T} v \left( -1 + TH^{p-2} k_0^{1-p} + TH^{p-2} k^{1-p} ((p-1)k_0^{-1} - n) \right) \geq 0 \end{aligned}$$

whenever  $(x, t) \in N$ . Finally, since

$$\sup_{1 < |x| < 1+k} |\nabla \tilde{h}(x, T)| \leq \frac{\tilde{H}}{k} \quad \text{and} \quad \tilde{h}(x, T) = 0 \quad \text{for all } x \in \partial B_1(0),$$

we obtain the upper bound for  $\tilde{h}(x, T)$  as well. □

## 7. Decay estimates and a change of variables

In this section we prove a lower bound ([Lemma 7.2](#)) and an upper bound ([Lemma 7.4](#)) on the decay of solutions. The following lemma, which is a change of variables, will be used in the proof of our decay estimates. The proof of the lemma follows from [[Kuusi 2008](#), Lemma 3.5].

**Lemma 7.1.** *Let  $u = u(x, t)$  be  $p$ -parabolic in  $\Omega \times (T_0, T_1)$ . Let  $C > 0$  be given and let*

$$f(\tau) = \frac{1}{C(p-2)} (\exp(C(p-2)\tau) - 1), \quad g(\eta) = (C(p-2)\eta + 1)^{1/(p-2)}$$

for  $p > 2$ , and

$$f(\tau) = \tau, \quad g(\eta) = \exp(C\eta)$$

for  $p = 2$ . Let  $w(x, \tau) = g(f(\tau))u(x, f(\tau))$ . Then  $w(x, \tau)$  is a (weak) solution to the equation,

$$\partial_\tau w = \Delta_p w + Cw$$

in  $\Omega \times (\tau_0, \tau_1)$ , where  $f(\tau_i) = T_i$ ,  $i \in \{0, 1\}$ .

**7A. A lower bound on the decay.** Using the classical subsolution constructed in [Lemma 6.1](#) and the change of variable outlined in [Lemma 7.1](#) we here prove the following lemma, which describes the optimal decay rate from below after a certain intrinsic waiting time. This lemma will be crucial when proving global  $C^{1,1}$ -estimates, see [Section 8](#), and when proving the local  $C^{1,1}$ -estimates, see [Section 9](#).

**Lemma 7.2.** *Let  $0 < \varrho \leq r/4$  and let  $g \in L^2(B_r(x_0))$  be a nonnegative function satisfying*

$$\int_{B_\varrho(x_0)} g \, dx \geq \lambda > 0.$$

Assume that  $\hat{h} \in C([t_0, \infty); L^2(B_r(x_0)))$  is a weak nonnegative solution solving the Cauchy problem

$$\begin{cases} \hat{h}_t - \Delta_p \hat{h} = 0 & \text{in } B_r(x_0) \times (t_0, \infty), \\ \hat{h} = g & \text{on } B_r(x_0) \times \{t_0\}. \end{cases}$$

Then there exist constants  $c_i \equiv c_i(\varrho/r, n, p)$ ,  $i \in \{1, 2\}$ , such that

$$\hat{h}(x, t) \geq \frac{\lambda}{c_1} \left( c_1(p-2) \frac{t-t_0}{\lambda^{2-p} r^p} + 1 \right)^{-1/(p-2)} \frac{d(x, \partial B_r(x_0))}{r}$$

whenever  $(x, t) \in B_r(x_0) \times (t_0 + c_2 \lambda^{2-p} r^p, \infty)$ . Furthermore, the constants  $c_i$ ,  $i \in \{1, 2\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* After scaling we may assume that  $x_0 = 0$ ,  $t_0 = 0$ ,  $r = 1$ , and  $\lambda = 1$ . Let

$$\varrho_0 := \min \left\{ \frac{1}{2} \frac{p}{n(p-2)} + 1, 2 \right\}^{(p-1)/p}.$$

Applying [Lemma 3.3](#) we find a time  $t^* \equiv t^*(n, p, \varrho/r)$  and a constant  $c^* \equiv c^*(n, p, \varrho/r)$  such that

$$\hat{h}(x, t^*) \geq \frac{1}{c^*} \quad \text{for all } x \in B_{1/\varrho_0}(0).$$

Set  $\bar{h}(x, t) = c^* \hat{h}(x/\varrho_0, t^* + t[c^*]^{p-2}/\varrho_0^p)$  and let  $w(x, \tau) = g(f(\tau)) \bar{h}(x, f(\tau))$ , where  $g$  and  $f$  are defined as in [Lemma 7.1](#). Then  $w(x, \tau)$  is a nonnegative weak solution to the equation

$$\partial_\tau w = \Delta_p w + Cw$$

in  $B_{\varrho_0}(0) \times (0, \infty)$  and  $w(x, 0) \geq 1$  for all  $x \in B_1(0)$ . In particular,  $w$  is a weak supersolution in  $B_{\varrho_0}(0) \times (0, \infty)$ . Now, [Theorem 3.2](#), [[Kuusi 2008](#), Corollary 3.6], [Lemma 7.1](#), and [[Kuusi 2008](#), Proposition 3.1] imply that we have, for a new constant  $\bar{c} \equiv \bar{c}(n, p) > 1$ ,

$$w(x, \tau) \geq \frac{1}{\bar{c}}, \quad (x, \tau) \in B_1(0) \times (0, \infty), \quad (7-1)$$

provided we choose  $C$  large enough in the definitions of  $f$  and  $g$  in [Lemma 7.1](#). Consider  $\hat{\tau} \geq 0$  arbitrary, let  $h$  be the classical subsolution of [Lemma 6.1](#) and let  $T$  be as in [Lemma 6.1](#). Then, simply using the intrinsic scaling, the comparison principle, and (7-1) we see that

$$w(x, \tau) \geq \frac{1}{\bar{c}} h(x, \bar{c}^{2-p}(\tau - \hat{\tau})) \quad (7-2)$$

whenever  $(x, \tau) \in (B_{\varrho_0}(0) \setminus B_1(0)) \times (\hat{\tau}, \hat{\tau} + \bar{c}^{p-2}T)$ . Since  $\hat{\tau} \geq 0$  is arbitrary we get from (7-2) and [Lemma 6.1](#) that there is a  $c \equiv c(n, p)$  such that

$$w(x, \tau) \geq \frac{1}{c} d(x, \partial B_{\varrho_0}(0)), \quad (x, \tau) \in B_{\varrho_0}(0) \times (\bar{c}^{p-2}T, \infty). \quad (7-3)$$

To complete the proof, it suffices to rephrase (7-3) in terms of  $\hat{h}(x, t)$ . □

**7B. An upper bound on the decay.** Working with solutions vanishing on the entire lateral boundary, we will make use of the following decay estimates.

**Lemma 7.3.** *Let  $u \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$  be a nonnegative weak subsolution in  $\Omega \times (0, T)$ . Then there exists a constant  $c = c(n, p, |\Omega|)$  such that*

$$\sup_{\Omega} u(\cdot, t) \leq \frac{c}{t^{n/\sigma}} \left( \int_{\Omega} u(x, 0) dx \right)^{p/\sigma}, \quad \sigma = n(p-2) + p.$$

The constant  $c$  is stable as  $p \rightarrow 2^+$ .

*Proof.* See [Kuusi and Parviainen 2009, Lemma 3.24] and use the  $L^1$ -contractivity (test with the function  $\min(1, u/\epsilon)$ ) and the comparison principle.  $\square$

In the following lemma we describe the optimal decay of the supremum of a solution vanishing on the entire lateral boundary; it follows from an iterative rescaling and comparison together with the decay estimate in Lemma 7.3.

**Lemma 7.4.** *Let  $u \in C([0, T]; L^2(\Omega)) \cap L^p(0, \infty; W_0^{1,p}(\Omega))$  be a nonnegative weak subsolution in  $\Omega \times (0, \infty)$ . Then there exist constants  $c_i \equiv c_i(p, n, |\Omega|)$ ,  $i \in \{1, 2\}$ , such that the following holds. Let*

$$\bar{\Lambda} := \int_{\Omega} u(x, 0) dx.$$

Then

$$\sup_{\Omega} u(\cdot, t) \leq c_1 \left( (p-2) \frac{\bar{\Lambda}^{p-2}}{c_1} t + 1 \right)^{-1/(p-2)} \bar{\Lambda}$$

whenever  $t > c_2 \bar{\Lambda}^{2-p}$ . The constants  $c_i$ ,  $i \in \{1, 2\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* Let  $w = w(x, t)$  solve the Dirichlet problem

$$\begin{cases} w_t - \Delta_p w = 0 & \text{in } \Omega \times (0, \infty), \\ w = 0 & \text{on } \partial\Omega \times [0, \infty) \end{cases} \quad (7-4)$$

and assume that

$$\int_{\Omega} w(x, 0) dx \leq 1. \quad (7-5)$$

Applying Lemma 7.3 to  $w$  we see that

$$\sup_{\Omega} w(\cdot, t) \leq \frac{c}{t^{n/\sigma}}$$

for some  $c = c(n, p, |\Omega|)$ , and for all  $t > 0$ . In particular, there exists  $t^* = t^*(n, p, |\Omega|) > 0$  such that

$$\sup_{\Omega} w(\cdot, t^*) \leq \frac{1}{2}. \quad (7-6)$$

To prove the lemma we will now use (7-6) in an iterative argument. In particular, consider the function

$$w_1(x, t) := \bar{\Lambda}^{-1} u(x, \bar{\Lambda}^{2-p} t).$$

Then  $w_1$  is a solution to (7-4) satisfying (7-5). Hence we have by (7-6) that

$$w_1(x, t^\star) \leq \frac{1}{2},$$

which after scaling back becomes

$$u(x, \bar{\Lambda}^{2-p} t^\star) \leq 2^{-1} \bar{\Lambda} \quad \text{whenever } x \in \Omega.$$

Next, consider the function

$$w_2(x, t) := (2^{-1} \bar{\Lambda})^{-1} u(x, \bar{\Lambda}^{2-p} t^\star + (2^{-1} \bar{\Lambda})^{2-p} t),$$

which again satisfies (7-4) and (7-5). Applying (7-6) to the function  $w_2$  we deduce, by elementary manipulations, that

$$\sup_{\Omega} u(\cdot, (\bar{\Lambda}^{2-p} + (2^{-1} \bar{\Lambda})^{2-p}) t^\star) \leq 2^{-2} \bar{\Lambda}.$$

Proceeding inductively we deduce that

$$\sup_{\Omega} u(\cdot, t_j) \leq 2^{-j} \bar{\Lambda},$$

where

$$t_j := t^\star \bar{\Lambda}^{2-p} \sum_{k=0}^{j-1} 2^{k(p-2)}, \quad j \in \{1, 2, \dots\}.$$

To complete the argument consider  $t \in (t_1, \infty)$  and let  $j$  be the largest  $j$  such that  $t_j \leq t$ . Then, by the comparison principle and by construction,

$$\sup_{x \in \Omega} u(\cdot, t) \leq 2^{-j} \bar{\Lambda}$$

and

$$t_j \leq t < t_{j+1}.$$

Since

$$2^{-j} = \left( \frac{(2^{p-2} - 1)}{t^\star} \bar{\Lambda}^{p-2} t_j + 1 \right)^{-1/(p-2)},$$

and  $2^{p-2} - 1 \geq \log(2)(p-2)$ , by retracing the argument we derive the conclusion of the lemma. Furthermore, the constants  $c_i$ ,  $i \in \{1, 2\}$ , are stable as  $p \rightarrow 2^+$ . In particular, we see that

$$((p-2)c_1^{-1}(n, p, |\Omega|) \bar{\Lambda}^{p-2} t + 1)^{-1/(p-2)} \rightarrow \exp(-c_1^{-1}(n, 2, |\Omega|)t). \quad \square$$

## 8. Global estimates in $C^{1,1}$ -domains

In this section we combine the optimal decay estimate established in [Lemma 8.1](#) together with the barrier function in [Lemma 6.4](#) to obtain the sharp decay estimate from above. Note that taking the initial data to be  $+\infty$  allows us to see that this is sharp with respect to the so-called ‘‘friendly giant’’; see for example [\[Kuusi et al. 2016\]](#).

**Lemma 8.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^{1,1}$ -domain satisfying the ball condition with radius  $r_0$ . Let  $u \in C([0, \infty); L^2(\Omega)) \cap L^p(0, \infty; W_0^{1,p}(\Omega))$  be a nonnegative  $p$ -parabolic function in  $\Omega \times (0, \infty)$ . Let*

$$\bar{\Lambda} := \int_{\Omega} u(x, 0) \, dx.$$

*Then there exists  $C_i \equiv C_i(\text{diam } \Omega / r_0, n, p)$ ,  $i \in \{1, 2\}$ , such that*

$$u(x, t) \leq C_1 \bar{\Lambda} ((p-2)C_1^{-1} \bar{\Lambda}^{p-2} t + 1)^{-1/(p-2)} d(x, \partial\Omega)$$

*whenever  $t > C_2 \bar{\Lambda}^{2-p}$ . Furthermore, the constants  $C_i$ ,  $i \in \{1, 2\}$ , are stable as  $p \rightarrow 2^+$ .*

*Proof.* By scaling we can without loss of generality assume that  $\bar{\Lambda} = 1$ . Let  $\hat{x}_0 \in \partial\Omega$  be an arbitrary point. Assume, for simplicity, that  $a_{2r_0}^-(\hat{x}_0) = 0$ , where  $a_{2r_0}^-(\hat{x}_0)$  is the exterior corkscrew point as in [Remark 2.3](#). Consider an arbitrary number  $\hat{t}$  such that  $\hat{t} > c_2$  (where  $c_2$  is from [Lemma 7.4](#)). Then, using [Lemma 7.4](#) we see that

$$u(x, t) \leq c_1 \left( \frac{p-2}{c_1} \hat{t} + 1 \right)^{1/(2-p)} =: \bar{\Lambda}_u(\hat{t}), \quad x \in \Omega, \quad t \geq \hat{t}. \quad (8-1)$$

Construct the function  $\tilde{h}$  in [Lemma 6.4](#) with the choices  $T = \bar{\Lambda}_u^{2-p}$  and  $H = \bar{\Lambda}_u$ ; then  $k_0$  from [Lemma 6.4](#) simplifies to

$$k_0 = \min \left\{ \bar{\Lambda}_u^{(2-p)/(p-1)} \bar{\Lambda}_u^{(p-2)/(p-1)}, \frac{p-1}{n} \right\} = \min \left\{ 1, \frac{p-1}{n} \right\}.$$

Consider now the function  $\hat{h}$  defined as

$$\hat{h}(x, t) = \tilde{h} \left( \frac{x}{r_0}, \frac{t - \hat{t}}{r_0^p} \right). \quad (8-2)$$

The function  $\hat{h}$  is a supersolution in

$$N := (B_{(1+1/k_0)r_0} \setminus \bar{B}_{r_0}) \times (\hat{t}, \hat{t} + \bar{\Lambda}_u^{2-p} r_0^p).$$

Thus, the comparison principle, [Definition 2.2](#), (8-1) and (8-2) imply that

$$u(x, t) \leq \hat{h}(x, t) \quad \text{in } N \cap \Omega_{\infty}.$$

Next, using the upper estimate (6-3) from [Lemma 6.4](#) we see that

$$u(x, \hat{t} + \bar{\Lambda}_u^{2-p} r_0^p) \leq C \bar{\Lambda}_u \left( \left| \frac{x}{r_0} \right| - 1 \right) \quad (8-3)$$

for a constant  $C = C(n, p)$ . As  $\hat{x}_0 \in \partial\Omega$  is arbitrary, we see, using (8-3) and (8-1), that

$$u(x, \hat{t} + \bar{\Lambda}_u^{2-p} r_0^p) \leq C d(x, \partial\Omega)$$

for a new constant  $C \equiv C(\text{diam } \Omega / r_0, n, p)$ . Furthermore, as  $\hat{t} > c_2$  is arbitrary, we see that if  $t > \bar{\Lambda}_u(c_2)^{2-p} r_0^p + c_2 := C_2$ , then

$$u(x, t) \leq C_1 ((p-2)C_1^{-1} t + 1)^{-1/(p-2)} d(x, \partial\Omega)$$

for a constant  $C_1 \equiv C_1(\text{diam } \Omega / r_0, n, p) > 1$ . □

The next lemma provides the corresponding lower bound.

**Lemma 8.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded connected  $C^{1,1}$ -domain satisfying the ball condition with radius  $r_0$ . Let  $u \in C([0, \infty); L^2(\Omega))$  be a nonnegative  $p$ -parabolic function in  $\Omega \times (0, \infty)$ . Suppose that there is a ball  $B_{4r_1}(x_1) \subset \Omega$ ,  $r_1 \in (0, r_0)$ , such that*

$$\bar{\Lambda} := \int_{B_{r_1}(x_1)} u(x, 0) \, dx > 0.$$

*Then there exist  $C_i \equiv C_i((\text{diam } \Omega)/r_0, r_1/r_0, n, p)$ ,  $i \in \{3, 4\}$ , such that*

$$u(x, t) \geq \frac{\bar{\Lambda}}{C_3} ((p-2)C_3 \bar{\Lambda}^{p-2} t + 1)^{-1/(p-2)} d(x, \partial\Omega)$$

*whenever  $x \in \Omega$  and  $t > C_4 \bar{\Lambda}^{2-p}$ . Furthermore, the constants  $C_i$ ,  $i \in \{3, 4\}$ , are stable as  $p \rightarrow 2^+$ .*

*Proof.* After scaling and translating we may assume that  $\bar{\Lambda} = 1$ ,  $x_1 = 0$  and  $r_0 = 1$ . Note that with these assumptions we have

$$\int_{B_{r_1}(0)} u(x, 0) \, dx = 1. \quad (8-4)$$

Define now the set

$$\Omega^\delta := \{x \in \Omega : d(x, \Omega) > \delta\}.$$

Since  $\Omega$  is connected and satisfies the ball condition with radius 1, we also obtain that  $\Omega^\delta$  is connected for  $\delta \in (0, \frac{1}{2}]$  and thus any two points in  $\Omega^\delta$  can be connected by a Harnack chain of balls of size  $\delta/4$  and with length depending only on  $n, p, \text{diam } \Omega$ , and  $\delta$ . By using [Lemma 3.3](#) and (8-4) we then find positive constants  $c^*$  and  $t^*$ , both depending only on  $n, p, \text{diam } \Omega$ , and  $r_1$ , such that

$$\inf_{x \in \Omega^{1/2}} u(x, t^*) \geq \frac{1}{c^*}.$$

[Lemma 7.2](#) then proves the result whenever  $x \in \Omega^1$ . Next, let  $y \in \Omega \setminus \Omega^1$  and let  $y^* \in \partial\Omega$  be such that  $d(y, \partial\Omega) = |y - y^*|$ . Since  $d(y, \partial\Omega) \leq 1$  and since the direction is unique (see [Remark 2.3](#)) we have that  $y = a_2 d(y, \partial\Omega)(y^*)$ . With this at hand we can consider the point  $a_2(y^*)$  (which is collinear with  $y, y^*$ ) satisfying

$$\int_{B_{1/4}(a_2(y^*))} u(x, t^*) \, dx \geq \frac{1}{c^*}.$$

Applying [Lemma 7.2](#) we see that

$$u(x, t) \geq \frac{1}{c^* c_1} (c_1 (p-2) [c^*]^{p-2} (t - t^*) + 1)^{-1/(p-2)} d(x, \partial B_1(a_2(y^*)))$$

whenever  $x \in B_1(a_2(y^*))$ ,  $t > t^*$ . Applying this for  $x = y$  completes the proof.  $\square$

**Remark 8.3.** Note that our tools are too rough to obtain the lower bound in [Lemma 8.2](#) independent of the distribution of the initial data. To remedy this, we assume that the initial data is positive in a region away from the boundary.



In the next theorem we combine the results of Lemmas 8.1 and 8.2 to obtain an elliptic-type global Harnack estimate.

**Theorem 8.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^{1,1}$ -domain satisfying the ball condition with radius  $r_0$ . Let  $u \in C([0, \infty); L^2(\Omega)) \cap L^p(0, \infty; W_0^{1,p}(\Omega))$  be a nonnegative  $p$ -parabolic function in  $\Omega \times (0, \infty)$ . Let*

$$\bar{\Lambda} := \int_{\Omega} u(x, 0) dx.$$

Assume that

$$\text{supp } u(\cdot, 0) \subset \Omega^\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}.$$

Then there are constants  $c_i \equiv c_i((\text{diam } \Omega)/r_0, \delta/r_0, n, p)$ ,  $i \in \{1, 2\}$ , such that

$$c_1^{-1} \leq \frac{u(x, t + \epsilon \bar{\Lambda}^{2-p})}{u(x, t)} \leq c_1$$

whenever  $\epsilon \in [0, 1]$ ,  $x \in \Omega$  and  $t \geq c_2 \bar{\Lambda}^{2-p}$ . Furthermore, the constants  $c_i$ ,  $i \in \{1, 2\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* In the following we will use the constants  $C_i$ ,  $i \in \{1, \dots, 4\}$ , introduced in Lemmas 8.1 and 8.2. Let  $t_0 = \max\{C_2, C_4\} \bar{\Lambda}^{2-p}$  and consider  $t \geq t_0$  and  $\epsilon \in (0, 1)$ . Then, using Lemmas 8.1 and 8.2 we see that

$$\begin{aligned} \frac{u(x, t + \epsilon \bar{\Lambda}^{2-p})}{u(x, t)} &\leq C_1 C_3 \left( \frac{(p-2)C_1^{-1}(t \bar{\Lambda}^{p-2} + \epsilon) + 1}{(p-2)C_2 t \bar{\Lambda}^{p-2} + 1} \right)^{-1/(p-2)}, \\ \frac{u(x, t + \epsilon \bar{\Lambda}^{2-p})}{u(x, t)} &\geq \frac{1}{C_1 C_3} \left( \frac{(p-2)C_3(t \bar{\Lambda}^{p-2} + \epsilon) + 1}{(p-2)C_1^{-1} t \bar{\Lambda}^{p-2} + 1} \right)^{-1/(p-2)}. \end{aligned}$$

**Theorem 8.5** follows from this by elementary manipulations. We omit further details.  $\square$

In the next theorem we use Lemmas 8.1 and 8.2 together with  $C^{1,\alpha}$  estimates for weak solutions to obtain a global boundary Harnack principle as well as Hölder continuity of ratios of solutions. The intrinsic time interval ensures that the estimate is  $p$ -stable.

**Theorem 8.5.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^{1,1}$ -domain satisfying the ball condition with radius  $r_0$ . Let  $u, v \in C([0, \infty); L^2(\Omega)) \cap L^p(0, \infty; W_0^{1,p}(\Omega))$  be nonnegative  $p$ -parabolic functions in  $\Omega \times (0, \infty)$ . Let*

$$\bar{\Lambda}_u = \int_{\Omega} u(x, 0) dx, \quad \bar{\Lambda}_v = \int_{\Omega} v(x, 0) dx.$$

Assume that the initial data is distributed as follows:

$$\text{supp } u(\cdot, 0), \text{supp } v(\cdot, 0) \subset \Omega^\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}.$$

Then there exists  $\bar{C}_1 \equiv \bar{C}_1((\text{diam } \Omega)/r_0, \delta/r_0, n, p)$  such that if  $\bar{C}_1 \leq T_- \leq T_+$  satisfy

$$T_- \min\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p} \leq T_+ \max\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p}, \quad (8-5)$$

the following holds. There exist  $\bar{C}_i \equiv \bar{C}_i((\text{diam } \Omega)/r_0, \delta/r_0, T_-, T_+, n, p)$ ,  $i \in \{2, 3\}$ , such that

$$\bar{C}_2^{-1} \frac{\bar{\Lambda}_u}{\bar{\Lambda}_v} \leq \frac{u(x, t)}{v(x, t)} \leq \bar{C}_2 \frac{\bar{\Lambda}_u}{\bar{\Lambda}_v} \quad (8-6)$$

whenever

$$(x, t) \in D := \Omega \times (T_- \min\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p}, T_+ \max\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p}).$$

Furthermore, there exists an exponent  $\sigma \equiv \sigma(n, p) \in (0, 1)$  such that

$$\left| \frac{u(x, t)}{v(x, t)} - \frac{u(y, s)}{v(y, s)} \right| \leq \bar{C}_3 \frac{\bar{\Lambda}_u}{\bar{\Lambda}_v} (|x - y| + \max\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{(2-p)/p} |t - s|^{1/p})^\sigma \quad (8-7)$$

whenever  $(x, t), (y, s) \in D$ . The constants  $\bar{C}_i$ ,  $i \in \{1, 2, 3\}$ , and  $\sigma$  are stable as  $p \rightarrow 2^+$ .

*Proof.* In the following we will let  $C_1, C_2, C_3$  and  $C_4$  be the constants in Lemmas 8.1 and 8.2. We begin by proving (8-6). Indeed, using Lemmas 8.1 and 8.2 we see that

$$\frac{\bar{\Lambda}_u}{\bar{\Lambda}_v} \frac{1}{C_1 C_3} \left( \frac{(p-2)C_1^{-1} \bar{\Lambda}_v^{p-2} t + 1}{(p-2)C_3 \bar{\Lambda}_u^{p-2} t + 1} \right)^{1/(p-2)} \leq \frac{u(x, t)}{v(x, t)}, \quad (8-8)$$

and

$$\frac{u(x, t)}{v(x, t)} \leq \frac{\bar{\Lambda}_u}{\bar{\Lambda}_v} C_1 C_3 \left( \frac{(p-2)C_3 \bar{\Lambda}_v^{p-2} t + 1}{(p-2)C_1^{-1} \bar{\Lambda}_u^{p-2} t + 1} \right)^{1/(p-2)} \quad (8-9)$$

whenever  $t \geq \max\{C_2, C_4\} \min\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p}$ . In particular we see that if  $T_- > \max\{C_2, C_4\}$ , and if  $T_-, T_+$  satisfy (8-5), then (8-6) holds with a constant  $\bar{C}_2$  depending only on  $(\text{diam } \Omega)/r_0, \delta/r_0, T_-, T_+, n, p$ .

To proceed, consider the rescaled  $p$ -parabolic functions

$$\tilde{u}(x, t) = \frac{u(x, \bar{\Lambda}_u^{2-p} t)}{\bar{\Lambda}_u}, \quad \tilde{v}(x, t) = \frac{v(x, \bar{\Lambda}_v^{2-p} t)}{\bar{\Lambda}_v}.$$

Using Lemma 7.4 for  $w \in \{\tilde{u}, \tilde{v}\}$  we get for  $t > c_2$  that

$$\sup_{x \in \Omega} w(x, t) \leq c_1 ((p-2)c_1^{-1} t + 1)^{-1/(p-2)} \leq c_1,$$

where  $c_1, c_2$  are as in Lemma 7.4. Thus we can apply [Lieberman 1993, Theorem 0.1] to conclude that there exist  $C$  and  $\sigma$ , depending on  $\Omega, p$  and  $n$ , such that, for  $w \in \{\tilde{u}, \tilde{v}\}$ ,

$$|\nabla w(x, t) - \nabla w(\tilde{x}, \tilde{t})| \leq C(|x - \tilde{x}| + |t - \tilde{t}|^{1/p})^\sigma$$

whenever  $(x, t), (\tilde{x}, \tilde{t}) \in \Omega \times (c_2, \infty)$ . In particular, arguing as in [Kuusi et al. 2014, (3.31), p. 2717], we have, for  $w \in \{\tilde{u}, \tilde{v}\}$ ,

$$\left| \frac{w(x, t)}{d(x, \partial\Omega)} - \frac{w(\tilde{x}, \tilde{t})}{d(\tilde{x}, \partial\Omega)} \right| \leq C(|x - \tilde{x}| + |t - \tilde{t}|^{1/p})^\sigma \quad (8-10)$$

whenever  $(x, t), (\tilde{x}, \tilde{t}) \in \Omega \times (c_2, \infty)$ . Scaling back to the original  $u, v$ , (8-10) becomes, with  $w \in \{u, v\}$ ,

$$\left| \frac{w(x, t)}{d(x, \partial\Omega)} - \frac{w(\tilde{x}, \tilde{t})}{d(\tilde{x}, \partial\Omega)} \right| \leq \bar{\Lambda}_w C (|x - \tilde{x}| + \bar{\Lambda}_w^{(p-2)/p} |t - \tilde{t}|^{1/p})^\sigma \quad (8-11)$$

whenever  $(x, t), (\tilde{x}, \tilde{t}) \in \Omega \times (c_2 \bar{\Lambda}_w^{p-2}, \infty)$ . Next, using the identity

$$\begin{aligned} & \frac{u(x, t)}{v(x, t)} - \frac{u(y, s)}{v(y, s)} \\ &= \frac{d(x, \partial\Omega)}{v(x, t)} \left( \frac{u(x, t)}{d(x, \partial\Omega)} - \frac{u(y, s)}{d(y, \partial\Omega)} \right) + \frac{u(y, s)}{v(y, s)} \frac{d(x, \partial\Omega)}{v(x, t)} \left( \frac{v(y, s)}{d(y, \partial\Omega)} - \frac{v(x, t)}{d(x, \partial\Omega)} \right), \end{aligned} \quad (8-12)$$

assuming that  $s, t \in (T_- \min\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p}, T_+ \max\{\bar{\Lambda}_u, \bar{\Lambda}_v\}^{2-p})$  and that  $T_- > c_2$  (where  $c_2$  is from Lemma 7.4), we can apply Lemmas 8.1 and 8.2 together with (8-6) and (8-11) in the identity (8-12) to obtain (8-7). This completes the proof of Theorem 8.5.  $\square$

**Remark 8.6.** Considering the estimates (8-8) and (8-9) in the proof of Theorem 8.5, we see that the nonlinearities dominate for large values of  $t$ . In particular, there exists a  $p$ -unstable constant  $C \equiv C((\text{diam } \Omega/r_0), \delta/r_0, n, p)$  such that

$$C^{-1} \leq \lim_{t \rightarrow \infty} \frac{u(x, t)}{v(x, t)} \leq C$$

whenever  $x \in \Omega$ . Note that  $C$  is independent of the initial data. This does not happen when  $p = 2$  and the effect is purely nonlinear.

**Remark 8.7.** Note that to prove Theorem 8.5 we rely on estimates established in Lemmas 8.1 and 8.2, instead of relying on the comparison principle, the Carleson estimate and the Harnack inequality, as in [Fabes et al. 1986, Theorem 2.1]. This is why our estimates from below depend on the distribution of the initial data. Furthermore, this falls fairly short of the result in [Fabes et al. 1986], but nonetheless we provide a  $p$ -stable version of the phenomena involved in our case.

## 9. Local estimates in $C^{1,1}$ -domains

In this section the main focus is to develop an intrinsic version of the boundary Harnack principle (1-2); see Section 9B. To do this, we first prove an upper and a lower decay rate estimate in the next section.

**9A. Upper and lower bound on the decay.** We begin with the upper bound, which follows by combining the barrier function from Lemma 6.4 together with the Carleson estimate Theorem 4.2. In the following,  $M$  will denote the NTA constant of the  $C^{1,1}$  domain  $\Omega$ ; see Remark 2.3.

**Lemma 9.1.** *Let  $u$  be a nonnegative solution in  $\Omega_T$ , where  $\Omega$  is a  $C^{1,1}$  domain satisfying the ball condition with radius  $r_0$ . Let  $x_0 \in \partial\Omega$  and  $0 < r \leq r_0$ . Let  $0 < \delta \leq \tilde{\delta} \leq 1$ . Assume that  $u(a_r(x_0), t_0) > 0$  for a fixed  $t_0 \in (0, T)$  and let*

$$\tau = \frac{C_4}{16} [C_5 u(a_r(x_0), t_0)]^{2-p} r^p,$$

where  $C_4$  and  $C_5$ , both depending on  $p, n$ , are as in [Theorem 3.9](#) and with  $t_0 > 5\tilde{\delta}^{p-1}\tau$ . Assume furthermore that  $u$  vanishes continuously on  $S_T \cap (B_r(x_0) \times (t_0 - 4\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau))$ . Then there exist constants  $c_i \equiv c_i(M, p, n)$ ,  $i \in \{8, 9\}$ , such that

$$\sup_Q u \leq \left(\frac{c_8}{\tilde{\delta}}\right)^{c_9/\delta} \frac{d(x_0, \partial\Omega)}{r} u(a_r(x_0), t),$$

where  $Q := (B_r(x_0) \cap \Omega) \times (t_0 - 2\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau)$ . Furthermore, the constants  $c_i$ ,  $i \in \{8, 9\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* Without loss of generality, we may after scaling assume that  $u(a_r(x_0), t_0) = 1$  and  $r = 4$ . Applying [Lemma 6.4](#) and [Theorem 4.2](#), with

$$k := \min\left\{\frac{p-1}{n}, \tilde{\delta}\left(\frac{C_4}{16}C_5^{2-p}\right)^{1/(p-1)}, 1\right\}, \quad H := \left(\frac{c_6}{\tilde{\delta}}\right)^{c_7/\delta},$$

with constants as in [Theorem 4.2](#) we get

$$u(x, t) \leq H$$

whenever  $(x, t) \in (B_4(x_0) \cap \Omega) \times (t_0 - 3\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau)$ . The comparison function, indexed by its initial time  $s_0$  and center point  $y_0$ , is

$$\hat{h}_{y_0, s_0}(x, t) = \tilde{h}(x - y_0, t - s_0)$$

with  $T = \tilde{\delta}^{p-1}\tau$ , where  $\tilde{h}$  is from [Lemma 6.4](#). Let  $y \in \partial\Omega \cap B_1(x_0)$  and consider  $y_0 = a_2(y)$ , an outer corkscrew point as in [Remark 2.3](#). Then, by the comparison principle

$$u(x, t) \leq \hat{h}_{y_0, s_0}(x, t), \tag{9-1}$$

in

$$(\Omega \cap [B_{1+k}(y_0) \setminus B_1(y_0)]) \times (s_0, s_0 + T)$$

whenever

$$(s_0, s_0 + T) \subset (t_0 - 3\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau).$$

From [Lemma 6.4](#) and (9-1) we have the estimate

$$u(x, t) \leq \frac{H \exp(2)}{k} (|y_0 - x| - 1)$$

whenever

$$(x, t) \in (\Omega \cap [B_{1+k}(y_0) \setminus B_1(y_0)]) \times (t_0 - 2\tilde{\delta}^{p-1}\tau, t_0 - \delta^{p-1}\tau),$$

and  $y_0 \in \partial\Omega \cap B_1(x_0)$ . From this the result follows by scaling back.  $\square$

The following lemma establishes a local lower bound on the decay, by combining the barrier from [Lemma 6.1](#) and the Harnack estimates in [Theorem 3.6](#).

**Lemma 9.2.** *Let  $u$  be a nonnegative solution in  $\Omega_T$ , where  $\Omega$  is a  $C^{1,1}$  domain satisfying the ball condition with radius  $r_0$ . Let  $x_0 \in \partial\Omega$  and let  $0 < r < r_0$  be fixed. Let  $A^- = (a_r(x_0), t_0)$ ,  $\theta_- = u(A^-)^{2-p}$  and  $t_0 \in (0, T)$ . There exist constants  $c_i \equiv c_i(M, p, n)$ ,  $i \in \{3, 4\}$ , such that if*

$$\theta_- r^p < t_0 \quad \text{and} \quad t_0 + 2c_4 \theta_- r^p < T,$$

then

$$\frac{1}{c_3} \frac{d(x, \partial\Omega)}{r} u(A^-) \leq u(x, t)$$

for  $x \in B_r(x_0) \cap \Omega$  and  $t_0 + c_4 \theta_- r^p < t < t_0 + 2c_4 \theta_- r^p$ . Furthermore, the constants  $c_i$ ,  $i \in \{3, 4\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* Set  $\lambda_- := u(A^-)$  and consider the scaled solution

$$v(y, s) = \frac{1}{\lambda_-} u(x_0 + ry, t_0 + s\lambda_-^{2-p} r^p).$$

Set also  $\tilde{\Omega} := \{y : x_0 + ry \in \Omega\}$  so that  $0 \in \partial\tilde{\Omega}$ . For the new function  $v$  we have the following situation: defining  $A_v^- = ((a_r(x_0) - x_0)/r, 0)$ ,

$$v(A_v^-) = 1, \quad d(A^-, \partial\tilde{\Omega}) = 1,$$

and  $v$  is a solution in  $\tilde{\Omega} \times (-1, \tilde{T})$ , where  $\tilde{T} := (T - t_0)\lambda_-^{p-2} r^{-p}$ . Since  $0 < r < r_0/4$  we know that  $\tilde{\Omega}$  satisfies the ball condition with radius 4. To continue, consider the set

$$D = \{y \in \tilde{\Omega} : d(y, B_1(0) \cap \partial\tilde{\Omega}) = d(y, \partial\tilde{\Omega}) = 1\}.$$

Note that  $D \subset B_2(0) \cap \tilde{\Omega}$  and that  $\sup\{d(a_1(y_0), y) : y \in D\} \leq 2$  for any  $y_0 \in \partial\tilde{\Omega} \cap B_1(0)$ . We obtain from the Harnack chain estimate in [Theorem 3.6](#) (applied with  $\delta = \min\{c_h^{(2-p)/p}, 1\}$  where  $c_h$  is from [Theorem 3.6](#)) that there is a  $\tilde{\tau}_1 > 0$  depending only on  $n, p, M$  such that

$$v(x, \tilde{\tau}_1) \geq \frac{1}{\tilde{c}_1}$$

whenever  $x \in \{y : d(y, D) < \frac{1}{4}\}$  provided  $\tilde{T} > \tilde{\tau}_1$ . Using [Lemma 7.2](#) (applied with  $r = 1$ ,  $\varrho = \frac{1}{4}$ ,  $g = v(\cdot, \tilde{\tau}_1)$ ,  $x_0 = \tilde{y} \in D$ ,  $t_0 = \tilde{\tau}_1$ ) for all points  $\tilde{y} \in D$ , we get

$$v(y, t) \geq \frac{1}{\tilde{c}_1 c_1} (c_1 \tilde{c}_1^{2-p} (p-2)(t - \tilde{\tau}_2) + c_1 c_2 (p-2) + 1)^{-1/(p-2)} d(y, \tilde{\Omega}) \quad (9-2)$$

whenever  $(y, t) \in (\tilde{\Omega} \cap B_1(0)) \times (\tilde{\tau}_2, \tilde{T})$ , with  $\tilde{\tau}_2 = \tilde{\tau}_1 + c_2 \tilde{c}_1^{p-2}$  provided  $\tilde{T} > \tilde{\tau}_2$ . Going back to  $\Omega$  and  $u$  gives us the result provided  $\tilde{T} > 2\tilde{\tau}_2$ .  $\square$

Combining [Lemmas 9.1](#) and [9.2](#) we obtain the joint estimate.

**Theorem 9.3.** *Let  $u$  be a nonnegative solution in  $\Omega_T$ , where  $\Omega$  is a  $C^{1,1}$  domain satisfying the ball condition with radius  $r_0$ . Let  $x_0 \in \partial\Omega$ ,  $t_0 \in (0, T)$ , and let  $0 < r < r_0$  be fixed. Let  $A^- = (a_r(x_0), t_0)$  and  $\theta_- = u(A^-)^{2-p}$ . There exist constants  $c_i \equiv c_i(M, p, n)$ ,  $i \in \{5, 6\}$ , such that if*

$$\theta_- r^p < t_0 \quad \text{and} \quad t_0 + 2c_4 \theta_- r^p < T,$$

then for  $A^+ = (a_r(x_0), t_0 + 2c_4\theta_-r^p)$  and  $\theta_+ = c_6^{-1}u(A^+)^{2-p}$  (where  $c_4$  is from [Lemma 9.2](#)), we have

$$5\theta_+ \leq \theta_-.$$

Furthermore, if  $u$  vanishes continuously on

$$S_T \cap (B_r(x_0) \times (t_0 + [2c_4\theta_- - 5\theta_+]r^p, t_0 + [2c_4\theta_- - \theta_+]r^p)),$$

then

$$\frac{1}{c_5} \frac{d(x, \partial\Omega)}{r} u(A^-) \leq u(x, t) \leq c_5 \frac{d(x, \partial\Omega)}{r} u(A^+)$$

for  $x \in B_r(x_0) \cap \Omega$  and  $t_0 + [2c_4\theta_- - 2\theta_+]r^p < t < t_0 + [2c_4\theta_- - \theta_+]r^p$ . Furthermore, the constants  $c_i$ ,  $i \in \{5, 6\}$ , are stable as  $p \rightarrow 2^+$ .

*Proof.* Rescale  $u$  as in the proof of [Lemma 9.2](#); let also  $\tilde{\tau}_2$  be as in the proof of [Lemma 9.2](#). Thus

$$\frac{1}{c_3} d(x, \partial\Omega) \leq v(x, t)$$

holds for  $(x, t) \in (B_r(x_0) \cap \Omega) \times (\tilde{\tau}_2, 2\tilde{\tau}_2)$ . Define  $\tau_+ = 2\tilde{\tau}_2$  and consider  $A_+ = (a_1(0), \tau_+)$ . Then using [\(9-2\)](#) we get

$$\begin{aligned} v(A_+)^{2-p} &\leq \left[ \frac{1}{\tilde{c}_1 c_1} (c_1 \tilde{c}_1^{2-p} (p-2)(t - \tilde{\tau}_2) + c_1 c_2 (p-2) + 1)^{-1/(p-2)} \right]^{2-p} \\ &= \frac{1}{(\tilde{c}_1 c_1)^{2-p}} [c_1 \tilde{c}_1^{2-p} (p-2) + c_1 c_2 (p-2) + 1] \\ &=: \tilde{c}_2. \end{aligned}$$

We will now apply [Lemma 9.1](#) with  $(r = 1, \delta^{p-1} = \min\{(16/(5\tilde{c}_2 C_4))C_5^{p-2}, 1\})$  and  $\tilde{\delta} = \delta$ , with  $C_4, C_5$  from [Lemma 9.1](#)). Doing this we see that

$$v(x, t) \leq \tilde{c}_3 d(x, \partial\tilde{\Omega}) v(A_+)$$

whenever  $(x, t) \in (B_1(0) \cap \Omega) \times (\tau_+ - v(A_+)^{2-p}, \tau_+ - v(A_+)^{2-p}/2)$ . Going back to  $\Omega$  and  $u$  gives us the result.  $\square$

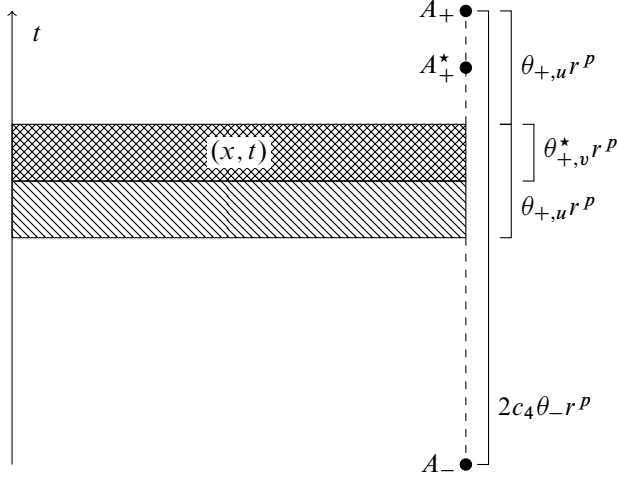
**9B. Local boundary Harnack estimate.** We are now ready to state and prove our local boundary Harnack principle; consult [Figure 1](#) for a schematic of the geometry.

**Theorem 9.4.** *Let  $u, v$  be two nonnegative solutions in  $\Omega_T$ , where  $\Omega$  is a  $C^{1,1}$ -domain satisfying the ball condition with radius  $r_0$ . Let  $x_0 \in \partial\Omega$ ,  $t_0 \in (0, T)$ , and let  $0 < r < r_0$  be fixed. Let  $A_- = (a_r(x_0), t_0)$  and assume that  $u(A_-) = v(A_-)$ . Let the constants  $c_i$ ,  $i \in \{4, 5, 6\}$  be as in [Lemmas 9.2](#) and [9.3](#). Let  $\theta_- = u(A_-)^{2-p}$ , and assume*

$$\theta_- r^p < t_0 \quad \text{and} \quad t_0 + 2c_4\theta_- r^p < T.$$

Set

$$A_+ = (a_r(x_0), t_0 + 2c_4\theta_- r^p), \quad \theta_{+,u} = c_6^{-1}u(A_+)^{2-p}.$$



**Figure 1.** The boxes  $\text{\textbackslash}\text{\textbackslash}\text{\textbackslash}$  and  $\text{\textbackslash}\text{\textbackslash}\text{\textbackslash}$  denote the regions where the right- and left-hand sides of [Theorem 9.4](#) hold respectively.

Assume that  $v(A_+) \geq u(A_+)$ . Then there exists a time  $t_+^*$ , depending on  $v$ , satisfying

$$\begin{aligned} t_+^* &\in (t_0 + (2c_4\theta_- - \theta_{+,u})r^p, t_0 + 2c_4\theta_-r^p), \\ A_+^* &= (a_r(x_0), t_+^*), \quad \theta_{+,v}^* = c_6^{-1}v(A_+^*)^{2-p} \end{aligned}$$

such that the following holds. If both  $u$  and  $v$  vanish continuously on

$$S_T \cap (B_r(x_0) \times (t_0 + [2c_4\theta_- - 5\theta_{+,u}]r^p, t_0 + [2c_4\theta_- - \theta_{+,u}]r^p)),$$

then

$$\frac{1}{c_5^2} \frac{u(A_-)}{v(A_+^*)} \leq \frac{u(x, t)}{v(x, t)} \leq c_5^2 \frac{u(A_+)}{v(A_-)}$$

whenever  $(x, t)$  belongs to the set

$$(B_r(x_0) \cap \Omega) \times (t_0 + [2c_4\theta_- - (\theta_{+,v}^* + \theta_{+,u})]r^p, t_0 + [2c_4\theta_- - \theta_{+,u}]r^p).$$

**Remark 9.5.** It should be noted that we cannot control the time  $t_+^*$  except which interval it lies in; it is a purely intrinsic parameter. Furthermore note that [Theorem 9.4](#) is equivalent to the boundary Harnack principle (1-2) when  $p = 2$ .

*Proof.* Let  $c_i$ ,  $i \in \{3, \dots, 6\}$ , be as in [Lemmas 9.2](#) and [9.3](#). By the assumptions, we know that  $\theta_{+,u} \geq \theta_{+,v} := c_6^{-1}v(A_+)^{2-p}$ . We then obtain by [Theorem 9.3](#), for  $x \in B_r(x_0) \cap \Omega$  and  $t_0 + (2c_4\theta_- - 2\theta_{+,u})r^p < t < t_0 + (2c_4\theta_- - \theta_{+,u})r^p$ , that

$$\frac{1}{c_5} \frac{d(x, \partial\Omega)}{r} u(A_-) \leq u(x, t) \leq c_5 \frac{d(x, \partial\Omega)}{r} u(A_+). \quad (9-3)$$

Using [Lemma 9.2](#) for  $x \in B_r(x_0) \cap \Omega$  and  $t_0 + c_4\theta_-r^p < t < t_0 + 2c_4\theta_-r^p$  we get that

$$\frac{1}{c_3} \frac{d(x, \partial\Omega)}{r} v(A_-) \leq v(x, t). \quad (9-4)$$

Now let  $t_+ = t_0 + 2c_4\theta_-r^p$  and let  $t_+^*$  be a time to be fixed such that  $t_+ - \theta_{+,u}r^p < t_+^* \leq t_+$ . First note that if  $t_+^* = t_+$  we have for  $\theta_{+,v}^* = c_6^{-1}(v(a_r(x_0), t_+^*))^{2-p}$ ,

$$t_+^* - \theta_{+,v}^*r^p \geq t_+ - \theta_{+,u}r^p;$$

furthermore, if  $t_+^* = t_+ - \theta_{+,u}^*r^p$  then

$$t_+^* - \theta_{+,v}^*r^p < t_+ - \theta_{+,u}r^p.$$

Thus by continuity there is a largest  $t_+^*$  such that

$$t_+^* - \theta_{+,v}^*r^p = t_+ - \theta_{+,u}r^p.$$

With  $t_+^*$  at hand we now apply [Lemma 9.1](#) (with the same  $\delta, \tilde{\delta}$  as in the proof of [Theorem 9.3](#)) combining it with (9-4) to get

$$\frac{1}{c_5} \frac{d(x, \partial\Omega)}{r} v(A_-) \leq v(x, t) \leq c_5 \frac{d(x, \partial\Omega)}{r} v(A_+^*) \quad (9-5)$$

for  $x \in B_r(x_0) \cap \Omega$  and  $t_0 + (2c_4\theta_- - (\theta_{+,u} + \theta_{+,v}^*))r^p < t < t_0 + (2c_4\theta_- - \theta_{+,u})r^p$ . Combining (9-3) and (9-5) we have completed the proof.  $\square$

**9C. Boundary measures in  $C^{1,1}$ -domains.** We conclude the section by describing the fine properties of the boundary measure defined in (1-6). The theorem below says that the induced measure is mutually absolutely continuous with respect to the surface measure of  $S_T$ .

**Theorem 9.6.** *Under the hypotheses of [Theorem 9.3](#),*

$$0 < \liminf_{\varrho \rightarrow 0} \frac{\mu_u(Q_\varrho(x, t))}{\varrho^{n+1}} \leq \limsup_{\varrho \rightarrow 0} \frac{\mu_u(Q_\varrho(x, t))}{\varrho^{n+1}} < +\infty,$$

where  $Q_\varrho(x, t) := B_\varrho(x) \times (t - \varrho^2, t)$ , whenever  $(x, t) \in V$ ,

$$V := (\partial\Omega \cap B_r(x_0)) \times (t_0 + (2c_4\theta_- - 2\theta_+)r^p, t_0 + (2c_4\theta_- - \theta_+)r^p).$$

*In particular,  $\mu_u$  is mutually absolutely continuous with respect to the surface measure of  $S_T$  on  $V$ .*

*Proof.* By [Theorem 9.3](#) we have

$$\lambda_- d(x, \partial\Omega) \leq u(x, t) \leq \lambda_+ d(x, \partial\Omega), \quad \lambda_\pm := c_5^{\pm 1} \frac{u(A_r^\pm)}{r} \quad (9-6)$$

whenever  $(x, t) \in Q$ , with

$$Q := (\Omega \cap B_r(x_0)) \times (t_0 + (2c_4\theta_- - 2\theta_+)r^p, t_0 + (2c_4\theta_- - \theta_+)r^p)$$



and  $\theta_{\pm}$  as in [Theorem 9.3](#). We now pick a point  $(y, s) \in S_T \cap \partial_p Q$ . Choose  $\varrho$  small enough so that  $U_{\varrho}(y, s) \cap (\Omega \times \mathbb{R})$  is contained in  $Q$ , where

$$U_{\varrho}(y, s) := B_{\varrho}(y) \times (s - \tilde{\tau}\varrho^2, s + \tilde{\tau}\varrho^2),$$

$\tilde{\tau} := \lambda_-^{2-p} \max\{8C_4C_5^{2-p}, 2(\tau_0 + \tau_1)\}$  and  $C_4, C_5$  and  $\tau_0, \tau_1$  are as in [Theorems 5.2](#) and [5.3](#). After a simple covering argument using [\(9-6\)](#), and [Theorems 5.2](#) and [5.3](#), we find a constant  $C \equiv C(p, n, M, \lambda_{\pm})$  such that

$$\frac{1}{C} \leq \frac{\mu_u(U_{\varrho/2}(y, s))}{\varrho^{n+1}} \leq C.$$

Taking a possibly larger  $C$ , and a smaller  $\varrho$ , this actually implies

$$\frac{1}{C} \leq \frac{\mu_u(Q_{\varrho}(y, s))}{\varrho^{n+1}} \leq C$$

uniformly for small enough  $\varrho$ . This proves the statement.  $\square$

**Remark 9.7.** Note that in the same region  $V$  as in [Theorem 9.6](#) we have that the measure is doubling. Moreover note that [Lemma 9.2](#) implies a Hopf-type result on this boundary cylinder  $V$ ; thus together with the fact that solutions are  $C^{1,\alpha}$  up to the boundary, we get that the logarithm of the normal derivative on the boundary is Hölder continuous. Now arguing as in [[Avelin et al. 2011](#), (1.7)–(1.10)] we get, for  $(x_0, t_0) \in V$  given, and  $\epsilon \in (0, 1)$ , that

$$\lim_{\varrho \rightarrow 0} \frac{\mu_u(Q_{\epsilon\varrho}(x_0, t_0))}{\mu_u(Q_{\varrho}(x_0, t_0))} = \epsilon^{n+1}.$$

In particular, the measure  $\mu_u$  is asymptotically optimal doubling.

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