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MAXIMAL GAIN OF REGULARITY IN VELOCITY AVERAGING LEMMAS

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We investigate new settings of velocity averaging lemmas in kinetic theory where a maximal gain of half a derivative is obtained. Specifically, we show that if the densities f and g in the transport equation $v\cdot\nabla_x f = g$ belong to $L^r_x L^{r'}_v$, where $2n/(n+1) < r \le 2$ and $n \ge 1$ is the dimension, then the velocity averages belong to $H^{1/2}_x$.

We further explore the setting where the densities belong to $L^{4/3}_x L^2_v$ and show, by completing the work initiated by Pierre-Emmanuel Jabin and Luis Vega on the subject, that velocity averages almost belong to $W_x^{n/(4(n-1)),4/3}$ in this case, in any dimension $n \ge 2$, which strongly indicates that velocity averages should almost belong to $W_x^{1/2,2n/(n+1)}$ whenever the densities belong to $L_x^{2n/(n+1)} L^2_v$.

These results and their proofs bear a strong resemblance to the famous and notoriously difficult problems of boundedness of Bochner–Riesz multipliers and Fourier restriction operators, and to smoothing conjectures for Schrödinger and wave equations, which suggests interesting links between kinetic theory, dispersive equations and harmonic analysis.

1. Introduction and main results

Velocity averaging lemmas are a category of regularity results concerning the kinetic transport equation

$$(\partial_t + v\cdot\nabla_x) f(t,x,v) = g(t,x,v), \tag{1-1}$$

where $(t,x,v) \in [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, or its stationary counterpart

$$v\cdot\nabla_x f(x,v) = g(x,v), \tag{1-2}$$

where $(x,v)\in\mathbb{R}^n\times\mathbb{R}^n$, with $n\ge 1$.

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Variants of the above equations are also relevant. Indeed, different spatial and velocity domains, as well as nonlinear velocity fields (consider the relativistic case), are sometimes studied. Nevertheless, for the sake of simplicity, we will focus exclusively on the Euclidean stationary setting (1-2), which, we believe, captures the essential features of kinetic transport (at least as far as velocity averaging is concerned). We refer the interested reader to Appendix C, where we establish an equivalence of velocity averaging lemmas for velocities in $\mathbb{R}^n$ and in $\mathbb{S}^{n-1}$. In particular, this provides a rather general method to adapt the results contained in the present work to settings where velocities belong to a manifold of codimension 1, which includes the nonstationary transport equation (1-1).

The classical velocity averaging lemma was established first in [Golse et al. 1988]. It asserts that if $f, g \in L^2_{x,v}$ satisfy the transport relation (1-2), then the velocity averages of f enjoy the regularization

$$\int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \in H_x^{\frac{1}{2}}$$

for any given $\varphi \in L^\infty_c(\mathbb{R}^n)$ (that is, any measurable function bounded almost everywhere with compact support). Note that such regularity results had already been suggested in weaker forms in [Agoshkov 1984; Golse et al. 1985].

An extension of this fundamental result to the $L^p_{x,v}$ setting, with $1 < p < \infty$, was also obtained in [Golse et al. 1988] and later substantially improved in [Bézard 1994; DeVore and Petrova 2001; DiPerna et al. 1991]. Generally speaking, such generalizations are deduced by interpolating the preceding $L^2_{x,v}$ case with the degenerate $L^1_{x,v}$ and $L^\infty_{x,v}$ cases. In this setting, it is established that, for any $\varphi \in L^\infty_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \in W_x^{s,p}$$

whenever $f, g \in L^p_{x,v}$, with $s = 1 - \frac{1}{p}$ if $p \leq 2$ and for any $0 \leq s < \frac{1}{p}$ if $p > 2$.

When $p \leq 2$, the optimality of the regularity index $s = 1 - \frac{1}{p}$ in the preceding result was shown in [Lions 1995] through a straightforward dimensional analysis. As for the case $p > 2$, it was also argued in that paper that the regularity of velocity averages cannot be improved beyond the value $s = \frac{1}{p}$, but this optimality argument remains incomplete, for it requires the use of a larger class of velocity weights $\varphi(v)$ with unbounded support. In fact, it turns out that, in general, the value $s = \frac{1}{p}$ is not optimal in the range $2 < p < \infty$, for it is possible to largely improve this regularity index beyond the value $s = \frac{1}{p}$ in dimension $n = 1$, as stated in the following one-dimensional theorem.

Theorem 1.1. *In dimension $n = 1$, let $f, g \in L^p_{x,v}$, with $2 < p < \infty$, be such that* (1-2) *holds true. Then,*

$$\int_{\mathbb{R}} f(x,v)\,\varphi(v)\,dv \in W_x^{s,p}$$

for all $0 \leq s < 1 - \frac{1}{p}$ and any $\varphi \in L^\infty_c(\mathbb{R})$.

This result clearly follows from the more general Theorem 4.3, by setting $p = r$ therein, which is established later on in Section 4.

The question of the optimality of the value $s = \frac{1}{p}$, when $p > 2$, in higher dimensions $n \geq 2$ was finally definitely settled in [DeVore and Petrova 2001, Theorem 1.3], where a remarkable construction of a convoluted counterexample shows the necessity of the constraint $s \leq \frac{1}{p}$ whenever $p > 2$ and $n \geq 2$. Note however that it remained so far unknown whether the endpoint value $s = \frac{1}{p}$ is admissible or not. It turns out that, as a byproduct of our methods, we are able to settle this question here by showing that the endpoint value $s = \frac{1}{p}$ is indeed admissible when $p > 2$ (see Theorem 3.6).

On the whole, the maximal gain of regularity, when $n \geq 2$, clearly happens for the value $p = 2$, where half a derivative is gained by averaging in velocity.

It is to be emphasized that the refined interpolation methods used in [DeVore and Petrova 2001] yield more precise results. Indeed, it is established therein that, for each $1 < p < \infty$, the velocity averages actually belong to the Besov space $B^s_{p,p}(\mathbb{R}^n)$ with $s = \min\{\frac{1}{p}, 1 - \frac{1}{p}\}$, which is smaller than $W^{s,p}(\mathbb{R}^n)$ for values $1 < p \leq 2$, and that this is optimal. Nevertheless, for the sake of simplicity, we will only focus here on standard Sobolev spaces and we will omit the more precise formulations of velocity averaging lemmas in Besov spaces, which can be easily deduced from our proofs if needed (we refer to the proof of Proposition 3.2 in Section 3 for some more details on this matter).

Numerous generalizations of velocity averaging lemmas are available. For instance, several settings where f and g belong to distinct spaces (possibly with different homogeneity) of different kinds (Besov, Sobolev, etc.) with mixed integrability and regularity in space and velocity have been considered in [Arsénio and Masmoudi 2014; Bézard 1994; DiPerna et al. 1991; Jabin and Vega 2004; Westdickenberg 2002]. Naturally, the ensuing gain of regularity on the velocity averages depends then on the different parameters used to characterize these function spaces. In these more general settings, the phenomena of dispersion (as discovered in [Castella and Perthame 1996]) and hypoellipticity (as discovered in [Bouchut 2002]; see also [Arsénio and Saint-Raymond 2011]) in kinetic transport equations come into play and, loosely speaking, interact with the regularization due to velocity averaging to produce new interesting results. We refer to [Arsénio and Masmoudi 2014; Westdickenberg 2002] and [Arsénio and Masmoudi 2014; Arsénio and Saint-Raymond 2011; Jabin and Vega 2004] for such results combining velocity averaging with the dispersive and hypoelliptic effects, respectively. Note that none of these phenomena is fully distinct from the others.

It was argued in [Arsénio and Masmoudi 2014; Westdickenberg 2002] that the influence of dispersion on velocity averaging produces a gain of integrability which can be interpreted, through Sobolev embeddings, as a regularity gain which is sometimes larger than half a derivative and even possibly close or equal to a whole derivative (note that the gain of regularity can never be larger than a whole derivative, for the transport operator is a differential operator of order 1). Furthermore, the hypoelliptic phenomenon may also produce a regularity gain close or equal to a whole derivative on the velocity average, see [Arsénio and Masmoudi 2014; Jabin and Vega 2004], but this requires assuming some regularity on f and g a priori.

In this article, we will exclusively focus on the gain of regularity due to velocity averaging, possibly combined with dispersion (without interpreting the gain of regularity through Sobolev embeddings as was done in [Arsénio and Masmoudi 2014; Westdickenberg 2002], though), and will mostly ignore the aforementioned effects produced by hypoellipticity that were analyzed in [Arsénio and Masmoudi

2014; Jabin and Vega 2004]. To this end, we will only consider settings where f and g belong to mixed Lebesgue spaces and no a priori regularity is assumed. In this case, it is largely agreed that the gain of regularity cannot exceed half a derivative in dimension $n \geq 2$ (but there is no proof of this general assertion, yet).

Thus, so far, the maximal gain of half a derivative is only known to be attained when f and g both belong to $L^2_{x,v}$. In the present work, we explore new settings of velocity averaging lemmas where a maximal gain of half a derivative is obtained. Our first main result shows that it is possible to gain exactly half a derivative even if f and g do not belong to $L^2_{x,v}$.

Theorem 1.2. *In any dimension $n \geq 1$, let $f, g \in L^r_x L^{r'}_v$, with $\frac{2n}{n+1} < r \leq 2$, be such that* (1-2) *holds true. Then,*

$$\int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \in H^{\frac{1}{2}}_x$$

for any $\varphi \in L^\infty_c(\mathbb{R}^n)$.

This result clearly follows from the more general Theorem 3.6 by setting $a = 2$ therein, which is established later on in Section 3. It is based on a TT^*-argument combined with the dispersion due to kinetic transport studied in Section 2 and velocity averaging.

Such a result had already been hinted at in [Jabin and Vega 2003; 2004], where it was established that, in two dimensions only ($n = 2$), velocity averages of f belong to H^s_x, for any $0 \leq s < \frac{1}{2}$, provided f and g belong to $L^{\frac{4}{3}}_x L^\infty_v$; see [Jabin and Vega 2004, Theorem 1.3].

In fact, in [Jabin and Vega 2003; 2004], the authors further identified another case which could potentially lead to a gain of almost half a derivative on the velocity averages. More precisely, they showed that, in two dimensions only ($n = 2$), velocity averages of f belong to $W^{s,\frac{4}{3}}_x$ for any $0 \leq s < \frac{1}{2}$, provided f and g belong to $L^{\frac{4}{3}}_x L^2_v$ and under the peculiar assumption that $g(x,v)\,\varphi(v)$ is an even function in v; see [Jabin and Vega 2004, Theorem 1.2]. The latter assumption is rather unnatural and it remained unclear whether this evenness condition could be removed or not.

By building upon the work from [Jabin and Vega 2004], combining our methods with the remarkable proof of Theorem 1.2 therein, we are able to bring a definitive answer to this two-dimensional problem, which is precisely the content of the following result.

Theorem 1.3. *In dimension $n = 2$, let $f, g \in L^{\frac{4}{3}}_x L^2_v$ be such that* (1-2) *holds true. Then,*

$$\int_{\mathbb{R}^2} f(x,v)\,\varphi(v)\,dv \in W^{s,\frac{4}{3}}_x$$

for all $0 \leq s < \frac{1}{2}$ and any $\varphi \in L^\infty_c(\mathbb{R}^2)$.

This result clearly follows from the more general Theorem 5.4, by setting $r = \frac{4}{3}$ therein, which is proved later on in Section 5. Its proof follows from the analysis of the boundedness of some adjoint transport operator on the dual space $L^4_x = (L^{\frac{4}{3}}_x)'$ and uses crucially the trivial fact that the exponent 4 is an even integer to control the square of this adjoint transport operator in L^2_x rather than the operator itself in L^4_x. This fact, among other characteristics of the proof, is strikingly reminiscent of the proofs of boundedness of Bochner–Riesz multipliers and Fourier restriction operators in two dimensions. We refer to [Grafakos 2009] for more on these subjects from harmonic analysis.

In higher dimensions, we extend the preceding result into the following theorem.

Theorem 1.4. *In any dimension* $n \geq 3$, *let* $f, g \in L^{\frac{4}{3}}_x L^2_v$ *be such that* (1-2) *holds true.*

Then,

$$\int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \in W^{s,\frac{4}{3}}_x$$

for all $0 \leq s < \frac{n}{4(n-1)}$ *and any* $\varphi \in L^\infty_c(\mathbb{R}^n)$.

This result clearly follows from the more general Theorem 6.8 by setting $r = \frac{4}{3}$ therein, which is proved later on in Section 6.

Observe that, employing rather general interpolation methods, it is possible to deduce a large variety of velocity averaging results, similar to those asserted in the above theorems, combining spaces for f and g with distinct integrabilities. We refer to [Arsénio and Masmoudi 2014] (see in particular the very general Theorem 4.7 therein) for such interpolation techniques.

It is likely that Theorem 1.4 may be largely improved. Indeed, note that a formal interpolation would yield

$$(L^1_x L^2_v, L^{\frac{2n}{n+1}}_x L^2_v)_{\frac{n}{2(n-1)}} = L^{\frac{4}{3}}_x L^2_v \quad \text{and} \quad (L^1_x, W^{\frac{1}{2},\frac{2n}{n+1}}_x)_{\frac{n}{2(n-1)}} = W^{\frac{n}{4(n-1)},\frac{4}{3}}_x, \tag{1-3}$$

whence formally extrapolating the above regularity result has us believe that, for any $\varphi \in L^\infty_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \in W^{s,\frac{2n}{n+1}}_x \tag{1-4}$$

for all $0 \leq s < \frac{1}{2}$, whenever $f, g \in L^{\frac{2n}{n+1}}_x L^2_v$ (see [Arsénio 2015] for a survey of velocity averaging lemmas and more on such conjectures; see also Figure 2 and the related comments following the proof of Theorem 6.8, below). In other words, Theorem 1.4 would follow from a formal interpolation of (1-4) with the degenerate L^1 case.

However, we do not know how to prove this estimate…

Finally, we would like to emphasize that, in this work, we investigate velocity averaging for its own sake, as a functional analytic study. Indeed, the search for maximal regularity in velocity averaging lemmas has already proved a challenging and interesting endeavor requiring diverse and original methods (extending beyond the classical settings of velocity averaging), producing interesting new results and leading to exciting research perspectives.

However, it should not be overlooked that velocity averaging lemmas also enjoy concrete applications to a wide variety of fundamental problems from kinetic theory. Such applications include, for instance, the existence of renormalized solutions to the Boltzmann equation [DiPerna and Lions 1989] and the convergence of such solutions to Leray solutions of the Navier–Stokes equations in a viscous incompressible hydrodynamic regime [Golse and Saint-Raymond 2004].

The investigation of sharp versions of averaging lemmas, such as the ones presented in this work, may lead to fundamental applications, as well. Indeed, we believe that such results may be very useful in establishing optimal regularity estimates in nonlinear conservation laws, through the study of their corresponding kinetic formulations. In particular, averaging lemmas with mixed integrability in x and v may be crucial in such attempts, for kinetic formulations are often based on densities which display distinct integrability or regularity properties in each variable.

We refer to [Lions et al. 1994a] (see Theorem 4 therein; see also [Lions et al. 1994b, Proposition 7] in the context of isentropic gas dynamics) for an early application of velocity averaging lemmas to kinetic formulations of scalar conservation laws, showing the existence of a regularizing phenomena as a truly nonlinear effect in hyperbolic equations. Nevertheless, the smoothness properties obtained through such applications have so far fallen short of the expected optimal regularity. In fact, other methods have already succeeded in establishing better results, see [Golse and Perthame 2013], which are sharp. However, one should keep in mind that the versions of velocity averaging lemmas used in these works were not sharp in the first place (for the kinetic formulation under consideration). In fact, it is likely that sharp versions of velocity averaging lemmas would yield sharp regularity properties in conservation laws, when kinetic formulations are available, which would largely expand the possibilities of reaching optimal regularity results in nonlinear conservation laws.

However, such research would require significant efforts and we will therefore not delve any further into this realm of applications, leaving it for subsequent works.

2. The transport operator and dispersive estimates

Let $f(x,v), g(x,v) \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_v)$ ($\mathcal{S}$ denotes the Schwartz space of rapidly decaying functions) be a solution of the transport equation (1-2). Then, introducing some cutoff function $\rho \in \mathcal{S}(\mathbb{R})$ such that $\rho(0) = 1$ and recalling the Fourier inversion formula

$$\rho(r) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{irs} \hat{\rho}(s)\, ds = \int_{\mathbb{R}} e^{-irs} \tilde{\rho}(s)\, ds,$$

where

$$\hat{\rho}(r) = \int_{\mathbb{R}} e^{-irs} \rho(s)\, ds,$$
$$\tilde{\rho}(r) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{irs} \rho(s)\, ds,$$

one can show that

$$f(x,v) = A_t f(x,v) + t B_t g(x,v), \tag{2-1}$$

with

$$A_t f(x,v) = \int_{\mathbb{R}} f(x - stv, v)\, \tilde{\rho}(s)\, ds,$$
$$B_t g(x,v) = \int_{\mathbb{R}} g(x - stv, v) \tilde{\tau}(s)\, ds,$$

where $\tau(s) = (1 - \rho(s))/(is)$ and $t \in \mathbb{R}$ is an interpolation parameter. We refer the reader to [Arsénio and Masmoudi 2014, Section 3] for full details on the derivation of this decomposition formula.

Further considering the Fourier transform in the space variable only,

$$\hat{f}(\eta, v) = \mathcal{F}_x f(\eta, v) = \int_{\mathbb{R}^n} e^{-i\eta\cdot x} f(x, v)\, dx,$$

it holds that

$$\begin{aligned} \mathcal{F}_x A_t f(\eta, v) &= \rho(t\eta\cdot v)\mathcal{F}_x f(\eta, v), \\ \mathcal{F}_x B_t g(\eta, v) &= \tau(t\eta\cdot v)\mathcal{F}_x g(\eta, v). \end{aligned} \tag{2-2}$$

Notice that τ is smooth near the origin, for $\rho(0) = 1$, and that

$$\tilde{\tau}(s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{isr}\frac{1-\rho(r)}{ir}\, dr = \mathbb{1}_{\{s\geq 0\}} - \int_{-\infty}^{s} \tilde{\rho}(\sigma)\, d\sigma. \tag{2-3}$$

In particular, $\tilde{\tau}$ is bounded pointwise and, if $\tilde{\rho}$ is compactly supported, so is $\tilde{\tau}$. Observe, however, that it is not possible to isolate the origin from the support of $\tilde{\tau}$.

Generally speaking, the estimates established in this work clearly apply to both operators A_t and B_t. Nevertheless, for the sake of simplicity, we only formulate our results in terms of the operator A_t. The corresponding results for B_t are easily deduced by replacing ρ by τ.

In this section, we study the dispersive properties of the operators A_t and B_t, which will serve in the proof of Theorem 3.6 below. To this end, we will use the following basic dispersive estimate established in [Castella and Perthame 1996]:

$$\|f(x - tv, v)\|_{L^p_x L^r_v} \leq \frac{1}{|t|^{n\left(\frac{1}{r}-\frac{1}{p}\right)}}\|f(x, v)\|_{L^r_x L^p_v} \quad \text{for all } 1 \leq r \leq p \leq \infty. \tag{2-4}$$

Our first dispersive estimate on A_t below is an elementary application of (2-4) to the operator A_t. We will not make any direct use of this simple result later on. It does, however, provide some insight into the dispersive properties of A_t and, therefore, we list it here for completeness.

Proposition 2.1. *For any given $1 \leq r \leq p \leq \infty$, the operator A_t satisfies the estimate*

$$\|A_t f\|_{L^p_x L^r_v} \leq \frac{1}{|t|^{n\left(\frac{1}{r}-\frac{1}{p}\right)}}\left\|\frac{\tilde{\rho}(s)}{s^{n\left(\frac{1}{r}-\frac{1}{p}\right)}}\right\|_{L^1}\|f\|_{L^r_x L^p_v}$$

for all $t \neq 0$.

Proof. This result is a simple extension of the standard dispersive estimate (2-4). Indeed, we have

$$\|A_t f\|_{L^p_x L^r_v} \leq \int_{\mathbb{R}}\|f(x - stv, v)\|_{L^p_x L^r_v}|\tilde{\rho}(s)|\, ds \leq \int_{\mathbb{R}}\frac{1}{|st|^{n\left(\frac{1}{r}-\frac{1}{p}\right)}}\|f(x, v)\|_{L^r_x L^p_v}|\tilde{\rho}(s)|\, ds. \qquad \square$$

Next, combining the dispersion of the free transport flow (2-4) with a TT^*-argument yields the following proposition.

Proposition 2.2. *Let $2 \leq a \leq \infty$, $2 < q \leq \infty$ and $1 \leq r \leq p \leq \infty$ be such that*

$$\frac{2}{a} = \frac{1}{p} + \frac{1}{r} \quad \text{and} \quad n\left(\frac{1}{r} - \frac{1}{p}\right) = \frac{2}{q}.$$

Then, the operator A_t satisfies the estimate

$$\|A_t f\|_{L^a_{x,v}} \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}\|_{L^{q'}} \|f\|_{L^r_x L^p_v}$$

for all $t \neq 0$, where $C > 0$ only depends on q.

Proof. First of all, notice that the case $q = \infty$, so that $a = p = r$, is obvious, with a constant $C = 1$. We may therefore assume, without any loss of generality, that $q < \infty$.

Thus, we estimate, using the dispersion (2-4),

$$\begin{aligned}
\|A_t f\|^2_{L^a_{x,v}} &= \||A_t f|^2\|_{L^{a/2}_{x,v}} \\
&= \left\| \int_{\mathbb{R}\times\mathbb{R}} f(x - stv, v)\, \tilde{\rho}(s)\, f(x - \sigma tv, v)\, \tilde{\rho}(\sigma)\, ds\, d\sigma \right\|_{L^{a/2}_{x,v}} \\
&\le \int_{\mathbb{R}\times\mathbb{R}} \|f(x,v) f(x - (\sigma - s)tv, v)\|_{L^{a/2}_{x,v}} |\tilde{\rho}(s)\tilde{\rho}(\sigma)|\, ds\, d\sigma \\
&\le \int_{\mathbb{R}\times\mathbb{R}} \|f(x,v)\|_{L^r_x L^p_v} \|f(x - (\sigma - s)tv, v)\|_{L^p_x L^r_v} |\tilde{\rho}(s)\, \tilde{\rho}(\sigma)|\, ds\, d\sigma \\
&\le \frac{1}{|t|^{\frac{2}{q}}} \|f\|^2_{L^r_x L^p_v} \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{|\sigma - s|^{\frac{2}{q}}} |\tilde{\rho}(s)\, \tilde{\rho}(\sigma)|\, ds\, d\sigma.
\end{aligned}$$

Hence, by virtue of the Hardy–Littlewood–Sobolev inequality,

$$\|A_t f\|_{L^a_{x,v}} \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}\|_{L^{q'}} \|f\|_{L^r_x L^p_v},$$

where $C > 0$ only depends on q. □

The preceding proposition only accepts parameters in the range $2 \le a \le \infty$. The next proposition handles the range $1 \le a \le 2$. It is obtained by interpolating the estimate from the preceding proposition with the degenerate L^1 case. Figure 1 represents the range of validity of the parameters $\frac{1}{p}$ and $\frac{1}{r}$ for both Propositions 2.2 and 2.3. More precisely, the shaded region therein delimited by the points $(0,0)$, $\left(\frac{1}{n}, 0\right)$, $\left(\frac{n+1}{2n}, \frac{n-1}{2n}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ is handled by Proposition 2.2, whereas the shaded region bounded by the points $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{n+1}{2n}, \frac{n-1}{2n}\right)$ and $(1,1)$ concerns Proposition 2.3.

Proposition 2.3. *Let $1 \le a \le 2$, $a' < q \le \infty$ and $1 \le r \le p \le \infty$ be such that*

$$\frac{2}{a} = \frac{1}{p} + \frac{1}{r} \quad \text{and} \quad n\left(\frac{1}{r} - \frac{1}{p}\right) = \frac{2}{q}.$$

Then, the operator A_t satisfies the estimate

$$\|A_t f\|_{L^a_{x,v}} \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}\|_{L^{q'}} \|f\|_{L^r_x L^p_v}$$

for all $t \neq 0$, where $C > 0$ only depends on q and a.

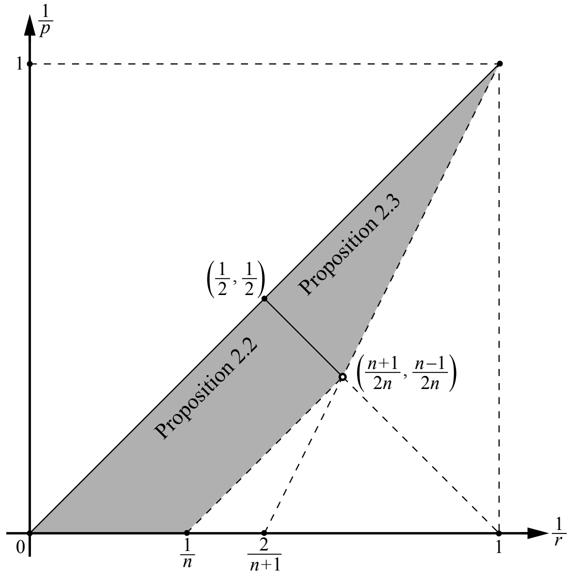

Figure 1. Range of validity of the parameters $\frac{1}{r}$ and $\frac{1}{p}$ in Propositions 2.2 and 2.3.

Proof. This result will follow from the interpolation of the case $a = 2$ from Proposition 2.2 and the trivial estimate

$$\|A_t f\|_{L^1_{x,v}} \leq \|\tilde{\rho}\|_{L^1} \|f\|_{L^1_{x,v}}. \tag{2-5}$$

Thus, without any loss of generality, we assume that $1 < a < 2$ and we define $0 < \theta < 1$, $2 < q_1 \leq \infty$ and $1 \leq r_1 \leq p_1 \leq \infty$ by

$$\theta = \frac{2}{a'} = \frac{1}{p'} + \frac{1}{r'}, \qquad p_1 = \theta r', \qquad r_1 = \theta p', \qquad q_1 = \theta q,$$

so that

$$\frac{1}{a} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{1} + \frac{\theta}{r_1},$$

and

$$1 = \frac{1}{p_1} + \frac{1}{r_1} \quad \text{and} \quad n\Big(\frac{1}{r_1} - \frac{1}{p_1}\Big) = \frac{2}{q_1}.$$

In particular, notice that, since $\frac{2}{q_1} < 1$, necessarily $1 < r_1 \leq p_1 < \infty$. On Figure 1, the point $\big(\frac{1}{r_1}, \frac{1}{p_1}\big)$ lies somewhere on the half open segment $\big[\big(\frac{1}{2}, \frac{1}{2}\big), \big(\frac{n+1}{2n}, \frac{n-1}{2n}\big)\big)$.

It follows then from Proposition 2.2 that

$$\|A_t f\|_{L^2_{x,v}} \leq \frac{C}{|t|^{\frac{1}{q_1}}} \|\tilde{\rho}\|_{L^{q_1'}} \|f\|_{L^{r_1}_x L^{p_1}_v}, \tag{2-6}$$

where $C > 0$ only depends on q_1.

Now, standard results from complex interpolation theory of Lebesgue spaces, see [Bergh and Löfström 1976, Section 5.1], establish that

$$(L^1_{x,v}, L^2_{x,v})_{[\theta]} = L^a_{x,v}, \quad (L^1_{x,v}, L^{r_1}_x L^{p_1}_v)_{[\theta]} = L^r_x L^p_v \quad \text{and} \quad (L^1, L^{q_1'})_{[\theta]} = L^{q'}.$$

Therefore, interpolating estimates (2-5) and (2-6) (these estimates remain valid for complex-valued functions), which are multilinear in $\tilde\rho$ and f (use the multilinear complex interpolation theorem [Bergh and Löfström 1976, Theorem 4.4.1]), we arrive at

$$\|A_t f\|_{L^a_{x,v}} \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde\rho\|_{L^{q'}} \|f\|_{L^r_x L^p_v},$$

where $C > 0$ only depends on q and a. □

Note that the adjoint operator of A_t satisfies $A_t^* = A_{-t}$. Combining Propositions 2.2 and 2.3 with a duality argument yields the following result.

Proposition 2.4. *Let* $1 \le a \le \infty$, $\max\{2, a\} < q \le \infty$ *and* $1 \le r \le p \le \infty$ *be such that*

$$\frac{2}{a} = \frac{1}{p} + \frac{1}{r} \quad \textit{and} \quad n\Big(\frac{1}{r} - \frac{1}{p}\Big) = \frac{2}{q}.$$

Then, the operator A_t *satisfies the estimate*

$$\|A_t f\|_{L^p_x L^r_v} \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde\rho\|_{L^{q'}} \|f\|_{L^a_{x,v}}$$

for all $t \neq 0$, *where* $C > 0$ *only depends on* q *and* a.

Proof. This result easily follows from a duality argument. Indeed, by Proposition 2.2 (if $1 \le a \le 2$) or Proposition 2.3 (if $2 \le a \le \infty$), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} A_t f(x,v) g(x,v)\,dx\,dv\right| &= \left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,v) A_{-t} g(x,v)\,dx\,dv\right| \\ &\le \|f\|_{L^a_{x,v}} \|A_{-t} g\|_{L^{a'}_{x,v}} \\ &\le \|f\|_{L^a_{x,v}} \frac{C}{|t|^{\frac{1}{q}}} \|\tilde\rho\|_{L^{q'}} \|g\|_{L^{p'}_x L^{r'}_v},\end{aligned}$$

where $C > 0$ only depends on q and a. Then, taking the supremum over all $g \in L^{p'}_x L^{r'}_v$ easily concludes the proof of the proposition. □

3. Dispersion and velocity averaging

We proceed now to combining the dispersive estimates from the previous section with the classical regularizing effects due to velocity averaging. This will eventually lead to our first main result Theorem 3.6.

To this end, we consider, for any $t \neq 0$ and $\varphi(v) \in L^\infty_c(\mathbb{R}^n)$, the velocity averaging operator T_t defined, for all $f(x,v) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, by

$$T_t f(x) = \int_{\mathbb{R}^n} A_t f(x,v)\,\varphi(v)\,dv.$$

In particular, for all $g(x) \in \mathcal{S}(\mathbb{R}^n)$, one has, by duality,

$$\int_{\mathbb{R}^n} T_t f(x) g(x)\,dx = \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,v) T_t^* g(x,v)\,dx\,dv,$$

where the adjoint operator T_t^* is defined by

$$T_t^* g(x,v) = A_{-t}(g(x)\,\varphi(v)) = \int_{\mathbb{R}} g(x+stv)\,\tilde{\rho}(s)\,ds\,\varphi(v).$$

We will also consider the operators T_t and T_t^* defined with B_t instead of A_t.

For clarity, throughout this section, we will always consider the same given velocity weight $\varphi(v) \in L_c^\infty(\mathbb{R}^n)$ and we will assume that its support is contained inside a closed ball of radius $R > 0$ centered at the origin.

We begin by applying the classical Hilbertian methods of velocity averaging from [Golse et al. 1988] to the operator T_t and its adjoint T_t^*. The resulting estimates are recorded in the following proposition. For the sake of completeness and convenience of the reader, we provide a complete justification of these results.

Proposition 3.1. *The operator T_t and its adjoint T_t^* satisfy the estimates*

$$\|(1-\Delta_x)^{\frac{1}{4}} T_t f\|_{L_x^2} \le C\left(\|\rho\|_{L^\infty}\,\|\varphi\|_{L_v^2} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}}\|\rho\|_{L^2}\,\|\varphi\|_{L_v^\infty}\right)\|f\|_{L_{x,v}^2},$$

$$\|(1-\Delta_x)^{\frac{1}{4}} T_t^* g\|_{L_{x,v}^2} \le C\left(\|\rho\|_{L^\infty}\,\|\varphi\|_{L_v^2} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}}\|\rho\|_{L^2}\,\|\varphi\|_{L_v^\infty}\right)\|g\|_{L_x^2}$$

for all $t \neq 0$, where $C > 0$ only depends on the dimension.

Proof. We deal with the estimate on the adjoint operator T_t^* first. Thus, it is readily seen, by Plancherel's theorem and using the Fourier representation (2-2) of A_t, that

$$\|T_t^* g\|_{L_{x,v}^2} = \frac{1}{(2\pi)^{\frac{n}{2}}}\|\rho(-t\eta\cdot v)\hat{g}(\eta)\varphi(v)\|_{L_{\eta,v}^2} \le \|\rho\|_{L^\infty}\,\|g\|_{L_x^2}\,\|\varphi\|_{L_v^2}. \tag{3-1}$$

Furthermore, using again Plancherel's theorem, we find that

$$\begin{aligned}\|(-\Delta_x)^{\frac{1}{4}} T_t^* g\|_{L_{x,v}^2} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\left\||\eta|^{\frac{1}{2}}\rho(-t\eta\cdot v)\hat{g}(\eta)\varphi(v)\right\|_{L_{\eta,v}^2} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\left\|\rho\left(-t\frac{\eta}{|\eta|}\cdot v\right)\hat{g}(\eta)\varphi\left(\frac{v}{|\eta|}\cdot\frac{\eta}{|\eta|}\frac{\eta}{|\eta|} + \left(v - v\cdot\frac{\eta}{|\eta|}\frac{\eta}{|\eta|}\right)\right)\right\|_{L_{\eta,v}^2},\end{aligned}$$

where we have rescaled the variable v by a factor $|\eta|$ in the direction $\frac{\eta}{|\eta|}$ only.

Then, writing $v' = (v_2, \dots, v_n)$ and recalling that the support of φ is contained in a closed ball of radius $R > 0$ centered at the origin, we deduce that

$$\begin{aligned}\|(-\Delta_x)^{\frac{1}{4}} T_t^* g\|_{L_{x,v}^2} &\le \|\rho(-tv_1)\mathbb{1}_{\{|v'|\le R\}}\|_{L_v^2}\,\|g\|_{L_x^2}\,\|\varphi\|_{L_v^\infty} \\ &= \left(\frac{|\mathbb{S}^{n-2}|}{n-1}\right)^{\frac{1}{2}}\frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}}\|\rho(s)\|_{L_s^2}\,\|g\|_{L_x^2}\,\|\varphi\|_{L_v^\infty}.\end{aligned} \tag{3-2}$$

Finally, combining estimates (3-1) and (3-2) establishes the estimate on T_t^*.

The estimate on T_t is then easily deduced from the estimate on T_t^* by a duality argument. ☐

Interpolating the preceding result with the degenerate L^1 case yields the following proposition.

Proposition 3.2. *For any given $1 \leq a \leq 2$, the operator T_t and its adjoint T_t^* satisfy the estimates*

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t f\|_{L^a_x} \leq C(\|\tilde{\rho}\|_{L^1}\|\varphi\|_{L^\infty_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|f\|_{L^a_{x,v}},$$

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^a_{x,v}} \leq C(\|\tilde{\rho}\|_{L^1}\|\varphi\|_{L^1_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|g\|_{L^a_x}$$

for all $t \neq 0$, where $s = 1 - \frac{1}{a}$ and $C > 0$ only depends on a and the dimension.

Proof. We deal with the estimate on T_t first.

It was established in [DeVore and Petrova 2001, Theorem 3.2] that the real interpolation space $(L^1, H^{\frac{1}{2}})_{2s,a}$, where $s = 1 - \frac{1}{a}$ with $1 < a < 2$, is precisely the Besov space $B^s_{a,a}$, which is continuously embedded into the classical fractional Sobolev space $W^{s,a}$; that is,

$$(L^1_x, H^{\frac{1}{2}}_x)_{2s,a} \subset W^{s,a}_x.$$

Note that it would be possible to formulate a better result by using below the smaller Besov space $B^s_{a,a}$, as in [DeVore and Petrova 2001]. However, for the sake of simplicity, we choose not to do so and stick to Sobolev spaces. We refer to [DeVore and Petrova 2001] for more details on this.

Next, it is well known from the real interpolation theory of Lebesgue spaces, see [Bergh and Löfström 1976, Theorem 5.2.1], that

$$(L^1_{x,v}, L^2_{x,v})_{2s,a} = L^a_{x,v}.$$

Therefore, the first part of this result easily follows from the real interpolation of the classical estimate on T_t from Proposition 3.1 with the case $p = 1$ of the simple estimate

$$\|T_t f\|_{L^p_x} \leq \|\tilde{\rho}\|_{L^1}\|\varphi\|_{L^{p'}_v}\|f\|_{L^p_{x,v}},$$

valid for any $1 \leq p \leq \infty$.

There only remains to establish the estimate on the adjoint operator T_t^*. To this end, note that T_t^* commutes with the differentiation in x so that the estimate on T_t^* from Proposition 3.1 can be recast as

$$\|T_t^* g\|_{L^2_{x,v}} \leq C\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)\|(1-\Delta_x)^{-\frac{1}{4}} g\|_{L^2_x},$$

where $C > 0$ only depends on the dimension.

We wish now to complex interpolate the preceding estimate with the elementary control

$$\|T_t^* g\|_{L^1_{x,v}} \leq \|\tilde{\rho}\|_{L^1}\|\varphi\|_{L^1_v}\|g\|_{L^1_x} \leq C\|\tilde{\rho}\|_{L^1}\|\varphi\|_{L^1_v}\|g\|_{h^1_x},$$

where h^1 denotes the local Hardy space; see [Runst and Sickel 1996, Section 2.1.2] for a definition.

To this end, we use the results from complex interpolation theory, see [Bergh and Löfström 1976, Theorem 5.1.1; Runst and Sickel 1996, Section 2.5.2],

$$(L^1_{x,v}, L^2_{x,v})_{[2s]} = L^a_{x,v} \quad \text{and} \quad (h^1_x, H^{-\frac{1}{2}}_x)_{[2s]} = W^{-s,a}_x,$$

to deduce that

$$\|T_t^* g\|_{L^a_{x,v}} \le C(\|\tilde\rho\|_{L^1}\|\varphi\|_{L^1_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac12}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|(1-\Delta_x)^{-\frac{s}{2}} g\|_{L^a_x}.$$

Finally, we easily conclude the proof of the proposition by using again that T_t^* commutes with the differentiation in x and replacing g by $(1-\Delta_x)^{\frac{s}{2}} g$ in the above estimate. □

Combining now the preceding proposition with a duality argument yields the following result.

Proposition 3.3. *For any given $2 \le a \le \infty$, the operator T_t and its adjoint T_t^* satisfy the estimates*

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t f\|_{L^a_x} \le C(\|\tilde\rho\|_{L^1}\|\varphi\|_{L^1_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac12}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|f\|_{L^a_{x,v}},$$

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^a_{x,v}} \le C(\|\tilde\rho\|_{L^1}\|\varphi\|_{L^\infty_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac12}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|g\|_{L^a_x}$$

for all $t \neq 0$, where $s = \frac{1}{a}$ and $C > 0$ only depends on a and the dimension.

Proof. These estimates follow straightforwardly from Proposition 3.2 through a duality argument.

Indeed, by Proposition 3.2, noticing that both T_t and T_t^* commute with differentiation in x, it holds that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}(1-\Delta_x)^{\frac{s}{2}} T_t f(x) g(x)\,dx\right| \\
&\qquad = \left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,v)(1-\Delta_x)^{\frac{s}{2}} T_t^* g(x,v)\,dx\,dv\right| \\
&\qquad \le \|f\|_{L^a_{x,v}}\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^{a'}_{x,v}} \\
&\qquad \le C(\|\tilde\rho\|_{L^1}\|\varphi\|_{L^1_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac12}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|f\|_{L^a_{x,v}}\|g\|_{L^{a'}_x},
\end{aligned}$$

where $C > 0$ only depends on a and the dimension. Then, taking the supremum over all $g \in L^{a'}_x$ easily concludes the proof of the first estimate on T_t.

Similarly, using Proposition 3.2 again, it holds that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n\times\mathbb{R}^n}(1-\Delta_x)^{\frac{s}{2}} T_t^* g(x,v) f(x,v)\,dx\,dv\right| \\
&\qquad = \left|\int_{\mathbb{R}^n} g(x)(1-\Delta_x)^{\frac{s}{2}} T_t f(x)\,dx\right| \\
&\qquad \le \|g\|_{L^a_x}\|(1-\Delta_x)^{\frac{s}{2}} T_t f\|_{L^{a'}_x} \\
&\qquad \le C(\|\tilde\rho\|_{L^1}\|\varphi\|_{L^\infty_v})^{1-2s}\left(\|\rho\|_{L^\infty}\|\varphi\|_{L^2_v} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac12}}\|\rho\|_{L^2}\|\varphi\|_{L^\infty_v}\right)^{2s}\|g\|_{L^a_x}\|f\|_{L^{a'}_{x,v}},
\end{aligned}$$

where $C > 0$ only depends on a and the dimension. Finally, taking the supremum over all $f \in L^{a'}_{x,v}$ easily concludes the proof of the proposition. □

From now on, in this section, we assume that the cutoff function $\rho(r)$ may be decomposed as a product $\rho(r) = \rho_1(r)\rho_2(r)$, so that $\tilde{\rho}(s) = \tilde{\rho}_1 * \tilde{\rho}_2(s)$. Naturally, we will denote by A_t^i, T_t^i and T_t^{i*}, where $i = 1, 2$, the respective operators A_t, T_t and T_t^* where we replace the cutoff ρ by ρ_i. It is then readily seen that

$$A_t = A_t^1 A_t^2 = A_t^2 A_t^1, \quad T_t = T_t^1 A_t^2 = T_t^2 A_t^1, \quad T_t^* = A_{-t}^2 T_t^{1*} = A_{-t}^1 T_t^{2*}. \tag{3-3}$$

As shown below, this useful trick allows us to combine the previous regularity results from this section with the dispersive estimates from Section 2 to obtain new estimates on the operators T_t and T_t^*.

Proposition 3.4. *Let* $1 \le a \le \infty$, $\max\{2, a\} < q \le \infty$ *and* $1 \le r \le p \le \infty$ *be such that*

$$\frac{2}{a} = \frac{1}{p} + \frac{1}{r} \quad \text{and} \quad n\Big(\frac{1}{r} - \frac{1}{p}\Big) = \frac{2}{q}.$$

Then, the operator T_t^* *satisfies the estimate*

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L_x^p L_v^r} \le \frac{C}{|t|^{\frac{1}{q}+s}} \|\tilde{\rho}_1\|_{L^{q'}} \|\tilde{\rho}_2\|_{L^1}^{1-2s} (\|\rho_2\|_{L^\infty} + \|\rho_2\|_{L^2})^{2s} \|g\|_{L_x^a}$$

for all $t \ne 0$ *such that* $|t| \le 1$, *where* $s = \min\{1 - \frac{1}{a}, \frac{1}{a}\}$ *and* $C > 0$ *only depends on* q, a, φ *and the dimension.*

Proof. We treat the case $1 \le a \le 2$ first, so that $q > 2$ and $s = 1 - \frac{1}{a}$. Writing $T_t^* = A_{-t}^1 T_t^{2*}$ and then successively employing Propositions 2.4 and 3.2, we find, noticing A_{-t}^1 commutes with differentiation in x, that

$$\begin{aligned}
&\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L_x^p L_v^r} \\
&\qquad = \|A_{-t}^1 (1-\Delta_x)^{\frac{s}{2}} T_t^{2*} g\|_{L_x^p L_v^r} \\
&\qquad \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}_1\|_{L^{q'}} \|(1-\Delta_x)^{\frac{s}{2}} T_t^{2*} g\|_{L_{x,v}^a} \\
&\qquad \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}_1\|_{L^{q'}} (\|\tilde{\rho}_2\|_{L^1} \|\varphi\|_{L_v^1})^{1-2s} \left(\|\rho_2\|_{L^\infty} \|\varphi\|_{L_v^2} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}} \|\rho_2\|_{L^2} \|\varphi\|_{L_v^\infty} \right)^{2s} \|g\|_{L_x^a},
\end{aligned}$$

where $C > 0$ only depends on q, a and the dimension. Since $|t| \le 1$, this concludes the proof of the proposition when $a \le 2$.

The case $a \ge 2$ is handled similarly. One now has that $q > a$ and $s = \frac{1}{a}$. Therefore, applying successively Propositions 2.4 and 3.3, we find that

$$\begin{aligned}
&\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L_x^p L_v^r} \\
&\qquad = \|A_{-t}^1 (1-\Delta_x)^{\frac{s}{2}} T_t^{2*} g\|_{L_x^p L_v^r} \\
&\qquad \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}_1\|_{L^{q'}} \|(1-\Delta_x)^{\frac{s}{2}} T_t^{2*} g\|_{L_{x,v}^a} \\
&\qquad \le \frac{C}{|t|^{\frac{1}{q}}} \|\tilde{\rho}_1\|_{L^{q'}} (\|\tilde{\rho}_2\|_{L^1} \|\varphi\|_{L_v^\infty})^{1-2s} \left(\|\rho_2\|_{L^\infty} \|\varphi\|_{L_v^2} + \frac{R^{\frac{n-1}{2}}}{|t|^{\frac{1}{2}}} \|\rho_2\|_{L^2} \|\varphi\|_{L_v^\infty} \right)^{2s} \|g\|_{L_x^a},
\end{aligned}$$

where $C > 0$ only depend on q, a and the dimension. Since $|t| \leq 1$, this concludes the proof of the proposition. ☐

Combining the previous result with a duality argument yields estimates on the operator T_t, which are contained in the following proposition.

Proposition 3.5. *Let $1 \leq a \leq \infty$, $\max\{2, a'\} < q \leq \infty$ and $1 \leq r \leq p \leq \infty$ be such that*

$$\frac{2}{a} = \frac{1}{p} + \frac{1}{r} \quad \text{and} \quad n\Big(\frac{1}{r} - \frac{1}{p}\Big) = \frac{2}{q}.$$

Then, the operator T_t satisfies the estimate

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t f\|_{L^a_x} \leq \frac{C}{|t|^{\frac{1}{q}+s}} \|\tilde\rho_1\|_{L^{q'}} \|\tilde\rho_2\|_{L^1}^{1-2s} (\|\rho_2\|_{L^\infty} + \|\rho_2\|_{L^2})^{2s} \|f\|_{L^r_x L^p_v} \tag{3-4}$$

for all $t \neq 0$ such that $|t| \leq 1$, where $s = \min\{1 - \frac{1}{a}, \frac{1}{a}\}$ and $C > 0$ only depends on q, a, φ and the dimension.

Proof. This estimate follows straightforwardly from Proposition 3.4 through a duality argument.

Indeed, using Proposition 3.4, we find, since T_t^* commutes with differentiation in x, that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} (1-\Delta_x)^{\frac{s}{2}} T_t f(x) g(x)\,dx\right| &= \left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,v)(1-\Delta_x)^{\frac{s}{2}} T_t^* g(x,v)\,dx\,dv\right| \\ &\leq \|f\|_{L^r_x L^p_v} \|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^{r'}_x L^{p'}_v} \\ &\leq \frac{C}{|t|^{\frac{1}{q}+s}} \|\tilde\rho_1\|_{L^{q'}} \|\tilde\rho_2\|_{L^1}^{1-2s} (\|\rho_2\|_{L^\infty} + \|\rho_2\|_{L^2})^{2s} \|f\|_{L^r_x L^p_v} \|g\|_{L^{a'}_x},\end{aligned}$$

where $C > 0$ only depends on q, a, φ and the dimension. Finally, taking the supremum over all $g \in L^{a'}_x$ easily concludes the proof of the proposition.

Note that, in order to deduce this result, we could just as well have applied here a combination of Propositions 2.2 and 2.3 with Propositions 3.3 and 3.2, respectively. ☐

We proceed now to the main theorem of this section. It contains Theorem 1.2 presented in the Introduction as special case (corresponding to the case $a = 2$ below) and provides a considerable extension of the classical velocity averaging lemma in $L^2_{x,v}$ (corresponding to the case $a = p = r = 2$ below). Indeed, observe that the case $a = 2$ therein yields a maximal gain of regularity of half a derivative on velocity averages for a variety of parameters, which was previously known to occur only in the classical $L^2_{x,v}$ setting.

Theorem 3.6. *In any dimension $n \geq 1$, let $1 \leq a \leq \infty$, $\max\{2, a'\} < q \leq \infty$ and $1 \leq r \leq p \leq \infty$ be such that*

$$\frac{2}{a} = \frac{1}{p} + \frac{1}{r} \quad \text{and} \quad n\Big(\frac{1}{r} - \frac{1}{p}\Big) = \frac{2}{q}.$$

Then, for any $f, g \in L^r_x(\mathbb{R}^n; L^p_v(\mathbb{R}^n))$ such that (1-2) holds true, one has

$$\int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \in W^{s,a}_x(\mathbb{R}^n)$$

for any $\varphi \in L^\infty_c(\mathbb{R}^n)$, *where* $s = \min\left\{1 - \frac{1}{a}, \frac{1}{a}\right\}$. *Furthermore, one has the estimate*

$$\left\| (1-\Delta_x)^{\frac{s}{2}} \int_{\mathbb{R}^n} f\varphi\, dv \right\|_{L^a_x} \le C(\|f\|_{L^r_x L^p_v} + \|g\|_{L^r_x L^p_v}),$$

where $C > 0$ *only depends on* φ *and constant parameters.*

Proof. We consider first the refined interpolation formula (2-1), which is valid for solutions of the transport equation (1-2), for some given cutoff $\rho \in \mathcal{S}(\mathbb{R})$. Clearly, further differentiating (2-1) in x and then averaging in v yields

$$\left\| (1-\Delta_x)^{\frac{s}{2}} \int_{\mathbb{R}^n} f\varphi\, dv \right\|_{L^a_x} \le \left\| (1-\Delta_x)^{\frac{s}{2}} \int_{\mathbb{R}^n} A_t f\varphi\, dv \right\|_{L^a_x} + t \left\| (1-\Delta_x)^{\frac{s}{2}} \int_{\mathbb{R}^n} B_t g\varphi\, dv \right\|_{L^a_x}. \tag{3-5}$$

We wish now to apply Proposition 3.5 to the preceding estimate. To this end, according to (3-3), we take the decompositions

$$\rho(r) = \rho_1(r)\rho_2(r) \quad \text{and} \quad \tau(r) = \frac{1-\rho(r)}{ir} = \tau_1(r)\tau_2(r),$$

where

$$\rho_1(r) = \frac{1}{(1+r^2)^{\frac{\beta}{2}}}, \qquad \tau_1(r) = \frac{1}{(1+r^2)^{\frac{\beta}{2}}},$$
$$\rho_2(r) = (1+r^2)^{\frac{\beta}{2}}\rho(r), \quad \tau_2(r) = (1+r^2)^{\frac{\beta}{2}}\tau(r)$$

for some fixed $\frac{1}{q} < \beta < \frac{1}{2}$. In view of the technical Lemma B.1 from Appendix B, it then holds that

$$\tilde{\rho}_1, \tilde{\tau}_1 \in L^{q'}, \quad \tilde{\rho}_2, \tilde{\tau}_2 \in L^1 \quad \text{and} \quad \rho_2, \tau_2 \in L^\infty \cap L^2.$$

All constants involving norms of the cutoff functions ρ_1, ρ_2, τ_1 and τ_2 in the right-hand side of (3-4) are therefore finite.

Thus, applying Proposition 3.5 to estimate (3-5), we conclude, for any $0 < t < 1$, that

$$\left\| (1-\Delta_x)^{\frac{s}{2}} \int_{\mathbb{R}^n} f\varphi\, dv \right\|_{L^a_x} \le C\left(\frac{1}{t^{\frac{1}{q}+s}} \|f\|_{L^r_x L^p_v} + t^{1-(\frac{1}{q}+s)} \|g\|_{L^r_x L^p_v} \right),$$

where $C > 0$ only depends on constant parameters. □

4. The one-dimensional case

In the previous section, by combining kinetic dispersion with velocity averaging, we have established, in Theorem 3.6, a whole new range of regularity results on the solutions of the kinetic transport equation (1-2). The results from Theorem 3.6 are valid in any dimension $n \ge 1$. In one dimension ($n = 1$), it turns out that it is possible to obtain more results for a wide range of parameters which are not covered by Theorem 3.6. This is due to the fact that, in one dimension, spatial frequencies are always parallel to velocities.

In the present section, we explore this one-dimensional setting, which provides a good test case for velocity averaging lemmas in mixed Lebesgue spaces and allows one to get familiar with the decompositions used in this work in a much simpler setting. It does not, however, set a road map for the remaining more involved sections concerning higher dimensions, for it heavily relies on the elementary structure of the transport equation in one dimension.

We use the same notation as in the previous sections.

Proposition 4.1. *In dimension $n = 1$, let $1 < p < \infty$, $1 \le r < \infty$, $0 \le s < \frac{1}{r}$ and $\rho \in \mathcal{S}(\mathbb{R})$ be such that $\tilde\rho$ has its support contained inside a ball of radius $r_0 > 0$ centered at the origin.*

Then, the operator T_t^* satisfies the estimate

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^p_x L^r_v} \le \frac{C}{|t|^s} (r_0 \|\tilde\rho\|_{L^\infty})^{1-\frac{1}{r}} (\|\tilde\rho\|_{L^1} + \|(-\Delta)^{\frac{sr}{2}} \tilde\rho\|_{L^1})^{\frac{1}{r}} \|g\|_{L^p_x}$$

for all $t \neq 0$ such that $|t| \le 1$, where $C > 0$ only depends on p, s and φ.

Proof. First, it is readily seen, for any $1 \le r \le p \le \infty$, that

$$\|T_t^* g\|_{L^p_x L^r_v} \le \|T_t^* g\|_{L^r_v L^p_x} \le \|\tilde\rho\|_{L^1} \|\varphi\|_{L^r_v} \|g\|_{L^p_x}. \tag{4-1}$$

When the restriction $r \le p$ is not satisfied, the above estimate fails and we need a more convoluted estimate to handle this case. To this end, we write that

$$\begin{aligned} |T_t^* g(x,v)| &= \left| \int_{\mathbb{R}} g(x+stv)\tilde\rho(s)\, ds \varphi(v) \right| = \left| \frac{1}{tv} \int_{\mathbb{R}} g(x+r)\tilde\rho\left(\frac{r}{tv}\right) dr\, \varphi(v) \right| \\ &\le \|\tilde\rho\|_{L^\infty} \left| \frac{1}{tv} \int_{\{|r| \le |tv| r_0\}} g(x+r)\, dr\, \varphi(v) \right| \le 2 r_0 \|\tilde\rho\|_{L^\infty} M g(x) |\varphi(v)|, \end{aligned}$$

where Mg denotes the Hardy–Littlewood maximal function of g defined by

$$Mg(x) = \sup_{\delta > 0} \frac{1}{2\delta} \int_{\{|y| \le \delta\}} |g(x-y)|\, dy.$$

Recall that the Hardy–Littlewood maximal operator $g \mapsto Mg$ is bounded over $L^p(\mathbb{R})$ for any $1 < p < \infty$, and maps $L^1(\mathbb{R})$ into the standard weak-L^1 space $L^{1,\infty}(\mathbb{R})$; see [Grafakos 2008, Theorem 2.1.6]. It therefore easily follows from the previous estimate that

$$\|T_t^* g\|_{L^{1,\infty}_x L^r_v} \le C r_0 \|\tilde\rho\|_{L^\infty} \|\varphi\|_{L^r_v} \|g\|_{L^1_x}$$

and

$$\|T_t^* g\|_{L^p_x L^r_v} \le C r_0 \|\tilde\rho\|_{L^\infty} \|\varphi\|_{L^r_v} \|g\|_{L^p_x} \tag{4-2}$$

for any $1 < p < \infty$ and $1 \le r \le \infty$, where $C > 0$ only depends on p.

Next, we further compute, exploiting the one-dimensional structure of the operators, for any $0 < \alpha < 1$,

$$\begin{aligned} (-\Delta_x)^{\frac{\alpha}{2}} T_t^* g(x,v) &= \int_{\mathbb{R}} (-\Delta_x)^{\frac{\alpha}{2}} g(x+stv)\tilde\rho(s)\, ds\, \varphi(v) \\ &= \frac{1}{|tv|^\alpha} \int_{\mathbb{R}} (-\Delta_s)^{\frac{\alpha}{2}} (g(x+stv))\tilde\rho(s)\, ds\, \varphi(v) \\ &= \frac{1}{|tv|^\alpha} \int_{\mathbb{R}} g(x+stv)(-\Delta)^{\frac{\alpha}{2}} \tilde\rho(s)\, ds\, \varphi(v), \end{aligned}$$

whence, for any $1 \leq p \leq \infty$,

$$\|(-\Delta_x)^{\frac{\alpha}{2}} T_t^* g\|_{L_x^p L_v^1} \leq \|(-\Delta_x)^{\frac{\alpha}{2}} T_t^* g\|_{L_v^1 L_x^p} \leq \frac{1}{|t|^\alpha} \|(-\Delta)^{\frac{\alpha}{2}} \tilde{\rho}\|_{L^1} \left\| \frac{1}{|v|^\alpha} \varphi \right\|_{L_v^1} \|g\|_{L_x^p},$$

which, when combined with (4-1), yields

$$\|(1-\Delta_x)^{\frac{\alpha}{2}} T_t^* g\|_{L_x^p L_v^1} \leq C \left(\|\tilde{\rho}\|_{L^1} \|\varphi\|_{L_v^1} + \frac{1}{|t|^\alpha} \|(-\Delta)^{\frac{\alpha}{2}} \tilde{\rho}\|_{L^1} \left\| \frac{1}{|v|^\alpha} \varphi \right\|_{L_v^1} \right) \|g\|_{L_x^p}, \tag{4-3}$$

where $C > 0$ only depends on p and α.

We wish now to interpolate the bound (4-2), where we set $r = \infty$, with (4-3). To this end, recalling that T_t^* commutes with differentiation in x, we first recast (4-3) as

$$\|T_t^* g\|_{L_x^p L_v^1} \leq C \left(\|\tilde{\rho}\|_{L^1} \|\varphi\|_{L_v^1} + \frac{1}{|t|^\alpha} \|(-\Delta)^{\frac{\alpha}{2}} \tilde{\rho}\|_{L^1} \left\| \frac{1}{|v|^\alpha} \varphi \right\|_{L_v^1} \right) \|g\|_{W_x^{-\alpha,p}}, \tag{4-4}$$

and then we use the standard results from complex interpolation theory, see [Bergh and Löfström 1976, Sections 5.1 and 6.4], valid for any $1 < p, r < \infty$,

$$(L_x^p L_v^\infty, L_x^p L_v^1)_{[\frac{1}{r}]} = L_x^p L_v^r \quad \text{and} \quad (L_x^p, W_x^{-\alpha,p})_{[\frac{1}{r}]} = W_x^{-\frac{\alpha}{r},p},$$

to deduce from the interpolation of (4-2) and (4-4) that

$$\|T_t^* g\|_{L_x^p L_v^r} \leq C (r_0 \|\tilde{\rho}\|_{L^\infty} \|\varphi\|_{L_v^\infty})^{1-\frac{1}{r}} \left(\|\tilde{\rho}\|_{L^1} \|\varphi\|_{L_v^1} + \frac{1}{|t|^\alpha} \|(-\Delta)^{\frac{\alpha}{2}} \tilde{\rho}\|_{L^1} \left\| \frac{1}{|v|^\alpha} \varphi \right\|_{L_v^1} \right)^{\frac{1}{r}} \|g\|_{W_x^{-\alpha/r,p}},$$

where $C > 0$ only depends on p and α. □

Note that it would be possible to improve the gain of regularity in the preceding proposition by assuming that the support of the velocity weight $\varphi(v)$ does not contain the origin. However, this is a rather unnatural setting which we prefer to avoid here.

Combining the previous result with a duality argument yields estimates on the operator T_t, which are contained in the following proposition.

Proposition 4.2. *In dimension $n = 1$, let $1 < p < \infty,\ 1 < r \leq \infty,\ 0 \leq s < 1 - \frac{1}{r}$ and $\rho \in \mathcal{S}(\mathbb{R})$ be such that $\tilde{\rho}$ has its support contained inside a ball of radius $r_0 > 0$ centered at the origin.*

Then, the operator T_t satisfies the estimate

$$\|(1-\Delta_x)^{\frac{s}{2}} T_t f\|_{L_x^p} \leq \frac{C}{|t|^s} (r_0 \|\tilde{\rho}\|_{L^\infty})^{\frac{1}{r}} (\|\tilde{\rho}\|_{L^1} + \|(-\Delta)^{\frac{sr'}{2}} \tilde{\rho}\|_{L^1})^{1-\frac{1}{r}} \|f\|_{L_x^p L_v^r} \tag{4-5}$$

for all $t \neq 0$ such that $|t| \leq 1$, where $C > 0$ only depends on p, s and φ.

Proof. This estimate follows straightforwardly from Proposition 4.1 through a duality argument.

Indeed, using Proposition 4.1, we find, since T_t^* commutes with differentiation in x, that

$$\begin{aligned}\left|\int_{\mathbb{R}}(1-\Delta_x)^{\frac{s}{2}}T_t f(x)g(x)\,dx\right| &= \left|\int_{\mathbb{R}\times\mathbb{R}} f(x,v)(1-\Delta_x)^{\frac{s}{2}}T_t^* g(x,v)\,dx\,dv\right| \\ &\le \|f\|_{L_x^p L_v^r}\|(1-\Delta_x)^{\frac{s}{2}}T_t^* g\|_{L_x^{p'}L_v^{r'}} \\ &\le \frac{C}{|t|^s}(r_0\|\tilde\rho\|_{L^\infty})^{\frac{1}{r}}(\|\tilde\rho\|_{L^1}+\|(-\Delta)^{\frac{sr'}{2}}\tilde\rho\|_{L^1})^{1-\frac{1}{r}}\|f\|_{L_x^pL_v^r}\|g\|_{L_x^{p'}},\end{aligned}$$

where $C>0$ only depends on p, s and φ. Finally, taking the supremum over all $g\in L_x^{p'}$ easily concludes the proof of the proposition. □

We proceed now to the main theorem of this section.

Theorem 4.3. *In dimension $n=1$, let $1<p<\infty$ and $1<r\le\infty$.*
Then, for any $f,g\in L_x^p(\mathbb{R};L_v^r(\mathbb{R}))$ such that (1-2) *holds true, one has*

$$\int_{\mathbb{R}} f(x,v)\,\varphi(v)\,dv\in W_x^{s,p}(\mathbb{R})$$

for any $\varphi\in L_c^\infty(\mathbb{R})$ and any $0\le s<1-\frac{1}{r}$. Furthermore, one has the estimate

$$\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}} f\varphi\,dv\right\|_{L_x^p}\le C(\|f\|_{L_x^pL_v^r}+\|g\|_{L_x^pL_v^r}),$$

where $C>0$ only depends on φ and constant parameters.

Proof. We consider first the refined interpolation formula (2-1), which is valid for solutions of the transport equation (1-2) for some given cutoff $\rho\in\mathcal{S}(\mathbb{R})$. Clearly, further differentiating (2-1) in x and then averaging in v yields

$$\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}} f\varphi\,dv\right\|_{L_x^p}\le\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}} A_t f\varphi\,dv\right\|_{L_x^p}+t\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}} B_t g\varphi\,dv\right\|_{L_x^p}. \tag{4-6}$$

We wish now to apply Proposition 4.2 to the preceding estimate. To this end, note that $\tilde\rho$ and all of its derivatives clearly are bounded pointwise and integrable. In order to apply that result, we also further need to ask that $\tilde\rho$ be compactly supported, which is always possible.

Next, in view of (2-3), notice that $\tilde\tau$ also is bounded pointwise, integrable and compactly supported. Therefore, there only remains to check that $(-\Delta)^{\frac{\alpha}{2}}\tilde\tau$ is integrable for any $0<\alpha<1$. This, in fact, easily follows from a direct application of the technical Lemma B.2 from Appendix B to

$$\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}}\tilde\tau]=|r|^\alpha\frac{1-\rho(r)}{ir}.$$

All constants involving norms of the cutoff functions ρ and τ in the right-hand side of (4-5) are therefore finite.

Thus, applying Proposition 4.2 to estimate (4-6), we conclude, for any $0<t<1$, that

$$\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}} f\varphi\,dv\right\|_{L_x^p}\le C\left(\frac{1}{t^s}\|f\|_{L_x^pL_v^r}+t^{1-s}\|g\|_{L_x^pL_v^r}\right),$$

where $C>0$ only depends on constant parameters. □

5. The two-dimensional case

Our study of the one-dimensional case in the previous section showed that it is possible to largely improve the classical velocity averaging results in that setting. In particular, we showed therein that the gain of regularity of velocity averages is, in some cases, substantially improved beyond the value $\frac{1}{2}$.

While such a general improvement is not achievable in higher dimensions ($n \geq 2$), in view of the counterexamples from [DeVore and Petrova 2001, Theorem 1.3] discussed in our Introduction, it is nevertheless possible, as shown below, to obtain new cases displaying a gain of regularity of velocity averages of almost half a derivative.

In two dimensions ($n = 2$), this was already strongly suggested in [Jabin and Vega 2004, Theorem 1.2]. Here, we build upon the work from that paper to obtain refined two-dimensional velocity averaging results displaying an almost maximal gain of regularity of half a derivative. In the next section, we will generalize these methods to higher dimensions ($n \geq 3$), without achieving a gain of half a derivative, though.

We define now, in any dimension $n \geq 1$, the velocity averaging operator on the sphere

$$S_t f(x) = \int_{\mathbb{S}^{n-1}} A_t f(x,v)\, dv = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} f(x - stv, v)\, \tilde{\rho}(s)\, ds\, dv,$$

and its adjoint operator

$$S_t^* g(x,v) = A_{-t}(g(x))(x,v) = \int_{\mathbb{R}} g(x + stv)\, \tilde{\rho}(s)\, ds,$$

so that

$$\int_{\mathbb{R}^n} S_t f(x) g(x)\, dx = \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} f(x,v) S_t^* g(x,v)\, dx\, dv.$$

We will also consider the operators S_t and S_t^* defined with B_t instead of A_t. These operators correspond to the kinetic transport equation (1-2) with velocities restricted to the sphere $v \in \mathbb{S}^{n-1}$ and are introduced here for mere convenience and simplicity of analysis later on.

This reduction to the sphere is possible here because the regularization phenomenon in the transport equation (1-2) comes from averaging in velocity directions rather than integration along velocity magnitudes. In fact, any bound established on S_t and S_t^* will yield a corresponding bound on T_t and T_t^*, respectively, as shown below in Proposition 5.3 (we also refer the reader to Appendix C for a discussion of the equivalence of velocity averaging lemmas with velocities in the full Euclidean space $\mathbb{R}^n$ and on the sphere $\mathbb{S}^{n-1}$).

Note that this is not true in general. For instance, in the two-dimensional time-dependent setting (1-1) with velocities restricted to the sphere $\mathbb{S}^1$ and $f, g \in L^2_{x,v}$, it was identified in [Bournaveas and Perthame 2001] that only a quarter of a derivative could be gained on the velocity average of f, whereas half a derivative is gained when velocities range in an open subset of $\mathbb{R}^2$ (which corresponds to a three-dimensional setting $(t,x) \in \mathbb{R}^{1+2}$ with velocities restricted to a manifold of dimension 2, much like the stationary case $x \in \mathbb{R}^3$ with $v \in \mathbb{S}^2$).

For completeness, we begin our analysis of the operators S_t and S_t^* by establishing their smoothing effect in L^2 employing the classical Hilbertian methods of velocity averaging from [Golse et al. 1988]. This result is valid in any dimension $n \geq 2$ and will also be used in the next section on higher-dimensional results.

Proposition 5.1. *In any dimension $n \geq 2$, the operator S_t and its adjoint S_t^* satisfy the estimates, for any $0 \leq s \leq \frac{1}{2}$,*

$$\|(1-\Delta_x)^{\frac{s}{2}} S_t f\|_{L^2_x} \leq C\left(\|\rho\|_{L^\infty} + \frac{1}{|t|^s}\left(\left\|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2} + \big\||r|^s\rho(r)\big\|_{L^\infty}\right)\right)\|f\|_{L^2_{x,v}},$$

$$\|(1-\Delta_x)^{\frac{s}{2}} S_t^* g\|_{L^2_{x,v}} \leq C\left(\|\rho\|_{L^\infty} + \frac{1}{|t|^s}\left(\left\|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2} + \big\||r|^s\rho(r)\big\|_{L^\infty}\right)\right)\|g\|_{L^2_x}$$

for all $t \neq 0$, where $C > 0$ only depends on the dimension.

Proof. This proof is almost identical to the general case of Proposition 3.1. Nevertheless, for later applications of this result, it is important to carefully keep track of the dependence of the constants on t and ρ.

We deal with the estimate on the adjoint operator S_t^* first. Thus, it is readily seen, by Plancherel's theorem, that

$$\|S_t^* g\|_{L^2_{x,v}} = \frac{1}{(2\pi)^{\frac{n}{2}}}\|\rho(-t\eta\cdot v)\hat{g}(\eta)\|_{L^2_{\eta,v}} \leq \|\rho\|_{L^\infty}\|g\|_{L^2_x}. \tag{5-1}$$

Furthermore, using again Plancherel's theorem, we find that

$$\begin{aligned}
\|(-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^2_{x,v}} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\big\||\eta|^s\rho(-t\eta\cdot v)\hat{g}(\eta)\big\|_{L^2_{\eta,v}} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{|\mathbb{S}^{n-2}|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}}\left\|\left(|\eta|^{2s-1}\int_{-|t\eta|}^{|t\eta|}|\rho(r)|^2\left(1-\left(\frac{r}{|t\eta|}\right)^2\right)^{\frac{n-3}{2}}dr\right)^{\frac{1}{2}}\hat{g}(\eta)\right\|_{L^2_\eta} \\
&\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{|\mathbb{S}^{n-2}|^{\frac{1}{2}}}{|t|^s}\left\|\left(\int_{-|t\eta|}^{|t\eta|}\left|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right|^2\left(1-\left(\frac{r}{|t\eta|}\right)^2\right)^{\frac{n-3}{2}}dr\right)^{\frac{1}{2}}\hat{g}(\eta)\right\|_{L^2_\eta},
\end{aligned}$$

with the convention that $|\mathbb{S}^0| = 2$ when $n = 2$.

Hence, if $n \geq 3$, we easily deduce that

$$\|(-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^2_{x,v}} \leq \frac{|\mathbb{S}^{n-2}|^{\frac{1}{2}}}{|t|^s}\left\|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2}\|g(x)\|_{L^2_x}.$$

In the two-dimensional case, the bound on the cutoff ρ is only slightly more involved. We estimate, in this case, for any $N > 0$, that

$$\begin{aligned}
\int_{-N}^{N}\left|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right|^2\frac{1}{\left(1-\left(\frac{r}{N}\right)^2\right)^{\frac{1}{2}}}dr &\leq \frac{2}{\sqrt{3}}\left\|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2}^2 + \int_{\frac{N}{2}}^{N}\left(\left|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right|^2 + \left|\frac{\rho(-r)}{|r|^{\frac{1}{2}-s}}\right|^2\right)\frac{1}{\left(1-\left(\frac{r}{N}\right)^2\right)^{\frac{1}{2}}}dr \\
&\leq \frac{2}{\sqrt{3}}\left\|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2}^2 + \frac{2}{3}\pi N\sup_{\frac{N}{2}\leq|r|\leq N}\left|\frac{\rho(r)}{|r|^{\frac{1}{2}-s}}\right|^2.
\end{aligned}$$

It therefore follows that, in any dimension $n \geq 2$,

$$\|(-\Delta_x)^{\frac{s}{2}} S_t^* g\|_{L^2_{x,v}} \leq \frac{C}{|t|^s} \left(\left\| \frac{\rho(r)}{|r|^{\frac{1}{2}-s}} \right\|_{L^2} + \big\| |r|^s \rho(r) \big\|_{L^\infty} \right) \|g\|_{L^2_x}. \tag{5-2}$$

Finally, combining estimates (5-1) and (5-2) establishes the estimate on S_t^*.

The estimate on S_t is then easily deduced from the estimate on S_t^* by a duality argument. □

At this stage, we need to further introduce a classical Littlewood–Paley decomposition, which will be used in our proofs. To this end, let $\psi_0(\eta), \psi(\eta) \in C_c^\infty(\mathbb{R}^n)$ be compactly supported smooth cutoff functions, whose supports satisfy

$$\operatorname{supp} \psi_0 \subset \{|\eta| \leq 1\} \quad \text{and} \quad \operatorname{supp} \psi \subset \left\{\tfrac{1}{2} \leq |\eta| \leq 2\right\},$$

and such that

$$\psi_0(\eta) + \sum_{k=0}^\infty \psi\left(\frac{\eta}{2^k}\right) = 1 \quad \text{for all } \eta \in \mathbb{R}^n.$$

For any tempered distribution $f(x) \in \mathcal{S}'(\mathbb{R}^n)$, we define the dyadic blocks $\Delta_0 f(x), \Delta_{2^k} f(x) \in \mathcal{S}(\mathbb{R}^n)$, for each $k \in \mathbb{Z}$, by

$$\Delta_0 f = \mathcal{F}^{-1}\psi_0(\eta)\mathcal{F} f \quad \text{and} \quad \Delta_{2^k} f = \mathcal{F}^{-1}\psi\left(\frac{\eta}{2^k}\right)\mathcal{F} f.$$

so that

$$f = \Delta_0 f + \sum_{k=0}^\infty \Delta_{2^k} f \quad \text{in } \mathcal{S}'(\mathbb{R}^n). \tag{5-3}$$

As in Section 3, from now on, we assume that the cutoff function $\rho(r)$ may be decomposed as a product $\rho(r) = \rho_1(r)\rho_2(r)$, so that $\tilde{\rho}(s) = \tilde{\rho}_1 * \tilde{\rho}_2(s)$. We recall from (3-3) that

$$A_t = A_t^1 A_t^2 = A_t^2 A_t^1, \quad S_t = S_t^1 A_t^2 = S_t^2 A_t^1, \quad S_t^* = A_{-t}^2 S_t^{1*} = A_{-t}^1 S_t^{2*}, \tag{5-4}$$

where A_t^i, S_t^i and S_t^{i*}, with $i = 1, 2$, denote the respective operators A_t, S_t and S_t^* with the cutoff ρ replaced by ρ_i.

As shown in the results below, a key idea here is to use this trick to gain integration in one dimension along v through the straightforward estimate

$$\begin{aligned} |S_t^* g(x,v)|^2 &= |A_{-t}^1 S_t^{2*} g(x,v)|^2 \\ &\leq \|\tilde{\rho}_1\|_{L^1} \int_{\mathbb{R}} |S_t^{2*} g(x + stv, v)|^2 \, |\tilde{\rho}_1(s)| \, ds \\ &\leq \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{[-r_1, r_1]} |S_t^{2*} g(x + stv, v)|^2 \, ds, \end{aligned} \tag{5-5}$$

where $\operatorname{supp} \tilde{\rho}_1 \subset [-r_1, r_1]$, for some $r_1 > 0$, and thus obtain new estimates on the adjoint operator S_t^*.

The next proposition contains an estimate which is central to the present two-dimensional setting. It strongly relies on the clever and elegant proof of Theorem 1.2 from [Jabin and Vega 2004], which it crucially improves by exploiting the structure of the operator A_t through the use of (5-5).

Proposition 5.2. *In dimension $n=2$, let $2 \le p \le 4$, $0 \le s < s_0 < \frac{1}{2}$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ have their supports contained inside balls of radii $r_1, r_2 > 0$, respectively, centered at the origin.*

Then, the operator S_t^* satisfies the estimate

$$\|(1-\Delta_x)^{\frac{s}{2}} S_t^* g\|_{L_x^p L_v^2} \le \frac{C}{|t|^{\frac{1}{2}}} \|\tilde{\rho}_1\|_{L^1}^{\frac{1}{2}} \|\tilde{\rho}_1\|_{L^\infty}^{\frac{1}{2}} (r_1+r_2)^{\frac{1}{4}} (1+r_1+r_2)^{\frac{1}{2}} \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde{\rho}_2\|_{L^1} \right) \|g\|_{L_x^p}$$

for all $t \neq 0$ such that $|t| \le 1$, where $C > 0$ only depends on fixed parameters.

Proof. First, notice that, for any $2 \le p \le \infty$,

$$\|S_t^* g\|_{L_x^p L_v^2} \le \|S_t^* g\|_{L_v^2 L_x^p} \le (2\pi)^{\frac{1}{2}} \|\tilde{\rho}\|_{L^1} \|g\|_{L_x^p}. \tag{5-6}$$

As for the regularity estimate, we employ the bound (5-5) and the standard Littlewood–Paley dyadic frequency decomposition previously introduced, to deduce, writing $g_k = \Delta_{2^k} g$ for convenience, for any $k \ge 0$, that

$$\begin{aligned}
\|S_t^* g_k\|_{L_x^4 L_v^2}^4 &= \int_{\mathbb{R}^2 \times \mathbb{S}^1} |S_t^* g_k(x, v_2)|^2 \left(\int_{\mathbb{S}^1} |S_t^* g_k(x, v_1)|^2 \, dv_1 \right) dx \, dv_2 \\
&\le \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{\mathbb{R}^2 \times \mathbb{S}^1} \left(\int_{[-r_1, r_1]} |S_t^{2*} g_k(x + s_2 t v_2, v_2)|^2 \, ds_2 \right) \\
&\qquad \times \left(\int_{\mathbb{S}^1} |S_t^* g_k(x, v_1)|^2 \, dv_1 \right) dx \, dv_2 \\
&= \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{\mathbb{R}^2 \times \mathbb{S}^1} |S_t^{2*} g_k(x, v_2)|^2 \\
&\qquad \times \left(\int_{[-r_1, r_1]} \int_{\mathbb{S}^1} |S_t^* g_k(x + s_2 t v_2, v_1)|^2 \, dv_1 \, ds_2 \right) dx \, dv_2 \\
&\le \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 \int_{\mathbb{R}^2 \times \mathbb{S}^1} |S_t^{2*} g_k(x, v_2)|^2 \\
&\qquad \times \left(\int_{[-r_1, r_1]^2} \int_{\mathbb{S}^1} |S_t^{2*} g_k(x + s_1 t v_1 + s_2 t v_2, v_1)|^2 \, dv_1 \, ds_1 \, ds_2 \right) dx \, dv_2 \\
&\le \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 \|S_t^{2*} g_k\|_{L_{x,v}^2}^2 \sup_{\substack{x \in \mathbb{R}^2 \\ v_2 \in \mathbb{S}^1}} I(x, v_2),
\end{aligned}$$

where

$$I(x, v_2) = \int_{[-r_1, r_1]^2} \int_{\mathbb{S}^1} |S_t^{2*} g_k(x + s_1 t v_1 + s_2 t v_2, v_1)|^2 \, dv_1 \, ds_1 \, ds_2.$$

Further using Proposition 5.1, we deduce that, for all $t > 0$ and any given $0 < s < \frac{1}{2}$,

$$\|S_t^* g_k\|_{L_x^4 L_v^2}^4 \le \frac{C}{|t|^{2s} 2^{k2s}} \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s}} \right\|_{L^2}^2 + \big\| |r|^s \rho_2(r) \big\|_{L^\infty}^2 \right) \|g\|_{L_x^2}^2 \sup_{\substack{x \in \mathbb{R}^2 \\ v_2 \in \mathbb{S}^1}} I(x, v_2). \tag{5-7}$$

We claim now that

$$\sup_{\substack{x\in\mathbb{R}^2\\ v_2\in\mathbb{S}^1}} I(x,v_2) \leq \frac{C(k+1)}{|t|2^{k2s}}(r_1+r_2)(1+r_1+r_2)^2\left(\left\|\frac{\rho_2(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2}^2+\|(1+|r|^s)\rho_2(r)\|_{L^\infty}^2+\|\tilde\rho_2\|_{L^1}^2\right)\|g\|_{L^\infty_x}^2, \tag{5-8}$$

where $C>0$ only depends on fixed parameters. In order to establish (5-8), we employ the change of variables $(s_1,s_2)\mapsto z=s_1tv_1+s_2tv_2$ whenever v_1 and v_2 form a basis, which holds almost everywhere. It is readily seen that the Jacobian determinant of this transformation is given by $t^2\sin\theta$, where $\theta\in[0,\pi]$ is the angle between v_1 and v_2 defined by $\cos\theta=v_1\cdot v_2$. Thus, noticing that $|z|=|s_1tv_1+s_2tv_2|\leq 2r_1|t|$, we infer

$$\begin{aligned} I(x,v_2) &= \int_{[-r_1,r_1]^2}\int_{\mathbb{S}^1}|S_t^{2*}g_k(x+s_1tv_1+s_2tv_2,v_1)|^2\,dv_1\,ds_1\,ds_2\\ &\leq \int_{\mathbb{R}^2\times\mathbb{S}^1}|S_t^{2*}g_k(x+z,v_1)|^2\frac{1}{t^2\sin\theta}\mathbb{1}_{\{|z|\leq 2r_1|t|,|z\cdot v_2^\perp|\leq r_1|t|\sin\theta\}}\,dz\,dv_1\\ &= \sum_{i=0}^\infty\int_{\mathbb{R}^2\times S_i}\left|\int_{\mathbb{R}}g_k(x+z+stv_1)\,\tilde\rho_2(s)\,ds\right|^2\frac{1}{t^2\sin\theta}\mathbb{1}_{\{|z|\leq 2r_1|t|,|z\cdot v_2^\perp|\leq r_1|t|\sin\theta\}}\,dz\,dv_1, \end{aligned} \tag{5-9}$$

where we have used the notation

$$v^\perp=\begin{pmatrix}v_1\\v_2\end{pmatrix}^\perp=\begin{pmatrix}-v_2\\v_1\end{pmatrix}$$

for any $v\in\mathbb{S}^1$, and have decomposed the domain of integration $v_1\in\mathbb{S}^1$ into

$$\mathbb{S}^1\setminus\{\sin\theta=0\}=\bigcup_{i=0}^\infty S_i,$$

with

$$S_i=\left\{v_1\in\mathbb{S}^1:\frac{1}{2^{i+1}}<|\sin\theta|\leq\frac{1}{2^i}\right\}.$$

Recall now that $\tilde\rho_2$ is supported inside a ball of radius $r_2>0$ centered at the origin. Therefore, we find that, in the last integrand above,

$$|(z+stv_1)\cdot v_2^\perp|\leq|z\cdot v_2^\perp|+|stv_1\cdot v_2^\perp|\leq(r_1+r_2)|t|\sin\theta\leq\frac{(r_1+r_2)|t|}{2^i},$$

and

$$|z+stv_1|\leq(2r_1+r_2)|t|.$$

Hence, considering a smooth cutoff function $\chi\in C_c^\infty(\mathbb{R})$ such that $\mathbb{1}_{\{|s|\leq 1\}}\leq\chi(s)\leq\mathbb{1}_{\{|s|\leq 2\}}$, setting for convenience $r_0=2(r_1+r_2)|t|$, and defining, for each given $x\in\mathbb{R}^2$, $v_2\in\mathbb{S}^1$ and $i\in\mathbb{N}$,

$$K_{x,v_2}^{i,k}(z)=g_k(x+z)\chi\left(\frac{|z\cdot v_2|}{r_0}\right)\chi\left(2^i\frac{|z\cdot v_2^\perp|}{r_0}\right),$$

we deduce from (5-9) that, using Proposition 5.1,

$$
\begin{aligned}
I(x,v_2) &\le \sum_{i=0}^{\infty} \int_{\mathbb{R}^2\times S_i} \left| \int_{\mathbb{R}} K^{i,k}_{x,v_2}(z+stv_1)\,\tilde\rho_2(s)\,ds \right|^2 \frac{1}{t^2\sin\theta}\,dz\,dv_1 \\
&\le \sum_{2^i\le r_0 2^{k2s}} \frac{2^{i+1}}{t^2} \int_{\mathbb{R}^2\times\mathbb{S}^1} |S_t^{2*}K^{i,k}_{x,v_2}(z,v_1)|^2\,dz\,dv_1 \\
&\qquad\qquad + \sum_{2^i> r_0 2^{k2s}} \frac{2^{i+1}}{t^2} \int_{\mathbb{R}^2\times S_i} |S_t^{2*}K^{i,k}_{x,v_2}(z,v_1)|^2\,dz\,dv_1 \\
&\le \sum_{2^i\le r_0 2^{k2s}} \frac{2^{i+1}}{t^2} \|S_t^{2*}K^{i,k}_{x,v_2}\|^2_{L^2_{z,v_1}} + \sum_{2^i> r_0 2^{k2s}} \frac{2^{i+1}}{t^2} |S_i|\,\|\tilde\rho_2\|^2_{L^1}\,\|K^{i,k}_{x,v_2}\|^2_{L^2_z} \\
&\le \frac{C}{|t|^{2s}} \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12-s}} \right\|^2_{L^2} + \|(1+|r|^s)\rho_2(r)\|^2_{L^\infty} \right) \sum_{2^i\le r_0 2^{k2s}} \frac{2^i}{t^2} \|(1-\Delta_z)^{-\frac{s}{2}} K^{i,k}_{x,v_2}\|^2_{L^2_z} \\
&\qquad\qquad + C \sum_{2^i> r_0 2^{k2s}} \frac{r_0^2}{t^2 2^i} \|\tilde\rho_2\|^2_{L^1}\,\|g_k\|^2_{L^\infty_x} \\
&\le \frac{C}{|t|^{2s}} \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12-s}} \right\|^2_{L^2} + \|(1+|r|^s)\rho_2(r)\|^2_{L^\infty} \right) \sum_{2^i\le r_0 2^{k2s}} \frac{2^i}{t^2} \|(1-\Delta_z)^{-\frac{s}{2}} K^{i,k}_{x,v_2}\|^2_{L^2_z} \\
&\qquad\qquad + C \frac{r_0}{t^2 2^{k2s}} \|\tilde\rho_2\|^2_{L^1}\,\|g\|^2_{L^\infty_x},
\end{aligned}
$$

where $C>0$ is an independent constant.

Next, a direct application of the technical Lemma A.1 from Appendix A to the preceding estimate yields

$$
\begin{aligned}
I(x,v_2) &\le \frac{C}{|t|^{2s}} \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12-s}} \right\|^2_{L^2} + \|(1+|r|^s)\rho_2(r)\|^2_{L^\infty} \right) \sum_{2^i\le r_0 2^{k2s}} \frac{r_0^2}{t^2 2^{k2s}} \|g\|^2_{L^\infty_x} + C \frac{r_0}{t^2 2^{k2s}} \|\tilde\rho_2\|^2_{L^1}\,\|g\|^2_{L^\infty_x} \\
&\le C \frac{(k+1)(r_1+r_2)^2 \log(2+r_1+r_2)}{|t|^{2s} 2^{k2s}} \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12-s}} \right\|^2_{L^2} + \|(1+|r|^s)\rho_2(r)\|^2_{L^\infty} \right) \|g\|^2_{L^\infty_x} + C \frac{r_1+r_2}{|t| 2^{k2s}} \|\tilde\rho_2\|^2_{L^1}\,\|g\|^2_{L^\infty_x},
\end{aligned}
$$

which establishes our claim (5-8).

Finally, combining estimates (5-7) and (5-8), we arrive at

$$
\begin{aligned}
\|S_t^* g_k\|^2_{L^4_x L^2_v} &\le \frac{C(k+1)^{\frac12}}{|t| 2^{k2s}} \|\tilde\rho_1\|_{L^1}\,\|\tilde\rho_1\|_{L^\infty} (r_1+r_2)^{\frac12} (1+r_1+r_2) \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12-s}} \right\|^2_{L^2} + \|(1+|r|^s)\rho_2(r)\|^2_{L^\infty} + \|\tilde\rho_2\|^2_{L^1} \right) \|g\|_{L^2_x}\,\|g\|_{L^\infty_x},
\end{aligned}
$$

where $C>0$ is an independent constant.

In order to conclude, we write $|g(x)| = \int_0^\infty \mathbb{1}_{\{|g(x)|\geq s\}}\,ds$ to deduce from the preceding estimate, assuming g is nonnegative, that

$$\begin{aligned}
\|S_t^* g_k\|_{L^4_x L^2_v} &\leq \int_0^\infty \|S_t^* \Delta_{2^k} \mathbb{1}_{\{|g(x)|\geq s\}}\|_{L^4_x L^2_v}\,ds \\
&\leq \frac{C(k+1)^{\frac14}}{|t|^{\frac12} 2^{ks}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac14}(1+r_1+r_2)^{\frac12} \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \int_0^\infty |\{|g(x)|\geq s\}|^{\frac14}\,ds \\
&\leq \frac{C(k+1)^{\frac14}}{|t|^{\frac12} 2^{ks}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac14}(1+r_1+r_2)^{\frac12} \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^{4,1}_x},
\end{aligned}$$

where $L^{4,1}_x$ denotes a standard Lorentz space; see [Bergh and Löfström 1976, Section 1.3] or [Grafakos 2008, Section 1.4] for definitions and properties of Lorentz spaces. When, g is signed, we arrive at the same estimate simply by decomposing $g = g^+ - g^-$ into its positive and negative parts, treating each contribution separately, and then noticing that

$$\begin{aligned}
\|g^+\|_{L^{4,1}_x} + \|g^-\|_{L^{4,1}_x} &\leq C \int_0^\infty \big|\{|g^+(x)|\geq s\}\big|^{\frac14} + \big|\{|g^-(x)|\geq s\}\big|^{\frac14}\,ds \\
&\leq C \int_0^\infty \left(\big|\{|g^+(x)|\geq s\}\big| + \big|\{|g^-(x)|\geq s\}\big| \right)^{\frac14} ds \\
&\leq C \int_0^\infty \big|\{|g(x)|\geq s\}\big|^{\frac14}\,ds \leq C\|g\|_{L^{4,1}_x}.
\end{aligned}$$

Moreover, by allowing an arbitrarily small loss of regularity, that is, by replacing $0 < s < \frac12$ by a slightly smaller value, it is possible to replace the Lorentz space $L^{4,1}_x$ by the standard Lebesgue space L^4_x in the right-hand side of the above estimate.

Therefore, on the whole, for any $0 \leq s < s_0 < \frac12$, we have established the estimate

$$\begin{aligned}
\|S_t^* g_k\|_{L^4_x L^2_v} &\leq \frac{C}{|t|^{\frac12} 2^{ks}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac14}(1+r_1+r_2)^{\frac12} \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^4_x},
\end{aligned}$$

where $C > 0$ only depends on constant parameters, which, when combined with the easy bound (5-6) for low frequencies, yields

$$\begin{aligned}
\|(1-\Delta_x)^{\frac{s}{2}} S_t^* g\|_{L^4_x L^2_v} &\leq \frac{C}{|t|^{\frac12}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac14}(1+r_1+r_2)^{\frac12} \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^4_x}. \qquad (5\text{-}10)
\end{aligned}$$

Finally, recalling from complex interpolation theory, see [Bergh and Löfström 1976, Section 5.1], that, for any $2 < p < 4$,

$$(L^2_x L^2_v, L^4_x L^2_v)_{[2-\frac{4}{p}]} = L^p_x L^2_v \quad \text{and} \quad (L^2_x, L^4_x)_{[2-\frac{4}{p}]} = L^p_x,$$

we conclude the proof of the proposition by interpolating the estimate (5-10) with the classical estimate on S_t^* from Proposition 5.1. □

Next, we utilize the previous result on the adjoint operator S_t^* to deduce corresponding estimates on T_t and T_t^*.

Proposition 5.3. *In dimension $n = 2$, let $\frac{4}{3} \le r \le 2$, $2 \le p \le 4$, $0 \le s < s_0 < \frac{1}{2}$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde\rho_1$ and $\tilde\rho_2$ have their supports contained inside balls of radii $r_1, r_2 > 0$, respectively, centered at the origin.*

Then, the operators T_t and T_t^* satisfy the estimates

$$\begin{aligned}\|(1-\Delta_x)^{\frac{s}{2}} T_t f\|_{L^r_x} \le \frac{C}{|t|^{\frac{1}{2}}} \|\varphi\|_{L^\infty} R^{\frac{1}{2}} \|\tilde\rho_1\|_{L^1}^{\frac{1}{2}} \|\tilde\rho_1\|_{L^\infty}^{\frac{1}{2}} (r_1+r_2)^{\frac{1}{4}} (1+r_1+r_2)^{\frac{1}{2}} \\ \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|f\|_{L^r_x L^2_v} \end{aligned} \tag{5-11}$$

and

$$\begin{aligned}\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L^p_x L^2_v} \le \frac{C}{|t|^{\frac{1}{2}}} \|\varphi\|_{L^\infty} R^{\frac{1}{2}} \|\tilde\rho_1\|_{L^1}^{\frac{1}{2}} \|\tilde\rho_1\|_{L^\infty}^{\frac{1}{2}} (r_1+r_2)^{\frac{1}{4}} (1+r_1+r_2)^{\frac{1}{2}} \\ \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^p_x} \end{aligned}$$

for all $t \ne 0$ such that $|t| \le 1$, where $C > 0$ only depends on fixed parameters.

Proof. It is readily seen that

$$T_t^* g(x,v) = \int_{\mathbb{R}} g(x+stv)\tilde\rho(s)\,ds\,\varphi(v) = \varphi(v) S^*_{|v|t} g\left(x, \frac{v}{|v|}\right).$$

Therefore, for any $2 \le p \le \infty$, we compute in polar coordinates, recalling that the support of the velocity weight $\varphi \in L^\infty_c(\mathbb{R}^2)$ is contained inside a closed ball of radius $R > 0$ centered at the origin,

$$\begin{aligned}\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g(x,v)\|_{L^p_x(\mathbb{R}^2; L^2_v(\mathbb{R}^2))} &= \left\| \left(\int_{\mathbb{R}^2} \left|(1-\Delta_x)^{\frac{s}{2}} T_t^* g(x,v)\right|^2 dv \right)^{\frac{1}{2}} \right\|_{L^p_x} \\ &= \left\| \int_0^\infty r \int_{\mathbb{S}^1} \left|\varphi(rv)(1-\Delta_x)^{\frac{s}{2}} S^*_{rt} g(x,v)\right|^2 dv\,dr \right\|_{L^{p/2}_x}^{\frac{1}{2}} \\ &\le \|\varphi\|_{L^\infty} \left(\int_0^R r \left\| \int_{\mathbb{S}^1} \left|(1-\Delta_x)^{\frac{s}{2}} S^*_{rt} g(x,v)\right|^2 dv \right\|_{L^{p/2}_x} dr \right)^{\frac{1}{2}} \\ &\le \|\varphi\|_{L^\infty} \left(\int_0^R r \|(1-\Delta_x)^{\frac{s}{2}} S^*_{rt} g(x,v)\|^2_{L^p_x(\mathbb{R}^2; L^2_v(\mathbb{S}^1))} dr \right)^{\frac{1}{2}}. \end{aligned}$$

Then, combining Proposition 5.2 with the above estimate, we find

$$\begin{aligned}\|(1-\Delta_x)^{\frac{s}{2}} T_t^* g\|_{L_x^p L_v^2} &\le \|\varphi\|_{L^\infty}\left(\int_0^R r\|(1-\Delta_x)^{\frac{s}{2}} S_{rt}^* g\|_{L_x^p L_v^2}^2\, dr\right)^{\frac{1}{2}} \\ &\le \frac{C}{|t|^{\frac{1}{2}}}\|\varphi\|_{L^\infty} R^{\frac{1}{2}}\|\tilde\rho_1\|_{L^1}^{\frac{1}{2}}\|\tilde\rho_1\|_{L^\infty}^{\frac{1}{2}}(r_1+r_2)^{\frac{1}{4}}(1+r_1+r_2)^{\frac{1}{2}} \\ &\qquad \times\left(\left\|\frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}}\right\|_{L^2}+\|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty}+\|\tilde\rho_2\|_{L^1}\right)\|g\|_{L_x^p},\end{aligned}$$

where $C>0$ is an independent constant, which establishes the estimate on T_t^*.

The estimate on T_t is then easily deduced from the estimate on T_t^* by a duality argument. □

We proceed now to the main theorem of this section. Note that an equivalent version of this result with spherical averages and an identical regularity gain can be readily obtained by applying the methods from Appendix C.

Theorem 5.4. *In dimension $n=2$, let $\frac{4}{3}\le r\le 2$.*

Then, for any $f, g\in L_x^r(\mathbb{R}^2; L_v^2(\mathbb{R}^2))$ such that (1-2) *holds true, one has*

$$\int_{\mathbb{R}^2} f(x,v)\,\varphi(v)\,dv\in W_x^{s,r}(\mathbb{R}^2)$$

for any $\varphi\in L_c^\infty(\mathbb{R}^2)$ and any $0\le s<\frac{1}{2}$. Furthermore, one has the estimate

$$\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}^2} f\varphi\,dv\right\|_{L_x^r}\le C(\|f\|_{L_x^r L_v^2}+\|g\|_{L_x^r L_v^2}),$$

where $C>0$ only depends on φ and constant parameters.

Proof. We consider first the refined interpolation formula (2-1), which is valid for solutions of the transport equation (1-2), for some given cutoff $\rho\in\mathcal{S}(\mathbb{R})$. Clearly, further differentiating (2-1) in x and then averaging in v yields

$$\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}^2} f\varphi\,dv\right\|_{L_x^r}\le\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}^2} A_t f\varphi\,dv\right\|_{L_x^r}+t\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}^2} B_t g\varphi\,dv\right\|_{L_x^r}. \quad (5\text{-}12)$$

We wish now to apply Proposition 5.3 to the preceding estimate. To this end, according to (5-4), we take the decompositions

$$\rho(r)=\rho_1(r)\rho_2(r)\quad\text{and}\quad\tau(r)=\frac{1-\rho(r)}{ir}=\tau_1(r)\tau_2(r),$$

where

$$\tilde\rho_1(r)\in C_c^\infty(\mathbb{R}),\quad \tau_1(r)=\frac{1}{(1+r^2)^{\frac{1}{4}}},$$
$$\tilde\rho_2(r)\in C_c^\infty(\mathbb{R}),\quad \tau_2(r)=(1+r^2)^{\frac{1}{4}}\tau(r).$$

Clearly, all constants involving norms of the cutoff functions ρ_1 and ρ_2 in the right-hand side of (5-11) are finite and we may therefore straightforwardly apply Proposition 5.3 to control the first term in the right-hand side of (5-12). However, the same is not so obviously true concerning the cutoff functions τ_1 and τ_2. The application of Proposition 5.3 to the second term in the right-hand side of (5-12) will therefore require some substantial technical work, which we present now.

To this end, we employ a homogeneous Littlewood–Paley frequency decomposition, see (5-3), of τ_1 and τ_2 to write

$$\tau = \Big(\sum_{j\in\mathbb{Z}} \Delta_{2^j}\tau_1\Big)\Big(\sum_{k\in\mathbb{Z}} \Delta_{2^k}\tau_2\Big) = \sum_{j\in\mathbb{Z}}(\Delta_{2^j}\tau_1)\tau_3^j + \sum_{j\in\mathbb{Z}}(\Delta_{2^j}\tau_2)\tau_4^j,$$

where

$$\tau_3^j = \sum_{\substack{k\in\mathbb{Z}\\ k\le j}} \Delta_{2^k}\tau_2 = \mathcal{F}^{-1}\Big[\psi_0\Big(\frac{r}{2^{j+1}}\Big)\Big] * \tau_2, \quad \tau_4^j = \sum_{\substack{k\in\mathbb{Z}\\ k< j}} \Delta_{2^k}\tau_1 = \mathcal{F}^{-1}\Big[\psi_0\Big(\frac{r}{2^{j}}\Big)\Big] * \tau_1.$$

In view of the linearity of the operator T_t with respect to the cutoffs ρ or τ, we only need to verify the finiteness of the constants in (5-11) with $\Delta_{2^j}\tau_1$ and τ_3^j playing the roles of ρ_1 and ρ_2, respectively, and then with $\Delta_{2^j}\tau_2$ and τ_4^j instead of ρ_1 and ρ_2, respectively. It is to be emphasized here that the ensuing bounds on the cutoffs will then depend on $j\in\mathbb{Z}$. In order to guarantee the boundedness of T_t, we will therefore need to make sure that our method eventually yields constants that are summable in $j\in\mathbb{Z}$.

We evaluate now the norms involved in the right-hand side of (5-11) where we replace ρ_1 by $\Delta_{2^j}\tau_1$ (or $\Delta_{2^j}\tau_2$) and ρ_2 by τ_3^j (or τ_4^j). The bounds on $\Delta_{2^j}\tau_2$ and τ_4^j are handled in a strictly similar manner and so we omit the corresponding details.

First, note that a direct application of Lemma B.3 from Appendix B together with the fact that τ_1 and τ_2 are smooth so that their Fourier transforms decay faster than any inverse power at infinity, shows that

$$\begin{aligned} |\widetilde{\Delta_{2^j}\tau_1}(r)| &\le \frac{C}{|r|^{\frac12}(1+|r|)^N}\mathbb{1}_{\{2^{j-1}\le|r|\le 2^{j+1}\}},\\ |\tilde{\tau}_3^j(r)| &\le \frac{C}{|r|^{\frac12}(1+|r|)^N}\mathbb{1}_{\{|r|\le 2^{j+1}\}} \end{aligned} \tag{5-13}$$

for any arbitrarily large $N>0$.

Furthermore, in view of Lemma B.4, it holds that each τ_3^j satisfies

$$|\tau_3^j(r)| \le \frac{C}{1+|r|^{\frac12}}$$

for some uniform $C>0$ independent of $j\in\mathbb{Z}$, whence, for any $0<s_0<\frac12$,

$$\frac{\tau_3^j(r)}{|r|^{\frac12-s_0}} \in L^2(\mathbb{R}) \quad\text{and}\quad (1+|r|^{s_0})\tau_3^j(r)\in L^\infty(\mathbb{R}), \tag{5-14}$$

uniformly in $j\in\mathbb{Z}$.

Therefore, using the bounds (5-13) and (5-14) to evaluate the terms involving $\rho_1=\Delta_{2^j}\tau_1$ and $\rho_2=\tau_3^j$ in the right-hand side of (5-11), we compute that the corresponding norm of the operator in (5-11) is no larger than a multiple of

$$\left(\frac{2^{\frac{j}{2}}}{1+2^{jN}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{\frac{j}{2}}(1+2^{jN})}\right)^{\frac{1}{2}}2^{\frac{j}{4}}(1+2^j)^{\frac{1}{2}} \le C\frac{2^{\frac{j}{4}}}{(1+2^j)^{N-\frac{1}{2}}},$$

which is summable over $j \in \mathbb{Z}$, provided $N > \frac{3}{4}$.

Thus, we conclude, according to Proposition 5.3, that the operators in the right-hand side of (5-12) are bounded.

It follows that, for any $0 < t < 1$,

$$\left\|(1-\Delta_x)^{\frac{s}{2}}\int_{\mathbb{R}^2} f\varphi\, dv\right\|_{L^r_x} \le C\left(\frac{1}{t^{\frac{1}{2}}}\|f\|_{L^r_x L^2_v} + t^{\frac{1}{2}}\|g\|_{L^r_x L^2_v}\right),$$

where $C > 0$ only depends on constant parameters. □

6. The higher-dimensional case

We move on now to the higher-dimensional case. More precisely, in the present section, we generalize the methods leading to Theorem 5.4 to establish an analog result valid in any dimension. Unfortunately, the ensuing result does not reach a maximal gain of regularity of half a derivative on the velocity averages, but only a gain of $\frac{n}{4(n-1)}$ derivatives, where $n \ge 3$ is the dimension. This drawback mainly stems from the fact that we work in the L^4_x setting, because our methods exploit the trivial fact that the exponent 4 is an even integer in order to control the square of some transport operator in L^2_x rather than the operator itself in L^4_x.

We begin with a few technical results. Loosely speaking, a key idea behind Proposition 5.2 consisted in noticing that $S_t^* g(x, v)$ is regular along the direction v and then using some duality argument in L^4_x to gain an integration variable in another nondegenerate direction. In higher dimensions, in order to carry out a similar strategy, we need to gain integration variables in $n-1$ nondegenerate directions. The next few lemmas will allow us to achieve such a dimensional build up of integration variables.

The following lemma generalizes estimate (5-8) from the proof of Proposition 5.2 and corresponds to a situation where we have already managed to build up the integration dimension all the way up to n (notice the n-dimensional integration in S in the estimate below).

Lemma 6.1. *In any dimension $n \ge 2$, let $0 < s < \frac{1}{2}$, $r_1 > 0$ and $\rho_2 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde{\rho}_2$ has its support contained inside a ball of radius $r_2 > 0$ centered at the origin.*

Then, denoting $S = (s_1, \dots, s_n) \in \mathbb{R}^n$, it holds that, for any $k \in \mathbb{N}$,

$$\sup_{\substack{x\in\mathbb{R}^n\\ v_n\in\mathbb{S}^{n-1}}} \int_{[-r_1,r_1]^n}\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}}\left|S_t^{2*}\Delta_{2^k} g\left(x+\sum_{j=1}^n s_j t v_j, v_1\right)\right|^2 dv_1\cdots dv_{n-1}\, dS$$
$$\le \frac{C(k+1)}{|t|2^{k2s}}(r_1+r_2)^{n-1}(1+r_1+r_2)^2\left(\left\|\frac{\rho_2(r)}{|r|^{\frac{1}{2}-s}}\right\|_{L^2}^2 + \|(1+|r|^s)\rho_2(r)\|_{L^\infty}^2 + \|\tilde{\rho}_2\|_{L^1}^2\right)\|g\|_{L^\infty_x}^2$$

for all $t \ne 0$ such that $|t| \le 1$, where $C > 0$ is independent of k, r_1, r_2 and ρ_2.

Proof. We assume that $\{v_2, \dots, v_n\}$ is a linearly independent set of vectors, which holds almost everywhere, and denote its span by H. Further let $u_1 \in \mathbb{R}^n$ be a unit vector orthogonal to H. The specific choice of unit vector is irrelevant, any such vector will do. Note that $\det(u_1, v_2, \dots, v_n) \neq 0$. Moreover, since $v_1 - (v_1 \cdot u_1)u_1$ belongs to H and recalling that the determinant is linear with respect to each of its column vectors, it holds that

$$\det(v_1, \dots, v_n) = v_1 \cdot u_1 \det(u_1, v_2, \dots, v_n).$$

We wish now to perform the change of variable $z = z(S) = \sum_{j=1}^n s_j t v_j$ in $\mathbb{R}^n$, whose Jacobian determinant is given by

$$\left|\frac{\partial z}{\partial S}\right| = t^n \det(v_1, \dots, v_n). \tag{6-1}$$

However, this operation becomes singular as v_1 approaches H, that is, as $v_1 \cdot u_1$ becomes small. Therefore, in order to deal with this degeneracy, we consider the following partition in v_1 of $\mathbb{S}^{n-1}$:

$$\mathbb{S}^{n-1} \setminus \{v_1 \cdot u_1 = 0\} = \bigcup_{i=0}^{\infty} S_i,$$

with

$$S_i = \left\{ v_1 \in \mathbb{S}^{n-1} : \frac{1}{2^{i+1}} |v_1 \cdot u_1| \leq \frac{1}{2^i} \right\}.$$

Then, defining $r_0 = n(r_1 + r_2)|t|$ and writing $g_k = \Delta_{2^k} g$, for convenience, one has the following straightforward estimate on $\bigcup_{2^i > r_0 2^{k2s}} S_i$:

$$\begin{aligned} \int_{\bigcup_{2^i > r_0 2^{k2s}} S_i} \left| S_t^{2*} g_k \left(x + \sum_{j=1}^n s_j t v_j, v_1 \right) \right|^2 dv_1 &\leq \sum_{2^i > r_0 2^{k2s}} |S_i| \, \|\tilde{\rho}_2\|_{L^1}^2 \, \|g\|_{L^\infty}^2 \\ &\leq \frac{C}{r_0 2^{k2s}} \|\tilde{\rho}_2\|_{L^1}^2 \, \|g\|_{L^\infty}^2, \end{aligned} \tag{6-2}$$

where $C > 0$ only depends on the dimension.

Now, on each domain S_i, with $2^i \leq r_0 2^{k2s}$, the Jacobian determinant (6-1) remains bounded away from zero. More precisely, for every $v_1 \in S_i$, it holds that

$$\begin{aligned} |t^n \det(u_1, v_2, \dots, v_n)| &\int_{[-r_1, r_1]^n} \left| S_t^{2*} g_k \left(x + \sum_{j=1}^n s_j t v_j, v_1 \right) \right|^2 dS \\ &= \frac{1}{|v_1 \cdot u_1|} \int_{z([-r_1, r_1]^n)} |S_t^{2*} g_k(x + z, v_1)|^2 \, dz \\ &\leq 2^{i+1} \int_{\left\{|z| \leq n r_1 |t|, |z \cdot u_1| \leq \frac{r_1 |t|}{2^i}\right\}} |S_t^{2*} g_k(x + z, v_1)|^2 \, dz \\ &= 2^{i+1} \int_{\left\{|z| \leq n r_1 |t|, |z \cdot u_1| \leq \frac{r_1 |t|}{2^i}\right\}} \left| \int_{\mathbb{R}} g_k(x + z + s t v_1) \tilde{\rho}_2(s) \, ds \right|^2 dz, \end{aligned} \tag{6-3}$$

where we have used that, for each $S \in [-r_1, r_1]^n$,

$$|z| = \left|\sum_{j=1}^{n} s_j t v_j\right| \le n r_1 |t| \quad \text{and} \quad |z \cdot u_1| = \left|\left(\sum_{j=1}^{n} s_j t v_j\right) \cdot u_1\right| = |s_1 t v_1 \cdot u_1| \le \frac{r_1 |t|}{2^i}.$$

Next, further notice that, whenever $|z| \le n r_1 |t|$, $|z \cdot u_1| \le r_1 |t|/2^i$, $|s| \le r_2$ and $v_1 \in S_i$, it holds that

$$|z + s t v_1| \le (n r_1 + r_2)|t| \quad \text{and} \quad |(z + s t v_1) \cdot u_1| \le \frac{(r_1 + r_2)|t|}{2^i}.$$

It therefore follows from (6-3) that

$$\begin{aligned}
&|t^n \det(u_1, v_2, \dots, v_n)| \int_{[-r_1, r_1]^n} \left| S_t^{2*} g_k\left(x + \sum_{j=1}^{n} s_j t v_j, v_1\right)\right|^2 dS \\
&\le 2^{i+1} \int_{\{|z| \le n r_1 |t|, |z \cdot u_1| \le \frac{r_1 |t|}{2^i}\}} \left| \int_{\mathbb{R}} g_k(x + z + s t v_1) \mathbb{1}_{\{|z + s t v_1| \le r_0, |(z + s t v_1) \cdot u_1| \le \frac{r_0}{2^i}\}} \tilde{\rho}_2(s)\, ds \right|^2 dz \\
&\le 2^{i+1} \int_{\mathbb{R}^n} |S_t^{2*} K_{x,u_1}^{i,k}(z, v_1)|^2\, dz, \qquad (6\text{-}4)
\end{aligned}$$

where

$$K_{x,u_1}^{i,k}(z) = g_k(x + z) \chi\left(\frac{|z - (z \cdot u_1) u_1|}{r_0}\right) \chi\left(2^i \frac{|z \cdot u_1|}{r_0}\right),$$

and $\chi \in C_c^\infty(\mathbb{R})$ is a smooth cutoff function such that $\mathbb{1}_{\{|s| \le 1\}} \le \chi(s) \le \mathbb{1}_{\{|s| \le 2\}}$.

Further integrating (6-4) in $v_1 \in S_i$ and then applying Proposition 5.1, we find that

$$\begin{aligned}
|t^n \det(u_1, v_2, \dots, v_n)| &\int_{S_i} \int_{[-r_1, r_1]^n} \left| S_t^{2*} g_k\left(x + \sum_{j=1}^{n} s_j t v_j, v_1\right)\right|^2 dS\, dv_1 \\
&\le 2^{i+1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |S_t^{2*} K_{x,u_1}^{i,k}(z, v_1)|^2\, dz\, dv_1 \\
&\le C \frac{2^i}{|t|^{2s}} \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2} - s}} \right\|_{L^2}^2 + \|(1 + |r|^s) \rho_2(r)\|_{L^\infty}^2 \right) \|(1 - \Delta_z)^{-\frac{s}{2}} K_{x,u_1}^{i,k}(z)\|_{L_z^2}^2,
\end{aligned}$$

where $C > 0$ only depends on the dimension. Moreover, a direct application of Lemma A.1 from Appendix A on paradifferential calculus yields

$$\|(1 - \Delta_z)^{-\frac{s}{2}} K_{x,u_1}^{i,k}(z)\|_{L_z^2}^2 \le \frac{C r_0^n}{2^i 2^{k2s}} \|g\|_{L^\infty}^2$$

for every $i, k \in \mathbb{N}$ such that $2^i \le r_0 2^{k2s}$, where $C > 0$ is independent of i, k and r_0, whence

$$\begin{aligned}
|t^n \det(u_1, v_2, \dots, v_n)| &\int_{S_i} \int_{[-r_1, r_1]^n} \left| S_t^{2*} g_k\left(x + \sum_{j=1}^{n} s_j t v_j, v_1\right)\right|^2 dS\, dv_1 \\
&\le C \frac{r_0^n}{|t|^{2s} 2^{k2s}} \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2} - s}} \right\|_{L^2}^2 + \|(1 + |r|^s) \rho_2(r)\|_{L^\infty}^2 \right) \|g\|_{L^\infty}^2, \qquad (6\text{-}5)
\end{aligned}$$

where $C > 0$ is independent of i, k, r_0 and ρ_2.

On the whole, combining (6-2), which is valid when $2^i > r_0 2^{k2s}$, with (6-5), which is valid when $2^i \le r_0 2^{k2s}$, we arrive at

$$\begin{aligned}
&\int_{[-r_1,r_1]^n}\int_{\mathbb{S}^{n-1}}\left|S_t^{2*}g_k\Big(x+\sum_{j=1}^n s_j t v_j, v_1\Big)\right|^2 dv_1\,dS \\
&\quad= \sum_{2^i\le r_0 2^{k2s}}\int_{[-r_1,r_1]^n}\int_{S_i}\left|S_t^{2*}g_k\Big(x+\sum_{j=1}^n s_j v_j, v_1\Big)\right|^2 dv_1\,dS \\
&\qquad\qquad + \int_{[-r_1,r_1]^n}\int_{\bigcup_{2^i>r_0 2^{k2s}} S_i}\left|S_t^{2*}g_k\Big(x+\sum_{j=1}^n s_j v_j, v_1\Big)\right|^2 dv_1\,dS \\
&\quad\le \frac{C}{|\det(u_1,v_2,\dots,v_n)|\,|t|^{2s}}\left(\left\|\frac{\rho_2(r)}{|r|^{\frac12-s}}\right\|_{L^2}^2+\|(1+|r|^s)\rho_2(r)\|_{L^\infty}^2\right) \\
&\qquad\qquad \times \sum_{2^i\le r_0 2^{k2s}}\frac{r_0^n}{|t|^n 2^{k2s}}\|g\|_{L^\infty}^2+\frac{Cr_1^n}{r_0 2^{k2s}}\|\tilde\rho_2\|_{L^1}^2\,\|g\|_{L^\infty}^2 \\
&\quad\le C\,\frac{(k+1)(r_1+r_2)^n\log(2+r_1+r_2)}{|\det(u_1,v_2,\dots,v_n)|\,|t|^{2s}2^{k2s}} \\
&\qquad\qquad \times\left(\left\|\frac{\rho_2(r)}{|r|^{\frac12-s}}\right\|_{L^2}^2+\|(1+|r|^s)\rho_2(r)\|_{L^\infty}^2\right)\|g\|_{L^\infty}^2+\frac{Cr_1^{n-1}}{|t|2^{k2s}}\|\tilde\rho_2\|_{L^1}^2\,\|g\|_{L^\infty}^2. \qquad (6\text{-}6)
\end{aligned}$$

Note that, when $n = 2$, the proof is then finished for $|\det(u_1, v_2)| = 1$. Therefore, when $n \ge 3$, there only remains to show that

$$\sup_{v_n\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}}\frac{1}{|\det(u_1,v_2,\dots,v_n)|}\,dv_2\cdots dv_{n-1}<\infty, \qquad (6\text{-}7)$$

which will clearly conclude the proof of the lemma upon integrating (6-6) in velocities $(v_2, \dots, v_{n-1})$ and combining the resulting estimate with (6-7).

In fact, the control (6-7) easily follows from a careful use of integration in spherical coordinates. Indeed, for each $2 \le j \le n-1$ and any choice of orthonormal vectors $\{u_{j+1}, \dots, u_n\}$, one has that (the unit vector u_1 is characterized here by the fact that it is orthogonal to the set $\{v_2, \dots, v_j, u_{j+1}, \dots, u_n\}$)

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}}\frac{1}{|\det(u_1,v_2,\dots,v_j,u_{j+1},\dots,u_n)|}\,dv_2\cdots dv_j \\
&\quad=\int_{\mathbb{S}^{j-1}\perp\{u_{j+1},\dots,u_n\}}\left(\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}}\frac{1}{|\det(u_1,v_2,\dots,v_{j-1},u_j,\dots,u_n)|}\,dv_2\cdots dv_{j-1}\right)du_j \\
&\qquad\qquad\times\int_0^\pi\cdots\int_0^\pi \sin^{n-3}\theta_n\cdots\sin^{j-2}\theta_{j+1}\,d\theta_{j+1}\cdots d\theta_n \\
&\quad\le C\sup_{\substack{u_j\in\mathbb{S}^{n-1}\\ u_j\cdot u_i=0\\ \text{for all } i=j+1,\dots,n}}\int_{\mathbb{S}^{n-1}}\cdots\int_{\mathbb{S}^{n-1}}\frac{1}{|\det(u_1,v_2,\dots,v_{j-1},u_j,\dots,u_n)|}\,dv_2\cdots dv_{j-1}.
\end{aligned}$$

Note that the unit vector u_1 above is also characterized by the fact that it is orthogonal to the set $\{v_2, \dots, v_{j-1}, u_j, \dots, u_n\}$. Hence, we deduce, for every $2 \le j \le n-1$, that

$$\sup_{\substack{u_{j+1},\dots,u_n \in \mathbb{S}^{n-1} \\ u_i \cdot u_k = 0 \text{ if } i \neq k}} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \frac{1}{|\det(u_1, v_2, \dots, v_j, u_{j+1}, \dots, u_n)|} \, dv_2 \cdots dv_j \le C \sup_{\substack{u_j,\dots,u_n \in \mathbb{S}^{n-1} \\ u_i \cdot u_k = 0 \text{ if } i \neq k}} \int_{\mathbb{S}^{n-1} \times \dots \times \mathbb{S}^{n-1}} \frac{1}{|\det(u_1, v_2, \dots, v_{j-1}, u_j, \dots, u_n)|} \, dv_2 \cdots dv_{j-1}.$$

Applying now the preceding estimate $n-2$ times to reduce iteratively the number of integrations over spheres, we find that

$$\begin{aligned} \sup_{v_n \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \frac{1}{|\det(u_1, v_2, \dots, v_n)|} \, dv_2 \cdots dv_{n-1} & \le C \sup_{\substack{u_j,\dots,u_n \in \mathbb{S}^{n-1} \\ u_i \cdot u_k = 0 \text{ if } i \neq k}} \int_{\mathbb{S}^{n-1} \times \dots \times \mathbb{S}^{n-1}} \frac{1}{|\det(u_1, v_2, \dots, v_{j-1}, u_j, \dots, u_n)|} \, dv_2 \cdots dv_{j-1} \\ & \le C \sup_{\substack{u_2,\dots,u_n \in \mathbb{S}^{n-1} \\ u_i \cdot u_k = 0 \text{ if } i \neq k}} \frac{1}{|\det(u_1, u_2, \dots, u_n)|}, \end{aligned}$$

where the unit vector u_1 is orthogonal to $\{u_2, \dots, u_n\}$, which implies

$$|\det(u_1, u_2, \dots, u_n)| = 1,$$

and thus establishes (6-7). □

For convenience, we introduce now, for any integer $N \ge 2$, setting $S = (s_2, \dots, s_{N-1}) \in \mathbb{R}^{N-2}$ and $V = (v_1, \dots, v_{N-1}) \in (\mathbb{S}^{n-1})^{N-1}$, the following nonlinear operator:

$$I_N g = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |S_t^* g(x, v_N)|^2 \left(\int_{[-r_1, r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right) \right|^2 dV \, dS \right) dv_N \, dx.$$

In particular, when $N = 2$, we have $I_2 g = \|S_t^* g\|^4_{L^4_x L^2_v}$.

Recall that, employing (5-5), it is possible to extract a one-dimensional integration from $S_t^* g(x, v_N)$ and $S_t^* g\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right)$ along v_N and v_1, respectively. Therefore, it is possible, at least formally, to gain an N-dimensional spatial integration in the above integrand by exploiting the integration along the variables s_j. Thus, loosely speaking, the number N represents the expected gain of spatial dimension on the domain of integration in I_N.

Prior to delving any further into our proofs, we take some time now to explain the general strategy behind the dimensional build up which will eventually allow us to apply Lemma 6.1 and establish the boundedness of $S_t^* : L^4_x \to W^{s,4}_x L^2_v$ for any $0 \le s < \frac{n}{4(n-1)}$, in Proposition 6.6, below.

More precisely, the aforementioned boundedness of S_t^* will be shown to follow from four properties of the nonlinear operator I_N:

• For $N = 2$,

$$I_2 h = \|S_t^* h\|^4_{L^4_x L^2_v}. \tag{6-8}$$

This property is a direct interpretation of the definition of I_2.

• For $N = n$ and any $0 < s < \frac{1}{2}$, assuming for simplicity that $h(x)$ has frequencies localized inside an annulus of inner and outer radii comparable to 2^k, with $k \in \mathbb{N}$,

$$I_n h \le \frac{C}{2^{k4s}} \|h\|^2_{L^2_x} \|h\|^2_{L^\infty_x}, \tag{6-9}$$

where $C > 0$ is independent of k. This estimate displays a gain of regularity, is a consequence of Lemma 6.1 and is established in Lemma 6.2, below.

• For any $N \ge 2$,

$$(I_N h)^2 \le \|S_t^* h\|^4_{L^4_x L^2_v} I_{2N-2} h, \tag{6-10}$$

which is a simple consequence of the Cauchy–Schwarz inequality (in x) followed by a careful change of variable. This estimate is established in Lemma 6.3, below.

• For any $N \ge 2$,

$$(I_N h)^2 \le C \|S^{2*} h\|^4_{L^4_{x,v}} I_{2N-1} h, \tag{6-11}$$

where $C > 0$ is an independent constant, which is a direct consequence of an application of (5-5) followed by a careful use of the Cauchy–Schwarz inequality with a change of variable. This estimate is established in Lemma 6.4, below.

The rule of the game of dimensional build up will then consist in employing estimates (6-10) and (6-11) to go from (6-8) to (6-9). In other words, by exploiting the mappings $N \mapsto 2N - 2$ and $N \mapsto 2N - 1$, for integers $N \ge 2$, we want to go from 2 to the dimension n. The fact that such a dimensional build up is actually possible is explained by the simple yet tricky Lemma 6.5, below.

Eventually, the appropriate combination of these estimates (and the handling of more technical difficulties) will give rise to the main result of this section, namely Theorem 6.8.

We proceed now with the actual preliminary results leading to Theorem 6.8.

For the sake of simplicity of notation, from now on, the variable S will denote the vector whose components are any number of integration variables $s_j \in [-r_1, r_1]$, whereas the variable V will denote the vector whose components are any number of integration variables $v_j \in \mathbb{S}^{n-1}$. At each step of our proofs, the exact meaning of S and V will be easily deduced from a careful inspection of the integrands and domains of integration.

Applying the preceding lemma combined with Proposition 5.1 to the above nonlinear operator I_N, when $N = n$ is the dimension, yields the following result.

Lemma 6.2. *In any dimension $n \ge 2$, let $0 < s < \frac{1}{2}$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ have their supports contained inside balls of radii $r_1, r_2 > 0$, respectively, centered at the origin.*

Then, it holds that, for any $k \in \mathbb{N}$,

$$I_n \Delta_{2^k} g \leq \frac{C(k+1)}{t^2 2^{k4s}} \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 (r_1+r_2)^{n-1}(1+r_1+r_2)^2 \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde{\rho}_2\|_{L^1} \right)^4 \|g\|_{L_x^2}^2 \|g\|_{L_x^\infty}^2$$

for all $t \neq 0$ *such that* $|t| \leq 1$, *where* $C > 0$ *only depends on fixed parameters.*

Proof. In view of the simple estimate (5-5), it holds that

$$\begin{aligned} I_n \Delta_{2^k} g &\leq \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \int_{[-r_1,r_1]} |S_t^{2*} \Delta_{2^k} g(x+s_n t v_n, v_n)|^2 \, ds_n \\ &\qquad \times \left(\int_{[-r_1,r_1]^{n-1}} \int_{(\mathbb{S}^{n-1})^{n-1}} \left| S_t^{2*} \Delta_{2^k} g\left(x + \sum_{j=1}^{n-1} s_j t v_j, v_1\right) \right|^2 dV\, dS \right) dv_n\, dx \\ &\leq \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |S_t^{2*} \Delta_{2^k} g(x, v_n)|^2 \\ &\qquad \times \left(\int_{[-r_1,r_1]^{n}} \int_{(\mathbb{S}^{n-1})^{n-1}} \left| S_t^{2*} \Delta_{2^k} g\left(x + \sum_{j=1}^{n} s_j t v_j, v_1\right) \right|^2 dV\, dS \right) dv_n\, dx \\ &\leq \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 \|S_t^{2*} \Delta_{2^k} g\|_{L_{x,v}^2}^2 \\ &\qquad \times \sup_{\substack{x \in \mathbb{R}^n \\ v_n \in \mathbb{S}^{n-1}}} \int_{[-r_1,r_1]^{n}} \int_{(\mathbb{S}^{n-1})^{n-1}} \left| S_t^{2*} \Delta_{2^k} g\left(x + \sum_{j=1}^{n} s_j t v_j, v_1\right) \right|^2 dV\, dS. \end{aligned}$$

Therefore, applying Proposition 5.1 and Lemma 6.1 to the preceding estimate yields

$$I_n \Delta_{2^k} g \leq \frac{C(k+1)}{t^2 2^{k4s}} \|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 (r_1+r_2)^{n-1}(1+r_1+r_2)^2 \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde{\rho}_2\|_{L^1} \right)^4 \|g\|_{L_x^2}^2 \|g\|_{L_x^\infty}^2. \qquad \square$$

The next result explains how to increase the expected dimension of the domain of integration in the nonlinear operator I_N from N to $2N-2$.

Lemma 6.3. *In any dimension* $n \geq 2$, *it holds that, for any integer* $N \geq 2$,

$$(I_N g)^2 \leq \|S_t^* g\|_{L_x^4 L_v^2}^4 I_{2N-2} g.$$

Proof. First, by the Cauchy–Schwarz inequality, we find

$$\begin{aligned} I_N g &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |S_t^* g(x, v_N)|^2 \, dv_N \right) \left(\int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right) \right|^2 dV\, dS \right) dx \\ &\leq \|S_t^* g\|_{L_x^4 L_v^2}^2 \left\| \int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right) \right|^2 dV\, dS \right\|_{L_x^2}, \end{aligned}$$

whence

$$(I_N g)^2 \le \|S_t^* g\|_{L_x^4 L_v^2}^4 \int_{\mathbb{R}^n} \left(\int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=N}^{2N-3} s_j t v_j, v_{2N-2}\right)\right|^2 dV\, dS \right) \times \left(\int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right)\right|^2 dV\, dS \right) dx.$$

Then, exploiting the integration in x to transfer the term $\sum_{j=N}^{2N-3} s_j t v_j$ in the above integrand, we deduce that

$$(I_N g)^2 \le \|S_t^* g\|_{L_x^4 L_v^2}^4 \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |S_t^* g(x, v_{2N-2})|^2 \times \left(\int_{[-r_1,r_1]^{2N-4}} \int_{(\mathbb{S}^{n-1})^{2N-3}} \left| S_t^* g\left(x + \sum_{j=2}^{2N-3} s_j t v_j, v_1\right)\right|^2 dV\, dS \right) dv_{2N-2}\, dx. \qquad \square$$

The next result explains how to increase the expected dimension of the domain of integration in the nonlinear operator I_N from N to $2N-1$.

Lemma 6.4. *In any dimension $n \ge 2$, let $\rho_1 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde{\rho}_1$ has its support contained inside a ball of radius $r_1 > 0$ centered at the origin.*

Then, it holds that, for any integer $N \ge 2$,

$$(I_N g)^2 \le 4 \|\tilde{\rho}_1\|_{L^1}^2 \, \|\tilde{\rho}_1\|_{L^\infty}^2 \, r_1 \, \|S_t^{2*} g\|_{L_{x,v}^4}^4 \, I_{2N-1} g.$$

Proof. First, in view of the simple estimate (5-5), one has

$$\begin{aligned} I_N g &\le \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \int_{[-r_1,r_1]} |S_t^{2*} g(x + s_N t v_N, v_N)|^2 \, ds_N \\ &\qquad \times \left(\int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N-1} s_j t v_j, v_1\right)\right|^2 dV\, dS \right) dv_N\, dx \\ &= \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |S_t^{2*} g(x, v_N)|^2 \\ &\qquad \times \left(\int_{[-r_1,r_1]^{N-1}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N} s_j t v_j, v_1\right)\right|^2 dV\, dS \right) dv_N\, dx \\ &= \|\tilde{\rho}_1\|_{L^1} \|\tilde{\rho}_1\|_{L^\infty} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \left(|S_t^{2*} g(x, v_N)|^2 + |S_t^{2*} g(x, -v_N)|^2 \right) \\ &\qquad \times \left(\int_{[0,r_1]} \int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\left(x + \sum_{j=2}^{N} s_j t v_j, v_1\right)\right|^2 dV\, dS\, ds_N \right) dv_N\, dx. \end{aligned}$$

Hence, by the Cauchy–Schwarz inequality, we find

$$
\begin{aligned}
(I_N g)^2 &\le 4\|\tilde\rho_1\|_{L^1}^2 \, \|\tilde\rho_1\|_{L^\infty}^2 \, \|S_t^{2*} g\|_{L^4_{x,v}}^4 \\
&\quad\times \left\| \int_{[0,r_1]} \int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\Big(x + \sum_{j=2}^{N} s_j t v_j, v_1\Big)\right|^2 dV\, dS\, ds_N \right\|_{L^2_{x,v_N}}^2 \\
&= 4\|\tilde\rho_1\|_{L^1}^2 \, \|\tilde\rho_1\|_{L^\infty}^2 \, \|S_t^{2*} g\|_{L^4_{x,v}}^4 \\
&\quad\times \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \left(\int_{[0,r_1]} \int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\Big(x + \sum_{j=N}^{2N-2} s_j t v_j, v_{2N-1}\Big)\right|^2 dV\, dS\, ds_N \right) \\
&\qquad\quad \times \left(\int_{[0,r_1]} \int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-1}} \left| S_t^* g\Big(x + \sum_{j=2}^{N} s_j t v_j, v_1\Big)\right|^2 dV\, dS\, ds_N \right) dv_N\, dx.
\end{aligned}
$$

Finally, exploiting the integration in x to transfer first the term $s_N t v_N$ and then the term $\sum_{j=N+1}^{2N-2} s_j t v_j$ in the above integrand, we deduce that

$$
\begin{aligned}
(I_N g)^2 &\le 4\|\tilde\rho_1\|_{L^1}^2 \, \|\tilde\rho_1\|_{L^\infty}^2 r_1 \|S_t^{2*} g\|_{L^4_{x,v}}^4 \\
&\quad\times \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \left(\int_{[-r_1,r_1]^{N-2}} \int_{(\mathbb{S}^{n-1})^{N-2}} \left| S_t^* g\Big(x + \sum_{j=N+1}^{2N-2} s_j t v_j, v_{2N-1}\Big)\right|^2 dV\, dS \right) \\
&\qquad\quad \times \left(\int_{[-r_1,r_1]^{N-1}} \int_{(\mathbb{S}^{n-1})^{N}} \left| S_t^* g\Big(x + \sum_{j=2}^{N} s_j t v_j, v_1\Big)\right|^2 dV\, dS \right) dv_{2N-1}\, dx \\
&= 4\|\tilde\rho_1\|_{L^1}^2 \, \|\tilde\rho_1\|_{L^\infty}^2 r_1 \|S_t^{2*} g\|_{L^4_{x,v}}^4 \\
&\quad\times \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |S_t^* g(x, v_{2N-1})|^2 \\
&\qquad\quad \times \left(\int_{[-r_1,r_1]^{2N-3}} \int_{(\mathbb{S}^{n-1})^{2N-2}} \left| S_t^* g\Big(x + \sum_{j=2}^{2N-2} s_j t v_j, v_1\Big)\right|^2 dV\, dS \right) dv_{2N-1}\, dx. \qquad \square
\end{aligned}
$$

The following result is a simple technical lemma which, at first, may seem somewhat unrelated but will prove very useful later on for building up dimensions in the proof of Proposition 6.6.

Lemma 6.5. *Let the mappings* $\Lambda_0, \Lambda_1 : \mathbb{N} \setminus \{0,1\} \to \mathbb{N} \setminus \{0,1\}$ *be defined by*

$$\Lambda_0 k = 2k - 2 \quad \text{and} \quad \Lambda_1 k = 2k - 1.$$

Then, for any integer $n \ge 3$, *there exists* $L \in \mathbb{N}$ *and* $a_0, a_1, \dots, a_L \in \{0,1\}$ *such that*

$$n = \Lambda_{a_0} \Lambda_{a_1} \cdots \Lambda_{a_L} 2,$$

and

$$n - 2 = \sum_{k=0}^{L} a_k 2^k.$$

Moreover, the above decomposition is unique provided $a_L = 1$.

Proof. We introduce first the auxiliary mappings $\widetilde{\Lambda}_0, \widetilde{\Lambda}_1 : \mathbb{N} \to \mathbb{N}$ defined by

$$\widetilde{\Lambda}_0 k = 2k \quad \text{and} \quad \widetilde{\Lambda}_1 k = 2k+1.$$

In particular, for any $k \in \mathbb{N}$, it holds that

$$(\widetilde{\Lambda}_0 k) + 2 = \Lambda_0(k+2) \quad \text{and} \quad (\widetilde{\Lambda}_1 k) + 2 = \Lambda_1(k+2).$$

Next, let $L \in \mathbb{N}$ and $a_0, a_1, \dots, a_L \in \{0,1\}$ be the parameters appearing in the dyadic decomposition of the positive integer $n-2$:

$$n-2 = \sum_{k=0}^{L} a_k 2^k.$$

Note that, assuming $a_L = 1$, the above choice of parameters is unique. Then, we have

$$n-2 = \widetilde{\Lambda}_{a_0}\left(\sum_{k=0}^{L-1} a_{k+1} 2^k\right) = \widetilde{\Lambda}_{a_0}\widetilde{\Lambda}_{a_1}\left(\sum_{k=0}^{L-2} a_{k+2} 2^k\right) = \cdots = \widetilde{\Lambda}_{a_0}\widetilde{\Lambda}_{a_1}\cdots\widetilde{\Lambda}_{a_L} 0.$$

It finally follows that

$$\begin{aligned} n &= 2 + \widetilde{\Lambda}_{a_0}\widetilde{\Lambda}_{a_1}\cdots\widetilde{\Lambda}_{a_L} 0 \\ &= \Lambda_{a_0}(2 + \widetilde{\Lambda}_{a_1}\cdots\widetilde{\Lambda}_{a_L} 0) \\ &= \Lambda_{a_0}\Lambda_{a_1}(2 + \widetilde{\Lambda}_{a_2}\cdots\widetilde{\Lambda}_{a_L} 0) = \cdots = \Lambda_{a_0}\Lambda_{a_1}\cdots\Lambda_{a_L} 2. \end{aligned}$$ □

Notice that, using the language of Lemma 6.5, it is possible to unify Lemmas 6.3 and 6.4 in the following estimate, for any $N \geq 2$:

$$(I_N g)^2 \leq (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^a \|S_t^* g\|_{L_x^4 L_v^2}^{4(1-a)} \|S_t^{2*} g\|_{L_{x,v}^4}^{4a} I_{\Lambda_a N} g, \tag{6-12}$$

where $a \in \{0,1\}$.

Now, appropriately combining Lemmas 6.2, 6.3 and 6.4, with the help of Lemma 6.5, we arrive at our main estimate on the operator S_t^*, which is recorded in the next proposition and generalizes Proposition 5.2 to higher dimensions.

Proposition 6.6. *In any dimension $n \geq 3$, let $2 \leq p \leq 4$, $0 \leq s < s_0 < \frac{1}{2}$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ have their supports contained inside balls of radii $r_1, r_2 > 0$, respectively, centered at the origin.*

Then, the operator S_t^* satisfies the estimate

$$\begin{aligned} \|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}} S_t^* g\|_{L_x^p L_v^2} \leq \frac{C}{|t|^{\frac{n}{4(n-1)}}} \|\tilde{\rho}_1\|_{L^1}^{\frac{1}{2}} \|\tilde{\rho}_1\|_{L^\infty}^{\frac{1}{2}} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde{\rho}_2\|_{L^1} \right) \|g\|_{L_x^p} \end{aligned}$$

for all $t \neq 0$ such that $|t| \leq 1$, where $C > 0$ only depends on fixed parameters.

Proof. First, notice that, for any $2 \le p \le \infty$,

$$\|S_t^* g\|_{L_x^p L_v^2} \le \|S_t^* g\|_{L_v^2 L_x^p} \le |\mathbb{S}^{n-1}|^{\frac{1}{2}} \|\tilde{\rho}\|_{L^1} \|g\|_{L_x^p}. \tag{6-13}$$

As for the regularity estimate, we employ the standard Littlewood–Paley dyadic frequency decomposition previously introduced to estimate $g_k = \Delta_{2^k} g$ for any $k \ge 0$.

To this end, we first decompose the dimension $n \ge 3$ according to Lemma 6.5,

$$n = \Lambda_{a_0} \Lambda_{a_1} \cdots \Lambda_{a_L} 2,$$

where $L \in \mathbb{N}$ and $a_0, a_1, \dots, a_L \in \{0, 1\}$, and then apply successively estimate (6-12) to deduce that

$$\begin{aligned}
\|S_t^* g_k\|_{L_x^4 L_v^2}^{2^{L+1}} &= (I_2 g_k)^{2^{L-1}} \\
&\le (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^{a_L 2^{L-2}} \|S_t^* g_k\|_{L_x^4 L_v^2}^{(1-a_L)2^L} \|S_t^{2*} g_k\|_{L_{x,v}^4}^{a_L 2^L} (I_{\Lambda_{a_L} 2} g_k)^{2^{L-2}} \\
&\le (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^{a_{L-1} 2^{L-3} + a_L 2^{L-2}} \|S_t^* g_k\|_{L_x^4 L_v^2}^{(1-a_{L-1})2^{L-1} + (1-a_L)2^L} \\
&\qquad \times \|S_t^{2*} g_k\|_{L_{x,v}^4}^{a_{L-1} 2^{L-1} + a_L 2^L} (I_{\Lambda_{a_{L-1}} \Lambda_{a_L} 2} g_k)^{2^{L-3}} \\
&\le \cdots \\
&\le (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^{\frac{1}{4} \sum_{j=0}^L a_j 2^j} \|S_t^* g_k\|_{L_x^4 L_v^2}^{\sum_{j=0}^L (1-a_j) 2^j} \\
&\qquad \times \|S_t^{2*} g_k\|_{L_{x,v}^4}^{\sum_{j=0}^L a_j 2^j} (I_{\Lambda_{a_0} \Lambda_{a_1} \cdots \Lambda_{a_L} 2} g_k)^{\frac{1}{4}} \\
&= (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^{\frac{n-2}{4}} \|S_t^* g_k\|_{L_x^4 L_v^2}^{2^{L+1} - (n-1)} \|S_t^{2*} g_k\|_{L_{x,v}^4}^{n-2} (I_n g_k)^{\frac{1}{4}}.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\|S_t^* g_k\|_{L_x^4 L_v^2}^{n-1} &\le (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^{\frac{n-2}{4}} \|S_t^{2*} g_k\|_{L_{x,v}^4}^{n-2} (I_n g_k)^{\frac{1}{4}} \\
&\le (4\|\tilde{\rho}_1\|_{L^1}^2 \|\tilde{\rho}_1\|_{L^\infty}^2 r_1)^{\frac{n-2}{4}} \|S_t^{2*} g_k\|_{L_{x,v}^2}^{\frac{n-2}{2}} \|S_t^{2*} g_k\|_{L_{x,v}^\infty}^{\frac{n-2}{2}} (I_n g_k)^{\frac{1}{4}}.
\end{aligned}$$

Next, further applying Proposition 5.1 and Lemma 6.2 to the preceding bound yields

$$\begin{aligned}
\|S_t^* g_k\|_{L_x^4 L_v^2}^{n-1} &\le \frac{C(k+1)^{\frac{1}{4}}}{|t|^{\frac{n}{4}} 2^{k \frac{n}{2} s}} \|\tilde{\rho}_1\|_{L^1}^{\frac{n-1}{2}} \|\tilde{\rho}_1\|_{L^\infty}^{\frac{n-1}{2}} (r_1 + r_2)^{\frac{2n-3}{4}} (1 + r_1 + r_2)^{\frac{1}{2}} \\
&\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde{\rho}_2\|_{L^1} \right)^{n-1} \|g\|_{L_x^2}^{\frac{n-1}{2}} \|g\|_{L_x^\infty}^{\frac{n-1}{2}},
\end{aligned}$$

where $C > 0$ is an independent constant.

The remainder of the demonstration follows the arguments from the end of the proof of Proposition 5.2, which we adapt to the present setting for completeness and convenience of the reader.

Thus, in order to conclude, we write $|g(x)| = \int_0^\infty \mathbb{1}_{\{|g(x)|\geq s\}}\, ds$ to deduce from the preceding estimate, assuming g is nonnegative, that

$$\begin{aligned}\|S_t^* g_k\|_{L^4_x L^2_v} &\leq \int_0^\infty \|S_t^* \Delta_{2^k} \mathbb{1}_{\{|g(x)|\geq s\}}\|_{L^4_x L^2_v}\, ds \\ &\leq \frac{C(k+1)^{\frac{1}{4(n-1)}}}{|t|^{\frac{n}{4(n-1)}} 2^{k\frac{n}{2(n-1)}s}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ &\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \int_0^\infty |\{|g(x)|\geq s\}|^{\frac14}\, ds \\ &\leq \frac{C(k+1)^{\frac{1}{4(n-1)}}}{|t|^{\frac{n}{4(n-1)}} 2^{k\frac{n}{2(n-1)}s}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ &\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s}} \right\|_{L^2} + \|(1+|r|^s)\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^{4,1}_x},\end{aligned}$$

where $L^{4,1}_x$ denotes a standard Lorentz space; see [Bergh and Löfström 1976, Section 1.3] or [Grafakos 2008, Section 1.4] for definitions and properties of Lorentz spaces. When, g is signed, we arrive at the same estimate simply by decomposing $g = g^+ - g^-$ into its positive and negative parts, treating each contribution separately, and then noticing that

$$\begin{aligned}\|g^+\|_{L^{4,1}_x} + \|g^-\|_{L^{4,1}_x} &\leq C \int_0^\infty \big|\{|g^+(x)|\geq s\}\big|^{\frac14} + \big|\{|g^-(x)|\geq s\}\big|^{\frac14}\, ds \\ &\leq C \int_0^\infty \big(\big|\{|g^+(x)|\geq s\}\big| + \big|\{|g^-(x)|\geq s\}\big|\big)^{\frac14}\, ds \\ &\leq C \int_0^\infty \big|\{|g(x)|\geq s\}\big|^{\frac14}\, ds \leq C\|g\|_{L^{4,1}_x}.\end{aligned}$$

Moreover, by allowing an arbitrarily small loss of regularity, that is, by replacing $0 < s < \frac12$ by a slightly smaller value, it is possible to replace the Lorentz space $L^{4,1}_x$ by the standard Lebesgue space L^4_x in the right-hand side of the above estimate.

Therefore, on the whole, for any $0 \leq s < s_0 < \frac12$, we have established the estimate

$$\begin{aligned}\|S_t^* g_k\|_{L^4_x L^2_v} &\leq \frac{C}{|t|^{\frac{n}{4(n-1)}} 2^{k\frac{n}{2(n-1)}s}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ &\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^4_x},\end{aligned}$$

where $C > 0$ only depends on constant parameters, which, when combined with the easy bound (6-13) for low frequencies, yields

$$\begin{aligned}\|(1-\Delta_x)^{\frac{ns}{4(n-1)}} S_t^* g\|_{L^4_x L^2_v} &\leq \frac{C}{|t|^{\frac{n}{4(n-1)}}} \|\tilde\rho_1\|_{L^1}^{\frac12} \|\tilde\rho_1\|_{L^\infty}^{\frac12} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ &\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac12 - s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L^4_x}. \qquad (6\text{-}14)\end{aligned}$$

Finally, since S_t^* commutes with differentiation in x and recalling from complex interpolation theory, see [Bergh and Löfström 1976, Sections 5.1 and 6.4], that, for any $2 < p < 4$,

$$(L_x^2 L_v^2, L_x^4 L_v^2)_{[2-\frac{4}{p}]} = L_x^p L_v^2 \quad \text{and} \quad (W_x^{-s,2}, W_x^{-s\frac{n}{2(n-1)},4})_{[2-\frac{4}{p}]} = W_x^{-s\frac{2n+p-4}{p(n-1)},p},$$

we conclude the proof of the proposition by interpolating the estimate (6-14) with the classical estimate on S_t^* from Proposition 5.1. □

Next, we utilize the previous result on the adjoint operator S_t^* to deduce corresponding estimates on T_t and T_t^*.

Proposition 6.7. *In any dimension $n \geq 3$, let $\frac{4}{3} \leq r \leq 2$, $2 \leq p \leq 4$, $0 \leq s < s_0 < \frac{1}{2}$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathbb{R})$ be such that $\tilde\rho_1$ and $\tilde\rho_2$ have their supports contained inside balls of radii $r_1, r_2 > 0$, respectively, centered at the origin.*

Then, the operators T_t and T_t^* satisfy the estimates

$$\begin{aligned} &\|(1-\Delta_x)^{\frac{(2n+r'-4)s}{2r'(n-1)}} T_t f\|_{L_x^r} \\ &\quad \leq \frac{C}{|t|^{\frac{n}{4(n-1)}}} \|\varphi\|_{L^\infty} R^{\frac{n(2n-3)}{4(n-1)}} \|\tilde\rho_1\|_{L^1}^{\frac{1}{2}} \|\tilde\rho_1\|_{L^\infty}^{\frac{1}{2}} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ &\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|f\|_{L_x^r L_v^2} \end{aligned} \tag{6-15}$$

and

$$\begin{aligned} &\|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}} T_t^* g\|_{L_x^p L_v^2} \\ &\quad \leq \frac{C}{|t|^{\frac{n}{4(n-1)}}} \|\varphi\|_{L^\infty} R^{\frac{n(2n-3)}{4(n-1)}} \|\tilde\rho_1\|_{L^1}^{\frac{1}{2}} \|\tilde\rho_1\|_{L^\infty}^{\frac{1}{2}} (r_1+r_2)^{\frac{2n-3}{4(n-1)}} (1+r_1+r_2)^{\frac{1}{2(n-1)}} \\ &\qquad \times \left(\left\| \frac{\rho_2(r)}{|r|^{\frac{1}{2}-s_0}} \right\|_{L^2} + \|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty} + \|\tilde\rho_2\|_{L^1} \right) \|g\|_{L_x^p} \end{aligned}$$

for all $t \neq 0$ such that $|t| \leq 1$, where $C > 0$ only depends on fixed parameters.

Proof. It is readily seen that

$$T_t^* g(x,v) = \int_{\mathbb{R}} g(x+stv)\tilde\rho(s)\, ds\, \varphi(v) = \varphi(v) S_{|v|t}^* g\left(x, \frac{v}{|v|}\right).$$

Therefore, for any $2 \leq p \leq \infty$, we compute in polar coordinates, recalling that the support of the velocity weight $\varphi \in L_c^\infty(\mathbb{R}^n)$ is contained inside a closed ball of radius $R > 0$ centered at the origin,

$$\begin{aligned} &\|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}} T_t^* g(x,v)\|_{L_x^p(\mathbb{R}^n; L_v^2(\mathbb{R}^n))} \\ &\qquad = \left\| \left(\int_{\mathbb{R}^n} |(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}} T_t^* g(x,v)|^2\, dv \right)^{\frac{1}{2}} \right\|_{L_x^p} \\ &\qquad = \left\| \int_0^\infty r^{n-1} \int_{\mathbb{S}^{n-1}} |\varphi(rv)(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}} S_{rt}^* g(x,v)|^2\, dv\, dr \right\|_{L_x^{p/2}}^{\frac{1}{2}} \end{aligned}$$

$$\leq \|\varphi\|_{L^\infty}\left(\int_0^R r^{n-1}\left\|\int_{\mathbb{S}^{n-1}}|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}}S_{rt}^*g(x,v)|^2\,dv\right\|_{L_x^{p/2}}dr\right)^{\frac12}$$
$$\leq \|\varphi\|_{L^\infty}\left(\int_0^R r^{n-1}\|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}}S_{rt}^*g(x,v)\|^2_{L_x^p(\mathbb{R}^n;L_v^2(\mathbb{S}^{n-1}))}\,dr\right)^{\frac12}.$$

Then, combining Proposition 6.6 with the above estimate, we find

$$\begin{aligned}
&\|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}}T_t^*g\|_{L_x^pL_v^2}\\
&\quad\leq \|\varphi\|_{L^\infty}\left(\int_0^R r^{n-1}\|(1-\Delta_x)^{\frac{(2n+p-4)s}{2p(n-1)}}S_{rt}^*g\|^2_{L_x^pL_v^2}\,dr\right)^{\frac12}\\
&\quad\leq \frac{C}{|t|^{\frac{n}{4(n-1)}}}\|\varphi\|_{L^\infty}R^{\frac{n(2n-3)}{4(n-1)}}\|\tilde\rho_1\|_{L^1}^{\frac12}\|\tilde\rho_1\|_{L^\infty}^{\frac12}(r_1+r_2)^{\frac{2n-3}{4(n-1)}}(1+r_1+r_2)^{\frac{1}{2(n-1)}}\\
&\qquad\qquad\times\left(\left\|\frac{\rho_2(r)}{|r|^{\frac12-s_0}}\right\|_{L^2}+\|(1+|r|^{s_0})\rho_2(r)\|_{L^\infty}+\|\tilde\rho_2\|_{L^1}\right)\|g\|_{L_x^p},
\end{aligned}$$

where $C>0$ is an independent constant, which establishes the estimate on T_t^*.

The estimate on T_t is then easily deduced from the estimate on T_t^* by a duality argument, which completes the proof of the proposition. □

We proceed now to the main theorem of this section.

Theorem 6.8. *In any dimension $n\geq 3$, let $\frac43\leq r\leq 2$.*

Then, for any $f,g\in L_x^r(\mathbb{R}^n;L_v^2(\mathbb{R}^n))$ such that (1-2) *holds true, one has*

$$\int_{\mathbb{R}^n}f(x,v)\,\varphi(v)\,dv\in W_x^{s,r}(\mathbb{R}^n)$$

for any $\varphi\in L_c^\infty(\mathbb{R}^n)$ and any

$$0\leq s<\frac12\Big(3-\frac4r\Big)+\frac{n}{4(n-1)}\Big(\frac4r-2\Big).$$

Furthermore, one has the estimate

$$\left\|(1-\Delta_x)^{\frac s2}\int_{\mathbb{R}^n}f\varphi\,dv\right\|_{L_x^r}\leq C(\|f\|_{L_x^rL_v^2}+\|g\|_{L_x^rL_v^2}),$$

where $C>0$ only depends on φ and constant parameters.

Proof. This demonstration follows the same ideas as the proof of Theorem 5.4. Nevertheless, for the sake of clarity and convenience of the reader, we provide a complete justification of this result.

We consider first the refined interpolation formula (2-1), which is valid for solutions of the transport equation (1-2), for some given cutoff $\rho\in\mathcal{S}(\mathbb{R})$. Clearly, further differentiating (2-1) in x and then averaging in v yields

$$\left\|(1-\Delta_x)^{\frac s2}\int_{\mathbb{R}^n}f\varphi\,dv\right\|_{L_x^r}\leq\left\|(1-\Delta_x)^{\frac s2}\int_{\mathbb{R}^n}A_tf\varphi\,dv\right\|_{L_x^r}+t\left\|(1-\Delta_x)^{\frac s2}\int_{\mathbb{R}^n}B_tg\varphi\,dv\right\|_{L_x^r}.\quad(6\text{-}16)$$

We wish now to apply Proposition 6.7 to the preceding estimate. To this end, according to (5-4), we take the decompositions

$$\rho(r) = \rho_1(r)\rho_2(r) \quad \text{and} \quad \tau(r) = \frac{1-\rho(r)}{ir} = \tau_1(r)\tau_2(r),$$

where

$$\begin{aligned} \tilde{\rho}_1(r) &\in C_c^\infty(\mathbb{R}), \quad \tau_1(r) = \frac{1}{(1+r^2)^{\frac{1}{4}}}, \\ \tilde{\rho}_2(r) &\in C_c^\infty(\mathbb{R}), \quad \tau_2(r) = (1+r^2)^{\frac{1}{4}}\tau(r). \end{aligned}$$

Clearly, all constants involving norms of the cutoff functions ρ_1 and ρ_2 in the right-hand side of (5-11) are finite and we may therefore straightforwardly apply Proposition 6.7 to control the first term in the right-hand side of (6-16). However, the same is not so obviously true concerning the cutoff functions τ_1 and τ_2. The application of Proposition 6.7 to the second term in the right-hand side of (6-16) will therefore require some substantial technical work, which we present now.

To this end, we employ a homogeneous Littlewood–Paley frequency decomposition, see (5-3), of τ_1 and τ_2 to write that

$$\tau = \Big(\sum_{j\in\mathbb{Z}} \Delta_{2^j}\tau_1\Big)\Big(\sum_{k\in\mathbb{Z}} \Delta_{2^k}\tau_2\Big) = \sum_{j\in\mathbb{Z}} (\Delta_{2^j}\tau_1)\tau_3^j + \sum_{j\in\mathbb{Z}} (\Delta_{2^j}\tau_2)\tau_4^j,$$

where

$$\tau_3^j = \sum_{\substack{k\in\mathbb{Z}\\ k\le j}} \Delta_{2^k}\tau_2 = \mathcal{F}^{-1}\Big[\psi_0\Big(\frac{r}{2^{j+1}}\Big)\Big] * \tau_2, \quad \tau_4^j = \sum_{\substack{k\in\mathbb{Z}\\ k< j}} \Delta_{2^k}\tau_1 = \mathcal{F}^{-1}\Big[\psi_0\Big(\frac{r}{2^{j}}\Big)\Big] * \tau_1.$$

In view of the linearity of the operator T_t with respect to the cutoffs ρ or τ, we only need to verify the finiteness of the constants in (6-15) with $\Delta_{2^j}\tau_1$ and τ_3^j playing the roles of ρ_1 and ρ_2, respectively, and then with $\Delta_{2^j}\tau_2$ and τ_4^j instead of ρ_1 and ρ_2, respectively. It is to be emphasized here that the ensuing bounds on the cutoffs will then depend on $j \in \mathbb{Z}$. In order to guarantee the boundedness of T_t, we will therefore need to make sure that our method eventually yields constants that are summable in $j \in \mathbb{Z}$.

We evaluate now the norms involved in the right-hand side of (6-15) where we replace ρ_1 by $\Delta_{2^j}\tau_1$ (or $\Delta_{2^j}\tau_2$) and ρ_2 by τ_3^j (or τ_4^j). The bounds on $\Delta_{2^j}\tau_2$ and τ_4^j are handled in a strictly similar manner and so we omit the corresponding details.

First, note that a direct application of Lemma B.3 from Appendix B together with the fact that τ_1 and τ_2 are smooth so that their Fourier transforms decay faster than any inverse power at infinity, shows that

$$\begin{aligned} |\widetilde{\Delta_{2^j}\tau_1}(r)| &\le \frac{C}{|r|^{\frac{1}{2}}(1+|r|)^N} \mathbb{1}_{\{2^{j-1}\le|r|\le 2^{j+1}\}}, \\ |\tilde{\tau}_3^j(r)| &\le \frac{C}{|r|^{\frac{1}{2}}(1+|r|)^N} \mathbb{1}_{\{|r|\le 2^{j+1}\}}, \end{aligned} \tag{6-17}$$

for any arbitrarily large $N > 0$.

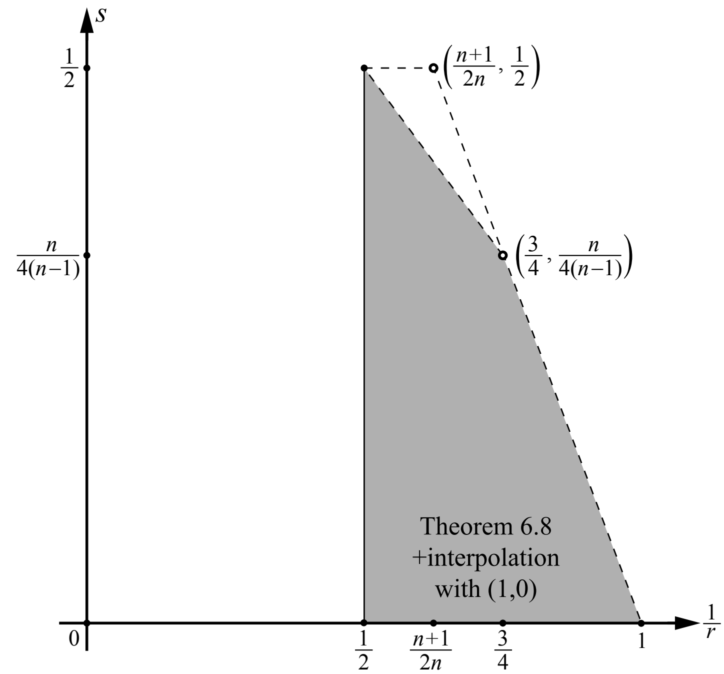

Figure 2. Range of validity of the parameters $\frac{1}{r}$ and s in Theorem 6.8 extended by interpolation with the degenerate L^1 case.

Furthermore, in view of Lemma B.4, it holds that each τ_3^j satisfies

$$|\tau_3^j(r)| \leq \frac{C}{1+|r|^{\frac{1}{2}}}$$

for some uniform $C > 0$ independent of $j \in \mathbb{Z}$, whence, for any $0 < s_0 < \frac{1}{2}$,

$$\frac{\tau_3^j(r)}{|r|^{\frac{1}{2}-s_0}} \in L^2(\mathbb{R}) \quad \text{and} \quad (1+|r|^{s_0})\tau_3^j(r) \in L^\infty(\mathbb{R}), \tag{6-18}$$

uniformly in $j \in \mathbb{Z}$.

Therefore, using the bounds (6-17) and (6-18) to evaluate the terms involving $\rho_1 = \Delta_{2^j}\tau_1$ and $\rho_2 = \tau_3^j$ in the right-hand side of (6-15), we compute that the corresponding norm of the operator in (6-15) is no larger than a multiple of

$$\left(\frac{2^{\frac{j}{2}}}{1+2^{jN}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{\frac{j}{2}}(1+2^{jN})}\right)^{\frac{1}{2}} 2^{j\,\frac{2n-3}{4(n-1)}}(1+2^j)^{\frac{1}{2(n-1)}} \leq C\frac{2^{j\,\frac{2n-3}{4(n-1)}}}{(1+2^j)^{N-\frac{1}{2(n-1)}}},$$

which is summable over $j \in \mathbb{Z}$, provided $N > \frac{2n-1}{4(n-1)}$.

Thus, we conclude, according to Proposition 6.7, that the operators in the right-hand side of (6-16) are bounded.

It follows that, for any $0 < t < 1$,

$$\left\| (1-\Delta_x)^{\frac{s}{2}} \int_{\mathbb{R}^n} f\varphi \, dv \right\|_{L^r_x} \le C\left(\frac{1}{t^{\frac{n}{4(n-1)}}} \|f\|_{L^r_x L^2_v} + t^{1-\frac{n}{4(n-1)}} \|g\|_{L^r_x L^2_v} \right),$$

where $C > 0$ only depends on constant parameters. □

As already mentioned at the end of our Introduction, it is possible that Theorem 6.8 may be largely improved. In fact, the formal interpolation result (1-3) seems to indicate that Theorem 6.8 should hold for all parameters $\frac{2n}{n+1} \le r \le 2$ and $1 \le s < \frac{1}{2}$. The range of parameters defined by $\frac{3}{4} \le r \le 2$ and

$$0 \le s < \frac{1}{2}\left(3 - \frac{4}{r}\right) + \frac{n}{4(n-1)}\left(\frac{4}{r} - 2\right)$$

would then be recovered by interpolation with the degenerate L^1 case.

Indeed, Figure 2 represents the range of validity of the parameters $\frac{1}{r}$ and s in Theorem 6.8 extended by interpolation with the degenerate L^1 case. More precisely, Theorem 6.8 handles the region bounded by the points $\left(\frac{1}{2}, 0\right)$, $\left(\frac{3}{4}, 0\right)$, $\left(\frac{3}{4}, \frac{n}{4(n-1)}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$, which yields the shaded region in Figure 2 when interpolated with the trivial L^1 case corresponding to the point $(1, 0)$. Observe that the points $\left(\frac{n+1}{2n}, \frac{1}{2}\right)$, $\left(\frac{3}{4}, \frac{n}{4(n-1)}\right)$ and $(1, 0)$ are all supported by the same line. It seems therefore natural to conjecture that a similar result should hold for all parameters encompassed by the area delimited by the points $\left(\frac{1}{2}, 0\right)$, $(1, 0)$, $\left(\frac{n+1}{2n}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$; see [Arsénio 2015] for more on such conjectures. This situation strongly resembles the corresponding existing conjectures for the boundedness of Bochner–Riesz multipliers and Fourier restriction operators.

Appendix A. Some paradifferential calculus

In this appendix, we record for reference a useful technical lemma. The proof of this lemma is based on classical methods from paradifferential calculus and paraproduct decompositions.

Lemma A.1. *Let $\chi_1 \in \mathcal{S}(\mathbb{R}^{n-1})$ and $\chi_2 \in \mathcal{S}(\mathbb{R})$. For each $i \in \mathbb{N}$ and $L > 0$, we define*

$$h_i^L(x) = \chi_1\left(\frac{x'}{L}\right)\chi_2\left(\frac{2^i x_n}{L}\right),$$

where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Further consider fixed parameters $s > 0$ and $0 < \lambda < 1$.

Then, for all $g \in L^\infty(\mathbb{R}^n)$, it holds that

$$\|(1-\Delta)^{-\frac{s}{2}}((\Delta_{2^k} g) h_i^L)\|_{L^2(\mathbb{R}^n)} \le \frac{C L^{\frac{n}{2}}}{2^{\frac{i}{2}} 2^{ks}} \|g\|_{L^\infty(\mathbb{R}^n)}$$

for every $i, k \in \mathbb{N}$ such that $2^i \le L 2^{\lambda k}$, where $C > 0$ is independent of i, k and L.

Proof. We first write a standard paraproduct decomposition (see (5-3) for the definition of dyadic blocks and the Littlewood–Paley decomposition):

$$(\Delta_{2^k} g) h_i^L = \Delta_{2^k} g \left(\Delta_0 h_i^L + \sum_{j=0}^{k-3} \Delta_{2^j} h_i^L \right) + \Delta_{2^k} g \sum_{j=k-2}^{k+2} \Delta_{2^j} h_i^L + \sum_{j=k+3}^{\infty} \Delta_{2^k} g \Delta_{2^j} h_i^L.$$

It is then easy to see that in the right-hand side above

(1) the first term has frequencies localized inside an annulus of inner radius 2^{k-2} and outer radius $9\cdot 2^{k-2}$,

(2) the second term has frequencies localized inside a ball of radius $5\cdot 2^{k+1}$,

(3) each summand in the third term has frequencies localized inside an annulus of inner radius 2^{j-2} and outer radius $9\cdot 2^{j-2}$.

Accordingly, we estimate that

$$\begin{aligned}
&\|(1-\Delta)^{-\frac{s}{2}}((\Delta_{2^k} g)h_i^L)\|_{L^2} \\
&\quad\le \frac{C}{2^{ks}}\|g\|_{L^\infty}\|h_i^L\|_{L^2} + C\|g\|_{L^\infty}\sum_{j=k-2}^{k+2}\|\Delta_{2^j}h_i^L\|_{L^2} + C\bigg(\sum_{j=k+3}^{\infty}\frac{1}{2^{js}}\bigg)\|g\|_{L^\infty}\|h_i^L\|_{L^2} \\
&\quad\le \frac{CL^{\frac{n}{2}}}{2^{\frac{i}{2}}2^{ks}}\|g\|_{L^\infty} + C\|g\|_{L^\infty}\sum_{j=k-2}^{k+2}\|\Delta_{2^j}h_i^L\|_{L^2}.
\end{aligned} \tag{A-1}$$

There only remains to control the terms $\|\Delta_{2^j}h_i^L\|_{L^2}$ above. To this end, noticing that

$$\begin{aligned}
\psi(\eta) &= \psi(\eta)(\mathbb{1}_{\{|\eta'|\ge\frac14\}} + \mathbb{1}_{\{|\eta'|<\frac14\}}) \\
&\le \mathbb{1}_{\{\frac14\le|\eta'|\le 2\}}\mathbb{1}_{\{|\eta_n|\le 2\}} + \mathbb{1}_{\{|\eta'|<\frac14\}}\mathbb{1}_{\{\frac14\le|\eta_n|\le 2\}} \le \mathbb{1}_{\{\frac14\le|\eta'|\le 2\}} + \mathbb{1}_{\{\frac14\le|\eta_n|\le 2\}},
\end{aligned}$$

where $\eta' = (\eta_1,\dots,\eta_{n-1})\in\mathbb{R}^{n-1}$, and using Plancherel's theorem, we obtain

$$\begin{aligned}
\|\Delta_{2^j}h_i^L\|_{L^2} &= \frac{L^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}2^i}\bigg\|\psi\Big(\frac{\eta}{L2^j}\Big)\hat\chi_1(\eta')\hat\chi_2\Big(\frac{\eta_n}{2^i}\Big)\bigg\|_{L^2_\eta} \\
&\le \frac{L^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}2^i}\|\mathbb{1}_{\{L2^{j-2}\le|\eta'|\le L2^{j+1}\}}\hat\chi_1(\eta')\|_{L^2_{\eta'}}\bigg\|\hat\chi_2\Big(\frac{\eta_n}{2^i}\Big)\bigg\|_{L^2_{\eta_n}} \\
&\qquad + \frac{L^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}2^i}\|\hat\chi_1(\eta')\|_{L^2_{\eta'}}\bigg\|\mathbb{1}_{\{L2^{j-2}\le|\eta_n|\le L2^{j+1}\}}\hat\chi_2\Big(\frac{\eta_n}{2^i}\Big)\bigg\|_{L^2_{\eta_n}} \\
&\le C\frac{L^{\frac{n}{2}}(L2^j)^{\frac{n-1}{2}}}{2^{\frac{i}{2}}}\|\mathbb{1}_{\{\frac14\le|\eta'|\le 2\}}\hat\chi_1(L2^j\eta')\|_{L^2_{\eta'}} \\
&\qquad + C\frac{L^{\frac{n}{2}}(L2^j)^{\frac12}}{2^i}\bigg\|\mathbb{1}_{\{\frac14\le|\eta_n|\le 2\}}\hat\chi_2\Big(\frac{L2^j}{2^i}\eta_n\Big)\bigg\|_{L^2_{\eta_n}}.
\end{aligned}$$

Hence, recalling that both $\hat\chi_1$ and $\hat\chi_2$ decay faster than any inverse power at infinity, we find, for any given large $N_1, N_2 > 0$,

$$\begin{aligned}
\|\Delta_{2^j}h_i^L\|_{L^2} &\le C\frac{L^{\frac{n}{2}}}{2^{\frac{i}{2}}(L2^j)^{N_1-\frac{n-1}{2}}} + C\frac{L^{\frac{n}{2}}2^{(N_2-\frac12)i}}{2^{\frac{i}{2}}(L2^j)^{N_2-\frac12}} \\
&\le C\frac{L^{\frac{n}{2}}}{2^{\frac{i}{2}}}\bigg(\Big(\frac{2^i}{L2^j}\Big)^{N_1-\frac{n-1}{2}} + \Big(\frac{2^i}{L2^j}\Big)^{N_2-\frac12}\bigg) \\
&\le C\frac{L^{\frac{n}{2}}}{2^{\frac{i}{2}}}\bigg(\frac{1}{2^{(N_1-\frac{n-1}{2})(1-\lambda)k}} + \frac{1}{2^{(N_2-\frac12)(1-\lambda)k}}\bigg),
\end{aligned}$$

so that, choosing N_1 and N_2 such that

$$N_1 - \frac{n-1}{2} = N_2 - \frac{1}{2} \geq \frac{s}{1-\lambda},$$

we get

$$\|\Delta_{2^j} h_i^L\|_{L^2} \leq C \frac{L^{\frac{n}{2}}}{2^{\frac{i}{2}} 2^{ks}}, \tag{A-2}$$

where $C > 0$ is independent of i, k and L.

On the whole, incorporating (A-2) into (A-1) yields

$$\|(1-\Delta)^{-\frac{s}{2}}((\Delta_{2^k} g) h_i^L)\|_{L^2} \leq \frac{C L^{\frac{n}{2}}}{2^{\frac{i}{2}} 2^{ks}} \|g\|_{L^\infty}. \qquad \square$$

Appendix B. Boundedness of Fourier transforms in L^p, with $1 \leq p < 2$

For reference, we show here a few handy criteria for establishing the boundedness in Lebesgue spaces L^p, with $1 \leq p < 2$, of Fourier transforms of given functions.

Lemma B.1. *Let $f(x) \in C^\alpha(\mathbb{R}^n)$, for some given $\alpha \in \mathbb{N}$, be such that*

$$\sup_{\substack{\gamma \in \mathbb{N}^n \\ |\gamma| \leq \alpha}} |x|^{|\gamma|} |\partial_x^\gamma f(x)| \leq \frac{C}{(1+|x|)^\lambda} \quad \text{for all } x \in \mathbb{R}^n, \tag{B-1}$$

for some $\lambda > 0$.

Then, the Fourier transform $\hat{f}$ belongs to $L^p(\mathbb{R}^n)$ for any $1 \leq p < 2$ satisfying

$$\alpha > n\left(\frac{1}{p} - \frac{1}{2}\right) \quad \text{and} \quad \lambda > n\left(1 - \frac{1}{p}\right). \tag{B-2}$$

In particular, for any given $1 \leq p < 2$ and any $\beta > \frac{1}{p'}$, the Fourier transform of $(1+|x|^2)^{-\frac{n\beta}{2}}$ belongs to $L^p(\mathbb{R}^n)$.

Proof. Let $\psi_0(x), \psi(x) \in C_c^\infty(\mathbb{R}^n)$ be compactly supported smooth cutoff functions, whose supports satisfy

$$\operatorname{supp} \psi_0 \subset \{|x| \leq 1\} \quad \text{and} \quad \operatorname{supp} \psi \subset \left\{\tfrac{1}{2} \leq |x| \leq 2\right\},$$

and such that

$$\psi_0(x) + \sum_{j=0}^{\infty} \psi\left(\frac{x}{2^j}\right) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

We define $g(\eta), h_j(\eta) \in \mathcal{S}(\mathbb{R}^n)$, for each $j \in \mathbb{N}$, by the inverse Fourier transforms

$$\tilde{g}(x) = \psi_0(x) f(x) \quad \text{and} \quad \tilde{h}_j(x) = \psi\left(\frac{x}{2^j}\right) f(x),$$

so that

$$\hat{f}(\eta) = g(\eta) + \sum_{j=0}^{\infty} h_j(\eta) \quad \text{in } \mathcal{S}'(\mathbb{R}^n). \tag{B-3}$$

Then, for any $1 \leq p < 2$ satisfying (B-2), so that $(2p\alpha)/(2-p) > n$, and by Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |g(\eta)|^p \, d\eta &= \int_{\mathbb{R}^n} (1+|\eta|)^{p\alpha} |g(\eta)|^p \frac{1}{(1+|\eta|)^{p\alpha}} \, d\eta \\ &\leq \left(\int_{\mathbb{R}^n} (1+|\eta|)^{2\alpha} |g(\eta)|^2 \, d\eta \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\eta|)^{\frac{2p\alpha}{2-p}}} \, d\eta \right)^{1-\frac{p}{2}} \\ &\leq C \left(\int_{\mathbb{R}^n} (1+|\eta|)^{2\alpha} |g(\eta)|^2 \, d\eta \right)^{\frac{p}{2}},\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_{\mathbb{R}^n} |h_j(\eta)|^p \, d\eta &= \int_{\mathbb{R}^n} (1+2^j|\eta|)^{p\alpha} |h_j(\eta)|^p \frac{1}{(1+2^j|\eta|)^{p\alpha}} \, d\eta \\ &\leq \left(\int_{\mathbb{R}^n} (1+2^j|\eta|)^{2\alpha} |h_j(\eta)|^2 \, d\eta \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1+2^j|\eta|)^{\frac{2p\alpha}{2-p}}} \, d\eta \right)^{1-\frac{p}{2}} \\ &\leq C 2^{-jn(1-\frac{p}{2})} \left(\int_{\mathbb{R}^n} (1+2^j|\eta|)^{2\alpha} |h_j(\eta)|^2 \, d\eta \right)^{\frac{p}{2}},\end{aligned}$$

where $C > 0$ only depends on p, α and the dimension.

Next, since

$$(1+2^j|\eta|)^\alpha \leq C\left(1 + 2^{j\alpha} \sum_{i=1}^n |\eta_i|^\alpha\right),$$

it follows from Plancherel's theorem that

$$\|g(\eta)\|_{L^p} \leq C\|(1+|\eta|)^\alpha g(\eta)\|_{L^2} \leq C \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|\leq\alpha}} \|\partial_x^\gamma(\psi_0(x) f(x))\|_{L^2} \leq C \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|\leq\alpha}} \|\mathbb{1}_{\{|x|\leq 1\}} \partial_x^\gamma f(x)\|_{L^2} < \infty.$$

and, further using (B-1),

$$\begin{aligned}2^{jn(\frac{1}{p}-\frac{1}{2})} \|h_j(\eta)\|_{L^p} &\leq C\|(1+2^j|\eta|)^\alpha h_j(\eta)\|_{L^2} \\ &\leq C \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|\leq\alpha}} 2^{j|\gamma|} \left\| \partial_x^\gamma \left(\psi\left(\frac{x}{2^j}\right) f(x) \right) \right\|_{L^2} \\ &\leq C \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|\leq\alpha}} 2^{j|\gamma|} \left\| \mathbb{1}_{\{2^{j-1}\leq|x|\leq 2^{j+1}\}} \partial_x^\gamma f(x) \right\|_{L^2} \\ &\leq C \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|\leq\alpha}} \left\| \mathbb{1}_{\{2^{j-1}\leq|x|\leq 2^{j+1}\}} |x|^{|\gamma|} \partial_x^\gamma f(x) \right\|_{L^2} \leq C \frac{2^{j\frac{n}{2}}}{1+2^{j\lambda}}.\end{aligned}$$

Hence, for any large $N \in \mathbb{N}$, since $\lambda - n\left(1-\frac{1}{p}\right) > 0$,

$$\sup_{N\in\mathbb{N}} \left\| g(\eta) + \sum_{j=0}^N h_j(\eta) \right\|_{L^p} \leq C \sum_{j=0}^\infty \frac{2^{jn(1-\frac{1}{p})}}{1+2^{j\lambda}} < \infty.$$

Therefore, according to (B-3), we deduce that the tempered distribution $\hat{f}$ coincides with the weak limit of functions uniformly bounded in L^p, which implies that $\hat{f} \in L^p$ for any $1 \le p < 2$ satisfying (B-2). □

Lemma B.2. *Let $f(x) \in C^\alpha(\mathbb{R}^n \setminus \{0\})$, for some given $\alpha \in \mathbb{N}$, be such that*

$$\sup_{\substack{\gamma \in \mathbb{N}^n \\ |\gamma| \le \alpha}} |x|^{|\gamma|} |\partial_x^\gamma f(x)| \le \frac{C|x|^\sigma}{(1+|x|)^{\lambda+\sigma}} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{B-4}$$

for some $\lambda > 0$ and $\sigma > -\lambda$.

Then, the Fourier transform $\hat{f}$ belongs to $L^p(\mathbb{R}^n)$ for any $1 \le p < 2$ satisfying

$$\alpha > n\left(\frac{1}{p} - \frac{1}{2}\right) \quad \text{and} \quad \lambda > n\left(1 - \frac{1}{p}\right) > -\sigma. \tag{B-5}$$

In particular, for any given $1 \le p < 2$ and any $\beta > \frac{1}{p'} > -\delta$, the Fourier transform of $|x|^{n\delta}(1+|x|^2)^{-\frac{n(\beta+\delta)}{2}}$ belongs to $L^p(\mathbb{R}^n)$.

Proof. Let $\psi(x) \in C_c^\infty(\mathbb{R}^n)$ be a compactly supported smooth cutoff function whose support satisfies

$$\operatorname{supp} \psi \subset \left\{\tfrac{1}{2} \le |x| \le 2\right\},$$

and such that

$$\sum_{j \in \mathbb{Z}} \psi\left(\frac{x}{2^j}\right) = 1 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

We define $h_j(\eta) \in \mathcal{S}(\mathbb{R}^n)$, for each $j \in \mathbb{Z}$, by the inverse Fourier transforms

$$\tilde{h}_j(x) = \psi\left(\frac{x}{2^j}\right) f(x),$$

so that

$$\hat{f}(\eta) = \sum_{j \in \mathbb{Z}} h_j(\eta) \quad \text{in } \mathcal{S}'(\mathbb{R}^n). \tag{B-6}$$

Then, for any $1 \le p < 2$ satisfying (B-5), so that $(2p\alpha)/(2-p) > n$, and by Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |h_j(\eta)|^p \, d\eta &= \int_{\mathbb{R}^n} (1+2^j|\eta|)^{p\alpha} |h_j(\eta)|^p \frac{1}{(1+2^j|\eta|)^{p\alpha}} \, d\eta \\
&\le \left(\int_{\mathbb{R}^n} (1+2^j|\eta|)^{2\alpha} |h_j(\eta)|^2 \, d\eta\right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1+2^j|\eta|)^{\frac{2p\alpha}{2-p}}} \, d\eta\right)^{1-\frac{p}{2}} \\
&\le C 2^{-jn\left(1-\frac{p}{2}\right)} \left(\int_{\mathbb{R}^n} (1+2^j|\eta|)^{2\alpha} |h_j(\eta)|^2 \, d\eta\right)^{\frac{p}{2}},
\end{aligned}$$

where $C > 0$ only depends on p, α and the dimension.

Next, since

$$(1+2^j|\eta|)^\alpha \le C\left(1 + 2^{j\alpha} \sum_{i=1}^n |\eta_i|^\alpha\right),$$

it follows from Plancherel's theorem and (B-4) that

$$\begin{aligned} 2^{jn(\frac{1}{p}-\frac{1}{2})}\|h_j(\eta)\|_{L^p} &\le C\|(1+2^j|\eta|)^\alpha h_j(\eta)\|_{L^2} \\ &\le C \sum_{\substack{\gamma\in\mathbb{N}^n\\ |\gamma|\le\alpha}} 2^{j|\gamma|}\left\|\partial_x^\gamma\left(\psi\left(\frac{x}{2^j}\right)f(x)\right)\right\|_{L^2} \\ &\le C \sum_{\substack{\gamma\in\mathbb{N}^n\\ |\gamma|\le\alpha}} 2^{j|\gamma|}\left\|\mathbb{1}_{\{2^{j-1}\le|x|\le 2^{j+1}\}}\partial_x^\gamma f(x)\right\|_{L^2} \\ &\le C \sum_{\substack{\gamma\in\mathbb{N}^n\\ |\gamma|\le\alpha}} \left\|\mathbb{1}_{\{2^{j-1}\le|x|\le 2^{j+1}\}}|x|^{|\gamma|}\partial_x^\gamma f(x)\right\|_{L^2} \le C\frac{2^{j(\frac{n}{2}+\sigma)}}{1+2^{j(\lambda+\sigma)}}. \end{aligned}$$

Hence, for any large $N\in\mathbb{N}$, since $\lambda-n\left(1-\frac{1}{p}\right)>0$ and $\sigma+n\left(1-\frac{1}{p}\right)>0$,

$$\sup_{N\in\mathbb{N}}\left\|\sum_{j=-N}^{N}h_j(\eta)\right\|_{L^p}\le C\sum_{j\in\mathbb{Z}}\frac{2^{j\left(n\left(1-\frac{1}{p}\right)+\sigma\right)}}{1+2^{j(\lambda+\sigma)}}<\infty.$$

Therefore, according to (B-6), we deduce that the tempered distribution $\hat{f}$ coincides with the weak limit of functions uniformly bounded in L^p, which implies that $\hat{f}\in L^p$ for any $1\le p<2$ satisfying (B-5). □

Lemma B.3. *Let $f(x)\in C^1(\mathbb{R}\setminus\{0\})$ be such that*

$$|f(x)|,\,|xf'(x)|\le\frac{C}{|x|^\lambda}\quad\text{for all } x\in\mathbb{R}^n\setminus\{0\},$$

for some $0<\lambda<1$.

Then, the Fourier transform $\hat{f}$ belongs to L^1+L^∞ and satisfies

$$|\hat{f}(\eta)|\le\frac{C}{|\eta|^{1-\lambda}}\quad\text{for almost every } \eta\in\mathbb{R}^n,$$

for some independent constant $C>0$.

Proof. Consider a cutoff $\chi\in C_c^\infty(\mathbb{R})$ such that $\mathbb{1}_{\{|x|\le 1\}}\le\chi(x)\le\mathbb{1}_{\{|x|\le 2\}}$. Then, on the one hand, the function $\chi(x)f(x)$ clearly is integrable so that its Fourier transform is bounded pointwise almost everywhere. On the other hand, the function $(1-\chi)(x)f(x)$ clearly verifies the hypotheses of Lemma B.1 so that its Fourier transform always coincides with an integrable function. This establishes that $\hat{f}\in L^1+L^\infty$.

Next, for any $t>0$, we have the estimate

$$\begin{aligned} |\hat{f}(\eta)| &= \left|\int_{\mathbb{R}} e^{-i\eta x}f(x)\,dx\right| \\ &\le \left|\int_{\mathbb{R}} e^{-i\eta x}\chi\left(\frac{x}{t}\right)f(x)\,dx\right| + \left|\int_{\mathbb{R}} e^{-i\eta x}(1-\chi)\left(\frac{x}{t}\right)f(x)\,dx\right| \end{aligned}$$

$$\leq \int_{\{|x|\leq 2t\}} |f(x)|\,dx + \frac{1}{|\eta|}\left|\int_{\mathbb{R}} e^{-i\eta x}\left((1-\chi)\left(\frac{x}{t}\right)f(x)\right)' dx\right|$$

$$\leq C\int_{\{|x|\leq 2t\}} \frac{1}{|x|^{\lambda}}\,dx + \frac{C}{t|\eta|}\int_{\{t\leq|x|\leq 2t\}} \frac{1}{|x|^{\lambda}}\,dx + \frac{C}{|\eta|}\int_{\{|x|\geq t\}} \frac{1}{|x|^{\lambda+1}}\,dx \leq C\left(t^{1-\lambda} + \frac{1}{|\eta| t^{\lambda}}\right).$$

Therefore, optimizing the preceding estimate in $t > 0$, which amounts to setting $t = \frac{1}{|\eta|}$ above, yields

$$|\hat{f}(\eta)| \leq \frac{C}{|\eta|^{1-\lambda}}. \qquad \square$$

Lemma B.4. *Let $f \in L^{\infty}(\mathbb{R})$ be such that*

$$|f(x)| \leq \frac{C}{1+|x|^{\alpha}} \quad \textit{for all } x \in \mathbb{R},$$

for some $0 \leq \alpha < 1$, and consider the convolution

$$f_R(x) = \int_{\mathbb{R}} R\chi(R(x-y))\,f(y)\,dy$$

for any $R > 0$, where $\chi \in \mathcal{S}(\mathbb{R})$.

Then, the convolution f_R also satisfies

$$|f_R(x)| \leq \frac{C}{1+|x|^{\alpha}} \quad \textit{for all } x \in \mathbb{R},$$

for some constant $C > 0$ independent of R.

Proof. Note first that

$$\|f_R\|_{L^{\infty}} \leq \|\chi\|_{L^1}\,\|f\|_{L^{\infty}}.$$

Therefore, we only have to consider values $|x| \geq 1$, say. Furthermore, by possibly replacing χ and f by $|\chi|$ and $|f|$, respectively, we may assume that χ and f are both nonnegative.

Then, for any $N > 1$, we estimate that

$$\begin{aligned} f_R(x) &= \int_{\{|y|\geq\frac{|x|}{2}\}} R\chi(R(x-y))\,f(y)\,dy + \int_{\{|y|<\frac{|x|}{2}\}} R\chi(R(x-y))\,f(y)\,dy \\ &\leq \int_{\{|y|\geq\frac{|x|}{2}\}} R\chi(R(x-y))\frac{C}{1+|y|^{\alpha}}\,dy + \int_{\{|y|<\frac{|x|}{2}\}} \frac{CR}{1+R^N|x-y|^N} f(y)\,dy \\ &\leq \frac{C}{1+|x|^{\alpha}}\int_{\{|y|\geq\frac{|x|}{2}\}} R\chi(R(x-y))\,dy + \frac{CR}{1+R^N|x|^N}\int_{\{|y|<\frac{|x|}{2}\}} \frac{1}{1+|y|^{\alpha}}\,dy \\ &\leq \frac{C}{1+|x|^{\alpha}} + \frac{CR}{1+R^N|x|^N}|x|^{1-\alpha}. \end{aligned}$$

Further noticing that

$$\sup_{R>0} \frac{R}{1+R^N|x|^N} \leq \max\left\{\sup_{0<R\leq\frac{1}{|x|}} R,\ \sup_{R>\frac{1}{|x|}} \frac{1}{R^{N-1}|x|^N}\right\} \leq \frac{1}{|x|},$$

we deduce

$$f_R(x) \leq \frac{C}{|x|^{\alpha}}. \qquad \square$$

Appendix C. Velocities restricted to a manifold of codimension 1

In this final independent appendix section, we explore the connection between averaging lemmas for the stationary transport equation (1-2) for velocities in the Euclidean space $v \in \mathbb{R}^n$ and averaging lemmas for the same equation for velocities lying in an appropriate manifold of codimension 1 in $\mathbb{R}^n$. Here, for simplicity, we only consider the case $v \in \mathbb{S}^{n-1}$. However, the elementary methods developed here can be used to establish similar connections with settings in other manifolds of codimension 1. In particular, this approach includes the time dependent case (1-1) where $(t,x) \in \mathbb{R}^{n+1}$ and $v \in \mathbb{R}^n$ and, thus, allows us to translate the main results contained in this work to several other interesting and relevant situations.

Proposition C.1. *Let $n \geq 2$, $s > 0$ and $1 \leq p,q,r \leq \infty$ be such that*

$$p \leq r, \quad s + n\Big(\frac{1}{p} - \frac{1}{r}\Big) \leq 1 \quad \textit{and} \quad \frac{1}{q} + s + \Big(\frac{1}{p} - \frac{1}{r}\Big) \leq 1, \tag{C-1}$$

and suppose that, for any $\varphi \in L_c^\infty(\mathbb{R}^n)$, there exists $C > 0$ such that one has the estimate

$$\left\| \int_{\mathbb{R}^n} f\varphi\, dv \right\|_{W_x^{s,r}} \leq C(\|f\|_{L_x^p L_v^q} + \|g\|_{L_x^p L_v^q}) \tag{C-2}$$

for any $f, g \in L_x^p(\mathbb{R}^n; L_v^q(\mathbb{R}^n))$ such that (1-2) holds true.

Then, for some other constant $C > 0$, one has the estimate

$$\left\| \int_{\mathbb{S}^{n-1}} f\, dv \right\|_{W_x^{s,r}} \leq C(\|f\|_{L_x^p L_v^q} + \|g\|_{L_x^p L_v^q}) \tag{C-3}$$

for any $f, g \in L_x^p(\mathbb{R}^n; L_v^q(\mathbb{S}^{n-1}))$ such that (1-2) holds true.

It is possible to show, though a dimensional analysis, that the restrictions (C-1) are in fact necessary in order that (C-2) may hold; see [Arsénio 2015, Section 4] for details.

Proof. We employ a strategy similar to the one used in [Arsénio and Saint-Raymond 2011, Appendix C] to go from the stationary case to a time-dependent setting. To this end, for any $f, g \in L_x^p(\mathbb{R}^n; L_v^q(\mathbb{S}^{n-1}))$ such that (1-2) holds true, we introduce an artificial radial dimension by defining, for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\tilde{f}(x,v) = f\left(x, \frac{v}{|v|}\right)\chi(|v|) \quad \text{and} \quad \tilde{g}(x,v) = |v| g\left(x, \frac{v}{|v|}\right)\chi(|v|)$$

for some given nonnegative cutoff function $\chi \in L_c^\infty(\mathbb{R})$.

Assuming that $\varphi(v) \equiv 1$ on the support of $\chi(|v|)$, it is then readily seen that

$$\int_{\mathbb{R}^n} \tilde{f}\varphi\, dv = \int_0^\infty \chi(r) r^{n-1}\, dr \int_{\mathbb{S}^{n-1}} f\, dv,$$

and

$$\begin{aligned} \|\tilde{f}\|_{L_v^q(\mathbb{R}^n)} &= \|r^{\frac{n-1}{q}} \chi(r)\|_{L_r^q([0,\infty))} \|f\|_{L_v^q(\mathbb{S}^{n-1})}, \\ \|\tilde{g}\|_{L_v^q(\mathbb{R}^n)} &= \|r^{\frac{n-1}{q}+1} \chi(r)\|_{L_r^q([0,\infty))} \|g\|_{L_v^q(\mathbb{S}^{n-1})}. \end{aligned}$$

Further observe that (1-2) also holds with $\tilde{f}$ and $\tilde{g}$ in place of f and g, respectively. Therefore, by plugging $\tilde{f}$ and $\tilde{g}$ into (C-2), we deduce the validity of estimate (C-3). □

A converse to the preceding proposition is also available.

Proposition C.2. *Let $n \geq 2$, $s > 0$ and $1 \leq p, q, r \leq \infty$ be such that*

$$p \leq r, \quad s + n\Big(\frac{1}{p} - \frac{1}{r}\Big) \leq 1 \quad \text{and} \quad \frac{1}{q} + s + \Big(\frac{1}{p} - \frac{1}{r}\Big) \leq 1, \tag{C-4}$$

and suppose that there exists $C > 0$ such that one has the estimate

$$\Big\| \int_{\mathbb{S}^{n-1}} f \, dv \Big\|_{W^{s,r}_x} \leq C(\|f\|_{L^p_x L^q_v} + \|g\|_{L^p_x L^q_v}) \tag{C-5}$$

for any $f, g \in L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))$ such that (1-2) *holds true.*

Then, for any $\varphi \in L^\infty_c(\mathbb{R}^n)$, there exists $C > 0$ such that one has the estimate

$$\begin{aligned} \Big\| \int_{\mathbb{R}^n} f\varphi \, dv \Big\|_{W^{s,r}_x} &\leq C(\|f\|_{L^p_x L^q_v} + \|g\|_{L^p_x L^q_v}) \quad \text{if } p \leq q, \\ \Big\| \int_{\mathbb{R}^n} f\varphi \, dv \Big\|_{W^{s,r}_x} &\leq C(\|f\|_{L^q_v L^p_x} + \|g\|_{L^q_v L^p_x}) \quad \text{if } p \geq q \end{aligned} \tag{C-6}$$

for any $f, g \in L^p_x(\mathbb{R}^n; L^q_v(\mathbb{R}^n))$ if $p \leq q$, or $f, g \in L^q_v(\mathbb{R}^n; L^p_x(\mathbb{R}^n))$ if $p \leq q$, such that (1-2) *holds true.*

Proof. We first assume that $p \leq q$. For any $f, g \in L^p_x(\mathbb{R}^n; L^q_v(\mathbb{R}^n))$ such that (1-2) holds true, we define, for all $(x, v) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, $\lambda > 0$ and $\varphi \in L^\infty_c(\mathbb{R}^n)$,

$$\tilde{f}_\lambda(x, v) = f(\lambda x, \lambda v)\, \varphi(\lambda v) \quad \text{and} \quad \tilde{g}_\lambda(x, v) = g(\lambda x, \lambda v)\, \varphi(\lambda v).$$

It is then readily seen that

$$\begin{aligned} \Big\| \int_{\mathbb{S}^{n-1}} \tilde{f}_\lambda \, dv \Big\|_{L^r_x} &= \lambda^{-\frac{n}{r}} \Big\| \int_{\mathbb{S}^{n-1}} f(x, \lambda v)\, \varphi(\lambda v) \, dv \Big\|_{L^r_x}, \\ \Big\| \int_{\mathbb{S}^{n-1}} \tilde{f}_\lambda \, dv \Big\|_{\dot{W}^{s,r}_x} &= \lambda^{s-\frac{n}{r}} \Big\| \int_{\mathbb{S}^{n-1}} f(x, \lambda v)\, \varphi(\lambda v) \, dv \Big\|_{\dot{W}^{s,r}_x} \end{aligned}$$

and

$$\begin{aligned} \|\tilde{f}_\lambda\|_{L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))} &= \lambda^{-\frac{n}{p}} \|f(x, \lambda v)\, \varphi(\lambda v)\|_{L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))}, \\ \|\tilde{g}_\lambda\|_{L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))} &= \lambda^{-\frac{n}{p}} \|g(x, \lambda v)\, \varphi(\lambda v)\|_{L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))}. \end{aligned}$$

Further observe that (1-2) also holds with $\tilde{f}_\lambda$ and $\tilde{g}_\lambda$ in place of f and g, respectively. Therefore, by plugging $\tilde{f}$ and $\tilde{g}$ into (C-5), we deduce that

$$\begin{aligned} \lambda^{n(\frac{1}{p}-\frac{1}{r})} \Big\| \int_{\mathbb{S}^{n-1}} f(x, \lambda v)\, \varphi(\lambda v) \, dv \Big\|_{L^r_x} + \lambda^{s+n(\frac{1}{p}-\frac{1}{r})} \Big\| \int_{\mathbb{S}^{n-1}} f(x, \lambda v)\, \varphi(\lambda v) \, dv \Big\|_{\dot{W}^{s,r}_x} \\ \leq C\big(\|f(x, \lambda v)\, \varphi(\lambda v)\|_{L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))} + \|g(x, \lambda v)\, \varphi(\lambda v)\|_{L^p_x(\mathbb{R}^n; L^q_v(\mathbb{S}^{n-1}))}\big). \end{aligned}$$

Next, recalling that φ is compactly supported within some large ball $B(0,R)$, say, noticing that $\lambda^{\frac{n-1}{q'}-s-n\left(\frac{1}{p}-\frac{1}{r}\right)} \in L^{q'}_\lambda([0,R])$, by (C-4), and then integrating the preceding estimate in λ over $[0,R]$, we find

$$\begin{aligned}
\left\| \int_{\mathbb{R}^n} f(x,v)\,\varphi(v)\,dv \right\|_{W^{s,r}_x} &\le \int_0^R \lambda^{n-1} \left\| \int_{\mathbb{S}^{n-1}} f(x,\lambda v)\,\varphi(\lambda v)\,dv \right\|_{W^{s,r}_x} d\lambda \\
&\le C \int_0^R \lambda^{n-1-s-n\left(\frac{1}{p}-\frac{1}{r}\right)} \| f(x,\lambda v)\,\varphi(\lambda v)\|_{L^p_x(\mathbb{R}^n;L^q_v(\mathbb{S}^{n-1}))}\,d\lambda \\
&\qquad + C \int_0^R \lambda^{n-1-s-n\left(\frac{1}{p}-\frac{1}{r}\right)} \| g(x,\lambda v)\,\varphi(\lambda v)\|_{L^p_x(\mathbb{R}^n;L^q_v(\mathbb{S}^{n-1}))}\,d\lambda \\
&\le C \left(\int_0^R \| f(x,\lambda v)\,\varphi(\lambda v)\|^q_{L^p_x(\mathbb{R}^n;L^q_v(\mathbb{S}^{n-1}))} \lambda^{n-1}\,d\lambda \right)^{\frac{1}{q}} \\
&\qquad + C \left(\int_0^R \| g(x,\lambda v)\,\varphi(\lambda v)\|^q_{L^p_x(\mathbb{R}^n;L^q_v(\mathbb{S}^{n-1}))} \lambda^{n-1}\,d\lambda \right)^{\frac{1}{q}} \\
&\le C \big(\|f\|_{L^p_x(\mathbb{R}^n;L^q_v(\mathbb{R}^n))} + \|g\|_{L^p_x(\mathbb{R}^n;L^q_v(\mathbb{R}^n))} \big),
\end{aligned}$$

which concludes the proof of (C-6), when $p \le q$.

The case $p \ge q$ is obtained similarly. □

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