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ON ASYMPTOTIC DYNAMICS FOR L^2 CRITICAL GENERALIZED KDV EQUATIONS WITH A SATURATED PERTURBATION

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We consider the L^2 critical gKdV equation with a saturated perturbation: $\partial_t u+(u_{xx}+u^5-\gamma u|u|^{q-1})_x=0$, where $q>5$ and $0<\gamma\ll 1$. For any initial data $u_0\in H^1$, the corresponding solution is always global and bounded in H^1. This equation has a family of solutions, and our goal is to classify the dynamics near solitons. Together with a suitable decay assumption, there are only three possibilities: (i) the solution converges asymptotically to a solitary wave whose H^1 norm is of size $\gamma^{-2/(q-1)}$ as $\gamma\to 0$; (ii) the solution is always in a small neighborhood of the modulated family of solitary waves, but blows down at $+\infty$; (iii) the solution leaves any small neighborhood of the modulated family of the solitary waves.

This extends the classification of the rigidity dynamics near the ground state for the unperturbed L^2 critical gKdV (corresponding to $\gamma=0$) by Martel, Merle and Raphaël. However, the blow-down behavior (ii) is completely new, and the dynamics of the saturated equation cannot be viewed as a perturbation of the L^2 critical dynamics of the unperturbed equation. This is the first example of classification of the dynamics near the ground state for a saturated equation in this context. The cases of L^2 critical NLS and L^2 supercritical gKdV, where similar classification results are expected, are completely open.

1. Introduction

1A. ***Setting of the problem.*** Let us consider the following Cauchy problem:

$$\begin{cases}\partial_t u+(u_{xx}+u^5-\gamma u|u|^{q-1})_x=0, & (t,x)\in[0,T)\times\mathbb{R},\\ u(0,x)=u_0(x)\in H^1(\mathbb{R}),\end{cases}\qquad (\text{gKdV}_\gamma)$$

with $q>5$ and $0<\gamma\ll 1$.

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The equation has two conservation laws, i.e., the mass and the energy:

$$M(u(t)) = \int u(t)^2 = M_0,$$
$$E(u(t)) = \frac{1}{2}\int u_x(t)^2 - \frac{1}{6}\int u(t)^6 + \frac{\gamma}{q+1}\int |u(t)|^{q+1} = E_0.$$

We can see that the solution of (gKdV$_\gamma$) is always global in time and bounded in H^1. First of all, (gKdV$_\gamma$) is locally well-posed in H^1 due to [Kato 1983; Kenig, Ponce and Vega 1993]; i.e., for any $u_0 \in H^1$, there exists a unique strong solution in $C([0,T), H^1)$ with either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T}\|u_x(t)\|_{L^2} = +\infty$. Since $\gamma > 0$, $q > 5$, the mass and energy conservation laws ensure that, for all $t \in [0,T)$,

$$\|u_x(t)\|_{L^2}^2 \lesssim |E_0| + \gamma^{-\frac{4}{q-5}} M_0 < +\infty,$$

so $T = +\infty$ and $u(t)$ is always bounded in H^1.

This equation does not have a standard scaling rule, but has the following pseudoscaling rule: for all $\lambda_0 > 0$, if $u(t,x)$ is a solution to (gKdV$_\gamma$), then

$$u_{\lambda_0}(t,x) = \lambda_0^{-\frac{1}{2}} u(\lambda_0^{-3}t, \lambda_0^{-1}x) \tag{1-1}$$

is a solution to

$$\begin{cases} \partial_t v + (v_{xx} + v^5 - \lambda_0^{-m}\gamma v|v|^{q-1})_x = 0, & (t,x) \in [0, \lambda_0^{-3}T) \times \mathbb{R}, \\ v(0,x) = \lambda_0^{-\frac{1}{2}} u_0(\lambda_0^{-1}x) \in H^1(\mathbb{R}), \end{cases}$$

with

$$m = \tfrac{1}{2}(q-5) > 0. \tag{1-2}$$

The pseudoscaling rule (1-1) leaves the L^2 norm of the initial data invariant.

There is a special class of solutions. We first introduce the ground state Q_ω for $0 \le \omega < \omega^* \ll 1$, which is the unique radial nonnegative solution with exponential decay to the ODE[1]

$$Q_\omega'' - Q_\omega + Q_\omega^5 - \omega Q_\omega |Q_\omega|^{q-1} = 0.$$

Then for all $\lambda_0 > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$ with $\lambda_0^{-m}\gamma < \omega^*$, the following is a solution to (gKdV$_\gamma$):

$$u(t,x) = \lambda_0^{-\frac{1}{2}} Q_{\lambda_0^{-m}\gamma}(\lambda_0^{-1}(x-x_0) - \lambda_0^{-3}(t-t_0)).$$

A solution of this type is called a *solitary wave* solution.

1B. ***On the critical problem with saturated perturbation.*** The saturated perturbation was first introduced for the nonlinear Schrödinger (NLS)

$$i\partial_t u + \Delta u + g(|u|^2)u = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d. \tag{NLS}$$

[1] The existence of such Q_ω was proved in [Berestycki and Lions 1983, Section 6], but in this paper we will give an alternative proof for the existence.

In many applications, the leading-order approximation of the nonlinearity, $g(s)$, is the power nonlinearity; i.e., $g(s) = \pm s^\sigma$. For example, $g(s) = s$ leads to the focusing cubic NLS equation, which appears in many contexts.

But such approximation may lead to nonphysical predictions. For example, from [Fibich 2015; Merle and Raphaël 2005; Merle, Raphaël and Szeftel 2010; Sulem and Sulem 1999], for NLS with critical or supercritical focusing nonlinearities (i.e., $g(s) = s^\sigma$ with $\sigma d \geq 2$), blow up may occur. However, this contradicts the experiments in the optical settings [Josserand and Rica 1997], which shows that there is no "singularity" and the solution always remains bounded.

One way to correct this model is to replace the power nonlinearities by saturated nonlinearities. A typical example[2] is $g(s) = s^\sigma - \gamma s^{\sigma+\delta}$, with $\delta > 0$, $\gamma > 0$. Similar to (gKdV$_\gamma$), in this case any H^1 solution to (NLS) is global in time and bounded in H^1.

On the other hand, the saturated perturbation is also related to the problem of continuation after blow up time. These kinds of problems arising in physics are poorly understood even at a formal level. One approach is to consider the solution $u_\varepsilon(t)$ to the following critical NLS with saturated perturbation:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u - \varepsilon |u|^q u = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases}$$

where

$$\frac{4}{d} < q < \frac{4}{d-2}.$$

Suppose the solution $u(t)$ to the unperturbed NLS (i.e., $\varepsilon = 0$) with initial data u_0, blows up in finite time $T < +\infty$. Then, it is easy to see that for all $\varepsilon > 0$, the solution $u_\varepsilon(t)$ exists globally in time, and for all $t < T$

$$\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t) \quad \text{in } H^1.$$

Now, we may consider the limit

$$\lim_{\varepsilon \to 0} u_\varepsilon(t), \quad t > T,$$

to see whether the limiting function exists and in what sense it satisfies the critical NLS. Such a construction for blow-up solutions using the Virial identity was given by Merle [1992a]. An alternative way to construct the approximate solution $u_\varepsilon(t)$ can also be found in [Merle 1989; 1992b; Merle, Raphaël and Szeftel 2013], but this only holds for very special cases. General constructions of this type are mostly open. In all cases, the asymptotic behavior of the approximate solution $u_\varepsilon(t)$ is crucial in the analysis.

Therefore, the asymptotic dynamics of dispersive equations with a saturated perturbation becomes a natural question.

1C. ***Results for L^2 critical gKdV equations.*** Let us recall some results for the following L^2 critical gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \tag{gKdV}$$

[2] See [Glasner and Allen-Flowers 2016; Marzuola, Raynor and Simpson 2010] for other kinds of saturated perturbations.

This equation is L^2 critical since, for all $\lambda > 0$,

$$u_\lambda(t,x) = \lambda^{-\frac{1}{2}} u(\lambda^{-3}t, \lambda^{-1}x)$$

is still a solution to (gKdV) and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$.

There is a special class of solutions, i.e., the solitary waves, which is given by

$$u(t,x) = \frac{1}{\lambda_0^{1/2}} Q\left(\frac{x - x_0 - \lambda_0^{-2}(t-t_0)}{\lambda_0}\right),$$

with

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}, \quad Q'' - Q + Q^5 = 0.$$

The function Q is called the ground state.

From variational arguments [Weinstein 1982], we know that if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution to (gKdV) is global in time and bounded in H^1, while for $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, blow up may occur. The blow up dynamics for solutions with slightly supercritical mass

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^* \tag{1-3}$$

has been developed in a series papers [Martel and Merle 2002a; 2002b; 2002c; Merle 2001]. In particular, they prove the existence of blow up solutions with negative energy, and give a specific description of the blow up dynamics and the formation of singularities.

Martel, Merle and Raphaël [2014; 2015a; 2015b] give an exclusive study of the asymptotic dynamics near the ground state Q.

More precisely, consider the initial data set

$$\mathcal{A}_{\alpha_0} = \left\{ u_0 \in H^1 \,\middle|\, u_0 = Q + \varepsilon_0,\ \|\varepsilon_0\|_{H^1} < \alpha_0,\ \int_{y>0} y^{10}\varepsilon_0^2 < 1 \right\},$$

and the L^2 tube around the solitary wave family

$$\mathcal{T}_{\alpha^*} = \left\{ u_0 \in H^1 \,\middle|\, \inf_{\lambda_0>0,\, x_0\in\mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{1/2}} Q\left(\frac{x-x_0}{\lambda_0}\right)\right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

Theorem 1.1. *For* $0 < \alpha_0 \ll \alpha^* \ll 1$ *and* $u_0 \in \mathcal{A}_{\alpha_0}$, *let* $u(t)$ *be the corresponding solution to* (gKdV), *and* $0 < T \leq +\infty$ *be the maximal lifetime. Then one of the following scenarios occurs*:

Blow up: *The solution* $u(t)$ *blows up in finite time* $0 < T < +\infty$ *with*

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T - t}, \quad \ell(u_0) > 0.$$

In addition, for all $t < T$, *we have* $u(t) \in \mathcal{T}_{\alpha^*}$.

Soliton: *The solution is global, and for all* $t < T = +\infty$, *we have* $u(t) \in \mathcal{T}_{\alpha^*}$. *In addition, there exist a constant* $\lambda_\infty > 0$ *and a* C^1 *function* $x(t)$ *such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) \to Q \quad \textit{in } H^1_{\mathrm{loc}} \quad \textit{as } t \to +\infty,$$
$$|\lambda_\infty - 1| \lesssim \delta(\alpha_0), \quad x(t) \sim \frac{t}{\lambda_\infty^2} \quad \textit{as } t \to +\infty.$$

Exit: For some finite time $0 < t^* < T$, *we have* $u(t^*) \notin \mathcal{T}_{\alpha^*}$.

Moreover, the blow-up and exit scenarios are stable by small perturbation in $\mathcal{A}_{\alpha_0}$.

Martel, Merle, Nakanishi and Raphaël [2016] proved that the initial data in $\mathcal{A}_{\alpha_0}$ which corresponds to the soliton regime is a codimension-one threshold submanifold between blow up and exit.

Theorem 1.2. *Let*

$$\mathcal{A}_{\alpha_0}^{\perp} = \left\{ \varepsilon_0 \in H^1 \;\middle|\; \|\varepsilon_0\|_{H^1} < \alpha_0,\ \int_{y>0} y^{10} \varepsilon_0^2 < 1,\ (\varepsilon_0, Q) = 0 \right\}.$$

Then there exist $\alpha_0 > 0$, $\beta_0 > 0$, *and a* C^1 *function* A,

$$\mathcal{A}_{\alpha_0}^{\perp} \to (-\beta_0, \beta_0),$$

such that for all $\gamma_0 \in \mathcal{A}_{\alpha_0}^{\perp}$ *and* $a \in [-\beta_0, \beta_0]$, *the solution of* (gKdV) *corresponding to* $u_0 = (1+a)Q + \gamma_0$ *satisfies*

- *the soliton regime if* $a = A(\gamma_0)$;
- *the blow-up regime if* $a > A(\gamma_0)$;
- *the exit regime if* $a < A(\gamma_0)$.

In particular, let

$$\mathcal{Q} = \{u_0 \in H^1 \mid \textit{there exists } \lambda_0, x_0 \textit{ such that } u_0 = \lambda_0^{-\frac{1}{2}} Q(\lambda_0^{-1}(x - x_0))\}.$$

Then there exists a small neighborhood $\mathcal{O}$ *of* $\mathcal{Q}$ *in* $H^1 \cap L^2(y_+^{10} dy)$ *and a codimension-one* C^1 *submanifold* $\mathcal{M}$ *of* $\mathcal{O}$ *such that* $\mathcal{Q} \subset \mathcal{M}$ *and for all* $u_0 \in \mathcal{O}$ *the corresponding solution of* (gKdV) *is in the soliton regime if and only if* $u_0 \in \mathcal{M}$.

1D. *Statement of the main result.* The aim of this paper is to classify the dynamics of (gKdV$_\gamma$) near the ground state Q for (gKdV), when γ is small enough. The main idea is that the defocusing term $\gamma u|u|^{q-1}$ has weaker nonlinear effect than the focusing term u^5. So, we may expect that (gKdV$_\gamma$) has similar separation behavior as (gKdV), when γ is small.

More precisely, we fix a small universal constant $\omega^* > 0$ (to ensure the existence of the ground state Q_ω), and then introduce the following L^2 tube around Q_γ:

$$\mathcal{T}_{\alpha^*,\gamma} = \left\{ u_0 \in H^1 \;\middle|\; \inf_{\lambda_0 > 0,\, \lambda_0^{-m}\gamma < \omega^*,\, x_0 \in \mathbb{R}} \left\| u_0 - \frac{1}{\lambda_0^{1/2}} Q_{\lambda_0^{-m}\gamma}\left(\frac{x - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}.$$

Then we have:

Theorem 1.3 (dynamics in $\mathcal{A}_{\alpha_0}$). *For all* $q > 5$, *there exists a constant* $0 < \alpha^*(q) \ll 1$ *such that if* $0 < \gamma \ll \alpha_0 \ll \alpha^* < \alpha^*(q)$, *then for all* $u_0 \in \mathcal{A}_{\alpha_0}$ *the corresponding solution* $u(t)$ *to* (gKdV$_\gamma$) *has one and only one of the following behaviors*:

Soliton: *For all* $t \in [0,+\infty)$, *we have* $u(t) \in \mathcal{T}_{\alpha^*,\gamma}$. *Moreover, there exist a constant* $\lambda_\infty \in (0,+\infty)$ *and a* C^1 *function* $x(t)$ *such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot +x(t)) \to Q_{\lambda_\infty^{-m}\gamma} \quad \text{in } H^1_{\text{loc}} \quad \text{as } t \to +\infty, \tag{1-4}$$

$$x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as } t \to +\infty. \tag{1-5}$$

Blow down: *For all* $t \in [0,+\infty)$, *we have* $u(t) \in \mathcal{T}_{\alpha^*,\gamma}$. *Moreover, there exist two* C^1 *functions* $\lambda(t)$ *and* $x(t)$ *such that*

$$\lambda^{\frac{1}{2}}(t)\, u(t, \lambda(t) \cdot +x(t)) \to Q \quad \text{in } H^1_{\text{loc}} \quad \text{as } t \to +\infty; \tag{1-6}$$

$$\lambda(t) \sim t^{\frac{2}{q+1}}, \quad x(t) \sim t^{\frac{q-3}{q+1}} \quad \text{as } t \to +\infty, \tag{1-7}$$

Exit: *There exists* $0 < t_\gamma^* < +\infty$ *such that* $u(t_\gamma^*) \notin \mathcal{T}_{\alpha^*,\gamma}$.

There exist solutions associated to each regime. Moreover, the soliton and exit regimes are stable under small perturbation in $\mathcal{A}_{\alpha_0}$.

Comments. (1) *Classification of the flow near the ground state.* Theorem 1.3 gives a detailed description of the flow near the ground state Q_γ of (gKdV$_\gamma$). This kind of problem has attracted considerable attention, especially for dispersive equations. See for example, [Nakanishi and Schlag 2011; 2012a; 2012b] for Klein–Gordon and mass-supercritical nonlinear Schrödinger equations; [Fibich, Merle and Raphaël 2006; Merle and Raphaël 2003; 2004; 2005; 2005; 2006; Raphaël 2005; Merle, Raphaël and Szeftel 2013] for mass-critical nonlinear Schrödinger equations; [Martel, Merle and Raphaël 2014; 2015a] for L^2 critical gKdV equations; [Kenig and Merle 2006; Duyckaerts and Merle 2009] for energy-critical nonlinear Schrödinger equations; [Kenig and Merle 2008; Duyckaerts and Merle 2008; Krieger, Nakanishi and Schlag 2013; 2014] for energy-critical wave equations; and [Collot, Merle and Raphaël 2017] for energy-critical nonlinear heat equations. Note that the fact that the blow-down regime near the ground state is a codimension-one threshold submanifold of initial data in $\mathcal{A}_{\alpha_0}$ could be proved much as in [Martel, Merle, Nakanishi and Raphaël 2016].

(2) *Asymptotic stability of solitons for* (gKdV$_\gamma$). Since the soliton regime is open, Theorem 1.3 also implies the asymptotic stability of the soliton Q_γ for (gKdV$_\gamma$) under some suitable decay assumption. Recall that from [Martel and Merle 2001], the soliton Q for the unperturbed critical gKdV equation is not stable in H^1.

(3) *Blow-down behaviors.* Theorem 1.3 shows that a saturated perturbation may lead to some chaotic behaviors (i.e., the blow-down behaviors), which does not seem to appear in the unperturbed case. Examples of solutions with a blow-down behavior were also found in [Donninger and Krieger 2013] for energy-critical wave equations. While for mass-critical NLS, the blow-down behavior can be obtained as the pseudoconformal transformation of the log-log regime.[3] However, Theorem 1.3 is the first time that this type of blow-down behavior is obtained in the context of a saturated perturbation. Furthermore, in

[3] See [Merle, Raphaël and Szeftel 2013, (1.16)], for example.

Theorem 1.3, the blow-down regime is a codimension-one threshold between two stable ones, which is in contrast with the mass-critical nonlinear Schrödinger case, where the blow-down regime is stable.

Now we consider the case when $\gamma \to 0$. As we mentioned before, the defocusing term $\gamma u|u|^{q-1}$ has weaker nonlinear effect than the focusing term u^5. So the results in Theorem 1.3 are expected to be a perturbation of the one in Theorem 1.1.

More precisely, we have:

Theorem 1.4. *Let us fix a nonlinearity $q > 5$, and choose $0 < \alpha_0 \ll \alpha^* < \alpha^*(q)$ as in Theorem 1.3. For all $u_0 \in \mathcal{A}_{\alpha_0}$, let $u(t)$ be the corresponding solution of* (gKdV), *and $u_\gamma(t)$ be the corresponding solution of* (gKdV$_\gamma$). *Then we have*:

(1) *If $u(t)$ is in the blow-up regime defined in Theorem 1.1, then there exists $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$ such that if $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$, then $u_\gamma(t)$ is in the soliton regime defined in Theorem 1.3. Moreover, there exist constants $d_i = d_i(u_0, q) > 0$, $i = 1, 2$, such that*
$$d_1 \gamma^{\frac{2}{q-1}} \leq \lambda_\infty \leq d_2 \gamma^{\frac{2}{q-1}}, \tag{1-8}$$
where λ_∞ is the constant defined in (1-4).

(2) *If $u(t)$ is in the exit regime defined in Theorem 1.1, then there exists $0 < \gamma(u_0, \alpha_0, \alpha^*, q) \ll \alpha_0$ such that if $0 < \gamma < \gamma(u_0, \alpha_0, \alpha^*, q)$, then $u_\gamma(t)$ is in the exit regime defined in Theorem 1.3.*

Remark 1.5. We can see from Theorem 1.4 that (gKdV$_\gamma$) is a perturbation of (gKdV) as $\gamma \to 0$: the soliton regime of (gKdV$_\gamma$) "converges" to the blow-up regime of (gKdV), and the exit regime "converges" to the exit regime of (gKdV).

Remark 1.6. Theorem 1.4 is the first result of this type for nonlinear dispersive equations. One may also expect similar results for the critical NLS or the slightly supercritical gKdV cases. But they are still completely open.

Indeed, for critical NLS, Malkin [1993] predicted a similar asymptotic behavior for the solution to the saturated problem of critical NLS in the log-log region. However, due to the different structures of NLS and gKdV, it seems hard to apply the strategy in this paper to the NLS case.

While for the slightly supercritical gKdV case, the stable self-similar blow-up dynamics is well-studied in [Lan 2016]. But, due to the fact that the self-similar profile constructed in [Koch 2015, Theorem 3] is not in the energy space H^1, we have to choose a suitable cut-off as an approximation of this profile. As a consequence, this generates some error terms that are hard to control, which makes it impossible to consider the saturated problem in this case. However, Strunk [2014] proved the local well-posedness result for supercritical gKdV in a space that contains the self-similar profile, which provides an alternative option for the saturated problems.

1E. *Notation.* For $0 \leq \omega < \omega^* \ll 1$, we let Q_ω be the unique nonnegative radial solution with exponential decay to the ODE
$$Q_\omega'' - Q_\omega + Q_\omega^5 - \omega Q_\omega |Q_\omega|^{q-1} = 0. \tag{1-9}$$

For simplicity, we define $Q = \mathcal{Q}_0$. Recall that we have

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}.$$

We also introduce the linearized operator at $\mathcal{Q}_\omega$:

$$L_\omega f = -f'' + f - 5\mathcal{Q}_\omega^4 f + q\omega|\mathcal{Q}_\omega|^{q-1} f.$$

Similarly, we define $L = L_0$.

Next, we introduce the scaling operator

$$\Lambda f = \tfrac{1}{2} f + y f'.$$

Then, for a given small constant α^*, we denote by $\delta(\alpha^*)$ a generic small constant with

$$\lim_{\alpha^* \to 0} \delta(\alpha^*) = 0.$$

Finally, we denote the L^2 scalar product by

$$(f, g) = \int f(x) g(x)\, dx.$$

1F. ***Outline of the proof.***

1F1. *Decomposition of the flow.* We are searching for solutions of the form

$$u(t,x) \sim \frac{1}{\lambda(t)^{1/2}} Q_{b(t),\omega(t)}\left(\frac{x - x(t)}{\lambda(t)}\right),$$

$$\omega = \frac{\gamma}{\lambda^m}, \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{\lambda_s}{\lambda} = -b, \quad \frac{x_s}{\lambda} = 1,$$

which lead to the modified self-similar equation

$$b\Lambda Q_{b,\omega} + (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})' = 0. \tag{1-10}$$

Formal computations show that b and ω must satisfy the condition

$$b_s + 2b^2 + c_0 \omega_s = 0,$$

where $c_0 = c_0(q) > 0$ is a universal constant.

Combining all the above, we get the formal finite-dimensional system

$$\begin{cases} \dfrac{ds}{dt} = \dfrac{1}{\lambda^3}, \quad \dfrac{\lambda_s}{\lambda} = -b, \quad \dfrac{x_s}{\lambda} = 1, \\ b_s + 2b^2 + c_0\omega_s = 0, \quad \omega = \dfrac{\gamma}{\lambda^m}. \end{cases} \tag{1-11}$$

By standard computations, it is easy to see that (1-11) has the following behavior. Let

$$L_0 = \frac{b(0)}{\lambda^2(0)} + \frac{m c_0 \gamma}{(m+2)\lambda^{m+2}(0)}.$$

We have:

(1) If $L_0 > 0$, then

$$b(t) \to 0, \quad \lambda(t) \to \left(\frac{m\gamma c_0}{(m+2)L_0}\right)^{\frac{1}{m+2}}, \quad x(t) \sim \left(\frac{(m+2)L_0}{m\gamma c_0}\right)^{\frac{2}{m+2}} t$$

as $t \to +\infty$, which corresponds to the soliton regime.

(2) If $L_0 = 0$, then

$$b(t) \to 0, \quad \lambda(t) \to +\infty, \quad x(t) \to +\infty$$

as $t \to +\infty$, which corresponds to the blow-down regime.

(3) If $L_0 < 0$, then

$$b(t) \to -\infty, \quad \lambda(t) \to +\infty$$

as $t \to +\infty$, which corresponds to the exit regime.

1F2. *Modulation theory.* Our first step is to find a solution to (1-10). But for our analysis, it is enough to consider a suitable approximation:[4]

$$Q_{b,\omega}(y) = \mathcal{Q}_\omega(y) + b\chi(|b|^\beta y)P_\omega(y).$$

As long as the solution remains in $\mathcal{T}_{\alpha^*,\gamma}$, we can introduce the geometrical decomposition

$$u(t) = \frac{1}{\lambda(t)^{1/2}}[Q_{b(t),\omega(t)} + \varepsilon(t)]\left(\frac{x - x(t)}{\lambda(t)}\right),$$

with $\omega(t) = \gamma/\lambda(t)^m$ and the error term satisfies some orthogonality conditions. Then the equations of the parameters are roughly speaking of the form

$$\frac{\lambda_s}{\lambda} + b = \frac{dJ_1}{ds} + O(\|\varepsilon\|^2_{H^1_{\text{loc}}}),$$
$$b_s + 2b^2 + c_0\omega_s = \frac{dJ_2}{ds} + O(\|\varepsilon\|^2_{H^1_{\text{loc}}}),$$

with

$$|J_i| \lesssim \|\varepsilon\|_{H^1_{\text{loc}}} + \int_{y>0} |\varepsilon|.$$

Therefore, a L^1 control on the right is needed, otherwise J_i will perturb the formal system (1-11).

1F3. *Monotonicity formula.* Our next step is to derive a control for $\|\varepsilon\|_{H^1_{\text{loc}}}$. Similar to [Martel, Merle and Raphaël 2014, Proposition 3.1], we introduce the nonlinear functional

$$\mathcal{F} \sim \int (\psi\varepsilon_y^2 + \varphi\varepsilon^2 - 5\varepsilon^2 Q_{b,\omega}^4\psi + q\omega\varepsilon^2|Q_{b,\omega}|^{q-1}\psi)$$

[4] See Section 2A for more details.

for some well-chosen weight functions (ψ, φ), which decay exponentially to the left and grow polynomially on the right. We will see from the choice of the orthogonality condition that the leading quadratic term of $\mathcal{F}$ is coercive:

$$\mathcal{F} \gtrsim \|\varepsilon\|^2_{H^1_{\mathrm{loc}}}.$$

Most importantly, we have the monotonicity formula

$$\frac{d}{ds}\left(\frac{\mathcal{F}}{\lambda^{2j}}\right) + \frac{\|\varepsilon\|^2_{H^1_{\mathrm{loc}}}}{\lambda^{2j}} \lesssim \frac{\omega^2 b^2 + b^4}{\lambda^{2j}}$$

for $j = 0, 1$. This formula is crucial in all three cases.

1F4. *Rigidity.* The selection of the dynamics depends on:

(1) For all t,

$$\left| b(t) + \frac{mc_0}{m+2}\omega(t) \right| \lesssim \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}} + b^2(t) + \omega^2(t).$$

(2) For some $t_1^* < T = +\infty$,

$$b(t_1^*) + \frac{mc_0}{m+2}\omega(t_1^*) \gg \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}} + b^2(t) + \omega^2(t).$$

(3) For some $t_1^* < T = +\infty$,

$$-b(t_1^*) - \frac{mc_0}{m+2}\omega(t_1^*) \gg \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}} + b^2(t) + \omega^2(t).$$

We will see that in the first case we have for all t,

$$|b(t)| \sim \omega(t) \gg \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}},$$

and in the second case we have

$$\omega(t) \gg |b(t)| \gg \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}}$$

for $t > t_2^* \geq t_1^*$ as long as $u(t)$ remains in $\mathcal{T}_{\alpha^*,\gamma}$. While in the third case, we have

$$-b(t) \gg \omega(t) \gg \|\varepsilon(t)\|^2_{H^1_{\mathrm{loc}}}$$

for $t > t_1^*$ as long as $u(t)$ remains in $\mathcal{T}_{\alpha^*,\gamma}$. Then reintegrating the modulation equations, we will see that these three cases correspond to the blow-down, soliton and exit regimes respectively.

Moreover, the condition on $b(t_1^*)$ and $\omega(t_1^*)$ which determines the soliton and exit regimes is an open condition to the initial data due to the continuity of the flow. On the other hand, it is easy to construct solutions, which belongs to the soliton and exit regimes respectively. Since, the initial data set $\mathcal{A}_{\alpha_0}$ is connected, we can see that there exist solutions corresponding to the blow-down regime.

1F5. *Proof of Theorem 1.4.* The proof of Theorem 1.4 is based on the fact that the separation condition for (gKdV$_\gamma$) is close to the separation condition for (gKdV) when $\gamma \to 0$. Then Theorem 1.4 follows immediately from a modified H^1 perturbation theory.[5]

[5] See [Killip, Kwon, Shao, and Visan 2012, Theorem 3.1] for the standard L^2 perturbation theory.

2. Nonlinear profile and decomposition of the flow

We will introduce the nonlinear profile and the geometrical decomposition similar to the one in [Martel, Merle and Raphaël 2014], which turns out to lead to the desired rigidity dynamics.

2A. *Structure of the linearized operator L_ω.* Denote by $\mathcal{Y}$ the set of smooth functions f such that for all $k \in \mathbb{N}$ there exist $r_k > 0$, $C_k > 0$ with

$$|\partial_y^k f(y)| \le C_k (1+|y|)^{r_k} e^{-|y|}. \tag{2-1}$$

Let us first recall some results about the linearized operator L.

Lemma 2.1 (properties of L [Martel and Merle 2001; Martel, Merle and Raphaël 2014; Weinstein 1985]). *The self-adjoint operator L (recall that we use the notation $L = L_0$, which was introduced in Section 1E) in L^2 has the following properties*:

(1) *Eigenfunction*: $LQ^3 = -8Q^3$, $LQ' = 0$, $\ker L = \{aQ' | a \in \mathbb{R}\}$.

(2) *Scaling*: $L(\Lambda Q) = -2Q$.

(3) *For any function $f \in L^2$ orthogonal to Q', there exists a unique $g \in H^2$ such that $Lg = f$ with $(g, Q') = 0$. Moreover, if f is even, then g is even, and if f is odd, then g is odd.*

(4) *If $f \in L^2$ such that $Lf \in \mathcal{Y}$, then $f \in \mathcal{Y}$.*

(5) *Coercivity*: *For all $f \in H^1$, if $(f, Q^3) = (f, Q') = 0$, then $(Lf, f) \ge (f, f)$. Moreover, there exists $\kappa_0 > 0$ such that for all $f \in H^1$,*

$$(Lf, f) \ge \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0}[(f, Q)^2 + (f, \Lambda Q)^2 + (f, y\Lambda Q)^2].$$

Proposition 2.2 (nonlocalized profiles [Martel, Merle and Raphaël 2014, Proposition 2.2]). *There exists a unique function P with $P' \in \mathcal{Y}$ such that*

$$(LP)' = \Lambda Q, \qquad \lim_{y\to-\infty} P(y) = \frac{1}{2}\int Q, \qquad |P(y)| \lesssim e^{-\frac{y}{2}} \quad \textit{for } y > 0, \tag{2-2}$$

$$(P, Q) = \frac{1}{16}\left(\int Q\right)^2, \quad (P, Q') = 0. \tag{2-3}$$

Now for the ground state $\mathcal{Q}_\omega$ and the linearized operator L_ω, we have the following properties:

Lemma 2.3. *For $0 < \omega < \omega^* \ll 1$, we have*:

(1) *Null space*: $\ker L_\omega = \{a\mathcal{Q}'_\omega \mid a \in \mathbb{R}\}$.

(2) *Pseudoscaling rule*: $L_\omega(\Lambda \mathcal{Q}_\omega) = -2\mathcal{Q}_\omega + \frac{1}{2}(q-5)\omega \mathcal{Q}_\omega^q$.

(3) *For any function $f \in L^2$ orthogonal to $\mathcal{Q}'_\omega$, there exists a unique $g \in H^2$ such that $L_\omega g = f$ with $(g, \mathcal{Q}'_\omega) = 0$. Moreover, if f is even, then g is even, and if f is odd, then g is odd.*

(4) *If $f \in L^2$ such that $L_\omega f \in \mathcal{Y}$, then $f \in \mathcal{Y}$.*

(5) *Let $Z_\omega = \partial \mathcal{Q}_\omega / \partial \omega$. Then $Z_\omega \in \mathcal{Y}$, and $L_\omega Z_\omega = -\mathcal{Q}_\omega |\mathcal{Q}_\omega|^{q-1}$.*

(6) *Coercivity*: *There exists a* $\kappa_0 > 0$ *such that for all* $f \in H^1$,

$$(L_\omega f, f) \geq \kappa_0 \|f\|_{H^1}^2 - \frac{1}{\kappa_0}[(f, \mathcal{Q}_\omega)^2 + (f, \Lambda\mathcal{Q}_\omega)^2 + (f, y\Lambda\mathcal{Q}_\omega)^2].$$

Proof. Part (1) follows from the same arguments as the proof of [Weinstein 1985, Proposition 2.8; 1986, Proposition 3.2]. Part (2) follows from direct computation. Part (3) is a direct corollary of (1), while for (4), from standard elliptic theory, we know that f is smooth and bounded. So we have $Lf \in \mathcal{Y}$, and from Lemma 2.1, we have $f \in \mathcal{Y}$.

Now we turn to the proof of (5). Differentiating (1-9), we obtain $L_\omega Z_\omega = -\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1}$. Since $\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} \in \mathcal{Y}$, if we can show that $Z_\omega \in L^2$, then we have $Z_\omega \in \mathcal{Y}$. To do this, we introduce the map

$$F : H_e^2 \times \mathbb{R} \mapsto L_e^2, \quad (u, \omega) \mapsto -u'' + u - u^5 + \omega u|u|^{q-1},$$

where H_e^2 and L_e^2 are the Banach spaces consisting of all H^2 and L^2 functions, respectively, which are even. Since $H^2(\mathbb{R})$ is continuously embedded into $L^\infty(\mathbb{R})$, the map F is well-defined.

We claim that there exists a small $\omega^* > 0$ such that if $0 \leq \omega < \omega^*$, then there exists a unique $u(\omega) \in H_e^2$ such that $F(u(\omega), \omega) = 0$. Since we have $F(Q, 0) = 0$, from implicit function theory, it only remains to show that the Fréchet derivative with respect to u, i.e., $\partial F/\partial u|_{(Q,0)} \in \mathcal{L}(H_e^2, L_e^2)$, is invertible and continuous. But it is easy to see that

$$\left.\frac{\partial F}{\partial u}\right|_{(Q,0)} = L,$$

which is invertible and continuous due to part (3) of Lemma 2.1. Hence, we obtain the existence of such $u(\omega)$. Moreover, since F is continuously differentiable with respect to both u and ω, we have $u(\omega)$ is continuously differentiable with respect to ω. In particular, we have $\partial u/\partial\omega \in H_e^2$. But from the uniqueness of $u(\omega)$, we must have $u(\omega) = \mathcal{Q}_\omega$. As a consequence, we have $Z_\omega = \partial\mathcal{Q}_\omega/\partial\omega = \partial u/\partial\omega \in H_e^2$, which concludes the proof of (5).

Finally, (6) follows immediately from a perturbation argument for part (5) of Lemma 2.1. More precisely, since $\mathcal{Q}_\omega$ is C^1 with respect to ω, we have, for all $f \in H^1$,

$$(L_\omega f, f) = (Lf, f) + O(\omega)\|f\|_{H^1}^2,$$

and

$$(f, \mathcal{Q}_\omega)^2 + (f, \Lambda\mathcal{Q}_\omega)^2 + (f, y\Lambda\mathcal{Q}_\omega)^2 = (f, Q)^2 + (f, \Lambda Q)^2 + (f, y\Lambda Q)^2 + O(\omega)\|f\|_{H^1}^2.$$

Together with part (5) of Lemma 2.1, we conclude the proof of part (6) of Lemma 2.3, which finishes the proof of Lemma 2.3. □

Proposition 2.4. *For* $0 < \omega < \omega^* \ll 1$, *there exists a smooth function* P_ω, *with* $P'_\omega \in \mathcal{Y}$, *such that*

$$(L_\omega P_\omega)' = \Lambda\mathcal{Q}_\omega, \quad \lim_{y\to-\infty} P_\omega(y) = \frac{1}{2}\int \mathcal{Q}_\omega, \tag{2-4}$$

$$(P_\omega, \mathcal{Q}'_\omega) = 0, \quad (P_\omega, \mathcal{Q}_\omega) = \frac{1}{16}\left(\int Q\right)^2 + F(\omega), \tag{2-5}$$

where F is a C^1 function with $F(0)=0$. Moreover there exist constants $C_0, C_1, \dots$, independent of ω, such that

$$|P_\omega(y)| + \left|\frac{\partial P_\omega}{\partial \omega}(y)\right| \le C_0 e^{-\frac{y}{2}} \quad \text{for all } y>0, \tag{2-6}$$

$$|P_\omega(y)| + \left|\frac{\partial P_\omega}{\partial \omega}(y)\right| \le C_0 \quad \text{for all } y \in \mathbb{R}, \tag{2-7}$$

$$|\partial_y^k P_\omega(y)| \le C_k e^{-\frac{|y|}{2}} \quad \text{for all } k \in \mathbb{N}_+,\ y \in \mathbb{R}. \tag{2-8}$$

Proof. The proof of Proposition 2.4 is almost parallel to Proposition 2.2. We look for a solution of the form $P_\omega = \widetilde{P}_\omega - \int_y^{+\infty} \Lambda Q_\omega$. The function $y \to \int_y^{+\infty} \Lambda Q_\omega$ is bounded and decays exponentially as $y \to +\infty$. Then, P_ω solves (2-4) if and only if $\widetilde{P}_\omega$ solves

$$(L_\omega \widetilde{P}_\omega)' = \Lambda Q_\omega + \left(L_\omega \int_y^{+\infty} \Lambda Q_\omega\right)' = R'_\omega,$$

where

$$R_\omega = (\Lambda Q_\omega)' - 5Q_\omega^4 \int_y^{+\infty} \Lambda Q_\omega + q\omega |Q_\omega|^{q-1} \int_y^{+\infty} \Lambda Q_\omega.$$

Note that $R_\omega \in \mathcal{Y}$. Since $(\Lambda Q_\omega, Q_\omega) = 0$ and $L_\omega Q'_\omega = 0$, we have $(R_\omega, Q'_\omega) = -(R'_\omega, Q_\omega) = 0$. Then from Lemma 2.3, there exists a unique $\widetilde{P}_\omega \in \mathcal{Y}$, orthogonal to Q'_ω, such that $L_\omega \widetilde{P}_\omega = R_\omega$. Then $P_\omega = \widetilde{P}_\omega - \int_y^{+\infty} \Lambda Q_\omega$ satisfies (2-4) with $(P_\omega, Q'_\omega) = 0$ and $\lim_{y \to -\infty} P_\omega(y) = \frac{1}{2}\int Q_\omega$. Moreover, we have

$$\begin{aligned} 2\int P_\omega Q_\omega &= -\int (L_\omega P_\omega)\Lambda Q_\omega + O(\omega) = \int \Lambda Q_\omega \int_y^{+\infty} \Lambda Q_\omega + O(\omega) \\ &= \frac{1}{2}\left(\int \Lambda Q_\omega\right)^2 + O(\omega) = \frac{1}{8}\left(\int Q\right)^2 + O(\omega). \end{aligned}$$

Let

$$F(\omega) = (P_\omega, Q_\omega) - \frac{1}{16}\left(\int Q\right)^2.$$

Then $F(0) = 0$.

Next we claim that $\partial \widetilde{P}_\omega / \partial \omega \in \mathcal{Y}$. Let us differentiate the equation $L_\omega \widetilde{P}_\omega = R_\omega$ to get

$$L_\omega\left(\frac{\partial \widetilde{P}_\omega}{\partial \omega}\right) = \frac{\partial R_\omega}{\partial \omega} - 20 Z_\omega Q_\omega^3 \widetilde{P}_\omega + q(q-1)\omega Z_\omega Q_\omega |Q_\omega|^{q-3} \widetilde{P}_\omega + q|Q_\omega|^{q-1} \widetilde{P}_\omega. \tag{2-9}$$

Since $Z_\omega \in \mathcal{Y}$, it is easy to check that $\partial R_\omega / \partial \omega \in \mathcal{Y}$. So Lemma 2.3 implies $\partial \widetilde{P}_\omega / \partial \omega \in \mathcal{Y}$.

Now it only remains to prove (2-6)–(2-8). But from [Berestycki and Lions 1983, Section 6], there exist constants $M_0, M_1, \dots$, independent of ω, such that for all $k \in \mathbb{N}$, $y \in \mathbb{R}$,

$$|\partial_y^k Q_\omega(y)| \le M_k e^{-\frac{2|y|}{3}}.$$

Together with (2-9) and the construction of P_ω, we obtain (2-6)–(2-8). It is easy to see that (2-6)–(2-8) also imply $F \in C^1$. □

Now, we proceed to a simple localization of the profile to avoid the nontrivial tail on the left. Let χ be a smooth function with $0 \le \chi \le 1,\ \chi' \ge 0,\ \chi(y) = 1$ if $y > -1$, and $\chi(y) = 0$ if $y < -2$. We fix

$$\beta = \tfrac{3}{4}, \tag{2-10}$$

and define the localized profile

$$\chi_b(y) = \chi(|b|^\beta y), \quad Q_{b,\omega}(y) = \mathcal{Q}_\omega + b\chi_b(y)P_\omega(y). \tag{2-11}$$

Lemma 2.5 (localized profiles). *For $|b| < b^* \ll 1,\ 0 < \omega < \omega^* \ll 1$, we have*:

(1) *Estimates on Q_b: For all $y \in \mathbb{R},\ k \in \mathbb{N}$,*

$$|Q_{b,\omega}(y)| \lesssim e^{-|y|} + |b|(\mathbf{1}_{[-2,0]}(|b|^\beta y) + e^{-\frac{|y|}{2}}), \tag{2-12}$$

$$|\partial_y^k Q_{b,\omega}(y)| \lesssim e^{-|y|} + |b|e^{-\frac{|y|}{2}} + |b|^{1+k\beta}\mathbf{1}_{[-2,-1]}(|b|^\beta y), \tag{2-13}$$

where $\mathbf{1}_I$ denotes the characteristic function of the interval I.

(2) *Equation of $Q_{b,w}$: Let*

$$-\Psi_{b,\omega} = b\Lambda Q_{b,\omega} + (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})'. \tag{2-14}$$

Then, for all $y \in \mathbb{R}$,

$$-\Psi_{b,\omega} = b^2((10\mathcal{Q}^3_\omega P^2_\omega)_y + \Lambda P_\omega) - \tfrac{1}{2}b^2(1-\chi_b)P_\omega + O\big(|b|^{1+\beta}\mathbf{1}_{[-2,-1]}(|b|^\beta y) + b^2(\omega + |b|)e^{-\frac{|y|}{2}}\big). \tag{2-15}$$

Moreover, we have

$$|\partial_y \Psi_{b,\omega}(y)| \lesssim |b|^{1+2\beta}\mathbf{1}_{[-2,-1]}(|b|^\beta y) + b^2 e^{-\frac{|y|}{2}}. \tag{2-16}$$

(3) *Mass and energy properties of $Q_{b,\omega}$:*

$$\left|\int Q^2_{b,\omega} - \left(\int \mathcal{Q}^2_\omega + 2b\int P_\omega \mathcal{Q}_\omega\right)\right| \lesssim |b|^{2-\beta}, \tag{2-17}$$

$$|E(Q_{b,\omega})| \lesssim |b| + \omega. \tag{2-18}$$

Proof. The proof of (1) follows immediately from the definition of $Q_{b,\omega}$ and Proposition 2.4. For (2), let us expand $Q_{b,\omega} = \mathcal{Q}_\omega + b\chi_b P_\omega$ in the expression of $\Psi_{b,\omega}$; using the fact that

$$\mathcal{Q}''_\omega - \mathcal{Q}_\omega + \mathcal{Q}^5_\omega - \omega\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} = 0, \quad (L_\omega P_\omega)' = \Lambda\mathcal{Q}_\omega,$$

we have

$$\begin{aligned}-\Psi_{b,\omega} = {} & b(1-\chi_b)\Lambda\mathcal{Q}_\omega + b\big(\chi_b''' P_\omega + 3\chi_b'' P_\omega' + 2\chi_b' P_\omega'' - \chi_b' P_\omega + 5\chi_b'\mathcal{Q}_\omega P_\omega - q\omega\chi_b'|\mathcal{Q}_\omega|^{q-1}P_\omega\big)\\ & + b^2\big((10\mathcal{Q}^3_\omega\chi_b^2 P^2_\omega)_y + P_\omega\Lambda\chi_b + \chi_b y P'_\omega\big)\\ & + b^3(10\mathcal{Q}^2_\omega\chi^3_b P^3_\omega)_y + b^4(5\mathcal{Q}_\omega\chi^4_b P^4_\omega)_y + b^5(\chi^5_b P^5_\omega)_y\\ & - \omega\big((\mathcal{Q}_\omega + b\chi_b P_\omega)|\mathcal{Q}_\omega + b\chi_b P_\omega|^{q-1} - \mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} - qb\chi_b P_\omega|\mathcal{Q}_\omega|^{q-1}\big)_y.\end{aligned}$$

We keep track of all terms up to b^2. Then (2-15) and (2-16) follow from the construction of the profile $Q_{b,\omega}$.

Finally, for (3), we have

$$\int \chi_b^2 P_\omega^2 \lesssim |b|^{-\beta}.$$

Then (2-17) follows from

$$\int Q_{b,\omega}^2 = \int \mathcal{Q}_\omega^2 + 2b\int \chi_b P_\omega \mathcal{Q}_\omega + b^2\int \chi_b^2 P_\omega^2.$$

While for (2-18), since $E(\mathcal{Q}_\omega) = O(\omega)$, we have

$$|E(Q_{b,\omega})| \lesssim |b| + |E(\mathcal{Q}_\omega)| \lesssim |b| + \omega. \qquad \square$$

2B. ***Geometrical decomposition and modulation estimates.*** In this paper we consider H^1 solutions to (gKdV$_\gamma$) a priori in the modulates tube $\mathcal{T}_{\alpha^*,\gamma}$ of functions near the soliton manifold. More precisely:

Lemma 2.6. *Assume that there exist $(\lambda_1(t), x_1(t)) \in ((\gamma/\omega^*)^{1/m}, +\infty) \times \mathbb{R}$ and $\varepsilon_1(t)$ such that for all $t \in [0, t_0)$, the solution $u(t)$ to (gKdV$_\gamma$) satisfies*

$$u(t,x) = \frac{1}{\lambda_1^{1/2}(t)}[\mathcal{Q}_{\omega_1(t)} + \varepsilon_1(t)]\left(\frac{x - x_1(t)}{\lambda_1(t)}\right), \tag{2-19}$$

with, for all $t \in [0, t_0)$,

$$\omega_1(t) + \|\varepsilon_1(t)\|_{L^2} \le \kappa \ll 1, \tag{2-20}$$

where

$$\omega_1(t) = \frac{\gamma}{\lambda_1^m(t)}.$$

Then we have:

(1) *There exist continuous functions $(\lambda(t), x(t), b(t)) \in (0, +\infty) \times \mathbb{R}^2$ such that for all $t \in [0, t_0)$,*

$$\varepsilon(t, y) = \lambda^{\frac{1}{2}}(t) u(t, \lambda(t)y + x(t)) - Q_{b(t),\omega(t)} \tag{2-21}$$

satisfies the orthogonality conditions

$$(\varepsilon(t), \mathcal{Q}_{\omega(t)}) = (\varepsilon(t), \Lambda \mathcal{Q}_{\omega(t)}) = (\varepsilon(t), y\Lambda \mathcal{Q}_{\omega(t)}) = 0, \tag{2-22}$$

where

$$\omega(t) = \frac{\gamma}{\lambda^m(t)}.$$

Moreover,

$$\omega(t) + \|\varepsilon(t)\|_{L^2} + |b(t)| + \left|1 - \frac{\lambda_1(t)}{\lambda(t)}\right| \lesssim \delta(\kappa), \tag{2-23}$$

$$\|\varepsilon(0)\|_{H^1} \lesssim \delta(\|\varepsilon_1(0)\|_{H^1}). \tag{2-24}$$

(2) *The parameters and error term depend continuously on the initial data. Consider a family of solutions $u_n(t)$, with $u_{0,n} \in \mathcal{A}_{\alpha_0}$, and $u_{0,n} \to u_0$ in H^1 as $n \to +\infty$. Let $(\lambda_n(t), b_n(t), x_n(t), \varepsilon_n(t))$ be the corresponding geometrical parameters and error terms of $u_n(t)$. Suppose the geometrical decompositions of $u_n(t)$ and $u(t)$ hold on $[0, T_0]$ for some $T_0 > 0$. Then for all $t \in [0, T_0]$, we have*

$$(\lambda_n(t), b_n(t), x_n(t), \varepsilon_n(t)) \xrightarrow{\mathbb{R}^3 \times H^1} (\lambda(t), b(t), x(t), \varepsilon(t)) \tag{2-25}$$

as $n \to +\infty$.

Proof. Lemma 2.6 is a standard consequence of the implicit function theorem. We leave the proof for Appendix A. □

Remark 2.7. Similar arguments have also been used in [Martel and Merle 2002a, Lemma 1; 2002c, Lemma 1; Martel, Merle and Raphaël 2014, Lemma 2.5; Merle 2001, Lemma 2] etc.

Remark 2.8. The smallness of $\omega(t)$ ensures that $\mathcal{Q}_{\omega(t)}$ and $Q_{b(t),\omega(t)}$ are both well-defined.

2C. *Modulation equation.* In the framework of Lemma 2.6, we introduce the rescaled variables (s, y)

$$y=\frac{x-x(t)}{\lambda(t)},\quad s=\int_0^t\frac{1}{\lambda^3(\tau)}\,d\tau. \tag{2-26}$$

Then, we have the following properties:

Proposition 2.9. *Assume for all* $t\in[0,t_0)$,

$$\omega(t)+\|\varepsilon(t)\|_{L^2}+\int\varepsilon_y^2e^{-\frac{3|y|}{2(q-2)}}\,dy\le\kappa \tag{2-27}$$

for some small universal constant $\kappa>0$. *Then the functions* $(\lambda(s),x(s),b(s))$ *are all* C^1 *and the following hold*:

(1) *Equation of* ε: *For all* $s\in[0,s_0)$,

$$\begin{aligned}\varepsilon_s-(L_\omega\varepsilon)_y+b\Lambda\varepsilon=\left(\frac{\lambda_s}{\lambda}+b\right)(\Lambda Q_{b,\omega}+\Lambda\varepsilon)+\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon)_y\\ -b_s\frac{\partial Q_{b,\omega}}{\partial b}-\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}+\Psi_{b,\omega}-(R_b(\varepsilon))_y-(R_{\mathrm{NL}}(\varepsilon))_y,\end{aligned} \tag{2-28}$$

where

$$\Psi_{b,\omega}=-b\Lambda Q_{b,\omega}-(Q''_{b,\omega}-Q_{b,\omega}+Q^5_{b,\omega}-\omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})', \tag{2-29}$$

$$R_b(\varepsilon)=5(Q^4_{b,\omega}-\mathcal{Q}^4_\omega)\varepsilon-q\omega(|Q_{b,\omega}|^{q-1}-|\mathcal{Q}_\omega|^{q-1})\varepsilon, \tag{2-30}$$

$$\begin{aligned}R_{\mathrm{NL}}(\varepsilon)=(\varepsilon+Q_{b,\omega})^5-5Q^4_{b,\omega}\varepsilon-Q^5_{b,\omega}\\ -\omega\big[(\varepsilon+Q_{b,\omega})|\varepsilon+Q_{b,\omega}|^{q-1}-q\varepsilon|Q_{b,\omega}|^{q-1}-Q_{b,\omega}|Q_{b,\omega}|^{q-1}\big].\end{aligned} \tag{2-31}$$

(2) *Estimate induced by the conservation laws*: *For* $s\in[0,s_0)$,

$$\|\varepsilon\|_{L^2}\lesssim|b|^{\frac14}+\omega^{\frac12}+\left|\int u_0^2-\int Q^2\right|^{\frac12}, \tag{2-32}$$

$$\frac{\|\varepsilon_y\|^2_{L^2}}{\lambda^2}\lesssim\frac{1}{\lambda^2}\left(\omega+|b|+\int\varepsilon^2e^{-\frac{|y|}{10}}\right)+\gamma\frac{\|\varepsilon_y\|^{m+2}_{L^2}}{\lambda^{m+2}}+|E_0|. \tag{2-33}$$

(3) *H^1 modulation equation: For all $s \in [0, s_0)$,*

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right| \lesssim \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|(\omega+|b|), \tag{2-34}$$

$$|b_s|+|\omega_s| \lesssim (\omega+|b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|\right]+\int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-35}$$

(4) *L^1 control on the right: Assume uniformly L^1 control on the right; that is, for all $t \in [0, t_0)$,*

$$\int_{y>0} |\varepsilon(t)| \lesssim \delta(\kappa). \tag{2-36}$$

Then the quantities J_1 and J_2 below are well-defined. Moreover, we have:

(a) *Law of λ: Let*

$$\rho_1(y)=\frac{4}{\left(\int Q\right)^2}\int_{-\infty}^{y} \Lambda Q, \quad J_1(s)=(\varepsilon(s), \rho_1), \tag{2-37}$$

where Q is the ground state for (gKdV). Then we have

$$\left|\frac{\lambda_s}{\lambda}+b-2\left((J_1)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_1\right)\right| \lesssim (\omega+|b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|\right]+\int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-38}$$

(b) *Law of b: Let*

$$\rho_2=\frac{16}{\left(\int Q\right)^2}\left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}\Lambda Q+P-\frac{1}{2}\int Q\right)-8\rho_1, \quad J_2(s)=(\varepsilon(s), \rho_2), \tag{2-39}$$

where P was introduced in Proposition 2.2. Then we have

$$\left|b_s+2b^2+\omega_s G'(\omega)+b\left((J_2)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right)\right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}}+(\omega+|b|)b^2, \tag{2-40}$$

where $G \in C^2$ with $G(0)=0,\ G'(0)=c_0>0$, for some universal constant c_0.

(c) *Law of b/λ^2: Let*

$$\rho=4\rho_1+\rho_2 \in \mathcal{Y}, \quad J(s)=(\varepsilon(s), \rho). \tag{2-41}$$

Then we have

$$\left|\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)+\frac{b}{\lambda^2}\left(J_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J\right)+\frac{\omega_s G'(\omega)}{\lambda^2}\right| \lesssim \frac{1}{\lambda^2}\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\right). \tag{2-42}$$

Remark 2.10. The proof of Proposition 2.9 follows almost the same procedure as [Martel, Merle and Raphaël 2014, Lemma 2.7]. It is important that there is no a priori assumption on the upper bound of $\lambda(t)$. This fact ensures that Proposition 2.9 can be used in all three regimes.[6]

Proof. (1) Equation (2-28) follows by direct computation from the equation of $u(t)$.

[6] We will see in Section 4 that we can't expect any (finite) upper bound on the scaling parameter $\lambda(t)$ in both the blow-down and exit cases.

(2) We write down the mass conservation law

$$\int Q_{b,\omega}^2 - \int Q^2 + \int \varepsilon^2 + 2(\varepsilon, Q_{b,\omega}) = \int u_0^2 - \int Q^2. \tag{2-43}$$

From (2-17) and the orthogonality condition (2-22), we have

$$\int \varepsilon^2 \lesssim |b| + \omega + |b|^{1-\beta}\|\varepsilon\|_{L^2} + \left|\int u_0^2 - \int Q^2\right|.$$

Then (2-32) follows from $\beta = \frac{3}{4}$.

Similarly, we use the energy conservation law and (2-18) to obtain

$$\begin{aligned}
2\lambda^2 E_0 &= 2E(Q_{b,\omega}) - 2\int \varepsilon(Q_{b,\omega})_{yy} + \int \varepsilon_y^2 - \frac{1}{3}\int [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6] \\
&\quad + \frac{2\omega}{q+1}\int [|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1}] \\
&= O(|b|+\omega) + \int \varepsilon_y^2 - 2\int \varepsilon[(Q_{b,\omega}-\mathcal{Q}_\omega)_{yy} + (Q_{b,\omega}^5 - \mathcal{Q}_\omega^5) + \omega(Q_{b,\omega}|Q_{b,\omega}|^{q-1} - \mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1})] \\
&\quad - \frac{1}{3}\int [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \\
&\quad + \frac{2\omega}{q+1}\int [|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}].
\end{aligned}$$

We estimate all terms in the above identity. By the definition of $Q_{b,\omega}$, we have

$$\begin{aligned}
&\left|\int \varepsilon[(Q_{b,\omega}-\mathcal{Q}_\omega)_{yy} + (Q_{b,\omega}^5 - \mathcal{Q}_\omega^5) + \omega(Q_{b,\omega}|Q_{b,\omega}|^{q-1} - \mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1})]\right| \\
&\qquad \lesssim |b|\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|^{1+2\beta}\int_{-2|b|^{-\beta}\le y\le 0} |\varepsilon| \\
&\qquad \lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}}.
\end{aligned}$$

For the nonlinear term, we use the Gagliardo–Nirenberg inequality to estimate

$$\begin{aligned}
\left|\int [(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5]\right| &\lesssim \int \varepsilon^2 \mathcal{Q}_\omega^4 + \int \varepsilon^6 + |b|\int \varepsilon^2 \\
&\lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b| + \|\varepsilon\|_{L^2}^4 \|\varepsilon_y\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
\left|\omega\int [|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\right| &\lesssim \omega\left(|b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int |\varepsilon|^{q+1}\right) \\
&\lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \frac{\gamma}{\lambda^m}\|\varepsilon\|_{L^2}^{\frac{q+3}{2}} \|\varepsilon_y\|_{L^2}^{m+2} \\
&\lesssim |b| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \gamma\frac{\|\varepsilon_y\|_{L^2}^{m+2}}{\lambda^m}.
\end{aligned}$$

Collecting all the estimates above, we obtain (2-33).

(3) Let us differentiate the orthogonality conditions

$$(\varepsilon(t), \Lambda \mathcal{Q}_{\omega(t)}) = (\varepsilon(t), y\Lambda \mathcal{Q}_{\omega(t)}) = 0.$$

Note that

$$\frac{d}{ds}(\varepsilon, \Lambda \mathcal{Q}_\omega) = (\varepsilon_s, \Lambda \mathcal{Q}_\omega) + \omega_s(\varepsilon, \Lambda Z_\omega),$$

where $Z_\omega = \partial \mathcal{Q}_\omega/\partial\omega \in \mathcal{Y}$. So we have

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda}+b\right) - \frac{(\varepsilon, L_\omega(\Lambda \mathcal{Q}_\omega)')}{\|\Lambda \mathcal{Q}_\omega\|_{L^2}^2}\right| + \left|\left(\frac{x_s}{\lambda}-1\right) - \frac{(\varepsilon, L_\omega(y\Lambda \mathcal{Q}_\omega)')}{\|\Lambda \mathcal{Q}_\omega\|_{L^2}^2}\right| \\ \lesssim \left(\left|\frac{\lambda_s}{\lambda}+b\right| + \left|\frac{x_s}{\lambda}-1\right| + |b|\right) \times \left(\omega + |b| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}\right) \\ + |b_s| + |\omega_s| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int \varepsilon^5 e^{-\frac{9|y|}{10}} + \int |\varepsilon|^q e^{-\frac{9|y|}{10}}.\end{aligned}$$

For the nonlinear term, we use Sobolev embedding and the a priori smallness (2-27),

$$\begin{aligned}\|\varepsilon e^{-\frac{|y|}{4}}\|_{L^\infty}^2 &\le \|\varepsilon e^{-\frac{3|y|}{4(q-2)}}\|_{L^\infty}^2 \\ &\lesssim \int (\partial_y \varepsilon^2 + \varepsilon^2) e^{-\frac{3|y|}{4(q-2)}} \\ &\ll 1,\end{aligned}$$

to estimate

$$\int \varepsilon^5 e^{-\frac{9|y|}{10}} + \int |\varepsilon|^q e^{-\frac{9|y|}{10}} \lesssim (\|\varepsilon e^{-\frac{|y|}{4}}\|_{L^\infty}^3 + \|\varepsilon e^{-\frac{3|y|}{4(q-2)}}\|_{L^\infty}^{q-2}) \int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-44}$$

Here we use the basic fact that $q > 5$.

For ω_s, we have

$$\omega_s = -m\omega \frac{\lambda_s}{\lambda} = m\omega b - m\omega\left(\frac{\lambda_s}{\lambda}+b\right). \tag{2-45}$$

The above estimates imply

$$\left|\frac{\lambda_s}{\lambda}+b\right| + \left|\frac{x_s}{\lambda}-1\right| \lesssim (\omega+|b|)|b| + |b_s| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} \tag{2-46}$$

and

$$\begin{aligned}\left|\left(\frac{\lambda_s}{\lambda}+b\right) - \frac{(\varepsilon, L_\omega(\Lambda \mathcal{Q}_\omega)')}{\|\Lambda \mathcal{Q}_\omega\|_{L^2}^2}\right| + \left|\left(\frac{x_s}{\lambda}-1\right) - \frac{(\varepsilon, L_\omega(y\Lambda \mathcal{Q}_\omega)')}{\|\Lambda \mathcal{Q}_\omega\|_{L^2}^2}\right| \\ \lesssim (\omega+|b|)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|\right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}.\end{aligned} \tag{2-47}$$

Next, let us differentiate the relation $(\varepsilon, \mathcal{Q}_\omega)=0$ to obtain

$$\begin{aligned}
0&=(\varepsilon,\mathcal{Q}_\omega)_s=(\varepsilon_s,\mathcal{Q}_\omega)+\left(\varepsilon,\omega_s\frac{\partial\mathcal{Q}_\omega}{\partial\omega}\right)\\
&=\omega_s\left(\varepsilon,\frac{\partial\mathcal{Q}_\omega}{\partial\omega}\right)-(\varepsilon,L_\omega(\mathcal{Q}_\omega'))-b(\Lambda\varepsilon,\mathcal{Q}_\omega)\\
&\quad+\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda Q_{b,\omega},\mathcal{Q}_\omega)+(\Lambda\varepsilon,\mathcal{Q}_\omega)]+\left(\frac{x_s}{\lambda}-1\right)[(Q_{b,\omega}',\mathcal{Q}_\omega)+(\varepsilon',\mathcal{Q}_\omega)]\\
&\quad-b_s[(P_\omega\chi_b,\mathcal{Q}_\omega)+(\beta y\chi_b',\mathcal{Q}_\omega)]-\omega_s\left(\mathcal{Q}_\omega,\frac{\partial Q_{b,\omega}}{\partial\omega}\right)+(\Psi_{b,\omega},\mathcal{Q}_\omega)+(R_b(\varepsilon)+R_{\mathrm{NL}}(\varepsilon),\mathcal{Q}_\omega'). \qquad (2\text{-}48)
\end{aligned}$$

Substituting the facts

$$(P_\omega\chi_b,\mathcal{Q}_\omega)+(\beta y\chi_b',\mathcal{Q}_\omega)=(P_\omega,\mathcal{Q}_\omega)+O(b^{10})\sim 1,$$

$$L_\omega\mathcal{Q}_\omega'=0,\quad (\mathcal{Q}_\omega,\Lambda\mathcal{Q}_\omega)=(\mathcal{Q}_\omega,\mathcal{Q}_\omega')=(\varepsilon,\Lambda\mathcal{Q}_\omega)=0,$$

$$|(R_b(\varepsilon)+R_{\mathrm{NL}}(\varepsilon),\mathcal{Q}_\omega')|\lesssim(\omega+|b|)\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+\int\varepsilon^2e^{-\frac{|y|}{10}}$$

and (2-15), (2-16), (2-44), (2-45) into (2-48), we obtain

$$|b_s|\lesssim\left(\omega+|b|+\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}\right)\left(\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|\right)+(\omega+|b|)\left[\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+|b|\right]+\int\varepsilon^2e^{-\frac{|y|}{10}}. \qquad (2\text{-}49)$$

Combining (2-45), (2-46) and (2-49), we get (2-34) and (2-35).

(4) First, we claim the sharp equation

$$\begin{aligned}
b_s+2b^2+\omega_sG'(\omega)-\frac{16b}{(\int Q)^2}\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon,L(\Lambda Q)')+20(\varepsilon,PQ^3Q')\right]&\\
=O\left(b^2(\omega+|b|)+\int\varepsilon^2e^{-\frac{|y|}{10}}\right)& \qquad (2\text{-}50)
\end{aligned}$$

holds. To prove this, we take the scalar product of (2-28) with $\mathcal{Q}_\omega$. We keep track of all terms up to b^2.

First, from (2-15), we have

$$\begin{aligned}
(\Psi_{b,\omega},\mathcal{Q}_\omega)&=-b^2((10P_\omega^2\mathcal{Q}_\omega^3)_y+\Lambda P_\omega,\mathcal{Q}_\omega)+O(b^2(|b|+\omega))\\
&=-b^2((10P^2Q^3)_y+\Lambda P,Q)+O(b^2(|b|+\omega))\\
&=-\tfrac18 b^2\|Q\|_{L^1}^2+O(b^2(|b|+\omega)), \qquad (2\text{-}51)
\end{aligned}$$

where for the last step we use the computation

$$\begin{aligned}
(\Lambda P,Q)&=-(P,\Lambda Q)=-(P,(LP)')=(P,(P''-P+5Q^4P)')\\
&=(P,P'''-P')+10\int Q^3Q'P^2,
\end{aligned}$$

and from Proposition 2.2, we obtain

$$((10P^2Q^3)_y + \Lambda P, Q) = \tfrac{1}{2} \lim_{y\to-\infty} P^2 = \tfrac{1}{8}\|Q\|_{L^1}^2.$$

Next, from Proposition 2.4, we have

$$\begin{aligned}\left(b_s \frac{\partial Q_{b,\omega}}{\partial b}, \mathcal{Q}_\omega\right) &= b_s((\chi_b + \beta y \chi_b')P_\omega, \mathcal{Q}_\omega) = b_s(P_\omega, \mathcal{Q}_\omega) + O(b^{10}) \\ &= \tfrac{1}{16} b_s \|Q\|_{L^1}^2 + F(\omega) b_s + O(b^{10}),\end{aligned} \tag{2-52}$$

where F is the C^1 function introduced in Proposition 2.4. From Lemma 2.3, we have

$$\begin{aligned}(Z_\omega, \mathcal{Q}_\omega) &= -\frac{1}{2}(L_\omega Z_\omega, \Lambda \mathcal{Q}_\omega) + O(\omega) = \frac{1}{2}\int (\Lambda \mathcal{Q}_\omega)\mathcal{Q}_\omega|\mathcal{Q}_\omega|^{q-1} + O(\omega) \\ &= \frac{q-1}{4(q+1)}\int |\mathcal{Q}_\omega|^{q+1} + O(\omega) > 0.\end{aligned}$$

Then from (2-35), we have

$$\left(\omega_s \frac{\partial Q_{b,\omega}}{\partial \omega}, \mathcal{Q}_\omega\right) = \omega_s \frac{1}{2}\frac{\partial \|\mathcal{Q}_\omega\|_{L^2}^2}{\partial \omega} + O(|b\omega_s|) = \omega_s \widetilde{G}'(\omega) + O\left(b^2(\omega+|b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right), \tag{2-53}$$

with $\widetilde{G}(\omega) = \frac{1}{2}(\|\mathcal{Q}_\omega\|_{L^2}^2 - \|Q\|_{L^2}^2)$. It is easy to check $\widetilde{G}(0) = 0$, $\widetilde{G} \in C^1$, and

$$\widetilde{G}'(0) = (Z_\omega, \mathcal{Q}_\omega)|_{\omega=0} = \frac{q-1}{4(q+1)}\int |Q|^{q+1} > 0.$$

Next, from Proposition 2.4 we have

$$|(Q'_{b,\omega} + \varepsilon_y, \mathcal{Q}_\omega)| \lesssim \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |(\mathcal{Q}'_\omega, \mathcal{Q}_\omega)| + |(P'_\omega, \mathcal{Q}_\omega)| + b^{10},$$

which together with (2-34) implies

$$\left|\left(\frac{x_s}{\lambda} - 1\right)(Q'_{b,\omega} + \varepsilon_y, \mathcal{Q}_\omega)\right| \lesssim b^2(\omega + |b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \tag{2-54}$$

For the small linear term, we have

$$\begin{aligned}\int R_b(\varepsilon)\mathcal{Q}'_\omega &= 20b\int P_\omega \mathcal{Q}_\omega^3 \mathcal{Q}'_\omega \varepsilon + |b|(\omega+|b|)O\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} \\ &= 20b\int PQ^3Q'\varepsilon + |b|(\omega+|b|)O\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}.\end{aligned} \tag{2-55}$$

Since the nonlinear term can be estimated with the help of (2-44), we then have

$$\begin{aligned}b_s + \frac{2b^2 + \omega_s \widetilde{G}'(\omega)}{1+H(\omega)} - \frac{16}{(1+H(\omega))(\int Q)^2}\left[(\Lambda Q_{b,\omega}, \mathcal{Q}_\omega)\left(\frac{\lambda_s}{\lambda} + b\right) + 20b(\varepsilon, PQ^3Q')\right] \\ = O\left(b^2(\omega+|b|) + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right),\end{aligned}$$

where

$$H(\omega)=\frac{16}{(\int Q)^2}F(\omega).$$

From (2-47) we have

$$\left|\left(\frac{\lambda_s}{\lambda}+b\right)-\frac{(\varepsilon,L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2}\right|\lesssim(\omega+|b|)\left[\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|\right]+\int\varepsilon^2e^{-\frac{|y|}{10}}.$$

Moreover, we have

$$|(\Lambda Q_{b,\omega},\mathcal{Q}_\omega)-b(\Lambda P,Q)|\lesssim b^{10}+|b(\Lambda P,Q)-b(\Lambda P_\omega,\mathcal{Q}_\omega)|\lesssim|b|(\omega+|b|).$$

We then conclude that

$$b_s+\frac{2b^2+\omega_s\widetilde{G}'(\omega)}{1+H(\omega)}-\frac{16b}{(1+H(\omega))(\int Q)^2}\left[\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon,L(\Lambda Q)')+20(\varepsilon,PQ^3Q')\right]$$
$$=O\left(b^2(\omega+|b|)+\int\varepsilon^2e^{-\frac{|y|}{10}}\right).\tag{2-56}$$

Finally, since $H\in C^1,\ H(0)=0$, it is enough to check that the function

$$G(\omega)=\int_0^\omega\frac{\widetilde{G}'(x)}{1+H(x)}\,dx$$

satisfies $G\in C^2,\ G(0)=0,\ G'(0)=c_0>0$. Then, (2-56) implies (2-50) immediately.

Now, we turn to the proof of (2-38), (2-40) and (2-42). For all $f\in\mathcal{Y}$, independent of s, $\left(\varepsilon,\int_{-\infty}^y f\right)$ is well-defined due to (2-36). Moreover, we have

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon,\int_{-\infty}^y f\right)=&-(\varepsilon,L_\omega f)+\left(\frac{\lambda_s}{\lambda}+b\right)\left(\Lambda Q_{b,\omega},\int_{-\infty}^y f\right)+\frac{\lambda_s}{\lambda}\left(\Lambda\varepsilon,\int_{-\infty}^y f\right)\\&-\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon,f)-\left(b_s\frac{\partial Q_{b,\omega}}{\partial b}+\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega},\int_{-\infty}^y f\right)\\&+\left(\Psi_{b,\omega},\int_{-\infty}^y f\right)+(R_b(\varepsilon)+R_{\mathrm{NL}}(\varepsilon),f).\end{aligned}$$

Using (2-34), (2-35), (2-44) and Proposition 2.4, we have

$$\begin{aligned}\frac{d}{ds}\left(\varepsilon,\int_{-\infty}^y f\right)=&-(\varepsilon,Lf)+\left(\frac{\lambda_s}{\lambda}+b\right)\left(\Lambda Q,\int_{-\infty}^y f\right)+\left(\frac{x_s}{\lambda}-1\right)(f,Q)\\&-\frac{1}{2}\frac{\lambda_s}{\lambda}\left(\varepsilon,\int_{-\infty}^y f\right)+O\left((|b|+\omega)\left[\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}\right]\right)\\&+O((|b|+\omega)|b|)+O\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right).\end{aligned}\tag{2-57}$$

Proof of (2-38). We apply (2-57) to $f = \Lambda Q$, using the facts

$$L\Lambda Q = -2Q, \quad \left(\Lambda Q, \int_{-\infty}^{y} \Lambda Q\right) = \frac{1}{8}\left(\int Q\right)^2, \quad \left(Q', \int_{-\infty}^{y} \Lambda Q\right) = 0,$$

to obtain

$$2(J_1)_s = \frac{16(\varepsilon, Q)}{\left(\int Q\right)^2} + \left(\frac{\lambda_s}{\lambda} + b\right) - \frac{\lambda_s}{\lambda} J_1 + O\left((|b| + \omega)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|\right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right).$$

Then (2-38) follows immediately from the orthogonality condition (2-22).

Proof of (2-40). We apply (2-57) to $f = \rho_2'$. Then from Lemma 2.1 and Proposition 2.2, we have

$$\begin{aligned}(\Lambda Q, \rho_2) &= \frac{16}{\left(\int Q\right)^2}\left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}}\Lambda Q + P - \frac{1}{2}\int Q, \Lambda Q\right) - \frac{32}{\left(\int Q\right)^2}\left(\Lambda Q, \int_{-\infty}^{y} \Lambda Q\right) \\ &= \frac{16}{\left(\int Q\right)^2}[(\Lambda P, Q) + (\Lambda Q, P)] + \frac{4\|Q\|_{L^1}^2}{\left(\int Q\right)^2} - \frac{16}{\left(\int Q\right)^2}\left(\int \Lambda Q\right)^2 = 0,\end{aligned}$$

and

$$(\rho', Q) = \frac{16}{\left(\int Q\right)^2}\left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}}(\Lambda Q)' + P', Q\right) - 8(\rho_1', Q).$$

Next, from

$$\begin{aligned}L(P') &= (LP)' + 20Q'Q^3P \\ &= \Lambda Q + 20Q'Q^3P\end{aligned}$$

and the orthogonality condition $(\varepsilon, \Lambda Q_\omega) = 0$, we have

$$\begin{aligned}(\varepsilon, L\rho_2') &= \frac{16}{\left(\int Q\right)^2}\left(\varepsilon, L\left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}}(\Lambda Q)' + P'\right]\right) - 8(\varepsilon, L\rho_1') \\ &= \frac{16}{\left(\int Q\right)^2}\left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q')\right] + O(\omega)\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}.\end{aligned}$$

Substituting all the above estimates into (2-57) with $f = \rho_2'$, we obtain

$$\begin{aligned}(J_2)_s = &-\frac{16}{\left(\int Q\right)^2}\left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q')\right] - \frac{1}{2}\frac{\lambda_s}{\lambda} J_2 \\ &+ O\left((|b| + \omega)\left[\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}} + |b|\right] + \int \varepsilon^2 e^{-\frac{|y|}{10}}\right).\end{aligned} \tag{2-58}$$

Then (2-40) follows from (2-50) and (2-58).

Proof of (2-42). From (2-38) and (2-42),

$$\begin{aligned}\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)&=\frac{b_s+2b^2}{\lambda^2}-\frac{2b}{\lambda^2}\left(\frac{\lambda_s}{\lambda}+b\right)\\&=-\frac{b}{\lambda^2}\left[(J_2)_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J_2\right]-\frac{2b}{\lambda^2}\left[2(J_1)_s+\frac{\lambda_s}{\lambda}J_1\right]-\frac{\omega_s G'(\omega)}{\lambda^2}+O\left(\frac{1}{\lambda^2}\left(\int\varepsilon^2e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\right)\right)\\&=-\frac{b}{\lambda^2}\left[J_s+\frac{1}{2}\frac{\lambda_s}{\lambda}J\right]-\frac{\omega_s G'(\omega)}{\lambda^2}+O\left(\frac{1}{\lambda^2}\left(\int\varepsilon^2e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\right)\right),\end{aligned}$$

which is exactly (2-42).

Finally, it is easy to check that $\lim_{|y|\to+\infty}\rho(y)=0$, which implies $\rho\in\mathcal{Y}$. □

3. Monotonicity formula

We will introduce the monotonicity tools developed in [Martel and Merle 2002c; Martel, Merle and Raphaël 2014]. This is the key technical argument of the analysis for solutions near the soliton.

3A. *Pointwise monotonicity.* Let $(\varphi_i)_{i=1,2}$, $\psi\in C^\infty(\mathbb{R})$ be such that

$$\varphi_i(y)=\begin{cases}e^y & \text{for } y<-1,\\ 1+y & \text{for } -\frac{1}{2}<y<\frac{1}{2},\\ y^i & \text{for } y>2,\end{cases}\qquad \varphi'(y)>0 \text{ for all } y\in\mathbb{R},\tag{3-1}$$

$$\psi(y)=\begin{cases}e^{2y} & \text{for } y<-1,\\ 1 & \text{for } y>-\frac{1}{2},\end{cases}\qquad \psi'(y)\geq 0 \text{ for all } y\in\mathbb{R}.\tag{3-2}$$

Let $B>100$ be a large universal constant to be chosen later. We then define the weight function

$$\psi_B(y)=\psi\left(\frac{y}{B}\right),\quad \varphi_{i,B}(y)=\varphi\left(\frac{y}{B}\right),\tag{3-3}$$

and the weighted Sobolev norm of ε

$$\mathcal{N}_i(s)=\int\left(\varepsilon_y^2(s,y)\psi_B(y)+\varepsilon^2(s,y)\varphi_{i,B}(y)\right)dy,\quad i=1,2,\tag{3-4}$$

$$\mathcal{N}_{i,\mathrm{loc}}(s)=\int\varepsilon^2(s,y)\varphi_{i,B}'(y)\,dy,\quad i=1,2.\tag{3-5}$$

Then we have the following monotonicity:

Proposition 3.1 (monotonicity formula). *There exist universal constants $\mu>0$, $B=B(q)>100$ and $0<\kappa\ll 1$ such that the following holds. Let $u(t)$ be a solution of* (gKdV$_\gamma$) *satisfying* (2-20) *on* $[0,t_0]$, *and hence the geometrical decomposition* (2-21) *holds on* $[0,t_0]$. *Let $s_0=s(t_0)$, and assume the following a priori bounds hold for all $s\in[0,s_0]$:*

(H1) *Scaling-invariant bound:*

$$\omega(s)+|b(s)|+\mathcal{N}_2(s)+\|\varepsilon(s)\|_{L^2}+\omega(s)\|\varepsilon_y(s)\|_{L^2}^m\leq\kappa.\tag{3-6}$$

(H2) *Bound related to H^1 scaling:*

$$\frac{\omega(s)+|b(s)|+\mathcal{N}_2(s)}{\lambda^2(s)} \le \kappa. \tag{3-7}$$

(H3) *L^2-weighted bound on the right:*

$$\int_{y>0} y^{10}\varepsilon^2(s,y)\,dy \le 50\left(1+\frac{1}{\lambda^{10}(s)}\right). \tag{3-8}$$

We define the Lyapounov functionals for $(i,j)\in\{1,2\}^2$ as

$$\mathcal{F}_{i,j} = \int\Big(\varepsilon_y^2\psi_B + (1+\mathcal{J}_{i,j})\varepsilon^2\varphi_{i,B} - \tfrac{1}{3}\psi_B[(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5] \\ + \frac{2\omega}{q+1}[|Q_{b,\omega}+\varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\psi_B\Big), \tag{3-9}$$

with[7]

$$\mathcal{J}_{i,j} = (1-J_1)^{-4(j-1)-2i} - 1. \tag{3-10}$$

Then the following estimates hold on $[0,s_0]$:

(1) *Scaling-invariant Lyapounov control*: *for $i=1,2$,*

$$\frac{d\mathcal{F}_{i,1}}{ds} + \mu\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}' \lesssim_B b^2(\omega^2+b^2). \tag{3-11}$$

(2) *H^1-scaling Lyapounov control*: *for $i=1,2$,*

$$\frac{d}{ds}\left(\frac{\mathcal{F}_{i,2}}{\lambda^2}\right) + \frac{\mu}{\lambda^2}\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}' \lesssim_B \frac{b^2(\omega^2+b^2)}{\lambda^2}. \tag{3-12}$$

(3) *Coercivity and pointwise bounds*: *there hold for all $(i,j)\in\{1,2\}^2$,*

$$\mathcal{N}_i \lesssim \mathcal{F}_{i,j} \lesssim \mathcal{N}_i, \tag{3-13}$$

$$|J_i| + |\mathcal{J}_{i,j}| \lesssim \mathcal{N}_2^{\frac{1}{2}}. \tag{3-14}$$

Remark 3.2. The proof of Proposition 3.1 is almost the same as that of [Martel, Merle and Raphaël 2014, Proposition 3.1]. The only difference here is the additional terms involving ω.

Remark 3.3. Similar to Proposition 2.9, we do not assume any a priori control on the upper bound of $\lambda(s)$ so that the monotonicity formula can be used in all three cases.

Remark 3.4. As mentioned in [Martel, Merle and Raphaël 2014, Proposition 3.1], the weight function ψ decays faster than φ_i on the left. As a result, $\mathcal{N}_2$ and $\mathcal{F}_{i,j}$ do not control $\int \varepsilon_y^2\varphi_{i,B}'$ (see Remark 3.5 of that paper for more details).

[7] Recall that J_1 was defined in (2-37).

Proof of Proposition 3.1. The proofs of (3-13) and (3-14) are exactly the same as [Martel, Merle and Raphaël 2014, Proposition 3.1]. We only need to prove (3-11) and (3-12). To do this, we compute directly to obtain that, for all $(i,j)\in\{1,2\}^2$,

$$\lambda^{2(j-1)}\left(\frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}}\right)_s=f_1+f_2+f_3+f_4+f_5, \tag{3-15}$$

where

$$\begin{aligned}
f_1&=2\int\left(\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon\right)(-(\psi_B\varepsilon)_y+\varepsilon\varphi_{i,B}-\psi_B\Delta_{b,\omega}(\varepsilon)),\\
f_2&=2\int\left(\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon\right)\varepsilon\mathcal{J}_{i,j}\varphi_{i,B},\\
f_3&=2\frac{\lambda_s}{\lambda}\int\Lambda\varepsilon(-(\psi_B\varepsilon_y)_y+(1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B}-\psi_B\Delta_{b,\omega}(\varepsilon))+(\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j},\\
f_4&=-2\int\psi_B(Q_{b,\omega})_s[\Delta_{b,\omega}-5\varepsilon Q_{b,\omega}^4+q\omega\varepsilon|Q_{b,\omega}|^{q-1}],\\
f_5&=\frac{2\omega_s}{q+1}\int[|Q_{b,\omega}+\varepsilon|^{q+1}-|Q_{b,\omega}|^{q+1}-(q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\psi_B,\\
\Delta_{b,\omega}(\varepsilon)&=(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5-\omega(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}+\omega Q_{b,\omega}|Q_{b,\omega}|^{q-1}.
\end{aligned}$$

Our goal is to show that for some $\mu_0>0$,

$$\frac{d}{ds}f_1\le-\mu_0\int((\varepsilon^2+\varepsilon^2)\varphi_{i,B}'+\varepsilon_{yy}^2\psi_B')+Cb^2(\omega^2+b^2), \tag{3-16}$$

$$\left|\frac{d}{ds}f_k\right|\le\frac{\mu_0}{10}\int((\varepsilon^2+\varepsilon^2)\varphi_{i,B}'+\varepsilon_{yy}^2\psi_B')+Cb^2(\omega^2+b^2)\quad\text{for } k=2,3,4,5. \tag{3-17}$$

The following properties will be used several times in this paper:[8]

$$|\varphi_i'''(y)|+|\varphi_i''(y)|+|\psi'''(y)|+|y\psi'(y)|+|\psi(y)|\lesssim\varphi_i'\lesssim\varphi_i\,\text{for all } y\in\mathbb{R}, \tag{3-18}$$

$$e^{|y|}(\psi(y)+|\psi'(y)|)\lesssim\varphi_i'\sim\varphi_i\quad\text{for all } y\in\left(-\infty,\tfrac{1}{2}\right], \tag{3-19}$$

$$\mathcal{N}_{1,\mathrm{loc}}\lesssim\mathcal{N}_{2,\mathrm{loc}}\lesssim\mathcal{N}_1\lesssim\mathcal{N}_2,\qquad\int\varepsilon^2\varphi_{1,B}\,dy\lesssim\mathcal{N}_{2,\mathrm{loc}}, \tag{3-20}$$

$$\int_{y>0}y^2\varepsilon^2(s)\lesssim\left(1+\frac{1}{\lambda^{\frac{10}{9}}(s)}\right)\mathcal{N}_{2,\mathrm{loc}}^{\frac{8}{9}}(s). \tag{3-21}$$

Control of f_1. First, we rewrite f_1 using the equation of ε in the form

$$\begin{aligned}
\varepsilon_s-\frac{\lambda_s}{\lambda}\Lambda\varepsilon&=(-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon))_y\\
&\quad+\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_{b,\omega}+\left(\frac{x_s}{\lambda}-1\right)(Q_{b,\omega}+\varepsilon)_y-b_s\frac{\partial Q_{b,\omega}}{\partial b}-\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}+\Psi_{b,\omega},
\end{aligned} \tag{3-22}$$

[8] See [Martel, Merle and Raphaël 2014, Section 3] for more details.

where $-\Psi_{b,\omega} = b\Lambda Q_{b,\omega} + (Q''_{b,\omega} - Q_{b,\omega} + Q^5_{b,\omega} - \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1})_y$. This yields

$$f_1 = f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5},$$

with

$$\begin{aligned}
f_{1,1} &= 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),\\
f_{1,2} &= 2\left(\frac{\lambda_s}{\lambda} + b\right)\int \Lambda Q_{b,\omega}(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),\\
f_{1,3} &= 2\left(\frac{x_s}{\lambda} - 1\right)\int (Q_{b,\omega} + \varepsilon)_y(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),\\
f_{1,4} &= -2\int \left(b_s\frac{\partial Q_{b,\omega}}{\partial b} + \omega_s\frac{\partial Q_{b,\omega}}{\partial \omega}\right)(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)),\\
f_{1,5} &= 2\int \Psi_{b,\omega}(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B\Delta_{b,\omega}(\varepsilon)).
\end{aligned}$$

For the term $f_{1,1}$, we integrate by parts to obtain a more manageable formula:

$$\begin{aligned}
f_{1,1} = 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))\psi_B&\\
+ 2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-\psi'_B\varepsilon_y + \varepsilon(\varphi_B - \psi_B)).&
\end{aligned}$$

We compute these terms separately. First, we have

$$\begin{aligned}
&2\int (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y(-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))\psi_B\\
&\quad= -\int \psi'_B(-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))^2\\
&\quad= -\int \psi'_B\big([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2\big) - \int \psi'_B(-\varepsilon_{yy} + \varepsilon)^2\\
&\quad= -\left[\int \psi'_B(\varepsilon_{yy}^2 + 2\varepsilon_y^2) + \varepsilon^2(\psi'_B - \psi'''_B)\right] - \int \psi'_B\big([-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon)]^2 - (-\varepsilon_{yy} + \varepsilon)^2\big).
\end{aligned}$$

Next, we integrate by parts to obtain

$$\begin{aligned}
&-2\int (\Delta_{b,\omega}(\varepsilon))_y(\varphi_{i,B} - \psi_B)\varepsilon\\
&\quad= -\frac{1}{3}\int (\varphi_{i,B} - \psi_B)'\big([(Q_{b,\omega} + \varepsilon)^6 - Q^6_{b,\omega} - 6\varepsilon Q^5_{b,\omega}] - 6\varepsilon[(Q_{b,\omega} + \varepsilon)^5 - Q^5_{b,\omega}]\big)\\
&\qquad - 2\int (\varphi_{i,B} - \psi_B)(Q_{b,\omega})_y[(Q_{b,\omega} + \varepsilon)^5 - Q^5_{b,\omega} - 5\varepsilon Q^4_{b,\omega}]\\
&\qquad + \frac{2\omega}{q+1}\int (\varphi_{i,B} - \psi_B)'\big([|Q_{b,\omega} + \varepsilon|^{q+1} - |Q_{b,\omega}|^{q+1} - (q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\\
&\qquad\qquad - (q+1)\varepsilon[(Q_{b,\omega} + \varepsilon)|Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\big)\\
&\qquad + 2\omega\int (\varphi_{i,B} - \psi_B)(Q_{b,\omega})_y\big[(Q_{b,\omega} + \varepsilon)|Q_{b,\omega} + \varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}|^{q-1} - q\varepsilon|Q_{b,\omega}|^{q-1}\big],
\end{aligned}$$

and

$$2\int(-\varepsilon_{yy}+\varepsilon)_y(-\psi_B'\varepsilon_y+\varepsilon(\varphi_{i,B}-\psi_B)) \\ = -2\Big[\int \psi_B'\varepsilon_{yy}^2+\int \varepsilon_y^2\big(\tfrac{3}{2}\varphi_{i,B}'-\tfrac{1}{2}\psi_B'-\tfrac{1}{2}\psi_B'''\big)+\int \varepsilon^2\big(\tfrac{1}{2}(\varphi_B-\psi_B)'-\tfrac{1}{2}(\varphi_B-\psi_B)'''\big)\Big].$$

Finally, by direct expansion, we have

$$\int(\Delta_{b,\omega}(\varepsilon))_y\psi_B'\varepsilon_y = 5\int \psi_B'\varepsilon_y\big((Q_{b,\omega})_y[(Q_{b,\omega}+\varepsilon)^4-Q_{b,\omega}^4]+\varepsilon_y(Q_{b,\omega}+\varepsilon)^4\big) \\ -q\omega\int \psi_B'\varepsilon_y\big((Q_{b,\omega})_y[|Q_{b,\omega}+\varepsilon|^{q-1}-|Q_{b,\omega}|^{q-1}]+\varepsilon_y|Q_{b,\omega}+\varepsilon|^{q-1}\big).$$

Collecting all the estimates above, we have

$$f_{1,1}=I+II,$$

where

$$\begin{aligned} I = &-\int[3\psi_B'\varepsilon_{yy}^2+(3\varphi_{i,B}'+\psi_B'-\psi_B''')\varepsilon_y^2+(\varphi_{i,B}'-\varphi_{i,B}''')\varepsilon^2] \\ &-2\int\big[\tfrac{1}{6}(Q_{b,\omega}+\varepsilon)^6-Q_{b,\omega}^6-6\varepsilon Q_{b,\omega}^5-\varepsilon[(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5]\big](\varphi_{i,B}'-\psi_B') \\ &+2\int[(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5-5\varepsilon Q_{b,\omega}^4](Q_{b,\omega})_y(\psi_B-\varphi_{i,B}) \\ &+10\int \psi_B'\varepsilon_y\big((Q_{b,\omega})_y[(Q_{b,\omega}+\varepsilon)^4-Q_{b,\omega}^4]+\varepsilon_y(Q_{b,\omega}+\varepsilon)^4\big) \\ &-\int \psi_B'\big([-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon)]^2-(-\varepsilon_{yy}+\varepsilon)^2\big) \\ = &\, I_1+I_2+I_3+I_4+I_5, \end{aligned}$$

and

$$\begin{aligned} II = 2\omega\int\Big[&\frac{|Q_{b,\omega}+\varepsilon|^{q+1}-|Q_{b,\omega}|^{q+1}-(q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}}{q+1} \\ &-\varepsilon[(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}-Q_{b,\omega}|Q_{b,\omega}|^{q-1}]\Big](\varphi_{i,B}'-\psi_B') \\ &-2\omega\int[(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}-Q_{b,\omega}|Q_{b,\omega}|^{q-1}-q\varepsilon|Q_{b,\omega}|^{q-1}](Q_{b,\omega})_y(\psi_B-\varphi_{i,B}) \\ &-2q\omega\int \psi_B'\varepsilon_y\big((Q_{b,\omega})_y[|Q_{b,\omega}+\varepsilon|^{q-1}-|Q_{b,\omega}|^{q-1}]+\varepsilon_y|Q_{b,\omega}+\varepsilon|^{q-1}\big). \end{aligned}$$

For I_k, $k=1,2,3,4$, we can use the same strategy as in [Martel, Merle and Raphaël 2014, Proposition 3.1] to obtain

$$\sum_{k=1}^{4} I_k \le -\mu_1\int(\varepsilon_{yy}^2\psi_B'+\varepsilon_y^2\varphi_{i,B}'+\varepsilon^2\varphi_{i,B}')+Cb^4 \tag{3-23}$$

for some universal constant $\mu_1>0$.

The idea is to split the integral into three parts. We denote by $I_k^<$, $I_k^\sim$ and $I_k^>$ the integration on $y<-\frac{B}{2}$, $|y|\le\frac{B}{2}$ and $y>\frac{B}{2}$, respectively, for $k=1,2,3,4$.

On the region $y<-\frac{B}{2}$, using the weighted Sobolev bound introduced in [Merle 2001, Lemma 6; Martel, Merle and Raphaël 2014, Proposition 3.1],

$$\|\varepsilon^2\sqrt{\varphi'_{i,B}}\|_{L^\infty}^2 \lesssim \|\varepsilon\|_{L^2}^2\left(\int\varepsilon_y^2\varphi'_{i,B}+\int\varepsilon^2\frac{(\varphi''_{i,B})^2}{\varphi'_B}\right)\lesssim\delta(\kappa)\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}, \tag{3-24}$$

we have

$$\begin{aligned}I_2^<+I_3^<+I_4^< &\lesssim_B \int(\varepsilon^6+\varepsilon^5Q'_{b,\omega}+\varepsilon_y^2\varepsilon^4)\varphi'_{i,B}+\|Q_{b,\omega}\|_{L^\infty(y<-\frac{B}{2})}\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}\\ &\lesssim\delta(\kappa)\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}.\end{aligned}$$

Hence we have

$$\sum_{k=1}^4 I_k^< \le -\mu_2\int_{y<-\frac{B}{2}}(\varepsilon_{yy}^2\psi'_B+\varepsilon_y^2\varphi'_{i,B}+\varepsilon^2\varphi'_{i,B}) \tag{3-25}$$

for some $\mu_2>0$.

For the region $|y|\le\frac{B}{2}$, we have

$$\sum_{k=1}^4 I_k^\sim = -\frac{1}{B}\int_{|y|<\frac{B}{2}}(3\varepsilon_y^2+\varepsilon^2-5Q^4\varepsilon^2+20yQ'Q^3\varepsilon^2)+O\left(\int_{|y|<\frac{B}{2}}(|b|+\omega)\varepsilon^2+\varepsilon^6\right).$$

We then introduce the following coercivity lemma:

Lemma 3.5 [Martel, Merle and Raphaël 2014, Lemma 3.4]. *There exist $B_0>100$, $\mu_3>0$ such that, for all $\varepsilon\in H^1$ and $B>B_0$, we have*

$$\int_{|y|<\frac{B}{2}}(3\varepsilon_y^2+\varepsilon^2-5Q^4\varepsilon^2+20yQ'Q^3\varepsilon^2)\ge\mu_3\int_{|y|<\frac{B}{2}}\varepsilon_y^2+\varepsilon^2-\frac{1}{B}\int\varepsilon^2e^{-\frac{|y|}{2}}.$$

The above lemma implies immediately that

$$\sum_{k=1}^4 I_k^\sim \le -\mu_2\int_{|y|<\frac{B}{2}}(\varepsilon_{yy}^2\psi'_B+\varepsilon_y^2\varphi'_{i,B}+\varepsilon^2\varphi'_{i,B}), \tag{3-26}$$

while for the region $y>\frac{B}{2}$, we have $\psi'_B=\psi'''_B\equiv 0$. We also have

$$\|\varepsilon\|_{L^\infty(y>\frac{B}{2})}^2\lesssim\|\varepsilon\|_{H^1(y>\frac{B}{2})}^2\lesssim\mathcal{N}_2\le\delta(\kappa).$$

Hence, we have

$$\sum_{k=2}^4 I_k^> \lesssim [\|Q_{b,\omega}\|_{L^\infty(y>\frac{B}{2})}+\|\varepsilon\|_{L^\infty(y>\frac{B}{2})}]\int(\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B},$$

which implies

$$\sum_{k=1}^4 I_k^> \le -\mu_2\int_{y>\frac{B}{2}}(+\varepsilon_y^2+\varepsilon^2)\varphi'_{i,B}. \tag{3-27}$$

Combining (3-25)–(3-27), we obtain (3-23).

Now we turn to the estimate of I_5. We have

$$|I_5| \lesssim \int \psi_B'(|\varepsilon_{yy}|+|\varepsilon|+|\varepsilon|^5+\omega|\varepsilon|^q)(|\varepsilon|^5+\omega|\varepsilon|^q+|Q_{b,\omega}\varepsilon|)$$
$$\leq \frac{\mu_1}{100}\int(\varepsilon_{yy}^2+\varepsilon^2)\psi_B'+C(\mu_1)\Big(\int Q_{b,\omega}^2\varepsilon^2\psi_B'+\int \varepsilon^{10}\psi_B'+\omega^2\int|\varepsilon|^{2q}\psi_B'\Big). \tag{3-28}$$

Combining (3-24) and the hypothesis (H1), we have

$$\int Q_{b,\omega}^2\varepsilon^2\psi_B' \lesssim \|Q_{b,\omega}\|^2_{L^\infty(y<-\frac{B}{2})}\int \varepsilon^2\varphi_{i,B}' \leq \frac{\mu_1}{500}\int(\varepsilon_{yy}^2+\varepsilon^2)\psi_B', \tag{3-29}$$

$$\int \varepsilon^{10}\psi_B' \lesssim \|\varepsilon^2(\psi_B')^{\frac14}\|^4_{L^\infty}\int\varepsilon^2 \lesssim \delta(\kappa)\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+\Big(\int\varepsilon_y^2(\psi_B')^{\frac12}\Big)^2, \tag{3-30}$$

and

$$\omega^2\int|\varepsilon|^{2q}\psi_B' \lesssim \omega^2\|\varepsilon^2(\psi_B')^{\frac14}\|^4_{L^\infty}\int|\varepsilon|^{2q-8} \lesssim \omega^2\|\varepsilon^2(\psi_B')^{\frac14}\|^4_{L^\infty}\|\varepsilon\|^{q-3}_{L^2}\|\varepsilon_y\|^{q-5}_{L^2}$$
$$\lesssim \delta(\kappa)\int(\varepsilon_y^2+\varepsilon^2)\varphi_{i,B}'+\Big(\int\varepsilon_y^2(\psi_B')^{\frac12}\Big)^2,$$

where we use the fact that $\omega\|\varepsilon_y\|^m_{L^2}\leq\kappa$ for the last inequality.

From $((\psi')^{\frac12})''\lesssim\varphi_i'$ and (H1), we have

$$\Big(\int\varepsilon_y^2(\psi_B')^{\frac12}\Big)^2 = \Big(-\int\varepsilon\varepsilon_{yy}(\psi_B')^{\frac12}+\frac12\int\varepsilon^2((\psi_B')^{\frac12})''\Big)^2$$
$$\lesssim \int\varepsilon^2\int\varepsilon_{yy}^2\psi_B'+\Big(\int\varepsilon^2\varphi_{i,B}'\Big)^2$$
$$\lesssim \delta(\kappa)\int(\varepsilon_{yy}^2\psi_B'+\varepsilon^2\varphi_{i,B}'). \tag{3-31}$$

Substituting (3-29)–(3-31) into (3-28), we have

$$|I_5| \lesssim \frac{\mu_1}{50}\Big(\int\varepsilon_{yy}^2\psi_B'+\int(\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}'\Big). \tag{3-32}$$

Now, we turn to the estimate of II. We write II in the form

$$II = II^< + II^>,$$

where $II^<$ and $II^>$ correspond to the integration on $y<-\frac{B}{2}$ and $y>-\frac{B}{2}$ respectively.

For $II^<$, using the fact that $\psi_B'\sim(\varphi_{i,B}')^2$ for $y<-\frac{B}{2}$, we have

$$|II^<| \lesssim \omega\Big(\int_{y<-\frac{B}{2}}(|\varepsilon|^{q+1}+|Q_{b,\omega}|^{q-1}\varepsilon^2)\varphi_{i,B}'+\int_{y<-\frac{B}{2}}|Q_{b,\omega}'|(|\varepsilon|^q+\varepsilon^2)\varphi_{i,B}\Big)$$
$$+\omega\int_{y<-\frac{B}{2}}\psi_B'|\varepsilon_y|\big(|\varepsilon|^{q-1}+|Q_{b,\omega}|^{q-2}|\varepsilon|+|\varepsilon_y||\varepsilon|^{q-1}+|\varepsilon_y||Q_{b,\omega}|^{q-1}\big)$$
$$\leq C(\mu_1)\omega\Big(\int\varphi_{i,B}'(|\varepsilon|^{q+1}+|\varepsilon|^q)+\int\psi_B'(\varepsilon_y^2|\varepsilon|^{q-1}+|\varepsilon_y||\varepsilon|^{q-1})\Big)+\frac{\mu_1}{500}\int_{y<-\frac{B}{2}}(\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}'.$$

We use (H1)–(H3) and the Gagliardo–Nirenberg inequality to estimate these terms separately. First, we have

$$\begin{aligned}\omega\int|\varepsilon|^{q+1}\varphi'_{i,B}&\lesssim\omega\|\varepsilon^2(\varphi'_{i,B})^{\frac12}\|^2_{L^\infty}\int|\varepsilon|^{q-3}\\&\lesssim\omega\biggl(\int\varepsilon^2\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\biggr)(\|\varepsilon_y\|_{L^2}^{\frac{q-5}{2}}\|\varepsilon\|_{L^2}^{\frac{q-1}{2}})\\&\lesssim(\omega\|\varepsilon_y\|^m_{L^2})\biggl(\int\varepsilon^2\biggr)^{\frac{q+3}{4}}\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B},\end{aligned}$$

and

$$\begin{aligned}\omega\int|\varepsilon|^q\varphi'_{i,B}&\lesssim\omega\big\|\varepsilon^2(\varphi'_{i,B})^{\frac12}\big\|^{\frac32}_{L^\infty}\big\||\varepsilon|^{\frac12}(\varphi'_{i,B})^{\frac14}\big\|_{L^4}\big\||\varepsilon|^{\frac{2q-7}{2}}\big\|_{L^{4/3}}\\&\lesssim\omega\biggl(\int\varepsilon^2\biggr)^{\frac34}\biggl(\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\biggr)\|\varepsilon\|_{L^2}^{\frac{q-2}{2}}\|\varepsilon_y\|_{L^2}^{\frac{q-5}{2}}\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}.\end{aligned}$$

From $\psi'\lesssim(\varphi_i')^2$ and (3-31), we also have

$$\begin{aligned}\omega\int\psi'_B\varepsilon_y^2|\varepsilon|^{q-1}&\lesssim\omega\|\varepsilon^2(\psi'_B)^{\frac14}\|^2_{L^\infty}\|\varepsilon\|^{q-5}_{L^\infty}\int\varepsilon_y^2(\psi'_B)^{\frac12}\\&\lesssim(\omega\|\varepsilon_y\|^m_{L^2})\|\varepsilon\|^{m+2}_{L^2}\biggl(\int(\varepsilon^2+\varepsilon_y^2)(\psi'_B)^{\frac12}\biggr)\int\varepsilon_y^2(\psi'_B)^{\frac12}\\&\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+\biggl(\int\varepsilon_y^2(\psi'_B)^{\frac12}\biggr)^2\\&\lesssim\delta(\kappa)\biggl(\int\varepsilon_{yy}^2\psi'_B+\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\biggr),\end{aligned}$$

and

$$\begin{aligned}\omega\int\psi'_B|\varepsilon_y||\varepsilon|^{q-1}&\lesssim\omega\|\varepsilon^2(\psi'_B)^{\frac14}\|^{\frac32}_{L^\infty}\|\varepsilon_y(\psi'_B)^{\frac14}\|_{L^2}\||\varepsilon|^{q-4}\|_{L^2}\\&\lesssim(\omega\|\varepsilon_y\|^m_{L^2})\|\varepsilon\|^{m+2}_{L^2}\biggl(\int(\varepsilon^2+\varepsilon_y^2)(\psi'_B)^{\frac12}\biggr)\biggl(\int\varepsilon_y^2(\psi'_B)^{\frac12}\biggr)^{\frac12}\\&\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+\biggl(\int\varepsilon_y^2(\psi'_B)^{\frac12}\biggr)^{\frac32}\\&\le\frac{\mu_1}{1000}\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+C(\mu_1)\biggl(\int\varepsilon_y^2(\psi'_B)^{\frac12}\biggr)^2\\&\le\frac{\mu_1}{500}\biggl(\int\varepsilon_{yy}^2\psi'_B+\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\biggr).\end{aligned}$$

In conclusion, we have

$$|II^<|\le\frac{\mu_1}{50}\biggl(\int\varepsilon_{yy}^2\psi'_B+\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}\biggr).\tag{3-33}$$

For $II^{>}$, we know that $\psi_B' \equiv 0$ for $y > -\frac{B}{2}$. Using Sobolev embedding, we have

$$\|\varepsilon\|^2_{L^\infty(y>-\frac{B}{2})} \lesssim \|\varepsilon\|_{L^2(y>-\frac{B}{2})}\|\varepsilon_y\|_{L^2(y>-\frac{B}{2})} \leq \mathcal{N}_2 \leq 1.$$

Thus, we have

$$|II^{>}| \lesssim \omega\biggl(\int \varepsilon^2\varphi_{i,B}' + \int (Q_{b,\omega})_y\varepsilon^2\varphi_{i,B}\biggr) \lesssim_B \delta(\kappa)\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}'. \tag{3-34}$$

Combining (3-23), (3-32), (3-33) and (3-34), we have

$$f_{1,1} \leq -\mu_0 \int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}' + Cb^4 \tag{3-35}$$

for some universal constant $\mu_0 > 0$.

Now, let us deal with $f_{1,2}$. It is easy to see that

$$f_{1,2} = \tilde{I} + \widetilde{II},$$

where

$$\tilde{I} = 2\biggl(\frac{\lambda_s}{\lambda}+b\biggr)\int \Lambda Q_{b,\omega}\bigl(-(\psi_B\varepsilon_y)_y + \varepsilon\varphi_{i,B} - \psi_B[(Q_{b,\omega}+\varepsilon)^5 - Q_{b,\omega}^5]\bigr),$$
$$\widetilde{II} = 2\omega\biggl(\frac{\lambda_s}{\lambda}+b\biggr)\int \Lambda Q_{b,\omega}\psi_B((Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1} - Q_{b,\omega}|Q_{b,\omega}^{q-1}|).$$

The term $\tilde{I}$ can be estimated by the same argument as in [Martel, Merle and Raphaël 2014, Proposition 3.1]. Thus, we have

$$|\tilde{I}| \leq \frac{\mu_0}{500}\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}' + Cb^2(\omega^2+b^2).$$

We mention here that the modulation estimate (2-34) in this paper is slightly different from [Martel, Merle and Raphaël 2014, (2.29)]; i.e., there is an additional term "$\omega|b|$" on the right-hand side of (2-34). This additional term results in the appearance of the term "ω^2b^2" on the right-hand side of the above inequality.

While for $\widetilde{II}$, we have

$$|\widetilde{II}| \lesssim \omega\biggl|\frac{\lambda_s}{\lambda}+b\biggr|\biggl(B^{\frac{1}{2}}\mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}} + \int |\varepsilon|^q\psi_B\biggr).$$

Using (2-34) and the strategy for $f_{1,1}$, we have

$$|\widetilde{II}| \leq \frac{\mu_0}{500}\int (\varepsilon^2+\varepsilon_y^2)\varphi_{i,B}' + Cb^2(\omega^2+b^2).$$

A similar argument can be applied to $f_{1,k}$, $k=3,4,5$. Together with (3-35), we conclude the proof of (3-16).

Control of f_2. For f_2, we integrate by parts, using (3-22) to get

$$f_2 = 2\mathcal{J}_{i,j}\int \varepsilon\varphi_{i,B}\Big[(-\varepsilon_{yy}+\varepsilon-\Delta_{b,\omega}(\varepsilon))_y+\Big(\frac{\lambda_s}{\lambda}+b\Big)\Lambda Q_{b,\omega} \\ +\Big(\frac{x_s}{\lambda}-1\Big)(Q_{b,\omega}+\varepsilon)_y - b_s\frac{\partial Q_{b,\omega}}{\partial b}-\omega_s\frac{\partial Q_{b,\omega}}{\partial\omega}+\Psi_{b,\omega}\Big].$$

We integrate by parts, estimating all terms like we did for f_1. Together with

$$|\mathcal{J}_{i,j}|\lesssim |J_1|\lesssim \mathcal{N}_2^{\frac{1}{2}}\lesssim\delta(\kappa),$$

we have

$$|f_2|\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+b^2(\omega^2+b^2). \tag{3-36}$$

Control of f_3. Recall that

$$f_3 = 2\frac{\lambda_s}{\lambda}\int\Lambda\varepsilon\big(-(\psi_B\varepsilon_y)_y+(1+\mathcal{J}_{i,j})\varepsilon\varphi_{i,B}-\psi_B\Delta_{b,\omega}(\varepsilon)\big)+(\mathcal{J}_{i,j})_s\int\varphi_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}\mathcal{F}_{i,j}.$$

Integrating by parts,[9] we have

$$f_3=\hat{I}+\hat{II},$$

where

$$\hat{I}=\frac{\lambda_s}{\lambda}\int[(2-2(j-1))\psi_B-y\psi'_B]\varepsilon_y^2 \\ -\frac{1}{3}\frac{\lambda_s}{\lambda}\int[(2-2(j-1))\psi_B-y\psi'_B][(Q_{b,\omega}+\varepsilon)^6-Q_{b,\omega}^6-6\varepsilon Q_{b,\omega}^5] \\ +2\frac{\lambda_s}{\lambda}\int\psi_B\Lambda Q_b[(Q_{b,\omega}+\varepsilon)^5-Q_{b,\omega}^5-5\varepsilon Q_{b,\omega}^4] \\ +(\mathcal{J}_{i,j})_s\int\varepsilon^2\varphi_{i,B}-\frac{\lambda_s}{\lambda}(1+\mathcal{J}_{i,j})\int y\varphi'_{i,B}\varepsilon^2-2(j-1)\frac{\lambda_s}{\lambda}(1+\mathcal{J}_{i,j})\int\varepsilon^2\varphi_{i,B},$$

and

$$\hat{II}=\frac{2\omega}{q+1}\frac{\lambda_s}{\lambda}\int\Big[\Big(\frac{q+3}{q-1}-2(j-1)\Big)\psi_B-y\psi'_B\Big]\times\big[|Q_{b,\omega}+\varepsilon|^{q+1}-|Q_{b,\omega}|^{q+1}-(q+1)\varepsilon Q_{b,\omega}|Q_{b,\omega}|^{q-1}\big] \\ -2\omega\frac{\lambda_s}{\lambda}\int\psi_B\Lambda Q_b\big[(Q_{b,\omega}+\varepsilon)|Q_{b,\omega}+\varepsilon|^{q-1}-Q_{b,\omega}|Q_{b,\omega}|^{q-1}-q\varepsilon|Q_{b,\omega}|^{q-1}\big].$$

Similarly, we can use the same strategy as in [Martel, Merle and Raphaël 2014, Proposition 3.1] to estimate I, which leads to

$$|\hat{I}|\lesssim\delta(\kappa)\int(\varepsilon^2+\varepsilon_y^2)\varphi'_{i,B}+b^2(\omega^2+b^2). \tag{3-37}$$

9 See [Martel, Merle and Raphaël 2014, Proposition 3.1, Step 5] and [Lan 2016, (5.22)] for more details.

More precisely, we can rewrite $\hat{I}$ as

$$\begin{aligned}\hat{I} &= \frac{\lambda_s}{\lambda}\int[2(2-j)\psi_B - y\psi_B']\varepsilon_y^2\\ &\quad -\frac{1}{3}\frac{\lambda_s}{\lambda}\int[2(2-j)\psi_B - y\psi_B']\times[(Q_{b,\omega}+\varepsilon)^6 - Q_{b,\omega}^6 - 6\varepsilon Q_{b,\omega}^5]\\ &\quad + 2\frac{\lambda_s}{\lambda}\int \psi_B \Lambda Q_b[(Q_{b,\omega}+\varepsilon)^5 - Q_{b,\omega}^5 - 5\varepsilon Q_{b,\omega}^4]\\ &\quad + \frac{1}{i}\left[(\mathcal{J}_{i,j})_s - 2(j-1)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right]\int (i\varphi_{i,B} - y\varphi_{i,B}')\varepsilon^2\\ &\quad + \frac{1}{i}\left[(\mathcal{J}_{i,j})_s - (2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right]\int y\varphi_{i,B}'\varepsilon^2\\ &= \hat{I}_1 + \hat{I}_2,\end{aligned}$$

where

$$\hat{I}_2 = \frac{1}{i}\left[(\mathcal{J}_{i,j})_s - (2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right]\int y\varphi_{i,B}'\varepsilon^2.$$

We also denote by $\hat{I}_k^<$, $\hat{I}_k^\sim$ and $\hat{I}_k^>$, $k = 1, 2$, the integration over $y < -\frac{B}{2}$, $|y| < \frac{B}{2}$ and $y > \frac{B}{2}$ respectively.

For integration over $|y| < \frac{B}{2}$, the estimate is straightforward, we have

$$|\hat{I}_1^\sim| + |\hat{I}_2^\sim| \lesssim \delta(\kappa)\int_{|y|<\frac{B}{2}} (\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}'.$$

While for $y < -\frac{B}{2}$, using (2-34), we have

$$\begin{aligned}|\hat{I}_1^<| + |\hat{I}_2^<| &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\int_{y<-\frac{B}{2}} (\psi_B + |y|\varphi_{i,B}' + \varphi_{i,B})(\varepsilon_y^2 + \varepsilon^2) + |y|\varphi_{i,B}'\varepsilon^2\\ &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\left[\int_{y<-\frac{B}{2}} \varepsilon_y^2\varphi_{i,B}' + \int_{y<-\frac{B}{2}} |y|\varphi_{i,B}'\varepsilon^2\right]\\ &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\left[\int \varepsilon_y^2\varphi_{i,B}' + \left(\int_{y<-\frac{B}{2}} y^{100}e^{\frac{y}{B}}\varepsilon^2\right)^{\frac{1}{100}}\left(\int_{y<-\frac{B}{2}} e^{\frac{y}{B}}\varepsilon^2\right)^{\frac{99}{100}}\right]\\ &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\times\left(\int \varepsilon_y^2\varphi_{i,B}' + \mathcal{N}_{i,\mathrm{loc}}^{\frac{99}{100}}\right) \le \delta(\kappa)\int(\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}' + Cb^4.\end{aligned}$$

Now, for $y > \frac{B}{2}$, we first have

$$i\varphi_{i,B} - y\varphi_{i,B}' = 0$$

for all $y > B$. Hence

$$|\hat{I}_1^>| \lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\int(\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}' \lesssim \delta(\kappa)\int(\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}'.$$

Next, for $\hat{I}_2^>$, we know from (2-38) that

$$\left|(\mathcal{J}_{i,j})_s - (2(j-1)+i)(1+\mathcal{J}_{i,j})\frac{\lambda_s}{\lambda}\right| = \frac{4(j-1)+2i}{(1-J_1)^{4(j-1)+2i+1}}\left|(J_1)_s - \frac{\lambda_s}{2\lambda}(1-J_1)\right| \lesssim |b| + \mathcal{N}_{i,\mathrm{loc}}.$$

Together with (3-7), (3-8) and (3-21), we have

$$
\begin{aligned}
|\hat{I}_2^{>}| &\lesssim (|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}})\Big(1 + \frac{1}{\lambda^{\frac{10}{9}}}\Big)\mathcal{N}_{i,\mathrm{loc}}^{\frac{8}{9}} \\
&\lesssim |b|(1 + \delta(\kappa)|b|^{-\frac{5}{9}})\mathcal{N}_{i,\mathrm{loc}}^{\frac{8}{9}} + \mathcal{N}_{i,\mathrm{loc}}(1 + \delta(\kappa)\mathcal{N}_{i,\mathrm{loc}}^{-\frac{5}{9}})\mathcal{N}_{i,\mathrm{loc}}^{\frac{8}{9}} \\
&\lesssim \delta(\kappa)\int (\varepsilon_y^2 + \varepsilon^2)\varphi_{i,B}' + b^2(\omega^2 + b^2).
\end{aligned}
$$

Combining the above estimates, we obtain (3-37).

Finally, for $\widehat{II}$, from the fact

$$
\psi_B + \left|\left(\frac{q+3}{q-1} - 2(j-1)\right)\psi_B - y\psi_B'\right| \lesssim_B \varphi_{i,B}',
$$

we have

$$
|\widehat{II}| \lesssim \omega\left|\frac{\lambda_s}{\lambda}\right| \int (|\varepsilon|^{q+1} + |\varepsilon|^q + \varepsilon^2)\varphi_{i,B}'.
$$

Using $|\lambda_s/\lambda| \lesssim \delta(\kappa)$ and the strategy for $f_{1,1}$, we have

$$
|\widehat{II}| \lesssim \delta(\kappa)\int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'.
$$

In conclusion, we have

$$
|f_3| \lesssim \delta(\kappa)\int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}' + b^2(\omega^2 + b^2). \tag{3-38}
$$

Control of f_4. From (2-5) and (3-12), we have

$$
|(Q_{b,\omega})_s| \lesssim |b_s|\left|\frac{\partial Q_{b,\omega}}{\partial b}\right| + |\omega_s|\left|\frac{\partial Q_{b,\omega}}{\partial \omega}\right| \le (\omega + |b|)(|b| + \mathcal{N}_{i,\mathrm{loc}}^{\frac{1}{2}}) \lesssim \delta(\kappa).
$$

Using the Sobolev bounds (3-24) and the strategy for $f_{1,1}$, we have

$$
|f_4| \lesssim \delta(\kappa)\left(\int (\omega|\varepsilon|^q + |\varepsilon|^5 + \varepsilon^2)\varphi_{i,B}'\right) \lesssim \delta(\kappa)\left(\int \varepsilon_{yy}^2\psi_B' + \int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'\right). \tag{3-39}
$$

Control of f_5. From (2-34) we know that

$$
|\omega_s| = m\omega\left|\frac{\lambda_s}{\lambda}\right| \lesssim \delta(\kappa).
$$

Thus, by the Sobolev bounds (3-24) and the strategy for $f_{1,1}$, we have

$$
|f_5| \lesssim \delta(\kappa)\int (\omega|\varepsilon|^{q+1} + \varepsilon^2)\varphi_{i,B}' \lesssim \delta(\kappa)\left(\int \varepsilon_{yy}^2\psi_B' + \int (\varepsilon^2 + \varepsilon_y^2)\varphi_{i,B}'\right). \tag{3-40}
$$

Combining (3-36), (3-38), (3-39) and (3-40), we conclude the proof of (3-17), and hence the proof of Proposition 3.1. □

3B. *Dynamical control of the tail on the right.* In order to close the bootstrap bound (H3), we need the dynamical control of the L^2 tail on the right introduced in [Martel, Merle and Raphaël 2014]. More precisely, we choose a smooth function

$$\varphi_{10}(y) = \begin{cases} 0 & \text{for } y < 0, \\ y^{10} & \text{for } y > 1, \end{cases} \qquad \varphi_{10}' \geq 0.$$

Then we have:

Lemma 3.6 (dynamical control of the tail on the right [Martel, Merle and Raphaël 2014]). *Under the assumption of Proposition 3.1, it holds that*

$$\frac{1}{\lambda^{10}} \frac{d}{ds}\left(\lambda^{10} \int \varphi_{10}\varepsilon^2\right) \lesssim_B \mathcal{N}_{1,\mathrm{loc}} + b^2. \tag{3-41}$$

Proof. The proof of Lemma 3.6 is exactly the same as [Martel, Merle and Raphaël 2014, Lemma 3.7].

More precisely, from (3-22), we have

$$\frac{1}{2}\frac{d}{ds}\int \varphi_{10}\varepsilon^2 = \int \varphi_{10}\varepsilon\left[\frac{\lambda_s}{\lambda}\Lambda\varepsilon + (-\varepsilon_{yy} + \varepsilon - \Delta_{b,\omega}(\varepsilon))_y + \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda Q_{b,\omega} + \left(\frac{x_s}{\lambda} - 1\right)(Q_{b,\omega} + \varepsilon)_y - b_s\frac{\partial Q_{b,\omega}}{\partial b} - \omega_s\frac{\partial Q_{b,\omega}}{\partial \omega} + \Psi_{b,\omega}\right],$$

where

$$\Delta_{b,\omega}(\varepsilon) = (Q_{b,\omega} + \varepsilon)^5 - Q_{b,\omega}^5 - \omega(Q_{b,\omega} + \varepsilon)|Q_{b,\omega} + \varepsilon|^{q-1} + \omega Q_{b,\omega}|Q_{b,\omega}|^{q-1}.$$

We integrate the linear term by parts using the fact that $y\varphi_{10}' = 10\varphi_{10}$ for $y \geq 1$, and $\varphi_{10}''' \ll \varphi_{10}'$ for y large, to obtain

$$\begin{aligned}\int \varphi_{10}\varepsilon\left[\frac{\lambda_s}{\lambda}\Lambda\varepsilon + (-\varepsilon_{yy}+)_y\right] &= -\frac{1}{2}\frac{\lambda_s}{\lambda}\int y\varphi_{10}'\varepsilon^2 - \frac{3}{2}\int \varepsilon_y^2\varphi_{10}' - \frac{1}{2}\int \varepsilon^2\varphi_{10}' + \frac{1}{2}\int \varepsilon^2\varphi_{10}''' \\ &\leq -5\frac{\lambda_s}{\lambda}\int \varphi_{10}\varepsilon^2 - \frac{3}{2}\int \varepsilon_y^2\varphi_{10}' - \frac{1}{2}\int \varepsilon^2\varphi_{10}' + C\mathcal{N}_{1,\mathrm{loc}}.\end{aligned}$$

Next, from (2-15), (2-34), and (2-35), it is easy to obtain

$$\left|\int \varphi_{10}\varepsilon\left[\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda Q_{b,\omega} + \left(\frac{x_s}{\lambda} - 1\right)(Q_{b,\omega} + \varepsilon)_y - b_s\frac{\partial Q_{b,\omega}}{\partial b} - \omega_s\frac{\partial Q_{b,\omega}}{\partial \omega} + \Psi_{b,\omega}\right]\right| \lesssim b^2 + \mathcal{N}_{1,\mathrm{loc}}.$$

While for the nonlinear term, we integrate by parts to remove all derivatives on ε to obtain

$$\left|\int \varphi_{10}\varepsilon[\Delta_{b,\omega}(\varepsilon)]_y\right| \lesssim \int \varphi_{10}\varepsilon^2 e^{-\frac{|y|}{2}}(\|\varepsilon\|_{L^\infty(y>0)}) + \int \varepsilon^6\varphi_{10}' + \omega\int |\varepsilon|^{q+1}\varphi_{10}' \lesssim \delta(\kappa)\int \varepsilon^2\varphi_{10}',$$

where we use the fact that $|Q_{b,\omega}| + |Q_{b,\omega}'| \lesssim e^{-|y|/2}$ for $y > 0$ and

$$\|\varepsilon\|_{L^\infty(y>0)} \lesssim \mathcal{N}_1 \ll 1.$$

Hence, we have

$$\frac{d}{ds}\int \varphi_{10}\varepsilon^2 + 10\frac{\lambda_s}{\lambda}\int \varphi_{10}\varepsilon^2 \lesssim b^2 + \mathcal{N}_{1,\mathrm{loc}},$$

which, together with Gronwall's inequality, implies (3-41) immediately. □

4. Rigidity of the dynamics in $\mathcal{A}_{\alpha_0}$ and proof of Theorem 1.3

We will classify the behavior of any solution with initial data in $\mathcal{A}_{\alpha_0}$, which directly implies Theorem 1.3. To begin, we define

$$t^* = \sup\{0 < t < +\infty \mid \text{for all } t' \in [0,t],\ u(t') \in \mathcal{T}_{\alpha^*,\gamma}\}. \tag{4-1}$$

Assume $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll 1$; then the condition on the initial data, i.e., $u_0 \in \mathcal{A}_{\alpha_0}$, implies $t^* > 0$.

Next, by Lemma 2.6, $u(t)$ admits the following geometrical decomposition on $[0, t^*]$:

$$u(t,x) = \frac{1}{\lambda^{1/2}(t)}[Q_{b(t),\omega(t)} + \varepsilon(t)]\left(\frac{x - x(t)}{\lambda(t)}\right).$$

The condition $u_0 \in \mathcal{A}_{\alpha_0}$ implies

$$\omega(0) + \|\varepsilon(0)\|_{H^1} + \omega(0)\|\varepsilon_y(0)\|_{L^2}^m + |b(0)| + |1 - \lambda(0)| \lesssim \delta(\alpha_0), \tag{4-2}$$

$$\int_{y>0} y^{10}\varepsilon^2(0)\,dy \le 2. \tag{4-3}$$

Using Hölder's inequality, we have

$$\mathcal{N}_2(0) \lesssim \delta(\alpha_0). \tag{4-4}$$

Then let us fix a $0 < \kappa \ll 1$ as in Propositions 2.9 and 3.1, and define

$$t^{**} = \sup\{0 < t < t^* \mid \text{(H1), (H2) and (H3) hold for all } t' \in [0,t]\}. \tag{4-5}$$

Note that from (4-2)–(4-4), we have $t^{**} > 0$. Let $s^* = s(t^*)$ and $s^{**} = s(t^{**})$.

4A. *Consequence of the monotonicity formula.*

We derive some crucial estimates from the monotonicity formula introduced in Section 3.

Lemma 4.1. *We have the following*:

(1) *Almost monotonicity of the localized Sobolev norm*: *There exists a universal constant $K_0 > 1$ such that, for $i = 1, 2$ and $0 \le s_1 < s_2 \le s^{**}$,*

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2}\int (\varepsilon_y^2(s,y) + \varepsilon^2(s,y))\varphi'_{i,B}(y)\,dy\,ds \le K_0\Big[\mathcal{N}_i(s_1) + \sup_{s\in[s_1,s_2]}|b(s)|^3 + \sup_{s\in[s_1,s_2]}\omega^3(s)\Big], \tag{4-6}$$

$$\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2}\frac{1}{\lambda^2(s)}\left[\left(\int(\varepsilon_y^2 + \varepsilon^2)(s)\varphi'_{i,B}\right) + b^2(s)(|b(s)| + \omega(s))\right]ds \le K_0\left(\frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \left[\frac{b^2(s_1) + \omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2) + \omega^2(s_2)}{\lambda^2(s_2)}\right]\right). \tag{4-7}$$

(2) *Control of b and ω*: *For all $0 \le s_1 < s_2 \le s^{**}$,*

$$\omega(s_2) + \int_{s_1}^{s_2} b^2(s)\,ds \lesssim \mathcal{N}_1(s_1) + \omega(s_1) + \sup_{s\in[s_1,s_2]}|b(s)|. \tag{4-8}$$

(3) *Control of* b/λ^2: *Let* $c_1 = (m/(m+2))G'(0) > 0$, *where* G *is the* C^2 *function introduced in* (2-40). *Then there exists a universal constant* $K_1 > 1$ *such that, for all* $0 \le s_1 < s_2 \le s^{**}$,

$$\left|\frac{b(s_2)+c_1\omega(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)+c_1\omega(s_1)}{\lambda^2(s_1)}\right| \le K_1\left(\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right). \tag{4-9}$$

(4) *Refined control of* λ: *Let* $\lambda_0(s) = \lambda(s)(1-J_1(s))^2$. *Then there exists a universal constant* $K_2 > 1$ *such that, for all* $s \in [0, s^{**}]$,

$$\left|\frac{(\lambda_0)_s}{\lambda_0} + b\right| \le K_2[\mathcal{N}_1 + (|b|+\omega)(\mathcal{N}_2^{\frac{1}{2}} + |b|)]. \tag{4-10}$$

Proof. Proof of (4-6) *and* (4-8). From (2-50), we have

$$\frac{d}{ds}G(\omega) + b^2 \le -b_s + C\mathcal{N}_{1,\mathrm{loc}}.$$

Integrating from s_1 to s_2, we have

$$G(\omega(s_2)) + \int_{s_1}^{s_2} b^2 \lesssim \int_{s_1}^{s_2} \mathcal{N}_{1,\mathrm{loc}} + G(\omega(s_1)) + |b(s_2)-b(s_1)| \lesssim \int_{s_1}^{s_2} \mathcal{N}_{1,\mathrm{loc}} + G(\omega(s_1)) + \sup_{s\in[s_1,s_2]} |b(s)|.$$

Since $G(\omega) \sim \omega$, we obtain (4-8).

Next, from the monotonicity formulas (3-11) and (3-13) we obtain

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2}\int (\varepsilon_y^2(s,y) + \varepsilon^2(s,y))\varphi'_{i,B}(y)\,dy\,ds \lesssim \mathcal{N}_i(s_1) + \Big[\sup_{s\in[s_1,s_2]} b^2(s) + \sup_{s\in[s_1,s_2]} \omega^2(s)\Big]\int_{s_1}^{s_2} b^2. \tag{4-11}$$

Combining (4-8) and (4-11), we obtain (4-6).

Proof of (4-7). First, from (2-50) and (2-35), we have

$$\begin{aligned} 2\int_{s_1}^{s_2} \frac{|b|^3}{\lambda^2} &\le \int_{s_1}^{s_2}\left[-\frac{|b|b_s - \omega_s G'(\omega)|b| + C\mathcal{N}_{1,\mathrm{loc}} + \delta(\kappa)|b|^3}{\lambda^2}\right] \\ &\le -\frac{1}{2}\frac{b|b|}{\lambda^2}\bigg|_{s_1}^{s_2} + O\left(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}} + \omega b^2}{\lambda^2}\right) + \delta(\kappa)\int_{s_1}^{s_2}\frac{|b|^3}{\lambda^2}. \end{aligned} \tag{4-12}$$

Recall that $\omega = \gamma/\lambda^m$. Then from (2-34) we have

$$\begin{aligned} \int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2} &= -\int_{s_1}^{s_2}\frac{\lambda_s}{\lambda}\frac{\omega b}{\lambda^2} + \int_{s_1}^{s_2}\frac{\omega b}{\lambda^2}\left(\frac{\lambda_s}{\lambda} + b\right) \\ &\le \frac{1}{m+2}\int_{s_1}^{s_2}\left(\frac{\omega}{\lambda^2}\right)_s b + \delta(\kappa)\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2} + O\left(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\right) \\ &= \frac{1}{m+2}\int_{s_1}^{s_2}\frac{\omega}{\lambda^2}(-b_s) + \delta(\kappa)\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2} \\ &\qquad + O\left(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} + \frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right). \end{aligned} \tag{4-13}$$

From (2-50) and (2-35), we have

$$\begin{aligned}\int_{s_1}^{s_2}\frac{\omega}{\lambda^2}(-b_s)&\le\int_{s_1}^{s_2}\frac{\omega}{\lambda^2}\Big[\Big(2+\frac{m}{10}\Big)b^2+\omega_s G'(\omega)+C(m)\mathcal{N}_{1,\mathrm{loc}}\Big]\\&\le\Big(2+\frac{m}{10}\Big)\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2}+\int_{s_1}^{s_2}\frac{\omega_s\omega G'(\omega)}{(\gamma/\omega)^{2/m}}\\&\qquad+O\Big(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}+\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\Big).\end{aligned}\tag{4-14}$$

From (2-34) and (2-35) again, we have

$$\Big|\int_{s_1}^{s_2}\frac{\omega_s\omega G'(\omega)}{(\gamma/\omega)^{2/m}}\Big|=\frac{|M(\omega(s_2))-M(\omega(s_1))|}{\gamma^{2/m}}\lesssim\frac{\omega^2(s_1)}{\lambda^2(s_1)}+\frac{\omega^2(s_2)}{\lambda^2(s_2)},\tag{4-15}$$

where

$$M(\omega)=\int_0^\omega x^{1+\frac{2}{m}}G'(x)\,dx\sim\omega^{2+\frac{2}{m}}.$$

Therefore, combining (4-13)–(4-15), we have

$$\begin{aligned}\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2}&\le\Big(\frac{2+\frac{1}{10}m}{m+2}+\delta(\kappa)\Big)\int_{s_1}^{s_2}\frac{\omega b^2}{\lambda^2}\\&\qquad+O\Big(\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}+\Big[\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\Big]\Big).\end{aligned}\tag{4-16}$$

Taking $\kappa>0$ small enough, from (4-12) and (4-16) we have

$$\int_{s_1}^{s_2}\frac{b^2(\omega+|b|)}{\lambda^2}\lesssim\int_{s_1}^{s_2}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}+\Big[\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\Big].\tag{4-17}$$

Now, integrating the monotonicity formula (3-12), we have

$$\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)}+\int_{s_1}^{s_2}\frac{1}{\lambda^2(s)}\Big[\Big(\int(\varepsilon_y^2+\varepsilon^2)(s)\varphi'_{i,B}\Big)\Big]\,ds\lesssim\frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)}+\delta(\kappa)\int_{s_1}^{s_2}\frac{b^2(s)(\omega(s)+|b(s)|)}{\lambda^2(s)}\,ds,$$

which implies (4-7) immediately.

Proof of (4-9). The proof of (4-9) is based on integrating the equation of b/λ^2, i.e., (2-42). More precisely, from (2-34), (2-42) and the fact that $|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$ (recall that J given by (2-41) is a well-localized L^2 scalar product), we have

$$\begin{aligned}\Big|\Big(\frac{b}{\lambda^2}e^J\Big)_s+\frac{\omega_s G'(\omega)}{\lambda^2}e^J\Big|&=\Big|\Big(\frac{b}{\lambda^2}\Big)_s+\frac{b}{\lambda^2}J_s+\frac{\omega_s G'(\omega)}{\lambda^2}\Big|e^J\\&\lesssim\Big|\frac{\lambda_s}{\lambda}\frac{b}{\lambda^2}J\Big|+O\Big(\frac{1}{\lambda^2}\Big(\int\varepsilon^2e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\Big)\Big)\\&\lesssim\frac{b^2}{\lambda^2}|J|+O\Big(\frac{1}{\lambda^2}\Big(\int\varepsilon^2e^{-\frac{|y|}{10}}+(\omega+|b|)b^2\Big)\Big)\\&\lesssim O\Big(\frac{1}{\lambda^2}(\mathcal{N}_{1,\mathrm{loc}}+(\omega+|b|)b^2)\Big).\end{aligned}$$

We integrate this estimate in time using (4-7) to get

$$\left|\left[\frac{b}{\lambda^2}e^J\right]_{s_1}^{s_2}+\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}e^J\right|\lesssim\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\left[\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right]. \tag{4-18}$$

Note that $|e^J-1|\le 2|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$. Together with (4-7), we have

$$\begin{aligned}\left[\frac{b}{\lambda^2}e^J\right]_{s_1}^{s_2}&=\frac{b}{\lambda^2}\bigg|_{s_1}^{s_2}+\left|\left[\frac{b}{\lambda^2}\mathcal{N}_{1,\mathrm{loc}}^{\frac{1}{2}}\right]_{s_1}^{s_2}\right|\\&=\frac{b}{\lambda^2}\bigg|_{s_1}^{s_2}+O\left(\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}\right).\end{aligned} \tag{4-19}$$

Next, from (2-35), (4-7) and $|J|\lesssim\mathcal{N}_{1,\mathrm{loc}}^{1/2}$, we have

$$\begin{aligned}\left|\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}(e^J-1)\right|&\lesssim\int_{s_1}^{s^2}\frac{(b^2+\omega^2)b^2+\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}\\&\lesssim\frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_1)+\omega^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)+\omega^2(s_2)}{\lambda^2(s_2)}.\end{aligned} \tag{4-20}$$

Finally, recall $\omega=\gamma/\lambda^m$, so we have

$$\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}=\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{(\gamma/\omega)^{2/m}}=\frac{\Sigma(\omega)}{\lambda^2}\bigg|_{s_1}^{s_2},$$

where

$$\Sigma(\omega):=\frac{1}{\omega^{2/m}}\int_0^\omega x^{\frac{2}{m}}G'(x)\,dx.$$

Recall that G is the C^2 function introduced in (2-40). We then have $\Sigma\in C^2$ and $c_1=\Sigma'(0)=(m/(m+2))G'(0)>0$. Hence, we have

$$\int_{s_1}^{s_2}\frac{\omega_s G'(\omega)}{\lambda^2}=\frac{c_1\omega}{\lambda^2}\bigg|_{s_1}^{s_2}+O\left(\frac{\omega^2(s_1)}{\lambda^2(s_1)}+\frac{\omega^2(s_2)}{\lambda^2(s_2)}\right). \tag{4-21}$$

Combining (4-18)–(4-21), we conclude the proof of (4-9).

Proof of (4-10). From (3-14), we have

$$\left|\frac{\lambda}{\lambda_0}-1\right|\lesssim|J_1|\lesssim\mathcal{N}_2^{\frac{1}{2}}\lesssim\delta(\kappa);$$

thus we obtain from (2-38)

$$\begin{aligned}\left|\frac{(\lambda_0)_s}{\lambda_0}+b\right|&=\left|\frac{1}{1-J_1}\left[(1-J_1)\frac{\lambda_s}{\lambda}+b-2(J_1)_s\right]-\frac{J_1}{1-J_1}b\right|\\&\lesssim\int\varepsilon^2e^{-\frac{|y|}{10}}+(|b|+\omega)(\mathcal{N}_2^{\frac{1}{2}}+|b|).\end{aligned}$$

This concludes the proof of (4-10), and hence the proof of Lemma 4.1. ☐

4B. ***Rigidity dynamics in $\mathcal{A}_{\alpha_0}$.*** In this part, we will give a specific classification for the asymptotic behavior of solution with initial data in $\mathcal{A}_{\alpha_0}$.

We first introduce the separation time t_1^*:

$$t_1^*=\begin{cases} 0 & \text{if } |b(0)+c_1\omega(0)|\geq C^*(\mathcal{N}_1(0)+b^2(0)+\omega^2(0)),\\ \sup\{0<t<t^* \mid \text{for all } t'\in[0,t],\ |b(t')+c_1\omega(t')|\leq C^*(\mathcal{N}_1(t')+b^2(t')+\omega^2(t'))\} & \text{else,}\end{cases} \tag{4-22}$$

where[10]

$$C^*=100(K_1+K_0K_2)>0. \tag{4-23}$$

Then we have:

Proposition 4.2 (rigidity dynamics). *There exist universal constants $0<\gamma\ll\alpha_0\ll\alpha^*\ll 1$ and $C^*>1$ such that the following hold. Let $u_0\in\mathcal{A}_{\alpha_0}$, and $u(t)$ be the corresponding solution to (gKdV$_\gamma$). Then we have*:

(1) *The following trichotomy holds*:

Blow down: *If $t_1^*=t^*$, then $t_1^*=t^*=T=+\infty$, with*

$$|b(t)|+\mathcal{N}_2(t)\to 0 \quad \text{as } t\to+\infty, \tag{4-24}$$

$$\lambda(t)\sim t^{\frac{2}{q+1}},\quad x(t)\sim t^{\frac{q-3}{q+1}} \quad \text{as } t\to+\infty. \tag{4-25}$$

Exit: *If $t_1^*<t^*$ with*

$$b(t_1^*)+c_1\omega(t_1^*)\leq -C^*(\mathcal{N}_1(t_1^*)+b^2(t_1^*)+\omega^2(t_1^*)),$$

then $t^*<T=+\infty$. In particular,

$$\inf_{\lambda_0>0,\,\lambda_0^{-m}\gamma<\omega^*,\,x_0\in\mathbb{R}}\left\|u(t^*)-\frac{1}{\lambda_0^{1/2}}Q_{\lambda_0^{-m}\gamma}\left(\frac{x-x_0}{\lambda_0}\right)\right\|_{L^2}=\alpha^*. \tag{4-26}$$

Moreover, we have

$$b(t^*)\leq -C(\alpha^*)<0,\quad \lambda(t^*)\geq\frac{C(\alpha^*)}{\delta(\alpha_0)}\gg 1. \tag{4-27}$$

Soliton: *If $t_1^*<t^*$ with*

$$b(t_1^*)+c_1\omega(t_1^*)\geq C^*(\mathcal{N}_1(t_1^*)+b^2(t_1^*)+\omega^2(t_1^*)),$$

then $t^*=T=+\infty$. Moreover, we have

$$\mathcal{N}_2(t)+|b(t)|\to 0\quad \text{as } t\to+\infty, \tag{4-28}$$

$$\lambda(t)=\lambda_\infty(1+o(1)),\quad x(t)=\frac{t}{\lambda_\infty^2}(1+o(1))\quad \text{as } t\to+\infty, \tag{4-29}$$

for some $\lambda_\infty\in(0,+\infty)$.

(2) *All of the three scenarios introduced in* (1) *are known to occur. Moreover, the initial data sets which lead to the soliton and exit cases are open in $\mathcal{A}_{\alpha_0}$ (under the topology of $H^1\cap L^2(y_+^{10}\,dy)$).*

[10] Recall that K_0, K_1, K_2 and c_1 were introduced in Lemma 4.1.

Remark 4.3. It is easy to see Proposition 4.2 implies Theorem 1.3 immediately.

Remark 4.4. The constant C^* chosen here is not sharp. We can replace it by some slightly different ones.

Proof of Proposition 4.2. The basic idea of the proof is to show that the assumptions (H1)–(H3) introduced in Proposition 3.1 hold for all $t \in [0, t^*)$ (i.e., as long as the solution is close to the soliton manifold). And then together with the estimates obtained in Lemma 4.1, we can show that the error term ε does not perturb the ODE system, and hence the parameters (b, λ, x) have the same asymptotic behavior as the formal system (1-11), which concludes the proof of Theorem 1.3.

In the blow-down and exit cases, this is done by improving the estimates in (H1)–(H3) on $[0, t^{**}]$ (recall that t^{**} is the largest time t such that (H1)–(H3) hold on $[0, t]$), and then a standard bootstrap argument shows that $t^{**} = t^*$, i.e., (H1)–(H3) hold on $[0, t^*)$, while in the soliton case it seems hard to improve all the estimates on $[0, t^{**}]$. But, fortunately, we can use a similar bootstrap argument to show that some modified assumptions (H1)$'$, (H2)$'$, (H3)$'$ hold on $[0, t^*]$, which is also enough to obtain the asymptotic behavior of the parameters.

I. The blow-down case. Assume that $t_1^* = t^*$; i.e., for all $t \in [0, t^*]$,

$$|b(t) + c_1\omega(t)| \leq C^*(\mathcal{N}_1(t) + b^2(t) + \omega^2(t)). \tag{4-30}$$

Step 1: Closing the bootstrap. We claim that $t^{**} = t^*$; i.e., the bootstrap assumptions (H1), (H2) and (H3) hold on $[0, t^*]$.

Indeed, we claim that for all $s \in [0, s^{**})$,

$$\omega(s) + |b(s)| + \|\varepsilon(s)\|_{L^2} + \mathcal{N}_2(s) \lesssim \delta(\alpha_0), \tag{4-31}$$

$$\lambda(s) \geq \tfrac{4}{5}, \tag{4-32}$$

$$\int_{y>0} y^{10}\varepsilon^2(s)\, dy \leq 5. \tag{4-33}$$

Then choosing α^*, α_0, γ such that $0 < \gamma \ll \alpha_0 \ll \alpha^* \ll \kappa$, we can see that (4-31)–(4-33) imply $t^{**} = t^*$ immediately.

First, from (4-30) we have, for all $s \in [0, s^{**})$,

$$b(s) \leq 4C^*\mathcal{N}_1(s) - |b(s)|, \tag{4-34}$$

$$|b(s)| \lesssim \mathcal{N}_1(s) + \omega(s), \tag{4-35}$$

$$\omega(s) \lesssim \mathcal{N}_1(s) + |b(s)|. \tag{4-36}$$

Then we apply (4-34) and (4-36) to (4-10) to obtain

$$\begin{aligned}\frac{(\lambda_0)_s}{\lambda_0} &\geq -b - \mathcal{N}_1 - C(\omega + |b|)(\mathcal{N}_2^{\frac{1}{2}} + |b|)\\ &\geq -5C^*\mathcal{N}_1 + |b| - \delta(\kappa)|b| \gtrsim -\mathcal{N}_1.\end{aligned}$$

Integrating this from s_1 to s_2 for some $0 \le s_1 < s_2 \le s^{**}$, and using (4-6), we have

$$\lambda(s_2) \ge \tfrac{9}{10}\lambda(s_1). \tag{4-37}$$

In particular, we know from (4-2) that for all $s \in [0, s^{**})$

$$\lambda(s) \ge \tfrac{9}{10}\lambda(0) \ge \tfrac{4}{5}. \tag{4-38}$$

By our choice of γ, we have

$$\omega(s) = \frac{\gamma}{\lambda^m(s)} \le 2^m \gamma \lesssim \delta(\alpha_0). \tag{4-39}$$

Next, from (4-4), (4-6) and (4-35), we have for all $s \in [0, s^{**})$

$$\mathcal{N}_2(s) \lesssim \mathcal{N}_2(0) + \sup_{s' \in [0,s]} \mathcal{N}_2^3(s') + \sup_{s' \in [0,s]} \omega^3(s'),$$

which together with (4-35) implies

$$|b(s)| + \mathcal{N}_2(s) \lesssim \delta(\alpha_0)$$

for all $s \in [0, s^{**})$. Then from (2-32) and the condition on the initial data, we obtain

$$\|\varepsilon(s)\|_{L^2} \lesssim \delta(\alpha_0). \tag{4-40}$$

From (2-33) and the condition on the initial data, we have

$$\frac{\|\varepsilon_y(s)\|_{L^2}^2}{\lambda^2(s)} \lesssim \delta(\alpha_0) + \frac{\|\varepsilon_y(s)\|_{L^2}^{m+2}}{\lambda^{m+2}(s)}.$$

Since $\|\varepsilon_y(0)\|_{L^2} \lesssim \delta(\alpha_0)$, $\lambda(0) \sim 1$, from a standard bootstrap argument we have

$$\frac{\|\varepsilon_y(s)\|_{L^2}^2}{\lambda^2(s)} \lesssim \delta(\alpha_0).$$

Thus, we have

$$\omega(s)\|\varepsilon_y(s)\|_{L^2}^m \lesssim \gamma \frac{\|\varepsilon_y(s)\|_{L^2}^m}{\lambda^m(s)} \lesssim \delta(\alpha_0). \tag{4-41}$$

Finally, let us integrate (3-41) from 0 to s, using (4-3), (4-6), (4-8), (4-37) and (4-38) to obtain

$$\begin{aligned}\int \varphi_{10}\varepsilon^2(s)\,dy &\le \frac{\lambda^{10}(0)}{\lambda^{10}(s)} \int \varphi_{10}\varepsilon^2(0)\,dy + C\int_0^s \frac{\lambda^{10}(s')}{\lambda^{10}(s)} (\mathcal{N}_{1,\mathrm{loc}}(s') + b^2(s'))\,ds' \\ &\le 3 + C\int_0^s (\mathcal{N}_{1,\mathrm{loc}}(s') + b^2(s'))\,ds' \le 3 + \delta(\kappa) < 5.\end{aligned}$$

We therefore conclude the proof of (4-31)–(4-33), and obtain $t^{**} = t^*$. Since $0 < \alpha_0 \ll \alpha^*$, the estimate (4-31) implies $t^{**} = t^* = T = +\infty$.

Step 2: Proof of (4-24) and (4-25). We first claim that $\lambda(t) \to +\infty$ as $t \to +\infty$. Let

$$S = \int_0^{+\infty} \frac{1}{\lambda^3(\tau)}\,d\tau \in (0, +\infty].$$

From (2-35), (4-6), (4-8) and (4-36) we have

$$\int_0^{+\infty} |\omega_t|\,dt = \int_0^S |\omega_s|\,ds \lesssim \int_0^S (\mathcal{N}_{2,\mathrm{loc}}(s) + b^2(s))\,ds < +\infty,$$
$$\int_0^{+\infty} \frac{\gamma^2}{\lambda^{3+2m}(t)}\,dt = \int_0^S \omega^2(s)\,ds \lesssim \int_0^S (\mathcal{N}_{2,\mathrm{loc}}(s) + b^2(s))\,ds < +\infty.$$

This leads to $\lambda(t) \to +\infty$ as $t \to +\infty$, or equivalently $\lim_{t\to+\infty} \omega(t) = 0$.

Next, we claim that $S = +\infty$. Otherwise, $b(s), \omega(s) \in L^1([0, S))$. Applying this to (4-10), we obtain

$$\frac{(\lambda_0)_s}{\lambda_0} \in L^1([0, S)).$$

But since $\lambda_0(s) \to +\infty$ as $s \to S$, we have

$$\left|\int_0^{S-\delta_0} \frac{(\lambda_0)_s}{\lambda_0}(s')\,ds'\right| = \left|\log\left(\frac{\lambda_0(S-\delta_0)}{\lambda_0(0)}\right)\right| \to +\infty$$

as $\delta_0 \to 0$, which leads to a contradiction.

Now we can prove (4-24) and (4-25). To do this, we claim that, for all $s \in [0, +\infty)$,

$$\lambda^m(s)\mathcal{N}_2(s) + \int_0^s \lambda^m(s')(\varepsilon^2(s') + \varepsilon_y^2(s'))\varphi'_{2,B}\,ds' \lesssim 1. \tag{4-42}$$

From (3-11) we have

$$\frac{1}{\lambda^m}\frac{d}{ds}(\lambda^m \mathcal{F}_{2,1}) \le -\mu \int (\varepsilon^2 + \varepsilon_y^2)\varphi'_{2,B} + O(b^4 + \omega^2 b^2) - m\frac{\lambda_s}{\lambda}\mathcal{F}_{2,1}. \tag{4-43}$$

From (2-34), (3-13), (3-21) and (4-38), we have

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}\mathcal{F}_{2,1}\right| &\lesssim (|b| + \mathcal{N}_{1,\mathrm{loc}}^{\frac{1}{2}})\left[\left(1 + \frac{1}{\lambda^{10/9}(s)}\right)\mathcal{N}_{2,\mathrm{loc}}^{\frac{8}{9}} + \int \varepsilon_y^2 \psi_B\right]\\ &\lesssim b^2 + \delta(\kappa)\int (\varepsilon^2 + \varepsilon_y^2)\varphi'_{2,B}.\end{aligned}$$

Substituting this into (4-43) and integrating from 0 to s, using (4-35) and (4-36), we have,

$$\begin{aligned}\lambda^m(s)\mathcal{N}_2(s) + \int_0^s \lambda^m(s')(\varepsilon^2(s') + \varepsilon_y^2(s'))\varphi'_{2,B}\,ds' &\lesssim \int_0^s \lambda^m(s')\omega^4(s')\,ds' + \delta(\kappa)\int_0^s \lambda^m(s')\mathcal{N}_1(s')\,ds'\\ &\lesssim \gamma\int_0^s \omega^3(s')\,ds' + \delta(\kappa)\int_0^s \lambda^m(s')\mathcal{N}_1(s')\,ds'\\ &\lesssim \gamma\int_0^s b^2(s')\,ds' + \delta(\kappa)\int_0^s \lambda^m(s')\mathcal{N}_1(s')\,ds'.\end{aligned}$$

Together with (4-8), we obtain (4-42).

Since $\lambda(s) \to +\infty$ as $s \to +\infty$, we have

$$\mathcal{N}_2(s) \lesssim \lambda^{-m}(s) \to 0 \quad \text{as } s \to +\infty. \tag{4-44}$$

Now, using (4-10), (4-30) and (4-35), we have

$$\left|-\frac{(\lambda_0)_s}{\lambda_0}+c_1\omega\right| \lesssim \left|\frac{(\lambda_0)_s}{\lambda_0}+b\right|+|b+c_1\omega|$$
$$\lesssim \mathcal{N}_1+b^2+\omega^2+(|b|+\omega)(\mathcal{N}_2^{\frac{1}{2}}+|b|) \lesssim \mathcal{N}_1+\delta(\kappa)\omega.$$

Multiplying the above inequality by λ_0^m and integrating from 0 to s, we obtain

$$-C\int_0^s \lambda_0^m\mathcal{N}_1+\tfrac{1}{2}c_1\gamma s \le \int_0^s (\lambda_0)_s\lambda_0^{m-1} \le C\int_0^s \lambda_0^m\mathcal{N}_1+2c_1\gamma s.$$

From (4-42) and $|1-\lambda/\lambda_0|\lesssim\delta(\kappa)$, we obtain

$$\lambda^m(s)\sim s \quad \text{as } s\to+\infty.$$

We then have

$$t(s)=\int_0^s \lambda^3(s')\,ds' \sim s^{\frac{m+3}{m}} = s^{\frac{q+1}{q-5}} \quad \text{as } s\to+\infty,$$

which implies

$$\lambda(t)\sim t^{\frac{2}{q+1}} \quad \text{as } t\to+\infty.$$

Next, from (4-30) and (4-35), we have

$$b(t)\to 0 \quad \text{as } t\to+\infty.$$

Finally, integrating (2-34), we obtain

$$x(t)\sim t^{\frac{q-3}{q+1}} \quad \text{as } t\to+\infty,$$

which concludes the proof of (4-24) and (4-25).

II. The exit case. Assume $t_1^*<t^*$ with

$$b(t_1^*)+c_1\omega(t_1^*)\le -C^*(\mathcal{N}_1(t_1^*)+b^2(t_1^*)+\omega^2(t_1^*)). \tag{4-45}$$

Step 1: Closing the bootstrap. First of all, following the same procedure as in the blow-down case, we have, for all $s\in[0,s_1^*]$,

$$\omega(s)+|b(s)|+\|\varepsilon(s)\|_{L^2}+\omega(s)\|\varepsilon_y(s)\|_{L^2}^m+\mathcal{N}_2(s)\lesssim\delta(\alpha_0), \tag{4-46}$$
$$\lambda(s)\ge\tfrac{4}{5}, \tag{4-47}$$
$$\int_{y>0} y^{10}\varepsilon^2(s)\,dy\le 5. \tag{4-48}$$

In particular, we have $t_1^*<t^{**}\le t^*$. Now, we claim $t^{**}=t^*<T=+\infty$.

To prove this, we use a standard bootstrap argument by improving (H1), (H2) and (H3) on $[t_1^*,t^{**}]$. Let

$$\ell^*=\frac{b(t_1^*)+c_1\omega(t_1^*)}{\lambda^2(t_1^*)}<0.$$

It is easy to see that $|\ell^*| \lesssim \delta(\alpha_0)$. Now we observe from (4-9) that, for all $s \in [s_1^*, s^{**})$,

$$2\ell^* - C^* \frac{b^2(s)+\omega^2(s)}{\lambda^2(s)} \le \frac{b(s)+c_1\omega(s)}{\lambda^2(s)} \le \frac{\ell^*}{2} + C^* \frac{b^2(s)+\omega^2(s)}{\lambda^2(s)},$$

which implies

$$-b(s) \gtrsim \omega(s) > 0, \tag{4-49}$$

$$3\ell^* - C\frac{\omega(s)}{\lambda^2(s)} \le \frac{b(s)}{\lambda^2(s)} \le \frac{\ell^*}{3} < 0. \tag{4-50}$$

We then observe from (4-10) and (4-49) that

$$\frac{(\lambda_0)_s}{\lambda_0} \gtrsim -\mathcal{N}_{1,\mathrm{loc}},$$

which after integration, yields the almost monotonicity:

$$\text{for all } s_1^* \le s_1 < s_2 \le s^{**}, \quad \lambda(s_2) \ge \tfrac{9}{10}\lambda(s_1) \ge \tfrac{1}{2}. \tag{4-51}$$

So we obtain for all $s \in [s_1^*, s^{**})$,

$$\omega(s) + \frac{\omega(s)}{\lambda^2(s)} \lesssim \gamma \lesssim \delta(\alpha_0).$$

Together with (4-7) and (4-50), we have, for all $s \in [s_1^*, s^{**})$,

$$\frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0),$$

which improves (H2) if we choose $\alpha_0 \ll \kappa$. Next, using the same strategy as in the blow-down case, we have, for all $s \in [s_1^*, s^{**})$,

$$\int \varphi_{10}\varepsilon^2(s)\,dy \le 7.$$

Then, (H3) is improved. It now only remains to improve (H1). Since for all $t \in [t_1^*, t^*)$ we have $u(t) \in \mathcal{T}_{\alpha^*,\gamma}$, following the argument in Lemma 2.6, for all $t \in [0, t^*)$ we have $|b(t)| \lesssim \delta(\alpha^*)$. By (2-32), (4-6), and (4-49), we have, for all $s \in [s_1^*, s^{**})$,

$$\omega(s) + \|\varepsilon(s)\|_{L^2} + \mathcal{N}_2(s) \lesssim \delta(\alpha^*).$$

Now, following from the same argument as for (4-41), we have

$$\omega(s)\|\varepsilon_y(s)\|_{L^2}^m \lesssim \delta(\alpha_0).$$

Then (H1) is improved, due to our choice of the universal constant, i.e., $\alpha^* \ll \kappa$.

In conclusion, we have proved $t^{**} = t^*$.

Step 2: Proof of (4-26) and (4-27). We first claim that the exit case occurs in finite time $t^* < +\infty$. Dividing (4-10) by λ_0^2, and using (4-49) to estimate on $[t_1^*, t^*)$,

$$-\frac{\ell^*}{3} - C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} \le (\lambda_0)_t \le -3\ell^* + C\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}.$$

Integrating from t_1^* to t, we get

$$\frac{|\ell^*|(t-t_1^*)}{3} - C_1\int_{t_1^*}^{t}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} \le \lambda_0(t)-\lambda_0(t_1^*) \le 3|\ell^*|(t-t_1^*) + C_2\int_{t_1^*}^{t}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2}.$$

From (4-51), we have

$$\int_{t_1^*}^{t}\frac{\mathcal{N}_{1,\mathrm{loc}}}{\lambda^2} = \int_{s_1^*}^{s}\lambda\mathcal{N}_{1,\mathrm{loc}} \lesssim \lambda(s)\int_{s_1^*}^{s}\mathcal{N}_{1,\mathrm{loc}} \lesssim \delta(\kappa)\lambda(t).$$

Thus, for all $t\in[t_1^*,t^*)$,

$$\tfrac{1}{4}(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*)) \le \lambda(t) \le 4(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*)).$$

Next, from (4-49), we have for all $t\in[t_1^*,t^*)$,

$$-100|\ell^*|(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*))^2 \le b(t) \le -\frac{|\ell^*|}{100}(|\ell^*|(t-t_1^*)+\lambda_0(t_1^*))^2.$$

If $t^*=T=+\infty$, then the above estimate leads to $b(t)\to-\infty$ as $t\to+\infty$, which contradicts the fact that $|b(t)|\lesssim\delta(\alpha^*)$ for all $t\in[t_1^*,t^*)$. Thus, we have $t^*<T=+\infty$.

Finally, since $0<t^*<+\infty$, by the definition of t^*, we must have $-b(t^*)\ge C(\alpha^*)>0$. While from (4-49), we have

$$\lambda^2(t^*) \ge \frac{1}{2}\frac{|b(t^*)|}{|\ell^*|} \gtrsim \frac{C(\alpha^*)}{\delta(\alpha_0)} \gg 1,$$

which concludes the proof of (4-26) and (4-27).

III. The soliton case. Assume $t_1^*<t^*$ with

$$b(t_1^*)+c_1\omega(t_1^*) \ge C^*(\mathcal{N}_1(t_1^*)+b^2(t_1^*)+\omega^2(t_1^*))>0. \tag{4-52}$$

Step 1: Estimates on the rescaled solution. Similar to the exit case, we have, for all $s\in[0,s_1^*]$,

$$\omega(s)+|b(s)|+\|\varepsilon(s)\|_{L^2}+\omega(s)\|\varepsilon_y(s)\|_{L^2}^m+\mathcal{N}_2(s)\lesssim\delta(\alpha_0), \tag{4-53}$$

$$\lambda(s)\ge\tfrac{4}{5}, \tag{4-54}$$

$$\int_{y>0} y^{10}\varepsilon^2(s)\,dy \le 5. \tag{4-55}$$

But here we can't directly prove that $t^{**}=t^*$ as we did in the exit case. The main difficulty is that we lack some control on the upper bound of $\lambda(t_1^*)$, which makes it hard to improve the bootstrap assumptions (H2) and (H3). However, we will see that the bootstrap assumptions (H2) and (H3) are related to the scaling symmetry of the problem. If we use the pseudoscaling rule (1-1) on $[t_1^*,t^*)$ to rescale $\lambda(t_1^*)$ to 1, then we can get the desired result. Roughly speaking, on $[t_1^*,t^*]$, the bootstrap assumptions (H2) and (H3) should be replaced by some other suitable assumptions (H2)′ and (H3)′.

More precisely, we introduce the following change of coordinates. For all $t \in [t_1^*, t^*)$, let

$$\bar{t} = \frac{t - t_1^*}{\lambda^3(t_1^*)}, \quad \bar{x} = \frac{x - x(t_1^*)}{\lambda(t_1^*)}, \quad \bar{\gamma} = \frac{\gamma}{\lambda^m(t_1^*)}, \quad \bar{t}^* = \frac{t^* - t_1^*}{\lambda^3(t_1^*)}, \tag{4-56}$$

$$\bar{u}(\bar{t}, \bar{x}) = \lambda^{\frac{1}{2}} u\big(\lambda^3(t_1^*)\bar{t} + t_1^*, \lambda(t_1^*)\bar{x} + x(t_1^*)\big). \tag{4-57}$$

Then, from the pseudoscaling rule (1-1), $\bar{u}(\bar{t}, \bar{x})$ is a solution to the Cauchy problem

$$\begin{cases} \partial_{\bar{t}}\bar{u} + (\bar{u}_{\bar{x}\bar{x}} + \bar{u}^5 - \bar{\gamma}\bar{u}|\bar{u}|^{q-1})_{\bar{x}} = 0, & (\bar{t}, \bar{x}) \in [0, \bar{t}^*) \times \mathbb{R}, \\ \bar{u}(0, \bar{x}) = Q_{b(t_1^*), \omega(t_1^*)}(\bar{x}) + \varepsilon(t_1^*, \bar{x}) \in H^1(\mathbb{R}). \end{cases} \tag{4-58}$$

Moreover, for all $\bar{t} \in [0, \bar{t}^*)$ we define

$$\bar{\varepsilon}(\bar{t}, y) = \varepsilon(\lambda^3(t_1^*)\bar{t} + t_1^*, y), \quad \bar{\lambda}(\bar{t}) = \frac{\lambda(\lambda^3(t_1^*)\bar{t} + t_1^*)}{\lambda(t_1^*)}, \quad \bar{\omega}(\bar{t}) = \frac{\bar{\gamma}}{\bar{\lambda}^m(\bar{t})}, \tag{4-59}$$

$$\bar{b}(\bar{t}) = b(\lambda^3(t_1^*)\bar{t} + t_1^*), \quad \bar{x}(\bar{t}) = \frac{x(\lambda^3(t_1^*)\bar{t} + t_1^*) - x(t_1^*)}{\lambda(t_1^*)}. \tag{4-60}$$

Then, from (2-21), it is easy to check that

$$\bar{u}(\bar{t}, \bar{x}) = \frac{1}{\bar{\lambda}^{1/2}(\bar{t})}[Q_{\bar{b}(\bar{t}), \bar{\omega}(\bar{t})} + \bar{\varepsilon}(\bar{t})]\left(\frac{\bar{x} - \bar{x}(\bar{t})}{\bar{\lambda}(\bar{t})}\right),$$

with

$$(\bar{\varepsilon}(\bar{s}), Q_{\bar{\omega}(\bar{s})}) = (\bar{\varepsilon}(\bar{s}), \Lambda Q_{\bar{\omega}(\bar{s})}) = (\bar{\varepsilon}(\bar{s}), \bar{y}\Lambda Q_{\bar{\omega}(\bar{s})}) = 0,$$

where $(\bar{s}, \bar{y})$ are the scaling-invariant variables

$$\bar{s} = \int_0^{\bar{t}} \frac{1}{\bar{\lambda}^3(\tau)}\, d\tau, \quad \bar{y} = \frac{\bar{x} - \bar{x}(\bar{t})}{\bar{\lambda}(\bar{t})}.$$

We then introduce the weighted Sobolev norms

$$\begin{aligned} \bar{\mathcal{N}}_i(\bar{s}) &= \int \big(\bar{\varepsilon}_{\bar{y}}^2(\bar{s}, \bar{y})\psi_B(\bar{y}) + \bar{\varepsilon}^2(\bar{s}, \bar{y})\varphi_{i,B}(\bar{y})\big)\, d\bar{y}, \\ \bar{\mathcal{N}}_{i,\mathrm{loc}}(\bar{s}) &= \int \bar{\varepsilon}^2(\bar{s}, \bar{y})\varphi_{i,B}'(\bar{y})\, d\bar{y}, \end{aligned} \tag{4-61}$$

where $\varphi_{i,B}$ and ψ_B are the weight functions introduced in Section 3.

From now on, for all $\bar{t} \in [0, \bar{t}^*)$, we let $t = \lambda^3(t_1^*)\bar{t} + t_1^*$. In this setting, we have $\bar{s}(\bar{t}) = s(t) - s_1^*$. Since the pseudoscaling rule (1-1) is L^2 invariant, we have

$$\bar{u}(\bar{t}) \in \mathcal{T}_{\alpha^*, \bar{\gamma}} \iff u(t) \in \mathcal{T}_{\alpha^*, \gamma},$$

which yields

$$\bar{t}^* = \sup\{0 < \bar{t} < +\infty \mid \text{for all } t' \in [0, \bar{t}],\ \bar{u}(t') \in \mathcal{T}_{\alpha^*, \bar{\gamma}}\}.$$

Next, let $\kappa > 0$ be the universal constant introduced in Proposition 2.9, Proposition 3.1 and Lemma 4.1. We then define the following bootstrap assumptions for the rescaled solution $\bar{u}(\bar{t}, \bar{x})$. For all $\bar{s} \in [0, \bar{s}(\bar{t}))$:

(H1)′ Scaling-invariant bound:

$$\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s}) + \|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|^m_{L^2} \le \kappa. \tag{4-62}$$

(H2)′ Bound related to H^1 scaling:

$$\frac{\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \le \kappa. \tag{4-63}$$

(H3)′ L^2 weighted bound on the right:

$$\int_{\bar{y}>0} \bar{y}^{10}\bar{\varepsilon}^2(\bar{s},\bar{y})\,d\bar{y} \le 50\left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right). \tag{4-64}$$

We define $\bar{t}^{**}$ as

$$\bar{t}^{**} = \sup\{0 < \bar{t} < \bar{t}^{*} \mid \text{(H1)}', \text{(H2)}' \text{ and (H3)}' \text{ hold for all } t' \in [0,\bar{t}]\}. \tag{4-65}$$

Our goal here is to prove that $\bar{t}^{**} = \bar{t}^{*} = +\infty$, which gives us the desired asymptotic behaviors.[11] Let $\bar{s}^{*} = \bar{s}(\bar{t}^{*})$, $\bar{s}^{**} = \bar{s}(\bar{t}^{**})$. Since

$$\bar{\lambda}(0) = 1, \quad \bar{x}(0) = 0, \quad \bar{b}(0) = b(t_1^*), \quad \bar{\omega}(0) = \omega(t_1^*), \quad \bar{\varepsilon}(0,\bar{y}) = \varepsilon(t_1^*,\bar{y}), \quad \bar{\gamma} \lesssim \gamma, \tag{4-66}$$

we know from (4-53)–(4-55), that $\bar{s}^{**} > 0$.

On the other hand, on $[0,\bar{s}^{**})$, all conditions of Propositions 2.9 and 3.1 and Lemmas 3.6 and 4.1 are satisfied for $\bar{u}(\bar{t},\bar{x})$. Repeating the same procedure, we have:

Lemma 4.5 (estimates for the rescaled solution). *For all $\bar{s} \in [0,\bar{s}^{**})$ or equivalently $s \in [s_1^*, s_1^* + \bar{s}^{**})$, all estimates of Propositions 2.9 and 3.1 and Lemmas 3.6 and 4.1 hold with*

$$(t, x, u, \gamma, \lambda(t), b(t), x(t), \omega(t), \varepsilon(t), s, y)$$

replaced by

$$(\bar{t}, \bar{x}, \bar{u}, \bar{\gamma}, \bar{\lambda}(\bar{t}), \bar{b}(\bar{t}), \bar{x}(\bar{t}), \bar{\omega}(\bar{t}), \bar{\varepsilon}(\bar{t}), \bar{s}, \bar{y}).$$

Remark 4.6. For simplicity, we skip the statement of these similar estimates for $\bar{u}$. We also refer to the equation number of the corresponding inequality for $u(t)$, when we need to use these estimates for $\bar{u}(\bar{t})$.

Step 2: Closing the bootstrap. In this part, we will close the bootstrap argument to show that $\bar{t}^{**} = \bar{t}^{*} = +\infty$. This is done through the following steps:

(1) We prove that for $\bar{t}$ large enough, we have $\bar{\omega}(\bar{t}) \gg |\bar{b}(\bar{t})|$, which coincides with the formal ODE system (1-11) in the soliton region, where we have $\omega(t)$ converges to a positive constant, while $b(t)$ converges to 0 as $t \to +\infty$. Indeed, if $|\bar{b}(\bar{t})| \gtrsim \bar{\omega}(\bar{t})$ holds for all $\bar{t} \in [0,\bar{t}^{**}]$, we will obtain finite time blow-up if $\bar{b}(0) > 0$ or exit behavior if $\bar{b}(0) < 0$. Both of them lead to a contradiction.

[11] Since $\lambda(t_1^*) \gtrsim 1$, we know that (H1) is equivalent to (H1)′ and (H2) is weaker than (H2)′, while (H3) is stronger than (H3)′. It is hard to determine whether $t^{**} = \lambda^3(t_1^*)\bar{t}^{**} + t_1^*$ holds.

(2) The hardest part of the analysis is to prove that the scaling parameter $\bar{\lambda}$ is bounded from both above and below for all $\bar{t} \in [0, \bar{t}^{**}]$. This is done by proving that[12]

$$\left(\frac{1}{\bar{\lambda}^2}\right)_s + C\bar{\gamma}\left(\frac{1}{\bar{\lambda}^2}\right)^{1+\frac{m}{2}} \sim \bar{\ell}^* > 0.$$

(3) The estimates of the rest of the terms can be done by arguments similar to those in the blow-down and exit regions.

Now we turn to the proof of $\bar{t}^{**} = \bar{t}^* = +\infty$. We first define

$$\bar{t}_2^* = \begin{cases} 0 & \text{if } |\bar{b}(0)| \le \frac{1}{100}c_1\bar{\omega}(0), \\ \sup\{0 < \bar{t} < \bar{t}^* \mid \text{for all } t' \in [0, \bar{t}],\ |\bar{b}(t')| \ge \frac{1}{100}c_1\bar{\omega}(t')\} & \text{else.} \end{cases}$$

Our first observation is that $\bar{t}_2^* < \bar{t}^*$. Otherwise, since $\bar{t}_2^* = \bar{t}^* \ge \bar{t}^{**} > 0$, we have, for all $\bar{t} \in [0, \bar{t}^{**})$, $\bar{b}(\bar{t}) \ne 0$.

If $\bar{b}(0) > 0$, we claim that $\bar{t}^{**} = \bar{t}_2^* = \bar{t}^* = +\infty$. To prove this, we need to improve (H1)′, (H2)′ and (H3)′ on $[0, \bar{t}^{**}]$. Indeed, from the definition of $\bar{t}_2^*$, we have

$$0 < \bar{\omega}(\bar{t}) \lesssim \bar{b}(\bar{t}) \tag{4-67}$$

for all $\bar{t} \in [0, \bar{t}^{**})$. Applying this to (4-10), we have

$$\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0} \le -\bar{b} + O(\bar{\mathcal{N}}_{2,\text{loc}}) + \delta(\kappa)|\bar{b}|.$$

Integrating this from 0 to $\bar{t}$ using (4-6) and the fact that $\bar{\lambda}(0) = 1$, we obtain the almost monotonicity

$$\text{for all } 0 \le \bar{s}_1 < \bar{s}_2 \le \bar{s}^{**}, \quad \bar{\lambda}(\bar{s}_2) \le \tfrac{10}{9}\bar{\lambda}(\bar{s}_1) \le \tfrac{5}{4}. \tag{4-68}$$

On the other hand, we learn from (4-9), (4-52) and (4-66), that for all $\bar{s} \in [0, \bar{s}^{**})$

$$\frac{99}{100}\bar{\ell}^* - K_1\frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \le \frac{\bar{b}(\bar{s}) + c_1\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \le \frac{101}{100}\bar{\ell}^* + K_1\frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})}, \tag{4-69}$$

where

$$0 < \bar{\ell}^* = \frac{\bar{b}(0) + c_1\bar{\omega}(0)}{\bar{\lambda}^2(0)} = b(t_1^*) + c_1\omega(t_1^*) \lesssim \delta(\alpha_0).$$

Together with (4-67), we have for all $\bar{s} \in [0, \bar{s}^{**})$

$$\frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \sim \bar{\ell}^* \lesssim \delta(\alpha_0), \quad \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \bar{\ell}^* \lesssim \delta(\alpha_0). \tag{4-70}$$

Then from (4-68), (4-6) and (4-7), we have for all $\bar{s} \in [0, \bar{s}^{**})$

$$\frac{\bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0), \quad \bar{\mathcal{N}}_2(\bar{s}) + \bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| \lesssim \bar{\lambda}^2(\bar{s})\bar{\ell}^* + \delta(\alpha_0) \le \delta(\alpha_0). \tag{4-71}$$

[12] See (4-88) and (4-90) for details.

Then, from (2-32), (4-53) and fact

$$\bar{u}(0,\bar{x}) = Q_{b(t_1^*),\omega(t_1^*)}(\bar{x}) + \varepsilon(t_1^*,\bar{x}),$$

we know that

$$\begin{aligned}\|\bar{\varepsilon}(\bar{s})\|_{L^2} &\lesssim \delta(\alpha_0) + \left|\int \bar{u}^2(0) - \int Q^2\right|^{\frac{1}{2}}\\ &\lesssim \delta(\alpha_0) + \|\varepsilon(t_1^*)\|_{L^2} + |b(t_1^*)|^{\frac{1}{2}} + \omega^{\frac{1}{2}}(t_1^*) \lesssim \delta(\alpha_0).\end{aligned} \tag{4-72}$$

Now, from (2-33) and (4-71), we have

$$\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m = \bar{\gamma}\frac{\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m}{\bar{\lambda}^m(\bar{s})} \lesssim \delta(\alpha_0) + \left(\bar{\gamma}\frac{\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m}{\bar{\lambda}^m(\bar{s})}\right)^{\frac{m+2}{2}} + |\bar{\gamma}^{\frac{2}{m}}\bar{E}(\bar{u}(0))|^{\frac{m}{2}},$$

where $\bar{E}(\bar{u}(0))$ is the energy of the Cauchy problem (4-58), i.e.,

$$\bar{E}(\bar{u}(0)) = \frac{1}{2}\int \bar{u}_{\bar{x}}^2(0) - \frac{1}{6}\int \bar{u}^6(0) + \frac{\bar{\gamma}}{q+1}\int |\bar{u}(0)|^{q+1}.$$

Since

$$\bar{u}(0,\bar{x}) = \lambda^{\frac{1}{2}}(t_1^*)u(t_1^*,\lambda(t_1^*)\bar{x} + x(t_1^*)),$$

from the energy conservation law of (gKdV$_\gamma$) and the condition on the initial data, we have

$$|\bar{\gamma}^{\frac{2}{m}}\bar{E}(\bar{u}(0))| = \left|\gamma^{\frac{2}{m}}\frac{\bar{E}(\bar{u}(0))}{\lambda^2(t_1^*)}\right| = |\gamma^{\frac{2}{m}}E(u(t_1^*))| = |\gamma^{\frac{2}{m}}E_0| \lesssim \delta(\alpha_0).$$

Thus, for all $\bar{s} \in [0,\bar{s}^{**})$, we have

$$\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0) + (\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m)^{1+\frac{m}{2}}.$$

From (4-53) and (4-66), we have

$$\bar{\omega}(0)\|\bar{\varepsilon}_{\bar{y}}(0)\|_{L^2}^m = \omega(s_1^*)\|\varepsilon_y(s_1^*)\|_{L^2}^m \lesssim \delta(\alpha_0).$$

Then a standard bootstrap argument leads to

$$\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0) \tag{4-73}$$

for all $\bar{s} \in [0,\bar{s}^{**})$.

Finally, integrating (3-41), using (4-6) and (4-68) we obtain

$$\begin{aligned}\int \varphi_{10}(\bar{y})\bar{\varepsilon}^2(\bar{s},\bar{y})\,d\bar{y} &\leq \frac{\bar{\lambda}^{10}(0)}{\bar{\lambda}^{10}(\bar{s})}\int \varphi_{10}(\bar{y})\bar{\varepsilon}^2(0,\bar{y})\,d\bar{y} + \frac{C}{\bar{\lambda}^{10}(\bar{s})}\int_0^{\bar{s}} \bar{\lambda}^{10}(\bar{\mathcal{N}}_{1,\mathrm{loc}} + \bar{b}^2)\\ &\leq \frac{1}{\bar{\lambda}^{10}(\bar{s})}\left[5 + C\bar{\lambda}^{10}(0)\int_0^{\bar{s}}(\bar{\mathcal{N}}_{1,\mathrm{loc}} + \bar{b}^2)\right] \leq \frac{5+\delta(\kappa)}{\bar{\lambda}^{10}(\bar{s})}.\end{aligned} \tag{4-74}$$

Combining (4-70)–(4-74), we conclude that $\bar{t}^{**} = \bar{t}^{*}$. Since all H^1 solutions of (4-58) are global in time, we must have $\bar{t}^{**} = \bar{t}^{*} = +\infty$, provided that $\alpha_0 \ll \alpha^*$. Now we substitute (4-70) into (4-10) to obtain

$$\frac{\bar{\ell}^*}{3} - C\frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2} \le -(\bar{\lambda}_0)_{\bar{t}} \le 3\bar{\ell}^* + C\frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2}.$$

Integrating in time, we have for all $\bar{t} \in [0, +\infty)$

$$0 < \bar{\lambda}_0(\bar{t}) \le \bar{\lambda}(0) - \frac{\bar{\ell}^* \bar{t}}{3} + C\int_0^{\bar{t}} \frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2}.$$

From (4-68) and (4-6) we have

$$\int_0^{\bar{t}} \frac{\overline{\mathcal{N}}_{1,\mathrm{loc}}}{\bar{\lambda}^2} = \int_0^{\bar{s}} \bar{\lambda}(\tau)\overline{\mathcal{N}}_{1,\mathrm{loc}}(\tau)\, d\tau \lesssim \int_0^{\bar{s}} \overline{\mathcal{N}}_{1,\mathrm{loc}}(\tau)\, d\tau \lesssim \delta(\kappa),$$

which implies that the solution blows up in finite time. This is a contradiction.

Now we consider the other case $\bar{b}(0) < 0$. We claim again that $\bar{t}_2^* = \bar{t}^{**} = \bar{t}^* = +\infty$. It is also done by improving the three bootstrap assumptions. First, we know from (4-9), (4-52) and (4-66) that (4-69) still holds in this case. And the definition of $\bar{t}_2^*$ implies

$$0 < \bar{\ell}^* \lesssim -\frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \sim \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}. \tag{4-75}$$

Then we apply the fact that $0 < \bar{\omega} \lesssim -\bar{b}$ to (4-10) to obtain

$$\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0} \ge -\tfrac{1}{2}\bar{b} - O(\overline{\mathcal{N}}_{2,\mathrm{loc}}).$$

Integrating in time we have

$$\text{for all } 0 \le \bar{s}_1 < \bar{s}_2 \le \bar{s}^{**}, \quad \bar{\lambda}(\bar{s}_2) \ge \tfrac{9}{10}\bar{\lambda}(\bar{s}_1) \ge \tfrac{4}{5}, \tag{4-76}$$

which yields for all $\bar{s} \in [0, \bar{s}^{**})$

$$\bar{\omega}(\bar{s}) + \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \bar{\gamma} \lesssim \delta(\alpha_0). \tag{4-77}$$

From (4-75), (4-6) and (4-7), we get

$$\overline{\mathcal{N}}_2(\bar{s}) + |\bar{b}(\bar{s})| + \frac{\overline{\mathcal{N}}_2(\bar{s}) + |\bar{b}(\bar{s})|}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0). \tag{4-78}$$

Using the same argument as we did for (4-72)–(4-74), we have

$$\|\bar{\varepsilon}(\bar{s})\|_{L^2} \lesssim \delta(\alpha_0), \quad \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m \lesssim \delta(\alpha_0), \quad \int \varphi_{10}\bar{\varepsilon}^2(\bar{s})\, d\bar{y} \le 7. \tag{4-79}$$

Combining (4-77)–(4-79), we conclude that $\bar{t}^{**} = \bar{t}^* = +\infty$. But from (4-75), we have

$$-\bar{b} \sim \bar{\omega}(\bar{s}) \gtrsim \bar{\gamma}^{\frac{2}{m+2}}(\bar{\ell}^*)^{\frac{m}{m+2}} > 0. \tag{4-80}$$

On the other hand, from (4-8), we have

$$\int_0^{\bar{s}^{**}} \bar{b}^2(s')\,ds' \lesssim 1.$$

The above two estimates imply

$$\bar{s}^{**} = \int_0^{+\infty} \frac{1}{\bar{\lambda}^3(\tau)}\,d\tau < +\infty,$$

which leads to $\bar{\lambda}(\bar{t}_n) \to +\infty$ as $n \to +\infty$ for some sequence $\bar{t}_n \to +\infty$ or equivalently $\lim_{n\to+\infty} \bar{\omega}(\bar{t}_n) = 0$. This contradicts (4-80).

In conclusion, we have proved that $\bar{t}_2^* < \bar{t}^*$ with

$$|\bar{b}(\bar{t}_2^*)| \le \tfrac{1}{100} c_1 \bar{\omega}(\bar{t}_2^*).$$

Let $\bar{s}_2^* = \bar{s}(\bar{t}_2^*)$. Repeating the same procedure as before, we have for all $\bar{s} \in [0, \bar{s}_2^*]$

$$\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \|\bar{\varepsilon}(\bar{s})\|_{L^2} + \bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m + \bar{\mathcal{N}}_2(\bar{s}) \lesssim \delta(\alpha_0), \tag{4-81}$$

$$\frac{\bar{\omega}(\bar{s}) + |\bar{b}(\bar{s})| + \bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})} \lesssim \delta(\alpha_0), \tag{4-82}$$

$$\int_{\bar{y}>0} \bar{y}^{10}\bar{\varepsilon}^2(\bar{s})\,d\bar{y} \le 7\left(1 + \frac{1}{\bar{\lambda}^{10}(\bar{s})}\right). \tag{4-83}$$

In particular, we have $\bar{t}_2^* < \bar{t}^{**} \le \bar{t}^*$. Similarly, we need to improve the three bootstrap assumptions on $[\bar{t}_2^*, \bar{t}^{**})$ to obtain $\bar{t}^{**} = \bar{t}^* = +\infty$.

First, it is easy to see that (4-69) holds on $[\bar{s}_2^*, \bar{s}^{**})$. So the definition of $\bar{s}_2^*$ yields[13]

$$\frac{19}{20}\bar{\ell}^* \le \frac{c_1\bar{\omega}(\bar{s}_2^*)}{\bar{\lambda}^2(\bar{s}_2^*)} \le \frac{21}{20}\bar{\ell}^*, \tag{4-84}$$

which implies

$$\frac{9}{10}\left(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\right)^{\frac{2}{m+2}} \le \frac{1}{\bar{\lambda}^2(\bar{s}_2^*)} \le \frac{11}{10}\left(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\right)^{\frac{2}{m+2}}. \tag{4-85}$$

Next, we let

$$C_1 = \frac{99}{100}c_1 < c_1, \quad C_2 = \frac{101}{100}c_1 > c_1.$$

Then, we learn from (4-69) that, for all $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$,

$$\begin{aligned}\frac{99}{100}\bar{\ell}^* &\le \frac{\bar{b}(\bar{s}) + C_2\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} - \frac{c_1}{100}\frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} + O\left(\frac{\bar{b}^2(\bar{s}) + \bar{\omega}^2(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right)\\ &\le \frac{\bar{b}(\bar{s}) + C_2\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} - \frac{c_1}{100}\frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})} + \delta(\kappa)\left(\left|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right| + \left|\frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right|\right),\end{aligned}$$

[13]Recall that $c_1 = G'(0) > 0$, where G is the C^2 function introduced in (2-40).

which implies[14]

$$\frac{49}{50}\bar{\ell}^* \le \frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}, \tag{4-86}$$

where

$$\bar{\omega}_0(\bar{s}) = \frac{\bar{\gamma}}{\bar{\lambda}_0^m(\bar{s})}.$$

Substituting (4-10) into (4-86), using (4-7) and the fact that[15]

$$\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} > 0,$$

we have

$$\begin{aligned}
\frac{49}{50}\bar{\ell}^* &\le \frac{101}{100}\bigg(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg) - \frac{1}{100}\bigg(\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg) - \frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\
&\qquad + \frac{101K_2}{100}\frac{\bar{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \delta(\kappa)\bigg(\bigg|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg| + \bigg|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg|\bigg) \\
&\le \frac{101}{100}\bigg(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg) - \frac{1}{100}\bigg(\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg) - \frac{c_1}{300}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} \\
&\qquad + \frac{101K_0K_2}{100}\frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}_0^2(0)} + \delta(\kappa)\bigg(\bigg|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg| + \bigg|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg|\bigg) \\
&\le \frac{101}{100}\bigg(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3} + \frac{C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg) + \frac{51K_0K_2}{50}\frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}^2(0)}.
\end{aligned} \tag{4-87}$$

From (4-52) and (4-66), we have

$$\bar{\ell}^* = \frac{\bar{b}(0) + c_1\bar{\omega}(0)}{\bar{\lambda}^2(0)} \ge 100(K_1 + K_0K_2)\frac{(\bar{\mathcal{N}}_1(0) + \bar{b}^2(0) + \bar{\omega}^2(0))}{\bar{\lambda}^2(0)}.$$

So (4-87) implies that for all $\bar{s} \in [\bar{s}_2^*, \bar{s}^{**})$,

$$\frac{1}{2}\bigg(\frac{1}{\bar{\lambda}_0^2}\bigg)_{\bar{s}} + C_2\bar{\gamma}\bigg(\frac{1}{\bar{\lambda}_0^2}\bigg)^{1+\frac{m}{2}} \ge \frac{9}{10}\bar{\ell}^*. \tag{4-88}$$

Similar to (4-86), we have

$$\frac{51}{50}\bar{\ell}^* \ge \frac{\bar{b}(\bar{s}) + C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} - \delta(\kappa)\bigg|\frac{\bar{b}(\bar{s}) + C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\bigg|, \tag{4-89}$$

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[14] Here we use the fact that $|1 - (\bar{\lambda}/\bar{\lambda}_0)| \lesssim |\bar{J}_1| \lesssim \delta(\kappa)$.

[15] This is a direct consequence of (4-86).

which leads to

$$\frac{51}{50}\bar{\ell}^* \geq \frac{99}{100}\left(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3}+\frac{C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right)+\frac{1}{100}\left(\frac{\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right)+\frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{99K_2}{100}\frac{\overline{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}-\delta(\kappa)\left(\left|\frac{\bar{b}(\bar{s})+C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|+\left|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|\right),$$

and

$$\frac{51}{50}\bar{\ell}^* \geq \frac{101}{100}\left(-\frac{(\bar{\lambda}_0)_{\bar{s}}}{\bar{\lambda}_0^3}+\frac{C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right)-\frac{1}{100}\left(\frac{\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right)+\frac{c_1}{200}\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})} + \frac{101K_2}{100}\frac{\overline{\mathcal{N}}_1(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}-\delta(\kappa)\left(\left|\frac{\bar{b}(\bar{s})+C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|+\left|\frac{\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\right|\right).$$

Using the same strategy as (4-87), and discussing the sign of $(\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s}))/\bar{\lambda}_0^2(\bar{s})$, we have

$$\frac{1}{2}\left(\frac{1}{\bar{\lambda}_0^2}\right)_{\bar{s}}+C_1\bar{\gamma}\left(\frac{1}{\bar{\lambda}_0^2}\right)^{1+\frac{m}{2}} \leq \frac{11}{10}\bar{\ell}^*. \tag{4-90}$$

Then we need following basic lemma:

Lemma 4.7. *Let* $F\colon [0,x_0)\to(0,+\infty)$ *be a* C^1 *function. Let* $\nu>0,\ L>0$ *be two positive constants. Then we have*:

(1) *If for all* $x\in[0,x_0)$

$$F_x+F^{1+\nu}\geq L,$$

then for all $x\in[0,x_0)$,

$$F(x)\geq \min(F(0),L^{\frac{1}{1+\nu}}).$$

(2) *If for all* $x\in[0,x_0)$

$$F_x+F^{1+\nu}\leq L,$$

then for all $x\in[0,x_0)$,

$$F(x)\leq \max(F(0),L^{\frac{1}{1+\nu}}).$$

It is easy to prove Lemma 4.7 by standard ODE theory. Now we apply Lemma 4.7 to (4-88) and (4-90) on $[\bar{s}_2^*,\bar{s}^{**})$, using (4-85) to obtain

$$\frac{90}{101}\left(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\right)^{\frac{2}{m+2}} \leq \frac{1}{\bar{\lambda}^2(\bar{s})} \leq \frac{10}{9}\left(\frac{\bar{\ell}^*}{c_1\bar{\gamma}}\right)^{\frac{2}{m+2}} \tag{4-91}$$

for all $\bar{s}\in[\bar{s}_2^*,\bar{s}^{**})$. This also implies that, for all $\bar{s}\in[\bar{s}_2^*,\bar{s}^{**})$,

$$\bar{\omega}(\bar{s})\sim\bar{\gamma}^{\frac{2}{m+2}}(\bar{\ell}^*)^{\frac{m}{m+2}}\lesssim\delta(\alpha_0),\qquad \frac{\bar{\omega}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\sim\bar{\ell}^*\lesssim\delta(\alpha_0). \tag{4-92}$$

From (4-86) and (4-89), we have

$$\frac{\bar{b}(\bar{s})+C_2\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\geq\frac{49}{50}\bar{\ell}^*,\quad \frac{\bar{b}(\bar{s})+C_1\bar{\omega}_0(\bar{s})}{\bar{\lambda}_0^2(\bar{s})}\leq 2\bar{\ell}^*;$$

together with (4-92), we have

$$\left|\frac{\bar{b}(\bar{s})}{\bar{\lambda}^2(\bar{s})}\right|\lesssim\bar{\ell}^*\lesssim\delta(\alpha_0),\quad |\bar{b}(\bar{s})|\lesssim\bar{\gamma}^{\frac{2}{m+2}}(\bar{\ell}^*)^{\frac{m}{m+2}}\lesssim\delta(\alpha_0). \tag{4-93}$$

Again, from the mass conservation law (2-32), the energy conservation law (2-33) and the almost monotonicity (4-6), (4-7), we have for all $\bar{s}\in[\bar{s}_2^*,\bar{s}^{**})$

$$\|\bar{\varepsilon}(\bar{s})\|_{L^2}+\bar{\omega}(\bar{s})\|\bar{\varepsilon}_{\bar{y}}(\bar{s})\|_{L^2}^m+\bar{\mathcal{N}}_2(\bar{s})+\frac{\bar{\mathcal{N}}_2(\bar{s})}{\bar{\lambda}^2(\bar{s})}\lesssim\delta(\alpha_0). \tag{4-94}$$

Finally, we learn from (4-91) that, for all $\bar{s}_2^*\leq\bar{s}_1<\bar{s}_2\leq\bar{s}^{**}$,

$$\frac{1}{4}<\left(\frac{81}{101}\right)^5\leq\left(\frac{\bar{\lambda}(\bar{s}_1)}{\bar{\lambda}(\bar{s}_2)}\right)^{10}\leq\left(\frac{101}{81}\right)^5<4.$$

Then for all $\bar{s}\in[\bar{s}_2^*,\bar{s}^{**})$, we integrate (3-41) from $\bar{s}_2^*$ to $\bar{s}$ to obtain

$$\begin{aligned}\int\varphi_{10}(\bar{y})\bar{\varepsilon}^2(\bar{s},\bar{y})\,d\bar{y}&\leq\frac{\bar{\lambda}^{10}(\bar{s}_2^*)}{\bar{\lambda}^{10}(\bar{s})}\int\varphi_{10}(\bar{y})\bar{\varepsilon}^2(\bar{s}_2^*,\bar{y})\,d\bar{y}+\frac{C}{\bar{\lambda}^{10}(\bar{s})}\int_{\bar{s}_2}^{\bar{s}}\bar{\lambda}^{10}(\bar{\mathcal{N}}_{1,\mathrm{loc}}+\bar{b}^2)\\&\leq\frac{\bar{\lambda}^{10}(\bar{s}_2^*)}{\bar{\lambda}^{10}(\bar{s})}\times 7\left(1+\frac{1}{\bar{\lambda}^{10}(\bar{s}_2^*)}\right)+4C\int_{\bar{s}_2}^{\bar{s}}(\bar{\mathcal{N}}_{1,\mathrm{loc}}+\bar{b}^2)\\&\leq 28\left(1+\frac{1}{\bar{\lambda}^{10}(\bar{s})}\right)+\delta(\kappa)<30\left(1+\frac{1}{\bar{\lambda}^{10}(\bar{s})}\right).\end{aligned} \tag{4-95}$$

Combining (4-92)–(4-95), we have improved (H1)′, (H2)′ and (H3)′; hence $\bar{t}^{**}=\bar{t}^*=+\infty$. This also implies $t^*=+\infty$.

Step 3: Proof of (4-28) and (4-29). Now it is sufficient to prove

$$|\bar{b}(\bar{t})|+\bar{\mathcal{N}}_2(\bar{t})\to 0,\quad \bar{\lambda}(\bar{t})\to\bar{\lambda}_\infty\in(0,+\infty)$$

as $\bar{t}\to+\infty$. First of all, from (4-91), we know that

$$\bar{s}^{**}=\bar{s}^*=\int_0^{+\infty}\frac{1}{\bar{\lambda}^3(\tau)}\,d\tau=+\infty.$$

Then we claim that $\bar{b}_{\bar{s}}\bar{b}\in L^1((0,+\infty))$. Indeed, from (2-50), we have

$$|\bar{b}\bar{b}_{\bar{s}}+\bar{\omega}_{\bar{s}}G'(\bar{\omega})\bar{b}|\lesssim\bar{b}^2+\int\bar{\varepsilon}^2e^{-\frac{|\bar{y}|}{10}}\in L^1((0,+\infty)).$$

From (2-34), we have

$$\bar{\omega}_{\bar{s}} G'(\bar{\omega})\bar{b} = m\bar{\omega}G'(\bar{\omega})\bar{b}^2 + O\left(\bar{\omega}\left|\bar{b}\left(\frac{\bar{\lambda}_{\bar{s}}}{\bar{\lambda}} + \bar{b}\right)\right|\right) = O\left(\bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}}\right).$$

The above two estimates imply

$$\int_0^{+\infty} |\bar{b}_{\bar{s}}\bar{b}(s')|\, ds' = \int_0^{+\infty} \tfrac{1}{2}|(\bar{b}^2)_{\bar{s}}| < +\infty.$$

Together with

$$\int_0^{+\infty} \bar{b}^2(\bar{s})\, d\bar{s} < +\infty,$$

we conclude that $\bar{b}(\bar{t}) \to 0$ as $\bar{t} \to +\infty$. Next, we use (2-50) again to obtain

$$|\bar{b}_{\bar{s}} + \bar{\omega}_{\bar{s}} G'(\bar{\omega})| \lesssim \bar{b}^2 + \int \bar{\varepsilon}^2 e^{-\frac{|\bar{y}|}{10}} \in L^1((0,+\infty)).$$

Thus, we have

$$\int_0^{+\infty} |(\bar{b} + G(\bar{\omega}))_{\bar{s}}(s')|\, ds' < +\infty.$$

We then know that $b(\bar{t}) + G(\bar{\omega}(\bar{t}))$ has a limit as $\bar{t} \to +\infty$. Since $\lim_{\bar{t}\to+\infty} \bar{b}(\bar{t}) = 0$, we obtain that $G(\bar{\omega}(\bar{t}))$ has a limit as $\bar{t} \to +\infty$. On the other hand, we have $G'(0) > 0$, $\bar{\omega}(\bar{t}) \ll 1$, so there exists a constant $\bar{\omega}_\infty > 0$ such that

$$\lim_{\bar{t}\to+\infty} \bar{\omega}(\bar{t}) = \bar{\omega}_\infty \sim \bar{\gamma}^{\frac{2}{m+2}} (\bar{\ell}^*)^{\frac{m}{m+2}},$$

or equivalently

$$\lim_{\bar{t}\to+\infty} \bar{\lambda}(\bar{t}) = \bar{\lambda}_\infty \sim \left(\frac{c_1\bar{\gamma}}{\bar{\ell}^*}\right)^{\frac{1}{m+2}}.$$

Let

$$\ell^* = \frac{b(t_1^*) + c_1\omega(t_1^*)}{\lambda^2(t_1^*)} > 0.$$

Recall that

$$\bar{\gamma} = \frac{\gamma}{\lambda^m(t_1^*)}, \quad \bar{\ell}^* = b(t_1^*) + c_1\omega(t_1^*), \quad \bar{\lambda}(\bar{t}) = \frac{\lambda(\lambda^3(t_1^*)\bar{t} + t_1^*)}{\lambda(t_1^*)}.$$

We obtain

$$\lim_{t\to+\infty} \lambda(t) = \lambda_\infty \sim \left(\frac{c_1\gamma}{\ell^*}\right)^{\frac{1}{m+2}}. \tag{4-96}$$

Next, the inequality (4-6) implies the existence of a sequence $\bar{s}_n$ such that

$$\overline{\mathcal{N}}_1(\bar{s}_n) \lesssim \int (\bar{\varepsilon}^2(\bar{s}_n) + \bar{\varepsilon}_{\bar{y}}^2(\bar{s}_n))\varphi'_{2,B} \to 0 \quad \text{as } n \to +\infty,$$

where $\lim_{n\to+\infty} \bar{s}_n = +\infty$. Using the monotonicity (4-11), we have

$$\overline{\mathcal{N}}_1(\bar{s}) \to 0 \quad \text{as } \bar{s} \to +\infty.$$

Together with (3-21) and (4-91), we obtain

$$\overline{\mathcal{N}}_2(\bar{t}) \to 0 \quad \text{as } \bar{t} \to +\infty,$$

which implies

$$\mathcal{N}_2(t) \to 0 \quad \text{as } t \to +\infty.$$

Finally, from (2-34), we have

$$\lambda^2(t)x_t(t) \sim 1 \quad \text{as } t \to +\infty,$$

which after integration implies

$$x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as } t \to +\infty.$$

We then conclude the proof of (4-28) and (4-29), and hence the proof of the first part of Proposition 4.2.

IV. Nonemptiness and stability. Now we give the proof of the second part of Proposition 4.2.

First, we show that the soliton and exit regimes are stable under small perturbation in $\mathcal{A}_{\alpha_0}$. From (2-25), we know that the parameters depend continuously on the initial data, which implies that the exit and soliton cases are both open in $\mathcal{A}_{\alpha_0}$, since the separation condition is an open condition of initial data in $\mathcal{A}_{\alpha_0}$.

Indeed, for all $u_0 \in \mathcal{A}_{\alpha_0}$, if the corresponding solution $u(t)$ to (gKdV$_\gamma$) belongs to the soliton regime, we let t_1^* be the separation time introduced in Proposition 4.2. For all $\tilde{u}_0 \in \mathcal{A}_{\alpha_0}$, close enough to u_0, we let $\tilde{u}(t)$ be the corresponding solution to (gKdV$_\gamma$), and $\tilde{b}(t)$, $\tilde{x}(t)$, $\tilde{\lambda}(t)$, $\tilde{\varepsilon}(t)$ be the corresponding geometrical parameters and error term. Then from local theory, we have $\sup_{t\in[0,t_1^*]} \|u(t)-\tilde{u}(t)\|_{H^1} \ll 1$, which together with (2-25), leads to

$$\tilde{b}(t_1^*) + c_1\tilde{\omega}(t_1^*) \geq \tfrac{999}{1000} C^*(\widetilde{\mathcal{N}}_1(t_1^*) + \tilde{b}^2(t_1^*) + \tilde{\omega}^2(t_1^*)).$$

So $\tilde{u}(t)$ must belong to the soliton regime. This implies the openness of soliton regime. The openness of the exit regime follows from the same argument.

Next, we claim that there exists initial data in $\mathcal{A}_{\alpha_0}$ such that the corresponding solution to (gKdV$_\gamma$) belongs to the soliton and exit regimes respectively. First, it is easy to check that the traveling wave solution

$$u(t,x) = Q_\gamma(x-t)$$

belongs to the soliton regime. On the other hand, from (2-43), we can see, in both the soliton and blow-down cases, we have

$$\|u_0\|_{L^2} \geq \|Q\|_{L^2}.$$

Hence, for initial data $u_0 \in \mathcal{A}_{\alpha_0}$ with[16] $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the corresponding solution must belong to the exit regime.

Finally, since the sets of initial data which lead to the soliton and exit regimes are both open and nonempty in $\mathcal{A}_{\alpha_0}$, together with the fact that $\mathcal{A}_{\alpha_0}$ is connected, we conclude that there exists $u_0 \in \mathcal{A}_{\alpha_0}$ such that the corresponding solution to (gKdV$_\gamma$) belongs to the blow-down regime. ☐

[16] Since we assume that $\gamma \ll \alpha_0$, such u_0 exists in $\mathcal{A}_{\alpha_0}$.

5. Proof of Theorem 1.4

In this part we will use the local Cauchy theory of generalized KdV equations developed in [Kenig, Ponce and Vega 1993] to prove Theorem 1.4.

5A. H^1 *perturbation theory.* First of all, let us introduce the following:

Lemma 5.1 [Kenig, Ponce and Vega 1993]. *The following linear estimates hold*:

(1) *For all* $u_0 \in H^1$,

$$\left\| \frac{\partial}{\partial x} W(t) u_0 \right\|_{L_x^\infty L_t^2(\mathbb{R})} + \|W(t)u_0\|_{L_x^5 L_t^{10}(\mathbb{R})} \lesssim \|u_0\|_{L^2}, \tag{5-1}$$

$$\|D_x^{\alpha_q} D_t^{\beta_q} W(t) u_0\|_{L_x^p L_t^r(I)} \lesssim \|D_x^{s_q} u_0\|_{L^2}, \tag{5-2}$$

where $q > 5$ *is the power of the defocusing nonlinear term of* (gKdV$_\gamma$), *and*

$$W(t)f = e^{-t\partial_x^3} f, \qquad s_q = \frac{1}{2} - \frac{2}{(q-1)},$$
$$\alpha_q = \frac{1}{10} - \frac{2}{5(q-1)}, \qquad \beta_q = \frac{3}{10} - \frac{6}{5(q-1)},$$
$$\frac{1}{p} = \frac{2}{5(q-1)} + \frac{1}{10}, \qquad \frac{1}{r} = \frac{3}{10} - \frac{4}{5(q-1)}.$$

(2) *For all well-localized* g, *we have*

$$\sup_{t \in I} \left\| \frac{\partial}{\partial x} \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^2} \lesssim \|g\|_{L_x^1 L_t^2(I)}, \tag{5-3}$$

$$\left\| \frac{\partial^2}{\partial x^2} \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^\infty L_t^2(I)} \lesssim \|g\|_{L_x^1 L_t^2(I)}, \tag{5-4}$$

$$\left\| \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^5 L_t^{10}(I)} \lesssim \|g\|_{L_x^{5/4} L_t^{10/9}(I)}, \tag{5-5}$$

$$\left\| D_x^{\alpha_q} D_t^{\beta_q} \int_0^t W(t-t') g(\cdot, t')\, dt' \right\|_{L_x^p L_t^r(I)} \lesssim \|g\|_{L_x^{p'} L_t^{r'}(I)}, \tag{5-6}$$

$$\|g\|_{L_x^{5(q-1)/4} L_t^{5(q-1)/2}} \lesssim \|D_x^{\alpha_q} D_t^{\beta_q} g\|_{L_x^p L_t^r}, \tag{5-7}$$

where

$$1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'}.$$

Proof. See Theorem 3.5, Corollary 3.8, Lemma 3.14, Lemma 3.15 and Corollary 3.16 in [Kenig, Ponce and Vega 1993] for the proofs of (1) and (2). □

Now we define the norms

$$\eta_I^1(w) = \|w\|_{L_x^5 L_t^{10}(I)}, \quad \eta_I^2(w) = \|w_x\|_{L_x^\infty L_t^2(I)}, \quad \eta_I^3(w) = \|D_x^{\alpha_q} D_t^{\beta_q} w\|_{L_x^p L_t^r(I)},$$

$$\Omega_I(w) = \max_{j=1,2}[\eta_I^j(w) + \eta_I^j(w_x)] + \eta_I^3(w),$$

$$\Delta_I(h) = \|h\|_{L_x^1 L_t^2(I)} + \|h_x\|_{L_x^{5/4} L_t^{10/9}(I)} + \|h_x\|_{L_x^1 L_t^2(I)} + \|h_{xx}\|_{L_x^{5/4} L_t^{10/9}(I)} + \|h_x\|_{L_x^{p'} L_t^{r'}(I)}$$

for all interval $I \subset \mathbb{R}$.

Then we have the following:

Proposition 5.2 (modified long-time H^1 perturbation theory). *Let I be an interval containing 0 and $\tilde{u}$ be an H^1 solution to*

$$\begin{cases} \partial_t \tilde{u} + (\partial_{xx}\tilde{u} + \tilde{u}^5 - \gamma \tilde{u}|\tilde{u}|^{q-1})_x = e_x, & (t,x) \in I \times \mathbb{R}, \\ \tilde{u}(0,x) = \tilde{u}_0 \in H^1. \end{cases} \tag{5-8}$$

Suppose we have

$$\sup_{t\in I} \|\tilde{u}(t)\|_{H^1} + \Omega_I(\tilde{u}) \le M$$

for some $M > 0$ independent of γ. Let $u_0 \in H^1$ be such that

$$\|u_0 - \tilde{u}_0\|_{H^1} + \|e\|_{L_x^1 L_t^2(I)} + \|e_x\|_{L_x^{5/4} L_t^{10/9}(I)} + \|e_x\|_{L_x^1 L_t^2(I)} + \|e_{xx}\|_{L_x^{5/4} L_t^{10/9}(I)} + \|e_x\|_{L_x^{p'} L_t^{r'}(I)} \le \varepsilon$$

for some small $0 < \varepsilon < \varepsilon_0(M)$. Then the solution of (gKdV$_\gamma$) with initial data u_0 satisfies

$$\sup_{t\in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(u - \tilde{u}) \le C(M)\varepsilon. \tag{5-9}$$

Remark 5.3. The perturbation theory still holds true if we replace H^1 by H^s, with $s \ge \frac{1}{2} - 2/(q-1) > 0$.

Proof of Proposition 5.2. Without loss of generality, we assume that $I = [0, T_0]$ for some $T_0 > 0$.

We first claim the following:

Lemma 5.4 (short-time perturbation theory). *Under the same notation as Proposition 5.2, if we assume in addition that $\Omega_I(\tilde{u}) \le \varepsilon_0$ for some small $0 < \varepsilon_0 = \varepsilon_1(M) \ll 1$, then there exists a constant $C_0(M)$ which depends only on M such that if $0 < \varepsilon < \varepsilon_0 = \varepsilon_1(M)$, then*

$$\sup_{t\in I} \|u - \tilde{u}\|_{H^1} + \Omega_I(u - \tilde{u}) \le C_0(M)\varepsilon. \tag{5-10}$$

We leave the proof of Lemma 5.4 for Appendix B.

Now we turn to the proof of Proposition 5.2. Let $\varepsilon_0 = \varepsilon_1(2M) > 0$ as in Lemma 5.4. We then choose $0 = t_0 < t_1 < \cdots < t_N = T_0$ (recall that we assume $I = [0, T_0]$) such that for all $j = 1, \ldots, N$,

$$\Omega_{[t_{j-1}, t_j]}(\tilde{u}) \le \varepsilon_0.$$

From a standard argument, we know that $N = N(M, \varepsilon_0) = N(M) > 0$. We use Lemma 5.4 on each interval $[t_{j-1}, t_j]$ to obtain

$$\sup_{t\in[t_{j-1},t_j]} \|u(t) - \tilde{u}(t)\|_{H^1} + \Omega_{[t_{j-1},t_j]}(\tilde{u}) \le C_0(M) \max(\varepsilon, \|u(t_{j-1}) - \tilde{u}(t_{j-1})\|_{H^1}).$$

Arguing by induction, using $\|u(0)-\tilde{u}(0)\|_{H^1}\le\varepsilon$, we have for all $j=1,\dots,N$,

$$\sup_{t\in[t_{j-1},t_j]}\|u(t)-\tilde{u}(t)\|_{H^1}+\Omega_{[t_{j-1},t_j]}(\tilde{u})\le C(j,M)\varepsilon.$$

Summarizing these estimates, we have

$$\begin{aligned}\sup_{t\in I}\|u-\tilde{u}\|_{H^1}+\Omega_I(\tilde{u})&\le\sum_{j=1}^{N}\sup_{t\in[t_{j-1},t_j]}\|u(t)-\tilde{u}(t)\|_{H^1}+\Omega_{[t_{j-1},t_j]}(\tilde{u})\\&\le\sum_{j=1}^{N}C(j,M)\varepsilon=C(M)\varepsilon,\end{aligned}$$

which concludes the proof of Proposition 5.2. □

5B. ***End of the proof of Theorem 1.4.*** Now for $0<\gamma\ll\alpha_0\ll\alpha^*\ll1$, we choose a $u_0\in\mathcal{A}_{\alpha_0/2}\subset\mathcal{A}_{\alpha_0}$ such that the corresponding solution $u(t)$ to (gKdV) belongs to the blow-up regime with blow-up time $T<+\infty$. Let $u_\gamma(t)$ be the corresponding solution to (gKdV$_\gamma$). From [Martel, Merle and Raphaël 2014, Section 4.4], we know that there exists a $0<T_1^*<T<+\infty$, geometrical parameters $(\lambda(t),b(t),x(t))$ and an error term $\varepsilon(t)$ such that the following geometrical decomposition holds on $[0,T_1^*]$:

$$u(t,x)=\frac{1}{\lambda(t)^{1/2}}[Q_{b(t)}+\varepsilon(t)]\left(\frac{x-x(t)}{\lambda(t)}\right),\tag{5-11}$$

with

$$(\varepsilon,Q)=(\varepsilon,\Lambda Q)=(\varepsilon,y\Lambda Q)=0.\tag{5-12}$$

Moreover, we have for all $t\in[0,T_1^*]$

$$\mathcal{N}_2(t)+\|\varepsilon(t)\|_{L^2}+|b(t)|+|1-\lambda(t)|\lesssim\delta(\alpha_0),\tag{5-13}$$

$$\int_{y>0}y^{10}\varepsilon^2(t,y)\,dy\le5,\tag{5-14}$$

$$b(T_1^*)\ge2C^*\mathcal{N}_1(T_1^*),\tag{5-15}$$

where C^* is the universal constant[17] introduced in Section 4B. One may easily check that C^* defined by (4-23) is independent of γ.

Next, we claim that there exists a constant $C(u_0,q)>1$ which depends only on u_0 and q such that

$$\sup_{t\in[0,T_1^*]}\|u(t)\|_{H^1}+\Omega_{[0,T_1^*]}(u)+\Delta_{[0,T_1^*]}(u|u|^{q-1})\le C(u_0,q)<+\infty.\tag{5-16}$$

Indeed, from [Kenig, Ponce and Vega 1993, Corollary 2.11] (taking $s=1$), we have

$$\eta^1_{[0,T_1^*]}(u)+\eta^1_{[0,T_1^*]}(u_x)+\eta^2_{[0,T_1^*]}(u)+\eta^2_{[0,T_1^*]}(u_x)\le C(u_0,q)<+\infty.$$

[17] The constant C^* chosen here might be different from the one in [Martel, Merle and Raphaël 2014, (4.23)]. But we can always replace C^* (both constants in this paper and in [Martel, Merle and Raphaël 2014]) by some larger universal constant.

Then, from Duhamel's principle, we have

$$u(t) = W(t)u_0 + \int_0^t (W(t-t')\partial_x(u^5))\,dt'.$$

Together with (5-2), (5-6) and the Gagliardo–Nirenberg inequality introduced in [Bahouri, Chemin and Danchin 2011, Theorem 2.44], we have

$$\begin{aligned}\eta^3_{[0,T_1^*]}(u) &\lesssim \|u_x u^4\|_{L_x^{p'}L_t^{r'}} + \|u_0\|_{H^1} \lesssim \|u\|^4_{L_x^5 L_t^{10}} \|u_x\|_{L_x^{p_0}L_t^{r_0}} + \|u_0\|_{H^1} \\ &\lesssim \|u\|^4_{L_x^5 L_t^{10}} \|D_x^{s_q} u\|^{1-s_q}_{L_x^5 L_t^{10}} \|D_x^{s_q} u_x\|^{s_q}_{L_x^\infty L_t^2} + \|u_0\|_{H^1} \\ &\lesssim \|u\|^4_{L_x^5 L_t^{10}} (\|u\|^{1-s_q}_{L_x^5 L_t^{10}} \|u_x\|^{s_q}_{L_x^5 L_t^{10}})^{1-s_q} (\|u_x\|^{1-s_q}_{L_x^\infty L_t^2} \|u_{xx}\|^{s_q}_{L_x^\infty L_t^2})^{s_q} + \|u_0\|_{H^1} \\ &\lesssim \left(\eta^1_{[0,T_1^*]}(u) + \eta^1_{[0,T_1^*]}(u_x) + \eta^2_{[0,T_1^*]}(u) + \eta^2_{[0,T_1^*]}(u_x)\right)^5 + \|u_0\|_{H^1},\end{aligned}$$

where

$$\frac{1}{p_0} = \frac{1}{10} - \frac{2}{5(q-1)}, \quad \frac{1}{r_0} = \frac{3}{10} + \frac{4}{5(q-1)}.$$

This implies $\Omega_{[0,T_1^*]}(u) \le C(u_0,q) < +\infty$.

Next, using the arguments in [Kenig, Ponce and Vega 1993, Section 6], we obtain

$$\Delta_{[0,T_1^*]}(u|u|^{q-1}) \lesssim (\Omega_{[0,T_1^*]}(u))^q \le C(u_0,q),$$

which yields (5-16).

Then we apply Proposition 5.2 to $u(t)$ and $u_\gamma(t)$, with $e = \gamma u|u|^{q-1}$. Note that from (5-16), we have

$$\Delta_{[0,T_1^*]}(e) < \gamma C(u_0,q) \le \gamma^{\frac{1}{2}} \ll \varepsilon_0(C(u_0,q)),$$

provided that $0 < \gamma < \gamma(u_0,\alpha_0,\alpha^*,q) \ll 1$. Then Proposition 5.2 implies that, for all $t \in [0,T_1^*]$, we have

$$\|u(t) - u_\gamma(t)\|_{H^1} \lesssim \gamma^{\frac{1}{2}}. \tag{5-17}$$

Combining with (5-11)–(5-14), we know that $u_\gamma(t) \in \mathcal{T}_{\alpha_0,\gamma}$ for all $t \in [0,T_1^*]$. This allows us to apply Lemma 2.6 to $u_\gamma(t)$ on $[0,T_1^*]$; i.e., there exist geometrical parameters $(b_\gamma(t),\lambda_\gamma(t),x_\gamma(t))$ and an error term $\varepsilon_\gamma(t)$, such that

$$u_\gamma(t,x) = \frac{1}{\lambda_\gamma(t)^{1/2}}[Q_{b_\gamma(t),\omega_\gamma(t)} + \varepsilon_\gamma(t)]\left(\frac{x - x_\gamma(t)}{\lambda_\gamma(t)}\right),$$

with

$$\omega_\gamma(t) = \frac{\gamma}{\lambda_\gamma^m(t)}.$$

Moreover, the orthogonality conditions (2-22) hold.

Now, from Lemma 2.6 and (5-17) we obtain that, for all $t \in [0,T_1^*]$,

$$\left|1 - \frac{\lambda(t)}{\lambda_\gamma(t)}\right| + |b_\gamma(t) - b(t)| + |x_\gamma(t) - x(t)| + \|\varepsilon_\gamma(t) - \varepsilon(t)\|_{H^1} \lesssim \delta(\gamma). \tag{5-18}$$

Together with (5-13)–(5-15), we have the following:

(1) For all $t \in [0, T_1^*]$, (4-53)–(4-55) hold for $u_\gamma(t)$.

(2) At the time $t = T_1^*$,
$$b_\gamma(T_1^*) + c_1\omega_\gamma(T_1^*) \geq C^*(\mathcal{N}_{1,\gamma}(T_1^*) + b_\gamma^2(T_1^*) + \omega_\gamma^2(T_1^*)),$$
where
$$\mathcal{N}_{i,\gamma}(t) = \int (\varepsilon_\gamma)_y^2 \psi_B + \varepsilon_\gamma^2 \varphi_{i,B}.$$

By the argument in Section 4, we know that $u_\gamma(t)$ belongs to the soliton regime introduced in Theorem 1.3. Moreover, we also obtain (1-8) from (4-96). This concludes the proof of the first part of Theorem 1.4.

The second part of Theorem 1.4 follows from exactly the same procedure. Thus, we complete the proof of Theorem 1.4.

Appendix A. Proof of the geometrical decomposition

We will give the proof of Lemma 2.6. We first introduce the following notation: for all suitable, $(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v)$
$$F_1(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (Q_{\tilde{\omega}}, \varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}), \tag{A-1}$$
$$F_2(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (\Lambda Q_{\tilde{\omega}}, \varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}), \tag{A-2}$$
$$F_3(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (y\Lambda Q_{\tilde{\omega}}, \varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}), \tag{A-3}$$
where
$$\varepsilon_{\tilde{\lambda},\tilde{x},\tilde{b},\tilde{\omega},v}(y) = \tilde{\lambda}^{\frac{1}{2}} v(\tilde{\lambda} y + \tilde{x}) - Q_{\tilde{b},\tilde{\omega}}(y).$$

We mention here that we don't assume
$$\tilde{\omega} = \frac{\gamma}{\tilde{\lambda}^m}.$$

At $(\tilde{\lambda}, \tilde{x}, \tilde{b}, \tilde{\omega}, v) = (1, 0, 0, 0, Q)$, we have
$$\left(\frac{\partial F_1}{\partial \tilde{\lambda}}, \frac{\partial F_1}{\partial \tilde{x}}, \frac{\partial F_1}{\partial \tilde{b}}\right) = \big((\Lambda Q, Q), (Q', Q), (P, Q)\big),$$
$$\left(\frac{\partial F_2}{\partial \tilde{\lambda}}, \frac{\partial F_2}{\partial \tilde{x}}, \frac{\partial F_2}{\partial \tilde{b}}\right) = \big((\Lambda Q, \Lambda Q), (Q', \Lambda Q), (P, \Lambda Q)\big),$$
$$\left(\frac{\partial F_3}{\partial \tilde{\lambda}}, \frac{\partial F_3}{\partial \tilde{x}}, \frac{\partial F_3}{\partial \tilde{b}}\right) = \big((\Lambda Q, y\Lambda Q), (Q', y\Lambda Q), (P, y\Lambda Q)\big).$$
Since
$$(\Lambda Q, Q) = (Q', Q) = (Q', \Lambda Q) = (\Lambda Q, y\Lambda Q) = 0,$$
$$(P, Q) \neq 0, \quad (\Lambda Q, \Lambda Q) \neq 0, \quad (Q', y\Lambda Q) \neq 0,$$

it is easy to see that the above Jacobian is not degenerate. Hence, from implicit function theory, we have: there exist unique continuous maps

$$(\tilde{\lambda}_0, \tilde{x}_0, \tilde{b}_0):\ (\tilde{\omega}, v) \mapsto (1-\delta, 1+\delta) \times (-\delta, \delta) \times (-\delta, \delta), \quad \delta > 0, \tag{A-4}$$

such that for all $\tilde{\omega} \ll 1$, $\|v - Q\|_{H^1} \ll 1$, it holds that

$$F_j\big(\tilde{\lambda}_0(\tilde{\omega}, v), \tilde{x}_0(\tilde{\omega}, v), \tilde{b}_0(\tilde{\omega}, v), \tilde{\omega}, v\big) = 0, \quad j = 1, 2, 3. \tag{A-5}$$

The uniqueness also implies that, for all $\tilde{\omega} \ll 1$, we have

$$\tilde{\lambda}_0(\tilde{\omega}, \mathcal{Q}_{\tilde{\omega}}) \equiv 1. \tag{A-6}$$

Next we fix a time $t \in [0, t_0)$ as in Lemma 2.6. For a solution $u(t)$ to (gKdV$_\gamma$) with

$$u(t,x) = \frac{1}{\lambda_1^{1/2}(t)}[\mathcal{Q}_{\omega_1(t)} + \varepsilon_1(t)]\left(\frac{x - x_1(t)}{\lambda_1(t)}\right),$$

and

$$\omega_1(t) = \frac{\gamma}{\lambda_1^m(t)} \ll 1,$$

we let

$$v(t, \cdot) = \lambda_1^{\frac{1}{2}}(t) u(t, \lambda_1(t) \cdot + x_1(t)) = \mathcal{Q}_{\omega_1(t)}(\cdot) + \varepsilon_1(t, \cdot).$$

Then we have $\|v(t,\cdot) - Q(\cdot)\|_{H^1} \ll 1$.

We claim that there exists a $\underline{\lambda}(t) > 0$ such that

$$\lambda_1(t)\tilde{\lambda}_0\left(\frac{\gamma}{\underline{\lambda}^m(t)}, v(t)\right) = \underline{\lambda}(t), \quad \frac{\gamma}{\underline{\lambda}^m(t)} \ll 1. \tag{A-7}$$

This is easily verified by implicit function theory. We let

$$M(\underline{\lambda}, v) = \underline{\lambda} - \lambda_1(t)\tilde{\lambda}_0\left(\frac{\gamma}{\underline{\lambda}^m}, v\right).$$

Then we have

$$M(\lambda_1(t), \mathcal{Q}_{\omega_1(t)}) = 0,$$
$$\frac{\partial M}{\partial \underline{\lambda}}\bigg|_{(\underline{\lambda}, v) = (\lambda_1(t), \mathcal{Q}_{\omega_1(t)})} = 1 + m\omega_1(t)\frac{\partial \tilde{\lambda}_0}{\partial \tilde{\omega}}(\omega_1(t), \mathcal{Q}_{\omega_1(t)}) > 0,$$

which implies (A-7) immediately.

Applying (A-4)–(A-7) to $v(t)$, we have

$$F_j\big(\tilde{\lambda}_0(\omega(t), v(t)), \tilde{x}_0(\omega(t), v(t)), \tilde{b}_0(\omega(t), v(t)), \omega(t), v(t)\big) = 0, \quad j = 1, 2, 3, \tag{A-8}$$
$$\lambda_1(t)\tilde{\lambda}_0(\omega(t), v(t)) = \underline{\lambda}(t), \tag{A-9}$$

where

$$\omega(t) = \frac{\gamma}{\underline{\lambda}^m(t)}.$$

Now, we let

$$\lambda(t)=\underline{\lambda}(t),\quad b(t)=\tilde{b}_0(\omega(t),v(t)),\quad x(t)=x_1(t)+\lambda_1(t)\tilde{x}_0(\omega(t),v(t)), \tag{A-10}$$

$$\omega(t)=\frac{\gamma}{\lambda^m(t)},\quad \varepsilon(t,y)=\lambda^{\frac{1}{2}}(t)u(t,\lambda(t)\cdot+x(t))-Q_{b(t),\omega(t)}. \tag{A-11}$$

We claim that this $(\lambda(t),x(t),b(t))$ satisfies the orthogonality conditions (2-22). Indeed, from (A-7)–(A-9), we have

$$\begin{aligned}
0&=F_1\big(\tilde{\lambda}_0(\omega(t),v(t)),\tilde{x}_0(\omega(t),v(t)),\tilde{b}_0(\omega(t),v(t)),\omega(t),v(t)\big)\\
&=\Big(\mathcal{Q}_{\omega(t)}(\cdot),\ \tilde{\lambda}_0^{\frac{1}{2}}(\omega(t),v(t))v\big(t,\tilde{\lambda}_0(\omega(t),v(t))\cdot+\tilde{x}_0(\omega(t),v(t))\big)-Q_{b(t),\omega(t)}(\cdot)\Big)\\
&=\Big(\mathcal{Q}_{\omega(t)}(\cdot),\ [\lambda_1(t)\tilde{\lambda}_0(\omega(t),v(t))]^{\frac{1}{2}}\\
&\qquad\times u\big(t,\lambda_1(t)[\tilde{\lambda}_0(\omega(t),v(t))\cdot+\tilde{x}_0(\omega(t),v(t))]+x_1(t)\big)-Q_{b(t),\omega(t)}(\cdot)\Big)\\
&=\big(\mathcal{Q}_{\omega(t)}(\cdot),\ \lambda^{\frac{1}{2}}(t)u(t,\lambda(t)\cdot+x(t))-Q_{b(t),\omega(t)}(\cdot)\big)\\
&=(\mathcal{Q}_{\omega(t)},\varepsilon(t)).
\end{aligned}$$

The other two orthogonality conditions can be verified similarly.

Finally, since the maps

$$(\tilde{\lambda}_0,\tilde{x}_0,\tilde{b}_0):\ (\tilde{\omega},v)\mapsto(1-\delta,1+\delta)\times(-\delta,\delta)\times(-\delta,\delta)$$

are continuous, the remaining part of Lemma 2.6 follows immediately.

Appendix B. Proof of Lemma 5.4

We give the proof of the modified short-time perturbation theory, i.e., Lemma 5.4.

First, we let $v(t,x)=u(t,x)-\tilde{u}(t,x)$, $S(t)=\Omega_{[0,t]}(v)$. We claim the following estimate holds true for all $t\in I$:

$$S(t)\lesssim_M \varepsilon+S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde{u})^4+\Omega_I(\tilde{u})^{q-1}). \tag{B-1}$$

Since $S(0)=0$ and $\Omega_I(\tilde{u})\le\varepsilon_0$, we know that Lemma 5.4 follows from a standard bootstrap argument. Now it only remains to prove (B-1).

First, by Duhamel's principle, we have

$$\begin{aligned}
v(t)&=W(t)(\tilde{u}_0-u_0)+\int_0^t\big(W(t-t')\partial_x[\tilde{u}^5-\gamma\tilde{u}|\tilde{u}|^{q-1}-(\tilde{u}+v)^5+\gamma(\tilde{u}+v)|\tilde{u}+v|^{q-1}-e]\big)\,dt'\\
&=v_L(t)+v_N(t).
\end{aligned}$$

For the linear part v_L, from Lemma 5.1, we have

$$\Omega_{[0,t]}(v_L)+\sup_{t'\in[0,t]}\|v_L\|_{H^1}\lesssim\|\tilde{u}_0-u_0\|_{H^1}\lesssim\varepsilon. \tag{B-2}$$

Now, for the nonlinear part v_N, we use Lemma 5.1 to estimate

$$\eta^1_{[0,t]}(v_N) \lesssim \|e_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \|(v+\tilde{u})^4(v+\tilde{u})_x - \tilde{u}^4\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \| |v+\tilde{u}|^{q-1}(v+\tilde{u})_x - |\tilde{u}|^{q-1}\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])}.$$

By Hölder's inequality, we have

$$\begin{aligned}\|(v+\tilde{u})^4(v+\tilde{u})_x - \tilde{u}^4\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} &\lesssim \|((v+\tilde{u})^4-\tilde{u}^4)\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}} + \|(v+\tilde{u})^4 v_x\|_{L_x^{5/4}L_t^{10/9}}\\ &\lesssim (\|\tilde{u}\|^3_{L_x^5L_t^{10}} + \|v\|^3_{L_x^5L_t^{10}})\|v\|_{L_x^5L_t^{10}}\|\tilde{u}_x\|_{L_x^\infty L_t^2} + \|v\|^4_{L_x^5L_t^{10}}(\|v_x\|_{L_x^\infty L_t^2} + \|\tilde{u}_x\|_{L_x^\infty L_t^2})\\ &\lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),\end{aligned}$$

and

$$\begin{aligned}\| |v+\tilde{u}|^{q-1}(v+\tilde{u})_x - |\tilde{u}|^{q-1}\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}([0,t])} &\lesssim \|(|v+\tilde{u}|^{q-1} - |\tilde{u}|^{q-1})\tilde{u}_x\|_{L_x^{5/4}L_t^{10/9}} + \| |v+\tilde{u}|^{q-1}v_x\|_{L_x^{5/4}L_t^{10/9}}\\ &\lesssim (\|\tilde{u}\|^{q-2}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}} + \|v\|^{q-2}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\|v\|_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}\|\tilde{u}_x\|_{L_x^\infty L_t^2}\\ &\qquad + \|v\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}(\|v_x\|_{L_x^\infty L_t^2} + \|\tilde{u}_x\|_{L_x^\infty L_t^2})\\ &\lesssim (\|D_x^{\alpha_q}D_t^{\beta_q}\tilde{u}\|^{q-2}_{L_x^pL_t^r} + \|D_x^{\alpha_q}D_t^{\beta_q}v\|^{q-2}_{L_x^pL_t^r})\|D_x^{\alpha_q}D_t^{\beta_q}v\|_{L_x^pL_t^r}\|\tilde{u}_x\|_{L_x^\infty L_t^2}\\ &\qquad + \|D_x^{\alpha_q}D_t^{\beta_q}v\|^{q-1}_{L_x^pL_t^r}(\|v_x\|_{L_x^\infty L_t^2} + \|\tilde{u}_x\|_{L_x^\infty L_t^2})\\ &\lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),\end{aligned}$$

where we used (5-7) for the last two inequalities. The above two estimates imply

$$\eta^1_{[0,t]}(v_N) \lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}) + \varepsilon. \tag{B-3}$$

Similarly, we have

$$\eta^1_{[0,t]}(\partial_x v_N) \lesssim \|e_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \|((v+\tilde{u})^5 - \tilde{u}^5)_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])} + \|((v+\tilde{u})|v+\tilde{u}|^{q-1} - \tilde{u}|\tilde{u}|^{q-1})_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])}.$$

By Hölder's inequality again, we have

$$\begin{aligned}\|((v+\tilde{u})^5 - \tilde{u}^5)_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])} &\lesssim \|(v+\tilde{u})^4 v_{xx}\|_{L_x^{5/4}L_t^{10/9}} + \|(v+\tilde{u})^3(v_x + 2\tilde{u}_x)v_x\|_{L_x^{5/4}L_t^{10/9}}\\ &\lesssim (\|\tilde{u}\|^4_{L_x^5L_t^{10}} + \|v\|^4_{L_x^5L_t^{10}})\|v_{xx}\|_{L_x^\infty L_t^2} + \|v\|^3_{L_x^5L_t^{10}}\|v_x\|_{L_x^\infty L_t^2}(\|v_x\|_{L_x^5L_t^{10}} + \|\tilde{u}_x\|_{L_x^5L_t^{10}})\\ &\lesssim S(t)(S(t)^4 + S(t)^{q-1} + \Omega_I(\tilde{u})^4 + \Omega_I(\tilde{u})^{q-1}),\end{aligned}$$

and

$$
\begin{aligned}
&\|((v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1})_{xx}\|_{L_x^{5/4}L_t^{10/9}([0,t])}\\
&\quad\lesssim \||v+\tilde u|^{q-1}v_{xx}\|_{L_x^{5/4}L_t^{10/9}}+\||v+\tilde u|^{q-2}(v_x+2\tilde u_x)v_x\|_{L_x^{5/4}L_t^{10/9}}\\
&\quad\lesssim (\|\tilde u\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}+\|v\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\|v_{xx}\|_{L_x^\infty L_t^2}\\
&\qquad\qquad+\|v\|^{q-2}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}\|v_x\|_{L_x^\infty L_t^2}(\|v_x+2\tilde u_x\|_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\\
&\quad\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1}).
\end{aligned}
$$

Collecting these estimates, we have

$$
\eta^1_{[0,t]}(\partial_x v_N)\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1})+\varepsilon. \tag{B-4}
$$

Next, using a similar strategy, we have

$$
\begin{aligned}
\eta^2_{[0,t]}(v_N)&\lesssim \|e\|_{L_x^1L_t^2([0,t])}+\|(v+\tilde u)^5-\tilde u^5\|_{L_x^1L_t^2([0,t])}+\|(v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1}\|_{L_x^1L_t^2([0,t])}\\
&\lesssim \varepsilon+(\|\tilde u\|^4_{L_x^5L_t^{10}}+\|v\|^4_{L_x^5L_t^{10}})\|v\|_{L_x^5L_t^{10}}\\
&\qquad+(\|v\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}}+\|\tilde u\|^{q-1}_{L_x^{5(q-1)/4}L_t^{5(q-1)/2}})\|v\|_{L_x^5L_t^{10}}\\
&\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1})+\varepsilon,
\end{aligned}\tag{B-5}
$$

and

$$
\begin{aligned}
\eta^2_{[0,t]}(\partial_x v_N)&\lesssim \|e_x\|_{L_x^1L_t^2([0,t])}+\|((v+\tilde u)^5-\tilde u^5)_x\|_{L_x^1L_t^2([0,t])}\\
&\qquad+\|((v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1})_x\|_{L_x^1L_t^2([0,t])}\\
&\lesssim \varepsilon+\|((v+\tilde u)^4-\tilde u^4)\tilde u_x\|_{L_x^1L_t^2([0,t])}+\|(v+\tilde u)^4v_x\|_{L_x^1L_t^2([0,t])}\\
&\qquad+\|(|v+\tilde u|^{q-1}-|\tilde u|^{q-1})\tilde u_x\|_{L_x^1L_t^2([0,t])}+\||v+\tilde u|^{q-1}v_x\|_{L_x^1L_t^2([0,t])}\\
&\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde u)^4+\Omega_I(\tilde u)^{q-1})+\varepsilon.
\end{aligned}\tag{B-6}
$$

Finally, we need to estimate $\eta^3_{[0,t]}(v_N)$. From Lemma 5.1, we have

$$
\begin{aligned}
\eta^3_{[0,t]}(v_N)&\lesssim \|e_x\|_{L_x^{p'}L_t^{r'}([0,t])}+\|((v+\tilde u)^5-\tilde u^5)_x\|_{L_x^{p'}L_t^{r'}([0,t])}\\
&\qquad+\|((v+\tilde u)|v+\tilde u|^{q-1}-\tilde u|\tilde u|^{q-1})_x\|_{L_x^{p'}L_t^{r'}([0,t])}\\
&\lesssim \varepsilon+\|((v+\tilde u)^4-\tilde u^4)\tilde u_x\|_{L_x^{p'}L_t^{r'}}+\|(v+\tilde u)^4v_x\|_{L_x^{p'}L_t^{r'}}\\
&\qquad+\|(|v+\tilde u|^{q-1}-|\tilde u|^{q-1})\tilde u_x\|_{L_x^{p'}L_t^{r'}}+\||v+\tilde u|^{q-1}v_x\|_{L_x^{p'}L_t^{r'}}.
\end{aligned}
$$

By similar technique to that used for (B-6), we have

$$
\begin{aligned}
&\|((v+\tilde u)^4-\tilde u^4)\tilde u_x\|_{L_x^{p'}L_t^{r'}}+\|(v+\tilde u)^4v_x\|_{L_x^{p'}L_t^{r'}}\\
&\qquad\lesssim \|(v+\tilde u)^4-\tilde u^4\|_{L_x^{5/4}L_t^{5/2}}\|\tilde u_x\|_{L_x^{p_0}L_t^{r_0}}+\|(v+\tilde u)^4\|_{L_x^{5/4}L_t^{5/2}}\|v_x\|_{L_x^{p_0}L_t^{r_0}}\\
&\qquad\lesssim \|v_x\|_{L_x^{p_0}L_t^{r_0}}(S(t)^4+\Omega_{[0,t]}(\tilde u)^4)+\|\tilde u_x\|_{L_x^{p_0}L_t^{r_0}}S(t)(S(t)^3+\Omega_{[0,t]}(\tilde u)^3),
\end{aligned}
$$

and

$$\|(|v+\tilde{u}|^{q-1}-|\tilde{u}|^{q-1})\tilde{u}_x\|_{L_x^{p'}L_t^{r'}}+\||v+\tilde{u}|^{q-1}v_x\|_{L_x^{p'}L_t^{r'}}$$
$$\lesssim \|v_x\|_{L_x^{p_0}L_t^{r_0}}(S(t)^{q-1}+\Omega_{[0,t]}(\tilde{u})^{q-1})+\|\tilde{u}_x\|_{L_x^{p_0}L_t^{r_0}}S(t)(S(t)^{q-2}+\Omega_{[0,t]}(\tilde{u})^{q-2}),$$

where

$$\frac{1}{p_0}=\frac{1}{10}-\frac{2}{5(q-1)},\qquad \frac{1}{r_0}=\frac{3}{10}+\frac{4}{5(q-1)}.$$

By the Gagliardo–Nirenberg inequality introduced in [Bahouri, Chemin and Danchin 2011, Theorem 2.44], we have

$$\|v_x\|_{L_x^{p_0}L_t^{r_0}}\lesssim \|D_x^{s_q}v\|_{L_x^5L_t^{10}}^{1-s_q}\|D_x^{s_q}v_x\|_{L_x^\infty L_t^2}^{s_q}$$
$$\lesssim (\|v\|_{L_x^5L_t^{10}}^{1-s_q}\|v_x\|_{L_x^5L_t^{10}}^{s_q})^{1-s_q}(\|v_x\|_{L_x^\infty L_t^2}^{1-s_q}\|v_{xx}\|_{L_x^\infty L_t^2}^{s_q})^{s_q}\lesssim S(t).$$

Similarly, we have

$$\|\tilde{u}_x\|_{L_x^{p_0}L_t^{r_0}}\lesssim \Omega_{[0,t]}(\tilde{u});$$

hence

$$\eta_{[0,t]}^3(v_N)\lesssim S(t)(S(t)^4+S(t)^{q-1}+\Omega_I(\tilde{u})^4+\Omega_I(\tilde{u})^{q-1})+\varepsilon. \tag{B-7}$$

Combining (B-2)–(B-7), we conclude the proof of (B-1), and hence the proof of Lemma 5.4.

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