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We consider a thin elastic sheet in the shape of a disk that is clamped at its boundary such that the displacement and the deformation gradient coincide with a conical deformation with no stretching there. These are the boundary conditions of a so-called “d-cone”. We define the free elastic energy as a variation of the von Kármán energy, which penalizes bending energy in  $L^p$  with  $p \in (2, \frac{8}{3})$  (instead of, as usual,  $p = 2$ ). We prove ansatz-free upper and lower bounds for the elastic energy that scale like  $h^{p/(p-1)}$ , where  $h$  is the thickness of the sheet.

## 1. Introduction

Strong deformations of thin elastic sheets under the influence of some external force have been a topic of considerable interest in the physics and engineering community over the last decades. These “postbuckling” phenomena are relevant on many length scales, e.g., for structural failure, for the design of protective structures, or in atomic-force microscopy of virus capsids and bacteria. In the physics literature, one finds numerous contributions that discuss the focusing of elastic energy in ridges and conical vertices; see [Cerde et al. 1999; Venkataramani 2004; Lobkovsky and Witten 1997]. The overview article [Witten 2007] contains a comprehensive review of the activities in that area of physics. However, quoting the seminal work [Lobkovsky et al. 1995], the “understanding of the strongly buckled state remains primitive”, and this fact has not changed fundamentally since the publication of that article more than 20 years ago.

In the mathematical literature on thin elastic sheets, there have been two major topics: On the one hand, there are the derivations of lower-dimensional models starting from three-dimensional finite elasticity [Ciarlet 1997; Le Dret and Raoult 1995; Friesecke et al. 2002; 2006]. On the other hand, there has been quite some effort to investigate the qualitative properties of plate models by determining the scaling behavior of the free elastic energy with respect to the small parameters in the model (such as the thickness of the sheet). Such scaling laws have been derived, e.g., in [Bella and Kohn 2014; Ben Belgacem et al. 2002; Bourne et al. 2017; Kohn and Nguyen 2013]. Building on the results from [Venkataramani 2004], it has been proved in [Conti and Maggi 2008] that the free energy per unit thickness of the so-called “single fold” scales like  $h^{5/3}$ , where  $h$  is the thickness of the sheet. This is also the conjectured scaling behavior for the confinement problem, which consists in determining the minimum of elastic energy necessary to fit a thin elastic sheet into a container whose size is smaller than the diameter of the sheet. The energy focusing in conical vertices has been investigated in [Brandman et al. 2013; Müller and Olbermann 2014],

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where the following has been proved: Consider a thin elastic sheet in the shape of a disc, and fix it at the boundary and at the center such that it agrees with a (nonflat) conical configuration there. Then the elastic energy scales like  $h^2 \log(1/h)$ . On a technical level, [Conti and Maggi 2008; Brandman et al. 2013; Müller and Olbermann 2014] consider an energy functional of the form

$$I_h(y) = \int_{\Omega} |Dy^T Dy - \text{Id}_{2 \times 2}|^2 + h^2 |D^2 y|^2 dx, \quad (1)$$

where  $\Omega \subset \mathbb{R}^2$  is the undeformed sheet,  $y : \Omega \rightarrow \mathbb{R}^3$  is the deformation, and  $\text{Id}_{2 \times 2}$  is the 2-by-2 identity matrix. The first term is the (nonconvex) membrane energy, and the second is the bending energy. If one manages to derive scaling laws for this two-dimensional model, then as a consequence, it is often the case that analogous results for three-dimensional elasticity are not difficult to derive as a corollary by the results from [Friecke et al. 2002]; see for example [Conti and Maggi 2008; Brandman et al. 2013]. Of course, the character of the variational problem heavily depends on the chosen boundary conditions.

While the mentioned articles have contributed a lot to the mathematical understanding of folds and vertices in thin sheets, they do not consider situations where the constraints prevent the sheet from adopting an isometric immersion with respect to the reference metric as its configuration, but do not prevent it from adopting a *short* map as its configuration. (We recall that a map  $y : \Omega \rightarrow \mathbb{R}^3$  is short if every path  $\gamma \subset \Omega$  is mapped to a shorter path  $y(\gamma) \subset \mathbb{R}^3$ .) Such a situation is characteristic of postbuckling, and in particular, the confinement problem.

The reason why short maps are problematic can be found in the famous Nash–Kuiper theorem [Nash 1954; Kuiper 1955a; 1955b]: if one is given a short map  $y_0 \in C^1(\Omega; \mathbb{R}^3)$  and  $\varepsilon > 0$ , then there exists an isometric immersion  $y \in C^1(\Omega; \mathbb{R}^3)$  with  $\|y - y_0\|_{C^0} < \varepsilon$ . This is relevant in the present context, since the difference between the induced metric and the flat reference metric is the leading-order term in the energy (1). Thus, if short maps are permissible, then there exists a vast amount of configurations with vanishing or very small membrane energy. One needs a principle that is capable of showing that all these maps are associated with a large amount of bending energy. As has recently been shown in [Lewicka and Pakzad 2017], this problem is not only encountered when dealing with the geometrically fully nonlinear plate model (1). It is also present in the von Kármán model, which we are going to treat here. In fact, the proof in [Lewicka and Pakzad 2017] is based on a suitable adaptation of the Nash–Kuiper argument to the von Kármán model.

Possibly the simplest example of a variational problem where isometric immersions are prohibited by the boundary conditions, but short maps are not, is given by a modification of the “conically constrained” sheets from [Brandman et al. 2013; Müller and Olbermann 2014]. The modification consists in considering clamped boundary conditions (for displacements and deformation gradients), and dropping the constraint on the deformation at the center of the sheet. This completely changes the character of the problem, and the method of proof from [Brandman et al. 2013; Müller and Olbermann 2014] breaks down.

This is the variational problem we will consider here, and we will prove an energy scaling law for it; see Theorem 2.1 below. There is one caveat: we penalize the bending energy in  $L^p$  with  $p \in (2, \frac{8}{3})$ , see (6), instead of, as would be dictated by a heuristic derivation of the von Kármán model from three-dimensional elasticity,  $p = 2$ . For a discussion of this modification, see Remark 2.2.

Our method of proof builds on the observations we made in [Olbermann 2016; 2017], where we proved scaling laws for an elastic sheet with a single disclination. The guiding principle is that the (linearized) Gauss curvature is controlled by both the membrane and the bending energy, in different function spaces. The boundary conditions can be used to show that the Gauss curvature is bounded from below in a certain space “in between” in the sense of interpolation. In the recent paper [Olbermann 2018], we showed that for the setting of [Olbermann 2016; 2017], it is not necessary to use interpolation, and lower bounds for the bending energy can be obtained by using the control over the membrane energy alone. The present setting with a flat reference metric however defines an interpolation-type problem for the Gauss curvature, and we hope that this approach can also yield results for similar variational problems.

This paper is structured as follows: In Section 2, we state our main result, Theorem 2.1. In Section 3, we collect some facts from the literature, concerning the Brouwer degree, Sobolev and Triebel–Lizorkin spaces, and interpolation theory. The proof of Theorem 2.1 is contained in Section 4.

**Notation.** We write  $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$  and  $S^1 = \partial B_1$ . When dealing with functions on  $S^1$ , we will identify  $S^1$  with the one-dimensional torus  $\mathbb{R}/(2\pi\mathbb{Z})$ . For  $x = (x_1, x_2) \in \mathbb{R}^2$ , we write  $\hat{x} = x/|x|$  and  $x^\perp = (-x_2, x_1)$ . In Section 2 below, we introduce a function  $\beta \in W^{2,p}(S^1)$  that can be considered as fixed for the rest of the paper. The symbol “ $C$ ” is used as follows: A statement such as “ $f \leq Cg$ ” is shorthand for “there exists a constant  $C > 0$  that only depends on  $\beta$  such that  $f \leq Cg$ ”. The value of  $C$  may change within the same line. For  $f \leq Cg$ , we also write  $f \lesssim g$ . The symmetrized gradient of a function  $u : U \rightarrow \mathbb{R}^2$  with  $U \subseteq \mathbb{R}^2$  is denoted by  $\text{sym } Du = \frac{1}{2}(Du + Du^T)$ .

## 2. Setting and statement of the main theorem

Let  $\beta \in W^{2,p}(S^1)$  with

$$\int_{S^1} (\beta^2(t) - \beta'^2(t)) dt = 0 \quad \text{and} \quad \int_{S^1} |\beta + \beta''| dt > 0. \quad (2)$$

Using the identification of  $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$  with the torus  $\mathbb{R}/(2\pi\mathbb{Z})$ , we define

$$\gamma(t) := -\frac{\beta^2(t)}{2} \quad \zeta(t) := \frac{1}{2} \int_0^t \beta^2(s) - \beta'^2(s) ds, \quad (3)$$

and we define  $u_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$u_\beta(x) \cdot \hat{x} := |x|\gamma(\hat{x}), \quad u_\beta(x) \cdot \hat{x}^\perp := |x|\zeta(\hat{x}), \quad (4)$$

Furthermore, we set

$$v_\beta(x) = |x|\beta(\hat{x}). \quad (5)$$

Note that the deformation defined by  $u_\beta, v_\beta$  is an isometric immersion in the von Kármán sense; i.e.,

$$\text{sym } Du_\beta + \frac{1}{2} Dv_\beta \otimes Dv_\beta = 0,$$

but  $D^2 v_\beta \notin L^p$  for  $p \geq 2$ . The set of allowed configurations is given by

$$\mathcal{A}_{\beta,p} := \{(u, v) \in W^{1,2}(B_1) \times W^{2,p}(B_1) : v = v_\beta, Dv = Dv_\beta \text{ and } u = u_\beta \text{ on } S^1\}.$$

The energy is given by a sum of membrane and bending energy,

$$I_{h,p}(u, v) = \|\operatorname{sym} Du + \tfrac{1}{2} Dv \otimes Dv\|_{L^2(B_1)}^2 + h^2 \|D^2 v\|_{L^p(B_1)}^2. \quad (6)$$

In the statement of our main theorem, the dual exponent  $p'$  is defined as usual by  $1/p + 1/p' = 1$ . We are going to prove:

**Theorem 2.1.** *Let  $p \in (2, \frac{8}{3})$ . Then there exists a constant  $C = C(\beta, p) > 0$  such that*

$$C^{-1} h^{p'} \leq \inf_{y \in \mathcal{A}_{\beta,p}} I_{h,p}(y) \leq C h^{p'}.$$

**Remark 2.2.** (i) The arguments of the energy functional  $(u, v) : B_1 \rightarrow \mathbb{R}^3$  can be thought of as the displacements of a deformation  $x \mapsto x + \varepsilon^2 u(x) + \varepsilon v(x) e_3$ , where  $\varepsilon$  is another small parameter (with  $h \ll \varepsilon$ ). The membrane energy geometrically corresponds to the deviation of the induced metric tensor from the flat Euclidean metric: the induced metric is given by  $(\operatorname{Id}_{2 \times 2} + \varepsilon^2 Du + \varepsilon e_3 \otimes Dv)^T (\operatorname{Id}_{2 \times 2} + \varepsilon^2 Du + \varepsilon e_3 \otimes Dv)$ , and the membrane term  $\operatorname{sym} Du + \frac{1}{2} Dv \otimes Dv$  is the leading-order term of the difference to the flat reference metric. We say that  $\det D^2 v$  is the “linearized Gauss curvature” since we have that the Gauss curvature is given by  $K = \varepsilon^2 \det D^2 v + o(\varepsilon^2)$ . Rigorously, the von Kármán energy, (6) with  $p = 2$ , has only been justified as a limit of three-dimensional finite elasticity for small deformations [Ciarlet 1980; Friesecke et al. 2006]. Nevertheless, it has a long and successful history of describing phenomena including moderate deformations.

(ii) The conditions on the boundary values in (2) are the von Kármán version of the requirement that the associated conical deformation defined by  $u_\beta, v_\beta$  has no membrane energy and is not contained in a plane.

(iii) The restriction to the range  $p \in (2, \frac{8}{3})$  is due to our method of proof, which is an application of the Gagliardo–Nirenberg inequality to the linearized Gauss curvature  $\det D^2 v$ . This interpolation inequality is only valid for that range. The standard von Kármán model is linear in the material response, and hence it penalizes the bending energy in  $L^2$ . In this case, one expects an energy scaling law of the form  $I_{h,2} \sim h^2 \log(1/h)$ , as is the case when the center of the sheet is fixed; see [Brandman et al. 2013; Müller and Olbermann 2014]. In order to obtain lower bounds for this case, one would have to show “additional regularity”, in the sense that one would need to control higher  $L^p$  norms of  $D^2 v$  by the  $L^2$  norm. One might hope that such estimates are possible, e.g., for minimizers of the functional. However, we do not know if this is possible.

(iv) We do not know if our method of proof can be adapted to prove an analogous result for the geometrically fully nonlinear plate model that is given by the energy  $\tilde{I}_{h,p} : W^{2,p}(B_1; \mathbb{R}^3) \rightarrow \mathbb{R}$ ,

$$\tilde{I}_{h,p}(y) = \int_{B_1} |Dy^T Dy - \operatorname{Id}_{2 \times 2}|^2 dx + h^2 \|D^2 y\|_{L^p(B_1)}^2.$$

The reason is that it seems much more complicated to obtain a good test function in  $W^{1,p}$  for  $\sum_i \det D^2 y_i$  (which is the appropriate linearization of Gauss curvature in that setting) that would yield a lower bound for this quantity in the Sobolev space  $W^{-1,p'}$ . In the von Kármán case, we can simply use the identity

$$(\operatorname{div} \psi)|_{Dv(x)} \det D^2 v(x) = \operatorname{div}(\psi(Dv(x)) \operatorname{cof} D^2 v(x))$$

and compute a lower bound for this quantity by Gauss' theorem, using the boundary values of  $Dv$ . In the case of  $y \in W^{2,p}(B_1; \mathbb{R}^3)$ , we cannot argue similarly component by component: only the *sum*  $\sum_i \det D^2 y_i$  is controlled by the energy. The task is to find a test function that (a) allows us to use Gauss' theorem and the boundary values to obtain a lower bound of order 1, and (b) is controlled in  $W^{1,p}$  by the bending energy. We have not found a way to do so.

### 3. Preliminaries

**The Brouwer degree.** At the heart of our proof of the lower bound for the energy is an interpolation estimate for the linearized Gauss curvature. This quantity can be thought of as a pull-back of the volume form on  $\mathbb{R}^2$  under the map  $Dv : B_1 \rightarrow \mathbb{R}^2$ . This is where the Brouwer degree becomes relevant, since integrals over the linearized Gauss curvature “downstairs” (on  $B_1$ ) can be expressed as integrals “upstairs” (on  $\mathbb{R}^2$ ) over the Brouwer degree of  $Dv$ .

For a bounded set  $U \subset \mathbb{R}^n$ ,  $f \in C^\infty(\bar{U}; \mathbb{R}^n)$  and  $y \in \mathbb{R}^n \setminus f(\partial U)$ , the Brouwer degree  $\deg(f, U, y)$  may be defined as follows: Let  $A_{y,f}$  denote the connected component of  $\mathbb{R}^n \setminus f(\partial U)$  that contains  $y$ , and let  $\mu$  be a smooth  $n$ -form on  $\mathbb{R}^n$  with support in  $A_{y,f}$  such that  $\int_{\mathbb{R}^n} \mu = 1$ . Then we set

$$\deg(f, U, y) = \int_U f^\# \mu,$$

where  $f^\#$  denotes the pull-back under  $f$ . By approximation with smooth functions,  $\deg(f, U, y)$  may be defined for every  $f \in C^0(\bar{U}; \mathbb{R}^n)$  and  $y \in \mathbb{R}^n \setminus f(\partial U)$ . If  $f \in W^{1,\infty}(\bar{U}; \mathbb{R}^n)$  and  $\mu$  is an  $n$ -form with regularity  $W^{1,\infty}$ , it follows straightforwardly from the definition that

$$\int_{\mathbb{R}^n} \deg(f, U, \cdot) \mu = \int_U f^\# \mu.$$

If  $\mu = \varphi \, dz$ , where  $dz$  is the canonical volume form on  $\mathbb{R}^n$ , this can be written as

$$\int_{\mathbb{R}^n} \varphi(z) \deg(f, U, z) \, dz = \int_U \varphi(f(x)) \det Df(x) \, dx. \quad (7)$$

If  $f \in C^1(U; \mathbb{R}^n)$ ,  $U$  has Lipschitz boundary and  $\mu$  is a smooth  $(n-1)$ -form on  $\mathbb{R}^n$ , then we have

$$\int_{\mathbb{R}^n} \deg(f, U, \cdot) \, d\mu = \int_U f^\#(d\mu) = \int_{\partial U} f^\# \mu. \quad (8)$$

It can be shown that  $y \mapsto \deg(f, U, y)$  is constant on the connected components of  $\mathbb{R}^n \setminus f(\partial U)$ . Finally, we are going to use the fact that  $\deg(f, U, y)$  only depends on  $f|_{\partial U}$ . Thus for every continuous function  $\tilde{f} : \partial U \rightarrow \mathbb{R}^n$ , and  $y \notin \tilde{f}(\partial U)$ , we may define

$$\deg^\partial(\tilde{f}, \partial U, y) = \deg(f, U, y),$$

where  $f$  is any continuous extension of  $\tilde{f}$  to  $\bar{U}$ . For more details (in particular for the proofs of the statements made here), see [Fonseca and Gangbo 1995].

**Function spaces.** Our main estimate for the Gauss curvature is a version of the Gagliardo–Nirenberg inequality for the spaces  $W^{-m,p}$  with  $m \in \mathbb{N}$  and  $p \in (1, \infty)$ . To define these spaces, let  $\Omega \subset \mathbb{R}^n$  be a bounded open set. For  $u \in L^1(\Omega)$  with compact support in  $\Omega$ , we set

$$\|u\|_{W_0^{m,p}(\Omega)} := \left( \int_{\Omega} |D^m u|^p \, dx \right)^{1/p}.$$

This defines a norm on the space  $W_0^{m,p}(\Omega)$  which is defined as the set of those  $u \in L^1(\Omega)$  that are compactly supported in  $\Omega$  and satisfy  $\|u\|_{W_0^{m,p}(\Omega)} < \infty$ . The dual space of  $W_0^{m,p}(\Omega)$  is denoted by  $W^{-m,p'}(\Omega)$ , where  $p'$  satisfies  $1/p + 1/p' = 1$ . The norm on  $W^{-m,p'}(\Omega)$  is given by

$$\|f\|_{W^{-m,p'}(\Omega)} = \sup\{\langle f, \varphi \rangle : \varphi \in W_0^{m,p}(\Omega), \|\varphi\|_{W_0^{m,p}(\Omega)} \leq 1\}.$$

Additionally, we define the space  $W^{m,p}(\mathbb{R}^n)$  as the completion of  $C_c^\infty(\mathbb{R}^n)$  under the norm

$$\|u\|_{W^{m,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |D^m u|^p \, dx \right)^{1/p}.$$

The Gagliardo–Nirenberg inequality that we want to prove is an interpolation inequality for the spaces  $W^{-m,p'}(\Omega)$ . In fact, the interpolation can be carried out in the spaces  $W^{m,p}$  (with  $m \geq 0$  and the understanding  $W^{0,p} \equiv L^p$ ). These will be derived by appealing to results from the literature, where one finds a well-developed interpolation theory for the Triebel–Lizorkin spaces  $F_{p,q}^s$ , which contains the appropriate interpolation between Lebesgue and Sobolev spaces as a special case.

Let  $\mathcal{D}'(\mathbb{R}^n)$  denote the space of temperate distributions on  $\mathbb{R}^n$ , and let  $\mathcal{F} : \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$  denote the Fourier transform. We briefly recall the Littlewood–Paley decomposition of temperate distributions: Let  $\eta_0 \in C_c^\infty(\mathbb{R}^n)$  be such that  $0 \leq \eta_0 \leq 1$ ,  $\eta_0(x) = 1$  for  $|x| \leq 1$ ,  $\eta_0(x) = 0$  for  $|x| \geq 2$ . Set  $\eta_j(x) = \eta_0(2^{-j}x) - \eta_0(2^{-j+1}x)$  for  $j \geq 1$ .

**Definition 3.1** [Triebel 1983, Chapter 2.3.1]. For  $-\infty < s < \infty$ ,  $0 < p, q < \infty$ , let

$$F_{p,q}^s(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : \|f\|_{F_{p,q}^s(\mathbb{R}^n)} := \left\| \sum_{j \in \mathbb{Z}} 2^{sj} \mathcal{F}^{-1} \eta_j \mathcal{F} f \right\|_{l^q L^p(\mathbb{R}^n)} < \infty\}.$$

The following special cases of the Triebel–Lizorkin spaces will be relevant for us (see [Triebel 1983, Sections 2.2.2 and 2.3.5]):

$$\begin{aligned} L^p(\mathbb{R}^n) &= F_{p,2}^0(\mathbb{R}^n), \\ W^{k,p}(\mathbb{R}^n) &= F_{p,2}^k(\mathbb{R}^n) \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{9}$$

Apart from their interpolation properties, the following embedding theorem will play a role in our proof:

**Theorem 3.2** [Triebel 1983, Theorem 2.7.1]. Suppose  $-\infty < s_1 < s_0 < \infty$ ,  $0 < p_0 < p_1 < \infty$  and  $0 < q_0, q_1 < \infty$  such that

$$s_1 - \frac{n}{p_1} = s_0 - \frac{n}{p_0}.$$

Then we have the continuous embedding

$$F_{p_0,q_0}^{s_0}(\mathbb{R}^n) \subset F_{p_1,q_1}^{s_1}(\mathbb{R}^n).$$

**Real interpolation.** We recall some basic facts concerning the real interpolation method. Let  $X_0, X_1$  be Banach spaces such that there exists a topological vector space  $Z$  with continuous embeddings  $X_0, X_1 \subset Z$ . In such a situation, let  $t > 0$  and  $x \in X_0 + X_1$ . We define

$$K(t, x) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0, x_1 \in X_1, x_0 + x_1 = x\}.$$

Let  $0 \leq \theta \leq 1$  and  $p \geq 1$ . The real interpolation space  $(X_0, X_1)_{\theta, p}$  is defined as

$$(X_0, X_1)_{\theta, p} = \{x \in X_0 + X_1 : \Phi_{\theta, p}(x) < \infty\},$$

where

$$\Phi_{\theta, p}(x) = \begin{cases} \left( \int_0^\infty |t^{-\theta} K(t, x)|^p \frac{dt}{t} \right)^{1/p} & \text{if } p < \infty, \\ \sup_{t>0} |t^{-\theta} K(t, x)| & \text{else.} \end{cases}$$

The interpolation space  $(X_0, X_1)_{\theta, p}$  is a normed space with the norm  $\Phi_{\theta, p}(x)$ . For every  $p < \infty$ , we have the continuous embedding

$$(X_0, X_1)_{\theta, p} \subset (X_0, X_1)_{\theta, \infty}. \quad (10)$$

For a proof, see, e.g., Chapter 1.3 of [Triebel 1978]. Concerning real interpolation of Triebel–Lizorkin spaces, we have the following theorem:

**Theorem 3.3** [Triebel 1978, Theorem 1 in Chapter 2.4.2]. *Let  $-\infty < s_0, s_1 < \infty$ ,  $1 < p_0, p_1, q_0, q_1 < \infty$ ,  $0 < \theta < 1$  and*

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

*Then we have*

$$(F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n))_{\theta, p} = F_{p, p}^s(\mathbb{R}^n).$$

#### 4. Proof of Theorem 2.1

A sketch of the proof of Theorem 2.1 goes as follows: As usual, the upper bound is provided by a conical construction that is smoothed on a ball around the origin with the appropriate length scale; see Lemma 4.1. At the heart of the lower bound, we have an interpolation inequality for the linearized Gauss curvature  $\det D^2 v$ . The Gagliardo–Nirenberg inequality [Nirenberg 1959] yields

$$\|\det D^2 v\|_{W^{-1, p'}(B_1)} \lesssim \|\det D^2 v\|_{W^{-2, 2}(B_1)}^{1-\alpha} \|\det D^2 v\|_{L^{p/2}(B_1)}^\alpha \quad (\text{formally}), \quad (11)$$

with  $\alpha \in [\frac{1}{2}, 1]$  determined by

$$\frac{2}{p'} - 1 = \left( \frac{2}{p/2} - 2 \right) \alpha + 1 - \alpha,$$

i.e.,

$$\alpha = \frac{2}{3p - 4}.$$

In (11), the left-hand side can be bounded from below using the boundary conditions and an argument involving the mapping degree. Namely, for an appropriately chosen test function  $\varphi \in C_c^\infty(\mathbb{R}^2)$ , we have

$$\int_{B_1} \varphi \circ Dv(x) \det D^2v \, dx = \int_{\mathbb{R}^2} \varphi(z) \deg(Dv, B_1, z) \, dz = O(1).$$

For the details see Lemma 4.2.

The exponents in (11) are chosen such that the terms on the right-hand side can be estimated by the energy,

$$\begin{aligned} \|\det D^2v\|_{W^{-2,2}} &\lesssim \|\operatorname{sym} Du + \tfrac{1}{2}Dv \otimes Dv\|_{L^2} \lesssim I_{h,p}(u, v)^{1/2}, \\ \|\det D^2v\|_{L^{p/2}} &\lesssim \|D^2v\|_{L^p}^2 \lesssim h^{-2} I_{h,p}(u, v). \end{aligned} \quad (12)$$

With these estimates, we obtain the desired lower bound.

Basically, all that remains is to prove the aforementioned lemmas, and justify (11). We could not find a proof of the Gagliardo–Nirenberg inequality for “negative orders of differentiation” in the literature. We believe that it holds true, and that a proof could be given using the machinery from [Triebel 1978]. However, in our case a shorter route exists, using the fact that  $v : B_1 \rightarrow \mathbb{R}$  has a natural extension to  $\mathbb{R}^2$  with vanishing membrane energy on  $\mathbb{R}^2 \setminus B_1$ , and existing results on interpolation of Sobolev and Triebel–Lizorkin spaces on  $\mathbb{R}^n$ ; see again [Triebel 1978].

Now we start with the proof.

**Lemma 4.1.** *We have*

$$\inf_{y \in \mathcal{A}_{\beta,p}} I_{h,p}(y) < Ch^{p'}.$$

*Proof.* Recall the definition of  $u_\beta, v_\beta$  from (3)–(5). Let  $\eta \in C^\infty([0, \infty))$  with  $\eta(t) = 0$  for  $t < \frac{1}{2}$ ,  $\eta(t) = 1$  for  $t \geq 1$ . We set

$$v_{\beta,h}(x) = \eta\left(\frac{|x|}{h^{p'/2}}\right) v_\beta(x).$$

Now we have

$$\left| \operatorname{sym} Du_\beta(x) + \tfrac{1}{2}Dv_{\beta,h}(x) \otimes Dv_{\beta,h}(x) \right| = \begin{cases} 0 & \text{if } |x| \geq h^{p'/2}, \\ O(1) & \text{else.} \end{cases}$$

Furthermore, we have

$$\int_{B_1} |D^2v_{\beta,h}|^p = \int_{B_1 \setminus B_{h^{p'/2}}} dx \left| \frac{\beta''(\hat{x}) + \beta(\hat{x})}{|x|} \right|^p + \int_{B_{h^{p'/2}}} O(h^{-p(p'/2)}) \, dx \lesssim h^{(2-p)p'/2}.$$

This implies

$$I_{h,p}(u_\beta, v_{\beta,h}) = \int_{B_1} \left| \operatorname{sym} Du_\beta(x) + \tfrac{1}{2}Dv_{\beta,h}(x) \otimes Dv_{\beta,h}(x) \right|^2 dx + h^2 \left( \int_{B_1} |D^2v_{\beta,h}|^p \, dx \right)^{2/p} \lesssim h^{p'}.$$

□

**Lemma 4.2.** *Assume that  $\beta \in W^{2,p}(S^1)$  with*

$$\int \beta^2(t) - \beta'^2(t) \, dt = 0 \quad \text{and} \quad \int |\beta + \beta''| \, dt \neq 0,$$

and let  $v_\beta$  be defined by (5). Then there exists  $\varphi_\beta \in C_c^\infty(\mathbb{R}^2)$  such that  $\text{supp } \varphi_\beta \cap Dv_\beta(S^1) = \emptyset$  and

$$\int_{\mathbb{R}^2} \varphi_\beta(z) \deg^\partial(Dv_\beta, S^1, z) dz > 0.$$

*Proof.* Step 1: reduction to the smooth case. We claim that we may assume  $\beta \in C^\infty(S^1)$ . Indeed, for every  $\varepsilon > 0$  we may choose  $\tilde{\beta} \in C^\infty(S^1; \mathbb{R}^2)$  such that

$$\|\beta - \tilde{\beta}\|_{W^{2,p}} < \varepsilon \quad \text{and} \quad \int |\tilde{\beta} + \tilde{\beta}''| dt \neq 0.$$

Additionally, we may choose  $\tilde{\beta}$  such that

$$\int_{S^1} (\tilde{\beta}^2 - \tilde{\beta}'^2) dt = 0.$$

We have

$$Dv_\beta = \beta(\hat{x})\hat{x} + \beta'(\hat{x})\hat{x}^\perp, \quad Dv_{\tilde{\beta}} = \tilde{\beta}(\hat{x})\hat{x} + \tilde{\beta}'(\hat{x})\hat{x}^\perp.$$

By the continuous embedding  $W^{2,p} \rightarrow C^1$ , we have that  $\|Dv_\beta - Dv_{\tilde{\beta}}\|_{C^0(S^1)}$  and hence we can also make  $\|\deg^\partial(Dv_\beta, S^1, \cdot) - \deg^\partial(Dv_{\tilde{\beta}}, S^1, \cdot)\|_{L^1(\mathbb{R}^2)}$  arbitrarily small by a suitable choice of  $\varepsilon$ . If we manage to show  $\deg^\partial(Dv_{\tilde{\beta}}, S^1, \cdot) \neq 0$  in  $L^1(\mathbb{R}^2)$ , then we have also proved the claim of the lemma. Hence, from now on we prove the claim of the lemma for  $\beta \in C^\infty(S^1)$ .

Step 2: taking the derivative of “deg”. For  $t \in S^1$ , let  $e_t = (\cos t, \sin t)$ . Let  $\gamma : S^1 \rightarrow \mathbb{R}^2$  be defined by

$$\gamma(t) = \beta(t)e_t + \beta'(t)e_t^\perp.$$

It is enough to show that  $\deg^\partial(Dv_\beta, S^1, \cdot) = \deg^\partial(\gamma, S^1, \cdot)$  is nonzero in  $L^1(\mathbb{R}^2)$ . By (8), we have for any smooth one-form  $\omega = \omega_1 dx_1 + \omega_2 dx_2$  on  $\mathbb{R}^2$ ,

$$\int_{\mathbb{R}^2} \deg^\partial(\gamma, S^1, \cdot) d\omega = \int_{S^1} \gamma^\# \omega.$$

If we show that the right-hand side is nonzero for some choice of  $\omega$ , we are done. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$x \mapsto \sum_{t \in \gamma^{-1}(x)} \gamma'(t) = \sum_{t \in \gamma^{-1}(x)} (\beta(t) + \beta''(t))e_t^\perp. \quad (13)$$

Then we have

$$(\gamma^\# \omega)(t) = (\omega_1(\gamma(t)), \omega_2(\gamma(t))) \cdot f(\gamma(t)) dt,$$

and we see that it suffices to show that  $f \neq 0$  on a set of positive  $\mathcal{H}^1$  measure to prove the claim of the lemma.

Step 3: proof of the lemma by contradiction. We assume that  $f = 0$   $\mathcal{H}^1$ -almost everywhere and show that this leads to a contradiction. Since  $\gamma'(t) = (\beta(t) + \beta''(t))e_t^\perp$ , we have that  $\gamma'(t) = 0$  if and only if  $\beta(t) + \beta''(t) = 0$ . Let  $U$  be an open interval such that  $\gamma' \neq 0$  on  $U$  and  $\gamma : U \rightarrow \gamma(U)$  is a diffeomorphism.

Our aim is now to show that up to  $\mathcal{H}^1$  null sets, we have

$$\gamma^{-1}(\gamma(U)) \setminus U = U + \pi,$$

where we are using the identification of  $S^1$  with  $\mathbb{R}/(2\pi\mathbb{Z})$ .

Since  $f = 0$   $\mathcal{H}^1$ -almost everywhere on  $\gamma(U)$  and by the explicit form (13) of  $f$ , there exists  $E_1 \subset S^1$  with  $\mathcal{H}^1(E_1) = 0$  such that

$$\gamma(U \setminus E_1) \subseteq \gamma(S^1 \setminus U). \quad (14)$$

Next let

$$E_2 := \{t \in S^1 : \gamma'(t) = 0\}, \quad A := \gamma(E_2).$$

By Sard's lemma, we have  $\mathcal{H}^1(A) = 0$ . Furthermore, let

$$E_3 := \gamma^{-1}(\gamma(U \setminus E_1) \setminus A) \setminus U,$$

and let  $E_4 \subseteq E_3$  be the set of points that are not of density 1, i.e.,

$$E_4 := \left\{ x \in E_3 : \liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{H}^1((x - \varepsilon, x + \varepsilon) \cap E_3)}{2\varepsilon} < 1 \right\}.$$

It is a well-known fact from measure theory that  $\mathcal{H}^1(E_4) = 0$ . Let  $E_5 := \gamma^{-1}(\gamma(E_4)) \cap U$ . Then also  $\mathcal{H}^1(E_5) = 0$ .

Now let  $p \in U \setminus (E_1 \cup E_5)$ . Then  $\gamma(p) \notin A \cup \gamma(E_4)$ , and by (14), we have

$$\gamma(p) \in \gamma(S^1 \setminus U) \setminus (A \cup \gamma(E_5)).$$

Hence there exists  $p' \in E_3 \setminus E_4$  with  $\gamma(p') = \gamma(p)$ . We may choose a sequence  $p'_k$ ,  $k \in \mathbb{N}$ , with  $p'_k \in E_3$ ,  $\gamma(p'_k) \neq \gamma(p')$ , and  $p'_k \rightarrow p'$ . Since  $\gamma|_U$  is a diffeomorphism, we may set

$$p_k := \gamma|_U^{-1}(\gamma(p'_k))$$

and obtain a sequence  $p_k \rightarrow p$  in  $U$ , with  $\gamma(p_k) \neq \gamma(p)$ . Now we have for every  $k$ ,

$$\frac{\gamma(p_k) - \gamma(p)}{|\gamma(p_k) - \gamma(p)|} = \frac{\gamma(p'_k) - \gamma(p')}{|\gamma(p'_k) - \gamma(p')|}.$$

Passing to a suitable subsequence and taking the limit  $k \rightarrow \infty$  in that equation, we obtain that the vectors  $\gamma'(p)$  and  $\gamma'(p')$  are parallel. Since

$$\gamma'(t) = (\beta(t) + \beta''(t))e_t^\perp$$

and  $p \neq p'$ , we must have  $e_p = -e_{p'}$ , and hence (using the identification of  $S^1$  with  $\mathbb{R}/(2\pi\mathbb{Z})$ )

$$p' = p + \pi.$$

Summarizing, we have shown that for  $\mathcal{H}^1$ -almost every  $p \in U$ , we have  $p + \pi \in \gamma^{-1}(\gamma(U)) \setminus U$ . We also may conclude that for every  $p' \in E_3 \setminus E_4$ , it holds that  $p' + \pi \in U$ . Hence, as desired, we have

$$\gamma^{-1}(\gamma(U)) \setminus U = U + \pi \quad \text{up to } \mathcal{H}^1 \text{ null sets.}$$

Since for every  $x \in S^1 \setminus E_2$  there exists a neighborhood  $U$  of  $x$  with the properties we have assumed above, we obtain that for  $\mathcal{H}^1$ -almost every  $t \in S^1 \setminus E_2$ , we have  $\gamma(t) = \gamma(t + \pi)$ . Hence,

$$\beta(t)e_t + \beta'(t)e_t^\perp = -\beta(t + \pi)e_t - \beta'(t + \pi)e_t^\perp,$$

which implies

$$\beta(t + \pi) = -\beta(t), \quad \beta'(t + \pi) = -\beta'(t) \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in S^1 \setminus E_2. \quad (15)$$

We claim that we even have

$$\beta(t + \pi) = -\beta(t) \quad \text{for } t \in S^1. \quad (16)$$

Indeed, let  $t \in S^1$ . If  $t \in \overline{S^1 \setminus E_2}$ , then the claim follows from (15). If  $t$  is in the interior of  $E_2$ , then let  $T \in \partial E_2$  such that  $(t, T) \subset E_2$ . Then we have that also  $(t + \pi, T + \pi) \subset E_2$ , and  $\beta(T + \pi) = -\beta(T)$ ,  $\beta'(T + \pi) = -\beta'(T)$ . The values of  $\beta(t)$ ,  $\beta(t + \pi)$  are then determined by the initial values of  $\beta$ ,  $\beta'$  at the points  $T, T + \pi$  and by the ODE  $\beta + \beta'' = 0$ . By the linearity and translation invariance of this initial value problem, we obtain  $\beta(t + \pi) = -\beta(t)$  as desired. This proves the claim (16).

By (16), we have  $\int_{S^1} \beta(t) dt = 0$ . By the Poincaré–Wirtinger inequality, we have that

$$\int_{S^1} (\beta^2 - \beta'^2) dt \leq 0,$$

with equality only if  $\beta$  is of the form  $\beta(t) = C \sin(t + \alpha)$  for some  $C, \alpha \in \mathbb{R}$ . Equality must hold true by assumption, which yields

$$\beta + \beta'' = 0 \quad \text{on } S^1,$$

in contradiction to our assumptions. This proves the lemma.  $\square$

**Lemma 4.3.** *Let  $p \in (2, \frac{8}{3})$ , and*

$$\theta = 1 - \frac{2}{3p-4}, \quad \frac{1}{q} = \frac{1-\theta}{p/(p-2)} + \frac{\theta}{2}.$$

*Then we have*

$$W^{1,p}(\mathbb{R}^2) \subset (L^{p/(p-2)}(\mathbb{R}^2), W^{2,2}(\mathbb{R}^2))_{\theta,q}.$$

*Proof.* By (9), we have  $L^{p/(p-2)}(\mathbb{R}^2) = F_{p/(p-2),2}^0(\mathbb{R}^2)$  and  $W^{2,2}(\mathbb{R}^2) = F_{2,2}^2(\mathbb{R}^2)$ . By Theorem 3.3, we obtain

$$(L^{p/(p-2)}(\mathbb{R}^2), W^{2,2}(\mathbb{R}^2))_{\theta,q} = F_{q,q}^{2\theta}(\mathbb{R}^2).$$

Finally, by Theorem 3.2, we have

$$W^{1,p}(\mathbb{R}^2) = F_{p,2}^1(\mathbb{R}^2) \subset F_{q,q}^{2\theta}(\mathbb{R}^2).$$

Note that the assumption  $s_1 < s_0$  in Theorem 3.2 is fulfilled by  $1 > 2\theta$ , which in turn is a consequence of  $p \in (2, \frac{8}{3})$ .  $\square$

In the next lemma, we use the following notation: for  $(u, v) \in \mathcal{A}_{\beta, p}$ , we let  $\bar{v} : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in B_1, \\ u_\beta(x) & \text{if } x \in \mathbb{R}^2 \setminus B_1, \end{cases} \quad \bar{v}(x) = \begin{cases} v(x) & \text{if } x \in B_1, \\ v_\beta(x) & \text{if } x \in \mathbb{R}^2 \setminus B_1, \end{cases}$$

where  $u_\beta, v_\beta$  are as defined in (3)–(5).

**Lemma 4.4.** *Let  $(u, v) \in \mathcal{A}_{\beta, p}$ . Then*

$$\|\det D^2 \bar{v}\|_{W^{-2,2}(\mathbb{R}^2)} \lesssim \|\operatorname{sym} Du + \frac{1}{2} Dv \otimes Dv\|_{L^2(B_1)}.$$

*Proof.* We write down the Hessian determinant of  $\bar{v}$  in its very weak form,

$$\det D^2 \bar{v} = (\bar{v}_{,1} \bar{v}_{,2})_{,12} - \frac{1}{2} (\bar{v}_{,1}^2)_{,22} - \frac{1}{2} (\bar{v}_{,2}^2)_{,11} = -\frac{1}{2} \operatorname{curl} \operatorname{curl} D\bar{v} \otimes D\bar{v}.$$

Here, we have introduced  $\operatorname{curl}(w_1, w_2) = w_{1,2} - w_{2,1}$ . (In the formula above,  $\operatorname{curl}$  is first applied in each row of the matrix  $D\bar{v} \otimes D\bar{v}$ , and then on the components of the resulting column vector.) Since we have  $\operatorname{curl} \operatorname{curl}(Dw^T + Dw) = 0$  for every  $w \in W^{1,2}(B_1; \mathbb{R}^2)$ , we obtain

$$\det D^2 \bar{v} = -\operatorname{curl} \operatorname{curl}(\operatorname{sym} D\bar{u} + \frac{1}{2} D\bar{v} \otimes D\bar{v}).$$

We note that

$$\operatorname{sym} D\bar{u} + \frac{1}{2} D\bar{v} \otimes D\bar{v} = 0 \quad \text{on } \mathbb{R}^2 \setminus B_1.$$

Hence for every  $\varphi \in W^{2,2}(\mathbb{R}^2)$ , we obtain by two integrations by parts, and the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_{\mathbb{R}^2} \det D^2 \bar{v} \varphi \, dx &= - \int_{\mathbb{R}^2} (\operatorname{sym} D\bar{u} + \frac{1}{2} D\bar{v} \otimes D\bar{v}) : \operatorname{cof} D^2 \varphi \, dx \\ &\leq \|\operatorname{sym} Du + \frac{1}{2} Dv \otimes Dv\|_{L^2(B_1)} \|\varphi\|_{W^{2,2}(\mathbb{R}^2)}. \end{aligned} \quad \square$$

*Proof of Theorem 2.1.* The upper bound has been proved in Lemma 4.1. It remains to prove the lower bound.

For any  $(u, v) \in \mathcal{A}_{\beta, p}$  we have  $Dv|_{S^1} = Dv_\beta|_{S^1}$ , and hence  $\deg(Dv, B_1, \cdot) = \deg^\partial(Dv, S^1, \cdot) = \deg^\partial(Dv_\beta, S^1, \cdot)$ . By Lemma 4.2, we may choose  $\varphi \in C_c^\infty(\mathbb{R}^2)$  such that  $\varphi \circ Dv \in W_0^{1,p}(B_1) \subset W^{1,p}(\mathbb{R}^2)$  and

$$\begin{aligned} 0 < C(\beta) &= \int_{\mathbb{R}^2} \varphi(z) \deg(Dv, B_1, z) \, dz \\ &= \int_{B_1} \det D^2 v(x) \varphi(Dv(x)) \, dx \\ &= \int_{\mathbb{R}^2} \det D^2 \bar{v}(x) \varphi(D\bar{v}(x)) \, dx, \end{aligned}$$

where we have used the notation introduced above Lemma 4.4, and the fact that  $\det D^2 \bar{v} = 0$  on  $\mathbb{R}^2 \setminus B_1$ . By Lemma 4.3,  $\psi := \varphi \circ Dv \in (L^{p/(p-2)}(\mathbb{R}^2), W^{2,2}(\mathbb{R}^2))_{\theta, q}$ . Hence by (10), there exist functions  $\psi_0 : \mathbb{R}^+ \rightarrow L^{p/(p-2)}(\mathbb{R}^2)$  and  $\psi_1 : \mathbb{R}^+ \rightarrow W^{2,2}(\mathbb{R}^2)$  such that  $\psi_0(t) + \psi_1(t) = \psi$  for all  $t \in \mathbb{R}^+$  and

$$t^{-\theta} \|\psi_0(t)\|_{L^{p/(p-2)}(\mathbb{R}^2)} + t^{1-\theta} \|\psi_1(t)\|_{W^{2,2}(\mathbb{R}^2)} \lesssim \|\psi\|_{W^{1,p}(\mathbb{R}^2)}.$$

Rearranging, we have for every  $t > 0$  that

$$\begin{aligned}\|\psi_0(t)\|_{L^{p/(p-2)}(\mathbb{R}^2)} &\lesssim t^\theta \|\psi\|_{W^{1,p}(\mathbb{R}^2)}, \\ \|\psi_1(t)\|_{W^{2,2}(\mathbb{R}^2)} &\lesssim t^{\theta-1} \|\psi\|_{W^{1,p}(\mathbb{R}^2)}.\end{aligned}$$

Now we fix the argument,

$$t := \frac{\|\det D^2 \bar{v}\|_{W^{-2,2}}}{\|\det D^2 \bar{v}\|_{L^{p/2}}},$$

and write  $\psi_0 = \psi_0(t)$ ,  $\psi_1 = \psi_1(t)$ . Hence we may estimate

$$\begin{aligned}C(\beta) &= \int_{\mathbb{R}^2} \det D^2 \bar{v}(x) \varphi(D \bar{v}(x)) \, dx \\ &\lesssim \|\det D^2 \bar{v}\|_{L^{p/2}} \|\psi_0\|_{L^{p/(p-2)}} + \|\det D^2 \bar{v}\|_{W^{-2,2}} \|\psi_1\|_{W^{2,2}} \\ &\lesssim \|\det D^2 \bar{v}\|_{W^{-2,2}}^\theta \|\det D^2 \bar{v}\|_{L^{p/2}}^{1-\theta} \|\psi\|_{W^{1,p}} \\ &\lesssim I_{h,p}^{\theta/2} (h^{-2} I_{h,p})^{1-\theta} (h^{-2} I_{h,p})^{1/2} \\ &\lesssim I_{h,p}^{(3-\theta)/2} h^{2\theta-3},\end{aligned}\tag{17}$$

where we have used Lemma 4.4 and the facts

$$|\det D^2 v| \leq |D^2 v|^2, \quad \det D^2 \bar{v} = 0 \quad \text{on } \mathbb{R}^2 \setminus B_1$$

to obtain the fourth line from the third. This implies

$$I_{h,p} \gtrsim h^{(6-4\theta)/(3-\theta)} = h^{p/(p-1)} = h^{p'},$$

which proves the theorem.  $\square$

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
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# ANALYSIS & PDE

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Boundary behavior of solutions to the parabolic $p$ -Laplace equation	1
BENNY AVELIN, TUOMO KUUSI and KAJ NYSTRÖM	
On asymptotic dynamics for $L^2$ critical generalized KdV equations with a saturated perturbation	43
YANG LAN	
On the stability of type II blowup for the 1-corotational energy-supercritical harmonic heat flow	113
TEJ-EDDINE GHOUL, SLIM IBRAHIM and VAN TIEN NGUYEN	
On propagation of higher space regularity for nonlinear Vlasov equations	189
DANIEL HAN-KWAN	
On a boundary value problem for conically deformed thin elastic sheets	245
HEINER OLBERMANN	



2157-5045(2019)12:1;1-P