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ON THE STABILITY OF TYPE II BLOWUP FOR THE 1-COROTATIONAL ENERGY-SUPERCRITICAL HARMONIC HEAT FLOW

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We consider the energy-supercritical harmonic heat flow from $\mathbb{R}^d$ into the d-sphere $\mathbb{S}^d$ with $d \geq 7$. Under an additional assumption of 1-corotational symmetry, the problem reduces to the one-dimensional semilinear heat equation

$$\partial_t u = \partial_r^2 u + \frac{(d-1)}{r}\partial_r u - \frac{(d-1)}{2r^2}\sin(2u).$$

We construct for this equation a family of $\mathcal{C}^\infty$ solutions which blow up in finite time via concentration of the universal profile

$$u(r,t) \sim Q\left(\frac{r}{\lambda(t)}\right),$$

where Q is the stationary solution of the equation and the speed is given by the quantized rates

$$\lambda(t) \sim c_u(T-t)^{\frac{\ell}{\gamma}}, \quad \ell \in \mathbb{N}^*,\ 2\ell > \gamma = \gamma(d) \in (1,2].$$

The construction relies on two arguments: the reduction of the problem to a finite-dimensional one thanks to a robust universal energy method and modulation techniques developed by Merle, Raphaël and Rodnianski (*Camb. J. Math.* **3**:4 (2015), 439–617) for the energy supercritical nonlinear Schrödinger equation and by Raphaël and Schweyer (*Anal. PDE* **7**:8 (2014), 1713–1805) for the energy critical harmonic heat flow. Then we proceed by contradiction to solve the finite-dimensional problem and conclude using the Brouwer fixed-point theorem. Moreover, our constructed solutions are in fact $(\ell-1)$-codimension stable under perturbations of the initial data. As a consequence, the case $\ell = 1$ corresponds to a stable type II blowup regime.

1. Introduction

We consider the harmonic map heat flow which is defined as the negative gradient flow of the Dirichlet energy of maps between manifolds. Indeed, if Φ is a map from $\mathbb{R}^d \times [0,T)$ to a compact Riemannian

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manifold $\mathcal{M} \subset \mathbb{R}^n$, with second fundamental form Υ, then Φ solves

$$\begin{cases} \partial_t \Phi - \Delta\Phi = \Upsilon(\Phi)(\nabla\Phi, \nabla\Phi), \\ \Phi(t=0) = \Phi_0. \end{cases} \tag{1-1}$$

We assume that the target manifold is the d-sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. Then, (1-1) becomes

$$\begin{cases} \partial_t \Phi - \Delta\Phi = |\nabla\Phi|^2\Phi, \\ \Phi(t=0) = \Phi_0. \end{cases} \tag{1-2}$$

We will study the problem (1-2) under an additional assumption of 1-corotational symmetry, namely that a solution of (1-2) takes the form

$$\Phi(x,t) = \begin{pmatrix} \cos(u(|x|,t)) \\ (x/|x|)\sin(u(|x|,t)) \end{pmatrix}. \tag{1-3}$$

Under this ansatz, the problem (1-2) reduces to the one-dimensional semilinear heat equation

$$\begin{cases} \partial_t u = \partial_r^2 u + \dfrac{(d-1)}{r}\partial_r u - \dfrac{(d-1)}{2r^2}\sin(2u), \\ u(t=0) = u_0, \end{cases} \tag{1-4}$$

where $u(t) : r \in \mathbb{R}_+ \to u(r,t) \in [0,\pi]$. The set of solutions to (1-4) is invariant by the scaling symmetry

$$u_\lambda(r,t) = u\left(\frac{r}{\lambda}, \frac{t}{\lambda^2}\right) \quad \text{for all } \lambda > 0.$$

The energy associated to (1-4) is given by

$$\mathcal{E}[u](t) = \int_0^{+\infty} \left(|\partial_r u|^2 + \frac{(d-1)}{r^2}\sin^2(u)\right) r^{d-1}\,dr, \tag{1-5}$$

which satisfies

$$\mathcal{E}[u_\lambda] = \lambda^{d-2}\mathcal{E}[u].$$

The criticality of the problem is reflected by the fact that the energy (1-5) is left invariant by the scaling property when $d = 2$; hence, the case $d \geq 3$ corresponds to the energy-supercritical case.

The problem (1-4) is locally well-posed for data which are close in L^∞ to a uniformly continuous map, see [Koch and Lamm 2012], or in BMO, see [Wang 2011]. Actually, Eells and Sampson [1964] introduced the harmonic map heat flow as a process to deform any smooth map Φ_0 into a harmonic map via (1-2). They also proved that the solution exists globally if the sectional curvature of the target manifold is negative. There exist other assumptions for the global existence; for example, assuming the image of the initial data u_0 is contained in a ball of radius $\pi/(2\sqrt{\kappa})$, where κ is an upper bound on the sectional curvature of the target manifold $\mathcal{M}$; see [Jost 1981; Lin and Wang 2008]. Without these assumptions, the solution $u(r,t)$ may develop singularities in some finite time; see, for example, [Coron and Ghidaglia 1989; Chen and Ding 1990] for $d \geq 3$, and [Chang, Ding and Ye 1992] for $d = 2$. In this case, we say that $u(r,t)$ blows up in a finite time $T < +\infty$ in the sense that

$$\lim_{t\to T} \|\nabla u(t)\|_{L^\infty} = +\infty.$$

Here we call T the blowup time of $u(x,t)$. The blowup has been divided by Struwe [1996] into two types:

$$u \text{ blows up with type I if } \limsup_{t\to T}(T-t)^{\frac{1}{2}}\|\nabla u(t)\|_{L^\infty} < +\infty,$$

$$u \text{ blows up with type II if } \limsup_{t\to T}(T-t)^{\frac{1}{2}}\|\nabla u(t)\|_{L^\infty} = +\infty.$$

Struwe [1988] showed that the type I singularities are asymptotically self-similar; that is, their profile is given by a smooth shrinking function

$$u(r,t) = \phi\left(\frac{r}{\sqrt{T-t}}\right) \quad \text{for all } t \in [0,T),$$

where ϕ solves the equation

$$\phi'' + \left(\frac{d-1}{y} + \frac{y}{2}\right)\phi' - \frac{d-1}{2y^2}\sin(2\phi) = 0. \tag{1-6}$$

Thus, the study of type I blowup reduces to the study of nonconstant solutions of (1-6).

When $3 \le d \le 6$, by using a shooting method, Fan [1999] proved that there exists an infinite sequence of globally regular solutions ϕ_n of (1-6) which are called "shrinkers" (corresponding to the existence of type I blowup solutions of (1-4)), where the integer index n denotes the number of intersections of the function ϕ_n with $\frac{\pi}{2}$. More detailed quantitative properties of such solutions were studied in [Biernat and Bizoń 2011], where the authors conjectured that ϕ_1 is linear stable and provided numerical evidence supporting that ϕ_1 corresponds to a generic profile of type I blowup. Very recently, Biernat, Donninger and Schörkhuber [2016] proved the existence of a stable self-similar blowup solution for $d = 3$. Since (1-2) is not time-reversible, there exists another family of self-similar solutions called "expanders", which were introduced in [Germain and Rupflin 2011]. These expanders have been recently proved to be nonlinearly stable in [Germain, Ghoul and Miura 2017]. To our knowledge, the question on the existence of type II blowup solutions for (1-4) remains open for $3 \le d \le 6$.

When $d \ge 7$, Bizoń and Wasserman [2015] proved that (1-4) has no self-similar shrinking solutions. According to [Struwe 1988], this result implies that in dimensions $d \ge 7$, all singularities for (1-4) must be of type II (see also [Biernat 2015] for a recent analysis of such singularities). Recently, Biernat and Seki [2016], via the matched asymptotic method developed in [Herrero and Velázquez 1994], constructed for (1-4) a countable family of type II blowup solutions, each characterized by a different blowup rate:

$$\lambda(t) \sim (T-t)^{\frac{\ell}{\gamma}} \quad \text{as } t \to T, \tag{1-7}$$

where $\ell \in \mathbb{N}^*$ such that $2\ell > \gamma$ and $\gamma = \gamma(d)$ is given by

$$\gamma(d) = \tfrac{1}{2}(d-2-\tilde{\gamma}) \in (1,2] \quad \text{for } d \ge 7, \tag{1-8}$$

where $\tilde{\gamma} = \sqrt{d^2-8d+8}$. The blowup rate (1-7) is in fact driven by the asymptotic behavior of a stationary solution of (1-4), say Q, which is the unique (up to scaling) solution of the equation

$$Q'' + \frac{(d-1)}{r}Q' - \frac{(d-1)}{2r^2}\sin(2Q) = 0, \quad Q(0) = 0, \;\; Q'(0) = 1, \tag{1-9}$$

and admits the behavior for r large

$$Q(r) = \frac{\pi}{2} - \frac{a_0}{r^\gamma} + \mathcal{O}\left(\frac{1}{r^{2+\gamma}}\right) \quad \text{for some } a_0 = a_0(d) > 0, \tag{1-10}$$

(see the Appendix in [Biernat 2015] for a proof of the existence of Q). Note that the case $2\ell = \gamma$ only happens in dimension $d = 7$. In this case, Biernat [2015] used the method of [Herrero and Velázquez 1994] and formally derived the blowup rate

$$\lambda(t) \sim \frac{(T-t)^{\frac{1}{2}}}{|\log(T-t)|} \quad \text{as } t \to T. \tag{1-11}$$

He also provided numerical evidence supporting that the case $\ell = 1$ in (1-7) or (1-11) corresponds to a generic blowup solution.

In the energy-critical case, i.e., $d = 2$, van den Berg, Hulshof and King [2003], through a formal analysis based on the matched asymptotic technique of [Herrero and Velázquez 1994], predicted that there are type II blowup solutions to (1-4) of the form

$$u(r,t) \sim Q\left(\frac{r}{\lambda(t)}\right),$$

where

$$Q(r) = 2\tan^{-1}(r) \tag{1-12}$$

is the unique (up to scaling) solution of (1-9), and the blowup speed is governed by the quantized rates:

$$\lambda(t) \sim \frac{(T-t)^\ell}{|\log(T-t)|^{\frac{2\ell}{2\ell-1}}} \qquad \text{for } \ell \in \mathbb{N}^*.$$

This result was later confirmed by Raphaël and Schweyer [2014b]. Note that the case $\ell = 1$ was treated in [Raphaël and Schweyer 2013] and corresponds to a stable blowup. In particular, in those papers, they adapted the strategy developed in [Raphaël and Rodnianski 2012; Merle, Raphaël and Rodnianski 2011] for the study of wave and Schrödinger maps to construct for (1-4) type II blowup solutions. Their method relies on a two-step procedure:

- Construction of a suitable approximate blowup profile through iterated resolutions of elliptic equations. The *tail computation* allows us to formally derive the blowup speed. As a matter of fact, the asymptotic behavior at infinity of the stationary solution (1-12) is an essential algebraic fact for their analysis, which in fact drives the derivation of the blowup law and the possibility of a blowup solution with Q profile.
- Implementation of a robust universal energy method to control the solution in the blowup regime through the derivation of suitable "Lyapunov" functionals involving critical Sobolev norms adapted to the linearized flow near the ground state, which relies on neither spectral estimates nor the maximum principle and may be easily applied to various settings.

In this work, by considering $d \geq 7$, we ask whether we can carry out the analysis of [Raphaël and Schweyer 2014b] for the energy-critical case $d = 2$ to the construction of blowup solutions for (1-4) in the case $d \geq 7$. It happens that the asymptotic behavior (1-10) is perfectly suitable to replace the explicit profile (1-12) for an implementation of the strategy of [Raphaël and Schweyer 2014b]. The following theorem is the main result of this paper.

Theorem 1.1 (existence of type II blowup solutions to (1-4) with prescribed behavior). *Let $d \geq 7$ and γ be defined as in (1-8), we fix an integer*

$$\ell \in \mathbb{N}^* \quad \textit{such that} \quad 2\ell > \gamma,$$

and an arbitrary Sobolev exponent

$$\mathfrak{s} \in \mathbb{N}, \qquad \mathfrak{s} = \mathfrak{s}(\ell) \to +\infty \quad \textit{as } \ell \to +\infty.$$

Then there exists a smooth corotational radially symmetric initial data u_0 such that the corresponding solution to (1-4) is of the form

$$u(r,t) = Q\left(\frac{r}{\lambda(t)}\right) + q\left(\frac{r}{\lambda(t)}, t\right), \tag{1-13}$$

where

$$\lambda(t) = c(u_0)(T-t)^{\frac{\ell}{\gamma}}(1 + o_{t\to T}(1)), \quad c(u_0) > 0, \tag{1-14}$$

and

$$\lim_{t\to T} \|\nabla^\sigma q(t)\|_{L^2} = 0 \quad \textit{for all } \sigma \in \left(\tfrac{d}{2} + 3, \mathfrak{s}\right]. \tag{1-15}$$

Moreover, the case $\ell = 1$ corresponds to a stable blowup regime.

Remark 1.2. Since $\gamma = 2$ for $d = 7$ and $\gamma \in (1,2)$ for $d \geq 8$, the condition $2\ell > \gamma$ means that $\ell \geq 2$ for $d = 7$ and $\ell \geq 1$ for $d \geq 8$. Note that the condition $2\ell > \gamma$ allows us to avoid the presence of logarithmic corrections in the construction of the approximate profile. In other words, the case $2\ell = \gamma$ (equivalent to $\ell = 1$ and $d = 7$) would involve an additional logarithmic gain related to the growth of the approximate profile at infinity, which turns out to be essential for the derivation of the speed (1-11). Although our analysis could be naturally extended to this case ($\ell = 1$ and $d = 7$) with some complicated computations, we hope to treat this case in a separate work.

Remark 1.3. The quantization of the blowup rate (1-14) is the same as the one obtained in [Biernat and Seki 2016]. Note that in that paper, they only claim the existence result of a type II blowup solution with the rate (1-14) and say nothing about the dynamical description of the solution. On the contrary, our result shows that the constructed solution blows up in finite time by concentration of a stationary state in the supercritical regime. Moreover, our constructed solution is in fact ($\ell-1$)-codimension stable in the sense that we will precise shortly.

Remark 1.4. Fix $\ell \in \mathbb{N}^*$ such that $2\ell > \gamma$, an integer $L \gg \ell$ and $\mathfrak{s} \sim L \gg 1$. Then our initial data is of the form

$$u_0 = Q_{b(0)} + \varepsilon_0, \tag{1-16}$$

where Q_b is a deformation of the ground state Q and $b = (b_1, \dots, b_L)$ corresponds to possible unstable directions of the flow in the $\dot{H}^{\mathfrak{s}}$ topology in a suitable neighborhood of Q. We will show that for all $\varepsilon_0 \in \dot{H}^\sigma \cap \dot{H}^{\mathfrak{s}}$ (for some $\sigma = \sigma(d) > \frac{d}{2}$) small enough, for all $(b_1(0), b_{\ell+1}(0), \dots, b_L(0))$ small enough, there exists a choice of unstable directions $(b_2(0), \dots, b_\ell(0))$ such that the solution of (1-4) with the data (1-16) satisfies the conclusion of Theorem 1.1. This implies that our constructed solution is $(\ell-1)$-codimension stable. In other words, the case $\ell = 1$ corresponds to a stable type II blowup regime, which is in agreement with numerical evidence given in [Biernat 2015].

Remark 1.5. The harmonic heat flow shares many features with the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \quad \text{in } \mathbb{R}^d. \tag{1-17}$$

Two important critical exponents appear when considering the dynamics of (1-17):

$$p_S = \frac{d+2}{d-2} \quad \text{and} \quad p_{JL} = \begin{cases} +\infty & \text{for } d \le 10, \\ 1 + \dfrac{4}{d - 4 - 2\sqrt{d-1}} & \text{for } d \ge 11 \end{cases}$$

correspond to the cases $d = 2$ and $d = 7$ in the study of (1-4) respectively.

When $1 < p \le p_S$, Giga and Kohn [1987] and Giga, Matsui and Sasayama [2004] showed that all blowup solutions are of type I. Here the type I blowup means that

$$\limsup_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty} < +\infty;$$

otherwise we say the blowup solution is of type II.

When $p = p_S$, Filippas, Herrero and Velázquez [2000] formally constructed for (1-17) type II blowup solutions in dimensions $3 \le d \le 6$; however, they could not do the same in dimensions $d \ge 7$. This formal result is partly confirmed by Schweyer [2012] in dimension $d = 4$. Interestingly, Collot, Merle and Raphaël [2017] showed that type II blowup is ruled out in dimension $d \ge 7$ near the solitary wave.

When $p_S < p < p_{JL}$, Matano and Merle [2004], see also [Mizoguchi 2004], proved that only type I blowup occurs in the radial setting.

When $p > p_{JL}$, Herrero and Velázquez [1994] formally derived the existence of type II blowup solutions with the quantized rates

$$\|u(t)\|_{L^\infty} \sim (T-t)^{\frac{2\ell}{(p-1)\alpha(d,p)}}, \quad \ell \in \mathbb{N}, \ 2\ell > \alpha.$$

The formal result was clarified in [Matano and Merle 2009; Mizoguchi 2007; Collot 2017]. The collection of these works yields a complete classification of the type II blowup scenario for the radially symmetric energy-supercritical case.

In comparison to the case of the semilinear heat equation (1-17), it might be possible to prove that all blowup solutions to (1-4) are of type I in dimensions $3 \le d \le 6$. However, due to the lack of monotonicity of the nonlinear term, the analysis of the harmonic heat flow (1-4) is much more difficult than the case of the semilinear heat equation (1-17) treated in [Matano and Merle 2004].

Let us briefly explain the main steps of the proof of Theorem 1.1, which follows the method of [Raphaël and Schweyer 2014b] treated for the critical case $d = 2$. This kind of method has been successfully applied for various nonlinear evolution equations, in particular in the dispersive setting for the nonlinear Schrödinger equation both in the mass-critical [Merle and Raphael 2005a; 2005b; 2004; 2003] and mass-supercritical [Merle, Raphaël and Rodnianski 2015] cases, the mass-critical gKdV equation [Martel, Merle and Raphaël 2015a; 2015b; 2014], the energy-critical [Duyckaerts, Kenig and Merle 2013; Hillairet and Raphaël 2012] and energy-supercritical [Collot 2018] wave equation, the two-dimensional critical geometric equations, the wave maps [Raphaël and Rodnianski 2012], the Schrödinger maps [Merle, Raphaël and Rodnianski 2013] and the harmonic heat flow [Raphaël and Schweyer 2013; 2014b], the semilinear heat equation (1-17) in the energy-critical [Schweyer 2012] and energy-supercritical [Collot 2017] cases, and the two-dimensional Keller–Segel model [Raphaël and Schweyer 2014a; Ghoul and Masmoudi 2016]. In all these works, the method relies on two arguments:

- Reduction of an infinite-dimensional problem to a finite-dimensional one, through the derivation of suitable Lyapunov functionals and the robust energy method as mentioned in the two-step procedure above.
- The control of the finite-dimensional problem thanks to a topological argument based on index theory.

Note that this kind of topological argument has proved to be successful also for the construction of type I blowup solutions for the semilinear heat equation (1-17) in [Bricmont and Kupiainen 1994; Merle and Zaag 1997; Nguyen and Zaag 2017] (see also [Nguyen and Zaag 2016] for the case of logarithmic perturbations, [Bressan 1990; 1992; Ghoul, Nguyen and Zaag 2017] for the exponential source and [Nouaili and Zaag 2015] for the complex-valued case), the Ginzburg–Landau equation in [Masmoudi and Zaag 2008] (see also [Zaag 1998] for an earlier work), a nonvariational parabolic system in [Ghoul, Nguyen and Zaag 2018] and the semilinear wave equation in [Côte and Zaag 2013].

For the reader's convenience and for a better explanation, we first introduce notation used throughout this paper.

Notation. For each $d \geq 7$, we define

$$\begin{cases} \hbar = \lfloor \frac{1}{2}\left(\frac{d}{2} - \gamma\right) \rfloor \in \mathbb{N}, \\ \delta = \frac{1}{2}\left(\frac{d}{2} - \gamma\right) - \hbar, \quad \delta \in (0,1), \end{cases} \tag{1-18}$$

where $\lfloor x \rfloor \in \mathbb{Z}$ stands for the integer part of x, which is defined by $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Note that $\delta \neq 0$. Indeed, if $\delta = 0$, then there is $m \in \mathbb{N}$ such that $2\gamma = d - 4m \in \mathbb{N}$. This only happens when $\gamma = 2$ or $\gamma = \frac{3}{2}$ because $\gamma \in (1,2]$. The case $\gamma = 2$ gives $d = 7$ and $m = \frac{3}{4} \notin \mathbb{N}$. The case $\gamma = \frac{3}{2}$ gives $d = \frac{17}{2} \notin \mathbb{N}$.

Given a large integer $L \gg 1$, we set

$$\Bbbk = L + \hbar + 1. \tag{1-19}$$

Given $b_1 > 0$ and $\lambda > 0$, we define

$$B_0 = \frac{1}{\sqrt{b_1}}, \qquad B_1 = B_0^{1+\eta}, \quad 0 < \eta \ll 1, \tag{1-20}$$

and

$$f_\lambda(r) = f(y) \quad \text{with } y = \frac{r}{\lambda}.$$

Let $\chi \in \mathcal{C}_0^\infty([0,+\infty))$ be a positive nonincreasing cutoff function with $\operatorname{supp}(\chi) \subset [0,2]$ and $\chi \equiv 1$ on $[0,1]$. For all $M > 0$, we define

$$\chi_M(y) = \chi\left(\frac{y}{M}\right). \tag{1-21}$$

We also introduce the differential operator

$$\Lambda f = y\partial_y f$$

and the Schrödinger operator

$$\mathscr{L} = -\partial_{yy} - \frac{(d-1)}{y}\partial_y + \frac{Z}{y^2}, \quad \text{with } Z(y) = (d-1)\cos(2Q(y)). \tag{1-22}$$

Strategy of the proof. We now summary the main ideas of the proof of Theorem 1.1, which follows the route map in [Raphaël and Schweyer 2014b; Merle, Raphaël and Rodnianski 2015]:

(i) *Renormalized flow and iterated resonances*. Following the scaling invariance of (1-4), let us make the change of variables

$$w(y,s) = u(r,t), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)},$$

which leads to the renormalized flow

$$\partial_s w = \partial_y^2 w + \frac{(d-1)}{y}\partial_y w - b_1 \Lambda w - \frac{(d-1)}{2y^2}\sin(2w), \quad b_1 = -\frac{\lambda_s}{\lambda}. \tag{1-23}$$

Assuming that the leading part of the solution $w(y,s)$ is given by the ground state profile Q admitting the asymptotic behavior (1-10), the remaining part is governed by the Schrödinger operator $\mathscr{L}$ defined by (1-22). The linear operator $\mathscr{L}$ admits the factorization (see Lemma 2.2 below)

$$\mathscr{L} = \mathscr{A}^*\mathscr{A}, \quad \mathscr{A} f = -\Lambda Q \partial_y\left(\frac{f}{\Lambda Q}\right), \quad \mathscr{A}^* f = \frac{1}{y^{d-1}\Lambda Q}\partial_y(y^{d-1}\Lambda Q f), \tag{1-24}$$

which directly implies

$$\mathscr{L}(\Lambda Q) = 0,$$

where from a direct computation,

$$\Lambda Q \sim \frac{c_0}{y^\gamma} \quad \text{as } y \to +\infty, \text{ with } \gamma \text{ defined in (1-8)}.$$

More generally, we can compute the kernel of the powers of $\mathscr{L}$ through the iterative scheme

$$\mathscr{L}T_{k+1} = -T_k, \quad T_0 = \Lambda Q, \tag{1-25}$$

which displays a nontrivial tail at infinity (see Lemma 2.9 below),

$$T_k(y) \sim c_k y^{2k-\gamma} \quad \text{for } y \gg 1. \tag{1-26}$$

(ii) *Tail dynamics*. Following the approach in [Raphaël and Schweyer 2014b], we look for a slowly modulated approximate solution to (1-23) of the form

$$w(y,s) = Q_{b(s)}(y),$$

where

$$b = (b_1,\dots,b_L), \quad Q_{b(s)}(y) = Q(y) + \sum_{i=1}^{L} b_i T_i(y) + \sum_{i=2}^{L+2} S_i(y) \tag{1-27}$$

with a priori bounds

$$b_i \sim b_1^i, \quad |S_i(y)| \lesssim b_1^i y^{2(i-1)-\gamma},$$

so that S_i is in some sense homogeneous of degree i in b_1, and behaves better than T_i at infinity. The construction of S_i with the above a priori bounds is possible for a specific choice of the universal dynamical system which drives the modes $(b_i)_{1\le i\le L}$. This is so-called the *tail computation*. Let us illustrate the procedure of the *tail computation*. We plug the decomposition (1-27) into (1-23) and choose the law for $(b_i)_{1\le i\le L}$ which cancels the leading-order terms at infinity:

- At the order $\mathcal{O}(b_1)$: We cannot adjust the law of b_1 for the first term[1] and obtain from (1-23),

$$b_1(\mathcal{L}T_1 + \Lambda Q) = 0.$$

- At the order $\mathcal{O}(b_1^2, b_2)$: We obtain

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 \mathscr{L} T_2 + \mathscr{L} S_2 = b_1^2 \,\mathrm{NL}_1(T_1, Q),$$

where $\mathrm{NL}_1(T_1, Q)$ corresponds to nonlinear interaction terms. Note from (1-26) and (1-25), we have

$$\Lambda T_1 \sim (2-\gamma) T_1 \quad \text{for } y \gg 1, \qquad \mathscr{L} T_2 = -T_1,$$

and thus,

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 \mathscr{L} T_2 \sim [(b_1)_s + (2-\gamma) b_1^2 - b_2] T_1.$$

Hence the leading-order growth for y large is canceled by the choice

$$(b_1)_s + (2-\gamma) b_1^2 - b_2 = 0.$$

We then solve for

$$\mathscr{L} S_2 = -b_1^2(\Lambda T_1 - (2-\gamma) T_1) + b_1^2 \,\mathrm{NL}_1(T_1, Q),$$

and check the improved decay

$$|S_2(y)| \lesssim b_1^2 y^{2-\gamma} \quad \text{for } y \gg 1.$$

- At the order $\mathcal{O}(b_1^{k+1}, b_{k+1})$: We obtain an elliptic equation of the form

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} \mathscr{L} T_{k+1} + \mathscr{L} S_{k+1} = b_1^{k+1} \,\mathrm{NL}_k(T_1,\dots,T_k, Q).$$

[1] If $(b_1)_s = -c_1 b_1$, then $-\lambda_s/\lambda \sim b_1 \sim e^{-c_1 s}$; hence after an integration in time, $|\log\lambda| \lesssim 1$ and there is no blowup.

From (1-26) and (1-25), we have

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} \mathscr{L} T_{k+1} \sim [(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}] T_k,$$

which leads to the choice

$$(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = 0$$

for the cancellation of the leading-order growth at infinity. We then solve for the remaining S_{k+1}-term and check that $|S_{k+1}(y)| \lesssim b_1^{k+1} y^{2k-\gamma}$ for y large. We refer to Proposition 2.11 for all the details of the *tail computation*.

(iii) *The universal system of ODEs*. The above procedure leads to the following universal system of ODEs after L iterations:

$$\begin{cases} (b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \ b_{L+1} = 0, \\ -\dfrac{\lambda_s}{\lambda} = b_1, \quad \dfrac{ds}{dt} = \dfrac{1}{\lambda^2}. \end{cases} \tag{1-28}$$

Unlike the critical case treated in [Raphaël and Schweyer 2014b], there is no further logarithmic correction to take into account. The set of solutions to (1-28) (see Lemma 2.13 below) is explicitly given by

$$\begin{cases} b_k^e(s) = \dfrac{c_k}{s^k}, \quad 1 \le k \le L, \\ c_1 = \dfrac{\ell}{2\ell-\gamma}, \quad \ell \in \mathbb{N}^*, \ 2\ell > \gamma, \\ c_{k+1} = -\dfrac{\gamma(\ell-k)}{2\ell-\gamma} c_k, \quad 1 \le k \le \ell-1, \ \ell \ge 2 \\ c_j = 0, \quad j \ge \ell+1, \\ \lambda(s) \sim s^{-\frac{\ell}{2\ell-\gamma}}. \end{cases} \tag{1-29}$$

In the original time variable t, this implies that $\lambda(t)$ goes to zero in finite time T with the asymptotic

$$\lambda(t) \sim (T-t)^{\frac{\ell}{\gamma}}.$$

Moreover, the linearized flow of (1-28) near the solution (1-29) is explicit and displays $\ell-1$ unstable directions (see Lemma 2.14 below). This implies that the case $\ell = 1$ corresponds to a stable type II blowup regime.

(iv) *Decomposition of the flow and modulation equations*. Let the approximate solution Q_b be given by (1-27), which by construction generates an approximate solution to the renormalized flow (1-23),

$$\Psi_b = \partial_s Q_b - \Delta Q_b + b \Lambda Q_b + \frac{(d-1)}{2y^2} \sin(2Q_b) = \mathrm{Mod}(t) + O(b_1^{2L+2}),$$

where the modulation equation term is roughly of the form

$$\mathrm{Mod}(t) = \sum_{i=1}^{L} [(b_i)_s + (2i-\gamma) b_1 b_i - b_{i+1}] T_i.$$

We localize Q_b in the zone $y \le B_1$ to avoid the irrelevant growing tails for $y \gg 1/\sqrt{b_1}$. We then take initial data of the form

$$u_0(y) = Q_{b(0)}(y) + q_0(y),$$

where q_0 is small in some suitable sense and $b(0)$ is chosen to be close to the exact solution (1-29). By a standard modulation argument, we introduce the decomposition of the flow

$$u(r,t) = w(y,s) = (Q_{b(s)} + q)(y,s) = (Q_{b(t)} + v)\left(\frac{r}{\lambda(t)}, t\right), \tag{1-30}$$

where $L+1$ modulation parameters $(b(t), \lambda(t))$ are chosen in order to manufacture the orthogonality conditions

$$\langle q, \mathscr{L}^i \Phi_M \rangle = 0, \quad 0 \le i \le L, \tag{1-31}$$

where Φ_M, see (3-4), is some fixed direction depending on some large constant M, generating an approximation of the kernel of the powers of $\mathscr{L}$. This orthogonal decomposition (1-30), which follows from the implicit function theorem, allows us to compute the modulation equations governing the parameters $(b(t), \lambda(t))$ (see Lemmas 4.2 and 4.3 below),

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{i=1}^{L} |(b_i)_s + (2i-\gamma) b_1 b_i - b_{i+1}| \lesssim \|q\|_{\mathrm{loc}} + b_1^{L+1+\nu(\delta,\eta)}, \tag{1-32}$$

where $\|q\|_{\mathrm{loc}}$ measures a spatially localized norm of the radiation q and $\nu(\delta,\eta) > 0$.

(v) *Control of Sobolev norms*. According to (1-32), we need to show that local norms of q are under control and do not perturb the dynamical system (1-28). This is achieved via high-order mixed energy estimates which provide controls of the Sobolev norms adapted to the linear flow and based on the powers of the linear operator $\mathscr{L}$. In particular, we have the following coercivity of the high energy under the orthogonality conditions (1-31) (see Lemma A.5):

$$\mathscr{E}_{2\Bbbk}(s) = \int |\mathscr{L}^{\Bbbk} q|^2 \gtrsim \int |\nabla^{2\Bbbk} q|^2 + \int \frac{|q|^2}{1+y^{4\Bbbk}},$$

where $\Bbbk$ is given by (1-19). Here the factorization (1-24) will help to simplify the proof. As in [Raphaël and Rodnianski 2012; Raphaël and Schweyer 2014b; Merle, Raphaël and Rodnianski 2015], the control of $\mathscr{E}_{2\Bbbk}$ is done through the use of the linearized equation in the original variables (r,t); i.e., we work with v in (1-30) and not q. The energy estimate is of the form (see Proposition 4.4)

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}\right\} \lesssim \frac{b_1^{2L+1+2\nu(\delta,\eta)}}{\lambda^{4\Bbbk-d}}, \quad \nu(\delta,\eta) > 0, \tag{1-33}$$

where the right-hand side is controlled by the size of the error Ψ_b in the construction of the approximate profile Q_b above. An integration of (1-33) in time by using initial smallness assumptions, $b_1 \sim b_1^e$ and $\lambda(s) \sim b_1^{\ell/(2\ell-\gamma)}$ yields the estimate

$$\int |\nabla^{2\Bbbk} q|^2 + \int \frac{|q|^2}{1+y^{4\Bbbk}} \lesssim \mathscr{E}_{2\Bbbk}(s) \lesssim b_1^{2L+2\nu(\delta,\eta)},$$

which is good enough to control the local norms of q and close the modulation equations (1-32).

Note that we also need to control lower energies $\mathscr{E}_{2m}$ for $\hbar+2\le m\le \Bbbk-1$ because the control of the high energy $\mathscr{E}_{2\Bbbk}$ alone is not enough to control a nonlinear term appearing in the linearized equation around Q_b. In particular, we exhibit a Lyapunov functional with the dynamical estimate

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2m}}{\lambda^{4m-d}}\right\}\lesssim \frac{b_1^{2(m-\hbar)-1+2\nu'(\delta,\eta)}}{\lambda^{4m-d}},\qquad \nu'(\delta,\eta)>0.$$

Then, an integration in time yields

$$\mathscr{E}_{2m}(s)\lesssim\begin{cases} b_1^{\frac{\ell}{2\ell-\gamma}(4m-d)} & \text{for } \hbar+2\le m\le \ell+\hbar,\\ b_1^{2(m-\hbar-1)+2\nu'(\delta,\eta)} & \text{for } \hbar+\ell+1\le m\le \Bbbk-1,\end{cases}$$

which is enough to control the nonlinear term. Let us remark that the condition $m\ge\hbar+2$ ensures $4m-d>0$ so that $\mathscr{E}_{2m}$ is always controlled. By the coercivity of $\mathscr{E}_{2m}$, this means that we are only able to control the Sobolev norms $\|\nabla^{2\sigma}q\|^2_{L^2}$ for $\sigma\ge\hbar+2$, resulting in the asymptotic (1-15).

The above scheme designs a bootstrap regime (see Definition 3.2 for a precise definition) which traps the blowup solution with speed (1-14). According to Lemmas 2.13 and 2.14, such a regime displays $\ell-1$ unstable modes $(b_2,\dots,b_\ell)$ which we can control through a topological argument based on the Brouwer fixed-point theorem (see the proof of Proposition 3.5), and the proof of Theorem 1.1 follows.

The paper is organized as follows. In Section 2, we give the construction of the approximate solution Q_b of (1-4) and derive estimates on the generated error term Ψ_b (Proposition 2.11), as well as its localization (Proposition 2.12). We also give in this section some elementary facts on the study of the system (1-28) (Lemmas 2.13 and 2.14). Section 3 is devoted to the proof of Theorem 1.1, assuming a main technical result (Proposition 3.6). In particular, we give the proof of the existence of the solution trapped in some shrinking set to zero (Proposition 3.5) such that the constructed solution satisfies the conclusion of Theorem 1.1. Readers not interested in technical details may stop there. In Section 4, we give the proof of Proposition 3.6 which gives the reduction of the problem to a finite-dimensional one, and this is the heart of our analysis.

2. Construction of an approximate profile

This section is devoted to the construction of a suitable approximate solution to (1-4) by using the same approach developed in [Raphaël and Rodnianski 2012]. Similar approaches can also be found in [Raphaël and Schweyer 2013; 2014a; Hillairet and Raphaël 2012; Schweyer 2012; Merle, Raphaël and Rodnianski 2015]. The key to this construction is the fact that the linearized operator $\mathscr{L}$ around Q is completely explicit in the radial setting thanks to the explicit formulas of the kernel elements.

Following the scaling invariance of (1-4), we introduce the change of variables

$$w(y,s)=u(r,t),\qquad y=\frac{r}{\lambda(t)},\qquad \frac{ds}{dt}=\frac{1}{\lambda^2(t)},\tag{2-1}$$

which leads to the renormalized flow

$$\partial_s w=\partial_y^2 w+\frac{(d-1)}{y}\partial_y w+\frac{\lambda_s}{\lambda}\Lambda w-\frac{(d-1)}{2y^2}\sin(2w),\tag{2-2}$$

where $\lambda_s = d\lambda/ds$. Noticing that in the setting (2-1) we have

$$\partial_r u(r,t) = \frac{1}{\lambda(t)}\partial_y w(y,s)$$

and since we deal with the finite-time blowup of the problem (1-4), we naturally impose the condition

$$\lambda(t) \to 0 \quad \text{as } t \to T$$

for some $T \in (0, +\infty)$. Hence, $\partial_r u(r,t)$ blows up in finite time T.

Let us assume that the leading part of the solution of (2-2) is given by the harmonic map Q, which is a unique solution (up to scaling) of the equation

$$Q'' + \frac{(d-1)}{y}Q' - \frac{(d-1)}{2y^2}\sin(2Q) = 0, \quad Q(0) = 0, \quad Q'(0) = 1. \tag{2-3}$$

We aim to construct an approximate solution of (2-2) close to Q. The natural way is to linearize (2-2) around Q, which generates the Schrödinger operator defined by (1-22). Let us now recall the main properties of $\mathscr{L}$ in the following subsection.

2A. *Structure of the linearized Hamiltonian.* We recall the main properties of the linearized Hamiltonian close to Q, which is the heart of both construction of the approximate profile and the derivation of the coercivity properties serving for the high Sobolev energy estimates. Let us start by recalling the following result from [Biernat 2015], which gives the asymptotic behavior of the harmonic map Q:

Lemma 2.1 (development of the harmonic map Q). *Let $d \geq 7$. There exists a unique solution Q to (2-3) which admits the following asymptotic behavior. For any $k \in \mathcal{N}^*$:*

(i) (*asymptotic behavior of Q*)

$$Q(y) = \begin{cases} y + \sum_{i=1}^{k} c_i y^{2i+1} + \mathcal{O}(y^{2k+3}) & \text{as } y \to 0, \\ \frac{\pi}{2} - \frac{a_0}{y^\gamma}\left[1 + \mathcal{O}\left(\frac{1}{y^2}\right) + \mathcal{O}\left(\frac{1}{y^{\tilde\gamma}}\right)\right] & \text{as } y \to +\infty, \end{cases} \tag{2-4}$$

where γ is defined in (1-8), $\tilde\gamma = \sqrt{d^2 - 8d + 8}$ and $a_0 = a_0(d) > 0$.

(ii) (*degeneracy*)

$$\Lambda Q > 0, \quad \Lambda Q(y) = \begin{cases} y + \sum_{i=1}^{k} c_i' y^{2i+1} + \mathcal{O}(y^{2k+3}) & \text{as } y \to 0, \\ \frac{a_0\gamma}{y^\gamma}\left[1 + \mathcal{O}\left(\frac{1}{y^2}\right) + \mathcal{O}\left(\frac{1}{y^{\tilde\gamma}}\right)\right] & \text{as } y \to +\infty. \end{cases} \tag{2-5}$$

Proof. The proof of (2-4) is done through the introduction of the variables $x = \log y$ and $v(x) = 2Q(y) - \pi$ and consists of the phase portrait analysis of the autonomous equation

$$v''(x) + (d-2)v'(x) + (d-2)\sin(v(x)) = 0.$$

All details of the proof can be found in [Biernat 2015, pages 184–185]. The proof of (2-5) directly follows from the expansion (2-4). □

The linearized operator $\mathscr{L}$ displays a remarkable structure given by the following lemma:

Lemma 2.2 (factorization of $\mathscr{L}$). *Let $d \geq 7$ and define the first-order operators*

$$\mathscr{A}w = -\partial_y w + \frac{V}{y}w = -\Lambda Q \partial_y\left(\frac{w}{\Lambda Q}\right), \tag{2-6}$$

$$\mathscr{A}^* w = \frac{1}{y^{d-1}}\partial_y(y^{d-1}w) + \frac{V}{y}w = \frac{1}{y^{d-1}\Lambda Q}\partial_y(y^{d-1}\Lambda Q w), \tag{2-7}$$

where

$$V(y) := \Lambda \log(\Lambda Q) = \begin{cases} 1 + \mathcal{O}(y^2) & \text{as } y \to 0, \\ -\gamma + \mathcal{O}\left(\frac{1}{y^2}\right) + \mathcal{O}\left(\frac{1}{y^{\tilde{\gamma}}}\right) & \text{as } y \to +\infty. \end{cases} \tag{2-8}$$

We have

$$\mathscr{L} = \mathscr{A}^*\mathscr{A}, \quad \widetilde{\mathscr{L}} = \mathscr{A}\mathscr{A}^*, \tag{2-9}$$

where $\widetilde{\mathscr{L}}$ stands for the conjugate Hamiltonian.

Remark 2.3. The adjoint operator $\mathscr{A}^*$ is defined with respect to the Lebesgue measure

$$\int_0^{+\infty} (\mathscr{A}u) w y^{d-1}\, dy = \int_0^{+\infty} u(\mathscr{A}^* w) y^{d-1}\, dy.$$

Remark 2.4. We have

$$\mathscr{L}(\Lambda w) = \Lambda(\mathscr{L}w) + 2\mathscr{L}w - \frac{\Lambda Z}{y^2}w. \tag{2-10}$$

Since $\mathscr{L}(\Lambda Q) = 0$, one can express the definition of Z through the potential V as

$$Z(y) = V^2 + \Lambda V + (d-2)V. \tag{2-11}$$

Let $\widetilde{Z}$ be defined by

$$\widetilde{\mathscr{L}} = -\partial_{yy} - \frac{d-1}{y}\partial_y + \frac{\widetilde{Z}}{y^2}. \tag{2-12}$$

Then, a direct computation yields

$$\widetilde{Z}(y) = (V+1)^2 + (d-2)(V+1) - \Lambda V. \tag{2-13}$$

From (2-6) and (2-7), we see that the kernels of $\mathscr{A}$ and $\mathscr{A}^*$ are explicit:

$$\mathscr{A}w = 0 \quad \text{if and only if} \quad w \in \operatorname{Span}(\Lambda Q),$$

$$\mathscr{A}^* w = 0 \quad \text{if and only if} \quad w \in \operatorname{Span}\left(\frac{1}{y^{d-1}\Lambda Q}\right).$$

Hence, the elements of the kernel of $\mathscr{L}$ are given by

$$\mathscr{L}w = 0 \quad \text{if and only if} \quad w \in \operatorname{Span}(\Lambda Q, \Gamma), \tag{2-14}$$

where Γ can be found from the Wronskian relation

$$\Gamma'\Lambda Q - \Gamma(\Lambda Q)' = \frac{1}{y^{d-1}}, \tag{2-15}$$

that is,

$$\Gamma(y) = \Lambda Q(y) \int_1^y \frac{d\xi}{\xi^{d-1}(\Lambda Q(\xi))^2},$$

which admits the asymptotic behavior

$$\Gamma(y) = \begin{cases} \dfrac{1}{dy^{d-1}} + \mathcal{O}(y) & \text{as } y \to 0, \\ \dfrac{1}{a_0\gamma(d-2-2\gamma)y^{d-2-\gamma}} + \mathcal{O}\left(\dfrac{1}{y^{d-\gamma}}\right) & \text{as } y \to +\infty. \end{cases} \tag{2-16}$$

From (2-14), we may invert $\mathscr{L}$ as follows:

$$\mathscr{L}^{-1} f = -\Gamma(y) \int_0^y f(x)\Lambda Q(x) x^{d-1}\,dx + \Lambda Q(y) \int_0^y f(x)\Gamma(x)x^{d-1}\,dx. \tag{2-17}$$

The factorization of $\mathscr{L}$ allows us to compute $\mathscr{L}^{-1}$ in an elementary two-step process that will help us to avoid tracking the cancellation in the formula (2-17) induced by the Wronskian relation when estimating the growth of $\mathscr{L}^{-1} f$. In particular, we have the following:

Lemma 2.5 (inversion of $\mathscr{L}$). *Let f be a $\mathcal{C}^\infty$ radially symmetric function and $w = \mathscr{L}^{-1} f$ be given by* (2-17). *Then*

$$\mathscr{L} w = f, \quad \mathscr{A} w = \frac{1}{y^{d-1}\Lambda Q} \int_0^y f(x)\Lambda Q(x)x^{d-1}\,dx, \quad w = -\Lambda Q \int_0^y \frac{\mathscr{A} w(x)}{\Lambda Q(x)}\,dx. \tag{2-18}$$

Proof. From the relation (2-15), we compute

$$\mathscr{A}\Gamma = -\frac{1}{y^{d-1}\Lambda Q}.$$

Applying $\mathscr{A}$ to (2-17) and using the cancellation $\mathscr{A}(\Lambda Q) = 0$, we obtain

$$\mathscr{A} w = \frac{1}{y^{d-1}\Lambda Q} \int_0^y f(x)\Lambda Q(x)x^{d-1}\,dx.$$

From the definition (2-6) of $\mathscr{A}$, we write

$$w = -\Lambda Q \int_0^y \frac{\mathscr{A} w}{\Lambda Q}\,dx. \qquad \square$$

2B. ***Admissible functions.*** We define a class of admissible functions which display a suitable behavior both at the origin and infinity.

Definition 2.6 (admissible function). Fix $\gamma > 0$, we say that a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is admissible of degree $(p_1, p_2) \in \mathbb{N} \times \mathbb{Z}$ if:

(i) f admits a Taylor expansion to all orders around the origin,

$$f(y) = \sum_{k=p_1}^{p} c_k y^{2k+1} + \mathcal{O}(y^{2p+3});$$

(ii) f and its derivatives admit the bounds, for $y \geq 1$,

$$\text{for all } k \in \mathbb{N}, \quad |\partial_y^k f(y)| \lesssim y^{2p_2-\gamma-k}.$$

Remark 2.7. By (2-5), ΛQ is admissible of degree $(0, 0)$.

Note that $\mathscr{L}$ naturally acts on the class of admissible functions in the following way:

Lemma 2.8 (action of $\mathscr{L}$ and $\mathscr{L}^{-1}$ on admissible functions). *Let f be an admissible function of degree $(p_1, p_2) \in \mathbb{N} \times \mathbb{Z}$. Then*:

(i) *Λf is admissible of degree (p_1, p_2).*

(ii) *$\mathscr{L} f$ is admissible of degree* $(\max\{0, p_1 - 1\}, p_2 - 1)$.

(iii) *$\mathscr{L}^{-1} f$ is admissible of degree $(p_1 + 1, p_2 + 1)$.*

Proof. (i)–(ii) This is simply a consequence of Definition 2.6.

(iii) We aim to prove that if f is admissible of degree (p_1, p_2), then $w = \mathscr{L}^{-1} f$ is admissible of degree $(p_1 + 1, p_2 + 1)$. To do so, we use Lemma 2.5 to estimate

- for $y \ll 1$,

$$\mathscr{A} w = \frac{1}{y^{d-1}\Lambda Q} \int_0^y f \Lambda Q x^{d-1}\, dx = \mathcal{O}\left(\frac{1}{y^d} \int_0^y x^{2p_1+1+d}\, dx\right) = \mathcal{O}(y^{2p_1+2}),$$

$$w = -\Lambda Q \int_0^y \frac{\mathscr{A} w}{\Lambda Q}\, dx = \mathcal{O}\left(y \int_0^y x^{2p_1+1}\, dx\right) = \mathcal{O}(y^{2(p_1+1)+1}),$$

- for $y \geq 1$,

$$\mathscr{A} w = \mathcal{O}\left(\frac{1}{y^{d-1-\gamma}} \int_0^y x^{2p_2-2\gamma+d-1}\, dx\right) = \mathcal{O}(y^{2p_2+1-\gamma}),$$

$$w = \mathcal{O}\left(\frac{1}{y^\gamma} \int_0^y x^{2p_2+1}\right) = \mathcal{O}(y^{2(p_2+1)-\gamma}).$$

From the last formula in (2-18) and (2-8), we estimate

$$\partial_y w = -\partial_y \Lambda Q \int_0^y \frac{\mathscr{A} w}{\Lambda Q}\, dx - \mathscr{A} w = -\frac{\partial_y \Lambda Q}{\Lambda Q} w - \mathscr{A} w = \mathcal{O}(y^{2(p_2+1)-\gamma-1}).$$

Using $\mathscr{L} w = f$, we get

$$\partial_{yy} w = \mathcal{O}\left(\frac{|\partial_y w|}{y} + \frac{|w|}{y^2} + |f|\right) = \mathcal{O}(y^{2(p_2+1)-\gamma-2}).$$

By taking radial derivatives of $\mathscr{L} w = f$, we obtain by induction

$$|\partial_y^k w| \lesssim y^{2(p_2+1)-\gamma-k}, \quad k \in \mathbb{N}, \ y \geq 1.$$

□

The following lemma is a consequence of Lemma 2.8:

Lemma 2.9 (generators of the kernel of $\mathscr{L}^k$). *Consider the sequence of profiles*

$$T_k = (-1)^k \mathscr{L}^{-k} \Lambda Q, \quad k \in \mathbb{N}. \tag{2-19}$$

Then:

(i) *T_k is admissible of degree (k, k) for $k \in \mathbb{N}$.*

(ii) *$\Lambda T_k - (2k - \gamma) T_k$ is admissible of degree $(k, k-1)$ for $k \in \mathbb{N}^*$.*

Proof. (i) We note from (2-5) that ΛQ is admissible of degree $(0, 0)$. By induction and part (iii) of Lemma 2.8, the conclusion then follows.

(ii) We proceed by induction. For $k = 1$, we explicitly compute $T_1 = -\mathscr{L}^{-1} \Lambda Q$ by using Lemma 2.5 and the expansion (2-5) to get

$$\text{for all } m \in \mathbb{N}, \quad \partial_y^m T_1(y) = e_{1,m} y^{2-\gamma-m} + \mathcal{O}(y^{-\gamma-m}) \quad \text{as } y \to +\infty.$$

By induction, one can easily check that $\partial_y^m \Lambda f = \Lambda \partial_y^m f + m \partial_y^m f$ for $m \in \mathbb{N}^*$. Hence,

$$\partial_y^m [\Lambda T_1 - (2-\gamma) T_1] = \Lambda \partial_y^m T_1 - (2 - \gamma - m) \partial_y^m T_1 = \mathcal{O}(y^{-\gamma-m}) \quad \text{as } y \to +\infty.$$

Since T_1 and ΛT_1 are admissible of degree $(1, 1)$, we deduce that $\Lambda T_1 - (2-\gamma) T_1$ is admissible of degree $(1, 0)$.

We now assume the claim for $k \geq 1$, namely that $\Lambda T_k - (2k - \gamma) T_k$ is admissible of degree $(k, k-1)$. Let us prove that $\Lambda T_{k+1} - (2(k+1) - \gamma) T_{k+1}$ is admissible of degree $(k+1, k)$. We use formula (2-10) and definition (2-19) to write

$$\begin{aligned} \mathscr{L}(\Lambda T_{k+1} - (2k + 2 - \gamma) T_{k+1}) &= \Lambda \mathscr{L} T_{k+1} - (2k - \gamma) \mathscr{L} T_{k+1} - \frac{\Lambda Z}{y^2} T_{k+1} \\ &= \Lambda T_k - (2k - \gamma) T_k - \frac{\Lambda Z}{y^2} T_{k+1}. \end{aligned} \tag{2-20}$$

From part (i), we know that T_{k+1} is admissible of degree $(k+1, k+1)$. From (2-11) and (2-8), one can check that $(\Lambda Z / y^2) T_{k+1}$ admits the asymptotic

$$\frac{\Lambda Z}{y^2} T_{k+1} = \mathcal{O}(y^{2k+1}) \quad \text{as } y \to 0,$$

and

$$\partial_y^j \left(\frac{\Lambda Z}{y^2} T_{k+1} \right) = \mathcal{O}(y^{2(k+1)-j-\gamma-3}) \ll y^{2(k-1)+j-\gamma} \quad \text{as } y \to +\infty.$$

Together with the induction hypothesis, we deduce that the right-hand side of (2-20) is admissible of degree $(k, k-1)$. The conclusion then follows by using part (iii) of Lemma 2.8. □

We end this subsection by introducing a simple notion of homogeneous admissible function.

Definition 2.10 (homogeneous admissible function). Let $L \gg 1$ be an integer and $m = (m_1, \dots, m_L) \in \mathbb{N}^L$. We say that a function $f(b, y)$ with $b = (b_1, \dots, b_L)$ is homogeneous of degree $(p_1, p_2, p_3) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ if it is a finite linear combination of monomials

$$\tilde{f}(y) \prod_{k=1}^{L} b_k^{m_k},$$

with $\tilde{f}(y)$ admissible of degree (p_1, p_2) in the sense of Definition 2.6 and

$$(m_1, \dots, m_L) \in \mathbb{N}^L, \quad \sum_{k=1}^{L} k m_k = p_3.$$

We set

$$\deg(f) := (p_1, p_2, p_3).$$

2C. *Slowly modulated blowup profile.* We use the explicit structure of the linearized operator $\mathscr{L}$ to construct an approximate blowup profile. In particular, we claim the following:

Proposition 2.11 (construction of the approximate profile). *Let $d \geq 7$ and $L \gg 1$ be an integer. Let $M > 0$ be a large enough universal constant. Then there exists a small enough universal constant $b^*(M, L) > 0$ such that the following holds true. Consider a $\mathcal{C}^1$ map*

$$b = (b_1, \dots, b_L) : [s_0, s_1] \mapsto (-b^*, b^*)^L,$$

with a priori bounds in $[s_0, s_1]$,

$$0 < b_1 < b^*, \qquad |b_k| \lesssim b_1^k, \quad 2 \leq k \leq L. \tag{2-21}$$

Then there exist homogeneous profiles

$$S_1 = 0, \qquad S_k = S_k(b, y), \quad 2 \leq k \leq L+2,$$

such that

$$Q_{b(s)}(y) = Q(y) + \sum_{k=1}^{L} b_k(s) T_k(y) + \sum_{k=2}^{L+2} S_k(b, y) \equiv Q(y) + \Theta_{b(s)}(y) \tag{2-22}$$

generates an approximate solution to the renormalized flow (2-2)

$$\partial_s Q_b - \partial_{yy} Q_b - \frac{(d-1)}{y} \partial_y Q_b + b_1 \Lambda Q_b + \frac{(d-1)}{2y^2} \sin(2Q_b) = \Psi_b + \mathrm{Mod}(t), \tag{2-23}$$

with the following properties:

(*i*) (*modulation equation*)

$$\mathrm{Mod}(t) = \sum_{k=1}^{L} [(b_k)_s + (2k - \gamma) b_1 b_k - b_{k+1}] \left[T_k + \sum_{j=k+1}^{L+2} \frac{\partial S_j}{\partial b_k} \right], \tag{2-24}$$

where we use the convention $b_j = 0$ for $j \geq L+1$.

(*ii*) (*estimate on the profiles*) *The profiles* $(S_k)_{2\le k\le L+2}$ *are homogeneous with*

$$\deg(S_k) = (k, k-1, k) \quad \textit{for } 2 \le k \le L+2,$$
$$\frac{\partial S_k}{\partial b_m} = 0 \quad \textit{for } 2 \le k \le m \le L.$$

(*iii*) (*estimate on the error* Ψ_b) *For all* $0 \le m \le L$, *we have*:

- (*global weight bound*)

$$\int_{y\le 2B_1} |\mathscr{L}^{\hbar+m+1}\Psi_b|^2 + \int_{y\le 2B_1} \frac{|\Psi_b|^2}{1+y^{4(\hbar+m+1)}} \lesssim b_1^{2m+4+2(1-\delta)-C_L\eta}, \tag{2-25}$$

  where $B_1, \hbar, \delta$ *are defined in* (1-20) *and* (1-18).

- (*improved local bound*)

$$\textit{For all } M \ge 1, \quad \int_{y\le 2M} |\mathscr{L}^{\hbar+m+1}\Psi_b|^2 \lesssim M^C b_1^{2L+6}. \tag{2-26}$$

Proof. We aim to construct the profiles $(S_k)_{2\le k\le L+2}$ such that $\Psi_b(y)$ defined from (2-23) has the *least possible growth* as $y \to +\infty$. The key to this construction is the fact that the structure of the linearized operator $\mathscr{L}$ defined in (1-22) is completely explicit in the radial sector thanks to the explicit formulas of the elements of the kernel. This procedure will lead to the leading-order modulation equation

$$(b_k)_s = -(2k-\gamma)b_1 b_k + b_{k+1} \quad \text{for } 1 \le k \le L, \tag{2-27}$$

which actually cancels the worst growth of S_k as $y \to +\infty$.

- Expansion of Ψ_b. From (2-23) and (2-3), we write

$$\begin{aligned}
\partial_s Q_b - \partial_{yy} Q_b - \frac{(d-1)}{y}\partial_y Q_b + b_1\Lambda Q_b + \frac{(d-1)}{2y^2}\sin(2Q_b)& \\
= b_1\Lambda Q + \partial_s\Theta_b - \partial_{yy}\Theta_b - \frac{(d-1)}{y}\partial_y\Theta_y + \frac{(d-1)}{y^2}\cos(2Q)\Theta_b + b_1\Lambda\Theta_b& \\
+ \frac{(d-1)}{2y^2}[\sin(2Q+2\Theta_b) - \sin(2Q) - 2\cos(2Q)\Theta_b]& \\
:= A_1 + A_2.&
\end{aligned}$$

Using the expression (2-22) of Θ_b and the definition (2-19) of T_k (note that $\mathscr{L}T_k = -T_{k-1}$ with the convention $T_0 = \Lambda Q$), we write

$$\begin{aligned}
A_1 &= b_1\Lambda Q + \sum_{k=1}^{L}[(b_k)_s T_k + b_k\mathscr{L}T_k + b_1 b_k \Lambda T_k] + \sum_{k=2}^{L+2}[\partial_s S_k + \mathscr{L}S_k + b_1\Lambda S_k] \\
&= \sum_{k=1}^{L}[(b_k)_s T_k - b_{k+1}T_k + b_1 b_k\Lambda T_k] + \sum_{k=2}^{L+2}[\partial_s S_k + \mathscr{L}S_k + b_1\Lambda S_k]
\end{aligned}$$

$$= \sum_{k=1}^{L} [(b_k)_s - b_{k+1} + (2k-\gamma) b_1 b_k] T_k + \sum_{k=1}^{L} [\mathscr{L} S_{k+1} + \partial_s S_k + b_1 b_k [\Lambda T_k - (2k-\gamma) T_k] + b_1 \Lambda S_k] + [\mathscr{L} S_{L+2} + \partial_s S_{L+1} + b_1 \Lambda S_{L+1}] + [\partial_s S_{L+2} + b_1 \Lambda S_{L+2}].$$

We now write

$$\partial_s S_k = \sum_{j=1}^{L} (b_j)_s \frac{\partial S_k}{\partial b_j} = \sum_{j=1}^{L} [(b_j)_s + (2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j} - \sum_{j=1}^{L} [(2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}.$$

Hence,

$$A_1 = \mathrm{Mod}(t) + \sum_{k=1}^{L+1} [\mathscr{L} S_{k+1} + E_k] + E_{L+2},$$

where for $k = 1, \dots, L$,

$$E_k = b_1 b_k [\Lambda T_k - (2k-\gamma) T_k] + b_1 \Lambda S_k - \sum_{j=1}^{k-1} [(2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}, \tag{2-28}$$

and for $k = L+1, L+2$,

$$E_k = b_1 \Lambda S_k - \sum_{j=1}^{L} [(2j-\gamma) b_1 b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}. \tag{2-29}$$

For the expansion of the nonlinear term A_2, let us set

$$f(x) = \sin(2x)$$

and use a Taylor expansion to write (see page 1740 in [Raphaël and Schweyer 2014b] for a similar computation)

$$A_2 = \frac{(d-1)}{2y^2} \left[\sum_{i=2}^{L+2} \frac{f^{(i)}(Q)}{i!} \Theta_b^i + R_2 \right] = \frac{(d-1)}{2y^2} \left[\sum_{i=2}^{L+2} P_i + R_1 + R_2 \right],$$

where

$$P_i = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j, |J|_2 = i} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \tag{2-30}$$

$$R_1 = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j, |J|_2 \geq L+3} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \tag{2-31}$$

$$R_2 = \frac{\Theta_b^{L+3}}{(L+2)!} \int_0^1 (1-\tau)^{L+2} f^{(L+3)}(Q + \tau \Theta_b) \, d\tau, \tag{2-32}$$

with $J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}) \in \mathbb{N}^{2L+1}$ and

$$|J|_1 = \sum_{k=1}^{L} i_k + \sum_{k=2}^{L+2} j_k, \quad |J|_2 = \sum_{k=1}^{L} k i_k + \sum_{k=2}^{L+2} k j_k. \tag{2-33}$$

In conclusion, we have

$$\Psi_b = \sum_{k=1}^{L+1} \left[\mathscr{L} S_{k+1} + E_k + \frac{(d-1)}{2y^2} P_{k+1} \right] + E_{L+2} + \frac{(d-1)}{2y^2}(R_1 + R_2). \tag{2-34}$$

• Construction of S_k. From the expression of Ψ_b given in (2-34), we construct iteratively the sequences of profiles $(S_k)_{1 \le k \le L+2}$ through the scheme

$$\begin{cases} S_1 = 0, \\ S_k = -\mathscr{L}^{-1} F_k, \quad 2 \le k \le L+2, \end{cases} \tag{2-35}$$

where

$$F_k = E_{k-1} + \frac{(d-1)}{2y^2} P_k \quad \text{for } 2 \le k \le L+2.$$

We claim by induction on k that F_k is homogeneous with

$$\deg(F_k) = (k-1, k-2, k) \quad \text{for } 2 \le k \le L+2, \tag{2-36}$$

and

$$\frac{\partial F_k}{\partial b_m} = 0 \quad \text{for } 2 \le k \le m \le L+2. \tag{2-37}$$

From item (iii) of Lemma 2.8 and (2-36), we deduce that S_k is homogeneous with

$$\deg(S_k) = (k, k-1, k) \quad \text{for } 2 \le k \le L+2,$$

and from (2-37), we get

$$\frac{\partial S_k}{\partial b_m} = 0 \quad \text{for } 2 \le k \le m \le L+2,$$

which is the conclusion of item (ii).

Let us now give the proof of (2-36) and (2-37). We proceed by induction.

Case $k = 2$: We compute explicitly from (2-28) and (2-30),

$$F_2 = E_1 + \frac{(d-1)}{2y^2} P_2 = b_1^2 \left[\Lambda T_1 - (2-\gamma) T_1 + \frac{(d-1) f''(Q)}{2y^2} T_1^2 \right],$$

which directly follows (2-37). From Lemma 2.9, we know that T_1 and $\Lambda T_1 - (2-\gamma) T_1$ are admissible of degrees $(1,1)$ and $(1,0)$ respectively. Using (2-4), one can check the bound

$$\text{for all } m, j \in \mathbb{N}^2, \quad \left| \partial_y^m \left(\frac{f^{(j)}(Q)}{y^2} \right) \right| \lesssim y^{-\gamma-2-m} \quad \text{as } y \to +\infty. \tag{2-38}$$

Since T_1 is admissible of degree $(1,1)$, we have

$$\text{for all } m \in \mathbb{N}, \quad |\partial_y^m (T_1^2)| \lesssim y^{4-2\gamma-m} \quad \text{as } y \to +\infty.$$

By the Leibniz rule and the fact that $2\gamma - 2 > 0$, we get

$$\text{for all } m, j \in \mathbb{N}^2, \qquad \left| \partial_y^m \left(\frac{f^{(j)}(Q)}{y^2} T_1^2 \right) \right| \lesssim y^{-\gamma - m - (2\gamma - 2)} \lesssim y^{-\gamma - m}.$$

We also have the expansion near the origin,

$$\frac{f^{(j)}(Q)}{y^2} T_1^2 = \sum_{i=2}^{k} c_i y^{2i+1} + \mathcal{O}(y^{2k+3}), \quad k \geq 1.$$

Hence, $(f''(Q)/y^2)T_1^2$ is admissible of degree $(2, 0)$, which concludes the proof of (2-36) for $k = 2$.

Case $k \to k+1$: Estimate (2-37) holds by direct inspection. Let us now assume that S_k is homogeneous of degree $(k, k-1, k)$ and prove that S_{k+1} is homogeneous of degree $(k+1, k, k+1)$. In particular, the claim immediately follows from part (iii) of Lemma 2.8 once we show that F_{k+1} is homogeneous with

$$\deg(F_{k+1}) = \deg\left(E_k + \frac{P_{k+1}}{y^2} \right) = (k, k-1, k+1). \tag{2-39}$$

From part (ii) of Lemma 2.9 and the a priori assumption (2-21), we see that $b_1 b_k (\Lambda T_k - (2k - \gamma) T_k)$ is homogeneous of degree $(k, k-1, k+1)$. From part (i) of Lemma 2.8 and the induction hypothesis, $b_1 \Lambda S_k$ is also homogeneous of degree $(k, k-1, k+1)$. By definition, $b_1 (\partial S_k / \partial b_1)$ is homogeneous and has the same degree as S_k. Thus,

$$\left((2j - \gamma) b_1 - \frac{b_2}{b_1} \right) \left(b_1 \frac{\partial S_k}{\partial b_1} \right)$$

is homogeneous of degree $(k, k-1, k+1)$. From definitions (2-28) and (2-29), we derive

$$\deg(E_k) = (k, k-1, k+1), \quad k \geq 1.$$

It remains to control the term P_{k+1}/y^2. From the definition (2-30), we see that P_{k+1}/y^2 is a linear combination of monomials of the form

$$M_J(y) = \frac{f^{(j)}(Q)}{y^2} \prod_{m=1}^{L} b_m^{i_m} T_m^{i_m} \prod_{m=2}^{L+2} S_m^{j_m},$$

with

$$J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}), \qquad |J|_1 = j, \quad |J|_2 = k+1, \quad 2 \leq j \leq k+1.$$

Recall from part (i) of Lemma 2.9 the bound

$$\text{for all } n \in \mathbb{N}, \quad |\partial_y^n T_m| \lesssim y^{2m - \gamma - n} \quad \text{as } y \to +\infty,$$

and from the induction hypothesis and the a priori bound (2-21),

$$\text{for all } n \in \mathbb{N}, \quad |\partial_y^n S_m| \lesssim b_1^m y^{2(m-1) - \gamma - n} \quad \text{as } y \to +\infty.$$

Together with the bound (2-38), we obtain the following bound at infinity:

$$|M_J| \lesssim b_1^{|J|_2} y^{2|J|_2-\gamma-|J|_1\gamma-2-2\sum_{m=2}^{L+2} j_m} \lesssim b_1^{k+1} y^{2(k-1)-\gamma}.$$

The control of $\partial_y^n M_J$ follows by the Leibniz rule and the above estimates. One can also check that M_J is of order $y^{2|J|_2+|J|_1-1}$ near the origin. This concludes the proof of (2-39) as well as part (ii) of Proposition 2.11.

- Estimate on Ψ_b. From (2-34) and (2-35), the expression of Ψ_b is now reduced to

$$\Psi_b = E_{L+2} + \frac{(d-1)}{y^2}(R_1 + R_2),$$

where E_{L+2}, R_1 and R_2 are given by (2-29), (2-31) and (2-32).

We start by estimating the E_{L+2} term defined by (2-29). Since S_{L+2} is homogeneous of degree $(L+2, L+1, L+2)$, so are ΛS_{L+2} and $b_1(\partial S_{L+2}/\partial b_1)$. It follows that E_{L+2} is homogeneous of degree $(L+2, L+1, L+3)$. Using part (ii) of Lemma 2.8 and the relation $d-2\gamma-4\hbar = 4\delta$, see (1-18), we estimate for all $0 \le m \le L$

$$\begin{aligned}\int_{y\le 2B_1} |\mathscr{L}^{\hbar+m+1} E_{L+2}|^2 &\lesssim b_1^{2L+6} \int_{y\le 2B_1} |y^{2(L+1)-\gamma-2(\hbar+m+1)}|^2 y^{d-1}\,dy \\ &\lesssim b_1^{2L+6} \int_{y\le 2B_1} y^{4(L-m+\delta)-1}\,dy \\ &\lesssim b_1^{(2L+6)-2(L-m+\delta)(1+\eta)} \lesssim b_1^{2m+4+2(1-\delta)-C_L\eta},\end{aligned}$$

where $\eta = \eta(L),\ 0 < \eta \ll 1$.

We now turn to the control of the term R_1/y^2, which is a linear combination of terms of the form, see (2-31),

$$\widetilde{M}_J = \frac{f^{(j)}(Q)}{y^2} \prod_{n=1}^{L} b_n^{i_n} T_n^{i_n} \prod_{n=2}^{L+2} S_k^{j_n},$$

with

$$J = (i_1, \ldots, i_L, j_2, \ldots, j_{L+2}), \qquad |J|_1 = j, \quad |J|_2 \ge L+3, \quad 2 \le j \le L+2.$$

Using the admissibility of T_n and the homogeneity of S_n, we get the bounds

$$|\widetilde{M}_J| \lesssim b_1^{L+3} y^{2|J|_2+j-1} \lesssim b_1^{L+3} y^{2L+6} \quad \text{as } y \to 0,$$

and

$$|\widetilde{M}_J| \lesssim b_1^{|J|_2} y^{2|J|_2-j\gamma-2-\gamma} \quad \text{as } y \to +\infty,$$

where we used the facts that $j \ge 2$ and $2 - j\gamma < 0$, and similarly for higher derivatives by the Leibniz rule. Thus, we obtain the round estimate for all $0 \le m \le L$,

$$\begin{aligned}\int_{y\le 2B_1} \left|\mathscr{L}^{\hbar+m+1}\left(\frac{R_1}{y^2}\right)\right|^2 &\lesssim b_1^{2|J|_2} \int_{y\le 2B_1} |y^{2|J|_2-j\gamma-\gamma-2-2(\hbar+m+1)}|^2 y^{d-1}\,dy \\ &\lesssim b_1^{2m+4+2(1-\delta)-C_L\eta}.\end{aligned}$$

The term R_2/y^2 is estimated exactly as the term R_1/y^2 using the definition (2-32). Similarly, the control of $\int_{y\le 2B_1}|\Psi_b|^2/(1+y^{4(\hbar+m+1)})$ is obtained along the exact same lines as above. This concludes the proof of (2-25). The local estimate (2-26) directly follows from the homogeneity of S_k and the admissibility of T_k. □

We now proceed to a simple localization of the profile Q_b to avoid the growth of tails in the region $y \ge 2B_1 \gg B_0$. More precisely, we claim the following:

Proposition 2.12 (estimates on the localized profile). *Under the assumptions of Proposition 2.11, we assume in addition the a priori bound*

$$|(b_1)_s| \lesssim b_1^2. \tag{2-40}$$

Consider the localized profile

$$\widetilde{Q}_{b(s)}(y) = Q(y) + \sum_{k=1}^{L} b_k \widetilde{T}_k + \sum_{k=2}^{L+2} \widetilde{S}_k \quad \textit{with } \widetilde{T}_k = \chi_{B_1} T_k,\ \ \widetilde{S}_k = \chi_{B_1} S_k, \tag{2-41}$$

where B_1 and χ_{B_1} are defined as in (1-20) and (1-21). Then

$$\partial_s \widetilde{Q}_b - \partial_{yy}\widetilde{Q}_b - \frac{(d-1)}{y}\partial_y \widetilde{Q}_b + b_1 \Lambda \widetilde{Q}_b + \frac{(d-1)}{2y^2}\sin(2\widetilde{Q}_b) = \widetilde{\Psi}_b + \chi_{B_1}\operatorname{Mod}(t), \tag{2-42}$$

where $\widetilde{\Psi}_b$ satisfies the bounds:

(i) (*large Sobolev bound*) *For all $0 \le m \le L-1$,*

$$\int |\mathscr{L}^{\hbar+m+1}\widetilde{\Psi}_b|^2 + \int \frac{|\mathscr{A}\mathscr{L}^{\hbar+m}\widetilde{\Psi}_b|^2}{1+y^2} + \int \frac{|\mathscr{L}^{\hbar+m}\widetilde{\Psi}_b|^2}{1+y^4} + \int \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(\hbar+m+1)}} \lesssim b_1^{2m+2+2(1-\delta)-C_L\eta}, \tag{2-43}$$

and

$$\int |\mathscr{L}^{\hbar+L+1}\widetilde{\Psi}_b|^2 + \int \frac{|\mathscr{A}\mathscr{L}^{\hbar+L}\widetilde{\Psi}_b|^2}{1+y^2} + \int \frac{|\mathscr{L}^{\hbar+L}\widetilde{\Psi}_b|^2}{1+y^4} + \int \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(\hbar+L+1)}} \lesssim b_1^{2L+2+2(1-\delta)(1+\eta)}, \tag{2-44}$$

where $\hbar$ and δ are defined by (1-18).

(ii) (*very local bound*) *For all $M \le \frac{1}{2}B_1$ and $0 \le m \le L$,*

$$\int_{y\le 2M} |\mathscr{L}^{\hbar+m+1}\widetilde{\Psi}_b|^2 \lesssim M^C b_1^{2L+6}. \tag{2-45}$$

(iii) (*refined local bound near B_0*) *For all $0 \le m \le L$,*

$$\int_{y\le 2B_0} |\mathscr{L}^{\hbar+m+1}\widetilde{\Psi}_b|^2 + \int_{y\le 2B_0} \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(\hbar+m+1)}} \lesssim b_1^{2m+4+2(1-\delta)-C_L\eta}. \tag{2-46}$$

Proof. By a direct computation, we have

$$\begin{aligned}
\partial_s \widetilde{Q}_b - \partial_{yy}\widetilde{Q}_b - \frac{(d-1)}{y}\partial_y \widetilde{Q}_b + b_1\Lambda\widetilde{Q}_b + \frac{(d-1)}{2y^2}\sin(2\widetilde{Q}_b) \\
= \chi_{B_1}\left[\partial_s Q_b - \partial_{yy}Q_b - \frac{(d-1)}{y}\partial_y Q_b + b_1\Lambda Q_b + \frac{(d-1)}{2y^2}\sin(2Q_b)\right] \\
+ \Theta_b\left[\partial_s\chi_{B_1} - \left(\partial_{yy}\chi_{B_1} + \frac{d-1}{y}\partial_y\chi_{B_1}\right) + b_1\Lambda\chi_{B_1}\right] - 2\partial_y\chi_{B_1}\partial_y\Theta_b + b_1(1-\chi_{B_1})\Lambda Q \\
+ \frac{(d-1)}{2y^2}[\sin(2\widetilde{Q}_b) - \sin(2Q) - \chi_{B_1}(\sin(2Q_b) - \sin(2Q))].
\end{aligned}$$

According to (2-23) and (2-42), we write

$$\widetilde{\Psi}_b = \chi_{B_1}\Psi_b + \widehat{\Psi}_b,$$

where

$$\begin{aligned}
\widehat{\Psi}_b = \underbrace{b_1(1-\chi_{B_1})\Lambda Q}_{\widehat{\Psi}_b^{(1)}} + \underbrace{\frac{(d-1)}{2y^2}[\sin(2\widetilde{Q}_b) - \sin(2Q) - \chi_{B_1}(\sin(2Q_b) - \sin(2Q))]}_{\widehat{\Psi}_b^{(2)}} \\
+ \underbrace{\Theta_b\left[\partial_s\chi_{B_1} - \left(\partial_{yy}\chi_{B_1} + \frac{d-1}{y}\partial_y\chi_{B_1}\right) + b_1\Lambda\chi_{B_1}\right] - 2\partial_y\chi_{B_1}\partial_y\Theta_b}_{\widehat{\Psi}_b^{(3)}}.
\end{aligned}$$

The contribution of the term $\chi_{b_1}\Psi_b$ to the bounds (2-43), (2-44), (2-45) and (2-46) follows in exactly the same way as in the proof of (2-25) and (2-26). We are therefore left to estimate the term $\widehat{\Psi}_b$. All the terms in the expression of $\widehat{\Psi}_b$ are localized in $B_1 \le y \le 2B_1$, except for the first one whose support is a subset of $\{y \ge B_1\}$. Hence, the estimates (2-45) and (2-46) directly follow from (2-26) and (2-25).

Let us now find the contribution of $\widehat{\Psi}_b$ to the bounds (2-43) and (2-44). We estimate

$$\text{for all } n \in \mathbb{N}, \quad \left|\frac{d^n}{dy^n}(1-\chi_{B_1})\Lambda Q\right| \lesssim \frac{1}{y^{\gamma+n}}\mathbf{1}_{y\ge B_1};$$

hence, using the relation $d - 2\gamma - 4\hbar = 4\delta$, see (1-18), and the definition (1-20) of B_1, we estimate for all $0 \le m \le L$,

$$\int |\mathscr{L}^{\hbar+m+1}\widehat{\Psi}_b^{(1)}|^2 \lesssim b_1^2 \int_{y\ge B_1} \frac{y^{d-1}}{y^{4(\hbar+m+1)+2\gamma}} \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)+2m\eta}.$$

For the nonlinear term $\widehat{\Psi}_b^{(2)}$, we note from the admissibility of T_k and the homogeneity of S_k that the T_k-terms dominate for $y \ge B_1$ in Θ_b. Thus, for $y \ge B_1$,

$$\text{for all } n \in \mathbb{N}, \quad |\partial_y^n\Theta_b| \lesssim \sum_{k=1}^{L} b_1^k y^{2k-\gamma-n}\mathbf{1}_{y\ge B_1}. \tag{2-47}$$

Using (2-47) and noting that $\widehat{\Psi}_b^{(2)}$ is localized in $B_1 \le y \le 2B_1$, we obtain the round bound

$$|\partial_y^n \widehat{\Psi}_b^{(2)}| \lesssim \sum_{k=1}^{L} b_1^k y^{2(k-1)-\gamma-n} \mathbf{1}_{B_1 \le y \le 2B_1}$$
$$\lesssim \frac{b_1}{y^{\gamma+n}} \sum_{k=1}^{L} b_1^{-(k-1)\eta} \mathbf{1}_{B_1 \le y \le 2B_1}.$$

We then estimate for $0 \le m \le L$,

$$\int |\mathscr{L}^{\hbar+m+1} \widehat{\Psi}_b^{(2)}| \lesssim b_1^2 \sum_{k=1}^{L} b_1^{-2(k-1)\eta} \int_{B_1 \le y \le 2B_1} \frac{y^{d-1}}{y^{4(\hbar+m+1)+2\gamma}}\, dy$$
$$\lesssim b_1^{2m+2+2(1-\delta)(1+\eta)} \sum_{k=1}^{L} b_1^{(2m-2k+2)\eta}.$$

To control $\widehat{\Psi}_b^{(3)}$, we first note from the definition (1-21) and the assumption (2-40) that

$$|\partial_s \chi_{B_1}| \lesssim \frac{(b_1)_s}{b_1} \frac{y}{B_1} \mathbf{1}_{B_1 \le y \le 2B_1} \lesssim b_1 \mathbf{1}_{B_1 \le y \le 2B_1}.$$

Using (2-47), we estimate for $0 \le m \le L$,

$$\int |\mathscr{L}^{\hbar+m+1} \widehat{\Psi}_b^{(3)}| \lesssim \sum_{k=1}^{L} b_1^2 b_1^{2k} \int_{B_1 \le y \le 2B_1} \frac{y^{d-1}}{y^{4(\hbar+m+1)+2\gamma-4k+2}}\, dy$$
$$\lesssim b_1^{2m+2+2(1-\delta)(1+\eta)} \sum_{k=1}^{L} b_1^{(2m-2k)\eta}.$$

Gathering all the bounds yields

$$\int |\mathscr{L}^{\hbar+m+1} \widehat{\Psi}_b|^2 \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)} \sum_{k=1}^{L} b_1^{(2m-2k)\eta} \lesssim b_1^{2m+2+2(1-\delta)(1+\eta)+2\eta(m-L)}.$$

The control of

$$\int \frac{|\mathscr{A}\mathscr{L}^{\hbar+m} \widetilde{\Psi}_b|^2}{1+y^2}, \quad \int \frac{|\mathscr{L}^{\hbar+m} \widetilde{\Psi}_b|^2}{1+y^4}, \quad \text{and} \quad \int \frac{|\widehat{\Psi}_b|^2}{1+y^{4(\hbar+m+1)}}$$

is obtained along the exact same lines as above. This concludes the proof of (2-43) and (2-44), as well as Proposition 2.12. □

2D. ***Study of the dynamical system for $b = (b_1, \dots, b_L)$.*** The construction of the Q_b profile formally leads to the finite-dimensional dynamical system for $b = (b_1, \dots, b_L)$ by setting to zero the inhomogeneous $\mathrm{Mod}(t)$ term given in (2-24):

$$(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \; b_{L+1} = 0. \tag{2-48}$$

Unlike the critical case ($d = 2$) treated in [Raphaël and Schweyer 2014b], there is no further logarithmic correction to be taken into account in the system (2-48). In particular, the system (2-48) admits explicit solutions and the linearized operator near these solutions is explicit.

Lemma 2.13 (solution to the system (2-48)). *Let*

$$\tfrac{1}{2}\gamma < \ell \ll L, \quad \ell \in \mathbb{N}^*,$$

and consider the sequence

$$\begin{cases} c_1 = \dfrac{\ell}{2\ell - \gamma}, \\ c_{k+1} = -\dfrac{\gamma(\ell - k)}{2\ell - \gamma} c_k, & 1 \le k \le \ell - 1, \\ c_{k+1} = 0, & k \ge \ell. \end{cases} \tag{2-49}$$

Then the explicit choice

$$b_k^e(s) = \frac{c_k}{s^k}, \quad s > 0, \ 1 \le k \le L, \tag{2-50}$$

is a solution to (2-48).

The proof of Lemma 2.13 directly follows from an explicit computation which is left to the reader. We claim that the linearized flow of (2-48) near the solution (2-50) is explicit and displays $\ell - 1$ unstable directions. Note that the stability is considered in the sense that

$$\sup_s s^k |b_k(s)| \le C_k, \quad 1 \le k \le L.$$

In particular, we have the following result which was proved in [Merle, Raphaël and Rodnianski 2015]:

Lemma 2.14 (linearization of (2-48) around (2-50)). *Let*

$$b_k(s) = b_k^e(s) + \frac{\mathcal{U}_k(s)}{s^k}, \quad 1 \le k \le \ell, \tag{2-51}$$

and note that $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_\ell)$. Then, for $1 \le k \le \ell - 1$,

$$(b_k)_s + (2k - \gamma) b_1 b_k - b_{k+1} = \frac{1}{s^{k+1}} [s(\mathcal{U}_k)_s - (A_\ell \mathcal{U})_k + \mathcal{O}(|\mathcal{U}|^2)], \tag{2-52}$$

$$(b_\ell)_s + (2\ell - \gamma) b_1 b_\ell = \frac{1}{s^{k+1}} [s(\mathcal{U}_\ell)_s - (A_\ell \mathcal{U})_\ell + \mathcal{O}(|\mathcal{U}|^2)], \tag{2-53}$$

where

$$A_\ell = (a_{i,j})_{1 \le i,j \le \ell} \quad \textit{with} \quad \begin{cases} a_{1,1} = \dfrac{\gamma(\ell - 1)}{2\ell - \gamma} - (2 - \gamma) c_1, \\ a_{i,i} = \dfrac{\gamma(\ell - i)}{2\ell - \gamma}, & 2 \le i \le \ell, \\ a_{i,i+1} = 1, & 1 \le i \le \ell - 1, \\ a_{1,i} = -(2i - \gamma) c_i, & 2 \le i \le \ell, \\ a_{i,j} = 0 & \text{otherwise}. \end{cases}$$

Moreover, A_ℓ is diagonalizable:

$$A_\ell = P_\ell^{-1} D_\ell P_\ell, \quad D_\ell = \operatorname{diag}\left\{-1, \frac{2\gamma}{2\ell-\gamma}, \frac{3\gamma}{2\ell-\gamma}, \dots, \frac{\ell\gamma}{2\ell-\gamma}\right\}. \tag{2-54}$$

Proof. Since we have an analogous system to the one in [Merle, Raphaël and Rodnianski 2015] and the proof is essentially the same as written there, we kindly refer the reader to Lemma 3.7 in that paper for all details of the proof. □

3. Proof of Theorem 1.1 assuming technical results

This section is devoted to the proof of Theorem 1.1. We proceed in three subsections:

- In the first subsection, we give an equivalent formulation of the linearization of the problem in the setting (1-30).
- In the second subsection, we prepare the initial data and define the shrinking set $\mathcal{S}_K$ (see Definition 3.2) such that the solution trapped in this set satisfies the conclusion of Theorem 1.1.
- In the third subsection, we give all arguments of the proof of the existence of solutions trapped in $\mathcal{S}_K$ (Proposition 3.5) assuming an important technical result (Proposition 3.6) whose proof is left to the next section. Then we conclude the proof of Theorem 1.1.

3A. ***Linearization of the problem.*** Let $L \gg 1$ be an integer and $s_0 \gg 1$. We introduce the renormalized variables

$$y = \frac{r}{\lambda(t)}, \quad s = s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)}, \tag{3-1}$$

and the decomposition

$$u(r,t) = w(y,s) = (\tilde{Q}_{b(s)} + q)(y,s) = (\tilde{Q}_{b(t)} + q)\left(\frac{r}{\lambda(t)}, t\right), \tag{3-2}$$

where $\tilde{Q}_b$ is constructed in Proposition 2.12 and the modulation parameters

$$\lambda(t) > 0, \quad b(t) = (b_1(t), \dots, b_L(t))$$

are determined from the $L+1$ orthogonality conditions

$$\langle q, \mathscr{L}^k \Phi_M \rangle = 0, \quad 0 \le k \le L, \tag{3-3}$$

where Φ_M is a fixed direction depending on some large constant M defined by

$$\Phi_M = \sum_{k=0}^{L} c_{k,M} \mathscr{L}^k (\chi_M \Lambda Q), \tag{3-4}$$

with

$$c_{0,M} = 1, \qquad c_{k,M} = (-1)^{k+1} \frac{\sum_{j=0}^{k-1} c_{j,M} \langle \chi_M \mathscr{L}^j (\chi_M \Lambda Q), T_k \rangle}{\langle \chi_M \Lambda Q, \Lambda Q \rangle}, \quad 1 \le k \le L. \tag{3-5}$$

Here, Φ_M is built to ensure the nondegeneracy

$$\langle \Phi_M, \Lambda Q\rangle = \langle \chi_M \Lambda Q, \Lambda Q\rangle \gtrsim M^{d-2\gamma} \tag{3-6}$$

and the cancellation

$$\langle \Phi_M, T_k\rangle = \sum_{j=0}^{k-1} c_{j,M} \langle \mathscr{L}^j(\chi_M \Lambda Q), T_k\rangle + c_{k,M}(-1)^k \langle \chi_M \Lambda Q, \Lambda Q\rangle = 0. \tag{3-7}$$

In particular, we have

$$\langle \mathscr{L}^i T_k, \Phi_M\rangle = (-1)^k \langle \chi_M \Lambda Q, \Lambda Q\rangle \delta_{i,k}, \quad 0 \le i, k \le L. \tag{3-8}$$

From (2-2), we see that q satisfies the equation

$$\partial_s q - \frac{\lambda_s}{\lambda}\Lambda q + \mathscr{L} q = -\widetilde{\Psi}_b - \widehat{\mathrm{Mod}} + \mathcal{H}(q) - \mathcal{N}(q) \equiv \mathcal{F}, \tag{3-9}$$

where

$$\widehat{\mathrm{Mod}} = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda \widetilde{Q}_b - \chi_{B_1}\,\mathrm{Mod}, \tag{3-10}$$

$\mathcal{H}$ is the linear part given by

$$\mathcal{H}(q) = \frac{(d-1)}{y^2}[\cos(2Q) - \cos(2\widetilde{Q}_b)]q, \tag{3-11}$$

and $\mathcal{N}$ is the purely nonlinear term

$$\mathcal{N}(q) = \frac{(d-1)}{2y^2}[\sin(2\widetilde{Q}_b + 2q) - \sin(2\widetilde{Q}_b) - 2q\cos(2\widetilde{Q}_b)]. \tag{3-12}$$

We also need to write (3-9) in the original variables. To do so, consider the rescaled linearized operator

$$\mathscr{L}_\lambda = -\partial_{rr} - \frac{(d-1)}{r}\partial_r + \frac{Z_\lambda}{r^2} \tag{3-13}$$

and the renormalized function

$$v(r,t) = q(y,s), \quad \partial_t v = \frac{1}{\lambda^2(t)}\left(\partial_s q - \frac{\lambda_s}{\lambda}\Lambda q\right)_\lambda.$$

Then from (3-9), v satisfies

$$\partial_t v + \mathscr{L}_\lambda v = \frac{1}{\lambda^2}\mathcal{F}_\lambda, \quad \mathcal{F}_\lambda(r,t) = \mathcal{F}(y,s). \tag{3-14}$$

Note that

$$\mathscr{L}_\lambda = \frac{1}{\lambda^2}\mathscr{L}.$$

3B. ***Preparation of the initial data.*** We now describe the set of initial data u_0 of the problem (1-4), as well as the initial data for (b,λ) leading to the blowup scenario of Theorem 1.1. Assume that $u_0 \in H^\infty(\mathbb{R}^d)$ satisfies

$$\|u_0 - Q\|_{\dot{H}^s} \ll 1 \quad \text{for } \tfrac{d}{2} \le s \le \Bbbk. \tag{3-15}$$

By continuity of the flow and a standard argument, the smallness assumption (3-15) is propagated on a small time interval $[0, t_1)$. Thus, the decomposition (3-2),

$$u(r,t) = (\tilde{Q}_{b(t)} + q)\left(\frac{r}{\lambda(t)}, t\right), \quad \lambda(t) > 0, \;\; b = (b_1, \dots, b_L), \tag{3-16}$$

can be uniquely defined on the interval $t \in [0, t_1]$.

The existence of the decomposition (3-16) is a standard consequence of the implicit function theorem and the explicit relations

$$\frac{\partial}{\partial \lambda}(\tilde{Q}_{b(t)})_\lambda, \frac{\partial}{\partial b_1}(\tilde{Q}_{b(t)})_\lambda, \dots, \frac{\partial}{\partial b_L}(\tilde{Q}_{b(t)})_\lambda \bigg|_{\lambda=1,\, b=0} = (\Lambda Q, T_1, \dots, T_L),$$

which implies the nondegeneracy of the Jacobian

$$\left| \left\langle \frac{\partial}{\partial(\lambda, b_j)}(\tilde{Q}_{b(t)})_\lambda, \mathscr{L}^i \Phi_M \right\rangle_{1 \le j \le L,\, 0 \le i \le L} \right|_{\lambda=1,\, b=0} = |\langle \chi_M \Lambda Q, \Lambda Q\rangle|^{L+1} \neq 0.$$

In fact, the decomposition (3-16) exists as long as $t < T$ and q remains small in the energy topology. We now set up the bootstrap for the control of the parameters (b, λ) and the radiation q. We will measure the regularity of the map through the following coercive norms of q:

$$\mathscr{E}_{2k} = \int |\mathscr{L}^k q|^2 \ge C(M) \sum_{m=0}^{k-1} \int \frac{|\mathscr{L}^m q|^2}{1 + y^{4(k-m)}} \quad \text{for } \hbar + 1 \le k \le \Bbbk. \tag{3-17}$$

Our construction is built on a careful choice of the initial data for the modulation parameter b and the radiation q at time $s = s_0$. In particular, we will choose it in the following way:

Definition 3.1 (choice of the initial data). Take η and δ as in (1-20) and (1-18). Let consider the variable

$$\mathcal{V} = P_\ell \mathcal{U}, \tag{3-18}$$

where $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_\ell)$ is introduced in the linearization (2-51), namely

$$\mathcal{U}_k = s^k b_k - c_k, \quad \text{with } c_k \text{ given by (2-49)},$$

and P_ℓ refers to the diagonalization (2-54) of A_ℓ.

Let $s_0 \ge 1$. We assume

- (smallness of the initial perturbation for the b_k-unstable modes)

$$|s_0^{\frac{\eta}{2}(1-\delta)} \mathcal{V}_k(s_0)| < 1 \quad \text{for } 2 \le k \le \ell, \tag{3-19}$$

- (smallness of the initial perturbation for the b_k-stable modes)

$$|s_0^{\frac{\eta}{2}(1-\delta)} \mathcal{V}_1(s_0)| < 1, \qquad |b_k(s_0)| < s_0^{-\frac{5\ell(2k-\gamma)}{2\ell-\gamma}} \quad \text{for } \ell + 1 \le k \le L, \tag{3-20}$$

- (smallness of the data)

$$\sum_{k=\hbar+2}^{\Bbbk} \mathscr{E}_{2k}(s_0) < s_0^{-\frac{10L\ell}{2\ell-\gamma}}, \tag{3-21}$$

- (normalization) up to a fixed rescaling, we may always assume

$$\lambda(s_0) = 1. \tag{3-22}$$

In particular, the initial data described in Definition 3.1 belongs to the following set which shrinks to zero as $s \to +\infty$:

Definition 3.2 (definition of the shrinking set). Take η and δ as in (1-20) and (1-18). For all $K \geq 1$ and $s \geq 1$, we define $\mathcal{S}_K(s)$ as the set of all $(b_1(s), \dots, b_L(s), q(s))$ such that

$$\begin{aligned}
&|\mathcal{V}_k(s)| \leq 10 s^{-\frac{\eta}{2}(1-\delta)} && \text{for } 1 \leq k \leq \ell,\\
&|b_k(s)| \leq s^{-k} && \text{for } \ell+1 \leq k \leq L,\\
&\mathscr{E}_{2\Bbbk}(s) \leq K s^{-(2L+2(1-\delta)(1+\eta))},\\
&\mathscr{E}_{2m}(s) \leq \begin{cases} K s^{-\frac{\ell}{2\ell-\gamma}(4m-d)} \\ s^{-2(m-\hbar-1)-2(1-\delta)+K\eta} \end{cases} && \begin{aligned}&\text{for } \hbar+2 \leq m \leq \ell+\hbar,\\ &\text{for } \ell+\hbar+1 \leq m \leq \Bbbk-1.\end{aligned}
\end{aligned}$$

Remark 3.3. From (2-51), the bounds given in Definition 3.2 imply that for η small enough,

$$b_1(s) \sim \frac{c_1}{s}, \quad |b_k(s)| \lesssim |b_1(s)|^k.$$

Hence, the choice of the initial data $(b(s_0), q(s_0))$ belongs in $\mathcal{S}_K(s_0)$ if s_0 is large enough.

Remark 3.4. The introduction of the high Sobolev norm $\mathscr{E}_{2\Bbbk}$ is reflected in the relation

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{k=1}^{L} |(b_k)_s + (2k-\gamma)b_1 b_k - b_{k+1}| \lesssim C(M)\sqrt{\mathscr{E}_{2\Bbbk}} + \text{l.o.t.}, \tag{3-23}$$

which is computed thanks to the $L+1$ orthogonality conditions (3-3) (see Lemmas 4.2 and 4.3 below).

3C. *Existence of solutions trapped in $\mathcal{S}_K(s)$ and conclusion of Theorem 1.1.* We claim the following proposition:

Proposition 3.5 (existence of solutions trapped in $\mathcal{S}_K(s)$). *There exists $K_1 \geq 1$ such that for $K \geq K_1$, there exists $s_{0,1}(K)$ such that for all $s_0 \geq s_{0,1}$, there exists initial data for the unstable modes*

$$(\mathcal{V}_2(s_0), \dots, \mathcal{V}_\ell(s_0)) \in [-s_0^{-\frac{\eta}{2}(1-\delta)}, s_0^{-\frac{\eta}{2}(1-\delta)}]^{\ell-1}$$

such that the corresponding solution $(b(s), q(s))$ is in $\mathcal{S}_K(s)$ for all $s \geq s_0$.

Let us briefly give the proof of Proposition 3.5. Let us consider $K \geq 1$ and $s_0 \geq 1$ and $(b(s_0), q(s_0))$ as in Definition 3.1. We introduce the exit time

$$s_* = s_*(b(s_0), q(s_0)) = \sup\{s \geq s_0 \text{ such that } (b(s), q(s)) \in \mathcal{S}_K(s)\},$$

and assume that for any choice of

$$(\mathcal{V}_2(s_0),\dots,\mathcal{V}_\ell(s_0)) \in [-s_0^{-\frac{\eta}{2}(1-\delta)}, s_0^{-\frac{\eta}{2}(1-\delta)}]^{\ell-1},$$

the exit time satisfies $s_* < +\infty$ and look for a contradiction. By the definition of $\mathcal{S}_K(s_*)$, at least one of the inequalities in that definition is an equality. Owing the following proposition, this can happen only for the components $(\mathcal{V}_2(s_*),\dots,\mathcal{V}_\ell(s_*))$. Precisely, we have the following result which is the heart of our analysis:

Proposition 3.6 (control of $(b(s), q(s))$ in $\mathcal{S}_K(s)$ by $(\mathcal{V}_2(s),\dots,\mathcal{V}_\ell(s))$). *There exists $K_2 \ge 1$ such that for each $K \ge K_2$, there exists $s_{0,2}(K) \ge 1$ such that for all $s_0 \ge s_{0,2}(k)$, the following holds: Given the initial data at $s = s_0$ as in Definition 3.1, if $(b(s), q(s)) \in \mathcal{S}_K(s)$ for all $s \in [s_0, s_1]$, with $(b(s_1), q(s_1)) \in \partial\mathcal{S}_K(s_1)$ for some $s_1 \ge s_0$, then:*

(i) (*reduction to a finite-dimensional problem*)

$$(\mathcal{V}_2(s_1),\dots,\mathcal{V}_\ell(s_1)) \in \partial\left[-\frac{K}{s_1^{\frac{\eta}{2}(1-\delta)}}, \frac{K}{s_1^{\frac{\eta}{2}(1-\delta)}}\right]^{\ell-1}.$$

(ii) (*transverse crossing*)

$$\frac{d}{ds}\left(\sum_{i=2}^{\ell} |s^{\frac{\eta}{2}(1-\delta)}\mathcal{V}_i(s)|^2\right)\bigg|_{s=s_1} > 0.$$

Let us assume Proposition 3.6 and continue the proof of Proposition 3.5. From part (i) of Proposition 3.6, we see that

$$(\mathcal{V}_2(s_*),\dots,\mathcal{V}_\ell(s_*)) \in \partial\left[-\frac{K}{s_*^{\frac{\eta}{2}(1-\delta)}}, \frac{K}{s_*^{\frac{\eta}{2}(1-\delta)}}\right]^{\ell-1},$$

and the mapping

$$\Upsilon : [-1,1]^{\ell-1} \to \partial([-1,1]^{\ell-1}),$$

$$s_0^{\frac{\eta}{2}(1-\delta)}(\mathcal{V}_2(s_0),\dots,\mathcal{V}_\ell(s_0)) \mapsto \frac{s_*^{\frac{\eta}{2}(1-\delta)}}{K}(\mathcal{V}_2(s_*),\dots,\mathcal{V}_\ell(s_*)),$$

is well-defined. Applying the transverse-crossing property given in part (ii) of Proposition 3.6, we see that $(b(s), q(s))$ leaves $\mathcal{S}_K(s)$ at $s = s_0$; hence, $s_* = s_0$. This is a contradiction since Υ is the identity map on the boundary sphere and it cannot be a continuous retraction of the unit ball. This concludes the proof of Proposition 3.5, assuming that Proposition 3.6 holds.

• Conclusion of Theorem 1.1 assuming Proposition 3.6. From Proposition 3.5, we know that there exists initial data $(b(s_0), q(s_0))$ such that

$$(b(s), q(s)) \in \mathcal{S}_K(s) \quad \text{for all } s \ge s_0.$$

From (4-57), (4-58), we have

$$-\lambda\lambda_t = c(u_0)\lambda^{\frac{2\ell-\gamma}{\ell}}[1+o(1)],$$

which yields

$$-\lambda^{1-\frac{2\ell-\gamma}{\ell}}\lambda_t = c(u_0)(1+o(1)).$$

We easily conclude that λ vanishes in finite time $T = T(u_0) < +\infty$ with the following behavior near the blowup time:

$$\lambda(t) = c(u_0)(1+o(1))(T-t)^{\frac{\ell}{\gamma}},$$

which is the conclusion of item (i) of Theorem 1.1.

For the control of the Sobolev norms, we observe from (B-3) and Definition 3.2 that

$$\text{for all } \hbar+2 \le m \le \Bbbk, \quad \int |\partial_y^{2m} q|^2 \lesssim \mathscr{E}_{2m} \to 0 \quad \text{as } s \to +\infty.$$

From the relation $d = 4\hbar + 4\delta + 2\gamma$, we deduce that

$$\text{for all } \sigma \in \left[\tfrac{d}{2}+3, 2\Bbbk\right], \quad \int |\nabla^\sigma q|^2 \to 0 \quad \text{as } s \to +\infty,$$

which yields (ii) of Theorem 1.1.

4. Reduction of the problem to a finite-dimensional one

We now prove Proposition 3.6, which is the heart of our analysis. We proceed in three separate subsections:

- In the first subsection, we derive the laws for the parameters (b, λ) thanks to the orthogonality condition (3-3) and the coercivity of the powers of $\mathscr{L}$.
- In the second subsection, we prove the main monotonicity tools for the control of the infinite-dimensional part of the solution. In particular, we derive a suitable Lyapunov functional for the $\mathscr{E}_{2\Bbbk}$ energy, as well as the monotonicity formula for the lower Sobolev energy.
- In the third subsection, we conclude the proof of Proposition 3.6 thanks to the identities obtained in the first two parts.

4A. *Modulation equations.* We derive here the modulation equations for (b, λ). The derivation is mainly based on the orthogonality (3-3) and the coercivity of the powers of $\mathscr{L}$. Let us start with elementary estimates relating to the fixed direction Φ_M.

Lemma 4.1 (estimate for Φ_M). *Given Φ_M as defined in (3-4), we have*

$$|c_{k,M}| \lesssim M^{2k} \quad \textit{for all } 1 \le k \le L,$$

$$\int |\Phi_M|^2 \lesssim M^{d-2\gamma}, \qquad \int |\mathscr{L}\Phi_M|^2 \lesssim M^{d-2\gamma-4}.$$

Proof. Arguing by induction, we assume that

$$|c_{j,M}| \lesssim M^{2j}, \quad 1 \le j \le k.$$

Using the fact that $\mathscr{L}^j T_i$ is admissible of degree $(\max\{0, i-j\}, i-j)$, we estimate from the definition (3-5),

$$|c_{k+1,M}| \lesssim \frac{1}{M^{d-2\gamma}} \sum_{j=0}^{k} M^{2j} \int |\chi_M \Lambda Q \mathscr{L}^j (T_{k+1})|$$
$$\lesssim \frac{1}{M^{d-2\gamma}} \sum_{j=0}^{k} M^{2j} \int_{y \le M} \frac{y^{d-1}}{y^{\gamma}} y^{2(k+1-j)-\gamma}\, dy \lesssim M^{2(k+1)}.$$

Using the estimate for $c_{k,M}$ yields

$$\int |\Phi_M|^2 \lesssim \int |\chi_M \Lambda Q|^2 + \sum_{j=1}^{L} |c_{j,M}|^2 \int |\mathscr{L}^j (\chi_M \Lambda Q)|^2 \lesssim M^{d-2\gamma-4},$$

and

$$\int |\mathscr{L}\Phi_M|^2 \lesssim \sum_{j=0}^{L} |c_{j,M}|^2 \int |\mathscr{L}^{j+1}(\chi_M \Lambda Q)|^2 \lesssim M^{d-2\gamma}. \qquad \square$$

From the orthogonality conditions (3-3) and (3-9), we claim the following:

Lemma 4.2 (modulation equations). *Take $\hbar$, δ and η as defined in (1-18) and (1-20). For $K \ge 1$, we assume that there is $s_0(K) \gg 1$ such that $(b(s), q(s)) \in \mathcal{S}_K(s)$ for $s \in [s_0, s_1]$ for some $s_1 \ge s_0$. Then, the following hold for $s \in [s_0, s_1]$:*

$$\sum_{k=1}^{L-1} |(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}| + \left| b_1 + \frac{\lambda_s}{\lambda} \right| \lesssim b_1^{L+1+(1-\delta)(1+\eta)}, \tag{4-1}$$

and

$$|(b_L)_s + (2L-\gamma) b_1 b_L| \lesssim \frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)}. \tag{4-2}$$

Proof. We start with the law for b_L. Let

$$D(t) = \left| b_1 + \frac{\lambda_s}{\lambda} \right| + \sum_{k=1}^{L} |(b_k)_s + (2k-\gamma) b_1 b_k - b_{k+1}|,$$

where we recall that $b_k \equiv 0$ if $k \ge L+1$.

Now, we take the inner product of (3-9) with $\mathscr{L}^L \Phi_M$ and use the orthogonality (3-3) to write

$$\langle \widehat{\mathrm{Mod}}(t), \mathscr{L}^L \Phi_M \rangle = -\langle \mathscr{L}^L \widetilde{\Psi}_b, \Phi_M \rangle - \langle \mathscr{L}^{L+1} q, \Phi_M \rangle - \left\langle -\frac{\lambda_s}{\lambda} \Lambda q - \mathcal{L}(q) + \mathcal{N}(q), \mathscr{L}^L \Phi_M \right\rangle. \tag{4-3}$$

From the definition (3-4), we see that Φ_M is localized in $y \le 2M$. From (3-10) and (2-24), we compute by using the identity (3-8),

$$\langle \widehat{\mathrm{Mod}}(t), \mathscr{L}^L \Phi_M \rangle = (-1)^L \langle \Lambda Q, \Phi_M \rangle [(b_L)_s + (2L-\gamma) b_1 b_L] + \mathcal{O}(M^C b_1 D(t)).$$

The error term is estimated by using (2-26) with $m = L - \hbar - 1$ and Lemma 4.1:

$$|\langle \mathscr{L}^L \widetilde{\Psi}_b, \Phi_M \rangle| \le \left(\int_{y \le 2M} |\mathscr{L}^L \widetilde{\Psi}_b|^2 \right)^{\frac{1}{2}} \left(\int_{y \le 2M} |\Phi_M|^2 \right)^{\frac{1}{2}} \lesssim M^C b_1^{L+3} \lesssim b_1^{L+1+(1-\delta)(1+\eta)}.$$

For the linear term, we apply Lemma A.5 with $k = \Bbbk - 1$:

$$\mathscr{E}_{2\Bbbk}(q) \gtrsim \int \frac{|\mathscr{L}^{L+1} q|^2}{y^4(1 + y^{4(\hbar - 1)})} \gtrsim \int \frac{|\mathscr{L}^{L+1} q|^2}{1 + y^{4\hbar}}.$$

Hence, the Cauchy–Schwarz inequality yields

$$|\langle \mathscr{L}^{L+1} q, \Phi_M \rangle| \lesssim M^{2\hbar} \left(\int \frac{|\mathscr{L}^{L+1} q|^2}{1 + y^{4\hbar}} \right)^{\frac{1}{2}} \left(\int |\Phi_M|^2 \right)^{\frac{1}{2}} \lesssim M^{2\hbar + \frac{d}{2} - \gamma} \sqrt{\mathscr{E}_{2\Bbbk}}.$$

The remaining terms are easily estimated by using the following bound coming from Lemma A.5 and Lemma A.4:

$$\mathscr{E}_{2\Bbbk}(q) \gtrsim \int \frac{|\mathscr{L} q|^2}{y^4(1 + y^{4(\Bbbk - 2)})} \gtrsim \int \frac{|\partial_y q|^2}{y^4(1 + y^{4(\Bbbk - 2) + 2})} + \int \frac{q^2}{y^6(1 + y^{4(\Bbbk - 2) + 2})}. \tag{4-4}$$

This implies

$$\left| \left\langle -\frac{\lambda_s}{\lambda} \Lambda q + \mathcal{L}(q) + \mathcal{N}(q), \mathscr{L}^L \Phi_M \right\rangle \right| \lesssim M^C b_1 (\sqrt{\mathscr{E}_{2\Bbbk}} + D(t)).$$

Putting all the above estimates into (4-3) and using (3-6) together with the relation (1-18), we arrive at

$$|(b_L)_s + (2L - \gamma) b_1 b_L| \lesssim \frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)} + M^C b_1 D(t). \tag{4-5}$$

For the modulation equations for b_k with $1 \le k \le L - 1$, we take the inner product of (3-9) with $\mathscr{L}^k \Phi_M$ and use the orthogonality (3-3) to write for $1 \le k \le L - 1$,

$$\langle \widehat{\mathrm{Mod}}(t), \mathscr{L}^k \Phi_M \rangle = -\langle \mathscr{L}^k \widetilde{\Psi}_b, \Phi_M \rangle - \left\langle -\frac{\lambda_s}{\lambda} \Lambda q - \mathcal{L}(q) + \mathcal{N}(q), \mathscr{L}^k \Phi_M \right\rangle.$$

Proceeding as for b_L, we end up with

$$|(b_k)_s + (2k - \gamma) b_1 b_k - b_{k+1}| \lesssim b_1^{L+1+(1-\delta)(1+\eta)} + M^C b_1 (\sqrt{\mathscr{E}_{2\Bbbk}} + D(t)). \tag{4-6}$$

Similarly, we have by taking the inner product of (3-9) with Φ_M,

$$\left| \frac{\lambda_s}{\lambda} + b_1 \right| \lesssim b_1^{L+1+(1-\delta)(1+\eta)} + M^C b_1 (\sqrt{\mathscr{E}_{2\Bbbk}} + D(t)). \tag{4-7}$$

From (4-5), (4-6) and (4-7), we obtain the round bound

$$D(t) \lesssim M^C \sqrt{\mathscr{E}_{2\Bbbk}} + b_1^{L+1+(1-\delta)(1+\eta)}.$$

The conclusion then follows by substituting this bound into (4-5), (4-6) and (4-7). □

From the bound for $\mathscr{E}_{\Bbbk}$ given in Definition 3.2 and the modulation equation (4-2), we only have the pointwise bound

$$|(b_L)_s + (2L-\gamma)b_1 b_L| \lesssim b_1^{L+(1-\delta)(1+\eta)},$$

which is not good enough to close the expected one

$$|(b_L)_s + (2L-\gamma)b_1 b_L| \ll b_1^{L+1}.$$

We claim that the main linear term can be removed up to an oscillation in time leading to the improved modulation equation for b_L as follows:

Lemma 4.3 (improved modulation equation for b_L). *Under the assumption of Lemma 4.2, the following bound holds for all $s \in [s_0, s_1]$:*

$$\left|(b_L)_s + (2L-\gamma)b_1 b_L + \frac{d}{ds}\left\{\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right\}\right| \lesssim \frac{1}{B_0^{2\delta}}\big[C(M)\sqrt{\mathcal{E}_{2\Bbbk}} + b_1^{L+1+(1-\delta)-C_L\eta}\big]. \tag{4-8}$$

Proof. We commute (3-9) with $\mathscr{L}^L$ and take the inner product with $\chi_{B_0}\Lambda Q$ to get

$$\begin{aligned}\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle &\left\{\frac{d}{ds}\left[\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right] - \langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle \frac{d}{ds}\left[\frac{1}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right]\right\} \\ &= \langle \mathscr{L}^L q, \Lambda Q\,\partial_s(\chi_{B_0})\rangle - \langle \mathscr{L}^{L+1} q, \chi_{B_0}\Lambda Q\rangle + \frac{\lambda_s}{\lambda}\langle \mathscr{L}^L \Lambda q, \chi_{B_0}\Lambda Q\rangle \\ &\quad - \langle \mathscr{L}^L \tilde{\Psi}_b, \chi_{B_0}\Lambda Q\rangle - \langle \mathscr{L}^L \widehat{\mathrm{Mod}}(t), \chi_{B_0}\Lambda Q\rangle + \langle \mathscr{L}^L(\mathcal{L}(q) - \mathcal{N}(q)), \chi_{B_0}\Lambda Q\rangle. \end{aligned} \tag{4-9}$$

We recall from (2-5) that

$$B_0^{d-2\gamma} \lesssim |\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle| \lesssim B_0^{d-2\gamma}. \tag{4-10}$$

Let us estimate the second term in the left-hand side of (4-9). We use Cauchy–Schwarz and Lemma A.5 to estimate

$$|\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle| \lesssim B_0^{2\hbar+2}\|\chi_{B_0}\Lambda Q\|_{L^2}\left(\int \frac{|\mathscr{L}^L q|^2}{1+y^{4\hbar+4}}\right)^{\frac{1}{2}} \lesssim B_0^{\frac{d}{2}-\gamma+2\hbar+2}\sqrt{\mathscr{E}_{2\Bbbk}}. \tag{4-11}$$

We write

$$\begin{aligned}\left|\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle \frac{d}{ds}\left[\frac{1}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right]\right| &\lesssim \frac{|\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle|}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle^2}\left|\frac{(b_1)_s}{b_1}\right| \int_{B_0 \le y \le 2B_0} |\Lambda Q|^2 \\ &\lesssim b_1 \frac{B_0^{\frac{d}{2}-\gamma+2\hbar+2}\sqrt{\mathscr{E}_{2\Bbbk}}}{B_0^{2d-4\gamma}} B_0^{d-2\gamma} \lesssim \frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{B_0^{2\delta}},\end{aligned}$$

where we used the relation (1-18).

For the first three terms in the right-hand side of (4-9), we use Cauchy–Schwarz, Lemma A.5 and the fact that $\mathscr{L}(\Lambda Q) = 0$ to find that

$$|\langle \mathscr{L}^L q, \Lambda Q \partial_s(\chi_{B_0})\rangle| \lesssim \left|\frac{(b_1)_s}{b_1}\right| \left(\int_{B_0 \le y \le 2B_0} (1+y^{4\hbar+4})|\Lambda Q|^2\right)^{\frac{1}{2}} \left(\int \frac{|\mathscr{L}^L q|^2}{1+y^{4\hbar+4}}\right)^{\frac{1}{2}}$$
$$\lesssim b_1 B_0^{\frac{d}{2}-\gamma+2\hbar+2}\sqrt{\mathscr{E}_{2\Bbbk}} \lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}},$$

$$|\langle \mathscr{L}^{L+1} q, \chi_{B_0}\Lambda Q\rangle| \lesssim \left(\int (1+y^{4\hbar})|\chi_{B_0}\Lambda Q|^2\right)^{\frac{1}{2}} \left(\int \frac{|\mathscr{L}^{L+1} q|^2}{1+y^{4\hbar}}\right)^{\frac{1}{2}} \lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}}$$

and

$$\left|\frac{\lambda_s}{\lambda}\langle \mathscr{L}^L \Lambda q, \chi_{B_0}\Lambda Q\rangle\right| \lesssim b_1 \left(\int (1+y^{4(L+\hbar)+2})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac{1}{2}} \left(\int \frac{|\partial_y q|^2}{1+y^{4(L+\hbar)+2}}\right)^{\frac{1}{2}}$$
$$\lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}}.$$

The error term is estimated by using (2-46):

$$|\langle \mathscr{L}^L \widetilde{\Psi}_b, \chi_{B_0}\Lambda Q\rangle| \lesssim \left(\int (1+y^{4(L+\hbar+1)})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac{1}{2}} \left(\int \frac{|\widetilde{\Psi}_b|^2}{1+y^{4(L+\hbar+1)}}\right)^{\frac{1}{2}}$$
$$\lesssim B_0^{\frac{d}{2}-\gamma+2\hbar+2} b_1^{L+2+(1-\delta)-C_L\eta}.$$

The last term in the right-hand side of (4-9) is estimated in the same way:

$$|\langle \mathscr{L}^L(\mathcal{L}(q) - \mathcal{N}(q)), \chi_{B_0}\Lambda Q\rangle| \lesssim \int |\mathcal{L}(q)\mathscr{L}^L(\chi_{B_0}\Lambda Q)| + \int |\mathcal{N}(q)\mathscr{L}^L(\chi_{B_0}\Lambda Q)|$$
$$\lesssim \left(\int \frac{|\mathcal{L}(q)|^2}{1+y^{4\Bbbk-4}}\right)^{\frac{1}{2}} \left(\int (1+y^{4\Bbbk-4})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac{1}{2}}$$
$$+ \left(\int \frac{|\mathcal{N}(q)|^2}{1+y^{4\Bbbk}}\right)^{\frac{1}{2}} \left(\int (1+y^{4\Bbbk})|\mathscr{L}^L(\chi_{B_0}\Lambda Q)|^2\right)^{\frac{1}{2}}$$
$$\lesssim B_0^{\frac{d}{2}-1-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}} + b_1 B_0^2 B_0^{\frac{d}{2}-1-\gamma+2\hbar}\sqrt{\mathscr{E}_{\Bbbk}}$$
$$\lesssim B_0^{\frac{d}{2}-\gamma+2\hbar}\sqrt{\mathscr{E}_{2\Bbbk}}.$$

For the remaining term, we recall that $\mathscr{L}(\Lambda Q) = 0$, $\mathscr{L}^L T_k = 0$ for $1 \le k \le L-1$, and $\mathscr{L}^L T_L = (-1)^L \Lambda Q$, from which

$$\mathscr{L}^L(T_k \chi_{B_1}) = -\mathscr{L}^L(T_k(1-\chi_{B_1})), \quad 1 \le k \le L-1.$$

From (3-10), (2-24) and the fact that $\chi_{B_0}(1-\chi_{B_1}) = 0$, we write

$$\left|\langle \mathscr{L}^L \widehat{\mathrm{Mod}}(t), \chi_{B_0}\Lambda Q\rangle - (-1)^L\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle[(b_L)_s + (2L-\gamma)b_1 b_L]\right|$$
$$\lesssim \sum_{k=1}^{L} |(b_k)_s + (2k-\gamma)b_1 b_L - b_{k+1}| \left|\left\langle \sum_{j=k+1}^{L+2} \frac{\partial \widetilde{S}_j}{\partial b_k}, \mathscr{L}^L(\chi_{B_0}\Lambda Q)\right\rangle\right| + \left|\frac{\lambda_s}{\lambda} + b_1\right| |\langle \Lambda \widetilde{\Theta}_b, \mathscr{L}^L(\chi_{B_0}\Lambda Q)\rangle|.$$

Recalling that T_k is admissible of degree (k,k) and S_k is homogeneous of degree $(k,k-1,k)$, we derive the round bounds for $y\sim B_0$:

$$|\Lambda\Theta_b|\lesssim b_1 y^{2-\gamma},\qquad \sum_{j=k+1}^{L+2}\left|\frac{\partial S_j}{\partial b_k}\right|\le\sum_{j=k+1}^{L+2} b_1^{j-k}y^{2(j-1)-\gamma}\lesssim b_1 y^{2k-\gamma}.$$

Thus, from Lemma 4.2, we derive the bound

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}+b_1\right||\langle\Lambda\widetilde{\Theta}_b,\mathscr{L}^L(\chi_{B_0}\Lambda Q)\rangle|+\sum_{k=1}^{L}|(b_k)_s+(2k-\gamma)b_1b_L-b_{k+1}|\left|\left\langle\sum_{j=k+1}^{L+2}\frac{\partial\widetilde{S}_j}{\partial b_k},\mathscr{L}^L(\chi_{B_0}\Lambda Q)\right\rangle\right|\\ \lesssim\left(C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)}\right)b_1\int_{B_0\le y\le 2B_0}\frac{y^{2L-\gamma}y^{d-1}}{y^{2L+\gamma}}\,dy\\ \lesssim\left(C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)}\right)b_1B_0^{d-2\gamma}.\end{aligned}$$

The equation (4-8) follows by gathering all the above estimates into (4-9), dividing both sides of (4-9) by $(-1)^L\langle\Lambda Q,\chi_{B_0}\Lambda Q\rangle$ and using the relation (1-18). ☐

4B. *Monotonicity.* We derive in this subsection the main monotonicity formula for $\mathscr{E}_{2k}$ for $\hbar+1\le k\le\Bbbk$. We claim the following which is the heart of this paper:

Proposition 4.4 (Lyapounov monotonicity for the high Sobolev norm). *We have*

$$\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}\left[1+\mathcal{O}(b_1^{\eta(1-\delta)})\right]\right\}\le\frac{b_1}{\lambda^{4\Bbbk-d+2}}\left[\frac{\mathscr{E}_{2\Bbbk}}{M^{2\delta}}+b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{2L+2(1-\delta)(1+\eta)}\right],\tag{4-12}$$

and for $\hbar+2\le m\le\Bbbk-1$,

$$\frac{d}{dt}\left\{\frac{\mathscr{E}_{2m}}{\lambda^{4m-d}}[1+\mathcal{O}(b_1)]\right\}\le\frac{b_1}{\lambda^{4m-d+2}}\left[b_1^{m-\hbar-1+(1-\delta)-C\eta}\sqrt{\mathscr{E}_{2m}}+b_1^{2(m-\hbar-1)+2(1-\delta)-C\eta}\right].\tag{4-13}$$

Proof. The proof uses some ideas developed in [Raphaël and Schweyer 2014b; Merle, Raphaël and Rodnianski 2015]. Because the proof of (4-13) follows exactly the same lines as for (4-12), we only deal with the proof of (4-12). Let us start the proof of (4-12).

Step 1: suitable derivatives and energy identity. For $k\in\mathbb{N}$, we define the suitable derivatives of q and v as follows:

$$q_{2k}=\mathscr{L}^kq,\quad q_{2k+1}=\mathscr{A}\mathscr{L}^kq,\quad v_{2k}=\mathscr{L}_\lambda^kv,\quad v_{2k+1}=\mathscr{A}_\lambda\mathscr{L}_\lambda^kv,\tag{4-14}$$

where $q=q(y,s)$ and $v=v(r,t)$ satisfy (3-9) and (3-14) respectively, the linearized operator $\mathscr{L}$ and $\mathscr{L}_\lambda$ are defined by (1-22) and (3-13), $\mathscr{A}$ and $\mathscr{A}^*$ are the first-order operators defined by (2-6) and (2-7), and

$$\mathscr{A}_\lambda f=-\partial_rf+\frac{V_\lambda}{r}f,\qquad\mathscr{A}_\lambda^*f=\frac{1}{r^{d-1}}\partial_r(r^{d-1}f)+\frac{V_\lambda}{r}f,$$

with $V=\Lambda\log\Lambda Q$ admitting the asymptotic behaviors as in (2-8).

With the notation (4-14), we note that

$$q_{2k+1}=\mathscr{A}q_{2k},\quad q_{2k+2}=\mathscr{A}^*q_{2k+1},\quad v_{2k+1}=\mathscr{A}_\lambda v_{2k},\quad v_{2k+2}=\mathscr{A}_\lambda^*v_{2k+1}.$$

Recall from Lemma 2.2, we have the factorization

$$\mathscr{L}=\mathscr{A}^*\mathscr{A},\quad \widetilde{\mathscr{L}}=\mathscr{A}\mathscr{A}^*,\quad \mathscr{L}_\lambda=\mathscr{A}_\lambda^*\mathscr{A}_\lambda,\quad \widetilde{\mathscr{L}}_\lambda=\mathscr{A}_\lambda\mathscr{A}_\lambda^*,$$

where

$$\widetilde{\mathscr{L}}=-\partial_{yy}-\frac{d-1}{y}\partial_y+\frac{\widetilde{Z}}{y^2},\tag{4-15}$$

$$\widetilde{\mathscr{L}}_\lambda=-\partial_{rr}-\frac{d-1}{r}\partial_r+\frac{\widetilde{Z}_\lambda}{r^2},\tag{4-16}$$

with $\widetilde{Z}$ expressed in terms of V as in (2-13).

We commute (3-14) with $\mathscr{L}_\lambda^{\Bbbk-1}$ and use the notation (4-14) to derive

$$\partial_t v_{2\Bbbk-2}+\mathscr{L}_\lambda v_{2\Bbbk-2}=[\partial_t,\mathscr{L}_\lambda^{\Bbbk-1}]v+\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right).\tag{4-17}$$

Now commuting this equation with $\mathscr{A}_\lambda$ yields

$$\partial_t v_{2\Bbbk-1}+\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}=\frac{\partial_t V_\lambda}{r}v_{2\Bbbk-2}+\mathscr{A}_\lambda[\partial_t,\mathscr{L}_\lambda^{\Bbbk-1}]v+\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right).\tag{4-18}$$

Since $\mathscr{L}_\lambda=(1/\lambda^2)\mathscr{L}$, we then have

$$\mathscr{L}_\lambda^k v=\frac{1}{\lambda^{2k}}\mathscr{L}^k q;$$

hence,

$$\int|\mathscr{L}_\lambda^k v|^2=\frac{1}{\lambda^{4k-d}}\int|\mathscr{L}^k q|^2.$$

Using the definition (4-16) of $\widetilde{\mathscr{L}}_\lambda$ and an integration by parts, we write

$$\begin{aligned}\frac12\frac{d}{dt}\left(\frac{1}{\lambda^{4\Bbbk-d}}\mathscr{E}_{2\Bbbk}\right)&=\frac12\frac{d}{dt}\int|\mathscr{L}_\lambda^{\Bbbk}v|^2=\frac12\frac{d}{dt}\int\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}v_{2\Bbbk-1}\\&=\int\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\partial_t v_{2\Bbbk-1}+\frac12\int\frac{\partial_t(\widetilde{Z}_\lambda)}{r^2}v_{2\Bbbk-1}^2\\&=\int\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\partial_t v_{2\Bbbk-1}+b_1\int\frac{(\Lambda\widetilde{Z})_\lambda}{2\lambda^2r^2}v_{2\Bbbk-1}^2-\left(\frac{\lambda_s}{\lambda}+b_1\right)\int\frac{(\Lambda\widetilde{Z})_\lambda}{2\lambda^2r^2}v_{2\Bbbk-1}^2.\end{aligned}$$

Using the definition (2-7) of $\mathscr{A}^*$ and an integration by parts together with the definition (2-13) of $\widetilde{Z}$, we write

$$\begin{aligned}\int\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}v_{2\Bbbk-1}\mathscr{A}_\lambda^* v_{2\Bbbk-1}&=\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int\frac{\Lambda V}{y}q_{2\Bbbk-1}\mathscr{A}^*q_{2\Bbbk-1}\\&=\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int\frac{\Lambda V(2V+d)-\Lambda^2V}{2y^2}q_{2\Bbbk-1}^2\\&=\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int\frac{\Lambda\widetilde{Z}}{2y^2}q_{2\Bbbk-1}^2=\int\frac{b_1(\Lambda\widetilde{Z})_\lambda}{2\lambda^2r^2}v_{2\Bbbk-1}^2.\end{aligned}$$

From (4-17), we write

$$\frac{d}{dt}\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}v_{2\Bbbk-2} = \int \frac{d}{dt}\Big(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\Big) v_{2\Bbbk-1}v_{2\Bbbk-2} + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\partial_t v_{2\Bbbk-1} + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\Big[-\mathscr{A}_\lambda^* v_{2\Bbbk-1} + [\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{L}_\lambda^{\Bbbk-1}\Big(\frac{1}{\lambda^2}\mathcal{F}_\lambda\Big)\Big].$$

Gathering all the above identities and using (4-18) yields the energy identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big\{\Big(\frac{1}{\lambda^{4\Bbbk-d}}\mathscr{E}_{2\Bbbk}\Big) + 2\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}v_{2\Bbbk-2}\Big\} &= -\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 - \Big(\frac{\lambda_s}{\lambda}+b_1\Big)\int \frac{(\Lambda \widetilde{Z})_\lambda}{2\lambda^2 r^2} v_{2\Bbbk-1}^2 - \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} \\ &\quad + \int \frac{d}{dt}\Big(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\Big) v_{2\Bbbk-1}v_{2\Bbbk-2} + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\Big[[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{L}_\lambda^{\Bbbk-1}\Big(\frac{1}{\lambda^2}\mathcal{F}_\lambda\Big)\Big] \\ &\quad + \int \Big(\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\Big)\Big[\frac{\partial_t V_\lambda}{r} v_{2\Bbbk-2} + \mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v + \mathscr{A}_\lambda \mathscr{L}_\lambda^{\Bbbk-1}\Big(\frac{1}{\lambda^2}\mathcal{F}_\lambda\Big)\Big]. \end{aligned} \tag{4-19}$$

We now estimate all terms in (4-19). The proof uses the coercivity estimate given in Lemma A.5. In particular, we shall apply Lemma A.5 with $k = \Bbbk - 1$ to get the estimate

$$\mathscr{E}_{2\Bbbk} \gtrsim \int \frac{|q_{2\Bbbk-1}|^2}{y^2} + \sum_{m=0}^{\Bbbk-1}\int \frac{|q_{2m}|^2}{y^4(1+y^{4(\Bbbk-1-m)})} + \sum_{m=0}^{\Bbbk-2}\int \frac{|q_{2m+1}|^2}{y^6(1+y^{4(\Bbbk-2-m)})}. \tag{4-20}$$

<u>Step 2</u>: control of the lower-order quadratic terms. Let us start with the second term in the left-hand side of (4-19). From (2-8) and (2-13), we have the round bound

$$|\Lambda \widetilde{Z}(y)| + |\Lambda V(y)| \lesssim \frac{y^2}{1+y^4} \quad \text{for all } y \in [0, +\infty). \tag{4-21}$$

Making a change of variables and using the Cauchy–Schwarz inequality together with (4-20), we estimate

$$\begin{aligned}\Big|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}v_{2\Bbbk-2}\Big| &= \Big|\frac{b_1}{\lambda^{4\Bbbk-d}}\int \frac{\Lambda V}{y} q_{2\Bbbk-1}q_{2\Bbbk-2}\Big| \\ &\lesssim \frac{b_1}{\lambda^{4\Bbbk-d}}\Big(\int \frac{|q_{2\Bbbk-1}|^2}{y^2}\Big)^{\frac{1}{2}}\Big(\int \frac{|q_{2\Bbbk-2}|^2}{1+y^4}\Big)^{\frac{1}{2}} \lesssim \frac{b_1}{\lambda^{4\Bbbk-d}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

Using (4-21), (4-1) and (4-20), we estimate

$$\begin{aligned}\Big|\Big(\frac{\lambda_s}{\lambda}+b_1\Big)\int \frac{(\Lambda \widetilde{Z})_\lambda}{\lambda^2 r} v_{2\Bbbk-1}^2\Big| &= \Big|\Big(\frac{\lambda_s}{\lambda}+b_1\Big)\frac{1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda \widetilde{Z}}{y^2} q_{2\Bbbk-1}^2\Big| \\ &\lesssim \frac{b_1^{L+1+(1-\delta)(1+\eta)}}{\lambda^{4\Bbbk-d+2}}\int \frac{q_{2\Bbbk-1}^2}{y^2} \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

For the third term in the right-hand side of (4-19), we write

$$\begin{aligned}\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\right| &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + 4\int\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\right)^2 v_{2\Bbbk-2}^2\\ &= \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{4b_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{|\Lambda V|^2}{y^2} q_{2\Bbbk-2}^2\\ &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

A direct computation yields the round bound

$$\left|\frac{d}{dt}\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2}\right)\right| \lesssim \frac{b_1^2}{\lambda^4}(|\Lambda V| + |\Lambda^2 V|).$$

Thus, we use (4-21), the Cauchy–Schwarz inequality and (4-20) to estimate

$$\begin{aligned}\left|\int \frac{d}{dt}\left(\frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}\right) v_{2\Bbbk-1}v_{2\Bbbk-2}\right| &\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{|\Lambda V| + |\Lambda^2 V|}{y}|q_{2\Bbbk-1}q_{2\Bbbk-2}|\\ &\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\left(\int \frac{q_{2\Bbbk-1}^2}{y^2}\right)^{\frac{1}{2}}\left(\int \frac{q_{2\Bbbk-2}^2}{1+y^4}\right)^{\frac{1}{2}}\\ &\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\int\left(\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}v_{2\Bbbk-2}\right)\frac{\partial_t V_\lambda}{r}v_{2\Bbbk-2}\right| &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{|\Lambda V|^2}{y^2}q_{2\Bbbk-2}^2\\ &\le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}\end{aligned}$$

and

$$\begin{aligned}&\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}v_{2\Bbbk-1}[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v\right| + \left|\int\left(\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r}v_{2\Bbbk-2}\right)\mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v\right|\\ &\qquad \le \frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + C\left(\frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk} + \int \frac{|[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2}{\lambda^2(1+y^2)} + \int |\mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2\right).\end{aligned}$$

We claim the bound

$$\int \frac{|[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2}{\lambda^2(1+y^2)} + \int |\mathscr{A}_\lambda[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}, \tag{4-22}$$

whose proof is left to Appendix C.

The collection of all the above estimates to (4-19) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\} \le -\frac{1}{4}\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk} \\ + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) \\ + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) \\ + \int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right).\end{aligned} \tag{4-23}$$

Step 3: further use of dissipation. We aim to estimate all terms in the right-hand side of (4-23). From (4-21), (4-20) and the Cauchy–Schwarz inequality, we write

$$\begin{aligned}\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-1}\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right| &= \left|\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda V}{y} q_{2\Bbbk-1}\mathscr{L}^{\Bbbk-1}\mathcal{F}\right| \\ &\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\int \frac{q_{2\Bbbk-1}^2}{y^2}\right)^{\frac{1}{2}}\left(\int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^4}\right)^{\frac{1}{2}} \\ &\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\sqrt{\mathscr{E}_{2\Bbbk}}\left(\int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^4}\right)^{\frac{1}{2}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2\Bbbk-2}\mathscr{A}_\lambda\mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right| &= \left|\frac{b_1}{\lambda^{4\Bbbk-d+2}}\int \frac{\Lambda V}{y} q_{2\Bbbk-2}\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}\right| \\ &\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\int \frac{q_{2\Bbbk-2}^2}{1+y^4}\right)^{\frac{1}{2}}\left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^2}\right)^{\frac{1}{2}} \\ &\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\sqrt{\mathscr{E}_{2\Bbbk}}\left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^2}\right)^{\frac{1}{2}}.\end{aligned}$$

For the last term in (4-23), let us introduce the function

$$\xi_L = \frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\widetilde{T}_L \tag{4-24}$$

and the decomposition

$$\mathcal{F} = \partial_s\xi_L + \mathcal{F}_0 + \mathcal{F}_1, \quad \mathcal{F}_0 = -\widetilde{\Psi}_b - \widehat{\mathrm{Mod}} - \partial_s\xi_L, \quad \mathcal{F}_1 = \mathcal{H}(q) - \mathcal{N}(q), \tag{4-25}$$

where $\widetilde{\Psi}_b$ is as referred to in (2-42), and $\widehat{\mathrm{Mod}}$, $\mathcal{H}(q)$ and $\mathcal{N}(q)$ are as defined in (3-10) (3-11) and (3-12) respectively. Actually, we introduced the decomposition (4-25) and ξ_L to take advantage of the improved bound obtained in Lemma 4.3. We now write

$$\begin{aligned}
&\int \widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}\mathscr{A}_\lambda \mathscr{L}_\lambda^{\Bbbk-1}\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\\
&\quad= \frac{1}{\lambda^{4\Bbbk-d+2}}\left(\int \mathscr{A}^* q_{2\Bbbk-1}\mathscr{L}^{\Bbbk}(\partial_s \xi_L) + \int \mathscr{A}^* q_{2\Bbbk-1}\mathscr{L}^{\Bbbk}\mathcal{F}_0 + \int \widetilde{\mathscr{L}} q_{2\Bbbk-1}\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\right)\\
&\quad\le \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}(\partial_s\xi_L) + \frac{C}{\lambda^{4\Bbbk-d+2}}\left(\int |\mathscr{L}^{\Bbbk} q|^2\right)^{\frac12}\left(\int |\mathscr{L}^{\Bbbk}\mathcal{F}_0|\right)^{\frac12}\\
&\qquad\qquad + \frac18 \int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{C}{\lambda^{4\Bbbk-d+2}}\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1|^2\\
&\quad= \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}(\partial_s\xi_L) + \frac18\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2\\
&\qquad\qquad + \frac{C}{\lambda^{4\Bbbk-d+2}}\left(\sqrt{\mathscr{E}_{2\Bbbk}}\|\mathscr{L}^{\Bbbk}\mathcal{F}_0\|_{L^2} + \|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\|_{L^2}^2\right).
\end{aligned}$$

Injecting all these bounds into (4-23) yields

$$\begin{aligned}
\frac12 \frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\} &\le -\frac18\int |\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2 + \frac{Cb_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk} + \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q\mathscr{L}^{\Bbbk}(\partial_s\xi_L)\\
&\quad + \frac{b_1}{\lambda^{4\Bbbk-d+2}}\sqrt{\mathscr{E}_{2\Bbbk}}\left[\left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^2}\right)^{\frac12} + \left(\int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}|^2}{1+y^4}\right)^{\frac12}\right]\\
&\quad + \frac{C}{\lambda^{4\Bbbk-d+2}}\left(\sqrt{\mathscr{E}_{2\Bbbk}}\|\mathscr{L}^{\Bbbk}\mathcal{F}_0\|_{L^2} + \|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\|_{L^2}^2\right). \qquad (4\text{-}26)
\end{aligned}$$

<u>Step 4</u>: estimates for $\widetilde{\Psi}_b$ term. Recall from (2-44) that we already have the following estimate for $\widetilde{\Psi}_b$:

$$\|\mathscr{L}^{\Bbbk}\widetilde{\Psi}_b\|_{L^2} + \left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\widetilde{\Psi}_b|^2}{1+y^2}\right)^{\frac12} + \left(\int \frac{|\mathscr{L}^{\Bbbk-1}\widetilde{\Psi}_b|^2}{1+y^4}\right)^{\frac12} \lesssim b_1^{L+1+(1-\delta)(1+\eta)}. \qquad (4\text{-}27)$$

<u>Step 5</u>: estimates for $\widehat{\mathrm{Mod}}$ term. We claim the following:

$$\left(\int \frac{|\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2}{1+y^4}\right)^{\frac12} + \left(\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2}{1+y^2}\right)^{\frac12} \lesssim b_1^{(1-\delta)(1+\eta)}\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)}\right), \qquad (4\text{-}28)$$

$$\left(\int |\mathscr{L}^{\Bbbk}\widetilde{\mathrm{Mod}}|^2\right)^{\frac12} \lesssim b_1\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{\eta(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}} + b_1^{L+1+(1-\delta)(1+\eta)}\right), \qquad (4\text{-}29)$$

where

$$\widetilde{\mathrm{Mod}} = \widehat{\mathrm{Mod}} + \partial_s\xi_L.$$

Let us prove (4-28). We only deal with the first term since the second term is estimated similarly. We recall from (3-10) the definition of $\widehat{\mathrm{Mod}}$:

$$\widehat{\mathrm{Mod}} = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-\gamma)b_1 b_i - b_{i+1}]\left(\widetilde{T}_i + \sum_{j=i+1}\frac{\partial S_j}{\partial b_i}\chi_{B_1}\right),$$

where $\widetilde{Q}_b$ is defined as in (2-41) and we know from Lemma 2.9 that T_i is admissible of degree (i,i) and from Proposition 2.11 that S_j is homogeneous of degree $(j, j-1, j)$.

Since $|b_j| \lesssim b_1^j$ and $\mathscr{L}\Lambda Q = 0$, we use Lemma 2.8 to estimate

$$\begin{aligned}\int \frac{|\mathscr{L}^{\Bbbk-1}\Lambda\widetilde{Q}_b|^2}{1+y^4} &\lesssim \sum_{i=1}^{L} b_i^2 \int \frac{|\mathscr{L}^{\Bbbk-1}\Lambda\widetilde{T}_i|^2}{1+y^4} + \sum_{i=2}^{L+2}\int \frac{|\mathscr{L}^{\Bbbk-1}\Lambda\widetilde{S}_i|^2}{1+y^4}\\ &\lesssim \sum_{i=1}^{L} b_1^{2i}\int_{y\le 2B_1}\frac{y^{d-1}\,dy}{1+y^{4(\Bbbk-i)+2\gamma}} + \sum_{i=2}^{L+1} b_1^{2i}\int_{y\le 2B_1}\frac{y^{d-1}\,dy}{1+y^{4(\Bbbk-i+1)+2\gamma}} + b_1^{2L+4}\int_{y\le 2B_1}\frac{y^{d-1}\,dy}{1+y^{4\hbar+2\gamma}}\\ &\lesssim b_1^2,\end{aligned}$$

where we used the algebra $4(\Bbbk-L)+2\gamma-d+1 = 5-4\delta > 1$.

Using the cancellation $\mathscr{L}^{\Bbbk}T_i = 0$ for $1\le i\le L$ and the admissibility of T_i, we estimate

$$\sum_{i=1}^{L}\int\frac{|\mathscr{L}^{\Bbbk-1}(\chi_{B_1}T_i)|^2}{1+y^4} \lesssim \sum_{i=1}^{L}\int_{B_1\le y\le 2B_1} y^{4(i-\Bbbk)-2\gamma+d-1}\,dy \lesssim b_1^{2(1-\delta)(1+\eta)}.$$

Using the homogeneity of S_j, we estimate for $1\le i\le L$,

$$\sum_{j=i+1}^{L+2}\int\frac{1}{1+y^4}\left|\mathscr{L}^{\Bbbk-1}\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right)\right|^2 \lesssim \sum_{j=i+1}^{L+2} b_1^{2(j-i)}\int_{B_1\le y\le 2B_1} y^{4(j-1-\Bbbk)-2\gamma}y^{d-1}\,dy \lesssim b_1^2,$$

provided that $\eta \le \frac{1}{\delta}-1$.

The collection of the above bounds together with (4-1) and (4-2) yields

$$\left(\int\frac{|\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2}{1+y^4}\right)^{\frac{1}{2}} \lesssim b_1^{(1-\delta)(1+\eta)}\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}} + b_1^{L+1+(1-\delta)(1+\eta)}\right).$$

The same estimate holds for $\left(\int|\mathscr{A}\mathscr{L}^{\Bbbk-1}\widehat{\mathrm{Mod}}|^2/(1+y^2)\right)^{1/2}$ by following the same lines as above. This concludes the proof of (4-28).

We now prove (4-29). Let us write

$$\begin{aligned}\widetilde{\mathrm{Mod}} = -\left(\frac{\lambda_s}{\lambda}+b_1\right)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L-1}[(b_i)_s + (2i-\gamma)b_1b_i - b_{i+1}]\widetilde{T}_i\\ + \sum_{i=1}^{L}[(b_i)_s+(2i-\gamma)b_1b_i - b_{i+1}]\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\\ + \left[(b_L)_s + (2i-\gamma)b_1b_L + \frac{d}{ds}\left\{\frac{\langle\mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle\Lambda Q,\chi_{B_0}\Lambda Q\rangle}\right\}\right]\widetilde{T}_L + \frac{\langle\mathscr{L}^L q,\chi_{B_0}\Lambda Q\rangle}{\langle\Lambda Q,\chi_{B_0}\Lambda Q\rangle}\partial_s\widetilde{T}_L.\end{aligned}$$

Proceeding as in the proof of (4-28) yields the estimate

$$\int |\mathscr{L}^{\Bbbk}\Lambda\widetilde{Q}_b|^2+\sum_{i=1}^{L-1}\int|\mathscr{L}^{\Bbbk}\widetilde{T}_i|^2+\sum_{i=1}^{L}\sum_{j=i+1}^{L+2}\int\left|\mathscr{L}^{\Bbbk}\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right)\right|^2\lesssim b_1^2,$$

and

$$\int|\mathscr{L}^{\Bbbk}\widetilde{T}_L|^2\lesssim b_1^{2(1-\delta)(1+\eta)}. \tag{4-30}$$

From (4-10) and (4-11), we have the bound

$$\left|\frac{\langle\mathscr{L}^L q,\chi_{B_0}\Lambda Q\rangle}{\langle\Lambda Q,\chi_{B_0}\Lambda Q\rangle}\right|\lesssim B_0^{2(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}=b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}. \tag{4-31}$$

We also have

$$\int|\mathscr{L}^{\Bbbk}(\partial_s\chi_{B_1}T_L)|^2\lesssim b_1^2\int_{B_1\le y\le 2B_1}\frac{y^{d-1}\,dy}{y^{4(\Bbbk-L)+2\gamma}}\lesssim b_1^2b_1^{2(1-\delta)(1+\eta)}.$$

The collection of the above bounds together with Lemmas 4.2 and 4.3 yields

$$\begin{aligned}\left(\int|\mathscr{L}^{\Bbbk}\widetilde{\mathrm{Mod}}|^2\right)^{\frac12}&\lesssim b_1\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}}+b_1^{L+1+(1-\delta)(1+\eta)}\right)\\&\quad+b_1^{(1-\delta)(1+\eta)}b_1^{\delta}(C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)})\\&\quad+b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}b_1b_1^{(1-\delta)(1+\eta)}\\&\lesssim b_1\left(\frac{\sqrt{\mathscr{E}_{2\Bbbk}}}{M^{2\delta}}+b_1^{\eta(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+1+(1-\delta)(1+\eta)}\right),\end{aligned}$$

which is the conclusion of (4-29).

Injecting the estimates (4-27), (4-28) and (4-29) into (4-26), we arrive at

$$\begin{aligned}\frac12\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\}&\le-\frac18\int|\widetilde{\mathscr{L}}_\lambda v_{2\Bbbk-1}|^2+\frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\frac{\mathscr{E}_{2\Bbbk}}{M^{2\delta}}+b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}+b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}\right)\\&\quad+\frac{b_1\sqrt{\mathscr{E}_{2\Bbbk}}}{\lambda^{4\Bbbk-d+2}}\left[\left(\int\frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1|^2}{1+y^2}\right)^{\frac12}+\left(\int\frac{|\mathscr{L}^{\Bbbk-1}\mathcal{F}_1|^2}{1+y^4}\right)^{\frac12}\right]\\&\quad+\frac{1}{\lambda^{4\Bbbk-d+2}}\|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{F}_1\|_{L^2}^2+\frac{1}{\lambda^{4\Bbbk-d+2}}\int\mathscr{L}^{\Bbbk}q\mathscr{L}^{\Bbbk}(\partial_s\xi_L).\end{aligned} \tag{4-32}$$

<u>Step 6</u>: estimates for the linear small term $\mathcal{H}(q)$. We claim

$$\int|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2+\int\frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2}{1+y^2}+\frac{|\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2}{1+y^4}\lesssim b_1^2\mathscr{E}_{2\Bbbk}. \tag{4-33}$$

We only deal with the estimate for the first term because the last two terms are estimated similarly. Let us rewrite from (3-11) the definition of $\mathcal{H}(q)$,

$$\mathcal{H}(q)=\Phi q\quad\text{with }\Phi=\frac{(d-1)}{y^2}[\cos(2Q)-\cos(2Q+2\widetilde{\Theta}_b)],$$

where

$$\widetilde{\Theta}_b = \sum_{i=1}^{L} b_i \widetilde{T}_i + \sum_{i=2}^{L+2} \widetilde{S}_i(b, y).$$

From the asymptotic behavior of Q given in (2-4), the admissibility of T_i and the homogeneity of S_i, we deduce that Φ is a regular function both at the origin and at infinity. We then apply the Leibniz rule (C-2) with $k = \Bbbk - 1$ and $\phi = \Phi$ to write

$$\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q) = \sum_{m=0}^{\Bbbk-1} [q_{2m+1}\Phi_{2\Bbbk-1,2m+1} + q_{2m}\Phi_{2\Bbbk-1,2m}],$$

where $\Phi_{2\Bbbk-1,i}$ with $0 \le i \le 2\Bbbk - 1$ are defined by the recurrence relation given in Lemma C.1. In particular, we have the estimate

$$|\Phi_{k,i}| \lesssim \frac{b_1}{1 + y^{\gamma+(k-i)}} \lesssim \frac{b_1}{1 + y^{1+k-i}} \quad \text{for all } k \ge 1,\ \ 0 \le i \le k.$$

Hence, we estimate from (4-20),

$$\begin{aligned}
\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{H}(q)|^2 &\lesssim \sum_{m=0}^{\Bbbk-1} \Big[\int |q_{2m+1}\Phi_{2\Bbbk-1,2m+1}|^2 + \int |q_{2m}\Phi_{2\Bbbk-1,2m}|^2\Big] \\
&\lesssim b_1^2 \sum_{m=0}^{\Bbbk-1} \Big[\int \frac{|q_{2m+1}|^2}{1 + y^{2+2(2\Bbbk-1-2m-1)}} + \int \frac{|q_{2m}|^2}{1 + y^{2+2(2\Bbbk-1-2m)}}\Big] \\
&\lesssim b_1^2 \sum_{m=0}^{\Bbbk-1} \Big[\int \frac{|q_{2m+1}|^2}{1 + y^{2+4(\Bbbk-1-m)}} + \int \frac{|q_{2m}|^2}{1 + y^{4+4(\Bbbk-1-m)}}\Big] \lesssim b_1^2 \mathscr{E}_{2\Bbbk}.
\end{aligned}$$

This concludes the proof of (4-33).

<u>Step 7</u>: estimates for the nonlinear term $\mathcal{N}(q)$. This is the most delicate point in the proof of (4-12). We claim the following:

$$\int |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2 \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}, \tag{4-34}$$

$$\int \frac{|\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2}{1 + y^2} + \int \frac{|\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2}{1 + y^4} \lesssim b_1^{2L+2+2(1-\delta)(1+\eta)}, \tag{4-35}$$

provided that η and $1/L$ are small enough. We only deal with the proof of (4-34) since the same proof holds for (4-35).

Control for $y < 1$. Let us rewrite from (3-12) the definition of $\mathcal{N}(q)$:

$$\mathcal{N}(q) = \frac{q^2}{y}\Phi \quad \text{with } \Phi = \Big[-\frac{(d-1)}{y} \int_0^1 (1-\tau)\sin(2\widetilde{Q}_b + 2\tau q)\, d\tau\Big].$$

From (B-2) and the admissibility of T_i, we write

$$\frac{q^2}{y} = \frac{1}{y}\left(\sum_{i=0}^{\Bbbk} c_i T_i(y) + r_q(y)\right)^2 = \sum_{i=0}^{\Bbbk-1} \tilde{c}_i y^{2i+1} + \tilde{r}_q \quad \text{for } y < 1, \tag{4-36}$$

where

$$|\tilde{c}_i| \lesssim \mathscr{E}_{2\Bbbk}, \qquad |\partial_y^j \tilde{r}_q(y)| \lesssim y^{2\Bbbk-\frac{d}{2}-j} |\ln y|^{\Bbbk} \mathscr{E}_{2\Bbbk}, \quad 0 \le j \le 2\Bbbk-1, \ \ y < 1.$$

Let $\tau \in [0, 1]$ and

$$v_\tau = \widetilde{Q}_b + \tau q.$$

We obtain from Proposition 2.11 and the expansion (B-2),

$$v_\tau = \sum_{i=0}^{\Bbbk-1} \hat{c}_i y^{2i+1} + \hat{r}_q,$$

with

$$|\hat{c}_i| \lesssim 1, \qquad |\partial_y^j \hat{r}_q| \lesssim y^{2\Bbbk-\frac{d}{2}-j} |\ln y|^{\Bbbk}, \quad 0 \le j \le 2\Bbbk-1, \ \ y < 1.$$

Together with the Taylor expansion of $\sin(x)$ at $x = 0$, we write

$$\Phi(q) = \sum_{i=0}^{\Bbbk-1} \bar{c}_i y^{2i} + \bar{r}_q, \tag{4-37}$$

with

$$|\bar{c}_i| \lesssim 1, \qquad |\partial_y^j \bar{r}_q| \lesssim y^{2\Bbbk-\frac{d}{2}-1-j} |\ln y|^{\Bbbk}, \quad 0 \le j \le 2\Bbbk-1, \ \ y < 1.$$

From (4-36) and (4-37), we have the expansion of $\mathcal{N}$ near the origin,

$$\mathcal{N}(q) = \sum_{i=0}^{\Bbbk-1} \hat{\tilde{c}}_i y^{2i+1} + \hat{\tilde{r}}_q,$$

with

$$|\hat{\tilde{c}}_i| \lesssim \mathscr{E}_{2\Bbbk}, \qquad |\partial_y^j \hat{\tilde{r}}_q| \lesssim y^{2\Bbbk-\frac{d}{2}-j} |\ln y|^{\Bbbk} \mathscr{E}_{2\Bbbk}, \quad 0 \le j \le 2\Bbbk-1, \ \ y < 1.$$

From the definitions of $\mathscr{A}$ and $\mathscr{A}^*$, see (2-6) and (2-7), one can check that for $y < 1$,

$$|\mathscr{A}\mathscr{L}^{\Bbbk-1} \hat{\tilde{r}}_q| \lesssim \sum_{i=0}^{2\Bbbk-1} \frac{\partial_y^i \hat{\tilde{r}}_q}{y^{2\Bbbk-1-i}} \lesssim \mathscr{E}_{2\Bbbk} \sum_{i=0}^{2\Bbbk-1} \frac{y^{2\Bbbk-\frac{d}{2}-i} |\ln y|^{\Bbbk}}{y^{2\Bbbk-1-i}} \lesssim y^{-\frac{d}{2}+1} |\ln y|^{\Bbbk} \mathscr{E}_{2\Bbbk}.$$

Note from the asymptotic behavior (2-8) of V that $\mathscr{A}(y) = \mathcal{O}(y^2)$ for $y < 1$, which implies

$$\left|\mathscr{A}\mathscr{L}^{\Bbbk-1}\left(\sum_{i=0}^{\Bbbk-1} \hat{\tilde{c}}_i y^{2i+1}\right)\right| \lesssim \sum_{i=0}^{\Bbbk-1} |\hat{\tilde{c}}_i| y^2 \lesssim y^2 \mathscr{E}_{2\Bbbk}.$$

We then conclude

$$\int_{y<1} |\mathscr{A}\mathscr{L}^{\Bbbk-1} \mathcal{N}(q)|^2 \lesssim \mathscr{E}_{2\Bbbk}^2 \int_{y<1} y |\ln y|^{2\Bbbk} \, dy \lesssim \mathscr{E}_{2\Bbbk}^2 \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}.$$

Control for $y \geq 1$. Let us rewrite the definition of $\mathcal{N}(q)$:

$$\mathcal{N}(q) = Z^2\psi, \quad Z = \frac{q}{y}, \quad \psi = -(d-1)\int_0^1 (1-\tau)\sin(2\widetilde{Q}_b + 2\tau q)\,d\tau. \tag{4-38}$$

Note from the definitions of $\mathscr{A}$ and $\mathscr{A}^*$ that

$$\text{for all } k \in \mathbb{N}, \quad |\mathscr{A}\mathscr{L}^k f| \lesssim \sum_{i=0}^{2k+1} \frac{|\partial_y^i f|}{y^{2k+1-i}},$$

from which and the Leibniz rule, we write

$$\begin{aligned}\int_{y\geq 1} |\mathscr{A}\mathscr{L}^{\Bbbk-1}\mathcal{N}(q)|^2 &\lesssim \sum_{k=0}^{2\Bbbk-1}\int_{y\geq 1}\frac{|\partial_y^k\mathcal{N}(q)|^2}{y^{4\Bbbk-2k-2}}\\ &\lesssim \sum_{k=0}^{2\Bbbk-1}\sum_{i=0}^{k}\int_{y\geq 1}\frac{|\partial_y^i Z^2|^2|\partial_y^{k-i}\psi|^2}{y^{4\Bbbk-2k-2}}\\ &\lesssim \sum_{k=0}^{2\Bbbk-1}\sum_{i=0}^{k}\sum_{m=0}^{i}\int_{y\geq 1}\frac{|\partial_y^m Z|^2|\partial_y^{i-m}Z|^2|\partial_y^{k-i}\psi|^2}{y^{4\Bbbk-2k-2}}.\end{aligned}$$

We aim to use the pointwise estimate (B-5) to prove that for $0 \leq k \leq 2\Bbbk-1$, $0 \leq i \leq k$ and $0 \leq m \leq i$,

$$A_{k,i,m} := \int_{y\geq 1}\frac{|\partial_y^m Z|^2|\partial_y^{i-m}Z|^2|\partial_y^{k-i}\psi|^2}{y^{4\Bbbk-2k-2}} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}, \tag{4-39}$$

which concludes the proof of (4-34).

To prove (4-39), we distinguish three cases:

Case I: $k=0$. Since $0 \leq m \leq i \leq k$, we have $k=i=m=0$. Although this is the simplest case, it gives us a basic idea to handle the other cases. From (4-38), it is obvious that $|\psi|$ is uniformly bounded. We write

$$A_{0,0,0} = \int_{y\geq 1}\frac{|q|^4|\psi|^2}{y^{4\Bbbk+2}}y^{d-1}\,dy \lesssim \int_{1\leq y\leq B_0}\frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy + \int_{y\geq B_0}\frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy.$$

Using (B-5), Definition 3.2, $b_1 \sim \frac{1}{s}$ and the fact that $d = 4\hbar + 2\gamma + 4\delta$, see (1-18), we estimate

$$\begin{aligned}\int_{1\leq y\leq B_0}\frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy &\lesssim \left\|\frac{y^{d-2}|q|^2}{y^{2(2\Bbbk-1)}}\right\|_{L^\infty(y>1)}\left\|\frac{y^{d-2}|q|^2}{y^{2(2\ell+2\hbar+3)}}\right\|_{L^\infty(y>1)}\int_{1\leq y\leq B_0} y^{4\ell+5-4\delta-2\gamma}\,dy\\ &\lesssim \mathscr{E}_{2\Bbbk}\mathscr{E}_{2(\ell+\hbar+2)}B_0^{4\ell+6-4\delta-2\gamma}\\ &\lesssim K b_1^{2L+2(1-\delta)(1+\eta)}b_1^{2(\ell+1)+2(1-\delta)-K\eta}b_1^{-2\ell-3+2\delta+\gamma}\\ &\lesssim K b_1^{2L+2(1-\delta)(1+\eta)}b_1^{1+\gamma-K\eta} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}.\end{aligned}$$

For the integral on the domain $y \geq B_0$, let us write

$$\int_{y\geq B_0} \frac{|q|^4}{y^{4\Bbbk+3-d}}\,dy \lesssim \left\| \frac{y^{d-2}|q|^2}{y^{2(2\Bbbk-2\ell-1)}} \right\|_{L^\infty(y>1)} \left\| \frac{y^{d-2}|q|^2}{y^{2(2\ell+2\hbar+1)}} \right\|_{L^\infty(y>1)} \int_{y\geq B_0} \frac{dy}{y^{4\delta+2\gamma-1}}$$
$$\lesssim \mathscr{E}_{2(\Bbbk-\ell)}\mathscr{E}_{2(\ell+\hbar+1)} B_0^{2-4\delta-2\gamma}$$
$$\lesssim b_1^{2(\Bbbk-\ell-\hbar-1)+2(1-\delta)-K\eta} b_1^{2\ell+2(1-\delta)-K\eta} b_1^{2\delta+\gamma-1}$$
$$\lesssim b_1^{2L+2(1-\delta)(1+\eta)} b_1^{1+\gamma-(K+2(1-\delta))\eta} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}.$$

This concludes the proof of (4-39) when $k = i = m = 0$.

Case II: $k \geq 1$ and $k = i$. We first use the Leibniz rule to write

$$\text{for all } l \in \mathbb{N}, \quad |\partial_y^l Z|^2 \lesssim \sum_{j=0}^{l} \frac{|\partial_y^j q|^2}{y^{2+2l-2j}}, \tag{4-40}$$

from which,

$$A_{k,k,m} \lesssim \sum_{j=0}^{m}\sum_{l=0}^{k-m} \int_{y\geq 1} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2}} y^{d-1}\,dy.$$

We claim that for all $(j,l) \in \mathbb{N}^2$ and $1 \leq j+l \leq 2\Bbbk-1$,

$$B_{j,l,0} := \int_{y\geq 1} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2}} y^{d-1}\,dy \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{(\gamma-1)}{2}}, \tag{4-41}$$

which immediately follows from (4-39) for the case when $k = i$.

To prove (4-41), we proceed as for the case $k = 0$ by splitting the integral in two parts as follows:

$$\begin{aligned}
&B_{j,l,0} \\
&= \int_{1\leq y\leq B_0} \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar+6}} y^{7-4\delta-2\gamma}\,dy + \int_{y\geq B_0} \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar}} \frac{dy}{y^{4\delta+2\gamma-1}} \\
&\lesssim \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar+6}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-4} + \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{4\Bbbk-2j-2l+4\hbar}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-1} \\
&= \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{2J_1-2j+2J_2-2l}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-4} + \left\| \frac{(y^{d-2}|\partial_y^j q|^2)(y^{d-2}|\partial_y^l q|^2)}{y^{2J_3-2j+2J_4-2l}} \right\|_{L^\infty(y\geq 1)} b_1^{2\delta+\gamma-1} \\
&:= B_{j,l,0,J_1,J_2} b_1^{2\delta+\gamma-4} + B_{j,l,0,J_3,J_4} b_1^{2\delta+\gamma-1},
\end{aligned}$$

where $J_n (n = 1,2,3,4)$ satisfy

$$J_1 + J_2 = 2\Bbbk + 2\hbar + 3, \quad J_3 + J_4 = 2\Bbbk + 2\hbar.$$

We now estimate $B_{j,l,0,J_1,J_2}$.

- If l is even, we take

$$J_2 = \begin{cases} l+2 & \text{if } l \le 2\Bbbk - 4, \\ l & \text{if } l = 2\Bbbk - 2. \end{cases}$$

This gives

$$2\hbar + 4 \le J_2 \le 2\Bbbk - 2, \quad 2\hbar + 5 \le J_1 = 2\Bbbk + 2\hbar + 3 - J_2 \le 2\Bbbk - 1.$$

Using (B-5), we have the estimate

$$B_{j,l,0,J_1,J_2} \lesssim \left\| \frac{y^{d-2}|\partial_y^j q|^2}{y^{2J_1-2j}} \right\|_{L^\infty(y\ge1)} \left\| \frac{y^{d-2}|\partial_y^l q|^2}{y^{2J_2-2l}} \right\|_{L^\infty(y\ge1)} \lesssim \mathscr{E}_{J_1+1}\sqrt{\mathscr{E}_{J_2}\mathscr{E}_{J_2+2}}.$$

- If l is odd, we simply take $J_2 = l+1$, which gives

$$2\hbar + 4 \le J_2 \le 2\Bbbk - 2, \quad 2\hbar + 5 \le J_1 \le 2\Bbbk - 1.$$

Hence,

$$B_{j,l,0,J_1,J_2} \lesssim \mathscr{E}_{J_1+1}\sqrt{\mathscr{E}_{J_2}\mathscr{E}_{J_2+2}}.$$

Recall from Definition 3.2 that for all even integers m in the range $2\hbar + 4 \le m \le 2\Bbbk$,

$$\mathscr{E}_m \le \begin{cases} b_1^{\frac{\ell}{2\ell-\gamma}(2m-d)} & \text{for } 2\hbar+4 \le m \le 2\hbar + 2\ell, \\ b_1^{m-2\hbar-2+2(1-\delta)-K\eta} & \text{for } 2\hbar + 2\ell + 2 \le m \le 2\Bbbk. \end{cases} \tag{4-42}$$

- If $J_1 + 1 \ge 2\hbar + 2\ell + 2$ and $J_2 \ge 2\hbar + 2\ell + 2$, then

$$B_{j,l,0,J_1,J_2} \lesssim b_1^{J_1+J_2-4\hbar-2+4(1-\delta)-2K\eta} \lesssim b_1^{2L+2+4(1-\delta)-K\eta}.$$

- If $J_1+1 \le 2\hbar+2\ell$, then $J_2 = 2\Bbbk + 2\hbar + 3 - J_1 \ge 2\Bbbk - 2\ell + 4 \ge 2\hbar + 2\ell + 2$ because $\Bbbk \gg \ell$. This implies

$$B_{j,l,0,J_1,J_2} \lesssim b_1^{\frac{\ell}{2\ell-\gamma}(2J_1+2-d)+J_2+1-2(\hbar+1)+2(1-\delta)-K\eta} \lesssim b_1^{2L+2+4(1-\delta)-K\eta}.$$

Hence, we obtain

$$B_{j,l,0,J_1,J_2} \lesssim b_1^{2L+2+4(1-\delta)-K\eta} \quad \text{for } J_1 + J_2 = 2\Bbbk + 2\hbar + 3.$$

Similarly, one can prove that

$$B_{j,l,0,J_3,J_4} \lesssim b_1^{2L-1+4(1-\delta)-K\eta} \quad \text{for } J_3 + J_4 = 2\Bbbk + 2\hbar.$$

Therefore,

$$\begin{aligned} B_{j,l,0} &\lesssim b_1^{2L+2+4(1-\delta)-K\eta} b_1^{2\delta+\gamma-4} + b_1^{2L-1+4(1-\delta)-K\eta} b_1^{2\delta+\gamma-1} \\ &\lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+(\gamma-1)-(K+2-2\delta)\eta} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{(\gamma-1)}{2}} \end{aligned}$$

for $\eta \le (\gamma-1)/(2(K+2-2\delta))$. This concludes the proof of (4-41) as well as (4-39) when $k = i$.

Case III: $k \ge 1$ and $k - i \ge 1$. Let us write from (4-39) and (4-40),

$$A_{k,m,i} \lesssim \sum_{j=0}^{m} \sum_{l=0}^{i-m} \int_{y\ge1} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2}} \frac{|\partial_y^{k-i}\psi|^2}{y^{-2(k-i)}}. \tag{4-43}$$

At this stage, we need to make precise the decay of $|\partial_y^n \psi|$ to archive the bound (4-39). To do so, let us recall that T_i is admissible of degree (i,i) (see Lemma 2.9) and S_i is homogeneous of degree $(i, i-1, i)$ (see Proposition 2.11). Together with (2-4), we estimate

$$\text{for all } j \geq 1, \quad |\partial_y^j \widetilde{Q}_b| \lesssim \frac{1}{y^{\gamma+j}} + \sum_{l=1}^{2L+2} \frac{b_1^l y^{2l}}{y^{\gamma+j}} \mathbf{1}_{\{y \leq 2B_1\}} \lesssim \frac{b_1^{-(2L+2)\eta}}{y^{\gamma+j}}.$$

Let $\tau \in [0,1]$ and $v_\tau = \widetilde{Q}_b + \tau q$. We use the Faà di Bruno formula to write

$$\begin{aligned}\text{for all } n \in \mathbb{N}, \quad |\partial_y^n \psi|^2 &\lesssim \int_0^1 \sum_{m^*=n} |\partial_{v_\tau}^{m_1+\cdots+m_n} \sin(v_\tau)|^2 \prod_{i=1}^n |\partial_y^i \widetilde{Q}_b + \partial_y^i q|^{2m_i}\, d\tau \\ &\lesssim \sum_{m^*=n} \prod_{i=1}^n \Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}} + |\partial_y^i q|^2 \Big)^{m_i}, \quad m^* = \sum_{i=1}^n i m_i.\end{aligned}$$

For $1 \leq y \leq B_0$, we use (B-5) to estimate

$$|\partial_y^i q|^2 = y^{4\Bbbk-2i-2} \left| \frac{\partial_y^i q}{y^{2\Bbbk-i-1}} \right|^2 \leq B_0^{4\Bbbk-2i-d} \mathscr{E}_{2\Bbbk} \leq b_1^{-C(K)\eta+i+\gamma} \leq \frac{b_1^{-C(K)\eta}}{y^{2\gamma+2i}},$$

from which, we have

$$|\partial_y^n \psi|^2 \lesssim \sum_{m^*=n} \prod_{i=1}^n \Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}} + \frac{b_1^{-C(K)\eta}}{y^{2\gamma+2i}} \Big)^{m_i} \lesssim \frac{b_1^{-C(K,L)\eta}}{y^{2\gamma+2n}} \quad \text{for all } 1 \leq y \leq B_0. \tag{4-44}$$

For $y \geq B_0$, we use again (B-5) to write for all $1 \leq n \leq 2\Bbbk - 1$,

$$\begin{aligned}|\partial_y^n \psi|^2 &\lesssim \sum_{m^*=n} \prod_{i=1}^{2\hbar+2\ell+1} \Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}} + y^{4\hbar+4\ell+2-2i} \left| \frac{\partial_y^i q}{y^{2\hbar+2\ell+1-i}} \right|^2 \Big)^{m_i} \prod_{i=2\hbar+2\ell+1}^{n} \Big(\frac{b_1^{-C(L)\eta}}{y^{2\gamma+2i}} + |\partial_y^i q|^2 \Big)^{m_i} \\ &\lesssim \sum_{m^*=n} \prod_{i=1}^{2\hbar+2\ell+1} (b_1^{-C(L)\eta+\gamma+i} + b_1^{-K\eta+i+\gamma} b_1^{2\ell+2(1-\delta)} y^{4\ell+4(1-\delta)})^{m_i} \\ &\qquad\qquad \times \prod_{i=2\hbar+2\ell+1}^{n} (b_1^{-C(L)\eta+\gamma+i} + b_1^{-K\eta+\gamma+i})^{m_i} \\ &\lesssim b_1^{-C(L,K)\eta+n+\gamma\sum_{i=1}^n m_i} (b_1 y^2)^{(2\ell+2(1-\delta))\sum_{i=1}^{2\hbar+2\ell+1} m_i} \quad \text{for all } y \geq B_0.\end{aligned} \tag{4-45}$$

Injecting (4-44) and (4-45) into (4-43), we arrive at

$$A_{k,i,m} \lesssim b_1^{-C\eta} \sum_{j=0}^{m} \sum_{l=0}^{i-m} \Big(\int_{1\leq y\leq B_0} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2+2\gamma}} + b_1^{\alpha} \int_{y\geq B_0} \frac{|\partial_y^j q|^2 |\partial_y^l q|^2}{y^{4\Bbbk-2j-2l+2-2\alpha}} \Big),$$

where $\alpha = k - i + (2\ell + 2(1-\delta))\sum_{i=1}^{2\hbar+2\ell+1} m_i$. Arguing as for the proof of (4-41), we end up with

$$A_{k,i,m} \lesssim b_1^{-C\eta} (b_1^{2L+1+\gamma+2(1-\delta)(1-\eta)+\frac{(\gamma-1)}{2}} + b_1^{2L+1+2(1-\delta)(1-\eta)+\frac{(\gamma-1)}{2}}) \lesssim b_1^{2L+1+2(1-\delta)(1-\eta)}$$

for η small enough. This finishes the proof of (4-39) as well as (4-34). Since the proof of (4-35) follows exactly the same lines as the proof of (4-34), we omit it.

Inserting (4-33), (4-34) and (4-35) into (4-32) and recalling from Definition 3.2 that

$$\mathscr{E}_{2\Bbbk} \le K b_1^{2L+2(1-\delta)(1+\eta)},$$

we arrive at

$$\frac{1}{2}\frac{d}{dt}\left\{\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}[1+\mathcal{O}(b_1)]\right\}$$
$$\lesssim \frac{b_1}{\lambda^{4\Bbbk-d+2}}\left(\frac{\mathscr{E}_{2\Bbbk}}{M^{2\delta}}+b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{2L+2(1-\delta)(1+\eta)}\right)+\frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}(\partial_s \xi_L). \tag{4-46}$$

<u>Step 8</u>: time oscillations. In this step, we want to find the contribution of the last term in (4-46) to the estimate (4-12). Let us write

$$\begin{aligned}\frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}(\partial_s \xi_L) &= \frac{d}{ds}\left\{\frac{1}{\lambda^{4\Bbbk-d+2}}\left[\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}\xi_L - \frac{1}{2}\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right]\right\}\\ &\quad + \frac{4\Bbbk-d+2}{\lambda^{4\Bbbk-d+2}}\frac{\lambda_s}{\lambda}\left[\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}\xi_L + \frac{1}{2}\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right]\\ &\quad - \frac{1}{\lambda^{4\Bbbk-d+2}}\int \mathscr{L}^{\Bbbk}(\partial_s q - \partial_s \xi_L)\mathscr{L}^{\Bbbk}\xi_L.\end{aligned} \tag{4-47}$$

From (4-30) and (4-31), we have

$$\int |\mathscr{L}^{\Bbbk}\xi_L|^2 \lesssim b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk}. \tag{4-48}$$

This implies

$$\begin{aligned}\left|\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}\xi_L\right| &\lesssim \left(\int |\mathscr{L}^{\Bbbk} q|^2\right)^{\frac{1}{2}}\left(\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right)^{\frac{1}{2}}\\ &\lesssim \sqrt{\mathscr{E}_{2\Bbbk}}\, b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}\, b_1^{(1-\delta)(1+\eta)} = b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}.\end{aligned}$$

Since $dt/ds = \lambda^2$, we then write

$$\frac{d}{ds}\left\{\frac{1}{\lambda^{4\Bbbk-d+2}}\left[\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}\xi_L - \frac{1}{2}\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right]\right\} = \frac{d}{dt}\left(\frac{\mathscr{E}_{2\Bbbk}}{\lambda^{4\Bbbk-d}}\mathcal{O}(b_1^{\eta(1-\delta)})\right). \tag{4-49}$$

Noting from (4-1) that $|\lambda_s/\lambda| \lesssim b_1$, this gives

$$\left|\frac{\lambda_s}{\lambda}\left[\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk}\xi_L + \frac{1}{2}\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right]\right| \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}. \tag{4-50}$$

For the last term in (4-47), we use (3-9) and the decomposition (4-25) to write

$$\begin{aligned}&\int \mathscr{L}^{\Bbbk}(\partial_s q - \partial_s \xi_L)\mathscr{L}^{\Bbbk}\xi_L\\ &\quad = \left[-\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk+1}\xi_L + \frac{\lambda_s}{\lambda}\int \Lambda q \mathscr{L}^{2\Bbbk}\xi_L\right] + \int \mathscr{L}^{\Bbbk}[-\widetilde{\Psi}_b - \widetilde{\mathrm{Mod}} + \mathcal{H}(q) + \mathcal{N}(q)]\mathscr{L}^{\Bbbk}\xi_L.\end{aligned} \tag{4-51}$$

Using (4-31), the admissibility of T_L and the fact that $\mathscr{L}^k T_i = 0$ if $i < k$, we estimate

$$\begin{aligned}\int |\mathscr{L}^{\Bbbk+1}\xi_L|^2 &\lesssim \left|\frac{\langle \mathscr{L}^L q, \chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q, \chi_{B_0}\Lambda Q\rangle}\right|^2 \int |\mathscr{L}^{\Bbbk+1}[(1-\chi_{B_1})T_L]|^2 \\ &\lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}\int_{y\geq B_1} y^{2(2L-\gamma-2(\Bbbk+1))}y^{d-1}\,dy \\ &\lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}b_1^{(4-2\delta)(1+\eta)} \lesssim b_1^2 b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk},\end{aligned}$$

from which we obtain

$$\left|\int \mathscr{L}^{\Bbbk} q \mathscr{L}^{\Bbbk+1}\xi_L\right| \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}.$$

Similarly, we have the estimate

$$\int (1+y^{4\Bbbk})|\mathscr{L}^{2\Bbbk}\xi_L|^2 \lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}\int_{y\geq B_1} y^{4\Bbbk}y^{2(2L-\gamma-4\Bbbk)}y^{d-1}\,dy \lesssim b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk};$$

hence, using (4-4) and (4-1), we get

$$\left|\frac{\lambda_s}{\lambda}\int \Lambda q \mathscr{L}^{2\Bbbk}\xi_L\right| \lesssim b_1\left(\int \frac{|\partial_y q|^2}{1+y^{4\Bbbk-2}}\right)^{\frac12}\left(\int (1+y^{4\Bbbk})|\mathscr{L}^{2\Bbbk}\xi_L|^2\right)^{\frac12} \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk}.$$

From (4-48), (4-27) and (4-29), we have

$$\begin{aligned}\left|\int \mathscr{L}^{\Bbbk}(\tilde{\Psi}_b+\widetilde{\mathrm{Mod}})\mathscr{L}^{\Bbbk}\xi_L\right| &\lesssim \left(\int |\mathscr{L}^{\Bbbk}\xi_L|^2\right)^{\frac12}\left(\int |\mathscr{L}^{\Bbbk}(\tilde{\Psi}_b+\widetilde{\mathrm{Mod}})|^2\right)^{\frac12} \\ &\lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk} + b_1 b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}.\end{aligned}$$

In the same manner, we have the estimate

$$\int (1+y^4)|\mathscr{L}^{\Bbbk+1}\xi_L|^2 \lesssim b_1^{-2(1-\delta)}\mathscr{E}_{2\Bbbk}\int_{y\geq B_1} y^4 y^{2(2L-\gamma-2(\Bbbk+1))}y^{d-1}\,dy \lesssim b_1^{2\eta(1-\delta)}\mathscr{E}_{2\Bbbk},$$

from which, together with (4-33) and (4-35), we get the bound

$$\begin{aligned}\left|\int \mathscr{L}^{\Bbbk-1}(\mathcal{H}(q)+\mathcal{N}(q))\mathscr{L}^{\Bbbk+1}\xi_L\right| &\lesssim \left(\int \frac{|\mathscr{L}^{\Bbbk-1}(\mathcal{H}(q)+\mathcal{N}(q))|^2}{1+y^4}\right)^{\frac12}\left(\int (1+y^4)|\mathscr{L}^{\Bbbk+1}\xi_L|^2\right)^{\frac12} \\ &\lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk} + b_1 b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}.\end{aligned}$$

Collecting these final bounds into (4-51) yields

$$\left|\int \mathscr{L}^{\Bbbk}(\partial_s q - \partial_s \xi_L)\mathscr{L}^{\Bbbk}\xi_L\right| \lesssim b_1 b_1^{\eta(1-\delta)}\mathscr{E}_{2\Bbbk} + b_1 b_1^{L+(1-\delta)(1+\eta)}\sqrt{\mathscr{E}_{2\Bbbk}}. \tag{4-52}$$

Substituting (4-47), (4-49), (4-50) and (4-52) into (4-46) concludes the proof of (4-12) as well as Proposition 4.4. □

4C. *Conclusion of Proposition 3.6.* We give the proof of Proposition 3.6 in this subsection in order to complete the proof of Theorem 1.1. Note that this section corresponds to Section 6.1 of [Merle, Raphaël and Rodnianski 2015]. Here we follow exactly the same lines as in that paper and no new ideas are needed. We divide the proof into two parts:

Part 1: reduction to a finite-dimensional problem. Assume that for a given $K > 0$ large and an initial time $s_0 \geq 1$ large, we have $(b(s), q(s)) \in \mathcal{S}_K(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$. By using (4-1), (4-8), (4-12) and (4-13), we derive new bounds on $\mathcal{V}_1(s)$, $b_k(s)$ for $\ell + 1 \leq k \leq L$ and $\mathscr{E}_{2(\hbar+m)}$ for $1 \leq m \leq L+1$, which are better than those defining $\mathcal{S}_K(s)$ (see Definition 3.2). It then remains to control $(\mathcal{V}_2(s), \dots, \mathcal{V}_\ell(s))$. This means that the problem is reduced to the control of a finite-dimensional function $(\mathcal{V}_2(s), \dots, \mathcal{V}_\ell(s))$, and then we get the conclusion (i) of Proposition 3.6.

Part 2: transverse crossing. We aim to prove that if $(\mathcal{V}_2(s), \dots, \mathcal{V}_\ell(s))$ touches

$$\partial \widehat{\mathcal{S}}_K(s) := \partial \left(-\frac{K}{s^{\frac{\eta}{2}}(1-\delta)}, \frac{K}{s^{\frac{\eta}{2}}(1-\delta)} \right)^{\ell-1}$$

at $s = s_1$, it actually leaves $\partial \widehat{\mathcal{S}}_K(s)$ at $s = s_1$ for $s_1 \geq s_0$, provided that s_0 is large enough. We then get the conclusion (ii) of Proposition 3.6.

<u>Part 1</u>: reduction to a finite-dimensional problem. We give the proof of item (i) of Proposition 3.6 in this part. Given $K > 0$, $s_0 \geq 1$ and the initial data at $s = s_0$ as in Definition 3.1, we assume for all $s \in [s_0, s_1]$, $(b(s), q(s)) \in \mathcal{S}_K(s)$ for some $s_1 \geq s_0$. We claim that for all $s \in [s_0, s_1]$,

$$|\mathcal{V}_1(s)| \leq s^{-\frac{\eta}{2}(1-\delta)}, \tag{4-53}$$

$$|b_k(s)| \lesssim s^{-(k+\eta(1-\delta))} \qquad \text{for } \ell+1 \leq k \leq L, \tag{4-54}$$

$$\mathscr{E}_{2m} \leq \begin{cases} \frac{1}{2} K s^{-\frac{\ell(4m-d)}{2\ell-\gamma}} & \text{for } \hbar+2 \leq m \leq \ell+\hbar, \\ \frac{1}{2} s^{-2(m-\hbar-1)-2(1-\delta)+K\eta} & \text{for } \ell+\hbar+1 \leq m \leq \Bbbk-1, \end{cases} \tag{4-55}$$

$$\mathscr{E}_{2\Bbbk} \leq \tfrac{1}{2} K s^{-(2L+2(1-\delta)(1+\eta))}, \tag{4-56}$$

Once these estimates are proved, it immediately follows from Definition 3.2 of $\mathcal{S}_K$ that if $(b(s_1), q(s_1)) \in \partial \mathcal{S}_K(s_1)$, then $(\mathcal{V}_2, \dots, \mathcal{V}_\ell))(s_1)$ must be in $\partial \widehat{\mathcal{S}}_K(s_1)$, which concludes the proof of Proposition 3.6(i).

Before going to the proof of (4-53)–(4-56), let us compute explicitly the scaling parameter λ. To do so, let us note from (2-51) and the a priori bound on $\mathcal{U}_1$ given in Definition 3.2

$$b_1(s) = \frac{c_1}{s} + \frac{\mathcal{U}_1}{s} = \frac{\ell}{(2\ell-\gamma)s} + \mathcal{O}\left(\frac{1}{s^{1+c\eta}} \right).$$

Using (4-1) yields

$$-\frac{\lambda_s}{\lambda} = \frac{\ell}{(2\ell-\gamma)s} + \mathcal{O}\left(\frac{1}{s^{1+c\eta}} \right), \tag{4-57}$$

from which we write

$$\left| \frac{d}{ds} \{ \log(s^{\frac{\ell}{2\ell-\gamma}} \lambda(s)) \} \right| \lesssim \frac{1}{s^{1+c\eta}}.$$

We now integrate by using the initial data value $\lambda(s_0) = 1$ to get

$$\lambda(s) = \left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\gamma}} [1 + \mathcal{O}(s^{-c\eta})] \quad \text{for } s_0 \gg 1. \tag{4-58}$$

This implies

$$s_0^{-\frac{\ell}{2\ell-\gamma}} \lesssim \frac{s^{-\frac{\ell}{2\ell-\gamma}}}{\lambda(s)} \lesssim s_0^{-\frac{\ell}{2\ell-\gamma}}. \tag{4-59}$$

Improved control of $\mathscr{E}_{2\Bbbk}$: We aim to use (4-12) to derive the improved bound (4-56). To do so, we inject the bound of $\mathscr{E}_{2\Bbbk}$ given in Definition 3.2 into the monotonicity formula (4-12) and integrate in time by using $\lambda(s_0) = 1$: For all $s \in [s_0, s_1)$,

$$\mathscr{E}_{2\Bbbk}(s) \le C\lambda(s)^{4\Bbbk-d}\left[\mathscr{E}_{2\Bbbk}(s_0) + \left(\frac{K}{M^{2\delta}} + \sqrt{K} + 1\right)\int_{s_0}^{s} \frac{\tau^{-(2L+1+2(1-\delta)(1+\eta))}}{\lambda(\tau)^{4\Bbbk-d}}\, d\tau\right].$$

Using (4-59), we estimate

$$\begin{aligned}\lambda(s)^{4\Bbbk-d}\int_{s_0}^{s} \frac{\tau^{-(2L+1+2(1-\delta)(1+\eta))}}{\lambda(\tau)^{4\Bbbk-d}}\, d\tau &\lesssim s^{-\frac{\ell(4\Bbbk-d)}{2\ell-\gamma}}\int_{s_0}^{s} \tau^{\frac{\ell(4\Bbbk-d)}{2\ell-\gamma}-(2L+1+2(1-\delta)(1+\eta))}\, d\tau\\ &\lesssim s^{-(2L+2(1-\delta)(1+\eta))}.\end{aligned}$$

Here we used the fact that the integral is divergent because

$$\frac{\ell(4\Bbbk-d)}{2\ell-\gamma} - [2L+1+2(1-\delta)(1+\eta)] = \frac{2\gamma L}{2\ell-\gamma} + \mathcal{O}_{L\to+\infty}(1) \gg -1.$$

Using again (4-59) and the initial bound (3-21), we estimate

$$\lambda(s)^{4\Bbbk-d}\mathscr{E}_{2\Bbbk}(s_0) \le \left(\frac{s_0}{s}\right)^{\frac{\ell(4\Bbbk-d)}{2\ell-\gamma}} s_0^{-\frac{10L\ell}{2\ell-\gamma}} \lesssim s^{-(2L+2(1-\delta)(1+\eta))}$$

for L large enough. Therefore, we obtain

$$\mathscr{E}_{2\Bbbk}(s) \le C\left(\frac{K}{M^{2\delta}} + \sqrt{K} + 1\right)s^{-(2L+2(1-\delta)(1+\eta))} \le \frac{K}{2}s^{-(2L+2(1-\delta)(1+\eta))}$$

for $K = K(M)$ large enough. This concludes the proof of (4-56).

Improved control of $\mathscr{E}_{2m}$: We can improve the control of $\mathscr{E}_{2m}$ by using the monotonicity formula (4-13). We distinguish two cases:

Case 1: $\hbar + 2 \le m \le \ell + \hbar$. From the bound of $\mathscr{E}_{2m}$ given in Definition 3.2 and $b_1(s) \sim \frac{1}{s}$, we integrate (4-13) in time s by using $\lambda(s_0) = 1$ to find that

$$\begin{aligned}\mathscr{E}_{2m}(s) \le C\lambda(s)^{4m-d}\bigg[\mathscr{E}_{2m}(s_0) + \sqrt{K}\int_{s_0}^{s} \frac{\tau^{-\frac{\ell}{2\ell-\gamma}(2m-\frac{d}{2})-(m-\hbar+1-\delta-C\eta)}}{\lambda(\tau)^{4m-d}}\, d\tau\\ + \int_{s_0}^{s} \frac{\tau^{-(2m-2\hbar-1+2(1-\delta)-C\eta)}}{\lambda(\tau)^{4m-d}}\, d\tau\bigg].\end{aligned}$$

Using the initial bound (3-21) and (4-59), we estimate

$$C\lambda(s)^{4m-d}\mathscr{E}_{2m}(s_0) \lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}$$

for s_0 large.

Using (4-59) and the identity

$$\begin{aligned}\frac{\ell}{2\ell-\gamma}\left(2m-\frac{d}{2}\right)-(m-\hbar+1-\delta-C\eta) &= -\frac{\gamma}{2}-1+C\eta+\frac{\gamma}{2\ell-\gamma}\left(m-\hbar-\delta-\frac{\gamma}{2}\right)\\ &\le -1-\frac{\gamma\delta}{2\ell-\gamma}+C\eta<-1,\end{aligned}$$

we estimate

$$\begin{aligned}\lambda(s)^{4m-d}\int_{s_0}^{s}\frac{\tau^{-\frac{\ell}{2\ell-\gamma}(2m-\frac{d}{2})-(m-\hbar+1-\delta-C\eta)}}{\lambda(\tau)^{4m-d}}\,d\tau &\lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}\int_{s_0}^{s}\tau^{\frac{\ell}{2\ell-\gamma}(2m-\frac{d}{2})-(m-\hbar+1-\delta-C\eta)}\,d\tau\\ &\lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}\int_{s_0}^{s}\frac{d\tau}{\tau^{1+\varepsilon}}\lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}.\end{aligned}$$

Similarly, thanks to the identity

$$\begin{aligned}&\frac{\ell}{2\ell-\gamma}(4m-d)-(2m-2\hbar-1+2(1-\delta)-C\eta)\\ &\qquad=-\gamma-1+C\eta+\frac{\gamma}{2\ell-\gamma}(2m-2\hbar-2\delta-\gamma)\le -1-\frac{2\gamma\delta}{2\ell-\gamma}+C\eta<-1,\end{aligned}$$

we obtain

$$\lambda(s)^{4m-d}\int_{s_0}^{s}\frac{\tau^{-(2m-2\hbar-1+2(1-\delta)-C\eta)}}{\lambda(\tau)^{4m-d}}\,d\tau\lesssim s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}.$$

Therefore, we deduce that

$$\mathscr{E}_{2m}(s)\le C(1+\sqrt{K})s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}\le\frac{K}{2}s^{-\frac{\ell}{2\ell-\gamma}(4m-d)}$$

for K large, which yields the improved bound (4-55) for $\hbar+2\le m\le\ell+\hbar$.

Case 2: $\ell+\hbar+1\le m\le \Bbbk-1$. Proceeding as in the previous case, we arrive at

$$\mathscr{E}_{2m}(s)\le C\lambda(s)^{4m-d}\left[\mathscr{E}_{2m}(s_0)+\int_{s_0}^{s}\frac{\tau^{-[2m-2\hbar-1+2(1-\delta)-(C+\frac{K}{2})\eta]}}{\lambda(\tau)^{4m-d}}\,d\tau\right].$$

From the identity

$$\begin{aligned}\frac{\ell}{2\ell-\gamma}(4m-d)-\left(2m-2\hbar-1+2(1-\delta)-\left(C+\frac{K}{2}\right)\eta\right)&=-\gamma-1+\left(C+\frac{K}{2}\right)\eta+\frac{\gamma}{2\ell-\gamma}(2m-2\hbar-2\delta-\gamma)\\ &\ge -1+\frac{2\gamma(1-\delta)}{2\ell-\gamma}+\left(C+\frac{K}{2}\right)\eta>-1,\end{aligned}\tag{4-60}$$

together with (4-59), we estimate

$$\lambda(s)^{4m-d}\int_{s_0}^{s}\frac{\tau^{-[2m-2\hbar-1+2(1-\delta)-(C+\frac{K}{2})\eta]}}{\lambda(\tau)^{4m-d}}\,d\tau \lesssim s^{-\frac{\ell(4m-d)}{2\ell-\gamma}}\int_{s_0}^{s}\tau^{\frac{\ell(4m-d)}{2\ell-\gamma}-[2m-2\hbar-1+2(1-\delta)-(C+\frac{K}{2})\eta]}\,d\tau \lesssim s^{-[2(m-\hbar-1)+2(1-\delta)-(C+\frac{K}{2})\eta]}\le \tfrac{1}{4}s^{-[2(m-\hbar-1)+2(1-\delta)-K\eta]}.$$

Using (4-60), (4-59) and the initial bound (3-21), we derive

$$C\lambda(s)^{4m-d}\mathscr{E}_{2m}(s_0)\lesssim s^{-\frac{\ell(4m-d)}{2\ell-\gamma}}\lesssim s^{-[2(m-\hbar-1)+2(1-\delta)-(C+\frac{K}{2})\eta]}\le \tfrac{1}{4}s^{-[2(m-\hbar-1)+2(1-\delta)-K\eta]}.$$

This concludes the proof of (4-55).

Control of the stable modes, b_k's. We now close the control of the stable modes $(b_{\ell+1},\dots,b_L)$; in particular, we prove (4-54). We first treat the case when $k=L$. Let

$$\tilde b_L = b_L+\frac{\langle \mathscr{L}^L q,\chi_{B_0}\Lambda Q\rangle}{\langle \Lambda Q,\chi_{B_0}\Lambda Q\rangle}.$$

Then from (4-31) and (4-56),

$$|\tilde b_L-b_L|\lesssim b_1^{-(1-\delta)}\sqrt{\mathscr{E}_{2\Bbbk}}\lesssim b_1^{L+\eta(1-\delta)},$$

and hence from the improved modulation equation (4-8),

$$|(\tilde b_L)_s+(2L-\gamma)b_1\tilde b_L|\lesssim b_1|\tilde b_L-b_L|+\frac{1}{B_0^{2\delta}}[C(M)\sqrt{\mathscr{E}_{2\Bbbk}}+b_1^{L+(1-\delta)}]\lesssim b_1^{L+1+\eta(1-\delta)}.$$

This implies

$$\left|\frac{d}{ds}\left\{\frac{\tilde b_L}{\lambda^{2L-\gamma}}\right\}\right|\lesssim\frac{b_1^{L+1+\eta(1-\delta)}}{\lambda^{2L-\gamma}}.$$

Integrating this identity in time from s_0 and recalling that $\lambda(s_0)=1$ yields

$$\tilde b_L(s)\lesssim C\lambda(s)^{2L-\gamma}\left(\tilde b_L(s_0)+\int_{s_0}^{s}\frac{b_1(\tau)^{L+1+\eta(1-\delta)}}{\lambda(\tau)^{2L-\gamma}}\,d\tau\right).$$

Using (4-31), $b_1(s)\sim\frac{1}{s}$, the initial bounds (3-20) and (3-21) together with (4-59), we estimate

$$\lambda(s)^{2L-\gamma}\tilde b_L(s_0)\lesssim\left(\frac{s_0}{s}\right)^{\frac{\ell(2L-\gamma)}{2\ell-\gamma}}\left(s_0^{-\frac{5\ell(2L-\gamma)}{2\ell-\gamma}}+s_0^{\eta(1-\delta)}s_0^{-\frac{5L\ell}{2\ell-\gamma}}\right)\lesssim s^{-L-\eta(1-\delta)}$$

and

$$\lambda(s)^{2L-\gamma}\int_{s_0}^{s}\frac{b_1(\tau)^{L+1+\eta(1-\delta)}}{\lambda(\tau)^{2L-\gamma}}\,d\tau\lesssim s^{-\frac{\ell(2L-\gamma)}{2\ell-\gamma}}\int_{s_0}^{s}\tau^{\frac{\ell(2L-\gamma)}{2\ell-\gamma}-L-1-\eta(1-\delta)}\,d\tau\lesssim s^{-L-\eta(1-\delta)}.$$

Therefore,

$$b_L(s)\lesssim|\tilde b_L(s)|+|\tilde b_L(s)-b_L(s)|\lesssim s^{-L-\eta(1-\delta)},$$

which concludes the proof of (4-54) for $k = L$. Now we will propagate this improvement that we found for the bound of b_L to all b_k for all $\ell + 1 \le k \le L-1$. To do so we do a descending induction where the initialization is for $k = L$. Assume the bound

$$|b_k| \lesssim b_1^{k+\eta(1-\delta)}$$

for $k+1$ and let's prove it for k. Indeed, from (4-1) and the induction bound, we have

$$\left|(b_k)_s - (2k-\gamma)\frac{\lambda_s}{\lambda} b_k\right| \lesssim b_1^{L+1} + |b_{k+1}| \lesssim b_1^{k+1+\eta(1-\delta)},$$

which implies

$$\left|\frac{d}{ds}\left\{\frac{b_k}{\lambda^{2k-\gamma}}\right\}\right| \lesssim \frac{b_1^{k+1+\eta(1-\delta)}}{\lambda^{2k-\gamma}}.$$

Integrating this identity in time as for the case $k = L$, we end up with

$$b_k(s) \lesssim C\lambda(s)^{2k-\gamma}\left(b_k(s_0) + \int_{s_0}^{s} \frac{b_1(\tau)^{k+1+\eta(1-\delta)}}{\lambda(\tau)^{2k-\gamma}}\, d\tau\right) \lesssim s^{-k-\eta(1-\delta)},$$

where we used the initial bound (3-20), (4-59) and $k \ge \ell+1$. This concludes the proof of (4-54).

Control of the stable mode $\mathcal{V}_1$. We recall from (2-51) and (3-18) that

$$b_k = b_k^e + \frac{\mathcal{U}_k}{s^k}, \quad 1 \le k \le \ell, \quad \mathcal{V} = P_\ell \mathcal{U},$$

where P_ℓ diagonalizes the matrix A_ℓ with spectrum (2-54). From (2-52), and (4-1), we estimate for $1 \le k \le \ell - 1$,

$$|s(\mathcal{U}_k)_s - (A_\ell \mathcal{U})_k| \lesssim s^{k+1}|(b_k)_s + (2k-\gamma)b_1 b_k - b_{k+1}| + |\mathcal{U}|^2 \lesssim s^{-L+k} + |\mathcal{U}|^2.$$

From (2-53), (4-1) and the improved bound (4-54), we have

$$|s(\mathcal{U}_\ell)_s - (A_\ell \mathcal{U})_\ell| \lesssim s^{\ell+1}\big(|(b_k)_s + (2k-\gamma)b_1 b_\ell - b_{\ell+1}| + |b_{\ell+1}|\big) + |\mathcal{U}|^2 \lesssim s^{-\eta(1-\delta)} + |\mathcal{U}|^2.$$

Using the diagonalization (2-54), we obtain

$$s\mathcal{V}_s = D_\ell \mathcal{V} + \mathcal{O}(s^{-\eta(1-\delta)}). \tag{4-61}$$

Using (2-54) again yields the control of the stable mode $\mathcal{V}_1$:

$$|(s\mathcal{V}_1)_s| \lesssim s^{-\eta(1-\delta)}.$$

Thus from the initial bound (3-20),

$$|s^{\eta(1-\delta)}\mathcal{V}_1(s)| \le \left(\frac{s_0}{s}\right)^{1-\eta(1-\delta)} s_0^{\eta(1-\eta)}\mathcal{V}_1(s_0) + 1 \lesssim s_0^{\eta(1-\delta)},$$

which yields (4-53) for $s_0 \ge s_0(\eta)$ large enough.

Part 2: transverse crossing. We give the proof of item (ii) of Proposition 3.6 in this part. We compute from (4-61) and (2-54) at the exit time $s = s_1$

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\left(\sum_{k=2}^{\ell} |s^{\frac{\eta}{2}(1-\delta)}\mathcal{V}_k(s)|^2\right)\bigg|_{s=s_1} &= \left(s^{\eta(1-\delta)-1}\sum_{k=2}^{\ell}\left[\frac{\eta}{2}(1-\delta)\mathcal{V}_k^2(s)+s\mathcal{V}_k(\mathcal{V}_k)_s\right]\right)\bigg|_{s=s_1} \\
&= \left(s^{\eta(1-\delta)-1}\left[\sum_{k=2}^{\ell}\left[\frac{k\gamma}{2k-\gamma}+\frac{\eta}{2}(1-\delta)\right]\mathcal{V}_k^2(s)+\mathcal{O}\left(\frac{1}{s^{\frac{3}{2}\eta(1-\delta)}}\right)\right]\right)\bigg|_{s=s_1} \\
&\geq \frac{1}{s_1}\left[c(d,\ell)\sum_{k=2}^{\ell}|s_1^{\frac{\eta}{2}(1-\delta)}\mathcal{V}_k(s_1)|^2+\mathcal{O}\left(\frac{1}{s_1^{\frac{\eta}{2}(1-\delta)}}\right)\right] \\
&\geq \frac{1}{s_1}\left[c(d,\ell)+\mathcal{O}\left(\frac{1}{s_1^{\frac{\eta}{2}(1-\delta)}}\right)\right] > 0,
\end{aligned}$$

where we used item (i) of Proposition 3.6 in the last step. This completes the proof of Proposition 3.6.

Appendix A: Coercivity of the adapted norms

We give in this section the coercivity estimates for the operator $\mathscr{L}$ as well as the iterates of $\mathscr{L}$ under some suitable orthogonality condition. We first recall the standard Hardy-type inequalities for the class of radially symmetric functions,

$$\mathcal{D}_{\mathrm{rad}} = \{f \in \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ with radial symmetry}\}.$$

For simplicity, we write

$$\int f := \int_0^{+\infty} f(y)y^{d-1}\,dy$$

and

$$D^k = \begin{cases} \Delta^m & \text{if } k = 2m, \\ \partial_y\Delta^m & \text{if } k = 2m+1. \end{cases}$$

We have the following:

Lemma A.1 (Hardy-type inequalities). *Let $d \geq 7$ and $f \in \mathcal{D}_{\mathrm{rad}}$. Then*:

(i) (*Hardy near the origin*)

$$\int_0^1 \frac{|\partial_y f|^2}{y^{2i}} \geq \frac{(d-2-2i)^2}{4}\int_0^1 \frac{f^2}{y^{2+2i}} - C(d)f^2(1), \quad i = 0, 1, 2.$$

(ii) (*Hardy away from the origin for the noncritical exponent*) *Let $\alpha > 0$, $\alpha \neq \frac{1}{2}(d-2)$. Then*

$$\int_1^{+\infty} \frac{|\partial_y f|^2}{y^{2\alpha}} \geq \left(\frac{d-(2\alpha+2)}{2}\right)^2\int_1^{+\infty} \frac{f^2}{y^{2+2\alpha}} - C(\alpha,d)f^2(1).$$

(iii) (*Hardy away from the origin for the critical exponent*) *Let* $\alpha = \frac{1}{2}(d-2)$. *Then*

$$\int_1^{+\infty} \frac{|\partial_y f|^2}{y^{2\alpha}} \geq \frac{1}{4}\int_1^{+\infty} \frac{f^2}{y^{2+2\alpha}(1+\log y)^2} - C(d) f^2(1).$$

(iv) (*general weighted Hardy*) *For any* $\mu > 0$, $k \geq 2$ *an integer and* $1 \leq j \leq k-1$,

$$\int \frac{|D^j f|^2}{1+y^{\mu+2(k-j)}} \lesssim_{j,\mu} \int \frac{|D^k f|^2}{1+y^{\mu}} + \int \frac{f^2}{1+y^{\mu+2k}}.$$

Proof. The proof can be found in [Merle, Raphaël and Rodnianski 2015, Lemma B.1]. ☐

From the Hardy-type inequalities, we derive the following coercivity of $\mathscr{A}^*$:

Lemma A.2 (weight coercivity of $\mathscr{A}^*$). *Let* $\alpha \geq 0$. *There exists* $c_\alpha > 0$ *such that for all* $f \in \mathcal{D}_{\text{rad}}$,

$$\int \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} \geq c_\alpha\biggl(\int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2\alpha})} + \int \frac{f^2}{y^{2i+2}(1+y^{2\alpha})}\biggr), \quad i = 0,1,2. \tag{A-1}$$

Proof. We proceed in two steps:

Step 1: subcoercive estimate for $\mathscr{A}^*$. We first prove the following subcoercive bound for $\mathscr{A}^*$: for $i = 0,1,2$ and $\alpha \geq 0$,

$$\int \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} \gtrsim \int \frac{f^2}{y^{2i+2}(1+y^{2\alpha})} + \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2\alpha})} - f^2(1) - \int \frac{f^2}{1+y^{2i+2\alpha+4}}. \tag{A-2}$$

From the definition (2-7) of $\mathscr{A}^*$ and the asymptotic of V given in (2-8), we use an integration by parts to estimate near the origin

$$\begin{aligned}\int_{y\leq 1} \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} &\gtrsim \int_{y\leq 1} \frac{1}{y^{2i}}\biggl|\partial_y f + \frac{d}{y} f + \mathcal{O}(|yf|)\biggr|^2 \\ &\gtrsim \int_{y\leq 1} \frac{|\partial_y f|^2}{y^{2i}} + d\int_{y\leq 1} \frac{\partial_y(f^2)}{y^{2i+1}} + d^2\int_{y\leq 1} \frac{f^2}{y^{2i+2}} + \mathcal{O}\biggl(\int_{y\leq 1} \frac{f^2}{y^{2i-2}}\biggr) \\ &\gtrsim \int_{y\leq 1} \frac{|\partial_y f|^2}{y^{2i}} + (2+2i)d\int_{y\leq 1} \frac{f^2}{y^{2i+2}} + df^2(1) + \mathcal{O}\biggl(\int_{y\leq 1} \frac{f^2}{y^{2i-2}}\biggr) \\ &\gtrsim \int_{y\leq 1}\biggl(\frac{|\partial_y f|^2}{y^{2i}} + \frac{f^2}{y^{2i+2}}\biggr) - \int_{y\leq 1} y^2 f^2.\end{aligned}$$

Away from the origin, we use (2-8) to estimate

$$\int_{y\geq 1} \frac{|\mathscr{A}^* f|^2}{y^{2i}(1+y^{2\alpha})} \gtrsim \int_{y\geq 1} \frac{1}{y^{2i+2\alpha}}\biggl(\partial_y f + \frac{d-1-\gamma}{y} f\biggr)^2 - \int_{y\geq 1} \frac{f^2}{y^{2i+2\alpha+4}}.$$

We make the change of variable $g = y^{d-1-\gamma} f$ and use the Hardy inequality given in part (ii) of Lemma A.1 to write

$$\int_{y\geq 1} \frac{|\partial_y (y^{d-1-\gamma} f)|^2}{y^{2i+2\alpha+2(d-1-\gamma)}}\, dy = \int_{y\geq 1} \frac{|\partial_y g|^2}{y^{2i+2\alpha+2(d-1-\gamma)}}\, dy \gtrsim \int_{y\geq 1} \frac{g^2}{y^{2i+2\alpha+2(d-1-\gamma)+2}}\, dy - g^2(1)$$
$$\gtrsim \int_{y\geq 1} \frac{f^2}{y^{2i+2\alpha+2}} - f^2(1).$$

Gathering the above bounds together with the trivial bound from (2-8),

$$\int_{y\geq 1} \frac{|\partial_y f|^2}{y^{2i+2\alpha}} \lesssim \int_{y\geq 1} \frac{|\mathscr{A}^* f|^2}{y^{2i+2\alpha}} + \int_{y\geq 1} \frac{f^2}{y^{2i+2\alpha+2}}$$

yields the subcoercivity (A-2).

<u>Step 2</u>: coercivity of $\mathscr{A}^*$. We now argue by contradiction to show the coercivity of $\mathscr{A}^*$. Assume that (A-1) does not hold. Up to a renormalization, we consider the sequence $f_n \in \mathcal{D}_{\mathrm{rad}}$ with

$$\int \frac{f_n^2}{y^{2i+2}(1+y^{2\alpha})} + \int \frac{|\partial_y f_n|^2}{y^{2i}(1+y^{2\alpha})} = 1 \quad \text{and} \quad \int \frac{|\mathscr{A}^* f_n|^2}{y^{2i}(1+y^{2\alpha})} \leq \frac{1}{n}. \tag{A-3}$$

This implies by (A-2),

$$f_n^2(1) + \int \frac{f_n^2}{1+y^{2i+2\alpha+4}} \gtrsim 1. \tag{A-4}$$

From (A-3), the sequence f_n is bounded in H^1_{loc}. Hence, from a standard diagonal extraction argument, there exists $f_\infty \in H^1_{\mathrm{loc}}$ such that up to a subsequence,

$$f_n \rightharpoonup f_\infty \quad \text{in } H^1_{\mathrm{loc}},$$

and from the local compactness of one-dimensional Sobolev embeddings

$$f_n \to f_\infty \quad \text{in } L^2_{\mathrm{loc}}, \qquad f_n(1) \to f_\infty(1).$$

This implies by (A-3) and (A-4),

$$f_\infty^2(1) + \int \frac{f_\infty^2}{1+y^{2i+2\alpha+4}} \gtrsim 1 \quad \text{and} \quad \int \frac{f_\infty^2}{y^{2i+2}(1+y^{2\alpha})} \lesssim 1, \tag{A-5}$$

which means that $f_\infty \neq 0$. On the other hand, from (A-3) and the lower semicontinuity of norms for the weak topology, we have

$$\mathscr{A}^* f_\infty = 0.$$

Hence,

$$f_\infty = \frac{\beta}{y^{d-1}\Lambda Q} \quad \text{for some } \beta \neq 0.$$

Since $\Lambda Q \sim y$ near the origin, we have

$$\int_{y\leq 1} \frac{f_\infty^2}{y^{2i+2}} \gtrsim \int_{y\leq 1} \frac{y^{d-1}}{y^{2d+2i+2}}\, dy = \int_{y\leq 1} \frac{dy}{y^{d+2i+3}} = +\infty,$$

which contradicts the a priori regularity of f_∞ given in (A-5). □

We also need the following subcoercivity of $\mathscr{A}$.

Lemma A.3 (weight coercivity of $\mathscr{A}$). *Let $p \geq 0$ and $i = 0, 1, 2$ such that $|2p + 2i - (d - 2 - 2\gamma)| \neq 0$, where $\gamma \in (1, 2]$ is defined by (1-8). We have*

$$\int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} \gtrsim \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2p})} + \int \frac{f^2}{y^{2i+2}(1+y^{2p})} - \left[f^2(1) + \int \frac{f^2}{1+y^{2i+2p+4}} \right]. \tag{A-6}$$

Assume in addition that

$$\langle f, \Phi_M \rangle = 0 \quad \text{if } 2i + 2p > d - 2\gamma - 2,$$

where Φ_M is defined in (3-4). Then we have

$$\int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} \gtrsim \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2p})} + \int \frac{f^2}{y^{2i+2}(1+y^{2p})}. \tag{A-7}$$

Proof. The proof is very similar to the proof of Lemma A.2. We proceed into two steps. The first step is to derive the subcoercive estimate (A-6). In the second step, we use a compactness argument to show the coercivity of $\mathscr{A}$ under a suitable condition.

Step 1: subcoercive estimate for $\mathscr{A}$. From the definition (2-6) of $\mathscr{A}$ and the asymptotic of V given in (2-8), we estimate near the origin

$$\begin{aligned}
\int_{y \leq 1} \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} &\gtrsim \int_{y \geq 1} \frac{1}{y^{2i}} \left| -\partial_y f + \frac{f}{y} + \mathcal{O}(|yf|) \right|^2 \\
&\gtrsim \int_{y \leq 1} \frac{|\partial_y f|^2}{y^{2i}} + \int_{y \leq 1} \frac{f^2}{y^{2i+2}} - \int_{y \leq 1} \frac{\partial_y (f^2)}{y^{2i+1}} - \int_{y \leq 1} \frac{f^2}{y^{2i-2}} \\
&\gtrsim \int_{y \leq 1} \frac{|\partial_y f|^2}{y^{2i}} + (d - 2i - 1) \int_{y \leq 1} \frac{f^2}{y^{2i+2}} - f^2(1) - \int_{y \leq 1} \frac{f^2}{y^{2i-2}} \\
&\gtrsim \int_{y \leq 1} \frac{|\partial_y f|^2}{y^{2i}} + \int_{y \leq 1} \frac{f^2}{y^{2i+2}} - f^2(1) - \int_{y \leq 1} y^2 f^2.
\end{aligned}$$

Away from the origin, we estimate from (2-8)

$$\int_{y \geq 1} \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2p})} \gtrsim \int_{y \geq 1} \frac{1}{y^{2i+2p}} \left(\partial_y f + \frac{\gamma}{y} f \right)^2 - \int_{y \geq 1} \frac{f^2}{y^{2i+2p+4}}.$$

We make the change of variable $g = y^\gamma f$. From the assumption $|2i + 2p - (d - 2 - 2\gamma)| \neq 0$, we use the Hardy inequality given in part (ii) of Lemma A.1 to write

$$\int_{y \geq 1} \frac{|\partial_y (y^\gamma f)|^2}{y^{2i+2p+2\gamma}} = \int_{y \geq 1} \frac{|\partial_y g|^2}{y^{2i+2p+2\gamma}} \gtrsim \int_{y \geq 1} \frac{g^2}{y^{2i+2p+2+2\gamma}} - g^2(1) \gtrsim \int_{y \geq 1} \frac{f^2}{y^{2i+2p+2}} - f^2(1).$$

Note also that we have the trivial bound from (2-8),

$$\int_{y \geq 1} \frac{|\mathscr{A} f|^2}{y^{2i+2p}} + \int_{y \geq 1} \frac{f^2}{y^{2i+2p+2}} \gtrsim \int_{y \geq 1} \frac{|\partial_y f|^2}{y^{2i+2p}}.$$

The collection of the above bounds yields the subcoercivity (A-6).

Step 2: coercivity of $\mathscr{A}$. Arguing as the proof of (A-1), we end up with the existence of $f_\infty \neq 0$ such that

$$\int \frac{f_\infty^2}{y^{2i+2}(1+y^{2p})} \lesssim 1 \quad \text{and} \quad \mathscr{A} f_\infty = 0.$$

Hence, from the definition (2-6) of $\mathscr{A}$, we have

$$f_\infty = \beta \Lambda Q \quad \text{for some } \beta \neq 0.$$

If $2i+2p > d-2\gamma-2$, we use the orthogonality condition to deduce that

$$0 = \langle f_\infty, \Phi_M \rangle = \beta \langle \Lambda Q, \chi_M \Lambda Q \rangle.$$

Thus, $\beta = 0$. If $2i+2p \le d-2\gamma-2$, we use the fact that $\Lambda Q \sim 1/y^\gamma$ as $y \to +\infty$ to estimate

$$\int_{y\ge 1} \frac{|\Lambda Q|^2 y^{d-1}\, dy}{y^{2i+2}(1+y^{2p})} \gtrsim \int_{y \ge 1} y^{d-1-2\gamma-2i-2p-2}\, dy \gtrsim \int_{y\ge 1} y^{-1}\, dy = +\infty,$$

which contradicts with the regularity of f_∞. □

From the coercivities of $\mathscr{A}$ and $\mathscr{A}^*$, we claim the following coercivity for $\mathscr{L}$:

Lemma A.4 (weighted coercivity of $\mathscr{L}$ under a suitable orthogonality condition). *Let $k \in \mathbb{N}$, $i = 0, 1, 2$, and $M = M(k)$ large enough. Then there exists $c_{M,k} > 0$ such that for all $f \in \mathcal{D}_{\mathrm{rad}}$ satisfying the orthogonality*

$$\langle f, \Phi_M \rangle = 0 \quad \text{if } 2i+2k > d-2\gamma-4,$$

where Φ_M is defined by (3-4) and $\hbar$ is given in (1-18), we have

$$\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} \ge c_{M,k} \int \left(\frac{|\partial_{yy} f|^2}{y^{2i}(1+y^{2k})} + \frac{|\partial_y f|^2}{y^{2i}(1+y^{2k+2})} + \frac{|f|^2}{y^{2i+2}(1+y^{2k+2})} \right), \tag{A-8}$$

and

$$\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} \ge c_{M,k} \int \left(\frac{|\mathscr{A} f|^2}{y^{2i+2}(1+y^{2k})} + \int \frac{|f|^2}{y^{2i}(1+y^{2k+4})} \right). \tag{A-9}$$

Proof. We proceed in two steps:

Step 1: subcoercivity of $\mathscr{L}$. We apply Lemma A.2 to $\mathscr{A} f$ with $\alpha = k$ and note that

$$\partial_y(\mathscr{A} f) = \mathscr{A}(\partial_y f) + \partial_y \left(\frac{V}{y} \right) f,$$

to write

$$\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} \gtrsim \int \frac{|\mathscr{A} f|^2}{y^{2i+2}(1+y^{2k})} + \int \frac{|\partial_y(\mathscr{A} f)|^2}{y^{2i}(1+y^{2k})} \tag{A-10}$$

$$\gtrsim \int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{|\partial_y(\mathscr{A} f)|^2}{y^{2i}(1+y^{2k})}$$

$$\gtrsim \int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{|\mathscr{A}(\partial_y f)|^2}{y^{2i}(1+y^{2k})} - \int \frac{|f|^2}{y^{2i+2}(1+y^{2k})}. \tag{A-11}$$

Applying Lemma A.3 to f with $p=k+1$ and noting that the condition $|2(k+1)+2i-(d-2-2\gamma)|\neq 0$ is always satisfied (if not, we have $d=4+2\sqrt{(k+1+i)^2+2}\notin\mathbb{N}$), we have

$$\int \frac{|\mathscr{A} f|^2}{y^{2i}(1+y^{2k+2})} \gtrsim \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{f^2}{y^{2i+2}(1+y^{2k+2})} - \left[f^2(1) + \int \frac{f^2}{1+y^{2k+2i+6}}\right].$$

We apply again Lemma A.3 to $\partial_y f$ with $p=k$ to estimate

$$\int \frac{|\mathscr{A}(\partial_y f)|^2}{y^{2i}(1+y^{2k})} \gtrsim \int \frac{|\partial_{yy} f|^2}{y^{2i}(1+y^{2k})} + \int \frac{|\partial_y f|^2}{y^{2i+2}(1+y^{2k})} - \left[|\partial_y f(1)|^2 + \int \frac{|\partial_y f|^2}{1+y^{2k+2i+4}}\right].$$

Injecting these bounds into (A-11) yields the subcoercive estimate for $\mathscr{L}$,

$$\int \frac{|\mathscr{L} f|^2}{y^{2i}(1+y^{2k})} \gtrsim \int \frac{|\partial_{yy} f|^2}{y^{2i}(1+y^{2k})} + \int \frac{|\partial_y f|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{f^2}{y^{2i+2}(1+y^{2k+2})} - \left[f^2(1) + |f_y(1)|^2 + \int \frac{|f_y|^2}{1+y^{2k+2i+4}} + \int \frac{f^2}{1+y^{2k+2i+6}}\right]. \quad \text{(A-12)}$$

<u>Step 2</u>: coercivity of $\mathscr{L}$. We argue by contradiction. Assume that (A-8) does not hold. Up to a renormalization, there exists a sequence of functions $f_n \in \mathcal{D}_{\mathrm{rad}}$ such that

$$\int \frac{|\mathscr{L} f_n|^2}{y^{2i}(1+y^{2k})} \leq \frac{1}{n}, \quad \int \frac{|\partial_{yy} f_n|^2}{y^{2i}(1+y^{2k})} + \int \frac{|\partial_y f_n|^2}{y^{2i}(1+y^{2k+2})} + \int \frac{|f_n|^2}{y^{2i+2}(1+y^{2k+2})} = 1. \quad \text{(A-13)}$$

This implies by (A-12),

$$f_n^2(1) + |\partial_y f_n(1)|^2 + \int \frac{|\partial_y f_n|^2}{1+y^{2k+2i+4}} + \int \frac{f_n^2}{y^2(1+y^{2k+2i+6})} \gtrsim 1. \quad \text{(A-14)}$$

From (A-13), the sequence f_n is bounded in H^2_{loc}. Hence, from a standard diagonal extraction argument, there exists $f_\infty \in H^2_{\mathrm{loc}}$ such that up to a subsequence,

$$f_n \rightharpoonup f_\infty \quad \text{in } H^2_{\mathrm{loc}},$$

and from the local compactness of one-dimensional Sobolev embeddings

$$f_n \to f_\infty \quad \text{in } H^1_{\mathrm{loc}},$$

and

$$f_n(1) \to f_\infty(1), \quad \partial_y f_n(1) \to \partial_y f_\infty(1).$$

This implies by (A-13) and (A-14),

$$f_\infty^2(1) + |\partial_y f_\infty(1)|^2 + \int \frac{|\partial_y f_\infty|^2}{1+y^{2k+2i+4}} + \int \frac{f_\infty^2}{y^2(1+y^{2k+2i+6})} \gtrsim 1,$$

which means that $f_\infty \neq 0$. On the other hand, from (A-13) and the lower semicontinuity of norms for the weak topology, we deduce that f_∞ is a nontrivial function in the kernel of $\mathscr{L}$, namely that

$$\mathscr{L} f_\infty = 0,$$

which implies

$$f_\infty = \mu\Gamma + \beta\Lambda Q,$$

where μ and β two real numbers.

From (A-13) and the lower semicontinuity, we have

$$\int \frac{f_\infty^2}{y^{2i+2}(1+y^{2k+2})} < +\infty.$$

Recall from (2-16) that $\Gamma \sim 1/y^{d-1}$ as $y \to 0$. This yields the estimate

$$\int_{y\le 1} \frac{\Gamma^2}{y^{2i+2}(1+y^{2k+2})} \gtrsim \int_{y\le 1} \frac{dy}{y^{2i+2+d-1}} = +\infty;$$

hence, $\mu = 0$.

From (2-5), we have $\Lambda Q \sim 1/y^\gamma$ as $y \to +\infty$. If $2i+2k \le d-2\gamma-4$, we have

$$\int_{y\ge 1} \frac{|\Lambda Q|^2 y^{d-1}\,dy}{y^{2i+2}(1+y^{2k+2})} \gtrsim \int_{y\ge 1} y^{d-1-2i-2k-4-2\gamma}\,dy \gtrsim \int_{y\ge 1} y^{-1}\,dy = +\infty;$$

hence, $\beta = 0$. If $2i+2k > d-2\gamma-4$, we use the orthogonality condition to deduce

$$0 = \langle f_\infty, \Phi_M\rangle = \beta\langle \Lambda Q, \chi_M \Lambda Q\rangle,$$

which yields $\beta = 0$; hence $f_\infty = 0$. The contradiction then follows and the coercivity (A-8) is proved. The estimate (A-9) simply follows from (A-8) and (A-10). □

We are now in a position to prove the coercivity of $\mathscr{L}^k$ under a suitable orthogonality condition. We claim the following:

Lemma A.5 (coercivity of the iterate of $\mathscr{L}$). *Let $k \in \mathbb{N}$ and $M = M(k)$ large enough. Then there exists $c_{M,k} > 0$ such that for all $f \in \mathcal{D}_{\mathrm{rad}}$ satisfying the orthogonality condition*

$$\langle f, \mathscr{L}^m \Phi_M\rangle = 0, \quad 0 \le m \le k-\hbar,$$

where $\hbar$ is defined as in (1-18), we have

$$\mathscr{E}_{2k+2}(f) = \int |\mathscr{L}^{k+1} f|^2 \ge c_{M,k}\left\{\int \frac{|\mathscr{A}(\mathscr{L}^k f)|^2}{y^2} + \sum_{m=0}^{k}\int \frac{|\mathscr{L}^m f|^2}{y^4(1+y^{4(k-m)})} + \sum_{m=0}^{k-1}\frac{|\mathscr{A}(\mathscr{L}^m f)|^2}{y^6(1+y^{4(k-m-1)})}\right\}. \tag{A-15}$$

Proof. We argue by induction on k. For $k=0$, we apply Lemma A.2 to $\mathscr{A}f$ with $i=0$ and $\alpha=0$, then Lemma A.3 to f with $i=1$ and $p=0$ to write

$$\mathscr{E}_2(f) = \int |\mathscr{L} f|^2 \gtrsim \int \frac{|\mathscr{A} f|^2}{y^2} \gtrsim \int \frac{|\mathscr{A} f|^2}{y^2} + \int \frac{f^2}{y^4}.$$

Note that we had to use the orthogonality condition $\langle f, \Phi_M\rangle$ when $\hbar = 0$. In fact, the case $\hbar = 0$ only happens when $d = 7$. In this case, the condition $2 > d-2\gamma-2$ is fulfilled when applying Lemma A.2 with $i=1$ and $p=0$.

We now assume the claim for $k \geq 0$ and prove it for $k+1$. We have the orthogonality condition

$$\langle f, \mathscr{L}^m \Phi_M \rangle = 0, \quad 0 \leq m \leq k+1-\hbar.$$

Let $g = \mathscr{L} f$, then we have

$$\langle g, \mathscr{L}^m \Phi_M \rangle = 0, \quad 0 \leq m \leq k-\hbar.$$

By induction hypothesis, we write

$$\begin{aligned}
\int |\mathscr{L}^{k+2} f|^2 &= \int |\mathscr{L}^{k+1} g|^2 \\
&\gtrsim \int \frac{|\mathscr{A}(\mathscr{L}^k g)|^2}{y^2} + \sum_{m=0}^{k} \int \frac{|\mathscr{L}^m g|^2}{y^4(1+y^{4(k-m)})} + \sum_{m=0}^{k-1} \frac{|\mathscr{A}(\mathscr{L}^m g)|^2}{y^6(1+y^{4(k-m-1)})} \\
&= \int \frac{|\mathscr{A}(\mathscr{L}^{k+1} f)|^2}{y^2} + \sum_{m=1}^{k+1} \int \frac{|\mathscr{L}^m f|^2}{y^4(1+y^{4(k+1-m)})} + \sum_{m=1}^{k} \frac{|\mathscr{A}(\mathscr{L}^m f)|^2}{y^6(1+y^{4(k-m)})}.
\end{aligned}$$

Note that we have the orthogonality condition $\langle f, \Phi_M \rangle = 0$ when $k \geq \hbar - 1$. The case $k \leq \hbar - 2$ implies

$$4 + 4k \leq 4 + 4\left(\frac{d}{4} - \frac{\gamma}{2} - \delta\right) - 8 \leq d - 2\gamma - 4.$$

Hence, we use the coercivity bound (A-9) to derive

$$\int \frac{|\mathscr{L} f|^2}{y^4(1+y^{4k})} \gtrsim \int \frac{|\mathscr{A} f|^2}{y^6(1+y^{4k})} + \int \frac{f^2}{y^4(1+y^{4k+4})},$$

which concludes the proof of Lemma A.5. □

Appendix B: Interpolation bounds

We derive in this section interpolation bounds on q which are the consequence of the coercivity property given in Lemma A.5. We have the following:

Lemma B.1 (interpolation bounds). (i) *Weighted bounds for q_i: for $1 \leq m \leq \Bbbk$,*

$$\int |q_{2m}|^2 + \sum_{i=0}^{2k-1} \int \frac{|q_i|^2}{y^2(1+y^{4m-2i-2})} \leq C(M)\mathscr{E}_{2m}. \tag{B-1}$$

(ii) *Development near the origin*:

$$q = \sum_{i=1}^{\Bbbk} c_i T_{\Bbbk-i} + r_q, \tag{B-2}$$

with bounds

$$|c_i| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}},$$
$$|\partial_y^j r_q| \lesssim y^{2\Bbbk - \frac{d}{2} - j} |\ln(y)|^{\Bbbk} \sqrt{\mathscr{E}_{2\Bbbk}}, \quad 0 \leq j \leq 2\Bbbk - 1, \ \ y < 1.$$

(iii) *Bounds near the origin for q_i and $\partial_y^i q$*: *for* $y \le \frac{1}{2}$,

$$|q_{2i}| + |\partial_y^{2i} q| \lesssim y^{-\frac{d}{2}+2} |\ln y|^{\Bbbk} \sqrt{\mathscr{E}_{2\Bbbk}} \quad \textit{for } 0 \le i \le \Bbbk - 1,$$
$$|q_{2i-1}| + |\partial_y^{2i-1} q| \lesssim y^{-\frac{d}{2}+1} |\ln y|^{\Bbbk} \sqrt{\mathscr{E}_{2\Bbbk}} \quad \textit{for } 1 \le i \le \Bbbk.$$

(iv) *Weighted bounds for $\partial_y^i q$*: *for* $1 \le m \le \Bbbk$,

$$\sum_{i=0}^{2m} \int \frac{|\partial_y^i q|^2}{1 + y^{4m-2i}} \lesssim \mathscr{E}_{2m}. \tag{B-3}$$

Moreover, let $(i, j) \in \mathbb{N} \times \mathbb{N}^*$ *with* $2 \le i + j \le 2\Bbbk$. *Then*

$$\int \frac{|\partial_y^i q|^2}{1 + y^{2j}} \lesssim \begin{cases} \mathscr{E}_{2m} & \textit{for } i + j = 2m,\ \ 1 \le m \le \Bbbk, \\ \sqrt{\mathscr{E}_{2m}} \sqrt{\mathscr{E}_{2(m+1)}} & \textit{for } i + j = 2m + 1,\ \ 1 \le m \le \Bbbk - 1. \end{cases} \tag{B-4}$$

(v) *Pointwise bound far away*: *Let* $(i, j) \in \mathbb{N} \times \mathbb{N}$ *with* $1 \le i + j \le 2\Bbbk - 1$. *We have for* $y \ge 1$,

$$\left| \frac{\partial_y^i q}{y^j} \right|^2 \lesssim \frac{1}{y^{d-2}} \begin{cases} \mathscr{E}_{2m} & \textit{for } i + j + 1 = 2m,\ \ 1 \le m \le \Bbbk, \\ \sqrt{\mathscr{E}_{2m}} \sqrt{\mathscr{E}_{2(m+1)}} & \textit{for } i + j = 2m,\ \ 1 \le m \le \Bbbk - 1. \end{cases} \tag{B-5}$$

Proof. (i) The estimate (B-1) directly follows from Lemma A.5.

(ii) For $1 \le m \le \Bbbk$, we claim that $q_{2\Bbbk-2m}$ admits the Taylor expansion at the origin

$$q_{2\Bbbk-2m} = \sum_{i=1}^{m} c_{i,m} T_{m-i} + r_{2m}, \tag{B-6}$$

with the bounds

$$|c_{i,m}| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}},$$
$$|\partial_y^j r_{2m}| \lesssim y^{2m-\frac{d}{2}-j} |\ln(y)|^m \sqrt{\mathscr{E}_{2\Bbbk}}, \quad 0 \le j \le 2m - 1,\ \ y < 1,$$

The expansion (B-2) then follows from (B-6) with $m = \Bbbk$.

We proceed by induction in m for the proof of (B-6). For $m = 1$, we write from the definition (2-7) of $\mathscr{A}^*$,

$$r_1(y) = q_{2\Bbbk-1}(y) = \frac{1}{y^{d-1} \Lambda Q} \int_0^y q_{2\Bbbk} \Lambda Q x^{d-1}\, dx + \frac{d_1}{y^{d-1} \Lambda Q}.$$

Note from (B-1) that $\int |q_{2\Bbbk-1}|^2 / y^2 \lesssim \mathscr{E}_{2\Bbbk}$ and from (2-5) that $\Lambda Q \sim y$ as $y \to 0$; we deduce that $d_1 = 0$. Using the Cauchy–Schwarz inequality, we derive the pointwise estimate

$$|r_1(y)| \le \frac{1}{y^d} \left(\int_0^y |q_{2\Bbbk}|^2 x^{d-1}\, dx \right)^{\frac{1}{2}} \left(\int_0^y x^2 x^{d-1}\, dx \right)^{\frac{1}{2}} \lesssim y^{-\frac{d}{2}+1} \sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

We remark that there exists $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2\Bbbk-1}(a)|^2 \lesssim \int_{y \le 1} |q_{2\Bbbk-1}|^2 \lesssim \mathscr{E}_{2\Bbbk}.$$

We then define

$$r_2(y) = -\Lambda Q \int_a^y \frac{r_1}{\Lambda Q}\,dx,$$

and obtain from the pointwise estimate of r_1,

$$|r_2(y)| \lesssim y y^{-\frac{d}{2}+1}\sqrt{\mathscr{E}_{2\Bbbk}} \int_a^y \frac{dx}{x} \lesssim y^{-\frac{d}{2}+2}|\ln(x)|\sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

By construction and the definition (2-6) of $\mathscr{A}$, we have

$$\mathscr{A} r_2 = r_1 = q_{2\Bbbk-1}, \quad \mathscr{L} r_2 = \mathscr{A}^* q_{2\Bbbk-1} = q_{2\Bbbk} = \mathscr{L} q_{2\Bbbk-2}.$$

Recall that $\mathrm{Span}(\mathscr{L}) = \{\Lambda Q, \Gamma\}$, where Γ admits the singular behavior (2-16). From (B-1), we have $\int |q_{2\Bbbk-2}|^2/y^4 \lesssim \mathscr{E}_{2\Bbbk} < +\infty$. This implies that there exists $c_2 \in \mathbb{R}$ such that

$$q_{2\Bbbk-2} = c_2 \Lambda Q + r_2.$$

Moreover, there exists $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2\Bbbk-2}(a)|^2 \lesssim \int_{|y|\le 1} |q_{2\Bbbk-2}|^2 \lesssim \mathscr{E}_{2\Bbbk},$$

which implies

$$|c_2| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}}, \quad |q_{2\Bbbk-2}| \lesssim y^{-\frac{d}{2}+2}|\ln(y)|\sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

Since $\mathscr{A} r_2 = r_1$, we then write from the definition (2-6) of $\mathscr{A}$,

$$|\partial_y r_2| \lesssim |r_1| + \left|\frac{r_2}{y}\right| \lesssim y^{-\frac{d}{2}+2}|\ln(y)|\sqrt{\mathscr{E}_{2\Bbbk}}, \quad y < 1.$$

This concludes the proof of (B-6) for $m = 1$.

We now assume that (B-6) holds for $m \ge 1$ and prove it for $m+1$. The term r_{2m} is built as follows:

$$r_{2m-1} = \frac{1}{y^{d-1}\Lambda Q}\int_0^y r_{2m-2}\Lambda Q x^{d-1}\,dx, \quad r_{2m} = -\Lambda Q \int_a^y \frac{r_{2m-1}}{\Lambda Q}\,dx, \quad a \in \left(\tfrac{1}{2}, 1\right).$$

We now use the induction hypothesis to estimate

$$\begin{aligned}
|r_{2m+1}| &= \left|\frac{1}{y^{d-1}\Lambda Q}\int_0^y r_{2m}\Lambda Q x^{d-1}\,dx\right| \\
&\lesssim \frac{1}{y^d}\sqrt{\mathscr{E}_{2\Bbbk}}\int_0^y x^{2m+\frac{d}{2}}|\ln(x)|^m\,dx \\
&\lesssim y^{2m-\frac{d}{2}}\sqrt{\mathscr{E}_{2\Bbbk}}\int_0^y |\ln(x)|^m\,dx \\
&\lesssim y^{2m-\frac{d}{2}+1}|\ln(y)|^m\sqrt{\mathscr{E}_{2\Bbbk}}.
\end{aligned}$$

Here we used the identity

$$I_m = \int_0^y [\ln(x)]^m\,dx \lesssim y|\ln(y)|^m, \quad m \ge 1, \ \ y < 1.$$

Indeed, we have $I_1 = \int_0^y \ln(x)\,dx = y\ln(y) - y \lesssim y|\ln(y)|$ for $y < 1$. Assuming the claim for $m \geq 1$, we use an integration by parts to estimate for $m+1$

$$\begin{aligned} I_{m+1} &= \int_0^y [\ln(x)]^m (x\ln(x) - x)'\,dx \\ &= y[\ln(y)]^{m+1} - y[\ln(y)]^m - m(I_m - I_{m-1}) \lesssim y|\ln(y)|^{m+1}. \end{aligned}$$

Using an integration by parts yields

$$\int_a^y \frac{[\ln(x)]^m}{x}\,dx = \frac{[\ln(y)]^{m+1} - [\ln(a)]^{m+1}}{m+1}.$$

Hence, we have the estimate

$$\begin{aligned} |r_{2m+2}| = \left|\Lambda Q \int_a^y \frac{r_{2m+1}}{\Lambda Q}\,dx\right| &\lesssim y^{2m-\frac{d}{2}+2}\sqrt{\mathscr{E}_{2\Bbbk}} \int_a^y \frac{|\ln(x)|^m}{x}\,dx \\ &\lesssim y^{2m-\frac{d}{2}+2}|\ln(y)|^{m+1}\sqrt{\mathscr{E}_{2\Bbbk}}. \end{aligned}$$

By construction, we have

$$\mathscr{A} r_{2m+2} = r_{2m+1}, \quad \mathscr{L} r_{2m+2} = r_{2m}.$$

From the induction hypothesis and the definition (2-19) of T_k, we write

$$\mathscr{L} q_{2\Bbbk-2(m+1)} = q_{2\Bbbk-2m} = \sum_{i=1}^m c_{i,m} T_{m-i} + r_{2m} = \sum_{i=1}^m c_{i,m} \mathscr{L} T_{m+1-i} + \mathscr{L} r_{2m+2}.$$

The singularity (2-16) of Γ at the origin and the bound $\int_{y\leq 1} |q_{2\Bbbk-2(m+1)}|^2/y^4 \lesssim \mathscr{E}_{2\Bbbk}$ allows us to deduce

$$q_{2\Bbbk-2(m+1)} = \sum_{i=1}^m c_{i,m} T_{m+1-i} + c_{2m+2}\Lambda Q + r_{2m+2}.$$

From (B-1), we see that there exists $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2\Bbbk-2(m+1)}(a)|^2 \lesssim \int_{y\leq 1} |q_{2\Bbbk-2(m+1)}|^2 \lesssim \mathscr{E}_{2\Bbbk}.$$

Together with the induction hypothesis $|c_{i,m}| \lesssim \sqrt{\mathscr{E}_{2\Bbbk}}$ and the pointwise estimate on r_{2m+2}, we get the bound $|c_{2m+2}| \leq \sqrt{\mathscr{E}_{2\Bbbk}}$.

A brute force computation using the definitions of $\mathscr{A}$ and $\mathscr{A}^*$ and the asymptotic behavior (2-8) ensure that for any function f,

$$\partial_y^j f = \sum_{i=0}^j P_{i,j} f_i, \quad |P_{i,j}| \lesssim \frac{1}{y^{j-i}}, \tag{B-7}$$

and we estimate

$$|\partial_y^j r_{2m+2}| \lesssim \sum_{i=0}^j \frac{|r_{2m+2-i}|}{y^{j-i}} \lesssim \sqrt{\mathscr{E}_{2\Bbbk}} \sum_{i=0}^j \frac{y^{2m+2-i-\frac{d}{2}}|\ln(y)|^{m+1}}{y^{j-i}} \lesssim y^{2m+2-\frac{d}{2}-j}|\ln(y)|^{m+1}\sqrt{\mathscr{E}_{2\Bbbk}}.$$

This concludes the proof of (B-6) as well as (B-2).

(iii) The proof of (iii) directly follows from (B-6).

(iv) We have from (B-7),

$$|\partial_y^k q| \lesssim \sum_{j=0}^{k} \frac{|q_j|}{y^{k-j}},$$

and thus, using (B-1) and the pointwise bounds given in part (iii) yields

$$\sum_{i=0}^{2m} \int \frac{|\partial_y^i q|^2}{1+y^{4m-2i}} \lesssim \mathscr{E}_{2m} + \sum_{i=0}^{2m-1} \int_{y<1} |\partial_y^i q|^2 + \sum_{i=0}^{2m-1} \int_{y>1} \frac{|\partial_y^i q|^2}{y^{4m-2i}}$$
$$\lesssim \mathscr{E}_{2m} + \mathscr{E}_{2\Bbbk} \int_{y<1} y|\ln y|^{\Bbbk}\,dy + \sum_{i=0}^{2m-1}\sum_{j=0}^{i} \int_{y>1} \frac{|q_j|^2}{y^{4m-2j}} \lesssim \mathscr{E}_{2m},$$

which concludes the proof of (B-3).

The estimate (B-4) simply follows from (B-3). Indeed, if $i+j=2m$ with $1 \le m \le \Bbbk$, we have

$$\int \frac{|\partial_y^i q|^2}{1+y^{2j}} = \int \frac{|\partial_y^i q|^2}{1+y^{4m-2i}} \lesssim \mathscr{E}_{2m}.$$

If $i+j=2m+1$ with $1 \le m \le \Bbbk - 1$, we write

$$\int \frac{|\partial_y^i q|^2}{1+y^{2j}} = \int \frac{|\partial_y^i q|^2}{1+y^{4m-2i+2}} \lesssim \left(\int \frac{|\partial_y^i q|^2}{1+y^{4m-2i}}\right)^{\frac{1}{2}} \left(\int \frac{|\partial_y^i q|^2}{1+y^{4m-2i+4}}\right)^{\frac{1}{2}}$$
$$\lesssim \sqrt{\mathscr{E}_{2m}}\sqrt{\mathscr{E}_{2(m+1)}}.$$

(v) Let $i, j \ge 0$ with $1 \le i+j \le 2\Bbbk - 1$. Then $2 \le i+j+1 \le 2\Bbbk$ and we conclude from (B-4) that for $y \ge 1$,

$$\left|\frac{\partial_y^i q}{y^j}\right|^2 \lesssim \left|\int_y^{+\infty} \partial_x\left(\frac{(\partial_x^i q)^2}{x^{2j}}\right) dx\right| \lesssim \frac{1}{y^{d-2}}\left\{\int_y^{+\infty} \frac{|\partial_x^i q|^2}{x^{2j+2}} + \int_y^{+\infty} \frac{|\partial_x^{i+1} q|^2}{x^{2j}}\right\}$$
$$\lesssim \frac{1}{y^{d-2}} \begin{cases} \mathscr{E}_{2m} & \text{for } i+j+1 = 2m, \quad 1 \le m \le \Bbbk, \\ \sqrt{\mathscr{E}_{2m}}\sqrt{\mathscr{E}_{2(m+1)}} & \text{for } i+j+1 = 2m+1, \quad 1 \le m \le \Bbbk - 1. \end{cases}$$ □

Appendix C: Proof of (4-22)

We give here the proof of (4-22). Before going to the proof, we need the following Leibniz rule for $\mathscr{L}^k$.

Lemma C.1 (Leibniz rule for $\mathscr{L}^k$). *Let ϕ be a smooth function and $k \in \mathbb{N}$, we have*

$$\mathscr{L}^{k+1}(\phi f) = \sum_{m=0}^{k+1} f_{2m}\phi_{2k+2,2m} + \sum_{m=0}^{k} f_{2m+1}\phi_{2k+2,2m+1}, \tag{C-1}$$

$$\mathscr{A}\mathscr{L}^k(\phi f) = \sum_{m=0}^{k} f_{2m+1}\phi_{2k+1,2m+1} + \sum_{m=0}^{k} f_{2m}\phi_{2k+1,2m}, \tag{C-2}$$

where for $k=0$,

$$\phi_{1,0}=-\partial_y\phi,\quad \phi_{1,1}=\phi,$$
$$\phi_{2,0}=-\partial_y^2\phi-\frac{d-1+2V}{y}\partial_y\phi,\quad \phi_{2,1}=2\partial_y\phi,\quad \phi_{2,2}=\phi,$$

and for $k\geq 1$,

$$\begin{aligned}
\phi_{2k+1,0}&=-\partial_y\phi_{2k,0},\\
\phi_{2k+1,2i}&=-\partial_y\phi_{2k,2i}-\phi_{2k,2i-1},\quad 1\leq i\leq k,\\
\phi_{2k+1,2i+1}&=\phi_{2k,2i}+\frac{d-1+2V}{y}\phi_{2k,2i+1}-\partial_y\phi_{2k,2i+1},\quad 0\leq i\leq k-1,\\
\phi_{2k+1,2k+1}&=\phi_{2k,2k}=\phi,\\
\phi_{2k+2,0}&=\partial_y\phi_{2k+1,0}+\frac{d-1+2V}{y}\phi_{2k+1,0},\\
\phi_{2k+2,2i}&=\phi_{2k+1,2i-1}+\partial_y\phi_{2k+1,2i}+\frac{d-1+2V}{y}\phi_{2k+1,2i},\quad 1\leq i\leq k,\\
\phi_{2k+2,2i+1}&=-\phi_{2k+1,2i}+\partial_y\phi_{2k+1,2i+1},\quad 0\leq i\leq k,\\
\phi_{2k+2,2k+2}&=\phi_{2k+1,2k+1}=\phi.
\end{aligned}$$

Proof. We use the relations

$$\mathscr{A}(\phi f)=\phi\mathscr{A}f-\partial_y\phi f,\quad \mathscr{A}^*(\phi f)=\phi\mathscr{A}^*f+\partial_y\phi f,$$
$$\mathscr{A}f+\mathscr{A}^*f=\frac{d-1+2V}{y}f$$

to compute

$$\begin{aligned}
\mathscr{A}(\phi f)&=f_1\phi+f(-\partial_y\phi),\\
\mathscr{L}(\phi f)&=\mathscr{A}^*\mathscr{A}(\phi f)=f_2\phi+f_1(2\partial_y\phi)+f\left(-\partial_y^2\phi-\frac{d-1+2V}{y}\partial_y\phi\right),
\end{aligned}$$

which is the conclusions of (C-1) and (C-2) for $k=0$.

Assume that (C-1) and (C-2) hold for $k\in\mathbb{N}$; let us compute for $k\to k+1$. Using (C-1), we write

$$\begin{aligned}
\mathscr{A}\mathscr{L}^{k+1}(\phi f)&=\sum_{m=0}^{k+1}\mathscr{A}[f_{2m}\phi_{2k+2,2m}]+\sum_{m=0}^{k}\left[-\mathscr{A}^*+\frac{d-1+2V}{y}\right]f_{2m+1}\phi_{2k+2,2m+1}\\
&=\sum_{m=0}^{k+1}\{f_{2m+1}\phi_{2k+2,2m}+f_{2m}(-\partial_y\phi_{2k+2,2m})\}\\
&\quad+\sum_{m=0}^{k}\bigg\{f_{2m+2}(-\phi_{2k+2,2m+1})+f_{2m+1}(-\partial_y\phi_{2k+2,2m+1})\\
&\qquad\qquad+f_{2m+1}\left(\frac{d-1+2V}{y}\phi_{2k+2,2m+1}\right)\bigg\}
\end{aligned}$$

$$= \sum_{m=0}^{k} f_{2m+1}\Big(\phi_{2k+2,2m} - \partial_y \phi_{2k+2,2m+1} + \frac{d-1+2V}{y}\phi_{2k+2,2m+1}\Big)$$
$$+ \sum_{m=1}^{k} f_{2m}(-\partial_y \phi_{2k+2,2m} - \phi_{2k+2,2m+1}) + f_{2k+3}\phi_{2k+2,2k+2} + f(-\partial_y \phi_{2k+2,0}),$$

which yields the recurrence relation for $\phi_{2k+3,j}$ with $0 \le j \le 2k+3$.

Similarly, we write $\mathscr{L}^{k+2}(\phi f) = \mathscr{A}^*[\mathscr{A}\mathscr{L}^{k+1}(\phi f)]$ and use the formula (C-2) with $k+1$ to obtain the recurrence relation for $\phi_{2k+4,j}$ with $0 \le j \le 2k+4$. □

Let us now give the proof of (4-22). By induction and the definition (3-13), we have

$$[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v = \sum_{m=0}^{\Bbbk-2} \mathscr{L}_\lambda^m([\partial_t, \mathscr{L}_\lambda]\mathscr{L}_\lambda^{\Bbbk-2-m}v) = \sum_{m=0}^{\Bbbk-2} \mathscr{L}_\lambda^m\Big(\frac{\partial_t Z_\lambda}{r^2}\mathscr{L}_\lambda^{\Bbbk-2-m}v\Big).$$

Noting that

$$\frac{\partial_t Z_\lambda}{r^2} = \frac{b_1 \Lambda Z}{\lambda^4 y^2},$$

we make a change of variables to obtain

$$\int \frac{1}{\lambda^2(1+y^2)}|[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 = \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\int \frac{1}{1+y^2}\Big|\sum_{m=0}^{\Bbbk-2}\mathscr{L}^m\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2-m}q\Big)\Big|^2$$
$$\lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\sum_{m=0}^{\Bbbk-2}\int \frac{1}{1+y^2}\Big|\mathscr{L}^m\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2-m}q\Big)\Big|^2.$$

For $m=0$, we use (4-21) and (4-20) to estimate

$$\int \frac{1}{1+y^2}\Big|\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2}q\Big)\Big|^2 \lesssim \int \frac{|q_{2\Bbbk-4}^2|}{1+y^{10}} \lesssim \mathscr{E}_{2\Bbbk}.$$

For $m = 1, \dots, \Bbbk-2$, we apply (C-1) with

$$\phi = \frac{\Lambda Z}{y^2} = \frac{(d-1)\Lambda\cos(2Q)}{y^2}$$

and note from (2-4) that

$$|\phi_{k,i}| \lesssim \frac{1}{1+y^{2\gamma+2+(2k-i)}} \lesssim \frac{1}{1+y^{4+(2k-i)}}, \quad k \in \mathbb{N}^*,\ 0 \le i \le 2k,$$

which yields

$$\int \frac{1}{1+y^2}\Big|\mathscr{L}^m\Big(\frac{\Lambda Z}{y^2}\mathscr{L}^{\Bbbk-2-m}q\Big)\Big|^2 \lesssim \sum_{i=0}^{2m}\int \frac{q_{2\Bbbk-4-2m-i}^2}{(1+y^{10+(4m-2i)})} \lesssim \mathscr{E}_{2\Bbbk}.$$

Thus,

$$\int \frac{1}{\lambda^2(1+y^2)}|[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.$$

Similarly, we use (C-2) to get the estimate

$$\int |\mathscr{A}[\partial_t, \mathscr{L}_\lambda^{\Bbbk-1}]v|^2 \lesssim \frac{b_1^2}{\lambda^{4\Bbbk-d+2}}\mathscr{E}_{2\Bbbk}.$$

This concludes the proof of (4-22).

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