ANALYSIS & PDE

Volume 12 No. 1 2019

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msp

ANALYSIS AND PDE
Vol. 12, No. 1, 2019
dx.doi.org/10.2140/apde.2019.12.189

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ON PROPAGATION OF HIGHER SPACE REGULARITY FOR NONLINEAR VLASOV EQUATIONS

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This work is concerned with the broad question of propagation of regularity for smooth solutions to nonlinear Vlasov equations. For a class of equations (that includes Vlasov–Poisson and relativistic Vlasov–Maxwell systems), we prove that higher regularity in space is propagated, locally in time, into higher regularity for the moments in velocity of the solution. This in turn can be translated into some anisotropic Sobolev higher regularity for the solution itself, which can be interpreted as a kind of weak propagation of space regularity. To this end, we adapt the methods introduced by D. Han-Kwan and F. Rousset (*Ann. Sci. École Norm. Sup.* **49**:6 (2016) 1445–1495) in the context of the quasineutral limit of the Vlasov–Poisson system.

1. Introduction

This paper is concerned with the broad question of propagation of regularity for smooth solutions to Vlasov equations of the general form

$$\partial_t f + a(v) \cdot \nabla_x f + F(t,x,v) \cdot \nabla_v f = 0, \tag{1-1}$$

set in the phase space $\mathbb{T}^d \times \mathbb{R}^d$ (with $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ endowed with normalized Lebesgue measure), where $F : \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a force field satisfying $\nabla_v \cdot F = 0$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ is an advection field satisfying suitable assumptions, $a(v) = v$ being the main example to be considered. The (scalar) function $f(t,x,v)$ may be understood as the distribution function of a family of particles, which can be, depending

MSC2010: 35Q83.
Keywords: kinetic transport equations, kinetic averaging lemmas.

on the physical context, e.g., electrons, ions in plasma physics, or stars in galactic dynamics. The choice of the periodic torus $\mathbb{T}^d$ is made for simplicity.

The two precise examples of equations we specifically have in mind are the Vlasov equations arising from a coupling with Poisson or Maxwell equations, in which case the resulting coupled system is called the Vlasov–Poisson or the relativistic Vlasov–Maxwell system (we will discuss as well several other models).

- The *Vlasov–Poisson* system — either the repulsive or the attractive version, the sign of the interaction here does not matter here — is given by

$$\begin{cases} \partial_t f + v\cdot\nabla_x f \pm E\cdot\nabla_v f = 0, \\ E(t,x) = -\nabla_x\phi(t,x), \\ -\Delta_x\phi = \displaystyle\int_{\mathbb{R}^d} f\,dv - \int_{\mathbb{T}^d\times\mathbb{R}^d} f\,dv\,dx, \\ f|_{t=0} = f_0. \end{cases} \tag{1-2}$$

In the repulsive version (that is, with the sign $+$ in the Vlasov equation), this system describes the dynamics of charged particles in a nonrelativistic plasma, with a self-induced electric field.

In the attractive version (that is, with the sign $-$ in the Vlasov equation), it describes the dynamics of stars or planets with gravitational interaction.

- The *relativistic Vlasov–Maxwell* system, in dimension $d=3$, is given by

$$\begin{cases} \partial_t f + \hat{v}\cdot\nabla_x f + F\cdot\nabla_v f = 0, \\ \hat{v} := \dfrac{v}{\sqrt{1+|v|^2/c^2}}, \quad F(t,x,v) := E(t,x) + \dfrac{1}{c}\hat{v}\times B(t,x), \\ \dfrac{1}{c}\partial_t B + \nabla_x\times E = 0, \quad \nabla_x\cdot E = \displaystyle\int_{\mathbb{R}^3} f\,dv - \int_{\mathbb{T}^3\times\mathbb{R}^3} f\,dv\,dx, \\ -\dfrac{1}{c}\partial_t E + \nabla_x\times B = \dfrac{1}{c}\displaystyle\int_{\mathbb{R}^3}\hat{v} f\,dv, \quad \nabla_x\cdot B = 0, \\ f|_{t=0} = f_0, \quad (E,B)|_{t=0} = (E_0,B_0), \end{cases} \tag{1-3}$$

in which the parameter c is the speed of light. There are also related versions of (1-3) in lower dimensions. This system describes the dynamics of charged particles in a relativistic plasma, with a self-induced electromagnetic field. We recall that the (repulsive) Vlasov–Poisson system can be derived from (1-3) in the nonrelativistic regime, that is to say, in the limit $c\to\infty$, as studied in [Asano and Ukai 1986; Degond 1986; Schaeffer 1986].

In this paper, we will consider weighted Sobolev norms and associated weighted Sobolev spaces (based on L^2), defined, for $k\in\mathbb{N}$, $r\in\mathbb{R}$, as

$$\|f\|_{\mathcal{H}^k_r} := \left(\sum_{|\alpha|+|\beta|\le k}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(1+|v|^2)^r\,|\partial_x^\alpha\partial_v^\beta f|^2\,dv\,dx\right)^{\frac{1}{2}}, \tag{1-4}$$

where for $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \{1, \dots, d\}^n$, we write

$$|\alpha| = n, \quad |\beta| = n,$$

and

$$\partial_x^\alpha := \partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_n}}, \quad \partial_v^\beta := \partial_{v_{\beta_1}} \cdots \partial_{v_{\beta_n}}.$$

As usual the notation H^s will stand for the standard Sobolev spaces, without weight.

It will be also useful to introduce the weighted $W^{k,\infty}$ space, whose norm is defined, for $k \in \mathbb{N}$, $r \in \mathbb{R}$, by

$$\|f\|_{\mathcal{W}_r^{k,\infty}} := \sum_{|\alpha|+|\beta| \le k} \|(1+|v|^2)^{\frac{r}{2}} \, \partial_x^\alpha \partial_v^\beta f\|_{L^\infty_{x,v}}. \tag{1-5}$$

For the Vlasov–Poisson or Vlasov–Maxwell couplings, given an initial condition f_0 satisfying

$$f_0 \in \mathcal{H}_r^n$$

for $n, r > 0$ large enough (and with a smooth enough initial force $F(0)$), it is standard that there exists a unique local solution $f(t) \in C(0, T; \mathcal{H}_r^n)$. Under fairly general assumptions on the advection field a and the force F, the same result can also be shown for (1-1), as we will soon see.

Let us now present the precise problem we tackle in this work. Assuming some higher *space* regularity such as

$$\partial_x^{n+1} f_0 \in \mathcal{H}_r^0 \quad (\text{or } \ \partial_x^p f_0 \in \mathcal{H}_r^0 \ \text{ for } p \ge n+1), \tag{1-6}$$

the question we ask is the following: is there also propagation of any higher regularity for the solution $f(t)$? A first remark to be made is that there is no hope of proving that this sole additional assumption implies that the solution $f(t)$ also satisfies $\partial_x^{n+1} f(t) \in \mathcal{H}_r^0$, even for small values of t. Indeed, regularity in x and v is intricately intertwined for solutions of the Vlasov equation, as can be seen from the representation of the solution using the method of characteristics.

For $s, t \ge 0$ and $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, we define as usual the characteristic curves $(X(s,t,x,v), V(s,t,x,v))$ as the solutions to the system of ODEs

$$\begin{cases} \dfrac{d}{ds} X(s,t,x,v) = a(V(s,t,x,v)), & X(t,t,x,v) = x, \\ \dfrac{d}{ds} V(s,t,x,v) = F(s, X(s,t,x,v), V(s,t,x,v)), & V(t,t,x,v) = v. \end{cases} \tag{1-7}$$

The existence and uniqueness of such curves are consequences of the Cauchy–Lipschitz theorem (assuming we deal with smooth forces). The method of characteristics asserts that one can represent the solution of (1-1) as

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)). \tag{1-8}$$

Therefore we see (except maybe in trivial cases such as $F \equiv 0$) that derivatives in x of $f(t)$ involve derivatives in x and in v of f_0, so that regularity in x only of f_0 cannot in general be propagated for $f(t)$. However, given some smooth test function $\psi(v)$ (the case $\psi = 1$ is already interesting), we can also wonder about the higher regularity of the moment $m_\psi(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\psi(v)\,dv$. Such moments, which can be interpreted as hydrodynamic quantities, are important objects in kinetic theory. We have the representation formula

$$m_\psi(t,x) = \int_{\mathbb{R}^d} f_0(X(0,t,x,v), V(0,t,x,v))\,\psi(v)\,dv.$$

We note that for t small enough, the map $v \mapsto V(s,t,x,v)$ is a diffeomorphism for all $s \in [0,t]$. Indeed for $s = t$ this map is the identity and integrating with respect to s the equation satisfied by $V(s,t,x,v)$, we note that for t small enough and $s \in [0,t]$, the map $v \mapsto V(s,t,x,v)$ is a small perturbation of the identity, hence our claim that it is a diffeomorphism. In particular the map $v \mapsto V(0,t,x,v)$ is a diffeomorphism and we denote by $V^{-1}(t,x,v)$ its inverse. Using this diffeomorphism as a change of variables (in v) we get, for t small enough,

$$m_\psi(t,x) = \int_{\mathbb{R}^d} f_0(X(0,t,x,V^{-1}(t,x,v)),v)\,\psi(V^{-1}(t,x,v))\,|\det D_v V(0,t,x,v)|^{-1}\,dv.$$

Thanks to this formula, at least formally, the Leibniz rule ensures that derivatives in x of the moment m_ψ only involve derivatives in x of f_0. Recalling the extra higher regularity (1-6), it seems maybe natural to expect that the moment m_ψ belongs to the Sobolev space H^{n+1} in x. In the case where F is a *fixed* external force, assumed to be very smooth, say C^∞ with respect to all variables, since t is fixed, the fact that $m_\psi(t,\cdot)$ belongs to H^{n+1}_x follows indeed from the Leibniz formula, using the fact the characteristic curves (X,V) inherit the C^∞ regularity of F.

However, this argument seems to break down in the case where F depends on the solution $f(t)$ itself, as the regularity of F is then tightly linked to that of f. Let us discuss for instance the Poisson case — the Maxwell case is actually worse in the sense that in the Vlasov–Poisson coupling, F gains, loosely speaking, one derivative in x compared to f. As already mentioned, the local Cauchy theory yields $f(t) \in C(0,T;\mathcal{H}^n_r)$, and we have $F \in C(0,T;H^{n+1}_x)$. Note then that when applying $n+1$ derivatives in x on m_ψ, one needs to apply $n+1$ derivatives in x on $|\det D_v V(0,t,x,v)|^{-1}$, which amounts to applying in total $n+2$ derivatives to $V(0,t,x,v)$. However, by (1-7), we observe that (X,V) inherits the same order of regularity as F, and therefore it does not seem licit to take as many as $n+2$ derivatives.

The goal of this work is to show that despite this apparent shortcoming, it is indeed possible to show for a fairly wide class of nonlinear Vlasov equations (including the Vlasov–Poisson and Vlasov–Maxwell system) a result of propagation of regularity in x for the moments, assuming higher-order space regularity for the initial condition. This in turn can be translated into some anisotropic Sobolev higher regularity for the solution itself, which can be interpreted as a kind of weak propagation of space regularity.

It turns out that the *lagrangian approach*, that is to say, the approach that we have just underlined, based on representation formulas using characteristics, is not adapted to answer this question. Instead we shall rely on an *eulerian approach*, which is based to a larger extent on the PDE itself, inspired by the recent work of the author in collaboration with F. Rousset on the quasineutral limit of the Vlasov–Poisson system [Han-Kwan and Rousset 2016; ≥ 2019]. The quasineutral limit is a singular limit which loosely consists in a penalization of the laplacian in the Poisson equation. The small parameter is the scaled Debye length, which appears to be very small in several usual plasma settings. The limit leads to *singular*

Vlasov equations, which display a loss of regularity of the force field compared to that of the distribution function. As a consequence, these equations are in general ill-posed in the sense of Hadamard; see [Bardos and Nouri 2012; Han-Kwan and Nguyen 2016]. This problem might therefore look quite different from the one considered here; the similarity comes from the fact that the justification of the quasineutral limit ultimately loosely comes down to the proof of a *uniform*[1] propagation of one order of higher regularity for moments of solutions of the Vlasov–Poisson equation. Note though that the analysis of [Han-Kwan and Rousset 2016; ≥ 2019] requires the introduction of pointwise Penrose stability conditions, and also relies on pseudodifferential tools, which will not be the case in this paper. As a matter of fact, the singular Vlasov equations which can be formally derived in the quasineutral limit will not enter the class of Vlasov equations we will deal with in this work, precisely because of the aforementioned loss of derivative.

The methodology of [Han-Kwan and Rousset 2016] was also used in the context of large time estimates for data close to stable equilibria for the Vlasov–Maxwell system in the nonrelativistic regime, in a recent work in collaboration with T. Nguyen and F. Rousset [Han-Kwan et al. 2017].

As a matter of fact, the approach can be considered as *semilagrangian*, in the sense that at some point we still rely on characteristics as in the lagrangian approach but at the level of the PDEs that arise after applying derivatives on the Vlasov equation, whereas in the lagrangian approach, derivatives are taken after using the representation of the solution by characteristics.

2. Main results

2A. *The abstract framework.* Let us now describe precisely the class of Vlasov equations we deal with. We consider in this work the abstract equation

$$\partial_t f + a(v)\cdot\nabla_x f + F\cdot\nabla_v f = 0, \tag{2-1}$$

with the following structural assumptions. Among all these assumptions, we highlight that the force depends on the distribution function itself, but only through some of its moments in velocity.

• *Assumptions on the advection field.* The map $a:\mathbb{R}^d\to\mathbb{R}^d$ is a one-to-one C^∞ function such that

$$|a(v)|\le C(1+|v|) \quad \text{for all } v\in\mathbb{R}^d, \tag{2-2}$$

$$\|\partial_v^\alpha a\|_{L^\infty}\le C_\alpha \quad \text{for all } |\alpha|\neq 0, \tag{2-3}$$

and its inverse a^{-1} (defined on $a(\mathbb{R}^d)$) satisfies, for some $\lambda>0$,

$$|\partial_v^\alpha a^{-1}(w)|\le C_\alpha(1+|a^{-1}(w)|)^{1+\lambda|\alpha|} \quad \text{for all } w\in a(\mathbb{R}^d),\ \text{ for all } \alpha. \tag{2-4}$$

• *Assumptions on the force field.* The vector field F is divergence-free in v (i.e., satisfies $\nabla_v\cdot F=0$) and we have the following decomposition for some $\ell\in\mathbb{N}^*$:

$$F(t,x,v)=\sum_{j=1}^{\ell} A_j(v)\,F^j(t,x). \tag{2-5}$$

[1] With respect to the scaled Debye length.

We assume that for all $j \in \{1, \dots, \ell\}$, A_j is a C^∞ scalar function satisfying

$$\|\partial_v^\alpha A_j\|_{L^\infty} \le C_\alpha \quad \text{for all } \alpha. \tag{2-6}$$

Furthermore, there exist C^∞ functions $\psi_1(v), \dots, \psi_r(v)$ with at most polynomial growth, i.e., there is $r_0 > 0$ such that

$$\|\psi_i(v)\|_{\mathcal{W}^{k,\infty}_{-r_0}} \le C_{i,k} \quad \text{for all } k \in \mathbb{N} \tag{2-7}$$

such that, defining

$$m_{\psi_i}(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\psi_i(v)\,dv$$

for all $j = 1, \dots, \ell$, the vector field F^j is *uniquely determined by these moments and the initial conditions*, through a map

$$\big((m_{\psi_i})_{i=1,\dots,r}, (F^j(0))_{j=1,\dots,\ell}\big) \mapsto F^j, \tag{2-8}$$

and for all large enough $n > 1 + d$, and all $t > 0$, we have

$$\|F^j\|_{L^2(0,t;H^n_x)} \le \Gamma_n^{(j)}\Bigg(t, \|m_{\psi_1}\|_{L^2(0,t;H^n_x)}, \dots, \|m_{\psi_r}\|_{L^2(0,t;H^n_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\Bigg), \tag{2-9}$$

$$\|F^j\|_{L^\infty(0,t;H^n_x)} \le \Gamma_n^{(j)\prime}\Bigg(t, \|m_{\psi_1}\|_{L^\infty(0,t;H^n_x)}, \dots, \|m_{\psi_r}\|_{L^\infty(0,t;H^n_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\Bigg), \tag{2-10}$$

where $\Gamma_n^{(j)}$, $\Gamma_n^{(j)\prime}$ are polynomial functions that are nonincreasing with respect to each of their arguments (the others being fixed nonnegative numbers).

Finally, the force field satisfies the following stability property. Let f and g be two solutions of (2-1), and denote by $F[f]$ and $F[g]$ their associated force fields. Assume that the initial conditions $(F^j(0))_{j=1,\dots,\ell}$ are the same. Then, we have for all $j = 1, \dots, \ell$,

$$\begin{aligned}
&\|F^j[f] - F^j[g]\|_{L^2(0,t;H^n_x)} \\
&\qquad \le \Gamma_n^{(j)\sharp}\Bigg(t, \bigg\|\int (f-g)\psi_i(v)\,dv\bigg\|_{L^2(0,t;H^n_x)}, \dots, \bigg\|\int (f-g)\psi_r(v)\,dv\bigg\|_{L^2(0,t;H^n_x)}\Bigg),
\end{aligned} \tag{2-11}$$

where $\Gamma_n^{(j)\sharp}$ is a polynomial function that is nonincreasing with respect to each of its arguments and such that $\Gamma_n^{(j)\prime}(0,\cdot) = 0$.

We shall explain later why both Vlasov–Poisson and relativistic Vlasov–Maxwell systems enter the abstract framework.

2B. ***Statement of the main results.*** The regularity and integrability indices that will be useful to handle such equations will depend on the dimension d, the maximal growth of the moments that intervene in the definition of F, which is r_0, and the parameter of growth of the inverse of a, which is λ; let us set

$$N := \tfrac{3}{2}d + 4, \quad R := \max\big(\tfrac{1}{2}d + 2(1+\lambda)(1+d) + r_0\big). \tag{2-12}$$

We use in the following statement the notation $\lfloor \cdot \rfloor$ for the floor function.

The main result proved in this paper is the following theorem.

Theorem 2.1. *Let $n \geq N$ and $r > R$. Let $n' > n$ be an integer such that $n > \lfloor \frac{1}{2} n' \rfloor + 1$. Assume that $f_0 \in \mathcal{H}^n_r$ and $F^j(0) \in H^{n'}_x$ for all $j \in \{1, \dots, \ell\}$. Assume furthermore that the initial data f_0 satisfies the following higher anisotropic regularity*:

$$\partial_x^{2(n - \lfloor \frac{1}{2} n' \rfloor + k)} \partial_x^\alpha \partial_v^\beta f_0 \in \mathcal{H}^0_r \quad \text{for all } |\alpha| + |\beta| = n' - n - k, \ \text{for all } k \in \left\{1, \dots, 2\lfloor \tfrac{1}{2} n' \rfloor - n\right\}. \tag{2-13}$$

Then there is $T > 0$ such that the following holds. There exists a unique solution $(f(t), F(t))$ with initial data $(f_0, F(0))$ to (2-1) such that $f(t) \in C(0, T; \mathcal{H}^n_r)$.

Moreover, for all test functions $\psi \in L^\infty(0, T; \mathcal{W}^{n',\infty}_{-r_0})$, we have

$$\left\| \int f \psi \, dv \right\|_{L^2(0,T;H^{n'}_x)} \leq \Lambda_\psi(T, M), \tag{2-14}$$

where Λ_ψ is a polynomial function and

$$M = \|f_0\|_{\mathcal{H}^n_r} + \sum_{j=1}^{\ell} \|F^j(0)\|_{H^{n'}_x} + \sum_{k=1}^{2\lfloor \frac{1}{2} n' \rfloor - n} \ \sum_{|\alpha|+|\beta| = n'-n-k} \|\partial_x^{2(n - \lfloor \frac{n'}{2} \rfloor + k)} \partial_x^\alpha \partial_v^\beta f_0\|_{\mathcal{H}^0_r}.$$

Thanks to (2-9), we immediately deduce from (2-14) that the force field satisfies as well the higher regularity

$$F^j \in L^2(0, T; H^{n'}_x).$$

Another consequence concerns the flow $(X, V) = (X(t, 0, x, v), V(t, 0, x, v))$ as defined in (1-7), for which we also obtain a higher regularity property.

Corollary 2.2. *For some $T' \leq T$, we have*

$$\partial^\gamma_{x,v}(X - x - tv, V - v) \in L^\infty(0, T'; L^\infty_v L^2_x) \quad \text{for all } |\gamma| \leq n'.$$

Remark 2.3. Some remarks about Theorem 2.1 are in order:

• In the case where $n = 2m - 1$ and $n' = n + 1 = 2m$, the assumption (2-13) is simply given by $\partial_x^{n+1} f_0 \in \mathcal{H}^0_r$ and we obtain the $L^2_t H^{n+1}_x$ smoothness of the moments: in other words this gives an answer to the question raised in the beginning of the Introduction. Note though that the regularity result we prove is not pointwise in t.

• Observe that it is required that the higher regularity index n' is not too large compared to n (i.e., $n > \lfloor \frac{1}{2} n' \rfloor + 1$); such a restriction is somehow reminiscent of a similar one appearing in the celebrated result of Bony [1981, Théorème 6.1] concerning the propagation of Sobolev microlocal regularity at characteristic points for general nonlinear PDEs. We remark however that the class of PDEs considered in this work does not enter the framework of [Bony 1981], in particular because of the "nonlocality" in velocity. We refer to Section 10 for some remarks and (counter-)examples in this direction.

• As a matter of fact, our result can be somehow interpreted as a kinetic (and nonlocal) analogue of Bony's aforementioned theorem.

• If it is ensured that the solution $(f(t), F(t))$ to (2-1) is *global*, (e.g., for the Vlasov–Poisson system in dimension $d \leq 3$, see [Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991; Batt and Rein 1991; Horst 1993]), we do not know if the higher propagation of regularity for the moments is global.

• Let us mention that in a somewhat different direction, a vector field method was devised in [Smulevici 2016] (see also [Fajman et al. 2017]) in order to prove time decay of moments for Vlasov equations set in unbounded spaces.

In the case where the force is one derivative smoother than the distribution function f itself (that is to say, when estimates (2-9) hold with $n-1$ instead of n in the right-hand side), the statement of Theorem 2.1 may be strengthened, insofar as one may ask only for derivatives in x in the regularity assumption (2-13). We refer to such a case as the transport/elliptic case, which includes in particular the Vlasov–Poisson system; see Theorem 9.1 in Section 9.

As already mentioned in the Introduction, the higher regularity for moments as obtained in Theorem 2.1 actually yields regularity for the solution itself (see [Gérard 1990] for a microlocal version of this fact, in the context of averaging lemmas) in anisotropic Sobolev spaces (as defined in [Hörmander 1976, Chapter II, Section 2.5]), which we first introduce.

Definition 2.4. Let $m, n \in \mathbb{R}$. The anisotropic Sobolev space $H^{m,n}_{x,v}$ is defined as

$$H^{m,n}_{x,v} := \{g \in \mathcal{S}'(\mathbb{T}^d \times \mathbb{R}^d) : (1+|k|^2)^{\frac{m}{2}}(1+|\eta|^2)^{\frac{n}{2}}\,\hat{g}(k,\eta) \in L^2(\mathbb{Z}^d \times \mathbb{R}^d)\},$$

where $\hat{g}$ stands for the Fourier transform[2] of g. We also define

$$H^{m,-\infty}_{x,v} := \bigcup_{p\in\mathbb{R}} H^{m,p}_{x,v}.$$

Corollary 2.5. *Consider the same assumptions and notation as in Theorem 2.1. We have*

$$f(t,x,v) \in L^2(0,T; H^{n',-\infty}_{x,v}).$$

Corollary 2.5 is a direct consequence of some estimates obtained in the proof of Theorem 2.1; we will provide a proof of this fact in Section 7. It is actually possible to give an estimate of a value of $p<0$ such that $f \in L^2(0,T; H^{n',p}_{x,v})$.

2C. *Overview of the proof.* We discuss in this section the ingredients, inspired by [Han-Kwan and Rousset 2016], leading to the higher propagation of regularity for the moments (the local well-posedness theory is fairly standard; see Section 3). We shall discuss here the case $n = 2k-1$ and $n' = n+1 = 2k$. To ease readability, we assume here that the dimension is $d = 1$ (in higher dimensions, the algebra is more involved but the basic principle is the same).

Taking derivatives. Since we intend to propagate regularity in space, the first step consists in understanding how to appropriately apply derivatives in x to the Vlasov equation (2-1).

[2]Where $\hat{g}(k,\eta) = 1/(2\pi)^d \int_{\mathbb{T}^d\times\mathbb{R}^d} g(x,v)e^{-ix\cdot k}e^{-iv\cdot\eta}\,dx\,dv$, although the convention that is chosen for the writing of the Fourier transform does not matter here.

We note that applying the operator ∂_x^α does not seem relevant, as it does not commute well with the operator $F\partial_v$: as a result it is not possible to obtain a closed equation bearing on $\partial_x^\alpha f$ without appealing to $\partial_x^\beta \partial_v^\gamma f$ for $\gamma \neq 0$, and therefore such an approach would require a control of derivatives in v which we do not have at initial time (this is of course reminiscent of the mixing in x and v that we have evoked in the Introduction).

The idea is to look for more appropriate differential operators, with nonconstant coefficients, satisfying the following three key properties:

- At initial time, they involve only derivatives in x.
- They enjoy good commutation properties with the transport operator, so that it is eventually possible to obtain closed systems involving these differential operators alone.
- They allow a good control of the Sobolev norm of the moments.

It turns out that second-order differential operators in x and v, with coefficients depending on the solution itself, will be appropriate. More precisely, we consider the operator

$$L := \partial_x^2 + \varphi(t,x)\partial_x\partial_v + \psi(t,x)\partial_v^2,$$

whose coefficients φ and ψ will depend on the force field F. Setting $\mathcal{T} := \partial_t + a(v)\partial_x + F\partial_v$ as the transport operator, we ask that the coefficients φ, ψ solve a semilinear system of the form

$$\begin{cases} \mathcal{T}\phi = 2\partial_x F + G_1(\phi,\psi,\partial_{x,v}F), \\ \mathcal{T}\psi = G_2(\phi,\psi,\partial_{x,v}F), \\ \varphi|_{t=0} = 0, \quad \psi|_{t=0} = 0, \end{cases}$$

where G_1, G_2 are polynomial functions of degree greater than or equal to 2; this corresponds to zero-order coupling terms. Note in particular that by definition, $L = \partial_x^2$ at time $t = 0$. The semilinear system is precisely chosen in order to cancel bad terms in the commutation between L and $\mathcal{T}$, so that for any function g,

$$L\mathcal{T}(g) = \mathcal{T}L(g) + (LF)\,\partial_v g + (La)\,\partial_x g + (\partial_v a)\varphi Lg.$$

Applying this identity to the solution f of the Vlasov equation (1-1), this yields

$$\mathcal{T}L(f) = -(LF)\,\partial_v f - (La)\,\partial_x f - (\partial_v a)\varphi Lf.$$

This formula will play a key role in the analysis. The main term (in terms of regularity issues) is $-\partial_x^2 F\partial_v f$, since the others involve either more regular quantities (we recall indeed that F and a are assumed to be smooth with respect to v), or the quantity Lf, which paves the way for a closed system involving only compositions of L applied to f. As a consequence, the operators obtained as compositions of L appear to be relevant for applying higher-order derivatives in x, since by construction:

- They require only a control of space regularity at initial time.

• Denoting by L_k the composition of k operators L, one can obtain that $L_k f$ satisfies an equation of the form

$$\mathcal{T}(L_k f) = A(L_k f) - (\partial_x^{2k} F)\,\partial_v f + G((\partial_{x,v}^\alpha f)_{|\alpha| \le 2k-1}), \tag{2-15}$$

where A, G are bounded linear operators. We note that this equation involves derivatives in v of the solution, but only of order $2k-1 = n$, which we control thanks to the local well-posedness theory. This can therefore be seen as a closed equation for $L_k f$.

• One can show that for any smooth test function ψ,

$$\int_{\mathbb{R}} (L_k f)\,\psi(t,x,v)\,dv = \int_{\mathbb{R}} (\partial_x^{2k} f)\,\psi(t,x,v)\,dv + \text{“controlled terms”}.$$

In the controlled terms, the overload of derivatives in v falling on f is transferred to ψ by an integration-by-parts argument.

All in all, this eventually shows that the L_k are indeed well-suited to study the regularity of moments. This step is fully developed in Section 4. There are two separate difficulties in order to complete this task: obtaining the right algebra as discussed here, and proving Sobolev estimates for all the involved objects.

(In the case where $n' > n+1$, we need to set up an induction argument, and this leads the study of successive systems of coupled kinetic transport equations, which build on the general equation (2-15).)

Propagation of regularity on moments. We then turn to the study of moments of the solutions to (2-15). This step is partly inspired from (and thus related to) the treatment of linear Landau damping in [Mouhot and Villani 2011].

We first use the method of characteristics to invert the operator $\mathcal{T} - A$. It is convenient at this stage to use changes of variables in velocity (introduced and studied in Section 5) in order to straighten characteristics and eventually, roughly speaking, come down from $\mathcal{T}$ to the free transport operator $\partial_t + a(v)\cdot\nabla_x$. To this end, it turns out to be efficient to introduce the change of variables $v \mapsto \Phi$ where Φ solves the Burgers' equation

$$\partial_t \Phi + a(\Phi)\cdot\nabla_x \Phi = F(\,\cdot\,, \Phi), \quad \Phi|_{t=0} = v,$$

where we can prove that Φ remains close to v in small time (in terms of Sobolev norms). The problem comes down to the understanding of the contribution of the term $-(\partial_x^{2k} F)\,\partial_v f$, and eventually roughly reduces to the study of an equation of the type

$$H_1(t,x) = \int_0^t \int_{\mathbb{R}} (\partial_x H_2)(s, x-(t-s)a(v))\,U(t,s,x,v)\,dv\,ds + \text{“controlled terms”},$$

where we know only that H_2 is controlled in $L^2(0,T;L^2_x)$ and U is smooth, and we seek a bound of H_1 in $L^2(0,T;L^2_x)$ (such an estimate corresponds to a control on the moments of $L_k f$). The integral in time is due to the use of Duhamel's formula, and the integral in v to the fact that we study moments in v. We observe that the operator in the right-hand side seems to feature a loss of derivative in x. However, we use a smoothing effect to overcome this apparent loss, which was proved in [Han-Kwan and Rousset

2016]. The outcome is the estimate

$$\left\| \int_0^t \int_{\mathbb{R}^d} (\nabla_x H_2)(s, x-(t-s)a(v))\, U(t,s,x,v)\, dv\, ds \right\|_{L^2(0,t;L^2_x)} \lesssim \|H_2\|_{L^2(0,t;L^2_x)} \sup_{0\le t,s\le T} \|U(t,s,\cdot)\|,$$

where $\|\cdot\|$ stands for a high-order weighted Sobolev norm (in x and v) which we will make precise later. As noted in [Han-Kwan and Rousset 2016], this is reminiscent of (but different from) classical *kinetic averaging lemmas*, as it loosely speaking involves the gain of one full derivative; we refer to Section 6 for a thorough discussion.

2D. *Content of the end of the paper.* The paper is then organised as follows: the proofs of Corollaries 2.2 and 2.5 are provided at the end of Section 7. In Section 8, we check the general assumptions for the Vlasov–Poisson and relativistic Vlasov–Maxwell equations, and discuss some extensions as well. As already mentioned, Section 9 is devoted to the particular case of the transport/elliptic case, for which Theorem 2.1 can be improved. We end the paper with the study of two examples that we cook up in order to discuss the regularity assumptions of Theorem 2.1.

We will prove Theorem 2.1 when n is odd, of the form $n = 2m-1$, and the higher regularity index n' is even of the form $n' = 2(m+p)$. The other cases follow by the same arguments. The requirement on n and n' is $m > p+2$. The assumption (2-13) in this case is given by

$$\partial_x^{2(m-p+k)} \partial_x^\alpha \partial_v^\beta f_0 \in \mathcal{H}^0_r \quad \text{for all } |\alpha|+|\beta| = 2p-k, \text{ for all } k = 0,\dots,2p. \tag{2-16}$$

3. Local well-posedness

We prove in this section a basic local Sobolev well-posedness result for (2-1). We start by recalling useful product estimates in weighted Sobolev spaces, taken from [Han-Kwan and Rousset 2016].

Lemma 3.1. *Let s be a nonnegative integer. Consider a smooth nonnegative function $\chi = \chi(v)$ that satisfies $|\partial^\alpha \chi| \le C_\alpha \chi$ for every multi-index α such that $|\alpha| \le s$:*

- *Consider two functions $f = f(x,v)$, $g = g(x,v)$; then we have for $k \ge \frac{1}{2}s$,*

$$\|\chi f g\|_{H^s_{x,v}} \lesssim \|f\|_{W^{k,\infty}_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|g\|_{W^{k,\infty}_{x,v}} \|\chi f\|_{H^s_{x,v}}. \tag{3-1}$$

- *Consider a function $E = E(x)$ and a function $F(x,v)$; then we have for any $s_0 > d$,*

$$\|\chi E F\|_{H^s_{x,v}} \lesssim \|E\|_{H^{s_0}_x} \|\chi F\|_{H^s_{x,v}} + \|E\|_{H^s_x} \|\chi F\|_{H^s_{x,v}}. \tag{3-2}$$

- *Consider a vector field $E = E(x)$, a function $A(v)$, and a function $f = f(x,v)$; then we have for any $s_0 > 1+d$ and for any multi-indices α, β such that $|\alpha|+|\beta| = s \ge 1$,*

$$\|\chi[\partial_x^\alpha \partial_v^\beta, A(v)E(x)\cdot\nabla_v] f\|_{L^2_{x,v}} \lesssim \|A\|_{W^{s,\infty}_v} (\|E\|_{H^{s_0}_x} \|\chi f\|_{H^s_{x,v}} + \|E\|_{H^s_x} \|\chi f\|_{H^s_{x,v}}). \tag{3-3}$$

- *Consider two functions $f = f(x,v)$, $g = g(x,v)$; then we have for multi-indices α, β with $|\alpha|+|\beta| \le s$,*

$$\|\partial^\alpha_{x,v} f\, \partial^\beta_{x,v} g\|_{L^2} \lesssim \left\| \frac{1}{\chi} f \right\|_{L^\infty_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|\chi g\|_{L^\infty_{x,v}} \left\| \frac{1}{\chi} f \right\|_{H^s_{x,v}}. \tag{3-4}$$

Proposition 3.2. *Let $n > d+1$ and $r > r_0 + \frac{1}{2}d$. Assume that $f_0 \in \mathcal{H}^n_r$ and $F^j(0) \in H^n_x$. Then there exists $T > 0$ such that there is a unique solution $(f(t), F(t))$ with initial data $(f_0, F(0))$ to (2-1) such that $f(t) \in C(0, T; \mathcal{H}^n_r)$ and $F^j(t) \in L^\infty(0, T; H^n_x)$.*

Proof of Proposition 3.2. The existence part follows from a standard iterative construction. We define recursively a sequence of distribution functions $(f_{(k)})_{k\in\mathbb{N}}$, denoting by $F_{(k)}$ the force field associated to $f_{(k)}$ and the initial condition $(F^j(0))$. Let us define

$$R_0 := \|f_0\|_{\mathcal{H}^n_r} + \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}.$$

We set $f_{(0)} := f_0$ and assume that $f_{(k)}$ is already constructed (with associated force field $F_{(k)}$), and is such that for some $T_k > 0$, we have $f_{(k)} \in C(0, T_k; \mathcal{H}^n_r)$, and

$$\|f_{(k)}\|_{L^\infty(0,T_k;\mathcal{H}^n_r)} \le 2R_0. \tag{3-5}$$

We define $f_{(k+1)}$ as the unique solution on $[0, T_k)$ to the equation

$$\partial_t f_{(k+1)} + a(v)\cdot\nabla_x f_{(k+1)} + F_{(k)}\cdot\nabla_v f_{(k+1)} = 0, \quad f_{(k+1)}|_{t=0} = f_0, \tag{3-6}$$

obtained by the method of characteristics.

Applying the operator $\partial^\alpha_x\partial^\beta_v$ to (3-6) for $|\alpha|+|\beta| \le n$ yields

$$(\partial_t + a(v)\cdot\nabla_x + F_{(k)}\cdot\nabla_v)(\partial^\alpha_x\partial^\beta_v f_{(k+1)}) + [\partial^\alpha_x\partial^\beta_v, a(v)\cdot\nabla_x + F_{(k)}\cdot\nabla_v] f_{(k+1)} = 0.$$

We then take the L^2 scalar product with $(1+|v|^2)^r\, \partial^\alpha_x\partial^\beta_v f_{(k+1)}$ and sum for all $|\alpha|+|\beta| \le n$. By using (2-3), we have

$$\sum_{|\alpha|+|\beta|\le n} \int \big|[\partial^\alpha_x\partial^\beta_v, a(v)\cdot\nabla_x] f_{(k+1)}\, \partial^\alpha_x\partial^\beta_v f_{(k+1)}\big|(1+|v|^2)^r\, dv\, dx \le \|f_{(k+1)}\|^2_{\mathcal{H}^n_r}.$$

Thanks to (2-6) and estimate (3-3) in Lemma 3.1 with $s = n$, $\chi(v) = (1+|v|^2)^{\frac{1}{2}r}$ and $s_0 = n$ (recall that $n > d+1$), we have for all $j \in \{1, \dots, \ell\}$,

$$\big\|\chi[\partial^\alpha_x\partial^\beta_v, A_j(v)F^j_{(k)}(x)\cdot\nabla_v] f_{(k+1)}\big\|_{L^2_{x,v}} \lesssim \|F^j_{(k)}\|_{H^n_x}\|f_{(k+1)}\|_{\mathcal{H}^n_r}.$$

Therefore by Cauchy–Schwarz, we get

$$\sum_{|\alpha|+|\beta|\le n} \int \big|[\partial^\alpha_x\partial^\beta_v, F_{(k)}\cdot\nabla_v] f_{(k+1)}\, \partial^\alpha_x\partial^\beta_v f_{(k+1)}\big|(1+|v|^2)^r\, dv\, dx \lesssim \|F^j_{(k)}\|_{H^n_x}\|f_{(k+1)}\|^2_{\mathcal{H}^n_r}.$$

Recalling that $\nabla_v \cdot F = 0$, we deduce that for all $t \in (0, T_k)$,

$$\frac{d}{dt}\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r} \lesssim \bigg(1 + \sum_{j=1}^{\ell}\|F^j_{(k)}\|_{H^n_x}\bigg)\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r}$$

so that

$$\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r} \lesssim \|f_0\|_{\mathcal{H}^n_r}\exp\bigg[C\int_0^t\bigg(1 + \sum_{j=1}^{\ell}\|F^j_{(k)}(s)\|_{H^n_x}\bigg)\,ds\bigg]. \tag{3-7}$$

We set

$$m_{\psi_i,(k)}(t,x) = \int_{\mathbb{R}^d} f_{(k)}(t,x,v)\,\psi_i(v)\,dv,$$

and by Cauchy–Schwarz and (2-7), we get, for $r' > \frac{1}{2}d$ such that $r \geq r_0 + r'$, which is possible thanks to the assumption $r > r_0 + \frac{1}{2}d$, that

$$\begin{aligned}
\|m_{\psi_i,(k)}\|_{L^2(0,t;H^n)} &= \sum_{|\alpha|\leq n} \left\| \left(\int_{\mathbb{R}^d} \partial_x^\alpha f_{(k)} \psi_i \, dv \right)^2 \right\|_{L^2(0,t;L^1_x)}^{\frac{1}{2}} \\
&\lesssim \sum_{|\alpha|\leq n} \left\| \left(\int_{\mathbb{R}^d} |\partial_x^\alpha f_{(k)}|^2 (1+|v|^2)^{r_0+r'} \, dv \right) \left(\int_{\mathbb{R}^d} \frac{|\psi_i|^2 \, dv}{(1+|v|^2)^{r_0+r'}} \right) \right\|_{L^2(0,t;L^1_x)}^{\frac{1}{2}} \\
&\lesssim \sum_{|\alpha|\leq n} \left\| \left(\int_{\mathbb{R}^d} |\partial_x^\alpha f_{(k)}|^2 (1+|v|^2)^{r_0+r'} \, dv \right) \left(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^{r'}} \right) \right\|_{L^2(0,t;L^1_x)}^{\frac{1}{2}} \\
&\lesssim \|f_{(k)}\|_{L^2(0,t;\mathcal{H}^n_r)}.
\end{aligned}$$

Therefore, by (2-9), denoting by $C > 0$ a generic constant that does not depend on t or k, we obtain

$$\begin{aligned}
&\|f_{(k+1)}(t)\|_{\mathcal{H}^n_r} \\
&\qquad \lesssim \|f_0\|_{\mathcal{H}^n_r} \exp\left[Ct + C\sqrt{t} \sum_{j=1}^{\ell} \|F^j_{(k)}\|_{L^2(0,t;H^n_x)} \right] \\
&\qquad \lesssim \|f_0\|_{\mathcal{H}^n_r} \exp\left[Ct + C\sqrt{t} \sum_{j=1}^{\ell} \Gamma^{(j)}_n \left(t, (\sqrt{t}\|m_{\psi_i,(k)}\|_{L^\infty(0,t;H^n_x)})_{i=1,\dots,r}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x} \right) \right] \\
&\qquad \lesssim \|f_0\|_{\mathcal{H}^n_r} \exp\left[Ct + C\sqrt{t} \sum_{j=1}^{\ell} \Gamma^{(j)}_n \left(t, \sqrt{t}\|f_{(k)}\|_{L^\infty(0,t;\mathcal{H}^n_r)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x} \right) \right].
\end{aligned}$$

We now observe that if we choose $T > 0$ small enough so that

$$R_0 \exp\left[CT + C\sqrt{T} \sum_{j=1}^{\ell} \Gamma^{(j)}_n (T, 2\sqrt{T} R_0, R_0) \right] < 2R_0, \tag{3-8}$$

and $T_k \geq T$, then,

$$\|f_{(k+1)}(t)\|_{L^\infty(0,T;\mathcal{H}^n_r)} \leq 2R_0. \tag{3-9}$$

Therefore, by induction, we obtain that for all $k \in \mathbb{N}$, we have $f_{(k)} \in C(0,T;\mathcal{H}^n_r)$, and

$$\|f_{(k)}\|_{L^\infty(0,T;\mathcal{H}^n_r)} \leq 2R_0. \tag{3-10}$$

For $k \in \mathbb{N} \setminus \{0\}$, we set $h_k := f_{(k+1)} - f_{(k)}$, which satisfies the equation

$$\partial_t h_k + a(v)\cdot\nabla_x h_k + F[f_k]\cdot\nabla_v h_k + (F[f_{(k)}] - F[f_{(k-1)}])\cdot\nabla_v f_k = 0. \tag{3-11}$$

By weighted L^2 estimates, proceeding as before, we get

$$\frac{d}{dt}\|h_k(t)\|^2_{\mathcal{H}^0_r} \lesssim \Big(1+\sum_{j=1}^{\ell}\|F^j[f_{(k)}]\|_{H^n_x}\Big)\|h_k(t)\|^2_{\mathcal{H}^0_r} + \|f_{(k)}\|_{\mathcal{H}^n_r}\sum_{j=1}^{\ell}\|F^j[f_{(k)}]-F^j[f_{(k-1)}]\|_{L^2_x}\|h_k(t)\|_{\mathcal{H}^0_r}.$$

Let $t\in(0,T)$. Integrating in time, applying Cauchy–Schwarz and using the stability property (2-11) and the uniform estimates (3-10) for $(f_{(k)})$, we obtain

$$\begin{aligned}
&\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)}\\
&\quad\lesssim \int_0^t\Big(1+\sum_{j=1}^{\ell}\|F^j[f_{(k)}]\|_{H^n_x}\Big)\|h_k(s)\|_{\mathcal{H}^0_r}\,ds + \sum_{j=1}^{\ell}\int_0^t \|F^j[f_{(k)}]-F^j[f_{(k-1)}]\|_{L^2_x}\,ds\\
&\quad\lesssim \sqrt{t}\Big[(\sqrt{t}+\|F^j[f_{(k)}]\|_{L^2(0,t;H^n_x)})\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)} + \sum_{j=1}^{\ell}\|F^j[f_{(k)}]-F^j[f_{(k-1)}]\|_{L^2(0,t;L^2_x)}\Big]\\
&\quad\lesssim \sqrt{t}\Big[\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)} + \sqrt{t}\sum_{i=1}^{r}\Gamma_n^{(j)\sharp}\Big(t,\Big(\sqrt{t}\Big\|\int (f_{(k)}-f_{(k-1)})\psi_i(v)\,dv\Big\|_{L^\infty(0,t;L^2_x)}\Big)_{i=1,\dots,r}\Big)\Big]\\
&\quad\lesssim \sqrt{t}\Big[\|h_k\|_{L^\infty(0,t;\mathcal{H}^0_r)} + \sqrt{t}\sum_{i=1}^{r}\Gamma_n^{(j)\sharp}(t,\sqrt{t}\|h_{k-1}\|_{L^\infty(0,t;\mathcal{H}^0_r)})\Big].
\end{aligned}$$

We can thus pick a small enough time $T'>0$, independently of k such that for all $k\in\mathbb{N}\setminus\{0\}$,

$$\|f_{(k+1)}-f_{(k)}\|_{L^\infty(0,T';\mathcal{H}^0_r)} \le \tfrac{1}{2}\|f_{(k)}-f_{(k-1)}\|_{L^\infty(0,T';\mathcal{H}^0_r)}.$$

We can therefore pass to the limit in (3-6) and find that the limit $(f, F[f])$ satisfies (in the sense of distributions)

$$\partial_t f + a(v)\cdot\nabla_x f + F[f]\cdot\nabla_v f = 0, \tag{3-12}$$

with the initial conditions $(f_0, F^j(0))$. We deduce from (3-12) that $f\in C^0(0,T';\mathcal{H}^n_r)$ and $\partial_t f\in L^2(0,T';\mathcal{H}^{n-1}_{r-1})$. Also, thanks to (2-10), we deduce $F^j\in L^\infty(0,T';H^n_x)$. That the equation is satisfied in a classical way follows from the smoothness of $(f, F[f)])$. Uniqueness is also a consequence of the contraction estimate. □

The main matter is now to obtain the higher regularity statement for the moments. To this end, we will focus only on the task of obtaining a priori estimates for *smooth* solutions of (2-1); setting

$$M := \|f_0\|_{\mathcal{H}^{2m-1}_r} + \sum_{k=0}^{2p}\sum_{|\alpha|+|\beta|=2p-k}\|\partial_x^{2(m-p+k)}\partial_x^\alpha\partial_v^\beta f_0\|_{\mathcal{H}^0_r} + \sum_{j=1}^{\ell}\|F^j(0)\|_{H^{2(m+p)}_x}, \tag{3-13}$$

we look for some time $T_0 > 0$ depending only on M such that given a smooth test function $\psi \in L^\infty(0, T_0; \mathcal{W}^{2(m+p),\infty}_{-r_0})$, the following estimate holds:

$$\left\| \int f \psi(v)\, dv \right\|_{L^2(0,T_0;H^{2(m+p)}_x)} \leq C_\psi \Lambda(T_0, M), \tag{3-14}$$

where Λ is a polynomial function which is nondecreasing with respect to each of its arguments, once the others are fixed nonnegative numbers. In what follows, the function Λ may change from line to line but will always refer to such a function.

Once a priori estimates such as (3-14) as are obtained, we apply them to the sequence of solutions built in the iteration scheme proving the existence of solutions in the proof of Proposition 3.2. Passing to the limit yields the higher regularity for the moments of the solution $f(t)$.

4. Differential operators

In this section, we introduce and study the second-order differential operators (with coefficients depending on t and x) that we use in order to apply derivatives in x on the Vlasov equation (2-1).

The basic operators are defined in (4-3) and the definition of the coefficients is provided in Lemma 4.1. By definition these operators involve only derivatives in x at initial time. The key algebraic result reflecting the good commutation properties of these operators with the transport operator is stated in Lemma 4.2.

The composition of these operators is then studied:

- In Lemma 4.4, it is shown that they are indeed well suited to study the regularity of moments, as after integration in v, they act like derivations in x only (plus remainders that we can control). The proof is quite technical as one needs to be careful of the limited available smoothness on the coefficients of the differential operators. Note that in the statement, one does assume some (limited) higher-order smoothness for the moments: this is in prevision of a forthcoming induction argument.

- In Lemmas 4.5 and 4.6, the equations satisfied by the functions obtained after composition of these operators is established. This is where the key algebraic Lemma 4.2 appears to be crucial. Whereas the formal computation is straightforward, here again, the proof appears to be quite technical in order to justify that remainders are indeed well controlled. One also needs to be careful in order to get some Sobolev regularity for the coefficients involved in the equations.

- As the systems of equations in Lemmas 4.5 and 4.6 are not closed, this invites one to study the system satisfied by a larger set of appropriate functions; this is the purpose of Lemmas 4.7 and 4.8 (whose proof is similar to that of Lemmas 4.5 and 4.6).

4A. ***Second-order operators.*** As in the Introduction, we set $\mathcal{T} := \partial_t + a(v) \cdot \nabla_x + F \cdot \nabla_v$ as the transport operator to ease readability.

Lemma 4.1. *Let $n > d + 1$. Assume that $(F^j) \in L^2(0, T'; H^n_x)$ with norm bounded by $\Lambda(T', M)$. There is $T \in (0, T')$ such that there exists a unique smooth solution $(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})_{i,j,k,l\in\{1,\dots,d\}}$ on $[0, T]$ of the* *system*

$$
\begin{cases}
\mathcal{T}\varphi^{i,j}_{k,l} = \sum_{k'} \partial_{v_{k'}} a(v)_k \, \psi^{i,j}_{k',l} + \sum_{k'} \partial_{v_{k'}} a(v)_k \, \psi^{i,j}_{l,k'} - \sum_{k',l',m} \partial_{v_{l'}} a(v)_m \, \varphi^{i,j}_{k',l'} \varphi^{k',m}_{k,l} \\
\qquad\qquad + \delta_{k,j} \, \partial_{x_i} F_l + \delta_{k,i} \, \partial_{x_j} F_l + \sum_{l'} \varphi^{i,j}_{k,l'} \, \partial_{v_{l'}} F_l, \\
\mathcal{T}\psi^{i,j}_{k,l} = - \sum_{k',l',m} \partial_{v_{l'}} a(v)_m \, \varphi^{i,j}_{k',l'} \psi^{k',m}_{k,l} + \varphi^{i,j}_{k,l} \, \partial_{x_k} F_k + \sum_{k'} \psi^{i,j}_{k',l} \partial_{v_{k'}} F_k + \sum_{l'} \psi^{i,j}_{k,l'} \, \partial_{v_{l'}} F_l, \\
\varphi^{i,j}_{k,l}|_{t=0} = \psi^{i,j}_{k,l}|_{t=0} = 0,
\end{cases}
\tag{4-1}
$$

where δ denotes the Kronecker function and $a(v)_k$ and F_k stand for the k-th coordinates of $a(v)$ and F. Moreover we have the following estimates:

$$
\begin{aligned}
&\sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{W^{p,\infty}_{x,v}} \lesssim \Lambda(T, M) \quad \text{for all } p < n - 1 - \tfrac{1}{2} d, \\
&\sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{\mathcal{H}^{n-1}_{-\tilde{r}}} \lesssim \Lambda(T, M) \quad \text{for all } \tilde{r} > \tfrac{1}{2} d.
\end{aligned}
\tag{4-2}
$$

We will not reproduce the proof of Lemma 4.1, since it follows, mutatis mutandis, that of Lemma 4.2 of [Han-Kwan and Rousset 2016]: System (4-1) is solved as a semilinear system of coupled kinetic transport equations. Note that we use the assumptions (2-3) on a and (2-6) on A to control the contribution of the additional linear and semilinear terms that appear compared to Lemma 4.2 of [Han-Kwan and Rousset 2016].

We introduce now the second-order operators

$$
L_{i,j} := \partial^2_{x_i,x_j} + \sum_{1 \le k,l \le d} (\varphi^{i,j}_{k,l} \, \partial_{x_k} \partial_{v_l} + \psi^{i,j}_{k,l} \, \partial^2_{v_k,v_l}) \quad \text{for all } i, j \in \{1, \dots, d\}.
\tag{4-3}
$$

We observe that by uniqueness of the solution of (4-1) and a symmetry argument, $L_{i,j} = L_{j,i}$.

One of the interests of the operators $L_{i,j}$ comes from the following lemma.

Lemma 4.2. *For all smooth functions f, we have the formula*

$$
\begin{aligned}
L_{i,j}\mathcal{T}(f) = \mathcal{T}L_{i,j}(f) + \Big(\partial^2_{x_i,x_j} F + \sum_{k,l} \varphi^{i,j}_{k,l} \, \partial_{x_k} \partial_{v_l} F + \psi^{i,j}_{k,l} \, \partial^2_{v_k,v_l} F\Big) \cdot \nabla_v f \\
+ \sum_{k,l} \psi^{i,j}_{k,l} \, \partial^2_{v_k,v_l} a(v) \cdot \nabla_x f + \sum_{k,l,m} \partial_{v_l} a(v)_m \, \varphi^{i,j}_{k,l} L_{k,m} f.
\end{aligned}
\tag{4-4}
$$

Remark 4.3. Formula (4-4) can also be written in a more synthetic form:

$$
L_{i,j}\mathcal{T}(f) = \mathcal{T}L_{i,j}(f) + (L_{i,j} F) \cdot \nabla_v f + (L_{i,j} a) \cdot \nabla_x f + \sum_{k,l,m} \partial_{v_l} a(v)_m \, \varphi^{i,j}_{k,l} L_{k,m} f.
$$

Proof of Lemma 4.2. We have by direct computations

$$\partial^2_{x_i x_j}(\mathcal{T} f) = \mathcal{T}(\partial^2_{x_i,x_j} f) + \partial^2_{x_i,x_j} F\cdot\nabla_v f + \partial_{x_i} F\cdot\nabla_v\partial_{x_j} f + \partial_{x_j} F\cdot\nabla_v\partial_{x_i} f,$$

$$\begin{aligned}\varphi^{i,j}_{k,l}\,\partial_{x_k}\partial_{v_l}(\mathcal{T} f) &= \mathcal{T}(\varphi^{i,j}_{k,l}\,\partial_{x_k}\partial_{v_l} f) - \mathcal{T}(\varphi^{i,j}_{k,l})\partial_{x_k}\partial_{v_l} f\\ &\quad + \varphi^{i,j}_{k,l}\big(\partial_{v_l}a(v)\cdot\nabla_x\partial_{x_k} f + \partial_{x_k} F\cdot\nabla_v\partial_{v_l} f + \partial_{v_l} F\cdot\nabla_v\partial_{x_k} f + \partial_{x_k}\partial_{v_l} F\cdot\nabla_v f\big),\end{aligned}$$

$$\begin{aligned}\psi^{i,j}_{k,l}\,\partial^2_{v_k,v_l}(\mathcal{T} f) &= \mathcal{T}(\psi^{i,j}_{k,l}\,\partial^2_{v_k,v_l} f) - \mathcal{T}(\psi^{i,j}_{k,l})\partial^2_{v_k,v_l} f\\ &\quad + \psi^{i,j}_{k,l}\big(\partial_{v_l}a(v)\cdot\nabla_x\partial_{v_k} f + \partial_{v_k}a(v)\cdot\nabla_x\partial_{v_l} f + \partial^2_{v_k,v_l}a(v)\cdot\nabla_v f\\ &\qquad + \partial_{v_k} F\cdot\nabla_v\partial_{v_l} f + \partial_{v_l} F\cdot\nabla_v\partial_{v_k} f + \partial^2_{v_k,v_l} F\cdot\nabla_v f\big).\end{aligned}$$

We can rewrite

$$\begin{aligned}\varphi^{i,j}_{k,l}\,\partial_{v_l}a(v)\cdot\nabla_x\partial_{x_k} f &= \varphi^{i,j}_{k,l}\sum_m \partial_{v_l}a(v)_m\,\partial_{x_m}\partial_{x_k} f\\ &= \varphi^{i,j}_{k,l}\sum_m \partial_{v_l}a(v)_m\Big(L_{k,m} f - \sum_{k',l'}(\varphi^{k,m}_{k',l'}\,\partial_{x_{k'}}\partial_{v_{l'}} + \psi^{k,m}_{k',l'}\,\partial^2_{v_{k'},v_{l'}}) f\Big),\end{aligned}$$

which gives

$$\begin{aligned}L_{i,j}\mathcal{T}(f) &= \mathcal{T} L_{i,j}(f) + \partial_{x_i x_j} F\cdot\nabla_v f\\ &\quad + \sum_{k,l}\big(\varphi^{i,j}_{k,l}\,\partial_{x_k}\partial_{v_l} F\cdot\nabla_v f + \psi^{i,j}_{k,l}\,\partial^2_{v_k,v_l} F\cdot\nabla_v f + \psi^{i,j}_{k,l}\,\partial^2_{v_k,v_l} a(v)\cdot\nabla_x f\big)\\ &\quad + \sum_{k,l,m}\partial_{v_l}a(v)_m\varphi^{i,j}_{k,l} L_{k,m} f\\ &\quad + \sum_{k,l}\partial_{x_k}\partial_{v_l} f\Big[-\mathcal{T}\varphi^{i,j}_{k,l} + \sum_{k'}\partial_{v_{k'}}a(v)_k\psi^{i,j}_{k',l} + \sum_{k'}\partial_{v_{k'}}a(v)_k\psi^{i,j}_{l,k'}\\ &\qquad\qquad - \sum_{k',l',m}\partial_{v_{l'}}a(v)_m\varphi^{i,j}_{k',l'}\varphi^{k',m}_{k,l} + \delta_{k,j}\,\partial_{x_i} F_l + \delta_{k,i}\,\partial_{x_j} F_l + \sum_{l'}\varphi^{i,j}_{k,l'}\,\partial_{v_{l'}} F_l\Big]\\ &\quad + \sum_{k,l}\partial^2_{v_k,v_l} f\Big[-\mathcal{T}\psi^{i,j}_{k,l} - \sum_{k',l',m}\partial_{v_{l'}}a(v)_m\varphi^{i,j}_{k',l'}\psi^{k',m}_{k,l} + \varphi^{i,j}_{k,l}\,\partial_{x_k} F_k\\ &\qquad\qquad + \sum_{k'}\psi^{i,j}_{k',l}\,\partial_{v_{k'}} F_k + \sum_{l'}\psi^{i,j}_{k,l'}\,\partial_{v_{l'}} F_l\Big].\end{aligned}$$

We therefore deduce (4-4), because $(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})$ solves (4-1). □

4B. ***Composition of the second-order operators.*** Relying on Lemma 4.2, we shall use the $L_{i,j}$ operators in order to apply derivatives to the solution f of the Vlasov equation (2-1).

Set for $I, J \in \{1,\dots,d\}^k$,

$$L^{I,J} := L_{i_1,j_1}\cdots L_{i_k,j_k}. \tag{4-5}$$

Let us also introduce the following useful notation. Given $I = (i_1,\dots,i_k)$ and $J = (j_1,\dots,j_k)$, we set

$$\alpha(I,J) := (i_1, j_1, \dots, i_k, j_k), \tag{4-6}$$

and

$$\partial_x^{\alpha(I,J)} = \partial_{x_{i_1}} \partial_{x_{j_1}} \cdots \partial_{x_{i_k}} \partial_{x_{j_k}}. \tag{4-7}$$

Note that by construction,

$$L^{I,J}|_{t=0} = \partial_x^{\alpha(I,J)}.$$

In what follows, f will systematically stand for the solution of (2-1), starting from f_0 satisfying the assumptions of Theorem 2.1.

4C. *Moments in v.* We study in this section the moments in v of the $L^{I,J} f$. Until the end of the section, the times $T > 0$ will be such that the solution to (2-1) satisfies

$$\|f\|_{L^\infty(0,T;\mathcal{H}_r^{2m-1})} \leq 2R_0,$$

thanks to Proposition 3.2.

Lemma 4.4. • *Let $k = 0, \dots, p$ and $I, J \in \{1, \dots, d\}^{m+k}$. Assume that the force field satisfies $F^j \in L^2(0, T; H_x^{2(m+k)-1})$ with norm bounded by $\Lambda(T, M)$. Assume that for all $n = 2m, \dots, 2(m+k)-1$, for all $\varphi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+2+n-2m,\infty})$ such that $\|\varphi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+2+n-2m,\infty})} \leq \Lambda(T, M)$, and all $|\alpha| = n$, we have*

$$\left\| \int_{\mathbb{R}^d} (\partial_x^\alpha f)\, \varphi(t, x, v)\, dv \right\|_{L^2(0,T;L_x^2)} \leq \Lambda(T, M). \tag{4-8}$$

Let $\psi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+2+2k,\infty})$ satisfy $\|\psi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+2+2k,\infty})} \leq \Lambda(T, M)$. We have

$$\int_{\mathbb{R}^d} L^{I,J} f\, \psi(t, x, v)\, dv = \int_{\mathbb{R}^d} \partial_x^{\alpha(I,J)} f\, \psi(t, x, v)\, dv + \mathfrak{R}_{I,J,\psi}, \tag{4-9}$$

where $\mathfrak{R}_{I,J,\psi}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2(0,T;L_x^2)} \leq \Lambda(T, M). \tag{4-10}$$

• *Let $k = 0, \dots, p-1$ and $I, J \in \{1, \dots, d\}^{m+k}$. Assume the force field satisfies $F^j \in L^2(0, T; H_x^{2(m+k)})$ with norm bounded by $\Lambda(T, M)$. Assume that for all $n = 2m, \dots, 2(m+k)$, for all $|\alpha| = n$, and for all $\varphi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+2+n-2m,\infty})$ such that $\|\varphi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+2+n-2m,\infty})} \leq \Lambda(T, M)$, we have*

$$\left\| \int_{\mathbb{R}^d} (\partial_x^\alpha f)\, \varphi(t, x, v)\, dv \right\|_{L^2(0,T;L_x^2)} \leq \Lambda(T, M). \tag{4-11}$$

Let $\psi \in L^\infty(0, T; \mathcal{W}_{-r_0}^{d+3+2k,\infty})$ satisfy $\|\psi\|_{L^\infty(0,T;\mathcal{W}_{-r_0}^{d+3+2k,\infty})} \leq \Lambda(T, M)$. Let $\partial = \partial_{x_i}$ or ∂_{v_i} for some $i \in \{1, \dots, d\}$. We have

$$\int_{\mathbb{R}^d} \partial L^{I,J} f\, \psi(t, x, v)\, dv = \int_{\mathbb{R}^d} \partial_x^{\alpha(I,J)} \partial f\, \psi(t, x, v)\, dv + \mathfrak{R}_{I,J,\psi}, \tag{4-12}$$

where $\mathfrak{R}_{I,J,\psi}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2(0,T;L_x^2)} \leq \Lambda(T, M). \tag{4-13}$$

This result will allow us to set up an induction argument: indeed, with the assumption (4-8) (resp. (4-11)) that corresponds to regularity of the moments up to order $2(m+k)-1$ (resp. $2(m+k)$), this will imply that controlling the moments of the $(L^{I,J}f)$ gives information on the regularity of the moments up to order $2(m+k)$ (resp. $2(m+k)+1$).

Proof of Lemma 4.4. Let us focus only on the first item (the proof of the second one is completely similar). Let $\psi \in L^\infty(0,T;\mathcal{W}^{d+2+2k,\infty}_{-r_0})$. The beginning of the proof closely follows that of Lemma 4.3 of [Han-Kwan and Rousset 2016]. At first, we can expand $f_{I,J} = L^{I,J} f$ in a more tractable form. Let us set for readability

$$U := (\varphi^{i_\alpha,j_\beta}_{k',l}, \psi^{i_\alpha,j_\beta}_{k',l})_{1\le k',l\le d,\, 1\le\alpha,\beta\le m+k}.$$

Then, by induction, we obtain

$$\begin{aligned} f_{I,J} &= \partial_x^{\alpha(I,J)} f + \sum_{s=0}^{2(m+k)-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} P^{k_0}_{s,e,\alpha}(U)\, P^{k_1}_{s,e,\alpha}(\partial U)\cdots P^{k_s}_{s,e,\alpha}(\partial^s U)\,\partial_v^e \partial^\alpha f \\ &=: \partial_x^{\alpha(I,J)} f + \sum_{s=0}^{2(m+k)-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} \mathcal{R}^{k_0,\dots,k_s}_{s,e,\alpha}, \end{aligned} \tag{4-14}$$

where the sum is taken on indices such that

$$\begin{gathered} |e| = 1, \quad |\alpha| = 2(m+k)-1-s, \\ k_0+k_1+\cdots+k_s \le m+k, \quad k_0 \ge 1, \quad k_1+2k_2+\cdots+sk_s = s, \end{gathered} \tag{4-15}$$

and for all $0 \le p \le s$, we have $P^{k_p}_{s,e,\alpha}(X)$ is a polynomial of degree smaller than k_p (we denote by $\partial^k U$ the vector made of all the partial derivatives of length k of all components of U). We can set

$$\mathfrak{R}_{I,J,\psi} = \int_{\mathbb{R}^d} \psi(\cdot,v) \sum_{s=0}^{2(m+k)-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} \mathcal{R}^{k_0,\dots,k_s}_{s,e,\alpha}\, dv,$$

so that we have to estimate $\int_{\mathbb{R}^d} \psi \mathcal{R}^{k_0,\dots,k_s}_{s,e,\alpha}\, dv$. All the following estimates are uniform in time for $t \in [0,T]$ and we shall dismiss the time parameter to ease readability.

We begin by estimating the terms for which $s \ge 2k+1$. Note that for all these terms the total number of derivatives applied to f is at most $2m-1$.

- When $s < 2(m+k) - \frac{1}{2}d - 2$, we can use estimate (4-2) in Lemma 4.1 to obtain

$$\| P^{k_0}_{s,e,\alpha}(U)\, P^{k_1}_{s,e,\alpha}(\partial U)\cdots P^{k_s}_{s,e,\alpha}(\partial^s U)\|_{L^\infty_{x,v}} \le \Lambda(T,M),$$

and hence using that

$$\sup_v |(1+|v|^2)^{-\frac{1}{2}r_0}\psi(\cdot,v)| \le \Lambda(T,M),$$

we obtain by Cauchy–Schwarz that since $r > r_0 + r'$ for some $r' > \frac{1}{2}d$, we have

$$\begin{aligned}\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} &\le \big\| \|(1+|v|^2)^{-\frac{1}{2}(r_0-r')}\psi\|_{L^2_v} \|(1+|v|^2)^{\frac{1}{2}(r_0+r')}\, \partial^e_v \partial^\alpha f\|_{L^2_v} \big\| \\ &\le \Lambda(T,M) \biggl(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^{r'}} \biggr)^{\frac{1}{2}} \|f\|_{\mathcal{H}^{2m-1}_r} \\ &\le \Lambda(T,M).\end{aligned}$$

• Let us now consider $s \ge 2(m+k)-2-\frac{1}{2}d$. We start with the case where in the sequence $(k_1,\dots,k_s)$, the largest index l such that $k_l \ne 0$ and $k_p = 0$ for every $p > l$ is such that $l > \frac{1}{2}s$. In this case, since $lk_l \le s$ has to hold, we necessarily have $k_l = 1$. Moreover, for the indices $p < l$ such that $k_p \ne 0$, we must have $p \le pk_p < \frac{1}{2}s$. Thus, we can use estimate (4-2) in Lemma 4.1 to bound $\|\partial^p U\|_{L^\infty_{x,v}}$ provided $\frac{1}{2}s \le 2(m+k)-\frac{1}{2}d-2$. Since $s \le 2(m+k)-2$, this is satisfied thanks to the assumption that $2m > 2+d$. We thus obtain

$$\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \le \Lambda(T,M) \left\| \int \psi\, \partial^l U\, \partial^e_v \partial^\alpha f\, dv \right\|_{L^2_x}.$$

Next, we can use the fact that

$$\begin{aligned}\left\| \int \psi\, \partial^l U\, \partial^e_v \partial^\alpha f \right\|_{L^2_x} &\lesssim \Lambda(T,M) \big\| \|(1+|v|^2)^{-\frac{1}{2}r}\, \partial^l U\|_{L^2_v} \|(1+|v|^2)^{\frac{1}{2}(r_0+r)}\, \partial^e_v \partial^\alpha f\|_{L^2_v} \big\|_{L^2_x} \\ &\lesssim \Lambda(T,M) \|U\|_{\mathcal{H}^{2m-2}_{-r}} \sup_x \|(1+|v|^2)^{\frac{1}{2}r}\, \partial^e_v \partial^\alpha f\|_{L^2_v}.\end{aligned}$$

By the Sobolev embedding in x, we have

$$\sup_x \|(1+|v|^2)^{\frac{1}{2}r}\, \partial^e_v \partial^\alpha f\|_{L^2_v} \lesssim \|f\|_{\mathcal{H}^{2m-1}_r}$$

as soon as $2m-1 > 1+|\alpha|+\frac{1}{2}d = 1+2(m+k)-1-s+\frac{1}{2}d$, which is equivalent to $s > 1+2k+\frac{1}{2}d$. Since we are in the case where $s \ge 2(m+k)-2-\frac{1}{2}d$, the condition is matched, thanks to the assumption $2m > 3+d$. Consequently, by using estimate (4-2) in Lemma 4.1, we obtain again that

$$\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \lesssim \Lambda(T,M).$$

Finally, it remains to handle the case where $k_l = 0$ for every $l > \frac{1}{2}s$. As above, we necessarily have $\frac{1}{2}s < 2(m+k)-\frac{1}{2}d-2$ and hence by using again estimate (4-2) in Lemma 4.1, we find

$$\|\partial^l U\|_{L^\infty_{x,v}} \le \Lambda(T,M), \quad l \le \tfrac{1}{2}s.$$

We deduce

$$\left\| \int \mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \le \Lambda(T,M) \|f\|_{\mathcal{H}^{2m-1}_r} \le \Lambda(T,M).$$

It remains to treat the cases corresponding to $s \le 2k$; that is to say, $\mathcal{R}^{k_0,\dots,k_s}_{e,s,\alpha}$ contains the maximal number of derivatives applied to f. This means that $|\alpha| = 2m-1,\dots,2(m+k)-1$ so that at least

$2m$ derivatives of f are involved. We define for readability the associated coefficient

$$\Gamma := \psi\, P^{k_0}_{s,e,\alpha}(U)\, P^{k_1}_{s,e,\alpha}(\partial U) \cdots P^{k_s}_{s,e,\alpha}(\partial^s U),$$

and we have to study the L^2_x norm of $\int \Gamma\, \partial^e_v \partial^\alpha f\, dv$.

First, assume that $|\alpha| \le 2(m+k)-2$ (which corresponds to $s \ge 1$). We note that for all $s' = 0, \dots, 2(m+k)-1-|\alpha|$, we have by Lemma 4.1 that

$$\|\partial^{s'} U\|_{W^{k,\infty}_{x,v}} \le \Lambda(T, M) \quad \text{for all } k < 2(m+k)-2-\tfrac{1}{2}d - s'.$$

Since $s' \le 2(m+k)-1-|\alpha|$, we have $2(m+k)-2-\frac{1}{2}d-s' \ge |\alpha|-\frac{1}{2}d-1 > d+2+|\alpha|+1-2m$ because $2m > \frac{3}{2}d+4$. Therefore

$$\|\partial^{s'} U\|_{W^{d+2+|\alpha|+1-2m,\infty}_{x,v}} \le \Lambda(T, M),$$

$$\|\Gamma\|_{\mathcal{W}^{d+2+|\alpha|+1-2m,\infty}_{-r_0}} \le \Lambda(T, M).$$

We can thus use the assumption (4-8) to obtain the bound

$$\left\| \int \Gamma\, \partial^e_v \partial^\alpha f\, dv \right\|_{L^2_x} \le \Lambda(T, M). \tag{4-16}$$

Assume finally that $|\alpha| = 2(m+k)-1$ (which corresponds to $s=0$); that is to say, $2(m+k)$ derivatives of f are involved. We can write, by integration by parts in v (relying on the fast decay of f and its derivatives at infinity)

$$\int_{\mathbb{R}^d} \Gamma\, \partial^e_v \partial^\alpha f\, dv = -\int_{\mathbb{R}^d} \partial^e_v \Gamma\, \partial^\alpha f\, dv.$$

We have

$$\|\partial^e_v \Gamma\|_{\mathcal{W}^{d+1+2k,\infty}_{-r_0}} \le \Lambda(T, M),$$

and we can use again (4-8) to obtain

$$\left\| \int \partial^e_v \Gamma\, \partial^\alpha f\, dv \right\|_{L^2_x} \le \Lambda(T, M).$$

In summary, we have proved

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2_x} \le \Lambda(T, M). \qquad \square$$

4D. *The equation satisfied by $L^{I,J} f$.* Using the algebraic identities of Lemma 4.2, we obtain:

Lemma 4.5. *For all $k = 0, \dots, p$, the following holds. Assume that $(F^j) \in L^2(0, T; H^{2(m+k)-1}_x)$ with norm bounded by $\Lambda(T, M)$. For all $I, J \in \{1, \dots, d\}^{m+k}$, we have*

$$\mathcal{T}(L^{I,J} f) + \partial_x^{\alpha(I,J)} F \cdot \nabla_v f = \sum_{r=m-k}^{m+k} \sum_{K,L \in \{1,\dots,d\}^r} \sum_{|\alpha|+|\beta|=m+k-r} \gamma^{I,J}_{K,L,\alpha,\beta}\, L^{K,L}\, \partial^\alpha_x \partial^\beta_v f + R_{I,J}, \tag{4-17}$$

where

- $\gamma^{I,J}_{K,L,\alpha,\beta}$ *are coefficients satisfying*

$$\|\gamma^{I,J}_{K,L,\alpha,\beta}\|_{L^2(0,T; W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T, M), \tag{4-18}$$

- $R_{I,J}$ *is a remainder satisfying*

$$\|R_{I,J}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} \lesssim \Lambda(T,M) \quad \textit{for all } \tilde r \le r - \tfrac{1}{2}d.$$

A version of this lemma was proved in [Han-Kwan and Rousset 2016] in the case $k=0$.

Lemma 4.5 will be useful in the induction argument to treat the case of even integers. For odd integers, we have the following result.

Lemma 4.6. *For all $k = 0,\dots,p-1$, the following holds. Assume that $(F^j) \in L^2(0,T;H_x^{2(m+k)})$ with norm bounded by $\Lambda(T,M)$. For all $I,J \in \{1,\dots,d\}^{m+k}$, and $i=1,\dots,d$, we have*

$$\mathcal{T}(L^{I,J}\,\partial_{x_i} f) + \partial_{x_i}\partial_x^{\alpha(I,J)} F\cdot\nabla_v f = \sum_{r=m-k-1}^{m+k}\;\sum_{K,L\in\{1,\dots,d\}^r}\;\sum_{|\alpha|+|\beta|=m+k+1-r} \gamma^{x_i,I,J}_{K,L,\alpha,\beta}\, L^{K,L}\,\partial_x^\alpha\partial_v^\beta f + R_{x_i,I,J}, \tag{4-19}$$

$$\mathcal{T}(L^{I,J}\,\partial_{v_i} f) + \partial_{v_i}\partial_x^{\alpha(I,J)} F\cdot\nabla_v f = \sum_{r=m-k-1}^{m+k}\;\sum_{K,L\in\{1,\dots,d\}^r}\;\sum_{|\alpha|+|\beta|=m+k+1-r} \gamma^{v_i,I,J}_{K,L,\alpha,\beta} L^{K,L}\,\partial_x^\alpha\partial_v^\beta f + R_{v_i,I,J}, \tag{4-20}$$

where

- $\gamma^{x_i I,J}_{K,L,\alpha,\beta}, \gamma^{v_i I,J}_{K,L,\alpha,\beta}$ *are coefficients satisfying*

$$\|\gamma^{x_i,I,J}_{K,L,\alpha,\beta}, \gamma^{v_i,I,J}_{K,L,\alpha,\beta}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M), \tag{4-21}$$

- $R_{x_i,I,J}$, $R_{v_i,I,J}$ *are remainders satisfying*

$$\|R_{x_i,I,J}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} + \|R_{v_i,I,J}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} \lesssim \Lambda(T,M) \quad \textit{for all } \tilde r \le r - \tfrac{1}{2}d.$$

4E. ***The equation satisfied by*** $L^{I,J}\,\partial_x^\alpha\partial_v^\beta f$**.** Lemma 4.5 invites us to search for a closed equation on $L^{I,J}\,\partial_x^\alpha\partial_v^\beta f$ for $k\in\{0,\dots,p\}$, $r\in\{m-k,\dots,m+k\}$, $I,J\in\{1,\dots,d\}^r$ and all $|\alpha|+|\beta| = m+k-r$ (and similarly for Lemma 4.6). This is the purpose of the next two lemmas.

Lemma 4.7. *Let $k=0,\dots,p$. Let $r = m-k,\dots,m+k$. Assume that $(F^j)\in L^2(0,T;H_x^{2(m+k)-1})$ with norm bounded by $\Lambda(T,M)$. For all $I,J\in\{1,\dots,d\}^r$ and all $|\alpha|+|\beta| = m+k-r$, we have*

$$\mathcal{T}(L^{I,J}\,\partial_x^\alpha\partial_v^\beta f) + \partial_x^\alpha\partial_v^\beta\partial_x^{\alpha(I,J)} F\cdot\nabla_v f = \sum_{r'=m-k}^{r}\;\sum_{K,L\in\{1,\dots,d\}^{r'}}\;\sum_{|\alpha'|+|\beta'|=m+k-r'} \gamma^{I,J,\alpha,\beta}_{K,L,\alpha',\beta'}\, L^{K,L}\,\partial_x^{\alpha'}\partial_v^{\beta'} f + R_{I,J,\alpha,\beta}, \tag{4-22}$$

where

- $\gamma^{I,J,\alpha,\beta}_{K,L,\alpha',\beta'}$ *are coefficients satisfying*

$$\|\gamma^{x_i,I,J}_{K,L,\alpha,\beta}, \gamma^{v_i,I,J}_{K,L,\alpha,\beta}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M), \tag{4-23}$$

- $R_{I,J,\alpha,\beta}$ *is a remainder satisfying*

$$\|R_{I,J,\alpha,\beta}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})} \lesssim \Lambda(T,M) \quad \textit{for all } \tilde r \le r - \tfrac{1}{2}d.$$

Lemma 4.8. *Let $k=0,\dots,p-1$. Let $r=m-k-1,\dots,m+k$. Assume that $(F^j)\in L^2(0,T;H_x^{2(m+k)})$ with norm bounded by $\Lambda(T,M)$. For all $I,J\in\{1,\dots,d\}^r$, and all $|\alpha|+|\beta|=m+k+1-r$, we have*

$$\mathcal{T}(L^{I,J}\,\partial_x^\alpha\partial_v^\beta f)+\partial_x^\alpha\partial_v^\beta\partial_x^{\alpha(I,J)}F\cdot\nabla_v f$$

$$=\sum_{r'=m-k-1}^{r}\;\sum_{K,L\in\{1,\dots,d\}^{r'}}\;\sum_{|\alpha'|+|\beta'|=m+k+1-r'}\gamma_{K,L,\alpha',\beta'}^{I,J,\alpha,\beta}\,L^{K,L}\,\partial_x^{\alpha'}\partial_v^{\beta'}f+R_{I,J,\alpha,\beta},\tag{4-24}$$

where

- $\gamma_{K,L,\alpha',\beta'}^{I,J,\alpha,\beta}$ *are coefficients satisfying*
$$\|\gamma_{K,L,\alpha,\beta}^{x_i,I,J},\gamma_{K,L,\alpha,\beta}^{v_i,I,J}\|_{L^2(0,T;W_{x,v}^{d+2,\infty})}\lesssim\Lambda(T,M),\tag{4-25}$$
- $R_{I,J,\alpha,\beta}$ *is a remainder satisfying*
$$\|R_{I,J,\alpha,\beta}\|_{L^\infty(0,T;\mathcal{H}^0_{\tilde r})}\lesssim\Lambda(T,M)\quad\textit{for all }\tilde r\le r-\tfrac12 d.$$

We observe that as wanted, Lemmas 4.7 and 4.8 provide *closed* systems of equations.

To conclude this section, we shall give the proofs of Lemmas 4.5 and 4.7 (the proofs of the remaining Lemmas 4.6 and 4.8 being very similar).

4F. *Proofs of Lemmas 4.5 and 4.7.*

Proof of Lemma 4.5. Let $\tilde r<r-\frac12 d$. Since $r>d$, we can assume, without loss of generality, that $\tilde r>\frac12 d$. We can write, by an induction argument relying on Lemma 4.2, that

$$\mathcal{T}(L^{I,J}f)=F_{I,J},$$

with the source term $F_{I,J}$ given by $F_{I,J}=-\sum_{i=1}^4 F_i$, where

$$F_1=\sum_{\ell=1}^{m+k-1}L_{i_1,j_1}\cdots L_{i_{m+k-\ell},j_{m+k-\ell}}\times\big((\partial^2_{x_{i_{m+k-\ell+1}},x_{j_{m+k-\ell+1}}}F)\cdot\nabla_v L_{i_{m+k-\ell+2},j_{m+k-\ell+2}}\cdots L_{i_{m+k},j_{m+k}}f\big),\tag{4-26}$$

$$F_2=\sum_{\ell=1}^{m+k-1}L_{i_1,j_1}\cdots L_{i_{m+k-\ell},j_{m+k-\ell}}\times\bigg(\bigg[\sum_{k,l}\varphi_{k,l}^{i_{m+k-\ell+1},j_{m+k-\ell+1}}\,\partial_{x_{i_{m+k-\ell+1}}}\partial_{v_{j_{m+k-\ell+1}}}F+\psi_{k,l}^{i_{m+k-\ell+1},j_{m+k-\ell+1}}\,\partial^2_{v_{i_{m+k-\ell+1}},v_{j_{m+k-\ell+1}}}F\bigg]\cdot\nabla_v L_{i_{m+k-\ell+2},j_{m+k-\ell+2}}\cdots L_{i_{m+k},j_{m+k}}f\bigg),\tag{4-27}$$

$$F_3=\sum_{\ell=1}^{m+k-1}L_{i_1,j_1}\cdots L_{i_{m+k-\ell},j_{m+k-\ell}}\times\bigg(\bigg[\sum_{k,l}\psi_{k,l}^{i_{m+k-\ell+1},j_{m+k-\ell+1}}\,\partial^2_{v_{i_{m+k-\ell+1}},v_{j_{m+k-\ell+1}}}a\bigg]\cdot\nabla_x L_{i_{m+k-\ell+2},j_{m+k-\ell+2}}\cdots L_{i_{m+k},j_{m+k}}f\bigg),\tag{4-28}$$

$$F_4 = \sum_{\ell=1}^{m+k-1} L_{i_1,j_1} \cdots L_{i_{m+k-\ell},j_{m+k-\ell}} \times \sum_{k',l',m'} \partial_{v_{l'}} a(v)_{m'}\, \varphi_{k',l'}^{i_{m+k-\ell+1},j_{m+k-\ell+1}} L_{k',m'} L_{i_{m+k-\ell+2},j_{m+k-\ell+2}} \cdots L_{i_{m+k},j_{m+k}} f. \tag{4-29}$$

We shall focus on the contribution of F_1. We have to estimate terms of the form

$$F_{1,\ell} = L^{m+k-\ell} G_\ell, \quad G_\ell = \partial^2 E \cdot \nabla_v L^{\ell-1}, \tag{4-30}$$

where we use the notation L^n for the composition of n operators of type $L_{i,j}$ (the exact combination of the operators involved in the composition does not matter here). Note that as in (4-14), we can write L^n in the form

$$L^n = \partial_x^{\alpha_n} + \sum_{s=0}^{2n-2} \sum_{e,\,\alpha,\,k_0,\dots,k_s} P_{s,e,\alpha}^{k_0}(U)\, P_{s,e,\alpha}^{k_1}(\partial U) \cdots P_{s,e,\alpha}^{k_s}(\partial^s U)\, \partial_v^e \partial^\alpha, \tag{4-31}$$

where for all $0 \le p \le s$, we have $P_{s,e,\alpha}^{k_p}(X)$ is a polynomial of degree smaller than k_p, the multi-index α_n has length $2n$ and the sum is taken on indices such that

$$|e| = 1, \quad |\alpha| = 2n-1-s, \quad k_0 + k_1 + \cdots + k_s \le n, \quad k_0 \ge 1, \quad k_1 + 2k_2 + \cdots + sk_s = s. \tag{4-32}$$

Let us first establish a general estimate, adapted from [Han-Kwan and Rousset 2016]. We set for any function $G(x,v)$,

$$J_p(G)(x,v) = \sum_{s,\,\beta,\,K \in E} J_{p,s,\beta,K}(G), \tag{4-33}$$

where $K = (k_0, \dots, k_s)$ and

$$J_{p,s,\beta,K}(G)(x,v) = P_{s,\beta}^{k_0}(U)\, P_{s,\beta}^{k_1}(\partial U) \cdots P_{s,\beta}^{k_s}(\partial^s U)\, \partial^\beta G, \tag{4-34}$$

where for all $0 \le r \le s$, we have $P_{s,\beta}^{k_r}(X)$ is a polynomial of degree smaller than k_r and the sum is taken over indices belonging to the set E defined by

$$|\beta| = p - s, \quad k_0 + k_1 + \cdots + k_s \le \tfrac{1}{2} p, \quad k_1 + 2k_2 + \cdots + sk_s = s, \quad 0 \le s \le p-2. \tag{4-35}$$

Lemma 4.9. *For $2(m+k) - 1 \ge p \ge 1$, $2m > d+3$, $\tilde{r} > \frac{1}{2}d$ and s, p, K satisfying (4-35), we have the estimate*

$$\|J_p(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M)\Big(\|G\|_{\mathcal{H}^p_{\tilde r}} + \sum_{\substack{l \ge 2(m+k) - \frac{1}{2}d - 2 \\ l + |\alpha| \le p,\ |\alpha| \ge 2}} \|\partial^l U\, \partial^\alpha G\|_{\mathcal{H}^0_{\tilde r}}\Big). \tag{4-36}$$

Proof of Lemma 4.9. For the terms in the sum such that $s < 2(m+k) - \frac{1}{2}d - 2$, we can use estimate (4-2) in Lemma 4.1 to obtain

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M)\|G\|_{\mathcal{H}^p_{\tilde r}}.$$

When $s \ge 2(m+k) - \frac{1}{2}d - 2$, we first consider the terms for which in the sequence $(k_1, \dots, k_s)$ the largest index l for which $k_l \ne 0$ is such that $l < 2(m+k) - \frac{1}{2}d - 2$. Then again thanks to estimate (4-2) in Lemma 4.1, we obtain

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M)\|G\|_{\mathcal{H}^p_{\tilde r}}.$$

When $l \ge 2(m+k)-\frac{1}{2}d-2$, we first observe that we necessarily have $k_l = 1$. Indeed if $k_l \ge 2$, because of (4-35), we must have $l \le \frac{1}{2}s$. This is possible only if $2(m+k)-\frac{1}{2}d-2 \le \frac{1}{2}p-2 \le \frac{1}{2}(2(m+k)-3)$, which corresponds to $m+k \le \frac{1}{2}d+1$, and hence this is impossible. Consequently $k_l = 1$. Moreover, we note that for the other indices $\tilde l$ for which $k_{\tilde l} \ne 0$, because of (4-35), we must have $\tilde l k_{\tilde l} \le s - l k_l$, so that

$$\tilde l \le s-l \le s-2(m+k)+\tfrac{1}{2}d+2 \le \tfrac{1}{2}d-1,$$

and we observe that $\frac{1}{2}d-1 < 2m-\frac{1}{2}d-2$. Consequently, by another use of estimate (4-2) in Lemma 4.1, we obtain

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_{\tilde r}} \le \Lambda(T,M) \sum_{\substack{l \ge 2(m+k)-\frac{1}{2}d-2 \\ l+|\alpha|\le p,\, |\alpha|\ge 2}} \|\partial^l U\, \partial^\alpha G\|_{\mathcal{H}^0_{\tilde r}}.$$

The fact that $|\alpha| \ge 2$ comes from (4-35). □

We shall now estimate $F_{1,\ell}$. Looking at the expansion of $L^{m+k-\ell}$ given by (4-31), we have to estimate terms of the form $J_p(G_\ell)$ with $p \le 2(m+k-\ell)$. Using (4-31), we decompose G_ℓ as

$$G_\ell = \partial^2 F \cdot \nabla_v L^{\ell-1} f = \partial^2 F \cdot \nabla_v (H_{\ell,+} + H_{\ell,-}) =: G_{\ell,+} + G_{\ell,-},$$

where:

- In $H_{\ell+}$, we gather all terms of the form (4-34), with $G = f$, such that $2k+1+|\beta| \ge 2\ell$. These terms may contribute to terms with at least $2m$ derivatives of f.
- On the other hand in $H_{\ell,-}$, the terms that arise correspond to $2k+1+|\beta| < 2\ell$, which involve at most $2m-1$ derivatives of f.

We first focus on the contribution of $G_{\ell,-}$; we define

$$F_{1,\ell,-} := L^{m+k-\ell} G_{\ell,-}.$$

Let us start with the case $\ell \ge \frac{1}{2}(m+k)$. We can use the decomposition (4-31), which means that we have to estimate terms of the form $J_p(G_{\ell,-})$ with $p \le 2(m+k-\ell) \le 2(m+k)-1$, and apply Lemma 4.9 to get

$$\begin{aligned} &\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \\ &\quad \le \Lambda(T,M)\Bigg(\|G_{\ell,-}\|_{L^2(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})} + \sum_{\substack{l \ge 2(m+k)-\frac{1}{2}d-2 \\ l+|\alpha|\le 2(m+k-\ell),\, |\alpha|\ge 2}} \|\partial^l U\, \partial^\alpha G_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})}\Bigg). \end{aligned} \tag{4-37}$$

We observe that in the right-hand side of (4-37), we have $l \le 2(m+k-\ell)-2 \le m+k-2$; consequently, since $2m-1 > d-1$, we have $l < 2(m+k)-\frac{1}{2}d-2$, and hence we can estimate $\|\partial^l U\|_{L^\infty}$ by using estimate (4-2) in Lemma 4.1. This yields

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M)\|G_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k-\ell)}_{\tilde r})}, \quad \ell \ge \tfrac{1}{2}(m+k).$$

Then we use estimate (3-2) in Lemma 3.1 with $s = 2(m+k-\ell)$ and $s_0 = d+1$, and the definition of $G_{\ell,-}$ to estimate $\|G_{\ell,-}\|_{\mathcal{H}_{\tilde r}^{2(m+k-\ell)}}$. Since $d+2 < 2m-1$ and $2(m+k-\ell)+2 \le 2(m+k)-1$ (because $\ell \ge \frac{1}{2}(m+k) \ge 2$), we obtain

$$\begin{aligned}\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)} &\le \Lambda(T,M)\big(\sup_j \|F^j\|_{L^2(0,T;H^{d+1})}\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^{2(m+k-\ell)})} \\ &\qquad\qquad + \sup_j \|F^j\|_{L^2(0,T;H^{2(m+k-\ell)+2})}\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^{2(m+k-\ell)})}\big) \\ &\le \Lambda(T,M)\sup_j \|F^j\|_{L^2(0,T;H^{2(m+k)-1})}\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^{2(m+k-\ell)})}. \end{aligned} \tag{4-38}$$

By using the regularity assumption on F^j, this yields

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)} \le \Lambda(T,M)\|\nabla_v H_{\ell,-}\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^{2(m+k-\ell)})}.$$

To estimate the above right-hand side, we need to estimate $\partial_{x,v}^\gamma H_{\ell,-}$ with $|\gamma| \le 2(m+k-\ell)+1$. Recalling the definition of $H_{\ell,-}$, by taking derivatives using the expression (4-31), we see that we have to estimate terms under the form $J_p(f)$ with $p \le 2m-1$. Using Lemma 4.9 one more time, we thus obtain

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)} \le \Lambda(T,M)\Big(\|f\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^{2m-1})} + \sum_{\substack{l \ge 2(m+k)-\frac{1}{2}d-2 \\ l+|\alpha| \le 2m-1,\, |\alpha|\ge 2}} \|\partial^l U\, \partial^\alpha f\|_{L^\infty(0,T;\mathcal{H}_{\tilde r}^0)}\Big).$$

To estimate the right-hand side, we argue as follows. Let $r' > \frac{1}{2}d$ be such that $\tilde r + r' \le r$. Since $|\alpha| \ge 2$ and $|\alpha|-2+l \le 2m-3$, we can use estimate (3-4) in Lemma 3.1 (taking $\chi(v) = (1+|v|^2)^{\frac{1}{2}r'}$) to obtain

$$\|\partial^l U(1+|v|^2)^{\frac{1}{2}\tilde r}\, \partial^\alpha f\|_{L^2_{x,v}} \lesssim \|U\|_{\mathcal{H}_{-r'}^{2m-3}}\|(1+|v|^2)^r\, \partial^2 f\|_{L^\infty} + \|U\|_{L^\infty}\|f\|_{\mathcal{H}_r^{2m-1}}. \tag{4-39}$$

By using again estimate (4-2) in Lemma 4.1 and the Sobolev embedding, we finally obtain

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)} \le \Lambda(T,M)\|f\|_{L^\infty(0,T;\mathcal{H}_r^{2m-1})} \le \Lambda(T,M), \quad \ell \ge \tfrac{1}{2}(m+k). \tag{4-40}$$

It remains to handle the case $\ell \le \frac{1}{2}(m+k)$. Note that necessarily, for these cases to be meaningful, we must have $2k+1 < 2\ell$. Assume first $\ell \ge 2$. We obtain again (4-37). We first need to estimate $\|\partial^2 F \cdot \nabla_v H_{\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^{2(m+k-\ell)})}$. We thus have to study

$$\|\partial^\beta \partial^2 F \cdot \nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)},$$

with $|\beta|+|\gamma| \le 2(m+k-\ell)$. Since $\ell \ge 2$, we have $|\beta|+2 \le 2(m+k-1)$. If $|\beta|+2 < 2(m+k)-1-\frac{1}{2}d$, then we get, by the Sobolev embedding, the bound

$$\begin{aligned}\|\partial^\beta \partial^2 F \cdot \nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)} &\le \sup_j \|\partial^\beta \partial^2 F^j\|_{L^2(0,T;L_x^\infty)}\|\nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}_{\tilde r}^0)} \\ &\le \sup_j \|F^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\|f\|_{L^2(0,T;\mathcal{H}_{\tilde r}^{2m-1})} \\ &\le \Lambda(T,M),\end{aligned}$$

recalling the definition of $H_{\ell,-}$. If $|\beta| \geq 2(m+k)-3-\frac{1}{2}d$, then $|\gamma| \leq 2(m+k-\ell)-2(m+k)+3+\frac{1}{2}d$ and thus the term $\nabla_v \partial^\gamma H_{\ell,-}$ involves at most $\frac{1}{2}d+2$ derivatives. Since $2m-1 > \frac{3}{2}d+2$, we have

$$\begin{aligned}\|\partial^\beta \partial^2 F \cdot \nabla_v \partial^\gamma H_{\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} &\leq \sup_j \|\partial^\beta \partial^2 F^j\|_{L^2(0,T;L^2_x)} \|H_{\ell,-}\|_{L^2(0,T;\mathcal{W}^{d/2+2,\infty}_{\tilde r})} \\ &\leq \sup_j \|F^j\|_{L^2(0,T;H^{2(m+k)-1}_x} \|f\|_{L^2(0,T;\mathcal{H}^{2m-1}_r)} \\ &\leq \Lambda(T,M).\end{aligned}$$

We also have to estimate terms in (4-37) of the form

$$\|\partial^l U\, \partial^\beta \partial^2 F\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}},$$

with $l \geq 2(m+k)-\frac{1}{2}d-2$ and $l+|\beta|+|\gamma| \leq 2(m+k-\ell)$. Note that this implies $|\beta| \leq 2(m+k-\ell)-l \leq \frac{1}{2}d+2-2\ell \leq \frac{1}{2}d$ since we have $\ell \geq 1$. In particular this yields $|\beta|+2 < 2m-1-\frac{1}{2}d$ since $2m > 3+\frac{1}{2}d$, and thus by using the Sobolev embedding and (2-9), we obtain

$$\begin{aligned}\|\partial^l U\, \partial^\beta \partial^2 F\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}} &\lesssim \sup_j \|F^j\|_{H^{2m-1}_x} \|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}} \\ &\lesssim (\|f\|_{\mathcal{H}^{2m-1}_r} + \sup_j \|F^j(0)\|_{H^{2m-1}_x}) \|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}} \\ &\leq \Lambda(T,M) \|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}}.\end{aligned}$$

Consequently, it remains to estimate $\|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}}$ for $l \geq 2(m+k)-\frac{1}{2}d-2$ and $l+|\gamma| \leq 2(m+k-\ell)$. By using again (4-31) and the definition of $H_{\ell,-}$, we can expand $\partial^\gamma \nabla_v H_{\ell,-}$ as terms of the form $J_p(f)$, with $p \leq 2(\ell-k)+|\gamma|-1$. Since we have $2(\ell-k)+|\gamma|-1 \leq 1+\frac{1}{2}d < 2(m+k)-\frac{1}{2}d-2$, we can use estimate (4-2) in Lemma 4.1 again to estimate all the terms in the expression of $J_p(f)$ involving U and its derivatives in L^∞. This yields

$$\|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}} \leq \Lambda(T,M) \sum_{\tilde\gamma} \|\partial^l U\, \partial^{\tilde\gamma} f\|_{\mathcal{H}^0_{\tilde r}},$$

with $|\tilde\gamma| \leq |\gamma|+2(\ell-k)-1$. Consequently, arguing as for (4-39), we obtain

$$\|\partial^l U\, \partial^\gamma \nabla_v H_{\ell,-}\|_{\mathcal{H}^0_{\tilde r}} \leq \Lambda(T,M)\big(\|U\|_{L^\infty} \|f\|_{\mathcal{H}^{2m-1}_r} + \|(1+|v|^2)^r f\|_{L^\infty_{x,v}} \|U\|_{\mathcal{H}^{2m-1}_{-r'}}\big),$$

where we recall $r' > \frac{1}{2}d$ and we conclude finally by invoking estimate (4-2) in Lemma 4.1 and the Sobolev embedding that

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \leq \Lambda(T,M), \quad 2 \leq \ell \leq \tfrac{1}{2}(m+k). \tag{4-41}$$

For the case $\ell = 1$ to be meaningful, k must be equal to 0. We set aside the term $\partial_x^{\alpha(I,J)} F \cdot \nabla_v f$, which appears in the formula (4-17), and we thus have to study the term

$$L_{i_1,j_1} \cdots L_{i_{m-1},j_{m-1}} (\partial^2_{x_{i_m},x_{j_m}} F \cdot \nabla_v f) - \partial_x^{\alpha(I,J)} F \cdot \nabla_v f.$$

We argue exactly as before to obtain a bound in $L^2(0,T;\mathcal{H}^0_{\tilde r})$ by $\Lambda(T,M)$ (note indeed that at most $2m-1$ derivatives of f and F are involved). Gathering all pieces together, we have thus proven that

$$\|F_{1,\ell,-}\|_{L^2(0,T;\mathcal{H}^0_{\tilde r})} \le \Lambda(T,M). \tag{4-42}$$

Let us now treat the contribution of $G_{\ell,+}$, which will give rise to terms involving $2m$ up to $2(m+k)$ derivatives of f. Let $j \in \{0,\dots,2k\}$. Let us describe the form of the terms involving derivatives of order $2m+j$ of f. We note that $2m+j-1 \ge 2m-1 > m+p-1 \ge m+k-1$. This means that such terms are necessarily of the form

$$\big(\partial_x^{\alpha^0}\partial_v^{\beta^0} L_{i_1,j_1}\cdots L_{i_k,j_k}\,\partial_x^{\alpha^k}\partial_v^{\beta^k}\cdots L_{i_{m+j-k},j_{m+j-k}}\,\partial_x^{\alpha^{m+j-k}}\partial_v^{\beta^{m+j-k}}\big)f, \tag{4-43}$$

with

$$\sum_{k=0}^{m+j-k} |\alpha^k|+|\beta^k| = 2k-j, \qquad \sum_{k=0}^{m+j-k} |\beta^k| \neq 0.$$

In order to have exactly $2m+j$ derivatives of f, this expression can be rewritten as $L^{K,L}\,\partial_x^\alpha\partial_v^\beta f$, where $|K|=|L|=m+j-k$ and $|\alpha|+|\beta|=2k-j$, $|\beta|\ge 1$. Indeed if derivatives fall on a coefficient of one of the L_{i_k,j_k}, then there are less than $2m+j$ derivatives of f.

We denote by $\gamma^{I,J,1}_{K,L,\alpha,\beta}$ the coefficient associated to such terms. Note that for $|\gamma| \le 2k-j-1$, we have $\partial_x^\gamma\partial^2 F^i \in L^2(0,T;H_x^{2m+j-2})$. Since we have $2m > \frac{3}{2}d+4$, we can bound this term in $L^2(0,T;W_x^{d+2,\infty})$ by the Sobolev embedding. Likewise, for $|\gamma|\le 2k-j-1$, since $2m+j-1-\frac{1}{2}d > d+2$ we have $\partial_{x,v}^\gamma U \in L^\infty(0,T;W_{x,v}^{d+2,\infty})$. All in all, we deduce

$$\|\gamma^{I,J,1}_{K,L,\alpha,\beta}\|_{L^2(0,T;W_{x,v}^{d+2,\infty})} \le \Lambda(T,M).$$

It remains to treat the other terms that all involve at most $2m-1$ derivatives on f. If $k\ge 1$, we set aside the term $\partial_x^{\alpha(I,J)}F\cdot\nabla_v f$ in (4-17), which corresponds to the case $\ell=1$ treated above (relevant when $k=0$).

The other terms can be considered as remainders that are uniformly bounded in $L^2(0,T;\mathcal{H}^0_{\tilde r})$, since at most $2m-1$ derivatives are involved on f and at most $2(m+k)-1$ derivatives are involved on F; these terms can be treated exactly as we treated the remainders in $G_{\ell,-}$.

The treatment of F_2, F_3, F_4 gives rise to similar terms and we omit it. □

Proof of Lemma 4.7. The proof is similar to the previous one. We shall only explain why the terms involving at least $2m$ derivatives of f are indeed of the form appearing in (4-22).

Let $k=0,\dots,p-1$, and $r=m+j$ for $j=-k-1,\dots,k$. We look for the terms involving $2m+l$ derivatives of f for $l=0,\dots,k+1+j$. Among the operators in $L^{I,J}$ there are exactly $2m+l-(m+k+1-r) = 2m+j+l-k-1$ derivatives to be applied on f. Since $m>p\ge k+1$, we have $2m+j+l-k-1 > m+j$. This means that these derivatives must be of the form $L^{K,L}\,\partial_x^\gamma\partial_v^\delta$, with $|K|=|L|=m+l-k-1$ and $|\gamma|+|\delta| = j-l+k+1$ (up to commutations between the differential operators as in (4-43), which is treated as in the previous proof). In the end, the terms involving $2m+l$ derivatives of f are thus necessarily of the form $L^{K,L}\,\partial_x^\gamma\partial_v^\delta f$, with

$$|K|=|L|=m+l-k-1, \quad |\gamma|+|\delta| = 2k+2-l,$$

as appearing in (4-22). □

Remark 4.10. An inspection of the proof reveals that the uniform regularity of the coefficients in (4-18), (4-21), (4-23), (4-25) can be improved to $L^2(0,T;W^{p,\infty}_{x,v})$ for all $p < 2m-2-\frac{1}{2}d$.

5. Burgers' equation and the semilagrangian approach

In this section, we explain the procedure to straighten the transport operator $\mathcal{T}$, which allows, loosely speaking, to come down to the operator $\partial_t + a(v)\cdot\nabla_v$. This relies on several changes of variables in velocity that we introduce now.

Let $\Phi(t,x,v)$ satisfy the Burgers' equation

$$\begin{cases} \partial_t \Phi + a(\Phi)\cdot\nabla_x \Phi = F(t,x,\Phi), \\ \Phi(0,x,v) = v. \end{cases} \tag{5-1}$$

The interest in introducing the vector field Φ comes from the following algebraic identity.

Lemma 5.1. *Given a smooth function g satisfying $\mathcal{T}g = R$, the function*

$$G(t,x,v) := g(t,x,\Phi(t,x,v))$$

solves the equation

$$\partial_t G + a(\Phi(t,x,v))\cdot\nabla_x G = R(t,x,\Phi(t,x,v)). \tag{5-2}$$

Proof of Lemma 5.1. We compute

$$\begin{aligned} \partial_t G &= (\partial_t g)(t,x,\Phi(t,x,v)) + \partial_t\Phi\cdot(\nabla_v g)(t,x,\Phi(t,x,v)), \\ a(\Phi)\cdot\nabla_x G &= a(\Phi)\cdot(\nabla_x g)(t,x,\Phi(t,x,v)) + [a(\Phi)\cdot\nabla_x\Phi]\cdot(\nabla_v g)(t,x,\Phi(t,x,v)). \end{aligned}$$

Since $\mathcal{T}g = R$, we have

$$\begin{aligned} &(\partial_t g)(t,x,\Phi(t,x,v)) + a(\Phi)\cdot(\nabla_x g)(t,x,\Phi(t,x,v)) \\ &\qquad = -F(t,x,\Phi)\cdot(\nabla_v g)(t,x,\Phi(t,x,v)) + R(t,x,\Phi(t,x,v)). \end{aligned}$$

From (5-1), we deduce (5-2) as claimed. ☐

In other words, the change of variables in velocity $v \mapsto \Phi(t,x,v)$ allows us to straighten the vector field $\mathcal{T}$.

We now provide a lemma concerning the existence, uniqueness and regularity of solutions of (5-1).

Lemma 5.2. *Assume that for all $j = 1,\dots,\ell$, we have $F^j \in L^2(0,T';H^n_x)$ with norm bounded by $\Lambda(T',M)$. There is $T \in (0,T']$ such that the following holds. There exists a unique solution $\Phi(t,x,v) \in C^0(0,T;W^{k,\infty}_{x,v})$ of (5-1) and we have the following estimates*:

$$\sup_{[0,T]}\sup_v \sum_{|\alpha|\le n} \|\partial^\alpha_{x,v}(\Phi - v)\|_{L^2_{x,v}} + \sup_{[0,T]}\|\Phi - v\|_{W^{k,\infty}_{x,v}} \lesssim \Lambda(T,M), \tag{5-3}$$

$$\sup_{[0,T]}\sup_v \sum_{|\alpha|\le n} \|\partial^\alpha_{x,v}(a(\Phi) - a(v))\|_{L^2_{x,v}} + \sup_{[0,T]}\|a(\Phi) - a(v)\|_{W^{k,\infty}_{x,v}} \lesssim \Lambda(T,M) \tag{5-4}$$

for all $k < n - \frac{1}{2}d$.

We shall not provide the proof of Lemma 5.2 as it follows closely the proof of Lemma 4.6 in [Han-Kwan and Rousset 2016]. Here the source is semilinear, whereas there it is linear; however, the proof is essentially the same (see also [Han-Kwan et al. 2017] for a similar issue).

We now introduce the characteristics associated to Φ, defined as the solution to

$$\begin{cases} \partial_t \mathrm{X}(t,s,x,v) = a(\Phi)(t, \mathrm{X}(t,s,x,v), v), \\ \mathrm{X}(s,s,x,v) = x, \end{cases} \tag{5-5}$$

and study the deviation of X from the (modified) free transport flow.[3]

Lemma 5.3. *Assume that for all $j = 1, \dots, \ell$, we have $F^j \in L^\infty(0, T'; H^n_x)$ with norm bounded by $\Lambda(T', M)$. There is $T \in (0, T']$ such that the following holds. For every $0 \le s, t \le T$, we can write*

$$\mathrm{X}(t,s,x,v) = x + (t-s)(a(v) + \widetilde{\mathrm{X}}(t,s,x,v)), \tag{5-6}$$

with $\widetilde{\mathrm{X}}$ that satisfies the estimate

$$\sup_{t,s\in[0,T]} \sup_v \sum_{|\alpha|\le n} \|\partial^\alpha_{x,v} \widetilde{\mathrm{X}}(t,s,x,v)\|_{L^2_x} + \sup_{t,s\in[0,T]} \|\widetilde{\mathrm{X}}(t,s,x,v)\|_{W^{k,\infty}_{x,v}} \lesssim \Lambda(T, M) \tag{5-7}$$

for all $k < n - \frac{1}{2}d$. Moreover, the map $x \mapsto x + (t-s)\widetilde{\mathrm{X}}(t,s,x,v)$ is a diffeomorphism, and there exists $\Psi(t,s,x,v)$ such that the identity

$$\mathrm{X}(t,s,x,\Psi(t,s,x,v)) = x + (t-s)a(v)$$

holds. Finally, we have the estimate

$$\sup_{t,s\in[0,T]} \left[\sup_v \sum_{|\alpha|\le n} \|\partial^\alpha_{x,v}(\Psi(t,s,x,v) - v)\|_{L^2_x} + \|\Psi(t,s,x,v) - v\|_{W^{k,\infty}_{x,v}} \right] \lesssim \Lambda(T, M) \tag{5-8}$$

for all $k < n - \frac{1}{2}d$.

Again, we will not reproduce the proof of Lemma 5.3 as it follows closely that of Lemma 5.1 in [Han-Kwan and Rousset 2016].

In what follows, the procedure will consist in applying derivatives on (2-1) using multiple combinations of the operators $L^{I,J}$ that were introduced and studied in the previous section. This yields systems of Vlasov equations with sources, such as (4-22) in Lemma 4.7. It is on these equations that we will apply the change of variables $v \mapsto \Phi(t,x,v)$ in order to straighten the transport operator $\mathcal{T}$. We then integrate along characteristics, which is why the $\mathrm{X}(t,s,x,v)$ are useful, and average in velocity to obtain equations bearing on moments. In these equations, it will be crucial to apply various changes of variables based on the $\widetilde{\mathrm{X}}$ and Ψ introduced in Lemma 5.3.

This is what we refer to as the *semilagrangian* approach.

[3] Note that the X introduced here is not the same as the X previously defined in (1-7).

6. Averaging operators

For $i \in \{1, \dots, d\}$ and a smooth function $U(t, s, x, v)$, we define the following integral operator $K_U^{(i)}$ acting on scalar functions $H(t, x)$:

$$K_U^{(i)}(H)(t,x) = \int_0^t \int_{\mathbb{R}^d} (\partial_{x_i} H)(s, x-(t-s)a(v))\, U(t,s,x,v)\, dv\, ds. \tag{6-1}$$

The integral operator K can be seen as a modified version of the operator $\mathsf{K}_{\mathsf{U}}^{(i)}$

$$\mathsf{K}_{\mathsf{U}}^{(i)}(H)(t,x) = \int_0^t \int_{\mathbb{R}^3} (\partial_{x_i} H)(s, x-(t-s)v)\, \mathsf{U}(t,s,x,v)\, dv\, ds$$

that was studied in [Han-Kwan and Rousset 2016].

6A. ***The smoothing estimate.*** We note that the operators $K_U^{(i)}$ and $\mathsf{K}_{\mathsf{U}}^{(i)}$ seem to feature a loss of derivative in x. It was nevertheless proved in [Han-Kwan and Rousset 2016, Proposition 5.1 and Remark 5.1] that for the operators $\mathsf{K}_{\mathsf{U}}^{(i)}$, this loss is only apparent, provided that U is sufficiently smooth in x, v and decaying in v: this is the content of the following theorem.

Theorem 6.1 [Han-Kwan and Rousset 2016]. *Let $k > 1 + d$ and $\sigma > \frac{1}{2}d$. For all $H \in L^2(0, T; L^2_x)$, and for all $i \in \{1, \dots, d\}$, we have*

$$\|\mathsf{K}_{\mathsf{U}}^{(i)}(H)\|_{L^2(0,T;L^2_x)} \lesssim \sup_{0 \le s,t \le T} \|\mathsf{U}(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \|H\|_{L^2(0,T;L^2_x)}. \tag{6-2}$$

Based on this result, we deduce the following smoothing estimate[4] for the operators $K_U^{(i)}$.

Proposition 6.2. *Let $k > 1 + d$ and $\sigma > \frac{1}{2}d$. For all $H \in L^2(0, T; L^2_x)$, and for all $i \in \{1, \dots, d\}$, we have*

$$\|K_U^{(i)}(H)\|_{L^2(0,T;L^2_x)} \lesssim \sup_{0 \le s,t \le T} \|U(t,s,\cdot)\|_{\mathcal{H}^k_{r_k}} \|H\|_{L^2(0,T;L^2_x)}, \tag{6-3}$$

with $r_k = \sigma + (1+\lambda)(d+k)$.

Proof of Proposition 6.2. To ease readability we set $\partial_x = \partial_{x_i}$ and we forget about the subscript i. We come down from the modified to the straight operator by using the change of variable $w := a(v)$. We get

$$\begin{aligned} K_U(H)(t,x) &= \int_0^t \int_{a(\mathbb{R}^d)} (\partial_x H)(s, x-(t-s)w)\, U(t,s,x,a^{-1}(w)) |\det Da(a^{-1}(w))|^{-1}\, dw\, ds \\ &= \mathsf{K}_{\mathsf{U}}(H)(t,x), \end{aligned}$$

with

$$\mathsf{U}(t,s,x,w) := U(t,s,x,a^{-1}(w)) |\det Da(a^{-1}(w))|^{-1} 1_{a(\mathbb{R}^d)}.$$

Let $k > 1 + d$ and $\sigma > \frac{1}{2}d$. By Theorem 6.1, we get

$$\|K_U(H)\|_{L^2([0,T];L^2_x)} = \|\mathsf{K}_{\mathsf{U}}(H)\|_{L^2([0,T];L^2_x)} \lesssim \sup_{0 \le s,t \le T} \|\mathsf{U}(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \|H\|_{L^2([0,T];L^2_x)}.$$

[4]A close version of this result is also stated in [Han-Kwan et al. 2017] for the special case $a(v) = \hat{v}$.

By the assumption on a, we have

$$|\partial_w^\alpha a^{-1}(w)| \lesssim (1+|a^{-1}(w)|)^{1+\lambda|\alpha|}.$$

In particular, we deduce

$$|\det Da(a^{-1}(w))|^{-1} \lesssim (1+|a^{-1}(w)|)^{d(1+\lambda)}.$$

As a consequence, we have, by the Faà di Bruno formula, and using the reverse change of variable $v = a^{-1}(w)$ and (2-3), that

$$\|\mathsf{U}(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \lesssim \|U(t,s,\cdot)\|_{\mathcal{H}^k_{\sigma+(d+k)(1+\lambda)}},$$

and hence the claimed estimate. □

6B. ***Intermission: a comparison to averaging lemmas.*** We end this section with a comparison of the smoothing estimate we have just shown, in the simple case where $a(v) = v$, which corresponds to Theorem 6.1, with kinetic averaging lemmas. Averaging lemmas were introduced in [Golse et al. 1985; 1988; Agoshkov 1984] and now generically stand for various smoothing effects in average for kinetic transport-type equations.[5] They proved over the years to be fundamental in several contexts of kinetic theory, as they provide compactness and regularity. There exist many versions of these, involving several different assumptions on the functional spaces, on the number of derivatives in v or in x in the source etc.; see, e.g., [DiPerna et al. 1991; Perthame and Souganidis 1998; Golse and Saint-Raymond 2002; Bouchut 2002; Jabin and Vega 2004; Jabin 2009; Arsénio and Saint-Raymond 2011; Arsénio and Masmoudi 2014] for more recent advances. The closest (to Theorem 6.1) analogue of averaging lemmas is arguably the following result.

Theorem 6.3 [Perthame and Souganidis 1998]. *Let $1 < p < \infty$. Let $f, g = (g_j)_{j=1,\dots,d} \in L^p_{t,x,v}$ satisfy the transport equation*

$$\partial_t f + v\cdot\nabla_x f = \sum_{j=1}^d \partial_{x_j}\partial_v^k g_j, \tag{6-4}$$

where k is an arbitrary multi-index. Let $\varphi(v)$ be a C^∞ compactly supported function and set

$$\rho_\varphi(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\varphi(v)\,dv.$$

Then we have, for all $\alpha \in \left[0, \min\left(\frac{1}{p}, \frac{1}{p'}\right)\right)$,

$$\|\rho_\varphi\|_{L^p_{t,x}} \le C_{d,p,\alpha,\varphi} \|f\|_{L^p_{t,x,v}}^{1-\frac{\alpha}{|k|+1}} \|g\|_{L^p_{t,x,v}}^{\frac{\alpha}{|k|+1}}. \tag{6-5}$$

Let us focus especially on the case $p = 2$, $|k| = 0$, in which case (6-5) actually also holds for $\alpha = \frac{1}{2}$. Theorem 6.1 can also be understood as a kind of averaging lemma for the moments in v of the kinetic

[5] Actually this can be embedded in a more general framework; see in particular [Gérard 1990; Gérard and Golse 1992; Gérard et al. 1996].

equation (6-4), in the special case where the source has the form

$$\sum_{j=1}^{d} \partial_{x_j} H_j(t,x)\, \partial_v^k \mathcal{U}_j(t,x,v), \tag{6-6}$$

where $\mathcal{U}_j$ is smooth in x and v, and the initial condition is $f|_{t=0} = 0$. Let $\varphi(t,x,v)$ be a smooth and decaying test function. Then by the method of characteristics,

$$f(t,x,v) = \int_0^t \sum_{j=1}^{d} \partial_{x_j} H_j(s, x-(t-s)v)\, \partial_v^k \mathcal{U}_j(s, x-(t-s)v, v)\, ds,$$

and thus

$$\rho_\varphi(t,x) = \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^{d} \partial_{x_j} H_j(s, x-(t-s)v)\, \partial_v^k \mathcal{U}_j(s, x-(t-s)v, v)\, \varphi(t,x,v)\, ds = \sum_{j=1}^{d} \mathsf{K}^{(j)}_{U_j}(H_j)(t,x),$$

setting $U_j(s,t,x,v) = \partial_v^k \mathcal{U}_j(s, x-(t-s)v, v)\, \varphi(t,x,v)$. The regularity assumption of Theorem 6.1 can be written as

$$\sup_{0\le s,t\le T} \|U_j(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} < \infty$$

for $k > 1+d$, $\sigma > \frac{1}{2}d$, and the consequence is

$$\|\rho_\psi\|_{L^2(0,t;L^2_x)} \lesssim \sup_{0\le s,t\le T} \sum_j \|U_j(t,s,\cdot)\|_{\mathcal{H}^k_\sigma} \|H_j\|_{L^2(0,t;L^2_x)}. \tag{6-7}$$

This estimate is not a consequence of Theorem 6.3. Indeed, note that it does not involve the L^2 norm of the solution f: somehow, this can be roughly seen as a version of Theorem 6.3 allowing $\alpha = 1$, whereas Theorem 6.3 only allows $\alpha \le \frac{1}{2}$, at the expense of asking for the structure assumption (6-6) on the source g and of considering a norm for the source that is more demanding than the L^2 norm of estimate (6-5).

Observe also that Theorem 6.1 does not require the test function in v to be decaying at infinity, as long as for all j, we have $\mathcal{U}_j$ in (6-6) is itself decaying sufficiently fast at infinity.

7. Proofs of Theorem 2.1 and Corollaries 2.2 and 2.5

We finally set up an induction argument, which relies on the machinery developed in the previous sections, and will ultimately lead to the proof of Theorem 2.1. We summarize the procedure below:

- By induction, we assume smoothness on the moments until order $n'-1$. We can first apply Lemma 4.1 to obtain the same smoothness for the coefficients of the operators $L_{i,j}$.
- We apply Lemma 4.7 or 4.8 in order to get the system of equations satisfied by $(L^{K,L}\partial_x^\alpha \partial_v^\beta f)$, which is of the abstract form

$$\mathcal{T}(\mathfrak{F}) + \mathfrak{A}\mathfrak{F} = \mathcal{B},$$

where $\mathfrak{A}$ is a matrix whose coefficients we control and $\mathcal{B}$ is the rest we need to control. Loosely speaking, $\mathcal{B}$ consists either of remainders we can control thanks to the induction assumption, and terms of the form $-\partial_x^{\alpha(K,L)} F \cdot \nabla_v f$ for $K, L \in \{1,\dots,d\}^{m+k}$, whose contribution is the main matter.

• We then invert the operator $\mathcal{T}+\mathfrak{A}$ in order to solve the equation. At this stage, after integration in velocity (remember that we are interested in the regularity of moments), we use the changes of variables introduced in Lemmas 5.1, 5.2 and 5.3.

• What is rather straightforward then is the study of the contribution of the initial data and of the remainder terms in $\mathcal{B}$. As already said, the contribution of the terms $-\partial_x^{\alpha(K,L)} F\cdot\nabla_v f$ is more serious and involves the study of integrals of the form

$$\int_0^t\int_{\mathbb{R}^d}(\partial_{x_i}\partial_x^{\alpha(K,L)}F)(s,x-(t-s)a(v))\,U(t,s,x,v)\,dv\,ds,$$

which seem to feature a loss of derivative in x. We recognize the integral operators introduced and studied in Section 6. This is where the smoothing estimate of Proposition 6.2 proves to be crucial.

7A. ***End of the proof of Theorem 2.1.*** For $\boldsymbol{n}\geq 2m-1$, let $\mathcal{P}(\boldsymbol{n})$ be the following statement:

There is $T>0$ such that for all test functions

$$\psi(t,x,v)\in L^\infty(0,T;\mathcal{W}^{d+2+n-2m,\infty}_{-r_0}),$$

setting for all $|\alpha|=n$,

$$m_{\psi,\alpha}(t,x)=\int_{\mathbb{R}^d}\partial_x^\alpha f(t,x,v)\,\psi(t,x,v)\,dv,$$

there exists Λ for which

$$\sum_{|\alpha|=n}\|m_{\psi,\alpha}\|_{L^2(0,T;L^2_x)}\lesssim\Lambda(T,M). \tag{7-1}$$

By Proposition 3.2, it is clear that $\mathcal{P}(2m-1)$ is verified.

Let $\boldsymbol{n}\in\{2m,\dots,2(m+p)\}$. Let us assume that $\boldsymbol{n}$ is even, of the form $2(m+k)$. We shall not proceed with the case where $\boldsymbol{n}$ is odd, as it follows by completely similar arguments. Assume that $\mathcal{P}(2m),\dots,\mathcal{P}(\boldsymbol{n}-1)$ are satisfied and let $T>0$ be a time on which the estimates (7-1) (for $2m,\dots,\boldsymbol{n}-1$) are satisfied. We shall prove that $\mathcal{P}(\boldsymbol{n})$ is also verified. Once this is done, we deduce by induction that $\mathcal{P}(2m),\dots,\mathcal{P}(2(m+p))$ are satisfied; we then deduce the required estimates (3-14).

Thanks to the property $\mathcal{P}(\boldsymbol{n}-1)$ applied to the $(\psi_j)_{j=1,\dots,r}$, and (2-9), we first have

$$\sum_{j=1}^{\ell}\|F^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\leq\Lambda(T,M). \tag{7-2}$$

We can therefore apply Lemma 4.1 and obtain a possible smaller time, still denoted by T, and operators $L_{i,j}$ with coefficients $(\varphi^{i,j}_{k,l},\psi^{i,j}_{k,l})_{i,j,k,l\in\{1,\dots,d\}}$ belonging to $L^\infty(0,T;\mathcal{H}^{2(m+k)-2}_{-\tilde r})$ for all $\tilde r>\frac{1}{2}d$, with uniform regularity

$$\|(\varphi^{i,j}_{k,l},\psi^{i,j}_{k,l})_{i,j,k,l}\|_{L^\infty(0,T;\mathcal{H}^{2(m+k)-2}_{-\tilde r})}\leq\Lambda(T,M).$$

Let us consider the vector (the precise ordering does not matter)

$$\mathfrak{F}=(L^{K,L}\partial_x^\alpha\partial_v^\beta f)_{r\in\{m-k,\dots,m+k\},\,K,L\in\{1,\dots,d\}^r,\,|\alpha|+|\beta|=m+k-r}. \tag{7-3}$$

By Lemma 4.7, it follows that $\mathfrak{F}$ satisfies the system

$$\mathcal{T}(\mathfrak{F}) + \mathfrak{A}\mathfrak{F} = \mathcal{B} + \mathfrak{R}, \tag{7-4}$$

where $\mathfrak{A}(t,x,v)$ is a matrix with coefficients in $L^2(0,T;W^{d+2,\infty}_{x,v})$, satisfying

$$\|\mathfrak{A}\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M). \tag{7-5}$$

(The term $\mathfrak{A}\mathfrak{F}$ encodes the contribution of the leading-order terms in the triple sum of the right-hand side of (4-22).) On the other hand, $\mathfrak{R}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}\|_{L^2(0,T;\mathcal{H}^0_{\tilde{r}})} \lesssim \Lambda(T,M) \tag{7-6}$$

for all $\tilde{r} < r - \frac{1}{2}d$ and $\mathcal{B}$ is defined as follows: all its components are equal to 0 except those corresponding to the components associated to some $K, L \in \{1,\dots,d\}^{m+k}$, in which case it is equal to

$$-\partial_x^{\alpha(K,L)} F \cdot \nabla_v f.$$

The next step consists in using the change of variables $v \mapsto \Phi(t,x,v)$, where Φ solves (5-1), in order to straighten the vector field $\mathcal{T}$; see Lemma 5.1. To this end, we use Lemma 5.2 (reduce again $T > 0$ if necessary) and use the notation $\circ\,\Phi$ to denote the composition in v with Φ. Setting $\mathcal{F} = \mathfrak{F}\circ\Phi$, we obtain

$$(\partial_t + a(\Phi)\cdot\nabla_x)\mathcal{F} + (\mathfrak{A}\circ\Phi)\mathcal{F} = \mathcal{B}\circ\Phi + \mathfrak{R}\circ\Phi. \tag{7-7}$$

Let $\mathcal{A}(s,t,x,v)$ be the operator, whose existence is ensured by the Cauchy–Lipschitz theorem, as the solution of the following *linear* ODE

$$\partial_s\mathcal{A}(s,t,x,v) = \mathfrak{A}(s,x,\Phi(s,x,v))\,A(s,t,x,v), \quad A(t,t,x,v) = \mathrm{Id}.$$

Thanks to (7-5), we also have

$$\|\mathcal{A}(\cdot,t,\cdot)\|_{L^\infty(0,T;W^{d+2,\infty}_{x,v})} + \|\partial_s\mathcal{A}(\cdot,t,\cdot)\|_{L^2(0,T;W^{d+2,\infty}_{x,v})} \lesssim \Lambda(T,M). \tag{7-8}$$

By the method of characteristics we get

$$\begin{aligned}\mathcal{F}(t,x,v) &= \mathcal{A}(t,0,x,v)\,\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)\\ &\quad + \int_0^t \mathcal{A}(t,s,x,v)\,\mathcal{B}\circ\Phi(s,\mathrm{X}(s,t,x,v),v)\,ds + \int_0^t \mathcal{A}(t,s,x,v)\,\mathfrak{R}\circ\Phi(s,\mathrm{X}(s,t,x,v),v)\,ds.\end{aligned} \tag{7-9}$$

Suppose $\psi(t,x,v) \in L^\infty(0,T;\mathcal{W}^{d+2+2k,\infty}_{-r_0})$. Then we multiply the representation formula (7-9) by $\psi(t,x,\Phi(t,x,v))|\det D_v\Phi(t,x,v)|$ and integrate in v to obtain

$$\int_{\mathbb{R}^d} \mathcal{F}(t,x,v)\,\psi(t,x,\Phi(t,x,v))|\det D_v\Phi(t,x,v)|\,dv = I_0 + I_1 + I_2, \tag{7-10}$$

with

$$
\begin{aligned}
I_0 &= \int_{\mathbb{R}^d} \mathcal{A}(t,0,x,v)\,\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)\,\psi\circ\Phi|\det D_v\Phi(t,x,v)|\,dv,\\
I_1 &= \int_0^t\int_{\mathbb{R}^d} \mathcal{A}(t,s,x,v)\,(\mathfrak{R}\circ\Phi)(s,\mathrm{X}(s,t,x,v),v)\,\psi\circ\Phi|\det D_v\Phi(t,x,v)|\,dv\,ds, \qquad (7\text{-}11)\\
I_2 &= \int_0^t\int_{\mathbb{R}^d} \mathcal{A}(t,s,x,v)\,(\mathcal{B}\circ\Phi)(s,\mathrm{X}(s,t,x,v),v)\,\psi\circ\Phi|\det D_v\Phi(t,x,v)|\,dv\,ds.
\end{aligned}
$$

By the change of variables $v\mapsto\Phi(t,x,v)$, we have

$$\int_{\mathbb{R}^d}\mathcal{F}(t,x,v)\,\psi(t,x,\Phi(t,x,v))|\det D_v\Phi(t,x,v)|\,dv=\int_{\mathbb{R}^d}\mathfrak{F}(t,x,v)\,\psi(t,x,v)\,dv.$$

Let us first study this term. Since $\mathcal{P}(2m),\dots,\mathcal{P}(2(m+k)-1)$ are satisfied, we can apply Lemma 4.4 (the assumption (4-8) is indeed verified), which yields, see (4-9) and (4-10), that for all $I,J\in\{1,\dots,d\}^{m+k}$,

$$\int_{\mathbb{R}^d} L^{I,J} f\,\psi(t,x,v)\,dv=\int_{\mathbb{R}^d}\partial_x^{\alpha(I,J)} f\,\psi(t,x,v)\,dv+\mathfrak{R}_{I,J,\psi},$$

where $\mathfrak{R}_{I,J,\psi}$ is a remainder satisfying the estimate

$$\|\mathfrak{R}_{I,J,\psi}\|_{L^2(0,T;L^2_x)}\le\Lambda(T,M).$$

Consequently, recalling the definition of $\mathfrak{F}$ in (7-3), if we are able to obtain the bound

$$\|I_0\|_{L^2(0,T;L^2_x)}+\|I_1\|_{L^2(0,T;L^2_x)}+\|I_2\|_{L^2(0,T;L^2_x)}\le\Lambda(T,M),$$

then we deduce the bound

$$\sum_{I,J}\left\|\int_{\mathbb{R}^d}\partial_x^{\alpha(I,J)} f\psi\,dv\right\|_{L^2(0,T;L^2_x)}\le\Lambda(T,M);$$

that is, we obtain the sought bound (7-1) at rank n.

7A1. *Study of I_0.* Let us begin by treating the contribution of the initial data, which corresponds to the term I_0. First by using estimate (5-3) in Lemma 5.2, the L^∞ bound for $\mathcal{A}$ in (7-8), and the estimate

$$\|(1+|v|^2)^{-\frac12 r_0}\psi\|_{L^\infty_{x,v}}\lesssim 1, \qquad (7\text{-}12)$$

we have for all $x\in\mathbb{T}^d$,

$$\left|\int_{\mathbb{R}^d}\mathcal{A}(t,0,x,v)\,\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)(1+|v|^2)^{\frac12 r_0}|\det D_v\Phi(t,x,v)|\,dv\right| \le\Lambda(T,M)\int|\mathcal{F}(0,\mathrm{X}(0,t,x,v),v)|(1+|v|^2)^{\frac12 r_0}\,dv.$$

Therefore, we get that

$$\|I_0\|_{L^2(0,T;L^2_x)}\le\Lambda(T,M)\left\|\int_{\mathbb{R}^d}\|\mathcal{F}(0,\mathrm{X}(0,t,\cdot,v),v)\|_{L^2_x}(1+|v|^2)^{\frac12 r_0}\,dv\right\|_{L^2(0,T)}.$$

By using the change of variable $y = \mathrm{X}(0,t,x,v) + ta(v) = x - t\widetilde{\mathrm{X}}(0,t,x,v)$ and Lemma 5.3, we obtain

$$\|\mathcal{F}(0,\mathrm{X}(0,t,\cdot,v),v)\|_{L^2_x} \le \Lambda(T,M)\|\mathcal{F}(0,\cdot - ta(v),v)\|_{L^2_x} \le \Lambda(T,M)\|\mathcal{F}(0,\cdot,v)\|_{L^2_x}$$

and hence, we deduce that since $r > r_0 + \frac{1}{2}d$, for some $r' > \frac{1}{2}d$, it holds that

$$\|I_0\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M)\biggl(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^{r'}}\biggr)^{\frac{1}{2}} \|\mathcal{F}(0)\|_{\mathcal{H}^0_r}.$$

By using the fact that at $t = 0$ we have $\Phi(0,x,v) = v$ and $L^{(K,L)}|_{t=0} = \partial_x^{\alpha(K,L)}$, we end up with

$$\|\mathcal{F}(0)\|_{\mathcal{H}^0_r} = \|\mathfrak{F}(0)\|_{\mathcal{H}^0_r} \le \Lambda(M) \sum_{j=m-k}^{m+k} \sum_{|\alpha|+|\beta|=m+k-j} \|\partial_x^{2j}\partial_x^{\alpha}\partial_v^{\beta} f_0\|_{\mathcal{H}^0_r},$$

and hence we finally obtain

$$\|I_0\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M).$$

7A2. *Study of* I_1. We treat the other remainder term I_1 in a similar fashion. Indeed, using again estimate (5-3) in Lemma 5.2, (7-8) and (7-12), we first get

$$\begin{aligned}\|I_1\|&_{L^2(0,T;L^2_x)} \\ &\le \Lambda(T,M)\biggl\|\int_0^t\int_{\mathbb{R}^d} \|\mathfrak{R}(s,\mathrm{X}(s,t,\cdot,v),\Phi(s,\mathrm{X}(s,t,\cdot,v),v))\|_{L^2_x}(1+|v|^2)^{\frac{1}{2}r_0}\,dv\,ds\biggr\|_{L^2(0,T)}.\end{aligned}$$

Thanks to the change of variable $x \mapsto \mathrm{X}(s,t,x,v)$ and to the estimates of Lemma 5.3, it follows that

$$\begin{aligned}\|I_1\|_{L^2(0,T;L^2_x)} &\le \Lambda(T,M)\biggl\|\int_0^t\int_{\mathbb{R}^d} \|\mathfrak{R}(s,\cdot,\Phi(s,\cdot,v))\|_{L^2_x}(1+|v|^2)^{\frac{1}{2}r_0}\,dv\,ds\biggr\|_{L^2(0,T)} \\ &\le \Lambda(T,M)\biggl\|\int_0^t \|(\mathfrak{R}\circ\Phi)(s)\|_{\mathcal{H}^0_{\tilde{r}}}\,ds\biggr\|_{L^2(0,T)} \\ &\le \Lambda(T,M)\,T\,\|\mathfrak{R}\circ\Phi\|_{L^2(0,T;\mathcal{H}^0_{\tilde{r}})},\end{aligned}$$

by choosing $\tilde{r} > r_0 + \frac{1}{2}d$, which is possible since $r > r_0 + d$. Using again the change of variables $v \mapsto \Phi(t,x,v)$, Lemma 5.2 and the estimate (7-6), we thus obtain

$$\|I_1\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M).$$

7A3. *Study of* I_2. The main matter thus concerns the contribution of the term I_2, which features an apparent loss of derivative in x. This is however not the case, thanks to Proposition 6.2. Let $K, L \in \{1,\dots,d\}^{m+k}$. Writing $\partial_x^{\alpha(K,L)} = \partial_x\partial_x^{\alpha'}$ with $|\alpha'| = |\alpha(K,L)| - 1$, we are led to study terms of the form (here F_i^j stands for the i-th coordinate of F^j)

$$\sum_{j=1}^{\ell}\int_0^t\int_{\mathbb{R}^d}(\partial_x\partial_x^{\alpha'}F_i^j)(s,\mathrm{X}(s,t,x,v))\,\psi(t,\mathrm{X}(s,t,x,v),\Phi(s,\mathrm{X}(s,t,x,v),v))\\ \times\mathcal{A}_{K,L}^{I,J}\big(t,s,,\mathrm{X}(s,t,x,v),\Phi(s,\mathrm{X}(s,t,x,v),v)\big)\,A_j(\Phi(s,\mathrm{X}(s,t,x,v),v))\\ \times\partial_{v_i}f\big(s,\mathrm{X}(s,t,x,v),\Phi(s,\mathrm{X}(s,t,x,v),v)\big)|\det D_v\Phi(t,x,v)|\,dv\,ds,$$

where $\|\mathcal{A}_{K,L}^{I,J}\|_{L^\infty(0,T;W^{d+2,\infty}_{x,v})}\le\Lambda(T,M)$.

We use the change of variables $v=\Psi(s,t,x,w)$ to rewrite this expression as $\sum_{j=1}^{\ell}K_{U_j}(\partial_x^{\alpha'}F_i^j)$, with

$$U_j(s,t,x,v)=A_j\big(\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\ \times\mathcal{A}_{K,L}^{I,J}\big(t,s,x-(t-s)\,a(v),\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\ \times\psi\big(t,x-(t-s)\,a(v),\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\ \times\partial_{v_i}f\big(s,x-(t-s)\,a(v),\Phi(s,x-(t-s)\,a(v),\Psi(s,t,x,v))\big)\\ \times|\det D_v\Phi(t,x,\Psi(s,t,x,v))|\,|\det D_v\Psi(s,t,x,v)|, \tag{7-13}$$

where we recall the operators K were introduced in Section 6. In order to apply Proposition 6.2, we have to estimate, s,t being fixed, U_j in $\mathcal{H}^{2+d}_{r'}$, with $r'>\frac{1}{2}d+2(1+\lambda)(1+d)$ and $r\ge r'+r_0$, which is possible since $r>R$ as defined in (2-12). First, by (2-3), (2-6), (7-8), (5-3) in Lemma 5.2 and estimate (5-8) in Lemma 5.3, we can uniformly bound in L^∞ all terms involving A_j, Φ, Ψ and their derivatives (since only at most $2+d$ derivatives can be involved). For ψ, we use

$$\|(1+|v|^2)^{-\frac{1}{2}r_0}\partial^\alpha\psi\|_{L^\infty_{x,v}}\lesssim 1\quad\text{for all }|\alpha|\le d+2.$$

We are therefore led to estimate integrals of the form

$$I=\left|\int_{\mathbb{T}^d\times\mathbb{R}^d}|g(x-(t-s)\,a(v),\Phi(s,x-(t-s)v,\Psi(s,t,x,v)))|^2(1+|v|^2)^{r_0+r'}\,dv\,dx\right|,$$

where $g=\partial^\alpha f$, $|\alpha|\le d+3$. To this end, we can use the change of variables $v\mapsto w=\Psi(s,t,x,v)$ and rely on estimate (5-7) in Lemma 5.3 to obtain the bound

$$I\le\Lambda(T,M)\int_{\mathbb{T}^d\times\mathbb{R}^d}|g(\mathrm{X}(s,t,x,w),\Phi(s,\mathrm{X}(s,t,x,w),w))|^2(1+|w|^2)^{r_0+r'}\,dx\,dw.$$

Next, arguing as for I_1, we can use successively the change of variable $x\mapsto y=\mathrm{X}(s,t,x,w)$ with the estimates of Lemma 5.3, and the change of variable $w\mapsto u=\Phi(s,y,w)$ with estimate (5-3) in Lemma 5.2, to finally obtain

$$I\le\Lambda(T,M)\|g\|^2_{\mathcal{H}^0_r}\le\Lambda(T,M)\|f\|^2_{\mathcal{H}^{2m-1}_r},$$

since $2m-1\ge d+3$ and $r>R$. As a result we obtain the bound

$$\sup_{s,t}\|U_j\|_{\mathcal{H}^{2+d}_{r'}}\le\Lambda(T,M)\|f\|^2_{\mathcal{H}^{2m-1}_r}\le\Lambda(T,M). \tag{7-14}$$

We can therefore apply Proposition 6.2 to get the bound

$$\begin{aligned}\|K_{U_j,i}(F_i^j)\|_{L^2(0,T;L^2_x)} &\lesssim \sup_{s,t}\|U_j\|_{\mathcal{H}^{2+d}_{r'}}\|F_i^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\\ &\le \Lambda(T,M)\|F_i^j\|_{L^2(0,T;H_x^{2(m+k)-1})}\\ &\le \Lambda(T,M), \end{aligned} \tag{7-15}$$

thanks to estimate (7-2). We deduce

$$\|I_2\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M)$$

and gathering all pieces together, we therefore obtain (7-1) at rank $\boldsymbol{n}$, and the induction argument is complete. Theorem 2.1 follows.

7B. ***Proof of Corollary 2.2.*** In order to prove the higher-order regularity for the characteristics, we proceed as in [Han-Kwan and Rousset 2016, Lemma 5.1].

By Theorem 2.1 and the assumption (2-9), we have for all $j = 1,\dots,\ell$,

$$F^j \in L^2(0,T;H_x^{n'})$$

and thus by Sobolev embedding, we deduce that for $k < n' - \frac{1}{2}d$,

$$F^j \in L^2(0,T;W_x^{k,\infty}). \tag{7-16}$$

We set

$$Z := (Y,W) := (X - tv - x, V - v).$$

Let us first prove that $Z \in L^\infty(0,T;W^{k,\infty}_{x,v})$ for $k < n' - \frac{1}{2}d$. Note that by the definition of (X,V), we know Z satisfies the equation

$$Z = \left(\int_0^t (Y+v)\,ds,\ \int_0^t \sum_{j=1}^{\ell} A_j(W+v)\,F^j(Y+x+tv)\,ds\right).$$

By (2-6) and (7-16), we obtain by induction (on the number of applied derivatives) that for $t \le T$,

$$\sup_{|\alpha|<n'-\frac{1}{2}d}\ \sup_{[0,t]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_{x,v}} \lesssim \int_0^t \lambda(s)\big(1+\sup_{|\alpha|<n'-\frac{1}{2}d}\ \sup_{[0,s]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_{x,v}}\big)\,ds,$$

where λ is a nonnegative function belonging to $L^2(0,T)$, with norm bounded by $\Lambda(T,M)$. We deduce our claim thanks to the Gronwall inequality, which yields

$$\sup_{|\alpha|<n'-\frac{1}{2}d}\ \sup_{[0,t]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_{x,v}} \le \sqrt{t}\,\Lambda(T,M). \tag{7-17}$$

We deduce in particular from this estimate that for $T' \in (0,T]$ small enough, for all $v \in \mathbb{R}^d$, the map $x \mapsto X(T',0,x,v)$ is a C^1 diffeomorphism.

Next, let us turn to the $L^\infty_t L^\infty_v L^2_x$ estimate. We set

$$\mathcal{N}(t) := \sup_{|\alpha|\le n'}\ \sup_{[0,t]}\|\partial^\alpha_{x,v}Z\|_{L^\infty_v L^2_x}.$$

By an application of the Faà di Bruno formula, we obtain

$$\mathcal{N}(t) \lesssim \sum_{j=1}^{\ell} \int_0^t \sum_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} \, ds,$$

with

$$J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} := \big\| |(D_v^{k_1} A_j) \circ V(s) (D_x^{k_2} F^j) \circ X(s)| \, |\partial^{\beta_1}_{x,v}(X,V)| \cdots |\partial^{\beta_{k_1+k_2}}_{x,v}(X,V)| \big\|_{L^\infty_v L^2_x},$$

and where the sum is taken only on indices such that $k_1 + k_2 =: k \leq |\alpha| \leq n'$, $\beta_1 + \cdots + \beta_k = |\alpha|$ with for every j, $|\beta_j| \geq 1$ and $|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_k|$.

Let us observe that in the sum, if $k_1 + k_2 = k \geq 2$, we necessarily have $|\beta_{k-1}| < n' - \frac{1}{2}d$. Indeed, otherwise, we would have $|\beta_1| + \cdots + |\beta_k| \geq 2n' - d$ and thus $n' \geq 2n' - d$, which means $n' \leq d$. This is impossible by assumption on n'. Next:

- If $k_2 < n' - \frac{1}{2}d$ and $k_1 + k_2 = k \geq 2$, we obtain thanks to the above observation and (7-17) that for $i = 1, \dots, k-1$,

$$\|\partial^{\beta_i}_{x,v}(X,V)\|_{L^\infty_{x,v}} \lesssim 1 + T + \|\partial^{\beta_i}_{x,v}(Z)\|_{L^\infty_{x,v}} \lesssim \Lambda(T, M). \tag{7-18}$$

Moreover, using (2-6), (7-16) we get

$$\begin{aligned} J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} &\leq \|D^{k_1} A_j\|_{L^\infty_{x,v}} \|D^{k_2} F^j\|_{L^\infty_{x,v}} \Big\| \prod_{i=1}^{k-1} \partial^{\beta_i}_{x,v}(X,V) \Big\|_{L^\infty_{x,v}} \|\partial^{\beta_k}_{x,v}(X,V)\|_{L^\infty_v L^2_x} \\ &\leq \Lambda(T,M) \|D^{k_2} F^j\|_{L^\infty_{x,v}} (1 + \mathcal{N}(s)). \end{aligned}$$

If $k = 1$, the above estimate is clearly also valid.

- If $k_2 \geq n' - \frac{1}{2}d$, we observe that for every i, we have $|\beta_i| \leq |\beta_k| \leq n' - (k-1) < \frac{1}{2}d$. In particular $|\beta_i| < n' - \frac{1}{2}d$ by assumption on n' and we have that (7-18) holds for all $i = 1, \dots, k$. This yields

$$\begin{aligned} J^j_{k_1,k_2,\beta_1,\dots,\beta_{k_1+k_2}} &\lesssim \|(D_v^{k_1} A_j) \circ V\|_{L^\infty_v L^2_x} \|(D_x^{k_2} F^j) \circ X\|_{L^\infty_v L^2_x} \Lambda(T,M) \\ &\lesssim \|D^{k_1}_{x,v} A_j\|_{L^\infty_v} \|(D_x^{k_2} F^j) \circ X\|_{L^\infty_v L^2_x} \Lambda(T,M) \\ &\lesssim \Lambda(T,M) \|(D_x^{k_2} F^j)\|_{L^2_x}. \end{aligned}$$

To get the last estimate, we restrict to $T' \leq T$ small enough so that we can use the change of variable $y = X(t, 0, x, v)$ when computing the L^2_x norm of $(D^{k_2}_{x,v} F^j) \circ X$.

By combining the above estimates, we obtain that for $t \leq T'$,

$$\mathcal{N}(t) \leq \sqrt{t} \Lambda(T,M) + \int_0^t \Lambda(T,M) \sup_j \|F^j(s)\|_{H^{n'}_x} \mathcal{N}(s) \, ds.$$

By using again (7-16) and the Gronwall inequality, we thus obtain that for $t \leq T'$,

$$\mathcal{N}(t) \lesssim \sqrt{t} \Lambda(T,M),$$

which concludes the proof of Corollary 2.2.

7C. *Proof of Corollary 2.5.* The idea, as in [Gérard 1990, Proposition 5.2], consists in applying Theorem 2.1 with the test function

$$\psi_\eta(v) = e^{-v\cdot\eta} \in W^{n',\infty}_{x,v},$$

where $\eta \in \mathbb{R}^d$ has to be seen as the Fourier variable in velocity. A close inspection of the proofs reveals that the conclusion of Theorem 2.1 can be refined into

$$\text{for all } \eta \in \mathbb{R}^d, \qquad \left\| \int f\psi_\eta \, dv \right\|_{L^2(0,T_0;H^{n'}_x)} \le \Lambda(T_0, M, \|\psi_\eta\|_{W^{n',\infty}_v}), \tag{7-19}$$

where Λ is a polynomial function. Moreover, $\|\psi_\eta\|_{W^{n',\infty}_v} \lesssim \Lambda'(|\eta|)$, where Λ' is also a polynomial function (of degree n'). Since

$$\frac{1}{(2\pi)^{\frac{1}{2}d}} \int f\psi_\eta \, dv = \mathcal{F}_v f(t,x,\eta),$$

we deduce from (7-19) that for some $p > 0$ taken large enough,

$$\|\hat{f}(t,k,\eta)(1+|k|^2)^{\frac{1}{2}n'}(1+|\eta|^2)^{-\frac{1}{2}p}\|_{L^2(0,T_0;L^2(\mathbb{Z}^d\times\mathbb{R}^d))} < \infty,$$

which means that $f \in L^2(0, T_0; H^{n',-p}_{x,v})$.

8. Application to classical models from physics

The goal of this section is to briefly explain why both Vlasov–Poisson and relativistic Vlasov–Maxwell systems enter the abstract framework, and thus why Theorem 2.1 (and its corollaries) apply to these classical models.

8A. *Vlasov–Poisson.* The Cauchy problem for the Vlasov–Poisson system (1-2) was studied (among many other references)

- for (global) weak solutions in [Arsen'ev 1975],
- for local strong solutions in [Ukai and Okabe 1978], and for global strong solutions in [Bardos and Degond 1985; Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991; Glassey 1996; Batt and Rein 1991; Horst 1993].

Let us check the following structural assumptions for (1-2).

• *Assumptions on the advection field.* In this model, $a(v) = v$, so that all required assumptions on a are straightforward properties. One can take $\lambda = 0$ in (2-4).

• *Assumptions on the force field.* For the force field F, we can write $\ell = 1$, $A_1 = 1$ and $F^1 = -\nabla_x\phi$, where ϕ is computed thanks to the moment of order 0 of f only; that is, $\psi_1 = 1$ (thus $r_0 = 0$) and

$$m_{\psi_1} = \int_{\mathbb{R}^d} f \, dv.$$

The assumption (2-9) follows straightforwardly from the Poisson equation, as for all $n \in \mathbb{N}$, it holds that

$$\text{for all } t \geq 0, \quad \|F^1(t)\|_{H^n_x} \lesssim \|m_{\psi_1}(t)\|_{H^{n-1}_x}.$$

We however do not need the smoothing effect due to the Poisson equation. It follows directly that both estimates (2-9) and (2-10) hold. The stability estimate (2-11) holds because of the same estimate, by linearity of the Poisson equation. It turns out that using the smoothing estimate, we can obtain a stronger version of Theorem 2.1: we embed this situation in what we refer to as transport/elliptic systems, and refer to Theorem 9.1 in Section 9.

Note also that the Vlasov–Poisson system with dynamics constrained on geodesics introduced in the context of stellar dynamics in [Diacu et al. 2016] enters the abstract framework as well (and in this model there is no smoothing of the force field).

8B. *Relativistic Vlasov–Maxwell.* The Cauchy problem for the relativistic Vlasov–Maxwell system (1-3) was studied (among many other references)

- for (global) weak solutions in [DiPerna and Lions 1989],
- for (local) strong solutions in [Wollman 1984; 1987; Degond 1986; Asano 1986; Glassey and Strauss 1986; 1987; Glassey 1996; Schaeffer 2004; Bouchut et al. 2003; Klainerman and Staffilani 2002; Pallard 2015; Luk and Strain 2016].

Let us check the following structural assumptions for (1-3).

• *Assumptions on the advection field.* In this model, $a(v) = \hat{v}$, and one can check by a straightforward induction that

$$\|\partial_v^\alpha \hat{v}\|_{L^\infty_v} \leq C_\alpha \quad \text{for all } \alpha.$$

We have $a(\mathbb{R}^d) = B(0, c)$ and the explicit formula

$$\text{for all } w \in B(0,c), \quad a^{-1}(w) = \frac{w}{\sqrt{1 - |w|^2/c^2}}.$$

It follows that one can take $\lambda = 2$ in (2-4).

• *Assumptions on the force field.* For the force field F, we observe that we can take $\ell = 4$ and write

$$A_1 = 1, \quad F^1 = E, \tag{8-1}$$

and setting $B = (B_1, B_2, B_3)$ in an orthonormal basis (e_1, e_2, e_3),

$$\begin{aligned} A_2 &= \hat{v}_1, \quad F^2 = B_2 e_3 - B_3 e_1, \\ A_3 &= \hat{v}_2, \quad F^3 = B_3 e_1 - B_1 e_3, \\ A_4 &= \hat{v}_3, \quad F^4 = B_1 e_2 - B_2 e_1. \end{aligned} \tag{8-2}$$

The electromagnetic field (E, B) is computed only from initial data (E_0, B_0) and the moments of order 0 and 1, which correspond to $\psi_1 = 1$, $\psi_2 = \hat{v}$ (so that $r_0 = 0$) and

$$m_{\psi_1} = \int_{\mathbb{R}^d} f \, dv, \quad m_{\psi_2} = \int_{\mathbb{R}^d} f \hat{v} \, dv.$$

The assumption (2-9) follows from classical energy estimates for Maxwell equations: we have for all $n \in \mathbb{N}$ and all $t \geq 0$,

$$\|(E, B)\|_{L^2(0,t;H^n_x)} \leq C_n t^{\frac{3}{2}} \sum_{i=1}^{2} \|m_{\psi_i}\|_{L^2(0,t;H^n_x)} + \|(E, B)(0)\|_{L^2(0,t;H^n_x)};$$

see, e.g., [Han-Kwan et al. 2017, Lemma 3.2]. The estimate (2-10) is proved similarly. The stability estimate (2-11) holds because of the same energy estimate, by linearity of the Maxwell equations.

8C. *Remarks.* Some remarks about possible generalizations of the abstract framework are in order:

• It is possible to add a smooth force, of C^k regularity with k large enough, and still adapt the results of Theorem 2.1, without significantly modifying the analysis. This allows one for instance to consider Vlasov–Poisson systems with a smooth external magnetic field.

• The so-called relativistic gravitational Vlasov–Poisson system (which may be relevant for galactic dynamics) enters the abstract framework as well, by a combination of the estimates of Section 8A and 8B (see, e.g., [Glassey and Schaeffer 1985; Hadžić and Rein 2007; Kiessling and Tahvildar-Zadeh 2008; Lemou et al. 2008] for some references about this system).

• The divergence-free (in v) condition for F is not an absolute requirement for the analysis. It may be dropped, but would sometimes necessitate introducing more complicated formulas. In particular, it is likely that fluid/kinetic systems for sprays such as Vlasov–Stokes or Vlasov–Navier–Stokes in dimension $d = 2$ enter this framework (or a slightly modified version of it) as well. We refer, e.g., to [Jabin 2000; Boudin et al. 2009; 2015; Desvillettes 2010] for some references about these equations. See also [Baranger and Desvillettes 2006; Moussa and Sueur 2013] for other fluid/kinetic systems.

• Note that the so-called nonrelativistic Vlasov–Maxwell system (that is system (1-3) with v replacing all occurrences of $\hat{v}$) does not enter the abstract framework. Indeed the assumption (2-6) is not satisfied. However, we claim that (2-6) is crucial only for having a good local well-posedness theory in $\mathcal{H}^n_r$ spaces. This means that without (2-6), we can still obtain a result similar to that of Theorem 2.1, except that we have to *assume* the existence of a solution of (2-1) with the required regularity. For the nonrelativistic Vlasov–Maxwell system, such solutions do exist, following [Asano 1986], which requires the introduction of Sobolev spaces with loss of integrability in velocity.

9. The case of transport/elliptic-type Vlasov equations

9A. *An improvement of Theorem 2.1.* Let us assume in this section that the following strengthened version of (2-9) is satisfied:

$$\|F^j\|_{L^2(0,t;H^n_x)} \leq \Gamma^{(j)}_n\Bigg(t, \|m_{\psi_1}\|_{L^2(0,t;H^{n-1}_x)}, \dots, \|m_{\psi_r}\|_{L^2(0,t;H^{n-1}_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\Bigg). \tag{9-1}$$

In other words, the force is smoothed out and gains one derivative compared to the distribution function. We refer to such a situation as the transport/elliptic-type case. This includes in particular the Vlasov–Poisson system. We then have the following version of Theorem 2.1. This is an improved version in the sense that the higher regularity we ask for is only regularity in x and not at all in v (compare (9-2) below to (2-13) in Theorem 2.1).

Theorem 9.1. *Let $n \geq N$ and $r > R$. Let $n' > n$ be an integer such that $n > \lfloor \frac{1}{2} n' \rfloor + d + 1$. Assume that $f_0 \in \mathcal{H}^n_r$ and $F^j(0) \in H^{n'}_x$ for all $j \in \{1, \dots, \ell\}$. Assume furthermore that the initial data f_0 satisfies the following higher space regularity*:

$$\partial^\alpha_x f_0 \in \mathcal{H}^0_r \quad \textit{for all } |\alpha| = n'. \tag{9-2}$$

Then there is $T > 0$ such that the following holds. There exists a unique solution $(f(t), F(t))$ with initial data $(f_0, F(0))$ to (2-1) such that $f(t) \in C(0, T; \mathcal{H}^n_r)$.

Moreover, for all test functions $\psi \in L^\infty(0, T; \mathcal{W}^{n',\infty}_{-r_0})$, we have

$$\int f \psi \, dv \in L^2(0, T; H^{n'}_x). \tag{9-3}$$

As in Corollary 2.5, we may deduce as well under the assumptions of Theorem 9.1 that

$$f \in L^2(0, T; H^{n',-\infty}_{x,v}). \tag{9-4}$$

Proof of Theorem 9.1. The beginning of the proof is the same as for Theorem 2.1 (of which we keep the notations). Let us set in this context

$$M := \|f_0\|_{\mathcal{H}^{2m-1}_r} + \sum_{k=0}^{2p} \sum_{|\alpha|=2m+k} \|\partial^\alpha_x f_0\|_{\mathcal{H}^0_r} + \sum_{j=1}^{\ell} \|F^j(0)\|_{H^{2(m+p)}_x}. \tag{9-5}$$

We proceed with the same induction argument, treating all terms similarly except for[6] the treatment of the term I_0, for which the following is an improvement of Section 7. The idea will be to use integration by parts in v to trade derivatives in v against derivatives in x, allowing us to obtain estimates depending on (9-5) (compared to (3-13) for Theorem 2.1).

First note using the smoothing estimate (9-1) that we improve (7-2) to

$$\sum_{j=1}^{\ell} \|F^j\|_{L^2(0,T;H^{2(m+k)}_x)} \leq \Lambda(T, M). \tag{9-6}$$

We can use this improved estimate with Remark 4.10 to deduce that the coefficients of $\mathfrak{A}$, as appearing in (7-4), satisfy the improved form of (7-5)

$$\|\mathfrak{A}\|_{L^2(0,T;W^{p,\infty}_{x,v})} \lesssim \Lambda(T, M) \quad \text{for all } p < 2m - \tfrac{1}{2}d - 1. \tag{9-7}$$

Therefore we deduce the improved form of (7-8):

$$\|\mathcal{A}(\cdot, t, \cdot)\|_{L^\infty(0,T;W^{p,\infty}_{x,v})} \lesssim \Lambda(T, M) \quad \text{for all } p < 2m - \tfrac{1}{2}d - 1. \tag{9-8}$$

[6] We also remark that in order to treat the term I_2, we do not absolutely need to use Proposition 6.2; we can indeed rely on the smoothing estimate (9-6) on the force instead and argue as we did for I_1. This observation will be useful later in order to treat other Vlasov models.

The treatment of I_0 then leads to the study of terms of the general form

$$J = \int_{\mathbb{R}^d} (\partial_x^\alpha \partial_v^\beta \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\, m(t,x,v)\, dv,$$

where, for $j = m-k, \dots, m+k$, we have $|\alpha| + |\beta| = m+k+j$, $|\alpha| \geq 2j$, and

$$\|m\|_{L^\infty(0,T;\mathcal{H}^N_{-r'-r_0})} \leq \Lambda(T, M)$$

for all $N < 2m - \frac{1}{2}d - 1$ and all $r' > \frac{1}{2}d$. If $|\beta| = 0$, there is nothing special to do, as only derivatives in x are involved, so let us assume that $|\beta| \geq 1$. We write $\partial_v^\beta = \partial_v^{\beta'} \partial_v$. We have

$$J = \int_{\mathbb{R}^d} \partial_v[(\partial_x^\alpha \partial_v^{\beta'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, m(t,x,v)\, dv \\ - \int_{\mathbb{R}^d} (\partial_x^\alpha \partial_v^{\beta'})(\partial_v \mathrm{X}(0,t,x,v) \cdot \nabla_x \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\, m(t,x,v)\, dv,$$

and thus by integration by parts in v, we get

$$J = -\int_{\mathbb{R}^d} [(\partial_x^\alpha \partial_v^{\beta'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, \partial_v m(t,x,v)\, dv \\ - \int_{\mathbb{R}^d} (\partial_x^\alpha \partial_v^{\beta'})(\partial_v \mathrm{X}(0,t,x,v) \cdot \nabla_x \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\, m(t,x,v)\, dv.$$

We therefore observe that this procedure allows us to trade derivatives in v for derivatives in x.

Assume now that one can write, for some $l \in \{1, \dots, |\beta|\}$,

$$J = \sum_{|\beta'| \leq l} \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \partial_v^{\beta'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, m_{\alpha',\beta'}(t,x,v)\, dv + R_l,$$

where

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^{N_l}_{-r'-r_0})} \leq \Lambda(T, M) \tag{9-9}$$

for all $N_l < 2m - \frac{1}{2}d - 1 - |\beta| + l$ and all $r' > \frac{1}{2}d$, and R_l is a remainder satisfying

$$\|R_l\|_{L^2(0,T;L^2_x)} \leq \Lambda(T, M).$$

Let us show that this property holds as well for at rank $l-1$. Following the same integration-by-parts argument as above, we may write

$$J - R_l = J_1 + J_2 + J_3,$$

where

$$J_1 = \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, m_{\alpha',0}(t,x,v)\, dv,$$

$$J_2 = -\sum_{|\beta'| \leq l} \sum_{\beta' = (\beta'', j)} \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'} \partial_v^{\beta''} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)]\, \partial_{v_j} m_{\alpha',\beta'}(t,x,v)\, dv,$$

$$J_3 = -\sum_{|\beta'| \leq l} \sum_{\beta' = (\beta'', j)} \sum_{|\alpha'| \leq |\alpha| + |\beta| - l} \int_{\mathbb{R}^d} (\partial_x^{\alpha'} \partial_v^{\beta''})(\partial_{v_j} \mathrm{X}(0,t,x,v) \cdot \nabla_x \mathcal{F})(0, \mathrm{X}(0,t,x,v), v) \\ \times m_{\alpha',\beta'}(t,x,v)\, dv.$$

The terms J_1 and J_2 have good forms already. For J_3, by using the Leibniz rule, we observe that we need to study terms of the form

$$\bar{J} = \int_{\mathbb{R}^d} \partial^{\gamma}_{x,v} \mathrm{X}(0,t,x,v)\, \partial_x^{\eta_1} \partial_v^{\eta_2} \mathcal{F}(0, \mathrm{X}(0,t,x,v), v)\, m_{\alpha',\beta'}(t,x,v)\, dv,$$

with $|\eta_2| \le |\beta''| = l-1,\ 1 \le |\eta_1| \le |\alpha'|+1$, and $|\gamma| = |\alpha'|+|\beta''|-|\eta_1|-|\eta_2|+2$.

Assume first that $|\eta_1|+|\eta_2| \le 2m-1$. If $|\eta_1|+|\eta_2| < 2m-1-d$, then by the Sobolev embedding we have the bound

$$\|(1+|v|^2)^{\frac{1}{2}r} (\partial_x^{\eta_1} \partial_v^{\eta_2} \mathcal{F})(0, \mathrm{X}(0,t,x,v), v)\|_{L^\infty_{x,v}} \le \|f_0\|_{\mathcal{H}^{2m-1}_r} \le \Lambda(M).$$

Since $0 < |\gamma| \le 2(m+k)$, we use (9-6) and Lemma 5.2 to get

$$\|\partial^\gamma \mathrm{X}\|_{L^\infty(0,T;L^\infty_v L^2_x)} \le \Lambda(T,M).$$

(This is where the elliptic estimate (9-1) is crucially used.) Furthermore, since $2m - \frac{1}{2}d - 1 - 2p > d$, we have the bound

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{W}^{0,\infty}_{-r'-r_0})} \le \Lambda(T,M)$$

for $r' > \frac{1}{2}d$ such that $r > r' + r_0 + d$. Therefore such terms satisfy the bound

$$\|\bar{J}\|_{L^2(0,T;L^2_{x,v})} \le \Lambda(T,M),$$

and thus can be put into the remainder R_{l-1}. If $|\eta_1|+|\eta_2| \ge 2m-1-d$, then $|\gamma| \le 2k+d+1$. Since $2m-d-1 > \frac{1}{2}d$, we can use $\|\partial^\gamma \mathrm{X}\|_{L^\infty(0,T;L^\infty_{x,v})} \le \Lambda(T,M)$ and again, arguing as in the treatment of I_0 in the proof of Theorem 2.1, such terms are remainders.

Otherwise $|\eta_1|+|\eta_2| \ge 2m$. Then we have $|\gamma| \le 2k$ and thus $2(m+k)-|\gamma| \ge 2m$. We set in this case $m_{\eta_1,\eta_2} := \partial^\gamma_{x,v} \mathrm{X}\, m_{\alpha',\beta'}$. In order to show that $\partial^\gamma_{x,v} \mathrm{X}\, m_{\alpha',\beta'}$ has the required regularity, we are led to study terms of the form

$$\tilde{J} = \|\partial^a_{x,v} \partial^\gamma_{x,v} \mathrm{X}\, \partial^b_{x,v} m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^0_{-r'-r_0})}, \quad |a|+|b| = N_{l-1},$$

for all $N_l < 2m - \frac{1}{2}d - 1 - |\beta| + l - 1$ and all $r' > \frac{1}{2}d$. Assume first that $|a| < 2m - \frac{1}{2}d$; then we have $|a|+|\gamma| < 2(m+k) - \frac{1}{2}d$ and we use estimate (5-7) in Lemma 5.2 to get $\|\partial^a_{x,v}\partial^\gamma_{x,v}\mathrm{X}\|_{L^\infty(0,T;L^\infty_{x,v})} \le \Lambda(T,M)$, and apply (9-9) to obtain the bound

$$\|\partial^b_{x,v} m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^0_{-r'-r_0})} \le \Lambda(T,M).$$

Otherwise, $|a| \ge 2m - \frac{1}{2}d$. Since we have $2(m+k)-|\gamma| \ge N_{l-1}$ for all $N_{l-1} < 2m-2-|\beta|+l$, we can use estimate (5-7) in Lemma 5.2 to get

$$\sum_{|a| \le N_{l-1}} \|\partial^a_{x,v}\partial^\gamma_{x,v}\mathrm{X}\|_{L^\infty(0,T;L^\infty_v L^2_x)} \le \Lambda(T,M).$$

Since $|b| = N_{l-1} - |a| \le N_{l-1} - 2m + \frac{1}{2}d$, we have $N_l - |b| \ge 2m + 1 - \frac{1}{2}d > d$. As a result, by (9-9) and the Sobolev embedding we get

$$\|\partial^b_{x,v} m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{W}^{0,\infty}_{-r'-r_0})} \le \Lambda(T,M).$$

In all cases, we have obtained

$$\tilde{J} \le \Lambda(T,M).$$

Therefore the corresponding terms of J_3 can be written in the form

$$\int_{\mathbb{R}^d} (\partial_x^{\eta_1}\partial_v^{\eta_2}\mathcal{F})(0,\mathrm{X}(0,t,x,v),v)\, m_{\eta_1,\eta_2}(t,x,v)\, dv,$$

with

$$\|m_{\eta_1,\eta_2}\|_{L^\infty(0,T;\mathcal{H}^{N_{l-1}}_{-r'-r_0})} \le \Lambda(T,M)$$

for all $N_{l-1} < 2m - 1 - |\beta| + (l-1)$ and $r' > \frac{1}{2}d$.

We conclude by induction that we can write at rank $l = 0$

$$J = \sum_{|\alpha'|\le m+k+j} \int_{\mathbb{R}^d} [(\partial_x^{\alpha'}\mathcal{F})(0,\mathrm{X}(0,t,x,v),v)]\, m_{\alpha',0}(t,x,v)\, dv + R,$$

with

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{H}^{N}_{-r'-r_0})} \le \Lambda(T,M)$$

for all $N < 2m - 1 - |\beta|$ and $r' > \frac{1}{2}d$, and $\|R\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M)$ is a remainder.

We then note that $2m - 2 - 2k > d$, so that

$$\|m_{\alpha',\beta'}\|_{L^\infty(0,T;\mathcal{W}^{0,\infty}_{-r'-r_0})} \le \Lambda(T,M).$$

Arguing as in the previous treatment of I_0 in the proof of Theorem 2.1, we finally conclude that

$$\|I_0\|_{L^2(0,T;L^2_x)} \le \Lambda(T,M) \sum_{j=m-k}^{m+k} \sum_{|\alpha|=m+k-j} \|\partial_x^\alpha f_0\|_{\mathcal{H}^0_r}. \tag{9-10}$$

This allows us to conclude the proof. □

As already noted in the proof of Theorem 9.1, we actually do not need to use Proposition 6.2 to treat the term I_2 in view of Theorem 9.1; we can indeed rely on the smoothing estimate (9-1) on the force instead. Furthermore, one can obtain L^∞_t estimates instead of the L^2_t theory that we have developed. This observation implies the following fact: replacing (9-1) by the slightly weaker estimate (in the sense that it is implied by (9-1))

$$\|F^j\|_{L^2(0,t;H^n_x)} \le \Gamma^{(j)}_n\left(t, \|m_{\psi_1}\|_{L^\infty(0,t;H^{n-1}_x)}, \dots, \|m_{\psi_r}\|_{L^\infty(0,t;H^{n-1}_x)}, \sum_{j=1}^{\ell} \|F^j(0)\|_{H^n_x}\right), \tag{9-11}$$

together with an associated stability estimate replacing (2-11) with L^∞_t norms instead of L^2_t for the moments on the right-hand side, Theorem 9.1 still holds. It suffices to estimate all terms (that is to say, the moments, $I_0, I_1, I_2, \dots$) in $L^\infty(0,T;L^2_x)$ instead of $L^2(0,T;L^2_x)$ as previously done. This remark is useful in particular to treat the so-called Vlasov–Darwin model from plasma physics, which we introduce in the following section.

9B. ***Vlasov–Darwin.*** The Vlasov–Darwin system is another model that allows one to describe the dynamics of charged particles in a plasma, which lies between Vlasov–Poisson and relativistic Vlasov Maxwell systems. Like Vlasov–Poisson, it can be derived from the Vlasov–Maxwell system in the nonrelativistic regime, that is to say, in the limit $c \to \infty$. The difference is that the Vlasov–Darwin system happens to be a higher-order approximation than the Vlasov–Poisson, see [Bauer and Kunze 2005]; in particular it retains self-induced magnetic effects that have disappeared completely in the Vlasov–Poisson dynamics. It is given by

$$\begin{cases} \partial_t f + \hat{v}\cdot\nabla_x f + \left(E + \frac{1}{c}\hat{v}\times B\right)\cdot\nabla_v f = 0, \\ E = -\nabla_x\phi - \frac{1}{c}\partial_t A, \quad B = \nabla_x \times A, \\ -\nabla_x\phi = \int_{\mathbb{R}^3} f\,dv - \int_{\mathbb{T}^3\times\mathbb{R}^3} f\,dv\,dx, \\ -\Delta_x A = \frac{1}{c}\mathbb{P}\int_{\mathbb{R}^3}\hat{v} f\,dv, \quad \nabla_x\cdot A = 0, \end{cases} \tag{9-12}$$

where $c > 0$ is the speed of light and $\mathbb{P}$ denotes the Leray projection. The Cauchy problem for the Vlasov–Darwin system (1-3) was studied (among many other references)

- for (global) weak solutions in [Pallard 2006],
- for strong solutions in [Pallard 2006; Seehafer 2008; Sospedra-Alfonso et al. 2012].

To embed this system into the abstract framework, we need to make the additional assumption that all initial conditions f_0 that are considered are a.e. nonnegative. By a standard property of the Vlasov equation, any associated solution $f(t)$ is also a.e. nonnegative.

- *Assumptions on the advection field.* In this model $a(v) = \hat{v}$, which is already treated for the relativistic Vlasov–Maxwell case.

- *Assumptions on the force field.* We have the decomposition (8-1)–(8-2) as well. Let us set

$$E = E_L + E_T, \quad E_L = \nabla_x\phi, \quad E_T = -\frac{1}{c}\partial_t A$$

and introduce

$$\psi_1 = 1, \quad \psi_2 = \hat{v}, \quad \psi_3 = \frac{\hat{v}\otimes\hat{v}}{\sqrt{1+|v|^2/c^2}}, \quad \psi_3 = \mathrm{Id} - m_{\psi_3}$$

(so that $r_0 = 0$) and

$$m_{\psi_i} = \int_{\mathbb{R}^d} \psi_i f\,dv,$$

where m_{ψ_3} and m_{ψ_4} are symmetric matrices. Since E_L and E_T derive from potentials solving a Poisson equation, we have

$$\text{for all } t \geq 0, \quad \|(E_L, B)(t)\|_{H_x^n} \lesssim \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{H_x^{n-1}},$$

and thus

$$\|(E_L, B)\|_{L^\infty(0,t;H_x^n)} \lesssim \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{L^\infty(0,t;H_x^{n-1})}.$$

For E_T, this is a little more subtle; this is where we need that $f(t) \geq 0$ a.e. As in [Pallard 2006, Lemma 2.10], we obtain that E_T satisfies the inhomogeneous elliptic equation

$$-\Delta E_T + \frac{1}{c} m_{\psi_4} E_T = -\frac{1}{c}(m_{\psi_4} E_L - m_{\psi_2} \times B - \nabla_x : m_{\psi_3}). \tag{9-13}$$

We fix the time $t \geq 0$, which is a parameter here (we take the L_t^∞ norm in the end). Let $n > d$. By [Pallard 2006, Lemma 2.10], which relies on the fact that m_{ψ_4} is actually a *semidefinite* symmetric matrix, it follows that (9-13) has a unique solution E_T in H_x^1, with the bound

$$\begin{aligned}
\|E_T\|_{H_x^1} &\lesssim \|m_{\psi_4} E_L\|_{H_x^{-1}} + \|m_{\psi_2} \times B\|_{H_x^{-1}} + \|\nabla : m_{\psi_3}\|_{H_x^{-1}} \\
&\lesssim \|m_{\psi_4} E_L\|_{L_x^2} + \|m_{\psi_2} \times B\|_{L_x^2} + \|m_{\psi_3}\|_{L_x^2} \\
&\lesssim \|m_{\psi_4}\|_{H_x^n} \|E_L\|_{H_x^n} + \|m_{\psi_2}\|_{H_x^n} \|B\|_{H_x^n} + \|m_{\psi_3}\|_{H_x^n} \\
&\lesssim \left(1 + \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{H_x^n}\right)(\|m_{\psi_4}\|_{H_x^n} + \|m_{\psi_2}\|_{H_x^n} + \|m_{\psi_3}\|_{H_x^n}).
\end{aligned}$$

Then assume by induction that we have a bound of the form

$$\text{for all } k = 1, \ldots, N, \quad \|E_T\|_{H_x^k} \lesssim \Gamma_k(\|m_{\psi_1}\|_{H_x^n}, \ldots, \|m_{\psi_4}\|_{H_x^n}) \tag{9-14}$$

for $N \leq n$, where Γ_k is a polynomial function. Assume first that $N < n - \frac{1}{2}d$. Let $|\alpha| = N$. We note that $\partial_x^\alpha E_T$ satisfies

$$-\Delta \partial_x^\alpha E_T + \frac{1}{c} m_{\psi_4} \partial_x^\alpha E_T = -\frac{1}{c} \partial_x^\alpha (m_{\psi_4} E_L - m_{\psi_2} \times B - \nabla_x : m_{\psi_3}) - [\partial_x^\alpha, m_{\psi_4}] E_T.$$

We have by standard tame Sobolev estimates

$$\begin{aligned}
&\left\| \partial_x^\alpha (m_{\psi_4} E_L - m_{\psi_2} \times B - \nabla_x : m_{\psi_3}) \right\|_{H_x^{-1}} \\
&\qquad \lesssim \left(1 + \sum_{i=1}^{2} \|m_{\psi_i}(t)\|_{H_x^n}\right)(\|m_{\psi_4}\|_{H_x^n} + \|m_{\psi_2}\|_{H_x^n} + \|m_{\psi_3}\|_{H_x^n}).
\end{aligned} \tag{9-15}$$

Since $N < n - \frac{1}{2}d$, we can use the Sobolev embedding to obtain

$$\begin{aligned}
\|[\partial_x^\alpha, m_{\psi_4}] E_T\|_{H_x^{-1}} &\lesssim \|m_{\psi_4}\|_{W_x^{N,\infty}} \|E_T\|_{H_x^N} \\
&\lesssim \|m_{\psi_4}\|_{H_x^n} \Gamma_k(\|m_{\psi_1}\|_{H_x^n}, \ldots, \|m_{\psi_4}\|_{H_x^n}).
\end{aligned}$$

We apply again the H^1_x estimate of [Pallard 2006, Lemma 2.10] to obtain a bound of the form

$$\|E_T\|_{H_x^{N+1}} \lesssim \Gamma_{N+1}(\|m_{\psi_1}\|_{H^n_x}, \dots, \|m_{\psi_4}\|_{H^n_x}).$$

We deduce by induction that for all $N < n - \frac{1}{2}d$,

$$\|E_T\|_{H_x^{N+1}} \lesssim \Gamma_{N+1}(\|m_{\psi_1}\|_{H^n_x}, \dots, \|m_{\psi_4}\|_{H^n_x}).$$

In particular, since $n > d$, we deduce

$$\|E_T\|_{L^\infty_x} \lesssim \Gamma(\|m_{\psi_1}\|_{H^n_x}, \dots, \|m_{\psi_4}\|_{H^n_x}). \tag{9-16}$$

Now assume we have (9-14) for some $N \le n$. We have the tame Sobolev estimate

$$\begin{aligned}\|[\partial_x^\alpha, m_{\psi_4}]E_T\|_{H_x^{-1}} &\lesssim \|m_{\psi_4}\|_{H^n}(\|E_T\|_{H_x^N} + \|E_T\|_{L^\infty_x})\\ &\lesssim \|m_{\psi_4}\|_{H^n_x}\Gamma_N(\|m_{\psi_1}\|_{H^n_x}, \dots, \|m_{\psi_4}\|_{H^n_x}),\end{aligned}$$

by (9-14) at rank N and (9-16). Thus using the H^1_x estimate of [Pallard 2006, Lemma 2.10], we obtain (9-14) at rank $N+1$. By induction, we conclude that

$$\|E_T\|_{L^\infty(0,T;H_x^{n+1})} \lesssim \Gamma_{n+1}(\|m_{\psi_1}\|_{L^\infty(0,T;H^n_x)}, \dots, \|m_{\psi_4}\|_{L^\infty(0,T;H^n_x)}),$$

which is an estimate of the requested form (9-11). A stability estimate of the same form also holds because of similar considerations.

10. On the regularity assumptions of Theorem 2.1

The goal of this short last section is to discuss the type of regularity assumptions which could be conceivable for proving propagation of higher regularity.

Example 1. Consider the free transport equation

$$\partial_t f + v\,\partial_x f = 0, \tag{10-1}$$

set in $\mathbb{R}\times\mathbb{R}$ to simplify the discussion. Let $\varphi(v)$ be a C^∞ function, with compact support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and such that $\int_{\mathbb{R}} \varphi\, dv = 0$. Let g be the piecewise continuous function defined by $g(x) = 1$ for $x \in [-1, 1]$ and 0 elsewhere. Observe that in the sense of distributions, we have $g'(x) = \delta_{x=-1} - \delta_{x=-1}$, where δ stands for the Dirac measure. We consider the initial condition

$$f|_{t=0} = g(x)\,\varphi(v) \in L^2_{x,v},$$

and the solution to (10-1) can be written as

$$f(t,x,v) = g(x-tv)\,\varphi(v).$$

It follows by explicit computations that $\rho(t,x) := \int_{\mathbb{R}} f\, dv$ satisfies

$$\begin{aligned}\partial_x\rho(t,x) &= \varphi\left(\frac{x+1}{t}\right) - \varphi\left(\frac{x-1}{t}\right),\\ \partial_x^k\rho(t,x) &= \frac{1}{t^{k-1}}\left(\varphi^{(k-1)}\left(\frac{x+1}{t}\right) - \varphi^{(k-1)}\left(\frac{x-1}{t}\right)\right) \quad \text{for all } k \in \mathbb{N}^*.\end{aligned}$$

We have for $t < 4$,

$$\|\partial_x^k \rho(t)\|^2_{L^2_x} = \frac{1}{t^{2(k-1)}} \left(\left\| \varphi^{(k-1)}\left(\frac{x+1}{t}\right) \right\|^2_{L^2_x} + \left\| \varphi^{(k-1)}\left(\frac{x-1}{t}\right) \right\|^2_{L^2_x} \right),$$

since φ is compactly supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and thus

$$\|\partial_x^k \rho(t)\|^2_{L^2_x} = \frac{2}{t^{2(k-1)-1}} \|\varphi^{(k-1)}\|^2_{L^2_x}.$$

We deduce that for any $T > 0$, we have $\rho \notin L^2(0, T; H^2_x)$. However, $\rho(0, x) = 0 \in H^k_x$ for all $k \in \mathbb{N}$.

This example shows that regularity of moments at initial time may not be propagated, and more precise information such as (2-13) is somehow required to obtain higher regularity for moments.

Example 2. Consider the equation

$$\partial_t f + v\, \partial_x f + F(t, x)\, \partial_v f = 0 \tag{10-2}$$

on $\mathbb{T} \times \mathbb{R}$, with

$$F(t, x) = \int_{\mathbb{R}} \psi(v) f(t, x, v)\, dv,$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ with compact support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. It is clear that (10-2) enters the abstract framework of this work.

We consider the initial condition

$$f|_{t=0} = f_0^{(1)} + f_0^{(2)},$$

where $f_0^{(2)}$ is a smooth nonnegative function, with support in $\mathbb{T} \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $f_0^{(1)}$ is a smooth nonnegative function, with support in $\mathbb{T} \times [1, 2]$.

Consider $f^{(1)}$ the solution of (10-2) associated to the initial condition $f_0^{(1)}$, and assume that it is defined on an interval $[0, T]$ for $T > 0$ small enough. Now define $f^{(2)}$ as the solution on $[0, T]$ of the *linear* kinetic transport equation

$$\partial_t f + v\, \partial_x f + \left(\int_{\mathbb{R}} \psi(v) f^{(1)}\, dv \right) \partial_v f = 0,$$

with initial condition $f_0^{(2)}$.

Because of the form of the force F, notably because ψ is localised in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we observe that up to reducing $T > 0$, the solution f on $[0, T]$ of (10-2) can be written as

$$f = f^{(1)} + f^{(2)},$$

since $T > 0$ can be chosen small enough so that the support in velocity of $f^{(2)}(t)$ is disjoint from that of ψ, and thus

$$\int_{\mathbb{R}} \psi(v)\, f^{(2)}(t)\, dv = 0.$$

Now let $k \in \mathbb{N}$ and assume that there is $(x_0, v_0) \in \mathbb{T} \times (1, 2)$ such that $f|_{t=0}(x_0, v_0)$ is not zero and is locally H^k around this point. Because of the assumptions on the supports, this is equivalent to asking that $f_0^{(2)}(x_0, v_0)$ is not zero and is locally H^k around this point. However, we can choose (independently of $f_0^{(2)}$) $f_0^{(1)}$ so that $\int_{\mathbb{R}} \psi(v) f^{(1)}\, dv$ is not H^k, in such a way that $f^{(2)}(t)$ (and thus $f(t)$) is not locally H^k around points of the form $(X(0,t,x_0,v_0), V(0,t,x_0,v_0))$, where (X,V) denotes the characteristics associated to F, as defined in (1-7).

This example shows that local regularity may not be propagated (along characteristics), contrary to what happens for the class of PDEs considered in [Bony 1981]. This is due to the "nonlocality" in velocity. Therefore a global regularity assumption is required in order to obtain propagation of higher regularity.

This example can (also) be slightly modified, in order to prove that a local version of (2-13) cannot either be propagated into higher local regularity of moments; see the next (and last) example.

Example 3. Consider the equation

$$\partial_t f + v\,\partial_x f + F\big(t, x + \tfrac{1}{4}\big)\,\partial_v f = 0 \tag{10-3}$$

on $\mathbb{T}\times\mathbb{R}$ (here we identify $\mathbb{T}$ with $[0,1)$ with periodic boundary conditions). Let us consider as in the previous example

$$F(t,x) = \int_{\mathbb{R}} \psi(v)\, f(t,x,v)\, dv.$$

We consider the initial condition

$$f|_{t=0} = f_0^{(1)} + f_0^{(2)},$$

where $f_0^{(1)}$ is a nonnegative function, with compact support in $\big[0,\frac{1}{8}\big]\times\mathbb{R}$, and $f_0^{(2)}$ is a nonnegative function, with compact support in $\big[\frac{1}{4},\frac{3}{8}\big]\times\mathbb{R}$.

Observe that because of the shift in the argument of the force, by looking at the supports in x, the solution $f^{(2)}$ associated to the initial condition $f_0^{(2)}$ is equal to $f_0^{(2)}(t, x - tv, v)$ on $[0,T]$ for $T>0$ small enough. Moreover, we have

$$\left(\int_{\mathbb{R}} \psi(v) f^{(2)}\big(t, x+\tfrac{1}{4}, v\big)\, dv\right)\partial_v f^{(2)} = 0.$$

Now define $f^{(1)}$ as the solution on $[0,T]$ of the *linear* kinetic transport equation

$$\partial_t f + v\,\partial_x f + \left(\int_{\mathbb{R}} \psi(v)\, f_0^{(2)}\big(x+\tfrac{1}{4}-tv, v\big)\, dv\right)\partial_v f = 0,$$

with initial condition $f_0^{(2)}$.

We observe that up to reducing $T>0$, the solution f on $[0,T]$ of (10-2) can be written as

$$f = f^{(1)} + f^{(2)}.$$

Indeed, by looking at the supports in x, we can impose $T>0$ small enough so that

$$\left(\int_{\mathbb{R}} \psi(v)\, f^{(1)}\big(t, x+\tfrac{1}{4}, v\big)\, dv\right)\partial_v f^{(2)} = 0,$$
$$\left(\int_{\mathbb{R}} \psi(v)\, f^{(1)}\big(t, x+\tfrac{1}{4}, v\big)\, dv\right)\partial_v f^{(1)} = 0.$$

Now let $k\in\mathbb{N}$ and assume that there is $x_0\in\big(0,\frac{1}{8}\big)$ such that $\int_{\mathbb{R}} f|_{t=0}(x_0,v)\, dv \neq 0$ and $f|_{t=0}$ is locally H^k_x around this point. This is equivalent to asking that $\int_{\mathbb{R}} f_0^{(1)}(x_0,v)\, dv \neq 0$ and $f_0^{(1)}$ is locally H^k_x

around this point. This corresponds to a local analogue of (2-13). However, we can choose (independently of $f_0^{(1)}$) $f_0^{(2)}$ so that $\int_{\mathbb{R}} \psi(v) f_0^{(2)}(x - tv, v)\, dv$ is not locally H^k, in such a way that the moments in velocity of $f^{(1)}(t)$ (and thus of $f(t)$) are not locally H_x^k around points of the form $X(0, t, x_0, v_0)$, for some $v_0 \in \mathbb{R}$.

Acknowledgements

The author would like to thank Frédéric Rousset, as this work wouldn't have been possible without [Han-Kwan and Rousset 2016], and Jérémie Szeftel for a stimulating discussion about the second-order operators introduced in that work and used in this one as well.

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Received 12 Oct 2017. Revised 19 Mar 2018. Accepted 19 Apr 2018.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 12 No. 1 2019