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A UNIFIED FLOW APPROACH TO SMOOTH, EVEN L_p-MINKOWSKI PROBLEMS

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We study long-time existence and asymptotic behavior for a class of anisotropic, expanding curvature flows. For this we adapt new curvature estimates, which were developed by Guan, Ren and Wang to treat some stationary prescribed curvature problems. As an application we give a unified flow approach to the existence of smooth, even L_p-Minkowski problems in $\mathbb{R}^{n+1}$ for $p > -n-1$.

1. Introduction

Consider a smooth, closed, strictly convex hypersurface M_0 in Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$, given by a smooth embedding $F_0 : M \to \mathbb{R}^{n+1}$. Suppose the origin is in the interior of the region enclosed by M_0. We study the long-time behavior of a family of hypersurfaces $\{M_t\}$ given by smooth maps $F : M \times [0, T) \to \mathbb{R}^{n+1}$ satisfying the initial value problem

$$\partial_t F(x,t) = \varphi(\nu(x,t)) \frac{(F(x,t) \cdot \nu(x,t))^{2-p}}{\mathcal{K}(x,t)} \nu(x,t), \quad F(\cdot, 0) = F_0(\cdot). \tag{1-1}$$

Here $\mathcal{K}(\cdot, t)$ and $\nu(\cdot, t)$ are the Gauss curvature and the outer unit normal vector of $M_t = F(M, t)$ and φ is a positive, smooth function on $\mathbb{S}^n$. Furthermore, T is the maximal time for which the solution exists.

For $p = 2$, $\varphi \equiv 1$, flow (1-1) was studied in [Schnürer 2006] in $\mathbb{R}^3$ and in [Gerhardt 2014] in higher dimensions. Both works rely on the reflection principle of [Chow and Gulliver 1996; McCoy 2003]. Their result is as follows: the volume-normalized flow evolves any M_0 in the C^∞-topology to an origin-centered sphere. For $p > 2$, $\varphi \equiv 1$, it follows from [Chow and Gulliver 1996, Theorem 3.1], see also [Tsai 2005, Example 1], that (1-1) evolves M_0, after rescaling to fixed volume, in the C^1-topology to an origin-centered sphere. We refer the reader to [Ivaki 2016] regarding a rather comprehensive list of previous works on this curvature flow. In particular, in either case $\varphi \not\equiv 1$ or $\varphi \equiv 1$, $-n-1 < p < 2$, we are not aware of any result in the literature on the asymptotic behavior of the flow. The following theorem was proved in [Ivaki 2016] regarding the case $p = -n-1$, $\varphi \equiv 1$; in this case the flow belongs to a family of centroaffine normal flows introduced in [Stancu 2012].

Let us write B for the unit ball of $\mathbb{R}^{n+1}$ and put

$$\tilde{K}_t := \left(\frac{V(B)}{V(K_t)} \right)^{\frac{1}{n+1}} K_t,$$

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where K_t denotes the convex body enclosed by M_t and $V(\cdot)$ is the $(n+1)$-dimensional Lebesgue measure.

Theorem [Ivaki 2016]. *Let $n \geq 2$, $p = -n-1$, $\varphi \equiv 1$ and suppose K_0 has its Santaló point at the origin; i.e.,*

$$\int_{\mathbb{S}^n} \frac{u}{h_{K_0}(u)^{n+2}} \, d\sigma(u) = 0.$$

Then there exists a unique solution $\{M_t\}$ of flow (1-1) such that $\widetilde{M}_t$ converges in C^∞ to an origin-centered ellipsoid.

Here h_{K_0} is the support function of K_0. A closed, convex hypersurface M_0 can be described in terms of its support function $h_{K_0} : \mathbb{S}^n \to \mathbb{R}$ defined by

$$h_{K_0}(u) = \sup\{u \cdot x : x \in M_0\}.$$

If M_0 is smooth and strictly convex, then $h_{K_0}(u) = u \cdot F_0(\nu^{-1}(u))$.

From the evolution equation of $F(\cdot, t)$ it follows that

$$h(\cdot, t) := h_{K_t}(\cdot) : \mathbb{S}^n \times [0, T) \to \mathbb{R}$$

evolves by

$$\partial_t h(u, t) = \varphi(u)(h^{2-p} S_n)(u, t), \tag{1-2}$$

where $S_n(u, t) = 1/\mathcal{K}(\nu^{-1}(u, t), t)$. A homothetic self-similar solution of this flow satisfies

$$h^{1-p} \det(\overline{\nabla}^2 h + \mathrm{Id}\, h) = \frac{c}{\varphi} \tag{1-3}$$

for some positive constant c. Here $\overline{\nabla}$ is the covariant derivative on $\mathbb{S}^n$. Note that $S_n = \det(\overline{\nabla}^2 h + \mathrm{Id}\, h)$.

We list the main results of the paper extending the previous-mentioned results.

Theorem 1. *Let $-n-1 < p < \infty$ and φ be a positive, smooth function on $\mathbb{S}^n$ that is even, i.e., $\varphi(u) = \varphi(-u)$. Suppose K_0 is origin-symmetric. There exists a unique origin-symmetric solution $\{M_t\}$ of (1-1) such that $\{\widetilde{M}_t\}$ converges for a subsequence of times in C^1 to a smooth, origin-symmetric, strictly convex solution of (1-3). Also, when $p \leq n+1$ the convergence is in C^∞, and if $p \geq 1$ the convergence holds for the full sequence.*

If $-n-1 < p \leq -n$, we can extend the result of the previous theorem by dropping the assumption that φ is even.

Theorem 2. *Let $-n-1 < p \leq -n$ and K_0 satisfy*

$$\int_{\mathbb{S}^n} \frac{u}{\varphi(u) h_{K_0}(u)^{1-p}} \, d\sigma(u) = 0.$$

There exists a unique solution $\{M_t\}$ of flow (1-1) such that $\{\widetilde{M}_t\}$ converges for a subsequence of times in C^∞ to a positive, smooth, strictly convex solution of (1-3).

Given any convex body K_0, there exists a vector $\vec{v}$ such that $K_0 + \vec{v}$ has the origin in its interior and it satisfies the assumption of the second theorem.

For $\varphi \equiv 1$ we prove the following theorem.

Theorem 3. *Let $1 \neq p > -n-1$, $\varphi \equiv 1$ and K_0 satisfy*

$$\int_{\mathbb{S}^n} \frac{u}{h_{K_0}(u)^{1-p}}\, d\sigma(u) = 0.$$

Then there exists a unique solution $\{M_t\}$ of (1-1) such that $\{\widetilde{M}_t\}$ converges in C^1 to the unit sphere. In addition, for $1 \neq p \leq n+1$ the convergence holds in C^∞.

For $p \neq n+1$, self-similar solutions to (1-1) are solutions of the L_p-Minkowski problem (1-4), and for $p = n+1$, a self-similar solution to (1-1) is a solution to the normalized L_{n+1}-Minkowski problem (1-5); we shall introduce them now.

The Minkowski problem deals with existence, uniqueness, regularity, and stability of closed convex hypersurfaces whose Gauss curvature (as a function of the outer normals) is preassigned. Major contributions to this problem were made by Minkowski [1897; 1903], Aleksandrov [1938; 1939; 1942], Fenchel and Jessen [1938], Lewy [1938a; 1938b], Nirenberg [1953], Calabi [1958], Pogorelov [1952; 1971], Cheng and Yau [1976], Caffarelli, Nirenberg, and Spruck [Caffarelli et al. 1984], and others. A generalization of the Minkowski problem known as the L_p-Minkowski problem was introduced in [Lutwak 1993], where for any $1 < p \neq n+1$ and a preassigned even Borel measure on $\mathbb{S}^n$ whose support does not lie in a great sphere of $\mathbb{S}^n$ the existence and uniqueness of the solution were proved. This generalization for $1 < p \neq n+1$ was further studied in [Lutwak and Oliker 1995], where they obtained the $C^{k,\alpha}$ regularity of the solution. Solutions to many cases of these generalized problems followed later in [Ai et al. 2001; Andrews 2000; 2002; 2003; Böröczky et al. 2013; Chen 2006; Chou and Wang 2006; Dou and Zhu 2012; Gage 1993; Gage and Li 1994; Guan and Lin 2000; Huang and Lu 2013; Jiang 2010; Jiang et al. 2011; Lu and Wang 2013; Lutwak et al. 2004; Stancu 1996; 2002; 2003; Umanskiy 2003; Zhu 2014; 2015a; 2015b].

For $p \neq n+1$, in the smooth category, the L_p-Minkowski problem asks, given a smooth, positive function $\varphi : \mathbb{S}^n \to \mathbb{R}$, if there exists a smooth, closed, strictly convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$ such that

$$\frac{h^{1-p}(\nu(x))}{\mathcal{K}(x)} = \frac{1}{\varphi(\nu(x))}, \tag{1-4}$$

where $x \in M_0$, h denotes the support function, $\mathcal{K}$ the Gauss curvature and ν the Gauss map $M_0 \to \mathbb{S}^n$. The *even* L_p-Minkowski problem requires, in addition, that φ is an even function. The case $p = 1$ is the original Minkowski problem.

The special case of $p = n+1$ is troubling since (1-4) might not have a solution. To remedy this, Lutwak, Yang and Zhang [Lutwak et al. 2004] introduced a normalized formulation of the L_{n+1}-Minkowski problem and they proved the existence and uniqueness of the solution for any prescribed even Borel measure on $\mathbb{S}^n$ whose support is not contained in a great sphere of $\mathbb{S}^n$. In the smooth category, the normalized L_{n+1}-Minkowski problem asks for the existence of a smooth, closed, strictly convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$ that solves

$$\frac{1}{h^n(\nu(x))\mathcal{K}(x)} = \frac{V(K_0)}{\varphi(\nu(x))}, \tag{1-5}$$

where K_0 is the convex body with the boundary M_0. In the rest of the paper, the L_p-Minkowski problem refers to either (1-4) or (1-5), and we avoid the word "normalized".

The existence and regularity of solutions to the L_p-Minkowski problem are rather comprehensively discussed in [Chou and Wang 2006] for $p > -n-1$. Our study on (1-1) provides an alternative variational treatment (based on curvature flow) of the even L_p-Minkowski problem. For $p = 1$, Chou and Wang [2000] treated the classical L_1-Minkowski problem in the smooth category by a logarithmic Gauss curvature flow. For $n = 1$ and $1 \neq p > -3$, the existence of solutions to the L_p-Minkowski problems follows from Andrews' results [1998] on the asymptotic behavior of a family of contracting and expanding flows of curves. Also, in higher dimensions, the existence of solutions to the L_p-Minkowski problems follows from [Andrews 2000] when $-n-1 < p \leq -n+1$ (a short proof of this is also given in [Ivaki 2015]) or when φ is even (i.e., $\varphi(u) = \varphi(-u)$) and $-n+1 < p < 1$. See also [Andrews 1999; Andrews et al. 2016; Guan and Ni 2017; Urbas 1998; 1999].

Using our results for the flows above, it is now a simple matter to give a new, unified proof of the smooth, even L_p-Minkowski problem for all ranges of $p > -n-1$.

Corollary 4. *Let $-n-1 < p < \infty$ and φ be a positive, smooth function on $\mathbb{S}^n$ that is even, i.e., $\varphi(u) = \varphi(-u)$. Then for $p \neq n+1$ there exists an origin-symmetric, smooth, strictly convex body such that (1-4) is satisfied. For $p = n+1$, there exists an origin-symmetric, smooth, strictly convex body such that (1-5) is satisfied.*

Proof. By the first part of Theorem 1 (only the convergence for a *subsequence* of times is needed), there exists a smooth, strictly convex body K with the volume of the unit ball and a constant $c > 0$ such that

$$\frac{h}{\mathcal{K}} = \frac{ch^p}{\varphi}.$$

Hence $c \int_{\mathbb{S}^n} h^p/\varphi \, d\sigma = (n+1)V(B^n)$. Thus there is a solution to

$$\frac{h^{1-p}(\nu(x))}{\mathcal{K}(x)} = \left(\frac{(n+1)V(B)}{\int_{\mathbb{S}^n} h^p/\varphi \, d\sigma} \right) \frac{1}{\varphi(\nu(x))}.$$

Now let us define

$$\lambda := \begin{cases} \left(\dfrac{\int_{\mathbb{S}^n} h^p/\varphi \, d\sigma}{(n+1)V(B)} \right)^{\frac{1}{n+1-p}}, & p \neq n+1, \\ \left(\dfrac{(n+1)V(B)}{V(K)\int_{\mathbb{S}^n} h^{n+1}/\varphi \, d\sigma} \right)^{\frac{1}{n+1}}, & p = n+1. \end{cases}$$

Therefore, λK solves the smooth, even L_p-Minkowski problem. □

Let us close this section with a brief outline of this paper. The main difficulty in proving convergence of the normalized solutions is in obtaining long-time existence. The issue arises from the time-dependent anisotropic factor (the support function). We believe in such generality, (1-1) serves as the first example where a time-dependent anisotropic factor is allowed. To prove long-time existence, we first obtain bounds on the Gauss curvature in Section 3.1. Using the well-known standard technique of [Tso 1985] we obtain upper bounds. We obtain lower bounds by applying the same technique to the evolution of the polar body as in [Ivaki 2015]. Controlling the principal curvatures requires estimates of higher derivatives of the speed, which is generally quite difficult due to the nonlinearity of the flow. In Section 3.2 we obtain these crucial estimates by adapting the remarkable C^2 estimates of Guan, Ren and Wang [Guan et al. 2015, (4.2)] for the prescribed curvature problem. Long-time existence then follows readily by standard arguments. Once it is proved that solutions to the flow exist until they expand to infinity uniformly in all directions, the method of [Ivaki 2016, Section 8] applies and yields convergence of the volume-normalized solutions in C^1 to self-similar solutions provided $p \neq 1$. Further work is required to establish convergence of normalized solutions if $p = 1$, and to prove convergence in C^∞ for $p \leq n+1$. This is accomplished in Section 4; see also Remark 10.

2. Basic evolution equations

Let $g = \{g_{ij}\}$ and $W = \{w_{ij}\}$ denote, in order, the induced metric and the second fundamental form of M. At every point in the hypersurface M choose a local orthonormal frame $\{e_1, \dots, e_n\}$.

We use the standard notation

$$w_i^j = g^{mj} w_{im}, \quad (w^2)_i^j = g^{mj} g^{rs} w_{ir} w_{sm}, \quad |W|^2 = g^{ij} g^{kl} w_{ik} w_{lj} = w_{ij} w^{ij}.$$

Here, $\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$.

We use semicolons to denote covariant derivatives. The following geometric formulas are well-known:

$$\nu_{;i} = w_i^k e_k, \qquad h_{;i} = w_i^k (F \cdot e_k),$$
$$\nu_{;ij} = g^{kl} w_{ij;l} e_k - w_i^l w_{lj} \nu, \quad h_{;ij} = w_{ij} - h w_i^l w_{lj} + F \cdot \nabla w_{ij}.$$

Note that above we considered the support function as a function on the boundary of the hypersurface; that is, at the point $x \in M$ we have

$$h(x) = F(x) \cdot \nu(x).$$

For convenience, let $\psi(x) = h^{2-p}(x) \varphi(\nu(x))$. The following evolution can be deduced in a standard manner; see for example [Gerhardt 2006].

Lemma 5. *The following evolution equations hold*:

$$\partial_t \nu = -\nabla\left(\frac{\psi}{\mathcal{K}}\right),$$

$$\begin{aligned}\partial_t w_i^j &= -\left(\frac{\psi}{\mathcal{K}}\right)_{;ik} g^{kj} - \left(\frac{\psi}{\mathcal{K}}\right) w_i^k w_k^j \\ &= \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{i;kl}^j + \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{kr} w_l^r w_i^j - (n+1) \frac{\psi}{\mathcal{K}} w_i^k w_k^j + \psi \frac{\mathcal{K}^{kl,rs}}{\mathcal{K}^2} g^{jm} w_{kl;i} w_{rs;m} \\ &\qquad - \frac{2\psi}{\mathcal{K}^3} g^{jm} \mathcal{K}_{;i} \mathcal{K}_{;m} + \frac{1}{\mathcal{K}^2} g^{jk} \mathcal{K}_{;k} \psi_{;i} + \frac{1}{\mathcal{K}^2} g^{jk} \psi_{;k} \mathcal{K}_{;i} - \frac{1}{\mathcal{K}} g^{jk} \psi_{;ik},\end{aligned}$$

$$\partial_t h = \psi \frac{\mathcal{K}^{ij}}{\mathcal{K}^2} h_{;ij} + \psi h \frac{\mathcal{K}^{ij}}{\mathcal{K}^2} w_i^l w_{lj} - (n-1) \frac{\psi}{\mathcal{K}} - \frac{1}{\mathcal{K}} F \cdot \nabla \psi.$$

3. Long-time existence

3.1. ***Lower and upper bounds on Gauss curvature.*** The proofs of the following two lemmas are similar to the proofs of [Ivaki 2015, Lemmas 4.1, 4.2]. For completeness, we give the proofs here. In this section we use $\bar{\nabla}$ to denote covariant derivatives on the sphere with respect to the standard metric.

The matrix of the radii of the curvature of a smooth, closed, strictly convex hypersurface is denoted by $\mathfrak{r} = [\mathfrak{r}_{ij}]$ and the entries of $\mathfrak{r}$ are considered as functions on the unit sphere. They can be expressed in terms of the support function as $\mathfrak{r}_{ij} := \bar{\nabla}^2_{ij} h + \bar{g}_{ij} h$, where $[\bar{g}_{ij}]$ is the standard metric on $\mathbb{S}^n$. Additionally, we recall that $S_n = \det[\mathfrak{r}_{ij}]/\det[\bar{g}_{ij}]$.

Lemma 6. *Let $\{M_t\}$ be a solution of (1-1) on $[0,t_1]$. If $c_2 \le h_{K_t} \le c_1$ on $[0,t_1]$, then $\mathcal{K} \le c_4$ on $[0,t_1]$. Here c_4 depends on $K_0, c_1, c_2, p, \varphi$ and t_1.*

Proof. We apply the maximum principle to the following auxiliary function defined on the unit sphere:

$$\Theta = \frac{\psi S_n}{2c_1 - h} = \frac{\partial_t h}{2c_1 - h}.$$

At any minimum of Θ we have

$$0 = \bar{\nabla}_i \Theta = \bar{\nabla}_i \left(\frac{\psi S_n}{2c_1 - h} \right) \quad \text{and} \quad \bar{\nabla}^2_{ij} \Theta \ge 0.$$

Therefore, we get

$$\frac{\bar{\nabla}_i (\psi S_n)}{2c_1 - h} = -\frac{\psi S_n \bar{\nabla}_i h}{(2c_1 - h)^2}$$

and

$$\bar{\nabla}^2_{ij} (\psi S_n) + \bar{g}_{ij} \psi S_n \ge \frac{-\psi S_n \mathfrak{r}_{ij} + 2c_1 \psi S_n \bar{g}_{ij}}{2c_1 - h}. \tag{3-1}$$

Differentiating Θ with respect to time yields

$$\partial_t \Theta = \frac{\psi S_n^{ij}}{2c_1 - h} (\bar{\nabla}^2_{ij} (\psi S_n) + \bar{g}_{ij} \psi S_n) + \frac{\psi^2 S_n^2}{(2c_1 - h)^2} (1 + (2-p) h^{-1} (2c_1 - h)),$$

where S_n^{ij} is the derivative of S_n with respect to the entry $\mathfrak{r}_{ij}$. By applying inequality (3-1) to the preceding identity we deduce

$$\partial_t \Theta \ge \Theta^2 (1 - n + 2c_1 \mathcal{H}) - c\Theta^2, \tag{3-2}$$

where

$$\mathcal{H} = S_n^{-1} S_n^{ij} \bar{g}_{ij}.$$

Therefore, we arrive at

$$\frac{(h^{2-p}/\mathcal{K})\varphi}{2c_1 - h}(t,u) \ge \left(ct + \frac{1}{\min\limits_{u \in \mathbb{S}^n} \frac{(h^{2-p}/\mathcal{K})\varphi}{2c_1 - h}(0,u)} \right)^{-1} \ge \left(ct_1 + \frac{1}{\min\limits_{u \in \mathbb{S}^n} \frac{(h^{2-p}/\mathcal{K})\varphi}{2c_1 - h}(0,u)} \right)^{-1}. \qquad \square$$

Lemma 7. *Let $\{M_t\}$ be a solution of (1-1) on $[0,t_1]$. If $c_1 \le h_{K_t} \le c_2$ on $[0,t_1]$, then $\mathcal{K} \ge 1/(a + bt^{-\frac{n}{n+1}})$ on $(0,t_1]$, where a and b depend only on c_1, c_2, p, φ. In particular, $\mathcal{K} \ge c_3$ on $[0,t_1]$ for a positive number c_3 that depends on $K_0, c_1, c_2, p, \varphi$ and is independent of t_1.*

Proof. Suppose K_t^* is the polar body[1] of K_t with respect to the origin. We furnish quantities associated with polar bodies with $*$. The polar bodies evolve by

$$\partial_t h^* = -\psi^* S_n^{*-1}, \quad h^*(\cdot, t) = h_{K_t^*}(\cdot),$$

where

$$\psi^* = \frac{(h^{*2} + |\bar\nabla h^*|^2)^{\frac{n+1+p}{2}}}{h^{*n+1}} \varphi\left(\frac{h^* u + \bar\nabla h^*}{\sqrt{h^{*2} + |\bar\nabla h^*|^2}}\right);$$

see Lemma 11 for the proof. In addition, we have $c_1' = 1/c_2 \le h^* \le 1/c_1 = c_2'$. We will show that the function

$$\Theta = \frac{\psi^* S_n^{*-1}}{h^* - c_1'/2}$$

remains bounded. At any maximal point of Θ,

$$0 = \bar\nabla_i \Theta = \bar\nabla_i \left(\frac{\psi^* S_n^{*-1}}{h^* - c_1'/2}\right) \quad \text{and} \quad \bar\nabla_{ij}^2 \Theta \le 0.$$

Hence, we obtain

$$\frac{\bar\nabla_i(\psi^* S_n^{*-1})}{h^* - c_1'/2} = \frac{\psi^* S_n^{*-1} \bar\nabla_i h^*}{(h^* - c_1'/2)^2}, \tag{3-3}$$

and consequently,

$$\bar\nabla_{ij}^2(\psi^* S_n^{*-1}) + \bar g_{ij} \psi^* S_n^{*-1} \le \frac{\psi^* S_n^{*-1} \mathfrak{r}_{ij}^* - (c_1'/2) \psi^* S_n^{*-1} \bar g_{ij}}{h^* - c_1'/2}. \tag{3-4}$$

Differentiating Θ with respect to time yields

$$\partial_t \Theta = \frac{\psi^* S_n^{*-2}}{h^* - c_1'/2} S_n^{*ij} (\bar\nabla_{ij}^2(\psi^* S_n^{*-1}) + \bar g_{ij} \psi^* S_n^{*-1}) + \frac{S_n^{*-1}}{h^* - c_1'/2} \partial_t \psi^* + \Theta^2.$$

On the other hand, in view of

$$|\partial_t h^*| = \psi^* S_n^{*-1}, \quad \|\bar\nabla \partial_t h^*\| = \|\bar\nabla(\psi^* S_n^{*-1})\| = \frac{\psi^* S_n^{*-1} \|\bar\nabla h^*\|}{h^* - c_1'/2}, \quad \|\bar\nabla h^*\| \le c_2',$$

where for the second equation we used (3-3), we have

$$\frac{S_n^{*-1}}{h^* - c_1'/2} \partial_t \psi^* \le c(n, p, c_1, c_2, \varphi) \Theta^2.$$

Employing this last inequality and inequality (3-4) we infer that, at any point where the maximum of Θ is reached, we have

$$\partial_t \Theta \le \Theta^2 \left(c' - \frac{c_1'}{2} \mathcal{H}^*\right). \tag{3-5}$$

[1] The polar body of a convex body K with the origin of $\mathbb{R}^{n+1}$ in its interior is the convex body defined by $K^* = \{x \in \mathbb{R}^{n+1} : x \cdot y \le 1 \text{ for all } y \in K\}$.

Moreover, we have

$$\mathcal{H}^* \geq n\left(\frac{h^*-c_1'/2}{\psi^* S_n^{*-1}}\right)^{-\frac{1}{n}}\left(\frac{\psi^*}{h^*-c_1'/2}\right)^{-\frac{1}{n}} \geq n\Theta^{\frac{1}{n}}\left(\frac{c''}{c_1'-c_1'/2}\right)^{-\frac{1}{n}}.$$

Therefore, we can rewrite the inequality (3-5) as

$$\partial_t\Theta \leq \Theta^2(c-c'\Theta^{\frac{1}{n}})$$

for positive constants c and c' depending only on p, c_1, c_2, φ. Hence,

$$\Theta \leq c + c' t^{-\frac{n}{n+1}} \tag{3-6}$$

for some positive constants depending only on p, c_1, c_2, φ. This follows from the claim below.

Claim. *Suppose f is a positive smooth function of t on $[0, t_1]$ that satisfies*

$$\frac{d}{dt} f \leq c_0 + c_1 f + c_2 f^2 - c_3 f^{2+p}, \tag{3-7}$$

where c_3, p are positive. There exist constant $c, c' > 0$ independent of the solution and depending only on c_0, c_1, c_2, c_3, p, such that $f \leq c + c' t^{-1/(p+1)}$ on $(0, t_1]$.

Proof of claim. Note that there exists $x_0 > 0$ such that $c_0 + c_1 x + c_2 x^2 - c_3 x^{2+p} < -c_3/2 x^{2+p}$ for $x > x_0$. If $f(0) \leq x_0$, then f may increase forward in time, but when f reaches x_0, then f must start decreasing (since the right-hand side of (3-7) becomes negative). Thus we may assume, without loss of generality, that $f(0) > x_0$. Therefore, $f > x_0$ on a maximal time interval $[0, t_0)$. On $[0, t_0)$ we can solve

$$\frac{d}{dt} f \leq -\frac{c_3}{2} f^{2+p}$$

to obtain

$$f \leq \left(c_3 \frac{p+1}{2t}\right)^{-\frac{1}{p+1}}.$$

At t_0 we have $c_0 + c_1 f + c_2 f^2 - c_3 f^{1+p} = -(c_3/2) f^{2+p}$ and $f = x_0$; therefore the right-hand side of (3-7) is still negative. So $f \leq f(t_0)$ on $[t_0, t_1]$. In conclusion,

$$f \leq \max\left\{\left(c_3 \frac{p+1}{2t}\right)^{-\frac{1}{p+1}}, x_0 = f(t_0)\right\} \leq c + c' t^{-\frac{1}{1+p}},$$

where c, c' do not depend on solutions. □

The inequality (3-6) implies that

$$S_n^{*-1} \leq a' + b' t^{-\frac{n}{n+1}} \tag{3-8}$$

for some a' and b' depending only on p, c_1, c_2, φ. Now we can use the argument given in [Ivaki and Stancu 2013, Lemma 2.3] to obtain the desired lower bound: For every $u \in \mathbb{S}^n$, there exists a unique $u^* \in \mathbb{S}^n$ such that

$$(S_n h^{n+2})(u)(S_n^* h^{*n+2})(u^*) = 1;$$

see [Hug 1996]. In view of this identity and (3-8) we conclude that on $(0,t_1]$ we have

$$\mathcal{K} \geq \frac{1}{a + bt^{-\frac{n}{n+1}}}$$

for some a and b depending only on p, c_1, c_2, φ. The lower bound for $\mathcal{K}$ on $[0,\delta]$ for a small enough $\delta > 0$ follows from the short-time existence of the flow. The lower bound for $\mathcal{K}$ on $[\delta, t_1]$ follows from the inequality $\mathcal{K} \geq 1/(a + b\delta^{-\frac{n}{n+1}})$. □

3.2. *Upper and lower bounds on principal curvatures.* To obtain upper and lower bounds on the principal curvatures, denoted by $\{\kappa_i\}_{i=1}^n$, we will consider the auxiliary function used by Guan, Ren and Wang [Guan et al. 2015, (4.2)] for a prescribed curvature problem.

Lemma 8. *Let $\{M_t\}$ be a solution of (1-1) on $[0,t_1]$. If $c_1 \leq h_{K_t} \leq c_2$ on $[0,t_1]$, then $c_5 \leq \kappa_i \leq c_6$ on $[0,t_1]$, where c_5 and c_6 depend on $K_0, c_1, c_2, p, \varphi$ and t_1.*

Proof. In view of Lemmas 6 and 7, it suffices to show that $\|W\|$ remains bounded on $[0,t_1]$. Consider the auxiliary function

$$\Theta = \tfrac{1}{2}\log(\|W\|^2) - \alpha \log h.$$

Assume without loss of generality that $c_1 > 1$, for otherwise we replace h by $2h/c_1$, which does not effect the evolution equation of Θ. Using the parabolic maximum principle we show that for some α large enough $\Theta(\cdot\,,t)$ is always negative on $[0,t_1]$. If the conclusion of the theorem is false, we may choose (x_0,t_0) with $t_0 > 0$ and such that $\Theta(x_0,t_0) = 0$, $\Theta(x,t_0) \leq 0$, and $\Theta(x,t) < 0$ for $t < t_0$. Then,

$$\begin{aligned}
0 &\leq \dot{\Theta} - \psi\frac{\mathcal{K}^{kl}}{\mathcal{K}^2}\Theta_{;kl} \\
&= -\frac{\psi}{\|W\|^2}\frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w^j_{i;k} w^i_{j;l} + \frac{2\psi}{\|W\|^4}\frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w^j_i w^s_r w^i_{j;k} w^r_{s;l} + \psi\frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{kr} w^r_l - (n+1)\psi\frac{(w^2)^j_i w^i_j}{\mathcal{K}\|W\|^2} \\
&\quad + \frac{\psi w^i_j}{\|W\|^2}\left(\frac{\mathcal{K}^{kl,rs}}{\mathcal{K}^2} w_{kl;i} g^{jp} w_{rs;p} - 2\frac{g^{jp}\mathcal{K}_{;i}\mathcal{K}_{;p}}{\mathcal{K}^3}\right) + \left(\frac{2}{\mathcal{K}^2} g^{jp}\psi_{;i}\mathcal{K}_{;p} - \frac{1}{\mathcal{K}} g^{jp}\psi_{;ip}\right)\frac{w^i_j}{\|W\|^2} \\
&\quad + (n-1)\frac{\alpha\psi}{h\mathcal{K}} + \frac{\alpha}{h\mathcal{K}}(F\cdot\nabla\psi) - \frac{\alpha\psi}{h^2}\frac{\mathcal{K}^{kl}}{\mathcal{K}^2} h_{;k}h_{;l} - \alpha\psi\frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{kr} w^r_l.
\end{aligned}$$

Pick normal coordinates around x_0 such that in (x_0,t_0) it holds that

$$g_{ij} = \delta_{ij}, \quad w_{ij} = w_{ii}\delta_{ij}.$$

At (x_0,t_0) we may write

$$\mathcal{K}^{kl,rs} w_{kl;i} w_{rs;i} = \mathcal{K}^{kk,ll} w_{kk;i} w_{ll;i} - \mathcal{K}^{kk,ll} w^2_{kl;i},$$

due to the relation

$$\mathcal{K}^{kl,rs} w_{kl;i} w_{rs;j} w^{ij} = \sum_i w_{ii}\left(\sum_{p,q}\frac{\partial^2\mathcal{K}}{\partial\kappa_p\partial\kappa_q} w_{pp;i} w_{qq;i} + \sum_{p\neq q}\frac{\frac{\partial\mathcal{K}}{\partial\kappa_p} - \frac{\partial\mathcal{K}}{\partial\kappa_q}}{\kappa_p - \kappa_q} w^2_{pq;i}\right); \tag{3-9}$$

see for example [Gerhardt 2006, Lemma 2.1.14]. We obtain after multiplication by $\mathcal{K}^2$ that

$$\begin{aligned}
0 \leq &-\frac{\psi}{\|W\|^2}\mathcal{K}^{ii}\sum_l w_{ll;i}^2-\frac{\psi}{\|W\|^2}\mathcal{K}^{ii}\sum_{p\neq q} w_{pq;i}^2+\frac{2\psi}{\|W\|^4}\mathcal{K}^{ii}\Big(\sum_j w_{jj}w_{jj;i}\Big)^2\\
&+\psi\mathcal{K}^{ii}w_{ii}^2-(n+1)\psi\mathcal{K}\sum_i\frac{w_{ii}^3}{\|W\|^2}+\frac{\psi}{\|W\|^2}\sum_i w_{ii}\Big(\mathcal{K}^{pp,qq}w_{pp;i}w_{qq;i}-\mathcal{K}^{pp,qq}w_{pq;i}^2-2\frac{(\mathcal{K}_{;i})^2}{\mathcal{K}}\Big)\\
&+\sum_i(2\psi_{;i}\mathcal{K}_{;i}-\mathcal{K}\psi_{;ii})\frac{w_{ii}}{\|W\|^2}+(n-1)\frac{\alpha\psi\mathcal{K}}{h}+\frac{\alpha\mathcal{K}}{h}(F\cdot\nabla\psi)-\frac{\alpha\psi}{h^2}\mathcal{K}^{kl}h_{;k}h_{;l}-\alpha\psi\mathcal{K}^{ii}w_{ii}^2.
\end{aligned}$$

At (x_0, t_0) we have

$$0=\Theta_{;k}=\sum_i\frac{w_{ii}w_{ii;k}}{\|W\|^2}-\alpha\frac{h_{;k}}{h}. \tag{3-10}$$

We may assume at x_0 that $w_{11}=\max\{w_{ii}: 1\leq i\leq n\}$. Therefore,

$$\Theta(x_0,t_0)=0 \quad\Longrightarrow\quad \frac{c_1^\alpha}{\sqrt{n}}\leq w_{11}\leq c_2^\alpha. \tag{3-11}$$

On the other hand, since ψ is bounded above and below in view of the hypotheses of the lemma, we obtain

$$\begin{aligned}
\psi_{;i}\leq C_0 w_{ii} \quad\Longrightarrow\quad 2\psi_{;i}\mathcal{K}_{;i}&\leq\frac{\varepsilon\psi}{c_4}(\mathcal{K}_{;i})^2+\frac{c_4C_0^2}{\psi\varepsilon}w_{ii}^2\\
&\leq\varepsilon\psi\frac{(\mathcal{K}_{;i})^2}{\mathcal{K}}+C(\varepsilon,K_0,\varphi,t_1)\psi w_{ii}^2,
\end{aligned}\tag{3-12}$$

where c_4 (depending on t_1) is from Lemma 6, and

$$\psi_{;ii}\geq -C-Cw_{ii}-Cw_{ii}^2+\sum_k w_{ii;k}d_\nu\psi(\partial_k). \tag{3-13}$$

Using (3-10) in (3-13) we obtain

$$\begin{aligned}
-\frac{\mathcal{K}}{\|W\|^2}\sum_i w_{ii}\psi_{;ii}&\leq\frac{\mathcal{K}}{\|W\|^2}\sum_i w_{ii}\Big(C+Cw_{ii}+Cw_{ii}^2-\sum_k w_{ii;k}d_\nu\psi(\partial_k)\Big)\\
&\leq\frac{\mathcal{K}}{\|W\|^2}\sum_i w_{ii}(C+Cw_{ii}+Cw_{ii}^2)-\frac{\alpha\mathcal{K}}{h}\sum_k h_{;k}d_\nu\psi(\partial_k)\\
&=\frac{\mathcal{K}}{\|W\|^2}\sum_i w_{ii}(C+Cw_{ii}+Cw_{ii}^2)-\frac{\alpha\mathcal{K}}{h}\sum_i w_{ii}(\partial_i\cdot F)d_\nu\psi(\partial_i)\\
&\leq\frac{\psi}{\|W\|^2}\sum_i w_{ii}(C+Cw_{ii}^2)-\frac{\alpha\mathcal{K}}{h}\sum_i w_{ii}(\partial_i\cdot F)d_\nu\psi(\partial_i).
\end{aligned}\tag{3-14}$$

For the last inequality, we used that $\mathcal{K}$ is bounded above and ψ is bounded below (so the constant C depends on K_0, φ, t_1).

Combining (3-10), (3-12) and (3-14) implies

$$
\begin{aligned}
0 \le & -\frac{\psi}{\|W\|^2}\mathcal{K}^{ii}\sum_l w_{ll;i}^2 - \frac{\psi}{\|W\|^2}\mathcal{K}^{ii}\sum_{p\neq q} w_{pq;i}^2 + \frac{2\psi}{\|W\|^4}\mathcal{K}^{ii}\Big(\sum_j w_{jj}w_{jj;i}\Big)^2 + \psi\mathcal{K}^{ii}w_{ii}^2 \\
& -(n+1)\psi\mathcal{K}\sum_i \frac{w_{ii}^3}{\|W\|^2} + \frac{\psi}{\|W\|^2}\sum_l w_{ll}\Big(\mathcal{K}^{pp,qq}w_{pp;l}w_{qq;l} - \mathcal{K}^{pp,qq}w_{pq;l}^2 - (2-\varepsilon)\frac{(\mathcal{K}_{;l})^2}{\mathcal{K}}\Big) \\
& + \frac{\psi}{\|W\|^2}\sum_i w_{ii}(C + Cw_{ii}^2) - \frac{\alpha\mathcal{K}}{h}\sum_i w_{ii}(\partial_i\cdot F)d_\nu\psi(\partial_i) + (n-1)\frac{\alpha\psi\mathcal{K}}{h} \\
& + \frac{\alpha\mathcal{K}}{h}\sum_s(\partial_s\cdot F)d_F\psi(\partial_s) + \frac{\alpha\mathcal{K}}{h}\sum_i w_{ii}(\partial_i\cdot F)d_\nu\psi(\partial_i) - \frac{\alpha\psi}{h^2}\mathcal{K}^{ii}w_{ii}^2(\partial_i\cdot F)^2 - \alpha\psi\mathcal{K}^{ii}w_{ii}^2 \\
\le & \frac{\psi}{\|W\|^2}\Big(\sum_l w_{ll}(C + Cw_{ll}^2) - n\mathcal{K}\sum_l w_{ll}^3 + \mathcal{K}^{ii}w_{ii}^2\|W\|^2\Big) \\
& + \alpha\psi\Big(\frac{n\mathcal{K}}{h} - \mathcal{K}^{ii}w_{ii}^2 - \frac{\mathcal{K}^{ii}w_{ii}^2(\partial_i\cdot F)^2}{h^2} + \frac{\mathcal{K}}{h\psi}\sum_s(\partial_s\cdot F)d_F\psi(\partial_s)\Big) \\
& - \psi\sum_i(A_i + B_i + C_i + D_i - E_i) - \frac{\alpha\psi\mathcal{K}}{h} - \psi\mathcal{K}\sum_i\frac{w_{ii}^3}{\|W\|^2},
\end{aligned} \tag{3-15}
$$

where C depends on $\varepsilon, K_0, \varphi, t_1$, and

$$A_i = \frac{2-\varepsilon}{\|W\|^2\mathcal{K}}w_{ii}(\mathcal{K}_{;i})^2 - \frac{w_{ii}}{\|W\|^2}\sum_{p,q}\mathcal{K}^{pp,qq}w_{pp;i}w_{qq;i},$$

$$B_i = \frac{2}{\|W\|^2}\sum_j w_{jj}\mathcal{K}^{jj,ii}w_{jj;i}^2, \qquad C_i = \frac{2}{\|W\|^2}\sum_{j\neq i}\mathcal{K}^{jj}w_{jj;i}^2,$$

$$D_i = \frac{1}{\|W\|^2}\mathcal{K}^{ii}\sum_j w_{jj;i}^2, \qquad E_i = \frac{2}{\|W\|^4}\mathcal{K}^{ii}\Big(\sum_j w_{jj}w_{jj;i}\Big)^2.$$

The terms B_i and C_i deserve some explanation. C_i comes from the second term in (3-15), which is given by

$$-\frac{\psi}{\|W\|^2}\sum_i\mathcal{K}^{ii}\sum_{p\neq q}w_{pq;i}^2 \le -\frac{\psi}{\|W\|^2}\sum_{p\neq q}\mathcal{K}^{pp}w_{pq;p}^2 - \frac{\psi}{\|W\|^2}\sum_{p\neq q}\mathcal{K}^{qq}w_{pq;q}^2,$$

which is exactly C_i due to the Codazzi equation.

The third line of (3-15) arises from (3-9). Since the second term in the bracket of (3-9) is negative and the hypersurface is convex, we can proceed in the same way as we derived C_i and just throw away all indices i which are neither p nor q. This gives term B_i. The first term in the big bracket goes into A_i.

In Corollary 14 of the Appendix we will present an adaption of the method developed in [Guan et al. 2015] to deal with the curvature derivative terms A_i, B_i, C_i, D_i, E_i. There we prove that we obtain the following alternative: there exist positive numbers $\delta_2, \dots, \delta_n$, which only depend on the dimension and bounds on the Gauss curvature, such that either

$$w_{ii} > \delta_i w_{11} \quad \text{for all } 2 \le i \le n$$

or

$$A_i + B_i + C_i + D_i - E_i \ge 0 \quad \text{for all } 1 \le i \le n.$$

By taking α large in (3-11), in the first case we get a contradiction to the bound on the Gauss curvature. In the second case, using also $\mathcal{K}^{ii} w_{ii}^2 = \mathcal{K} \sum_i w_{ii}$, (3-15) yields

$$0 \le \frac{\psi}{\|W\|^2}\biggl(\sum_l w_{ll}(C + C w_{ll}^2) - n\mathcal{K}\sum_l w_{ll}^3\biggr) - (\alpha-1)\mathcal{K}\psi\sum_i w_{ii} \\ + \alpha\psi\biggl((n-1)\frac{\mathcal{K}}{h} - \frac{\mathcal{K}}{h^2}\sum_i w_{ii}(\partial_i \cdot F)^2 + \frac{\mathcal{K}}{h\psi}\sum_l (\partial_l \cdot F) d_F\psi(\partial_l)\biggr).$$

Consequently we obtain

$$0 \le \frac{C(\varepsilon, K_0, \varphi, t_1) w_{11}^3}{\|W\|^2} - (\alpha-1)\mathcal{K}\psi w_{11} + C(K_0, \varphi, t_1)\alpha,$$

where we discarded $-(\alpha-1)\mathcal{K}\psi\sum_{i\ne 1} w_{ii} \le 0$ and used the bounds on h, ψ and $\mathcal{K}$ to bound w_{11} in terms of w_{11}^3.

Now take α such that $(\alpha-1)\mathcal{K}\psi \ge C(\varepsilon, K_0, \varphi, t_1) + 1$. Therefore, in view of (3-11)

$$\begin{aligned} 0 &\le \frac{C(\varepsilon, K_0, \varphi, t_1) w_{11}^3}{\|W\|^2} - (\alpha-1)\mathcal{K}\psi w_{11} + C(K_0, \varphi, t_1)\alpha \\ &\le C(\varepsilon, K_0, \varphi, t_1)\biggl(\frac{w_{11}^2}{\|W\|^2} - 1\biggr) w_{11} - w_{11} + C(K_0, \varphi, t_1)\alpha \\ &\le -\frac{c_1^\alpha}{\sqrt{n}} + C(K_0, \varphi, t_1)\alpha. \end{aligned} \tag{3-16}$$

Taking α large enough yields a contradiction. □

Proposition 9. *The solution to (1-1) satisfies* $\lim_{t\to T} \max h_{K_t} = \infty$.

Proof. First, let $p \ge n+1$. In this case, by comparing with suitable outer balls, the flow exists on $[0, \infty)$. For $p > n+1$, consider an origin-centered ball B_r such that $K_0 \supseteq B_r$. Then $K_t \supseteq B_{r(t)}$, where

$$r(t) = ((\min h_{K_0})^{p-n-1} + t(p-n-1)\min\varphi)^{\frac{1}{p-n-1}}$$

and $B_{r(t)}$ expands to infinity as t approaches ∞. For $p = n+1$, $K_t \supseteq B_{r(t)}$ with $r(t) = e^{t\min\varphi}\min h_{K_0}$ and $B_{r(t)}$ expands to infinity as t approaches ∞.

Second, if $p < n+1$, then the flow exists only on a finite-time interval. If $\max h_{K_t} < \infty$, then by Lemmas 6, 7 and 8, the evolution equation (1-1) is uniformly parabolic on $[0, T)$. Thus, the result of [Krylov and Safonov 1980] and standard parabolic theory allow us to extend the solution smoothly past time T, contradicting its maximality. □

4. Convergence of normalized solutions

4.1. ***Convergence in C^1, $1 \neq p > -n-1$.*** By the proof of [Ivaki 2016, Corollary 7.5], there exist r, R such that

$$0 < r \leq h_{\widetilde{K}_t} \leq R < \infty. \tag{4-1}$$

Therefore, a subsequence of $\{\widetilde{K}_{t_k}\}$ converges in the Hausdorff distance to a limiting shape $\widetilde{K}_\infty$ with the origin in its interior. The argument of [Ivaki 2016, Section 8.1] implies

$$\varphi h^{1-p}_{\widetilde{K}_\infty} f_{\widetilde{K}_\infty} = c,$$

where $f_{\widetilde{K}_\infty}$ is the positive continuous curvature function of $\widetilde{K}_\infty$ and c is some positive constant. By [Chou and Wang 2006, Proposition 1.2], $\widetilde{K}_\infty$ is smooth and strictly convex. The C^1 convergence follows, which is purely geometric and does not depend on the evolution equation, from [Andrews 1997, Lemma 13].

Remark 10. Section 4.1 completes the discussion on the existence of solutions to the smooth, even L_p-Minkowski problems in $\mathbb{R}^{n+1}$ for $1 \neq p > -n-1$. The next section discusses the C^∞ convergence when $1 \neq p \leq n+1$, and also when $p = 1$ and solutions are origin-symmetric. We mention that in the latter case, by the proof of [Ivaki 2016, Corollary 7.5], the estimate (4-1) still holds.

4.2. ***Convergence in C^∞.*** By [Ivaki 2016, Lemma 9.2], there is a uniform upper bound on the Gauss curvature of the normalized solution when $p \leq n+1$. In the following, we first obtain a uniform lower bound on the Gauss curvature of the normalized solution $\widetilde{K}_t$.

Let $h : \mathbb{S}^n \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (1-2). Then for each $\lambda > 0$, $\bar{h}$ defined by

$$\bar{h} : \mathbb{S}^n \times [0, T/\lambda^{\frac{1+n-p}{n+1}}) \to \mathbb{R}^{n+1},$$
$$\bar{h}(u,t) = \lambda^{\frac{1}{n+1}} h(u, \lambda^{\frac{1+n-p}{n+1}} t),$$

is also a solution of evolution equation (1-2) but with the initial data $\lambda^{\frac{1}{n+1}} h(\cdot, 0)$.

For each *fixed* time $t \in [0, T)$, define $\bar{h}$ a solution of (1-2) as follows:

$$\bar{h}(u,\tau) = \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1}{n+1}} h\left(u, t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}} \tau\right).$$

Note that $\bar{h}(\cdot, 0)$ is the support function of $(V(B)/V(K_t))^{\frac{1}{n+1}} K_t$; therefore,

$$r \leq \bar{h}(u, 0) \leq R.$$

Write $\overline{K}_\tau$ for the convex body associated with $\bar{h}(\cdot, \tau)$ and let B_c denote the ball of radius c centered at the origin. Since B_R encloses $\overline{K}_0$, the comparison principle implies that B_{2R} will enclose $\overline{K}_\tau$ for $\tau \in [0, \delta]$, where δ depends only on p, R, ψ. By the first statement of Lemma 7 applied to $\bar{h}$, there is a uniform lower bound (depending only on r, R, p, φ) on the Gauss curvature of $\overline{K}_{\frac{\delta}{2}}$.

On the other hand, the volume of $\bar{K}_{\frac{\delta}{2}}$ is bounded above by $V(B_{2R})$; therefore,

$$\frac{V(B)}{V(B_{2R})} \le c_t := \frac{V(K_t)}{V(K_{t+(\frac{V(B)}{V(K_t)})^{(1+n-p)/(n+1)}\frac{\delta}{2}})} \le 1$$

for all $t \in [0, T)$. Consequently,

$$\left(\frac{V(B)}{V(K_{t+(\frac{V(B)}{V(K_t)})^{(1+n-p)/(n+1)}\frac{\delta}{2}})}\right)^{\frac{1}{n+1}} h\left(u, t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}} \frac{\delta}{2}\right) = c_t^{\frac{1}{n+1}} \bar{h}\left(\cdot, \frac{\delta}{2}\right)$$

has Gauss curvature bounded below for all $t \in [0, T)$.

Now we show that for every $\tilde{t} \in \left[(V(B)/V(K_0))^{\frac{1+n-p}{n+1}} \frac{\delta}{2}, T\right)$, we can find $t \in [0, T)$ such that

$$\tilde{t} = t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}} \frac{\delta}{2}.$$

Define $f(t) = t + (V(B)/V(K_t))^{\frac{1+n-p}{n+1}} \frac{\delta}{2} - \tilde{t}$ on $[0, T)$. Then f is continuous, and

$$\begin{cases} f(T) = T - \tilde{t} > 0, & p < n+1, \\ f(\infty) = \infty, & p = n+1, \\ f(0) \le 0, & p \le n+1. \end{cases}$$

The claim follows.

Next we obtain uniform lower and upper bounds on the principal curvatures of the normalized solution. Consider the convex bodies $\widetilde{K}_\tau := (V(B)/V(K_t))^{\frac{1}{n+1}} K_t$, where

$$\tau(t) := \int_0^t \left(\frac{V(K_s)}{V(B)}\right)^{\frac{1+n-p}{n+1}} ds.$$

Let us furnish all geometric quantities associated with $\widetilde{K}_\tau$ by a tilde. The evolution equation of $\tilde{h}_\tau$ is given by

$$\partial_\tau \tilde{h}_\tau = \varphi \tilde{h}^{2-p} \widetilde{S}_n - \frac{\int_{\mathbb{S}^n} \varphi \tilde{h}^{2-p} \widetilde{S}_n^2 \, d\sigma}{(n+1)V(B)} \tilde{h}.$$

Since $(\int_{\mathbb{S}^n} \varphi \tilde{h}^{2-p} \widetilde{S}_n^2 \, d\sigma)/((n+1)V(B))$ is uniformly bounded above, applying the maximum principle to $\Theta = \frac{1}{2}\log(\|\widetilde{W}\|^2) - \alpha \log \tilde{h}$, and arguing as in the proof of Lemma 8, we see that $\|\widetilde{W}\|$ has a uniform upper bound. This in turn, in view of our lower and upper bounds on the Gauss curvature of $\widetilde{K}_\tau$, implies that we have uniform lower and upper bounds on the principal curvatures of $\widetilde{K}_\tau$. Higher-order regularity estimates and convergence in C^∞ for a subsequence of $\{\widetilde{K}_\tau\}$ follow from [Krylov and Safonov 1980], standard parabolic theory and the Arzelà-Ascoli theorem. The convergence for the full sequence when $p \ge 1$ follows from the uniqueness of the self-similar solutions to (1-3); see [Lutwak 1993; Chou and Wang 2006]. Moreover, note that when $\varphi \equiv 1$ and $-n-1 < p < 1$, by the result of [Brendle et al. 2017], the limit is the unit sphere.

Appendix

Evolution of polar bodies. Let K be a smooth, strictly convex body with the origin in its interior. Suppose ∂K, the boundary of K, is parametrized by the radial function $r = r(u) : \mathbb{S}^n \to \mathbb{R}$. The metric $[g_{ij}]$, unit normal ν, support function h, and the second fundamental form $[w_{ij}]$ of ∂K can be written in terms of r and its partial derivatives as follows:

(a) $g_{ij} = r^2 \bar{g}_{ij} + \bar{\nabla}_i r \bar{\nabla}_j r$.

(b) $\nu = (ru - \bar{\nabla} r)/\sqrt{r^2 + \|\bar{\nabla} r\|^2}$.

(c) $h = r^2/\sqrt{r^2 + \|\bar{\nabla} r\|^2}$.

(d) $w_{ij} = (-r\bar{\nabla}^2_{ij} r + 2\bar{\nabla}_i r \bar{\nabla}_j r + r^2 \bar{g}_{ij})/\sqrt{r^2 + \|\bar{\nabla} r\|^2}$.

Since $1/r$ is the support function of K^*, see, e.g., [Schneider 2014, p. 57], we can calculate the entries of $[\mathfrak{r}^*_{ij}]$:

$$\mathfrak{r}^*_{ij} = \bar{\nabla}^2_{ij} \frac{1}{r} + \frac{1}{r} \bar{g}_{ij} = \frac{-r\bar{\nabla}^2_{ij} r + 2\bar{\nabla}_i r \bar{\nabla}_j r + r^2 \bar{g}_{ij}}{r^3}.$$

Thus, using (d) we get

$$\mathfrak{r}^*_{ij} = \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} w_{ij}.$$

Lemma 11. *As K_t evolve by (1-2), their polars K_t^* evolve as follows*:

$$\partial_t h^* = -\varphi\left(\frac{h^* u + \bar{\nabla} h^*}{\sqrt{h^{*2} + |\bar{\nabla} h^*|^2}}\right) \frac{(h^{*2} + |\bar{\nabla} h^*|^2)^{\frac{n+1+p}{2}}}{h^{*n+1} S_n^*}, \quad h^*(\cdot, t) := h_{K_t^*}(\cdot).$$

Proof. To obtain the evolution equation of $h_{K_t^*}$, we first need to parametrize M_t over the unit sphere

$$F = r(u(\cdot, t), t) u(\cdot, t) : \mathbb{S}^n \to \mathbb{R}^{n+1},$$

where $r(u(\cdot, t), t)$ is the radial function of M_t in the direction $u(\cdot, t)$. Note that

$$\partial_t r = \varphi \frac{h^{2-p}}{\mathcal{K}} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r},$$

and

$$\mathcal{K} = \frac{\det w_{ij}}{\det g_{ij}}, \quad \frac{1}{S_n^*} = \frac{\det \bar{g}_{ij}}{\det \mathfrak{r}^*_{ij}}, \quad \frac{\det \bar{g}_{ij}}{\det g_{ij}} = \frac{1}{r^{2n-2}(r^2 + \|\bar{\nabla} r\|^2)}, \quad h = \frac{1}{\sqrt{h^{*2} + \|\bar{\nabla} h^*\|^2}}.$$

Now we calculate

$$\begin{aligned}\partial_t h^* = \partial_t \frac{1}{r} &= -\frac{h^{2-p}}{\mathcal{K}} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} \varphi(\nu) \\ &= -h^{2-p} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} \frac{\det g_{ij}}{\det w_{ij}} \varphi(\nu)\end{aligned}$$

$$= -h^{2-p}\frac{\sqrt{r^2+\|\bar{\nabla} r\|^2}}{r^3}\frac{\det \bar{g}_{ij}}{\det \mathfrak{r}^*_{ij}}\frac{\det g_{ij}}{\det \bar{g}_{ij}}\frac{\det \mathfrak{r}^*_{ij}}{\det w_{ij}}\varphi(\nu)$$
$$= -\left(\frac{\sqrt{r^2+\|\bar{\nabla} r\|^2}}{r^3}\right)^{n+1}\frac{r^{2n-2}(r^2+\|\bar{\nabla} r\|^2)}{(h^{*2}+\|\bar{\nabla} h^*\|^2)^{\frac{2-p}{2}}}\frac{\varphi(\nu)}{S^*_n}.$$

Replacing r by $1/h^*$ and taking into account (b) finishes the proof. ☐

Estimates for curvature derivatives. For convenience we present some of the main ideas regarding how one can prove the alternative in Lemma 8 about balancing the curvature derivatives. This method was used in [Guan et al. 2015] for a similar stationary prescribed curvature equation. Recall that

$$A_i = \frac{2-\varepsilon}{\|W\|^2\mathcal{K}}w_{ii}(\mathcal{K}_{;i})^2 - \frac{w_{ii}}{\|W\|^2}\sum_{p,q}\mathcal{K}^{pp,qq}w_{pp;i}w_{qq;i},$$

$$B_i = \frac{2}{\|W\|^2}\sum_j w_{jj}\mathcal{K}^{jj,ii}w^2_{jj;i}, \quad C_i = \frac{2}{\|W\|^2}\sum_{j\neq i}\mathcal{K}^{jj}w^2_{jj;i},$$

$$D_i = \frac{1}{\|W\|^2}\mathcal{K}^{ii}\sum_j w^2_{jj;i}, \qquad E_i = \frac{2}{\|W\|^4}\mathcal{K}^{ii}\Bigl(\sum_j w_{jj}w_{jj;i}\Bigr)^2.$$

Note that the term A_i looks slightly different from the term A_i in [Guan et al. 2015, p. 1309], where the $\mathcal{K}$ is not present in the denominator. We have to define A_i in the way we did, because due to the inverse nature of the curvature flow equation we obtain an extra good derivative term. This allows us to choose the constant in A_i as $2-\varepsilon$, whereas a large constant was required in [Guan et al. 2015] (denoted by K there). Fortunately the proofs of Lemmas 4.2 and 4.3 in that paper also work for sufficiently small ε. The remaining terms B_i, C_i, D_i, E_i are all identical to those in [Guan et al. 2015].

In the following σ_k denotes the k-th elementary symmetric function of principal curvatures. We begin by recalling the following special case ($k=n$) of inequality (2.4) from [Guan et al. 2015, Lemma 2.2], which can be deduced easily by differentiating

$$G = \left(\frac{\sigma_n}{\sigma_l}\right)^{\frac{1}{n-l}}$$

twice, using the concavity of G and applying the Schwarz inequality. For any $\delta > 0$, $1 \le i \le n$ and $1 \le l < n$ we have

$$-\mathcal{K}^{pp,qq}w_{pp;i}w_{qq;i} + \left(1-\frac{1}{n-l}+\frac{1}{(n-l)\delta}\right)\frac{(\mathcal{K}_{;i})^2}{\mathcal{K}} \ge \left(1+\frac{1-\delta}{n-l}\right)\frac{\mathcal{K}((\sigma_l)_{;i})^2}{\sigma_l^2} - \frac{\mathcal{K}}{\sigma_l}\sigma_l^{pp,qq}w_{pp;i}w_{qq;i}.$$

In particular, by taking $\delta = 1/(2-\varepsilon)$, we have

$$(2-\varepsilon)\frac{(\mathcal{K}_{;i})^2}{\mathcal{K}} - \mathcal{K}^{pp,qq}w_{pp;i}w_{qq;i} \ge \left[1+\frac{1-\varepsilon}{(n-1)(2-\varepsilon)}\right]\frac{\mathcal{K}((\sigma_l)_{;i})^2}{\sigma_l^2} - \frac{\mathcal{K}\sigma_l^{pp,qq}w_{pp;i}w_{qq;i}}{\sigma_l}, \quad \text{(A-1)}$$

provided $(2-\varepsilon) > 1$, i.e., $0 < \varepsilon < 1$.

Lemma 12. *For each $i \neq 1$, if $\sqrt{3}\kappa_i \leq \kappa_1$, we have*

$$A_i + B_i + C_i + D_i - E_i \geq 0.$$

Proof. Note that from (A-1) with $l = 1$, it follows that $A_i \geq 0$ since $\sigma_1^{pp,qq} = 0$. The proof that $B_i + C_i + D_i - E_i \geq 0$ can literally be taken from [Guan et al. 2015, Lemma 4.2], starting with (4.10) of that paper. □

In the following proof we will write $\sigma_n = \mathcal{K}$ for better comparability with [Guan et al. 2015, Lemma 4.3]. Also denote by $\sigma_k(\kappa \mid i)$ the k-th elementary symmetric polynomial in the variables $\kappa_1, \dots, \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_n$ and $\sigma_k(\kappa \mid ij)$ accordingly.

Lemma 13. *For $\lambda = 1, \dots, n-1$ suppose there exists some $\delta \leq 1$ such that $\kappa_\lambda/\kappa_1 \geq \delta$. There exists a sufficiently small positive constant δ' depending on δ, ε and the bounds for $\mathcal{K}$ such that if $\kappa_{\lambda+1}/\kappa_1 \leq \delta'$, we have*

$$A_i + B_i + C_i + D_i - E_i \geq 0 \quad \textit{for } i = 1, \dots, \lambda.$$

Proof. This corresponds to [Guan et al. 2015, Lemma 4.3]. We highlight the main estimates in this proof. First of all, from [Guan et al. 2015, (4.16), (4.17)] one can extract the following estimate:

$$\begin{aligned}\|W\|^4(B_i + C_i + D_i - E_i) &\geq \|W\|^2 \sum_{j \neq i} (\sigma_{n-1}(\kappa \mid j) - 2\sigma_{n-1}(\kappa \mid ij)) w_{jj;i}^2 - w_{ii}^2 \sigma_n^{ii} w_{ii;i}^2 \\ &= \|W\|^2 \sum_{j \neq i} \sigma_{n-1}(\kappa \mid j) w_{jj;i}^2 - w_{ii}^2 \sigma_n^{ii} w_{ii;i}^2, \end{aligned} \tag{A-2}$$

since $\sigma_{n-1}(\kappa \mid ij) = 0$.

Now we show the right-hand side of (A-2) is dominated by $\|W\|^4 A_i$. From (A-1) we get for all $1 \leq \lambda < n$ and for all $1 \leq i \leq n$:

$$\begin{aligned} A_i &= \frac{(2-\varepsilon) w_{ii}}{\|W\|^2 \sigma_n} ((\sigma_n)_{;i})^2 - \frac{w_{ii}}{\|W\|^2} \sum_{p,q} \sigma_n^{pp,qq} w_{pp;i} w_{qq;i} \\ &\geq \frac{w_{ii}}{\|W\|^2} \left(1 + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)}\right) \frac{\sigma_n ((\sigma_\lambda)_{;i})^2}{\sigma_\lambda^2} - \frac{w_{ii}}{\|W\|^2} \frac{\sigma_n \sum_{p,q} \sigma_\lambda^{pp,qq} w_{pp;i} w_{qq;i}}{\sigma_\lambda} \\ &= \frac{w_{ii}\sigma_n}{\|W\|^2 \sigma_\lambda^2} \Bigg[\left(1 + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)}\right) \sum_a (\sigma_\lambda^{aa} w_{aa;i})^2 + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)} \sum_{a \neq b} \sigma_\lambda^{aa} \sigma_\lambda^{bb} w_{aa;i} w_{bb;i} \\ &\qquad + \sum_{a \neq b} (\sigma_\lambda^{aa} \sigma_\lambda^{bb} - \sigma_\lambda \sigma_\lambda^{aa,bb}) w_{aa;i} w_{bb;i} \Bigg]. \end{aligned} \tag{A-3}$$

For sufficiently small δ' and $\lambda = 1$ the simple estimates [Guan et al. 2015, (4.19), (4.20)] give

$$\|W\|^4 A_i \geq w_{ii}^2 \sigma_n^{ii} w_{11;i}^2 - C_\epsilon w_{ii} \sum_{a \neq 1} w_{aa;i}^2. \tag{A-4}$$

Combining this with (A-2) for $i = 1$ yields,

$$\begin{aligned}\|W\|^2(A_1+B_1+C_1+D_1-E_1) &\geq \sum_{j\neq 1}\sigma_{n-1}(\kappa\mid j)w_{jj;1}^2-\frac{C_\epsilon}{w_{11}}\sum_{j\neq 1}w_{jj;1}^2\\ &= \sum_{j\neq 1}\left(\frac{\sigma_n}{w_{jj}}-\frac{C_\epsilon}{w_{11}}\right)w_{jj;1}^2\\ &\geq \sum_{j\neq 1}\left(\frac{\sigma_n}{\delta' w_{11}}-\frac{C_\epsilon}{w_{11}}\right)w_{jj;1}^2, \end{aligned}\tag{A-5}$$

which is nonnegative for δ' sufficiently small. Hence the lemma is true in the case $\lambda = 1$.

For $\lambda > 1$ the series of elementary estimates [Guan et al. 2015, (4.22)–(4.27)] gives

$$\|W\|^4 A_i \geq w_{ii}^2\sigma_n^{ii}\sum_{a\leq\lambda}w_{aa;i}^2-\frac{w_{ii}C_\epsilon}{\delta^2}\sum_{a>\lambda}w_{aa;i}^2,$$

after adapting ϵ if necessary and choosing δ' sufficiently small again. Combining this last inequality with (A-2) for $1 \leq i \leq \lambda$ yields

$$\begin{aligned}\|W\|^2(A_i+B_i+C_i+D_i-E_i) &\geq \sum_{j\neq i}\sigma_{n-1}(\kappa\mid j)w_{jj;i}^2-\frac{C_\epsilon}{w_{ii}\delta^2}\sum_{j>\lambda}w_{jj;i}^2\\ &\geq \sum_{j>\lambda}\left(\sigma_{n-1}(\kappa\mid j)-\frac{C_\epsilon}{w_{ii}\delta^2}\right)w_{jj;i}^2\\ &\geq \sum_{j>\lambda}\left(\frac{\sigma_n}{w_{11}\delta'}-\frac{C_\epsilon}{w_{ii}\delta^2}\right)w_{jj;i}^2, \end{aligned}\tag{A-6}$$

which is nonnegative for small δ' for the same reason as in (A-5). □

Corollary 14. *There exist positive numbers $\delta_2, \dots, \delta_n$, depending only on the dimension, on ϵ and on the bounds for the Gauss curvature, such that either*

$$\kappa_i > \delta_i\kappa_1 \quad \textit{for all } 2 \leq i \leq n \tag{A-7}$$

or

$$A_i + B_i + C_i + D_i - E_i \geq 0 \quad \textit{for all } 1 \leq i \leq n. \tag{A-8}$$

Proof. Choosing $\lambda = 1$ and $\delta = 1$ in Lemma 13 yields the existence of δ' with the following property: if $\kappa_2/\kappa_1 \leq \delta'$, then

$$A_1 + B_1 + C_1 + D_1 - E_1 \geq 0.$$

Note that $\kappa_i \leq \kappa_2$ for $i \geq 2$. Choose $\delta_2 = \min\{\delta', 1/\sqrt{3}\}$. Therefore, in view of Lemma 12, $\kappa_2/\kappa_1 \leq \delta_2$ implies

$$A_i + B_i + C_i + D_i - E_i \geq 0 \quad \text{for all } i \geq 2.$$

We now apply induction, assuming we have constructed $\delta_2, \dots, \delta_j$. We may assume $\kappa_i > \delta_i \kappa_1$ for $2 \le i \le j$; otherwise $A_i + B_i + C_i + D_i - E_i \ge 0$ is already true for $2 \le i \le n$. Choose $\delta = \delta_j$ and $\lambda = j$ in Lemma 13 to get a δ' so that if $\kappa_{j+1} \le \delta' \kappa_1$, then $A_i + B_i + C_i + D_i - E_i \ge 0$ holds for $1 \le i \le j$. Now in view of Lemma 12, taking $\delta_{j+1} = \min\{\delta', 1/\sqrt{3}\}$ gives $A_i + B_i + C_i + D_i - E_i \ge 0$ for $j \le i \le n$. $\square$

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