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ON THE EXISTENCE AND STABILITY OF BLOWUP FOR WAVE MAPS INTO A NEGATIVELY CURVED TARGET

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We consider wave maps on $(1+d)$-dimensional Minkowski space. For each dimension $d \geq 8$ we construct a negatively curved, d-dimensional target manifold that allows for the existence of a self-similar wave map which provides a stable blowup mechanism for the corresponding Cauchy problem.

1. Introduction

We consider the Cauchy problem for a wave map from the Minkowski spacetime $(\mathbb{R}^{1,d}, \eta)$ into a warped product manifold $N^d = \mathbb{R}^+ \times_g \mathbb{S}^{d-1}$ with metric h; see, e.g., [O'Neill 1983; Tachikawa 1985] for a definition. The metric h has the form

$$h = du^2 + g(u)^2\, d\theta^2, \tag{1-1}$$

where $(u, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$ is the natural polar coordinate system on N^d, $d\theta^2$ is the standard metric on $\mathbb{S}^{d-1}$ and

$$g \in C^\infty(\mathbb{R}), \quad g \text{ is odd}, \quad g'(0) = 1, \quad g > 0 \text{ on } (0, \infty). \tag{1-2}$$

Furthermore, we endow the Minkowski space with standard spherical coordinates $(t, r, \omega) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{S}^{d-1}$. The metric η thereby becomes

$$\eta = -dt^2 + dr^2 + r^2\, d\omega^2. \tag{1-3}$$

In this setting, a map $U : (\mathbb{R}^{1,d}, \eta) \to (N^d, h)$ can be written as

$$U(t, r, \omega) = (u(t, r, \omega), \theta(t, r, \omega)).$$

We restrict our attention to the special subclass of so-called 1-equivariant or corotational maps where

$$u(t, r, \omega) = u(t, r) \quad \text{and} \quad \theta(t, r, \omega) = \omega.$$

Under this ansatz the wave maps equation for U reduces to the single semilinear radial wave equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\,\partial_r\right) u(t, r) + \frac{d-1}{r^2} g(u(t, r)) g'(u(t, r)) = 0; \tag{1-4}$$

see, e.g., [Shatah and Tahvildar-Zadeh 1994].

It is not hard to see that the Cauchy problem for (1-4) is locally well-posed for sufficiently smooth data and even the low-regularity theory is well understood [Shatah and Tahvildar-Zadeh 1994]. Consequently,

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the interesting questions concern the global Cauchy problem and in particular, the formation of singularities in finite time. There is by now a sizable literature on blowup for wave maps which we cannot review here in its entirety. Let it suffice to say that the energy-critical case $d = 2$ attracted particular attention; see, e.g., [Bizoń et al. 2001; Struwe 2003; Krieger et al. 2008; Rodnianski and Sterbenz 2010; Raphaël and Rodnianski 2012; Sterbenz and Tataru 2010a; 2010b; Krieger and Schlag 2012; Côte et al. 2015a; 2015b; Côte 2015; Gao and Krieger 2015; Lawrie and Oh 2016] for recent contributions. In supercritical dimensions $d \geq 3$ the existence of self-similar solutions is typical [Shatah 1988; Turok and Spergel 1990; Cazenave et al. 1998; Bizoń 2000; Bizoń and Biernat 2015] and stability results for blowup were obtained in [Bizoń et al. 2000; Donninger 2011; Donninger et al. 2012; Bizoń and Biernat 2015; Biernat et al. 2017; Chatzikaleas et al. 2017]. For nonexistence of type II blowup see [Dodson and Lawrie 2015]. Note, however, that there exists nonself-similar blowup in sufficiently high dimensions [Ghoul et al. 2018].

According to a heuristic principle, one typically has finite-time blowup if the curvature of the target is positive. For negatively curved targets, on the other hand, one expects global well-posedness. A notable exception to that rule is provided by the construction of a self-similar solution for a negatively curved target for $d = 7$ in [Cazenave et al. 1998], which indicates that the situation is more subtle. Here we show that the example from that paper is not a peculiarity. We construct suitable target manifolds for any dimension $d \geq 8$ that allow for the existence of an *explicit* self-similar solution. Moreover, we claim that the corresponding self-similar blowup is nonlinearly asymptotically stable under small perturbations of the initial data. In the case $d = 9$ we prove this claim rigorously. This provides the first example of *stable* blowup for wave maps into a negatively curved target.

1A. ***Self-similar solutions.*** In order to look for self-similar solutions, we first observe that (1-4) has the natural scaling symmetry

$$u(t,r) \mapsto u_\lambda(t,r) := u\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), \quad \lambda > 0, \tag{1-5}$$

in the sense that if u solves (1-4) then u_λ solves it, too. Consequently, it is natural to look for solutions of the form $u(t,r) = \phi(r/t)$. Taking into account the time-translation and reflection symmetries of (1-4), we arrive at the slightly more general ansatz

$$u(t,r) = \phi(\rho), \quad \rho = \frac{r}{T-t}, \tag{1-6}$$

where the free parameter $T > 0$ is the blowup time. By plugging the ansatz (1-6) into (1-4) we obtain the ordinary differential equation

$$(1-\rho^2)\phi''(\rho) + \left(\frac{d-1}{\rho} - 2\rho\right)\phi'(\rho) - \frac{(d-1)g(\phi(\rho))g'(\phi(\rho))}{\rho^2} = 0. \tag{1-7}$$

By recasting (1-7) into an integral equation and then using a fixed-point argument, one can show that any solution to (1-7) that vanishes together with its first derivative at $\rho = 0$ is identically zero near $\rho = 0$. Therefore, any nontrivial smooth solution ϕ to (1-7) for which $\phi(0) = 0$ must have $\phi'(0) \neq 0$, and since

$$\frac{\partial}{\partial r}\phi\left(\frac{r}{T-t}\right)\bigg|_{r=0} = \frac{\phi'(0)}{T-t}, \tag{1-8}$$

such a ϕ gives rise to a smooth solution of (1-4) which suffers a gradient blowup at the origin in finite time. Furthermore, due to finite speed of propagation, this type of singularity arises from smooth, compactly supported initial data. In the following, we restrict ourselves to the study of the solution in the backward lightcone of the singularity,

$$\mathcal{C}_T := \{(t,r) : t \in [0,T),\ r \in [0, T-t]\}. \tag{1-9}$$

Note that in terms of the coordinate ρ, $\mathcal{C}_T$ corresponds to the interval $[0, 1]$. Consequently, we look for solutions of (1-7) that belong to $C^\infty[0, 1]$.

2. Existence of blowup for a negatively curved target manifold

We construct for every $d \geq 8$ a negatively curved d-dimensional Riemannian manifold (N^d, h) which allows for a wave map $U : (\mathbb{R}^{1,d}, \eta) \to (N^d, h)$ that starts off smooth and blows up in finite time. We do this by a suitable choice of the function g that defines the metric on N^d by means of (1-1). To begin with, we restrict ourselves to small u and set

$$g(u) := u\sqrt{1 + 7u^2 - (23d - 170)u^4}. \tag{2-1}$$

Clearly, g is odd and smooth locally around the origin. Furthermore, $g(u) > 0$ for small $u > 0$ and $g'(0) = 1$; see (1-2). In addition, for $d \geq 8$, the metric (1-1) makes the manifold N^d negatively curved locally around $u = 0$; see Proposition 2.1. Next, (1-4) takes the form

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right)u(t,r) + \frac{(d-1)[u(t,r) + 14u(t,r)^3 - 3(23d-170)u(t,r)^5]}{r^2} = 0, \tag{2-2}$$

and the corresponding ordinary differential equation (1-7) becomes

$$(1-\rho^2)\phi''(\rho) + \left(\frac{d-1}{\rho} - 2\rho\right)\phi'(\rho) - \frac{(d-1)[\phi(\rho) + 14\phi(\rho)^3 - 3(23d-170)\phi(\rho)^5]}{\rho^2} = 0. \tag{2-3}$$

As already discussed, any nonzero function $\phi \in C^\infty[0, 1]$ that solves (2-3) and vanishes at $\rho = 0$ yields a classical solution to (2-2) that blows up in finite time. In fact, (2-3) has an *explicit* formal solution

$$\phi_0(\rho) = \frac{a\rho}{\sqrt{b - \rho^2}}, \tag{2-4}$$

where

$$a = \sqrt{\frac{d}{E(d)}}, \quad b = 1 + \frac{d}{2} - \frac{7d(d-1)}{E(d)}$$

and

$$E(d) = \sqrt{(46d^2 - 291d - 49)(d-1)} + 7(d-1).$$

Furthermore, if $d \geq 8$ then $E(d)$ is positive and $b > 1$, which makes ϕ_0 a smooth and increasing function on $[0, 1]$. Now we have the following result.

Proposition 2.1. *For each $d \geq 8$ there exists an $\varepsilon > 0$ and a function $g : \mathbb{R} \to \mathbb{R}$ satisfying (1-2) such that $g(u) = u\sqrt{1 + 7u^2 - (23d - 170)u^4}$ for $|u| < \phi_0(1) + \varepsilon$ and the manifold N^d with metric given by (1-1) has all sectional curvatures negative.*

The proof is somewhat lengthy but elementary and therefore postponed to Appendix A.

We define

$$u^T(t,r) := \phi_0\left(\frac{r}{T-t}\right), \quad (t,r) \in \mathcal{C}_T. \tag{2-5}$$

Note that $|\phi_0(\rho)| \leq \phi_0(1)$ for all $\rho \in [0, 1]$ and thus,

$$U^T(t,r,\omega) := (u^T(t,r), \omega)$$

is a wave map from $\mathcal{C}_T \subset \mathbb{R}^{1,d}$ to (N^d, h). By finite speed of propagation we obtain the following result.

Theorem 2.2. *For every $d \geq 8$ there exists a d-dimensional, negatively curved Riemannian manifold N^d such that the Cauchy problem for wave maps from Minkowski space $\mathbb{R}^{1,d}$ into N^d admits a solution which develops from smooth Cauchy data of compact support and forms a singularity in finite time.*

Remark 2.3. Our focus in this work was on functions g which lead to polynomial-type nonlinearities gg' in (1-4). Since

$$\frac{d}{du}(g(u)^2) = 2g(u)g'(u),$$

this is equivalent to g^2 being an even polynomial. The lowest-degree even polynomial g^2 which, through the metric (1-1), yields negative curvature (locally around the pole $u = 0$) on the target manifold N^d is of the form

$$g(u)^2 = u^2 + c_1 u^4 + c_2 u^6 \tag{2-6}$$

for an appropriate choice of $c_1, c_2 \in \mathbb{R}$. Furthermore, the function g given by (2-6) gives rise to a (formal) solution to (1-7) of the form (2-4). This solution in turn yields a bona fide self-similar blowup in the corresponding wave maps equation only if it is smooth on $[0, 1]$ and the corresponding function u^T from (2-5) stays inside the negatively curved neighborhood of the pole $u = 0$ whose metric is given by (2-6). The construction of such solutions, by a proper choice of coefficients c_1 and c_2 in (2-6), is in fact possible only for $d \geq 8$. There is, of course, some freedom in the choice of c_1 and c_2, and the one we made in (2-1) was led by the objective of "minimizing" their dependence on d by allowing them to depend at most linearly on it.

In order to determine the role of the solution u^T for generic evolutions, it is necessary to investigate its stability under perturbations. In fact, we claim that for any $d \geq 8$, the self-similar solution (2-5) exhibits *stable* blowup; i.e., there is an open set of radial initial data that give rise to solutions which approach u^T in $\mathcal{C}_T$ as $t \to T^-$. The rest of the paper is devoted to the proof of this stability property. We emphasize that the fact that the solutions we constructed are explicit is crucial for our approach to their stability analysis (see the proof of Proposition 3.7). Due to certain technical difficulties (see Remark 3.17) we restrict ourselves to the lowest odd dimension $d = 9$.

3. Stability of blowup

From now on we fix $d = 9$. In view of (1-4) and (2-1), we consider the Cauchy problem

$$\begin{cases} \left(\partial_t^2 - \partial_r^2 - \frac{8}{r}\,\partial_r\right) u(t,r) + \frac{8[u(t,r) + 14u(t,r)^3 - 111u(t,r)^5]}{r^2} = 0, & (t,r) \in \mathcal{C}_T, \\ u(0,r) = u_0(r), \quad \partial_0 u(0,r) = u_1(r), & r \in [0,T]. \end{cases} \tag{3-1}$$

The restriction to the backward lightcone $\mathcal{C}_T$ is possible and natural by finite speed of propagation. Furthermore, to ensure regularity of the solution at the origin $r = 0$, we impose the boundary condition

$$u(t,0) = 0 \quad \text{for } t \in [0,T). \tag{3-2}$$

The blowup solution (2-4) now becomes

$$u^T(t,r) = \phi_0(\rho) = \frac{3\rho}{\sqrt{2(155 - 74\rho^2)}}, \quad \text{where } \rho = \frac{r}{T-t}. \tag{3-3}$$

Note that by construction, the wave map evolution for the target manifold N^9 is given by (3-1), provided that $|u(t,r)| \le \phi_0(1) + \varepsilon_1$ for some small $\varepsilon_1 > 0$. We are only interested in the evolution in the backward lightcone of the point of blowup and therefore study (3-1) with no a priori restriction on the size of u. A posteriori we show that the solutions we construct stay below $\phi_0(1) + \varepsilon_1$.

Note further that (3-1) can be viewed as a nonlinear wave equation with polynomial nonlinearity. Indeed, the boundary condition (3-2) allows for a change of variable $u(t,r) = r v(t,r)$ which leads to an eleven-dimensional radial wave equation in v,

$$\left(\partial_t^2 - \partial_r^2 - \frac{10}{r}\,\partial_r\right) v(t,r) = -8[14v(t,r)^3 - 111r^2 v(t,r)^5]. \tag{3-4}$$

In fact, this is the point of view we adopt here. In particular, the nonlinear term in (3-1) becomes smooth and therefore admits a uniform Lipschitz estimate needed for a contraction mapping argument; see Lemma 3.12. We also remark that (3-4), in spite of its defocusing character (at least for small values of v), admits an explicit self-similar blowup solution. This is in stark contrast to the cubic defocusing wave equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{10}{r}\partial_r\right) v(t,r) = -v(t,r)^3$$

for which no self-similar solutions exist. The self-similar blowup in (3-4) can therefore be understood as a consequence of the presence of the focusing quintic term which dominates the dynamics for large initial data.

3A. ***Main result.*** We start by intuitively describing the main result. We fix $T_0 > 0$ and prescribe initial data $u[0]$ that are close to $u^{T_0}[0]$ on a ball of radius slightly larger than T_0. Here and throughout the paper we use the abbreviation $u[t] := (u(t,\cdot), \partial_t u(t,\cdot))$. Then we prove the existence of a particular T near T_0 for which the solution u converges to u^T inside the backward lightcone $\mathcal{C}_T$ in a norm adapted to the blowup behavior of u^T. For the precise statement of the main result we use Definitions 3.4 and 3.5.

Theorem 3.1. *Fix $T_0 > 0$. There exist constants $M, \delta, \varepsilon > 0$ such that for any radial initial data $u[0]$ satisfying*

$$\big\| |\cdot|^{-1}\big(u[0](|\cdot|) - u^{T_0}[0](|\cdot|)\big)\big\|_{H^6(\mathbb{B}^{11}_{T_0+\delta})\times H^5(\mathbb{B}^{11}_{T_0+\delta})} \leq \frac{\delta}{M} \tag{3-5}$$

the following statements hold:

(i) *The blowup time at the origin $T := T_{u[0]}$ belongs to the interval $[T_0 - \delta, T_0 + \delta]$.*

(ii) *The solution $u : \mathcal{C}_T \to \mathbb{R}$ to (3-1) satisfies*

$$(T-t)^{-\frac{9}{2}+k}\big\| |\cdot|^{-1}(u(t,|\cdot|) - u^T(t,|\cdot|))\big\|_{\dot{H}^k(\mathbb{B}^{11}_{T-t})} \leq \delta(T-t)^{\varepsilon}, \tag{3-6}$$

$$(T-t)^{-\frac{7}{2}+l}\big\| |\cdot|^{-1}(\partial_t u(t,|\cdot|) - \partial_t u^T(t,|\cdot|))\big\|_{\dot{H}^l(\mathbb{B}^{11}_{T-t})} \leq \delta(T-t)^{\varepsilon} \tag{3-7}$$

for integers $0 \leq k \leq 6$ and $0 \leq l \leq 5$. Furthermore,

$$\|u(t,\cdot) - u^T(t,\cdot)\|_{L^\infty(0,T-t)} \leq \delta(T-t)^{\varepsilon}. \tag{3-8}$$

Remark 3.2. The normalizing factor on the left-hand side of (3-6) and (3-7) appears naturally as it reflects the behavior of the self-similar solution u^T in the respective Sobolev norm; i.e.,

$$\big\| |\cdot|^{-1} u^T(t,|\cdot|)\big\|_{\dot{H}^k(\mathbb{B}^{11}_{T-t})} = \left\| |\cdot|^{-1}\phi_0\left(\frac{|\cdot|}{T-t}\right)\right\|_{\dot{H}^k(\mathbb{B}^{11}_{T-t})} = (T-t)^{\frac{9}{2}-k}\big\| |\cdot|^{-1}\phi_0(|\cdot|)\big\|_{\dot{H}^k(\mathbb{B}^{11}_1)}$$

and

$$\big\| |\cdot|^{-1}\,\partial_t u_T(t,|\cdot|)\big\|_{\dot{H}^l(\mathbb{B}^{11}_{T-t})} = (T-t)^{-2}\left\| \phi_0'\left(\frac{|\cdot|}{T-t}\right)\right\|_{\dot{H}^l(\mathbb{B}^{11}_{T-t})} = (T-t)^{\frac{7}{2}-l}\|\phi_0'(|\cdot|)\|_{\dot{H}^l(\mathbb{B}^{11}_1)}.$$

Remark 3.3. Since ϕ_0 is monotonically increasing on $[0,1]$, we have

$$\|u^T(t,\cdot)\|_{L^\infty(0,T-t)} = \max_{\rho\in[0,1]}\|\phi_0(\rho)\| = \phi_0(1). \tag{3-9}$$

Therefore, given $\varepsilon_1 > 0$, it follows from (3-8) and (3-9) that δ can be chosen small enough so that

$$\|u(t,\cdot)\|_{L^\infty(0,T-t)} \leq \|u(t,\cdot) - u^T(t,\cdot)\|_{L^\infty(0,T-t)} + \|u^T(t,\cdot)\|_{L^\infty(0,T-t)} \leq \varepsilon_1 + \phi_0(1).$$

Hence, for $t < T$ the solution $u(t,r)$ stays inside a neighborhood of $u = 0$ where the metric is given by (2-1); i.e., the portion of the target manifold that participates in the dynamics of the blowup solution is described by the metric (2-1).

3B. *Outline of the proof.* We use the method developed in the series of papers [Donninger 2011; 2014; 2017; Donninger and Schörkhuber 2012; 2014; 2016; 2017; Costin et al. 2017]. First, we introduce the rescaled variables

$$v_1(t,r) := \frac{T-t}{r}\,u(t,r), \quad v_2(t,r) := \frac{(T-t)^2}{r}\,\partial_t u(t,r). \tag{3-10}$$

Division by r is justified by the boundary condition (3-2) and the presence of the prefactors involving $T-t$ has to do with the change of variables we subsequently introduce. That is, we introduce similarity coordinates (τ,ρ) defined by

$$\tau := -\log(T-t)+\log T, \quad \rho := \frac{r}{T-t}, \tag{3-11}$$

and set

$$\psi_j(\tau,\rho) := v_j(T(1-e^{-\tau}), Te^{-\tau}\rho) \tag{3-12}$$

for $j=1,2$. As a consequence, (3-4) can be written as an abstract evolution equation,

$$\partial_\tau \Psi(\tau) = \boldsymbol{L}_0\Psi(\tau)+\boldsymbol{M}(\Psi(\tau)), \tag{3-13}$$

where $\Psi(\tau)=(\psi_1(\tau,\cdot),\psi_2(\tau,\cdot))$, $\boldsymbol{L}_0$ is the spatial part of the radial wave operator in the new coordinates, and $\boldsymbol{M}(\Psi(\tau))$ consists of the remaining nonlinear terms. The benefit of passing to the new variables (3-11) and (3-12) is that the backward lightcone $\mathcal{C}_T$ is transformed into a cylinder

$$\mathcal{C} := \{(\tau,\rho) : \tau\in[0,\infty),\ \rho\in[0,1]\},$$

the rescaled self-similar blowup solution u^T becomes a τ-independent function Ψ_{res} (this justifies the presence of t-dependent prefactors in (3-10)), and the problem of stability of blowup transforms into the problem of asymptotic stability of a static solution. We subsequently follow the standard approach for studying the stability of steady-state solutions and plug the ansatz $\Psi(\tau)=\Psi_{\text{res}}+\Phi(\tau)$ into (3-13). This leads to an evolution equation in Φ,

$$\partial_\tau \Phi(\tau) = \boldsymbol{L}_0\Phi(\tau)+\boldsymbol{L}'\Phi(\tau)+\boldsymbol{N}(\Phi(\tau)), \tag{3-14}$$

where $\boldsymbol{L}'$ is the Fréchet derivative of $\boldsymbol{M}$ at Ψ_{res} and $\boldsymbol{N}(\Phi(\tau))$ is the nonlinear remainder. We then proceed by studying (3-14) as an ordinary differential equation in a Hilbert space with the norm

$$\|\boldsymbol{u}\|^2 = \|(u_1,u_2)\|^2 := \|u_1(|\cdot|)\|^2_{H^6(\mathbb{B}^{11})}+\|u_2(|\cdot|)\|^2_{H^5(\mathbb{B}^{11})}. \tag{3-15}$$

However, passing to new variables also comes with a price. Namely, the radial wave operator $\boldsymbol{L}_0$ is not self-adjoint. Nonetheless, we establish well-posedness of the linearized problem (that is, (3-14) with $\boldsymbol{N}$ removed) by using methods from semigroup theory. In particular, we use a norm equivalent to (3-15) and the Lumer–Phillips theorem to show that $\boldsymbol{L}_0$ generates a semigroup $(\boldsymbol{S}_0(\tau))_{\tau\geq 0}$ with a negative growth bound. This in particular allows for locating the spectrum of $\boldsymbol{L}_0$. Furthermore, $\boldsymbol{L}'$ is compact so $\boldsymbol{L} := \boldsymbol{L}_0+\boldsymbol{L}'$ generates a strongly continuous semigroup $(\boldsymbol{S}(\tau))_{\tau\geq 0}$ and well-posedness of the linearized problem follows.

The stability of the solution u^T follows from a decay estimate on the semigroup $\boldsymbol{S}(\tau)$. To obtain such an estimate we exploit the relation between the growth bound of a semigroup and the location of the spectrum of its generator. We therefore study $\sigma(\boldsymbol{L})$ which, thanks to the compactness of $\boldsymbol{L}'$, amounts to studying the eigenvalue problem $(\lambda-\boldsymbol{L})\boldsymbol{u}=0$. We subsequently show that $\sigma(\boldsymbol{L})$ is contained in the left half-plane except for the point $\lambda=1$. However, this unstable eigenvalue corresponds to an apparent instability and we later use it to fix the blowup time. We therefore proceed by defining a spectral projection $\boldsymbol{P}$ onto the unstable space and study the semigroup $\boldsymbol{S}(\tau)$ restricted to $\operatorname{rg}(1-\boldsymbol{P})$. Furthermore, we establish a uniform bound on the resolvent $\boldsymbol{R_L}(\lambda)$ and invoke the Gearhart–Prüss theorem to obtain a negative growth bound on $(1-\boldsymbol{P})\boldsymbol{S}(\tau)$.

Appealing to Duhamel's principle, we rewrite (3-14) in the integral form

$$\Phi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{U}(\boldsymbol{v},T) + \int_0^\tau \boldsymbol{S}(\tau-s)\boldsymbol{N}(\Phi(s))\,ds, \tag{3-16}$$

where $\boldsymbol{U}(\boldsymbol{v},T)$ represents the rescaled initial data. We remark that the parameter T does not appear in the equation itself but in the initial data only. To obtain a decaying solution to (3-16) we suppress the unstable part of $\boldsymbol{S}(\tau)$ by introducing a correction term

$$\boldsymbol{C}(\Phi,\boldsymbol{U}(\boldsymbol{v},T)) := \boldsymbol{P}\Big(\boldsymbol{U}(\boldsymbol{v},T) + \int_0^\infty e^{-s}\boldsymbol{N}(\Phi(s))\,ds\Big)$$

into (3-16). That is, we consider the modified equation

$$\Phi(\tau) = \boldsymbol{S}(\tau)\big(\boldsymbol{U}(\boldsymbol{v},T) - \boldsymbol{C}(\Phi,\boldsymbol{U}(\boldsymbol{v},T))\big) + \int_0^\tau \boldsymbol{S}(\tau-s)\boldsymbol{N}(\Phi(s))\,ds. \tag{3-17}$$

We subsequently prove that for a fixed T_0 and small enough initial data $\boldsymbol{v}$, every T close to T_0 yields a unique solution to (3-17) that decays to zero at the linear decay rate. In other words, we prove the existence of a solution curve to (3-17) parametrized by T inside a small neighborhood of T_0, provided $\boldsymbol{v}$ is small enough.

Finally, we use the very presence of the unstable eigenvalue $\lambda = 1$ to prove the existence of a particular T near T_0 for which $\boldsymbol{C}(\Phi,\boldsymbol{U}(\boldsymbol{v},T)) = 0$ and hence obtain a decaying solution to (3-16) which, when translated back to the original coordinates, implies the main result.

3C. *Notation.* We denote by $\mathbb{B}^d_R$ the d-dimensional open ball of radius R centered at the origin. For brevity we let $\mathbb{B}^d := \mathbb{B}^d_1$. We write 2-component vector quantities in boldface, e.g., $\boldsymbol{u} = (u_1,u_2)$. By $\mathcal{B}(\mathcal{H})$ we denote the space of bounded operators on the Hilbert space $\mathcal{H}$. We denote by $\sigma(\boldsymbol{L})$ and $\sigma_p(\boldsymbol{L})$ the spectrum and the point spectrum, respectively, of a linear operator $\boldsymbol{L}$. Also, we denote by $\rho(\boldsymbol{L})$ the resolvent set $\mathbb{C}\setminus\sigma(\boldsymbol{L})$ and use the convention $\boldsymbol{R_L}(\lambda) := (\lambda-\boldsymbol{L})^{-1}$, $\lambda\in\rho(\boldsymbol{L})$, for the resolvent operator. We use the symbol $\lesssim$ with the standard meaning: $a \lesssim b$ if there exists a positive constant c, independent of a, b, such that $a \le cb$. Also, $a \simeq b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

3D. *Similarity coordinates and cylinder formulation.* After introducing the similarity coordinates

$$\tau := -\log(T-t) + \log T, \qquad \rho := \frac{r}{T-t},$$

and the rescaled variables

$$v_1(t,r) := \frac{T-t}{r}\,u(t,r), \qquad v_2(t,r) := \frac{(T-t)^2}{r}\,\partial_t u(t,r),$$
$$\psi_j(\tau,\rho) = v_j(T(1-e^{-\tau}), T\rho e^{-\tau}), \qquad j = 1,2,$$

we obtain from (3-1) the first-order system

$$\begin{bmatrix}\partial_\tau\psi_1\\ \partial_\tau\psi_2\end{bmatrix} = \begin{bmatrix} -\rho\,\partial_\rho\psi_1 - \psi_1 + \psi_2 \\ \partial_\rho^2\psi_1 + (10/\rho)\,\partial_\rho\psi_1 - \rho\partial_\rho\psi_2 - 2\psi_2\end{bmatrix} - \begin{bmatrix} 0 \\ 8(14\psi_1^3 - 111\rho^2\psi_1^5)\end{bmatrix} \tag{3-18}$$

for $(\tau,\rho)\in\mathcal{C}$. Furthermore, the initial data become

$$\begin{bmatrix}\psi_1(0,\rho)\\ \psi_2(0,\rho)\end{bmatrix}=\frac{1}{\rho}\begin{bmatrix}u_0(T\rho)\\ Tu_1(T\rho)\end{bmatrix}=\frac{1}{\rho}\begin{bmatrix}u^{T_0}(0,T\rho)\\ T\partial_0 u^{T_0}(0,T\rho)\end{bmatrix}+\frac{1}{\rho}\begin{bmatrix}F(T\rho)\\ TG(T\rho)\end{bmatrix}, \tag{3-19}$$

where T_0 is a fixed parameter and

$$F:=u_0-u^{T_0}(0,\cdot),\quad G:=u_1-\partial_0 u^{T_0}(0,\cdot).$$

In addition, we have the regularity conditions

$$\partial_\rho\psi_1(\tau,\rho)|_{\rho=0}=\partial_\rho\psi_2(\tau,\rho)|_{\rho=0}=0$$

for $\tau\ge 0$. Note further that we are studying the dynamics around u^{T_0} for a fixed T_0 and thus, it is natural to split the initial data as in (3-19). The parameter T is assumed to be close to T_0 and will be fixed later. As a consequence, the proximity of the initial data to $u^{T_0}[0]$ is measured by $\boldsymbol{v}:=(F,G)$.

3E. ***Perturbations of the blowup solution.*** For convenience, we set

$$\Psi(\tau)(\rho):=\begin{bmatrix}\psi_1(\tau,\rho)\\ \psi_2(\tau,\rho)\end{bmatrix}.$$

In the rescaled variables the blowup solution u^T becomes τ-independent, i.e.,

$$\begin{bmatrix}((T-t)/r)u^T(t,r)\\ ((T-t)^2/r)\partial_t u^T(t,r)\end{bmatrix}=\begin{bmatrix}(1/\rho)\phi_0(\rho)\\ \phi_0'(\rho)\end{bmatrix}=:\Psi_{\text{res}}(\tau)(\rho).$$

We proceed by studying the dynamics of (3-18) around Ψ_{res}. Our aim is to prove the asymptotic stability of Ψ_{res}, which in turn translates into the appropriate notion of stability of u^T. We therefore follow the standard method and plug the ansatz $\Psi=\Psi_{\text{res}}+\Phi$ into (3-18), where $\Phi(\tau)(\rho):=(\varphi_1(\tau,\rho),\varphi_2(\tau,\rho))$. This leads to an evolution equation for the perturbation Φ,

$$\begin{cases}\partial_\tau\Phi(\tau)=\widetilde{\boldsymbol{L}}\Phi(\tau)+\boldsymbol{N}(\Phi(\tau)),\\ \quad\Phi(0)=\boldsymbol{U}(\boldsymbol{v},T),\end{cases} \tag{3-20}$$

where $\widetilde{\boldsymbol{L}}$ and $\boldsymbol{N}$ are spatial operators and $\boldsymbol{U}(\boldsymbol{v},T)$ are the initial data. More precisely, $\widetilde{\boldsymbol{L}}:=\widetilde{\boldsymbol{L}}_0+\boldsymbol{L}'$, where

$$\widetilde{\boldsymbol{L}}_0\boldsymbol{u}(\rho):=\begin{bmatrix}-\rho u_1'(\rho)-u_1(\rho)+u_2(\rho)\\ u_1''(\rho)+(10/\rho)u_1'(\rho)-\rho u_2'(\rho)-2u_2(\rho)\end{bmatrix}, \tag{3-21}$$

$$\boldsymbol{L}'\boldsymbol{u}(\rho):=\begin{bmatrix}0\\ W(\rho,\phi_0(\rho))u_1(\rho)\end{bmatrix}, \tag{3-22}$$

$$\boldsymbol{N}(\boldsymbol{u})(\rho):=\begin{bmatrix}0\\ N(\rho,u_1(\rho))\end{bmatrix} \tag{3-23}$$

for a 2-component function $\boldsymbol{u}(\rho) = (u_1(\rho), u_2(\rho))$, where

$$\begin{aligned} N(\rho, u_1(\rho)) &= -\frac{8}{\rho^3}[n(\phi_0(\rho) + \rho u_1(\rho)) - n(\phi_0(\rho)) - n'(\phi_0(\rho))\rho u_1(\rho)], \\ W(\rho, \phi_0(\rho)) &= -\frac{8}{\rho^2} n'(\phi_0(\rho)) \quad \text{for } n(x) = 14x^3 - 111x^5. \end{aligned} \tag{3-24}$$

Also, we write the initial data as

$$\Phi(0)(\rho) = \boldsymbol{U}(\boldsymbol{v}, T)(\rho) = \begin{bmatrix} (1/\rho)\phi_0((T/T_0)\rho) \\ (T^2/T_0^2)\phi_0'((T/T_0)\rho) \end{bmatrix} - \begin{bmatrix} (1/\rho)\phi_0(\rho) \\ \phi_0'(\rho) \end{bmatrix} + \boldsymbol{V}(\boldsymbol{v}, T)(\rho), \tag{3-25}$$

where

$$\boldsymbol{V}(\boldsymbol{v}, T)(\rho) := \begin{bmatrix} (1/\rho)F(T\rho) \\ (T/\rho)G(T\rho) \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} F \\ G \end{bmatrix}.$$

3F. ***Strong lightcone solutions and blowup time at the origin.*** To proceed, we need the notion of a solution to the problem (3-20). In Section 3G we introduce the space

$$\mathcal{H} := H^6_{\text{rad}}(\mathbb{B}^{11}) \times H^5_{\text{rad}}(\mathbb{B}^{11})$$

and prove that the closure of the operator $\widetilde{\boldsymbol{L}}$, defined on a suitable domain, generates a strongly continuous semigroup $\boldsymbol{S}(\tau)$ on $\mathcal{H}$. Consequently, we formulate the problem (3-20) as an abstract integral equation via Duhamel's formula,

$$\Phi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\tau \boldsymbol{S}(\tau - s)\boldsymbol{N}(\Phi(s))\, ds. \tag{3-26}$$

This in particular establishes the well-posedness of the problem (3-20) in $\mathcal{H}$. We are now in the position to introduce the following definitions.

Definition 3.4. We say that $u : \mathcal{C}_T \to \mathbb{R}$ is a *solution* to (3-1) if the corresponding $\Phi : [0, \infty) \to \mathcal{H}$ belongs to $C([0, \infty); \mathcal{H})$ and satisfies (3-26) for all $\tau \geq 0$.

Definition 3.5. For the radial initial data (u_0, u_1) we define $\mathcal{T}(u_0, u_1)$ as the set of all $T > 0$ such that there exists a solution $u : \mathcal{C}_T \to \mathbb{R}$ to (3-1). We call

$$T_{(u_0,u_1)} := \sup(\mathcal{T}(u_0, u_1) \cup \{0\}) \tag{3-27}$$

the *blowup time at the origin*.

3G. ***Functional setting.*** We consider radial Sobolev functions $\hat{u} : \mathbb{B}^{11}_R \to \mathbb{C}$, i.e., $\hat{u}(\xi) = u(|\xi|)$ for $\xi \in \mathbb{B}^{11}_R$ and some $u : [0, R) \to \mathbb{C}$. We furthermore define

$$u \in H^m_{\text{rad}}(\mathbb{B}^{11}_R) \quad \text{if and only if} \quad \hat{u} \in H^m(\mathbb{B}^{11}_R) := W^{m,2}(\mathbb{B}^{11}_R).$$

With the norm

$$\|u\|_{H^m_{\text{rad}}(\mathbb{B}^{11}_R)} := \|\hat{u}\|_{H^m(\mathbb{B}^{11}_R)},$$

$H^m_{\mathrm{rad}}(\mathbb{B}^{11}_R)$ becomes a Banach space. In the rest of this paper we do not distinguish between u and $\hat{u}$. Now we define the Hilbert space

$$\mathcal{H} := H^6_{\mathrm{rad}}(\mathbb{B}^{11}) \times H^5_{\mathrm{rad}}(\mathbb{B}^{11}),$$

with the induced norm

$$\|\boldsymbol{u}\|^2 = \|(u_1, u_2)\|^2 := \|u_1\|^2_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})} + \|u_2\|^2_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})}.$$

3H. *Well-posedness of the linearized equation.* To establish well-posedness of the problem (3-20) we start by defining the domain of the free operator $\widetilde{\boldsymbol{L}}_0$; see (3-21). We follow [Donninger and Schörkhuber 2017] and let

$$\mathcal{D}(\widetilde{\boldsymbol{L}}_0) := \{\boldsymbol{u} \in C^\infty(0,1)^2 \cap \mathcal{H} : w_1 \in C^3[0,1],\ w_1''(0) = 0,\ w_2 \in C^2[0,1]\},$$

where

$$w_j(\rho) := D_{11} u_j(\rho) := \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^4 (\rho^9 u_j(\rho)) = \sum_{n=0}^{4} c_n \rho^{n+1} u_j^{(n)}(\rho)$$

for certain positive constants c_n, $\rho \in [0,1]$, and $j = 1, 2$. Since $C^\infty(\overline{\mathbb{B}^{11}})$ is dense in $H^m(\mathbb{B}^{11})$,

$$C^\infty_{\mathrm{even}}[0,1]^2 := \{\boldsymbol{u} \in C^\infty[0,1]^2 : \boldsymbol{u}^{(2k+1)}(0) = 0,\ k = 0, 1, 2, \dots\} \subset \mathcal{D}(\widetilde{\boldsymbol{L}}_0)$$

is dense in $\mathcal{H}$, which in turn implies that $\widetilde{\boldsymbol{L}}_0$ is densely defined on $\mathcal{H}$. Furthermore, we have the following result.

Proposition 3.6. *The operator $\widetilde{\boldsymbol{L}}_0 : \mathcal{D}(\widetilde{\boldsymbol{L}}_0) \subset \mathcal{H} \to \mathcal{H}$ is closable and its closure $\boldsymbol{L}_0 : \mathcal{D}(\boldsymbol{L}_0) \subset \mathcal{H} \to \mathcal{H}$ generates a strongly continuous one-parameter semigroup $(\boldsymbol{S}_0(\tau))_{\tau \geq 0}$ of bounded operators on $\mathcal{H}$ satisfying the growth estimate*

$$\|\boldsymbol{S}_0(\tau)\| \leq M e^{-\tau} \tag{3-28}$$

for all $\tau \geq 0$ and some $M > 0$. Furthermore, the operator $\boldsymbol{L} := \boldsymbol{L}_0 + \boldsymbol{L}' : \mathcal{D}(\boldsymbol{L}) \subset \mathcal{H} \to \mathcal{H}$, $\mathcal{D}(\boldsymbol{L}) = \mathcal{D}(\boldsymbol{L}_0)$, is the generator of a strongly continuous semigroup $(\boldsymbol{S}(\tau))_{\tau \geq 0}$ on $\mathcal{H}$ and $\boldsymbol{L}' : \mathcal{H} \to \mathcal{H}$ is compact.

Proof. The proof essentially follows the one of Proposition 3.1 in [Chatzikaleas et al. 2017] for $d = 9$. ☐

3I. *The spectrum of the free operator.* By exploiting the relation between the growth bound of a semigroup and the spectral bound of its generator, we can locate the spectrum of the operator $\boldsymbol{L}_0$. Namely, according to [Engel and Nagel 2000, p. 55, Theorem 1.10] the estimate (3-28) implies

$$\sigma(\boldsymbol{L}_0) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -1\}. \tag{3-29}$$

3J. *The spectrum of the full linear operator.* To understand the properties of the semigroup $\boldsymbol{S}(\tau)$ we investigate the spectrum of the full linear operator $\boldsymbol{L}$. First of all, we remark that $\lambda = 1$ is an eigenvalue of $\boldsymbol{L}$ (see Section 3K), which is an artifact of the freedom of choice of the parameter T; see, e.g., [Costin et al. 2017] for a discussion on this. What is more, $\lambda = 1$ is the only spectral point of $\boldsymbol{L}$ with a nonnegative real part. To prove this we first focus on the point spectrum.

Proposition 3.7. *We have*

$$\sigma_p(\boldsymbol{L}) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{1\}. \tag{3-30}$$

Proof. We argue by contradiction and assume there exists a $\lambda \in \sigma_p(\boldsymbol{L}) \setminus \{1\}$ with $\operatorname{Re}\lambda \geq 0$. This means that there exists a $\boldsymbol{u} = (u_1, u_2) \in \mathcal{D}(\boldsymbol{L}) \setminus \{0\}$ such that $\boldsymbol{u} \in \ker(\lambda - \boldsymbol{L})$. The spectral equation $(\lambda - \boldsymbol{L})\boldsymbol{u} = 0$ implies that the first component u_1 satisfies the equation

$$(1-\rho^2)u_1''(\rho) + \left(\frac{10}{\rho} - 2(\lambda+2)\rho\right)u_1'(\rho) - (\lambda+1)(\lambda+2)u_1(\rho) - V(\rho)u_1(\rho) = 0 \tag{3-31}$$

for $\rho \in (0,1)$, where

$$V(\rho) := -W(\rho, \phi_0(\rho)) = \frac{8n'(\phi_0(\rho))}{\rho^2} = -\frac{54(3737\rho^2 - 4340)}{(155 - 74\rho^2)^2}.$$

Since $\boldsymbol{u} \in \mathcal{H}$, we know u_1 must be an element of $H^6_{\mathrm{rad}}(\mathbb{B}^{11})$. From the smoothness of the coefficients in (3-31) we have an a priori regularity $u_1 \in C^\infty(0,1)$. In fact, we claim that $u_1 \in C^\infty[0,1]$. To show this, we use the Frobenius method. Namely, both $\rho = 0$ and $\rho = 1$ are regular singularities of (3-31) and Frobenius' theory gives a series form of solutions locally around singular points.

The Frobenius indices at $\rho = 0$ are $s_1 = 0$ and $s_2 = -9$. Therefore, two independent solutions of (3-31) have the form

$$u_1^1(\rho) = \sum_{i=0}^{\infty} a_i \rho^i \quad \text{and} \quad u_1^2(\rho) = C\log(\rho)u_1^1(\rho) + \rho^{-9}\sum_{i=0}^{\infty} b_i \rho^i$$

for some constant $C \in \mathbb{C}$ and $a_0 = b_0 = 1$. Since $u_1^1(\rho)$ is analytic at $\rho = 0$ and $u_1^2(\rho)$ does not belong to $H^6_{\mathrm{rad}}(\mathbb{B}^{11})$, we conclude that u_1 is a multiple of u_1^1 and therefore, $u_1 \in C^\infty[0,1)$.

The Frobenius indices at $\rho = 1$ are $s_1 = 0$ and $s_2 = 4 - \lambda$, and we distinguish different cases. If $4 - \lambda \notin \mathbb{Z}$ then the two linearly independent solutions are

$$u_1^1(\rho) = \sum_{i=0}^{\infty} a_i (1-\rho)^i \quad \text{and} \quad u_1^2(\rho) = (1-\rho)^{4-\lambda}\sum_{i=0}^{\infty} b_i (1-\rho)^i$$

with $a_0 = b_0 = 1$. Since $u_1^1(\rho)$ is analytic at $\rho = 1$ and u_1^2 does not belong to $H^6_{\mathrm{rad}}(\mathbb{B}^{11})$, we conclude that $u_1 \in C^\infty[0,1]$. If $4 - \lambda \in \mathbb{N}_0$, then the fundamental solutions around $\rho = 1$ are of the form

$$u_1^1(\rho) = (1-\rho)^{4-\lambda}\sum_{i=0}^{\infty} a_i (1-\rho)^i \quad \text{and} \quad u_1^2(\rho) = \sum_{i=0}^{\infty} b_i (1-\rho)^i + C\log(1-\rho)u_1^1(\rho),$$

with $a_0 = b_0 = 1$. Since $u_1^1(\rho)$ is analytic at $\rho = 1$ and u_1^2 does not belong to $H^6_{\mathrm{rad}}(\mathbb{B}^{11})$ unless $C = 0$, we again conclude that $u_1 \in C^\infty[0,1]$. Finally, if $4 - \lambda$ is a negative integer, the linearly independent solutions around $\rho = 1$ are

$$u_1^1(\rho) = \sum_{i=0}^{\infty} a_i (1-\rho)^i \quad \text{and} \quad u_1^2(\rho) = (1-\rho)^{4-\lambda}\sum_{i=0}^{\infty} b_i (1-\rho)^i + C\log(1-\rho)u_1^1(\rho),$$

with $a_0 = b_0 = 1$. Once again, since $u_1^1(\rho)$ is analytic at $\rho = 1$ and u_1^2 is not a member of $H^6_{\text{rad}}(\mathbb{B}^{11})$, we infer that $u_1 \in C^\infty[0, 1]$.

To obtain the desired contradiction, it remains to prove that (3-31) does not have a solution in $C^\infty[0, 1]$ for $\operatorname{Re}\lambda \geq 0$ and $\lambda \neq 1$. This claim is called the *mode stability* of the solution u^T. A general approach to proving mode stability of explicit self-similar blowup solutions to nonlinear wave equations of the type (1-4) was developed in [Costin et al. 2016; 2017]. We argue here along the lines of [Costin et al. 2017]. Also, for the rest of the proof, we follow the terminology of that paper. Namely, we call $\lambda \in \mathbb{C}$ an *eigenvalue* if it yields a $C^\infty[0, 1]$ solution to the equation in question. Also, if an eigenvalue λ satisfies $\operatorname{Re}\lambda \geq 0$ we say it is *unstable*; otherwise we call it *stable*. Our aim is therefore to prove that, apart from $\lambda = 1$, there are no unstable eigenvalues of the problem (3-31).

First of all, we make the substitution $v(\rho) = \rho u_1(\rho)$. This leads to the equation

$$(1-\rho^2)v''(\rho) + \left(\frac{8}{\rho} - 2(\lambda+1)\rho\right)v'(\rho) - \lambda(\lambda+1)v(\rho) - \widehat{V}(\rho)v(\rho) = 0, \tag{3-32}$$

where

$$\widehat{V}(\rho) := -\frac{10(15799\rho^4 - 5084\rho^2 - 19220)}{\rho^2(155-74\rho^2)^2}.$$

Now we formulate the corresponding supersymmetric problem,

$$(1-\rho^2)\tilde{v}''(\rho) + \left(\frac{8}{\rho} - 2(\lambda+1)\rho\right)\tilde{v}'(\rho) - (\lambda+2)(\lambda-1)\tilde{v}(\rho) - \widetilde{V}(\rho)\tilde{v}(\rho) = 0, \tag{3-33}$$

where

$$\widetilde{V}(\rho) := -\frac{18(3737\rho^4 + 5735\rho^2 - 24025)}{\rho^2(155-74\rho^2)^2};$$

see [Costin et al. 2017, Section 3.2] for the derivation. We claim that, apart from $\lambda = 1$, (3-32) and (3-33) have the same set of unstable eigenvalues. This is proved by a straightforward adaptation of the proof of Proposition 3.1 in [Costin et al. 2017].

To establish the nonexistence of unstable eigenvalues of the supersymmetric problem (3-33) we follow the proof of Theorem 4.1 in [Costin et al. 2017]. We start by introducing the change of variables

$$x = \rho^2, \quad \tilde{v}(\rho) = \frac{x}{\sqrt{155-74x}}y(x). \tag{3-34}$$

Equation (3-33) transforms into Heun's equation in its canonical form,

$$y''(x) + \left(\frac{13}{2x} + \frac{\lambda-3}{x-1} - \frac{74}{74x-155}\right)y'(x) + \frac{74\lambda(\lambda+3)x - (155\lambda^2 + 775\lambda + 1656)}{4x(x-1)(74x-155)}y(x) = 0. \tag{3-35}$$

Note that (3-34) preserves the analyticity of solutions at 0 and 1, and consequently, (3-33) and (3-35) have the same set of eigenvalues. The Frobenius indices of (3-35) at $x = 0$ are $s_1 = 0$ and $s_2 = -\frac{11}{2}$, so its normalized analytic solution at $x = 0$ is given by the power series

$$\sum_{n=0}^{\infty} a_n(\lambda)x^n, \quad a_0(\lambda) = 1. \tag{3-36}$$

The strategy is to study the asymptotic behavior of the coefficients $a_n(\lambda)$ as $n \to \infty$. More precisely, we prove that if $\lambda \in \overline{\mathbb{H}}$[1] then $\lim_{n\to\infty} a_{n+1}(\lambda)/a_n(\lambda) = 1$. Since $x = 1$ is the only singular point of (3-35) on the unit circle, it follows that the solution given by the series (3-36) is not analytic at $x = 1$.

First, we obtain the recurrence relation for coefficients $\{a_n(\lambda)\}_{n\in\mathbb{N}_0}$. By inserting (3-36) into (3-35) we get

$$310(2n+15)(n+2)a_{n+2}(\lambda) \\ = [155\lambda(\lambda+4n+9)+2(458n^2+2357n+2727)]a_{n+1}(\lambda) - 74(\lambda+2n+3)(\lambda+2n)a_n(\lambda),$$

where $a_{-1}(\lambda) = 0$ and $a_0(\lambda) = 1$, or, written differently,

$$a_{n+2}(\lambda) = A_n(\lambda)\, a_{n+1}(\lambda) + B_n(\lambda)\, a_n(\lambda), \tag{3-37}$$

where

$$A_n(\lambda) = \frac{155\lambda(\lambda+4n+9)+2(458n^2+2357n+2727)}{310(2n+15)(n+2)}$$

and

$$B_n(\lambda) = \frac{-37(\lambda+2n+3)(\lambda+2n)}{155(2n+15)(n+2)}.$$

We now let

$$r_n(\lambda) = \frac{a_{n+1}(\lambda)}{a_n(\lambda)}, \tag{3-38}$$

and thereby transform (3-37) into

$$r_{n+1}(\lambda) = A_n(\lambda) + \frac{B_n(\lambda)}{r_n(\lambda)}, \tag{3-39}$$

with the initial condition

$$r_0(\lambda) = \frac{a_1(\lambda)}{a_0(\lambda)} = A_{-1}(\lambda) = \tfrac{1}{26}\lambda^2 + \tfrac{5}{26}\lambda + \tfrac{828}{2015}.$$

Analogous to Lemma 4.2 in [Costin et al. 2017] we have that, given $\lambda \in \overline{\mathbb{H}}$, either

$$\lim_{n\to\infty} r_n(\lambda) = 1 \tag{3-40}$$

or

$$\lim_{n\to\infty} r_n(\lambda) = \tfrac{74}{155}. \tag{3-41}$$

Our aim is to prove that (3-40) holds throughout $\overline{\mathbb{H}}$. We do that by approximately solving (3-39) for $\lambda \in \overline{\mathbb{H}}$. Namely, we define an approximate solution (also called a quasisolution)

$$\tilde{r}_n(\lambda) = \frac{\lambda^2}{4n^2+28n+27} + \frac{\lambda}{n+7} + \frac{2n+12}{2n+23}$$

to (3-39); see [Costin et al. 2016, Section 4.1] for a discussion on how to obtain such an expression. Subsequently, we let

$$\delta_n(\lambda) = \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} - 1 \tag{3-42}$$

[1] Here, as in [Costin et al. 2017], $\overline{\mathbb{H}}$ denotes the closed complex right half-plane.

and from (3-39) we get the recurrence relation

$$\delta_{n+1} = \varepsilon_n - C_n \frac{\delta_n}{1+\delta_n} \tag{3-43}$$

for δ_n, where

$$\varepsilon_n = \frac{A_n \tilde{r}_n + B_n}{\tilde{r}_n \tilde{r}_{n+1}} - 1 \quad \text{and} \quad C_n = \frac{B_n}{\tilde{r}_n \tilde{r}_{n+1}}. \tag{3-44}$$

Now, for all $\lambda \in \overline{\mathbb{H}}$ and $n \geq 7$ we have the bounds

$$|\delta_7(\lambda)| \leq \tfrac{1}{3}, \quad |\varepsilon_n(\lambda)| \leq \tfrac{1}{12}, \quad |C_n(\lambda)| \leq \tfrac{1}{2}. \tag{3-45}$$

The last two inequalities above are proved in the same way as the corresponding ones in Lemma 4.4 in [Costin et al. 2017]. However, the proof of the first one needs to be slightly adjusted and we provide it in the appendix; see Proposition B.1. Next, by a simple inductive argument we conclude from (3-43) and (3-45) that

$$|\delta_n(\lambda)| \leq \tfrac{1}{3} \quad \text{for all } n \geq 7 \text{ and } \lambda \in \overline{\mathbb{H}}. \tag{3-46}$$

Since for any fixed $\lambda \in \overline{\mathbb{H}}$ we have $\lim_{n\to\infty} \tilde{r}_n(\lambda) = 1$, (3-46) and (3-42) exclude the case (3-41). Hence, (3-40) holds throughout $\overline{\mathbb{H}}$ and we conclude that there are no unstable eigenvalues of the supersymmetric problem (3-33), thus arriving at a contradiction and thereby completing the proof of the proposition. □

Remark 3.8. Apart from $\lambda = 1$ the point spectrum of the operator $\boldsymbol{L}$ is completely contained in the open left half-plane. It is natural to try to locate the eigenvalues that are closest to the imaginary axis, as their location is typically related to the rate of convergence to the blowup solution u^T. Our numerical calculations indicate that $-0.98 \pm 3.76\,i$ is the approximate location of the pair of (complex conjugate) stable eigenvalues with the largest real parts. It is interesting to contrast this with the analogous spectral problems for equivariant wave maps into the sphere and Yang–Mills fields, where all eigenvalues appear to be real; see [Bizoń and Biernat 2015].

Corollary 3.9. *We have*

$$\sigma(\boldsymbol{L}) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{1\}.$$

Proof. Assume there exists a $\lambda \in \sigma(\boldsymbol{L}) \setminus \{1\}$ with $\operatorname{Re}\lambda \geq 0$. From (3-29) we see that λ is contained in the resolvent set of $\boldsymbol{L}_0$. Therefore, we have the identity

$$\lambda - \boldsymbol{L} = [1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)](\lambda - \boldsymbol{L}_0). \tag{3-47}$$

This implies that $1 \in \sigma(\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))$ and since $\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$ is compact, it follows that $1 \in \sigma_p(\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))$. Thus, there exists a nontrivial $\boldsymbol{f} \in \mathcal{H}$ such that $[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]\boldsymbol{f} = 0$. Consequently, $\boldsymbol{u} := \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f} \neq 0$ satisfies $(\lambda - \boldsymbol{L})\boldsymbol{u} = 0$ and thus, $\lambda \in \sigma_p(\boldsymbol{L})$, but this is in conflict with Proposition 3.7. □

3K. ***The eigenspace of the isolated eigenvalue.*** In this section, we prove that the (geometric) eigenspace of the isolated eigenvalue $\lambda = 1$ for the full linear operator $\boldsymbol{L}$ is spanned by

$$\boldsymbol{g}(\rho) := \begin{bmatrix} g_1(\rho) \\ g_2(\rho) \end{bmatrix} = \begin{bmatrix} \phi_0'(\rho) \\ \rho\phi_0''(\rho) + 2\phi_0'(\rho) \end{bmatrix}. \tag{3-48}$$

Namely, we are looking for all $\boldsymbol{u} = (u_1, u_2) \in \mathcal{D}(\boldsymbol{L}) \setminus \{0\}$ which belong to $\ker(1 - \boldsymbol{L})$. A straightforward calculation shows that the spectral equation $(1-\boldsymbol{L})\boldsymbol{u} = 0$ is equivalent to the system of ordinary differential equations

$$\begin{cases} u_2(\rho) = \rho u_1'(\rho) + 2u_1(\rho), \\ (1-\rho^2)u_1''(\rho) + (10/\rho - 6\rho)u_1'(\rho) - \big(6 + (8/\rho^2)n'(\phi_0(\rho))\big)u_1(\rho) = 0 \end{cases} \tag{3-49}$$

for $\rho \in (0, 1)$. One can easily verify that a fundamental system of the second equation is given by the functions $\phi_0'(\rho)$ and $\rho^{-9}A(\rho)$, where $A(\rho)$ is analytic and nonvanishing at $\rho = 0$. We can therefore write the general solution to the second equation as

$$u_1(\rho) = C_1\phi_0'(\rho) + C_2\frac{A(\rho)}{\rho^9}.$$

The condition $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L})$ requires u_1 to lie in the Sobolev space $H^6_{\text{rad}}(\mathbb{B}^{11})$. Since $\phi_0' \in C^\infty[0, 1]$, this requirement yields $C_2 = 0$ which, according to the first equation in (3-49), gives $\boldsymbol{u} = C_1\boldsymbol{g}$. In conclusion,

$$\ker(1 - \boldsymbol{L}) = \langle \boldsymbol{g} \rangle, \tag{3-50}$$

as initially claimed.

3L. ***Time evolution of the linearized problem.*** To get around the spurious instability on the linear level, we use the fact that $\lambda = 1$ is isolated to introduce a (nonorthogonal) spectral projection $\boldsymbol{P}$ and study the subspace semigroup $\boldsymbol{S}(\tau)(1 - \boldsymbol{P})$. From Corollary 3.9 we then infer that the spectrum of its generator is contained in the left half-plane. This does not necessarily imply the desired decay on $\boldsymbol{S}(\tau)(1 - \boldsymbol{P})$. We nonetheless establish such a decay by first proving uniform boundedness of the resolvent of $\boldsymbol{L}$ in a half-plane that strictly contains $\overline{\mathbb{H}}$ and then using the Gearhart–Prüss theorem. For this purpose, we define

$$\Omega_{\varepsilon,R} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -1 + \varepsilon, |\lambda| \geq R\}$$

for $\varepsilon, R > 0$.

Proposition 3.10. *Let $\varepsilon > 0$. Then there exists a constant $R_\varepsilon > 0$ such that the resolvent $\boldsymbol{R_L}$ exists on $\Omega_{\varepsilon,R_\varepsilon}$ and satisfies*

$$\|\boldsymbol{R_L}(\lambda)\| \leq \frac{2}{\varepsilon}$$

for all $\lambda \in \Omega_{\varepsilon,R_\varepsilon}$.

Proof. Fix $\varepsilon > 0$ and take $\lambda \in \Omega_{\varepsilon,R}$ for an arbitrary $R \geq 2$. Then $\lambda \in \rho(\boldsymbol{L}_0)$ and the identity (3-47) holds. The proof proceeds as follows. For large enough R, we show that the operator $1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$ is invertible in $\Omega_{\varepsilon,R}$ and $\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$ and $[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}$ are uniformly norm bounded there. Via (3-47) this implies the desired bound on $\boldsymbol{R_L}(\lambda)$.

First of all, semigroup theory yields the estimate

$$\|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| \leq \frac{1}{\operatorname{Re}\lambda + 1}; \tag{3-51}$$

see [Engel and Nagel 2000, p. 55, Theorem 1.10]. Next, by a Neumann series argument, the operator $1-\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$ is invertible if $\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|<1$. To prove smallness of $\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$, we recall the definition of $\boldsymbol{L}'$, (3-22),

$$\boldsymbol{L}'\boldsymbol{u}(\rho):=\begin{bmatrix}0\\ \widetilde{W}(\rho)u_1(\rho)\end{bmatrix},\qquad \widetilde{W}(\rho)=-\frac{8}{\rho^2}n'(\phi_0(\rho))\quad\text{for } n(x)=14x^3-111x^5.$$

Let $\boldsymbol{u}=\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}$ or, equivalently, $(\lambda-\boldsymbol{L}_0)\boldsymbol{u}=\boldsymbol{f}$. The latter equation implies

$$(\lambda+1)u_1(\rho)=u_2(\rho)-\rho u_1'(\rho)+f_1(\rho).$$

Now we use Lemma 4.1 from [Donninger and Schörkhuber 2017] and $\|\widetilde{W}^{(k)}\|_{L^\infty(0,1)}\lesssim 1$ for all $k\in\{0,1,\dots,5\}$ to obtain

$$\begin{aligned}|\lambda+1|\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}\|=|\lambda+1|\|\boldsymbol{L}'\boldsymbol{u}\|&\simeq\|\widetilde{W}(u_2-(\cdot)u_1'+f_1)\|_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})}\\&\lesssim\|u_2\|_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})}+\|(\cdot)u_1'\|_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})}+\|f_1\|_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})}\\&\lesssim\|u_2\|_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})}+\|u_1\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})}+\|f_1\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})}\\&\lesssim\|\boldsymbol{u}\|+\|\boldsymbol{f}\|\lesssim\left(\frac{1}{\operatorname{Re}\lambda+1}+1\right)\|\boldsymbol{f}\|\lesssim\|\boldsymbol{f}\|,\end{aligned}$$

where we used (3-51). In other words,

$$\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|\lesssim\frac{1}{|\lambda+1|}\le\frac{1}{|\lambda|-1}\le\frac{1}{R-1}$$

and by choosing R sufficiently large, we can achieve $\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|\le\frac{1}{2}$. As a consequence, $[1-\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}$ exists for $\lambda\in\Omega_{\varepsilon,R_\varepsilon}$ and we obtain the bound

$$\begin{aligned}\|\boldsymbol{R}_{\boldsymbol{L}}(\lambda)\|&=\|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)[1-\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}\|\\&\le\|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|\|[1-\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}\|\\&\le\|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|\sum_{n=0}^{\infty}\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|^n\le\frac{2}{\varepsilon}.\end{aligned}$$

□

We now show the existence of a projection $\boldsymbol{P}$ which decomposes the Hilbert space $\mathcal{H}$ into a stable and an unstable subspace and furthermore prove that data from the stable subspace lead to solutions that decay exponentially in time. We also remark that it is crucial to ensure that rank $\boldsymbol{P}=1$, i.e., that $\boldsymbol{g}$ is the only unstable direction in $\mathcal{H}$.

Proposition 3.11. *There exists a projection operator*

$$\boldsymbol{P}\in\mathcal{B}(\mathcal{H}),\qquad \boldsymbol{P}:\mathcal{H}\to\langle\boldsymbol{g}\rangle,$$

which commutes with the semigroup $(\boldsymbol{S}(\tau))_{\tau\ge0}$. In addition, we have

$$\boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{f}=e^\tau\boldsymbol{P}\boldsymbol{f}\tag{3-52}$$

and there are constants $C, \varepsilon > 0$ *such that*

$$\|(1-\boldsymbol{P})\boldsymbol{S}(\tau)\boldsymbol{f}\| \le Ce^{-\varepsilon\tau}\|(1-\boldsymbol{P})\boldsymbol{f}\| \tag{3-53}$$

for all $\boldsymbol{f} \in \mathcal{H}$ *and* $\tau \ge 0$.

Proof. By Proposition 3.7, the eigenvalue $\lambda = 1$ of the operator $\boldsymbol{L}$ is isolated. We therefore introduce the spectral projection

$$\boldsymbol{P} : \mathcal{H} \to \mathcal{H}, \quad \boldsymbol{P} := \frac{1}{2\pi i}\int_\gamma \boldsymbol{R}_{\boldsymbol{L}}(\mu)\,d\mu,$$

where γ is a positively oriented circle around $\lambda = 1$. The radius of the circle is chosen small enough so that γ is completely contained inside the resolvent set of $\boldsymbol{L}$ and such that the interior of γ contains no spectral points of $\boldsymbol{L}$ other than $\lambda = 1$. The projection $\boldsymbol{P}$ commutes with the operator $\boldsymbol{L}$ and therefore with the semigroup $\boldsymbol{S}(\tau)$. Moreover, the Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M} := \operatorname{rg} \boldsymbol{P}$ and $\mathcal{N} := \operatorname{rg}(1-\boldsymbol{P}) = \ker \boldsymbol{P}$. Also, the spaces $\mathcal{M}$ and $\mathcal{N}$ reduce the operator $\boldsymbol{L}$, which is therefore decomposed into $\boldsymbol{L}_{\mathcal{M}}$ and $\boldsymbol{L}_{\mathcal{N}}$. The spectra of these operators are given by

$$\sigma(\boldsymbol{L}_{\mathcal{N}}) = \sigma(\boldsymbol{L}) \setminus \{1\}, \quad \sigma(\boldsymbol{L}_{\mathcal{M}}) = \{1\}. \tag{3-54}$$

We refer the reader to [Kato 1980, Chapter III, Section 6.4] for these standard results.

To proceed with the proof we show that $\operatorname{rank} \boldsymbol{P} := \dim \operatorname{rg} \boldsymbol{P} < +\infty$. We argue by contradiction and assume that $\operatorname{rank} \boldsymbol{P} = +\infty$. This means that $\lambda = 1$ belongs to the essential spectrum of $\boldsymbol{L}$; see [Kato 1980, p. 239, Theorem 5.28]. But according to Proposition 3.6 the operator $\boldsymbol{L}_0 = \boldsymbol{L} - \boldsymbol{L}'$ is a compact perturbation of $\boldsymbol{L}$, and due to the stability of the essential spectrum under compact perturbations we conclude that $\lambda = 1$ is a spectral point of $\boldsymbol{L}_0$. However, this is in conflict with (3-29), and therefore $\operatorname{rank} \boldsymbol{P} < +\infty$.

Now we prove that $\langle \boldsymbol{g} \rangle = \operatorname{rg} \boldsymbol{P}$. From the definition of the projection $\boldsymbol{P}$ we have $\boldsymbol{P}\boldsymbol{g} = \boldsymbol{g}$. Therefore $\langle \boldsymbol{g} \rangle \subseteq \operatorname{rg} \boldsymbol{P}$ and it remains to prove the reverse inclusion. From the fact that the operator $1 - \boldsymbol{L}_{\mathcal{M}}$ acts on the finite-dimensional Hilbert space $\mathcal{M} = \operatorname{rg} \boldsymbol{P}$ and (3-54) we infer that $\lambda = 0$ is the only spectral point of $1 - \boldsymbol{L}_{\mathcal{M}}$. Hence, $1 - \boldsymbol{L}_{\mathcal{M}}$ is nilpotent; i.e., there exists a $k \in \mathbb{N}$ such that

$$(1 - \boldsymbol{L}_{\mathcal{M}})^k \boldsymbol{u} = 0$$

for all $\boldsymbol{u} \in \operatorname{rg} \boldsymbol{P}$ and we assume k to be minimal. Due to (3-50) the claim follows immediately for $k = 1$. We therefore assume that $k \ge 2$. This implies the existence of a nontrivial function $\boldsymbol{u} \in \operatorname{rg} \boldsymbol{P} \subseteq \mathcal{D}(\boldsymbol{L})$ such that $(1 - \boldsymbol{L}_{\mathcal{M}})\boldsymbol{u}$ is nonzero and belongs to $\ker(1 - \boldsymbol{L}_{\mathcal{M}}) \subseteq \ker(1 - \boldsymbol{L}) = \langle \boldsymbol{g} \rangle$. Therefore $(1 - \boldsymbol{L})\boldsymbol{u} = \alpha \boldsymbol{g}$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. For convenience and without loss of generality, we set $\alpha = -1$. By a straightforward computation we see that the first component of $\boldsymbol{u}$ satisfies the differential equation

$$(1-\rho^2)u_1''(\rho) + \left(\frac{10}{\rho} - 6\rho\right)u_1'(\rho) - \left(6 + \frac{8}{\rho^2}n'(\phi_0(\rho))\right)u_1(\rho) = G(\rho) \tag{3-55}$$

for $\rho \in (0, 1)$, where

$$G(\rho) := 2\rho\phi_0''(\rho) + 5\phi_0'(\rho), \quad \rho \in [0, 1].$$

To find a general solution to (3-55) we first observe that

$$\hat{u}_1(\rho) := g_1(\rho) = \phi_0'(\rho), \quad \rho \in (0,1),$$

is a particular solution to the homogeneous equation

$$(1-\rho^2)u_1''(\rho) + \left(\frac{10}{\rho} - 6\rho\right)u_1'(\rho) - \left(6 + \frac{8}{\rho^2}n'(\phi_0(\rho))\right)u_1(\rho) = 0;$$

see (3-48) and (3-49). Note that the Wronskian for the equation above is

$$\mathcal{W}(\rho) := \frac{(1-\rho^2)^2}{\rho^{10}}.$$

Therefore, another linearly independent solution is

$$\hat{u}_2(\rho) := \hat{u}_1(\rho)\int_\rho^1 \frac{(1-x^2)^2}{x^{10}} \frac{1}{\phi_0'(x)^2}\,dx$$

for all $\rho \in (0,1)$. Note that near $\rho = 0$ we have the expansion

$$\hat{u}_2(\rho) = \frac{1}{\rho^9}\sum_{j=0}^\infty a_j \rho^j, \quad a_0 \neq 0,$$

as already indicated in Section 3K. Furthermore, we have

$$\hat{u}_2(\rho) = (1-\rho)^3 \sum_{j=0}^\infty b_j (1-\rho)^j, \quad b_0 \neq 0,$$

near $\rho = 1$. Now, by the variation-of-constants formula we see that the general solution to (3-55) can be written as

$$u_1(\rho) = c_1\hat{u}_1(\rho) + c_2\hat{u}_2(\rho) + \hat{u}_2(\rho)\int_0^\rho \frac{\hat{u}_1(y)G(y)y^{10}}{(1-y^2)^3}\,dy - \hat{u}_1(\rho)\int_0^\rho \frac{\hat{u}_2(y)G(y)y^{10}}{(1-y^2)^3}\,dy$$

for some constants $c_1, c_2 \in \mathbb{C}$ and for all $\rho \in (0,1)$. The fact that $u_1 \in H^6_{\mathrm{rad}}(\mathbb{B}^{11})$ implies $c_2 = 0$ as $\hat{u}_2$ has a ninth-order pole at $\rho = 0$. Therefore

$$u_1(\rho) = c_1\hat{u}_1(\rho) + \hat{u}_2(\rho)\int_0^\rho \frac{\hat{u}_1(y)G(y)y^{10}}{(1-y^2)^3}\,dy - \hat{u}_1(\rho)\int_0^\rho \frac{\hat{u}_2(y)G(y)y^{10}}{(1-y^2)^3}\,dy. \tag{3-56}$$

The last term in (3-56) is smooth on $[0,1]$. To analyze the second term, we set

$$\mathcal{I}(\rho) := \hat{u}_2(\rho)\int_0^\rho \frac{F(y)}{(1-y)^3}\,dy,$$

where

$$F(y) := \frac{\hat{u}_1(y)G(y)y^{10}}{(1+y)^3} = \frac{y^{10}(2y\phi_0'(y)\phi_0''(y) + 5\phi_0'(y)^2)}{(1+y)^3}.$$

By a direct calculation we get $F''(1) \neq 0$ and thus, the expansion of $\mathcal{I}(\rho)$ near $\rho = 1$ contains a term of the form $(1-\rho)^3\log(1-\rho)$. Consequently, $\mathcal{I}^{(4)} \notin L^2\left(\frac{1}{2}, 1\right)$, which is a contradiction to $u_1 \in H^6_{\mathrm{rad}}(\mathbb{B}^{11})$.

Finally we prove (3-52) and (3-53). Note that (3-52) follows from the fact that $\lambda = 1$ is an eigenvalue of the operator $\boldsymbol{L}$ with eigenfunction $\boldsymbol{g}$ and $\operatorname{rg} \boldsymbol{P} = \langle \boldsymbol{g} \rangle$. Next, from Corollary 3.9 and Proposition 3.10 we deduce the existence of constants $D, \varepsilon > 0$ such that

$$\|\boldsymbol{R_L}(\lambda)(1-\boldsymbol{P})\| \leq D$$

for all complex λ with $\operatorname{Re}\lambda > -\varepsilon$. Thus, (3-53) follows from the Gearhart–Prüss theorem; see [Engel and Nagel 2000, p. 302, Theorem 1.11]. □

3M. *Estimates for the nonlinearity.* In the next section we employ a fixed-point argument to prove the existence of decaying solutions to (3-26) for small initial data. To accomplish that, we need a Lipschitz-type estimate for the nonlinear operator $\boldsymbol{N}$; see (3-23). We first define

$$\mathcal{B}_\delta := \{\boldsymbol{u} \in \mathcal{H} : \|\boldsymbol{u}\| = \|(u_1, u_2)\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11}) \times H^5_{\mathrm{rad}}(\mathbb{B}^{11})} \leq \delta\}.$$

Lemma 3.12. *Let $\delta > 0$. For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{B}_\delta$, we have*

$$\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)\|\boldsymbol{u} - \boldsymbol{v}\|. \tag{3-57}$$

Remark 3.13. From this lemma we infer the estimate

$$\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim \delta\|\boldsymbol{u} - \boldsymbol{v}\|. \tag{3-58}$$

Therefore, the implied Lipschitz constant in (3-58) can be made as small as needed by adjusting the size of δ.

Proof. Based on (3-23) and (3-3), the difference $N(\rho, u) - N(\rho, v)$ can be written as

$$N(\rho, u) - N(\rho, v) = \sum_{j=1}^{4} n_j(\rho^2)(u^{j+1} - v^{j+1}), \tag{3-59}$$

where $n_j \in C^\infty[0, 1]$. For $\delta > 0$, we have $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{B}_\delta$, and due to the bilinear estimate

$$\|f_1 f_2\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})} \lesssim \|f_1\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})}\|f_2\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})},$$

we have

$$\begin{aligned}
\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| &= \|N(\cdot, u_1) - N(\cdot, v_1)\|_{H^5_{\mathrm{rad}}(\mathbb{B}^{11})} \\
&\leq \|N(\cdot, u_1) - N(\cdot, v_1)\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})} \\
&\lesssim \sum_{j=1}^{4} \|n_j((\cdot)^2)\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})}\|u_1^{j+1} - v_1^{j+1}\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})} \\
&\lesssim (\|u_1\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})} + \|v_1\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})})\|u_1 - v_1\|_{H^6_{\mathrm{rad}}(\mathbb{B}^{11})} \\
&\leq (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)\|\boldsymbol{u} - \boldsymbol{v}\|.
\end{aligned}$$

□

3N. ***The abstract nonlinear Cauchy problem.*** In this section we treat the existence and uniqueness of solutions to (3-20) for small initial data. According to Definition 3.4 we study the integral equation

$$\boldsymbol{\Phi}(\tau) = \boldsymbol{S}(\tau)\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\tau \boldsymbol{S}(\tau - s)\boldsymbol{N}(\Phi(s))\,ds \tag{3-60}$$

for $\tau \geq 0$ and $\boldsymbol{v} \in \mathcal{H}$ small. In order to employ a fixed-point argument, we introduce the necessary definitions. First, we define a Banach space

$$\mathcal{X} := \{\Phi \in C([0,\infty), \mathcal{H}) : \|\Phi\|_{\mathcal{X}} := \sup_{\tau > 0} e^{\varepsilon\tau}\|\Phi(\tau)\| < \infty\}, \tag{3-61}$$

where ε is sufficiently small and fixed. We denote by $\mathcal{X}_\delta$ the closed ball in $\mathcal{X}$ with radius δ; that is,

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \leq \delta\}. \tag{3-62}$$

Finally, we define the correction term

$$\boldsymbol{C}(\Phi, \boldsymbol{u}) := \boldsymbol{P}\left(\boldsymbol{u} + \int_0^\infty e^{-s}\boldsymbol{N}(\Phi(s))\,ds\right),$$

and set

$$\boldsymbol{K}(\Phi, \boldsymbol{u})(\tau) := \boldsymbol{S}(\tau)(\boldsymbol{u} - \boldsymbol{C}(\Phi, \boldsymbol{u})) + \int_0^\tau \boldsymbol{S}(\tau - s)\boldsymbol{N}(\Phi(s))\,ds.$$

The correction term serves the purpose of suppressing the exponential growth of the semigroup $\boldsymbol{S}(\tau)$ on the unstable space. We have the following result.

Theorem 3.14. *There exist constants $\delta, C > 0$ such that for every $\boldsymbol{u} \in \mathcal{H}$ which satisfies $\|\boldsymbol{u}\| \leq \delta/C$, there exists a unique $\Phi_{\boldsymbol{u}} \in \mathcal{X}_\delta$ such that*

$$\Phi_{\boldsymbol{u}} = \boldsymbol{K}(\Phi_{\boldsymbol{u}}, \boldsymbol{u}). \tag{3-63}$$

In addition, the solution $\Phi_{\boldsymbol{u}}$ is unique in the whole space $\mathcal{X}$ and the solution map $\boldsymbol{u} \mapsto \Phi_{\boldsymbol{u}}$ is Lipschitz continuous.

The proof coincides with the one of Theorem 3.7 in [Chatzikaleas et al. 2017].

We now study the initial data $\boldsymbol{U}(\boldsymbol{v}, T)$, see (3-25), and prove its continuity in T near T_0. For that reason we define

$$\mathcal{H}^R := H^6_{\mathrm{rad}} \times H^5_{\mathrm{rad}}(\mathbb{B}^{11}_R),$$

with the induced norm

$$\|\boldsymbol{w}\|^2_{\mathcal{H}^R} = \|w_1(|\cdot|)\|^2_{H^6(\mathbb{B}^{11}_R)} + \|w_2(|\cdot|)\|^2_{H^5(\mathbb{B}^{11}_R)}.$$

Lemma 3.15. *Fix $T_0 > 0$. Let $|\cdot|^{-1}\boldsymbol{v} \in \mathcal{H}^{T_0+\delta}$ for δ positive and sufficiently small. Then the map*

$$T \mapsto \boldsymbol{U}(\boldsymbol{v}, T) : [T_0 - \delta, T_0 + \delta] \to \mathcal{H}$$

is continuous. Furthermore, for all $T \in [T_0 - \delta, T_0 + \delta]$,

$$\||\cdot|^{-1}\boldsymbol{v}\|_{\mathcal{H}^{T_0+\delta}} \leq \delta \quad \Longrightarrow \quad \|\boldsymbol{U}(\boldsymbol{v}, T)\| \lesssim \delta.$$

Proof. We prove the result for $T_0 = 1$ only, as the general case is treated similarly. Assume $|\cdot|^{-1}\boldsymbol{v} \in \mathcal{H}^{1+\delta}$ for δ positive but less than $\frac{1}{2}T_0 = \frac{1}{2}$. We first introduce some auxiliary facts. Namely, by scaling we see that for $f \in H^6_{\text{rad}}(\mathbb{B}^{11}_{1+\delta})$ and $T \in [1-\delta, 1+\delta]$,

$$\|f(|T\,\cdot|)\|_{H^6(\mathbb{B}^{11}_1)} \lesssim \|f(|\cdot|)\|_{H^6(\mathbb{B}^{11}_{1+\delta})}.$$

Furthermore, from the density of $C^\infty_{\text{even}}[0, 1+\delta]$ in $H^6_{\text{rad}}(\mathbb{B}^{11}_{1+\delta})$ we conclude that given $\varepsilon > 0$, there exists a $\tilde{v}_1 \in C^\infty_{\text{even}}[0, 1+\delta]$ such that $\||\cdot|^{-1}v_1(|\cdot|) - \tilde{v}_1(|\cdot|)\|_{H^6_{\text{rad}}(\mathbb{B}^{11}_{1+\delta})} < \varepsilon$. Also, the functions $(1/T\rho)\phi_0(T\rho)$ and $\tilde{v}(T\rho)$ are smooth on $[0, 1]$ for $T \in [1-\delta, 1+\delta]$. Therefore,

$$\lim_{T \to \widetilde{T}} \big\||T\,\cdot|^{-1}\big(\phi_0(|T\,\cdot|) - \phi_0(|\widetilde{T}\,\cdot|)\big)\big\|_{H^6(\mathbb{B}^{11})} + \|\tilde{v}_1(|T\,\cdot|) - \tilde{v}_1(|\widetilde{T}\,\cdot|)\|_{H^6(\mathbb{B}^{11})} = 0. \tag{3-64}$$

Using these facts, we prove the continuity of the first component of the map $T \to \boldsymbol{U}(\boldsymbol{v}, T)$. Namely, given $\varepsilon > 0$, there exists a $\tilde{v}_1 \in C^\infty_{\text{even}}[0, 1+\delta]$ such that for $T, \widetilde{T} \in [1-\delta, 1+\delta]$ we have

$$\begin{aligned}
&\big\|[\boldsymbol{U}(\boldsymbol{v}, T)]_1 - [\boldsymbol{U}(\boldsymbol{v}, \widetilde{T})]_1\big\|_{H^6(\mathbb{B}^{11})} \\
&\qquad = \big\||\cdot|^{-1}\phi_0(|T\,\cdot|) + |\cdot|^{-1}v_1(|T\,\cdot|) - |\cdot|^{-1}\phi_0(|\widetilde{T}\,\cdot|) - |\cdot|^{-1}v_1(|\widetilde{T}\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} \\
&\qquad \lesssim \big\||\cdot|^{-1}(\phi_0(|T\,\cdot|) - \phi_0(|\widetilde{T}\,\cdot|))\big\|_{H^6(\mathbb{B}^{11})} + \big\||T\,\cdot|^{-1}v_1(|T\,\cdot|) - \tilde{v}_1(|T\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} \\
&\qquad\qquad + \big\|\tilde{v}_1(|T\,\cdot|) - \tilde{v}_1(|\widetilde{T}\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} + \big\||T\,\cdot|^{-1}\tilde{v}_1(|\widetilde{T}\,\cdot|) - v_1(|\widetilde{T}\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} \\
&\qquad \lesssim \big\||T\,\cdot|^{-1}(\phi_0(|T\,\cdot|) - \phi_0(|\widetilde{T}\,\cdot|))\big\|_{H^6(\mathbb{B}^{11})} + \big\||\cdot|^{-1}v_1(|\cdot|) - \tilde{v}_1(|\cdot|)\big\|_{H^6(\mathbb{B}^{11}_{1+\delta})} \\
&\qquad\qquad + \big\|\tilde{v}_1(|T\,\cdot|) - \tilde{v}_1(|\widetilde{T}\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} \\
&\qquad \le \big\||T\,\cdot|^{-1}\big(\phi_0(|T\,\cdot|) - \phi_0(|\widetilde{T}\,\cdot|)\big)\big\|_{H^6(\mathbb{B}^{11})} + \big\|\tilde{v}_1(|T\,\cdot|) - \tilde{v}_1(|\widetilde{T}\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} + \varepsilon,
\end{aligned}$$

This together with (3-64) implies that $[\boldsymbol{U}(\boldsymbol{v}, T)]_1$ is continuous. The second component is treated analogously. Now, given $\||\cdot|^{-1}\boldsymbol{v}\|_{\mathcal{H}^{1+\delta}} \le \delta$ and $T \in [1-\delta, 1+\delta]$, we have

$$\begin{aligned}
\|[\boldsymbol{U}(\boldsymbol{v}, T)]_1\|_{H^6(\mathbb{B}^{11})} &= \big\||\cdot|^{-1}\phi_0(|T\,\cdot|) - |\cdot|^{-1}\phi_0(|\cdot|) + |\cdot|^{-1}v_1(|T\,\cdot|)\big\|_{H^6(\mathbb{B}^{11})} \\
&\lesssim |T-1| + \||\cdot|^{-1}v_1\|_{H^6(\mathbb{B}^{11}_{1+\delta})} \lesssim \delta.
\end{aligned}$$

We obtain a similar estimate for the second component and finally deduce that

$$\|\boldsymbol{U}(\boldsymbol{v}, T)\| \lesssim \delta. \qquad \square$$

As already mentioned, the unstable eigenvalue $\lambda = 1$ is present due to the freedom of choice of the parameter T, and is therefore not considered a "real" instability of the linear problem. The following theorem is the precise version of this statement. Namely, for a given T_0 and small enough initial data $\boldsymbol{v}$, there exists a $T_{\boldsymbol{v}}$ close to T_0 that makes the correction term $\boldsymbol{C}(\Phi_{\boldsymbol{U}(\boldsymbol{v},T_{\boldsymbol{v}})}, \boldsymbol{U}(\boldsymbol{v}, T_{\boldsymbol{v}}))$ vanish. This in turn allows for proving the existence and uniqueness of an exponentially decaying solution to (3-60).

Theorem 3.16. *Fix $T_0 > 0$. Then there exist $\delta, M > 0$ such that for any $\boldsymbol{v}$ that satisfies*

$$\||\cdot|^{-1}\boldsymbol{v}\|_{\mathcal{H}^{T_0+\delta}} \le \frac{\delta}{M}$$

there exists a $T \in [T_0 - \delta, T_0 + \delta]$ *and a function* $\Phi \in \mathcal{X}_\delta$ *which satisfies*

$$\Phi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{U}(\boldsymbol{v}, T) + \int_0^\tau \boldsymbol{S}(\tau - s)\boldsymbol{N}(\Phi(s))\, ds \tag{3-65}$$

for all $\tau > 0$. *Moreover*, Φ *is the unique solution of this equation in* $C([0,\infty), \mathcal{H})$.

Proof. Let $T_0 > 0$ be fixed. We first prove that for any T in a small neighborhood of T_0 and small enough initial data $\boldsymbol{v}$ there exists a unique solution to (3-63) for $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{v}, T)$. From Lemma 3.15 we deduce the existence of sufficiently small δ and sufficiently large $M > 0$ so that for every $T \in [T_0 - \delta, T_0 + \delta]$, $\||\cdot|^{-1}\boldsymbol{v}\|_{\mathcal{H}^{T_0+\delta}} \le \delta/M$ implies $\|\boldsymbol{U}(\boldsymbol{v}, T)\|_{\mathcal{H}} \le \delta/C$ for a large enough $C > 0$. Via Theorem 3.14 this yields the unique solution to (3-63) for every T in the designated range. It remains to show that for small enough $\boldsymbol{v}$, there exists a particular $T_{\boldsymbol{v}} \in [T_0 - \delta, T_0 + \delta]$ that makes the correction term vanish, i.e., $\boldsymbol{C}(\Phi_{\boldsymbol{U}(\boldsymbol{v},T_{\boldsymbol{v}})}, \boldsymbol{U}(\boldsymbol{v}, T_{\boldsymbol{v}})) = 0$. Since $\boldsymbol{C}$ has values in $\operatorname{rg} \boldsymbol{P} = \langle \boldsymbol{g} \rangle$, the latter is equivalent to the existence of a $T_{\boldsymbol{v}} \in [T_0 - \delta, T_0 + \delta]$ such that

$$\bigl\langle \boldsymbol{C}(\Phi_{\boldsymbol{U}(\boldsymbol{v},T_{\boldsymbol{v}})}, \boldsymbol{U}(\boldsymbol{v}, T_{\boldsymbol{v}})), \boldsymbol{g} \bigr\rangle_{\mathcal{H}} = 0. \tag{3-66}$$

By definition, we have

$$\partial_T \begin{bmatrix} (1/\rho)\phi_0((T/T_0)\rho) \\ (T^2/T_0^2)\phi_0'((T/T_0)\rho) \end{bmatrix}\Bigg|_{T=T_0} = \frac{\boldsymbol{g}(\rho)}{T_0}$$

and this yields the expansion

$$\bigl\langle \boldsymbol{C}(\Phi_{\boldsymbol{U}(\boldsymbol{v},T_{\boldsymbol{v}})}, \boldsymbol{U}(\boldsymbol{v}, T)), \boldsymbol{g} \bigr\rangle_{\mathcal{H}} = \frac{\|\boldsymbol{g}\|^2}{T_0}(T - T_0) + O((T-T_0)^2) + O\biggl(\frac{\delta}{M} T^0\biggr) + O(\delta^2 T^0).$$

A simple fixed-point argument now proves (3-66); see [Donninger and Schörkhuber 2017, Theorem 4.15] for full details. ☐

Proof of Theorem 3.1. Fix $T_0 > 0$ and assume the radial initial data $u[0]$ satisfy

$$\bigl\||\cdot|^{-1}(u[0] - u^{T_0}[0])\bigr\|_{H^6(\mathbb{B}^{11}_{T_0+\delta}) \times H^5(\mathbb{B}^{11}_{T_0+\delta})} \le \frac{\delta}{M_0^2}$$

with $\delta, M_0 > 0$ to be chosen later. We set $\boldsymbol{v} := u[0] - u^{T_0}[0]$; see Section 3E. Then we have

$$\||\cdot|^{-1}\boldsymbol{v}\|_{\mathcal{H}^{T_0+\delta}} = \bigl\||\cdot|^{-1}(u[0] - u^{T_0}[0])\bigr\|_{\mathcal{H}^{T_0+\delta}} \le \frac{\delta}{M_0^2}.$$

Now, upon choosing $\delta > 0$ sufficiently small and $M_0 > 0$ sufficiently large, Theorem 3.16 yields a $T \in [T_0 - \delta/M_0, T_0 + \delta/M_0] \subset [1-\delta, 1+\delta]$ such that there exists a unique solution $\Phi = (\varphi_1, \varphi_2) \in \mathcal{X}$ to (3-65) with $\|\Phi(\tau)\| \le (\delta/M_0)e^{-2\varepsilon\tau}$ for all $\tau \ge 0$ and some $\varepsilon > 0$. Therefore, by construction,

$$u(t,r) = u^T(t,r) + \frac{r}{T-t}\varphi_1\biggl(\log\frac{T}{T-t}, \frac{r}{T-t}\biggr)$$

solves the original wave maps (3-1). Moreover,

$$\partial_t u(t,r) = \partial_t u^T(t,r) + \frac{r}{(T-t)^2}\varphi_2\biggl(\log\frac{T}{T-t}, \frac{r}{T-t}\biggr).$$

Therefore,

$$
\begin{aligned}
(T-t)^{k-\frac{9}{2}} \big\| |\cdot|^{-1}(u(t,|\cdot|)-u^T(t,|\cdot|)) \big\|_{\dot H^k(\mathbb{B}^{11}_{T-t})} &= (T-t)^{k-\frac{11}{2}} \left\| \varphi_1\left(\log\frac{T}{T-t}, \frac{|\cdot|}{T-t}\right)\right\|_{\dot H^k(\mathbb{B}^{11}_{T-t})} \\
&= \left\| \varphi_1\left(\log\frac{T}{T-t}, |\cdot|\right)\right\|_{\dot H^k(\mathbb{B}^{11})} \le \left\| \Phi\left(\log\frac{T}{T-t}\right)\right\|_{\mathcal{H}} \\
&\le \frac{\delta}{M_0}(T-t)^{2\varepsilon}
\end{aligned}
$$

for all $t \in [0,T)$ and any integer $0 \le k \le 6$. Furthermore,

$$
\begin{aligned}
&(T-t)^{l-\frac{7}{2}} \big\| |\cdot|^{-1}(\partial_t u(t,|\cdot|) - \partial_t u^T(t,|\cdot|)) \big\|_{\dot H^l(\mathbb{B}^{11}_{T-t})} \\
&\qquad = (T-t)^{l-\frac{11}{2}} \left\| \varphi_2\left(\log\frac{T}{T-t}, \frac{|\cdot|}{T-t}\right)\right\|_{\dot H^l(\mathbb{B}^{11}_{T-t})} \\
&\qquad = \left\| \varphi_2\left(\log\frac{T}{T-t}, |\cdot|\right)\right\|_{\dot H^l(\mathbb{B}^{11})} \le \left\| \Phi\left(\log\frac{T}{T-t}\right)\right\|_{\mathcal{H}} \le \frac{\delta}{M_0}(T-t)^{2\varepsilon}
\end{aligned}
$$

for all $l = 0, 1, \dots, 5$. Finally, by Sobolev embedding we infer

$$
\begin{aligned}
\|u(t,\cdot)-u^T(t,\cdot)\|_{L^\infty(0,T-t)} &\le (T-t)\big\| |\cdot|^{-1}(u(t,|\cdot|)-u^T(t,|\cdot|)) \big\|_{L^\infty(0,T-t)} \\
&\lesssim (T-t)\big\| |\cdot|^{-1}(u(t,|\cdot|)-u^T(t,|\cdot|)) \big\|_{H^{11/2+\varepsilon}(\mathbb{B}^{11}_{T-t})} \\
&\lesssim \frac{\delta}{M_0}(T-t)^{\varepsilon}
\end{aligned}
$$

and this finishes the proof by setting $M := M_0^2$. ☐

Remark 3.17. Based on [Donninger and Schörkhuber 2017; Chatzikaleas et al. 2017], the analogue of Theorem 3.1 in any odd dimension $d \ge 11$ follows from the mode stability of the solution u^T. However, a nontrivial adjustment of the method of the proof of Proposition 3.7 is required in order to establish the analogous result for all higher odd d simultaneously. This will be addressed in a forthcoming publication.

Appendix A: Proof of Proposition 2.1

A straightforward computation shows that all sectional curvatures of the manifold N^d are given by either

$$
\text{(i)}\ \ \frac{-g''(u)}{g(u)} \quad \text{or} \quad \text{(ii)}\ \ \frac{1-g'(u)^2}{g(u)^2}. \tag{A-1}
$$

We first show that the two expressions above are negative provided $d \ge 8$ and $u \in I := [0, \phi_0(1)]$. For convenience we let $d = e + 8$. We now have

$$
\frac{g''(u)}{g(u)} = \frac{6(23e+14)^2u^6 - 63(23e+14)u^4 - 2(115e+21)u^2 + 21}{[(23e+14)u^4 - 7u^2 - 1]^2}. \tag{A-2}
$$

Denote the numerator in the above expression by $N(e,u)$. To show that the first quantity in (A-1) is negative it suffices to prove that $N(e,u) > 0$ for $(e,u) \in [0,\infty) \times I$. To that end, it is enough to show that for any fixed $e \geq 0$ the following inequalities hold:

$$\text{(i)}\ \ N(e,0) > 0, \quad \text{(ii)}\ \ N(e,\phi_0(1)) > 0 \quad \text{and} \quad \text{(iii)}\ \ \partial_u^2 N(e,u) < 0\ \text{ for } u \in I. \tag{A-3}$$

We start by proving the third claim above. Note that it is enough to show that

$$\text{(i)}\ \ \partial_u^2 N(e,0) < 0, \quad \text{and} \quad \text{(ii)}\ \ \partial_u^3 N(e,u) \leq 0\ \text{ for } u \in I. \tag{A-4}$$

To establish (A-4) we need the following:

$$\partial_u^2 N(e,u) = 4[45(23e+14)^2 u^4 - 189(23e+14)u^2 - 115e - 21], \tag{A-5}$$

$$\partial_u^3 N(e,u) = 72(23e+14)u[10(23e+14)u^2 - 21], \tag{A-6}$$

$$\partial_u^5 N(e,u) = 4320u(23e+14)^2. \tag{A-7}$$

Equation (A-5) gives $\partial_u^2 N(e,0) = -4(115e+21)$ and the first claim in (A-4) follows. From (A-7) we see that $\partial_u^3 N(e,u)$ is convex for $u \in I$. Therefore, since $\partial_u^3 N(e,0) = 0$ it is enough to show that

$$\partial_u^3 N(e,\phi_0(1)) \leq 0 \tag{A-8}$$

for the second claim in (A-4) to hold. To establish this inequality, we first use definition (2-4) to compute

$$\phi_0(1) = \left(\frac{2}{\sqrt{(e+7)(46e^2+445e+567)} - 7(e+7)}\right)^{\frac{1}{2}}.$$

Now, according to (A-6), it is enough to prove that $10(23e+14)\phi_0(1)^2 - 21 < 0$ for (A-8) to hold. This inequality is equivalent to $441e^2 - 925e + 1316 > 0$, which clearly holds for all $e \geq 0$. This concludes the proof of the third claim in (A-3). Since the first claim in (A-3) is obviously true, it is left to prove that $N(e,\phi_0(1)) > 0$. To that end we first compute

$$N(e,\phi_0(1)) = \frac{2(P(e)\sqrt{Q(e)} - R(e))}{[\sqrt{Q(e)} - 7(e+7)]^3}, \tag{A-9}$$

where

$$P(e) = 7(69e^3 + 1831e^2 + 11500e + 17094),$$

$$Q(e) = (e+7)(46e^2+445e+567),$$

$$R(e) = 20723e^4 + 433338e^3 + 3077307e^2 + 8566502e + 7537866.$$

The denominator in (A-9) is positive if and only of $Q(e)^2 - 49(e+7)^2 > 0$. This is equivalent to $2(e+8)(e+7)(23e+14) > 0$, which is manifestly true for $e \geq 0$. The numerator in (A-9) is positive if and only if $P(e)^2 Q(e) - R(e)^2 > 0$, which is equivalent to $2(23e+14)^2 S(e) > 0$, where

$$S(e) = 10143e^7 + 289189e^6 + 2979735e^5 + 12402439e^4 + 11046366e^3 - 30567884e^2 + 15651132e + 22614480.$$

The positivity of $S(e)$ is easily shown; for example we have

$$12402439e^4 + 22614480 > 30567884e^2.$$

The positivity of $N(e,\phi_0(1))$ follows.

Now we turn to proving that the second expression in (A-1) is negative for $d \geq 8$ and $u \in I$. Since $g''(u)/g(u)$ is positive for $u \in I$ and $g(u) > 0$ for small positive values of u, we conclude that both g'' and g are positive on $(0, \phi_0(1)]$. Consequently

$$g'(u) - 1 = g'(u) - g'(0) = \int_0^u g''(t)\,dt > 0 \quad \text{for } u \in (0, \phi_0(1)].$$

Hence $g'(u)^2 - 1 > 0$ and therefore

$$\frac{1 - g'(u)^2}{g(u)^2} < 0$$

for $u \in (0, \phi_0(1)]$. Additionally, by direct computation we see that

$$\frac{1 - g'(0)^2}{g(0)^2} = -21 < 0.$$

Finally, for each $d \geq 8$ we infer the existence of $\varepsilon > 0$ for which both expressions in (A-1) are negative provided $|u| < \phi_0(1) + \varepsilon$. For $|u| \geq \phi_0(1) + \varepsilon$, the function $g(u)$ can be easily modified so that it satisfies (1-2) and both expressions in (A-1) remain negative.

Appendix B: Estimate for δ_7

Proposition B.1. *For δ_7 defined in (3-42) and $\lambda \in \overline{\mathbb{H}}$ we have*

$$|\delta_7(\lambda)| \leq \tfrac{1}{3}. \tag{B-1}$$

Proof. Following the proof of Lemma 4.3 in [Costin et al. 2017] we show that r_7 and $(\tilde{r}_7)^{-1}$ are analytic in $\overline{\mathbb{H}}$. This implies that δ_7 is also analytic there. Furthermore, being a rational function, δ_7 is evidently polynomially bounded in $\overline{\mathbb{H}}$. Therefore, according to the Phragmén–Lindelöf principle,[2] it suffices to prove that (B-1) holds on the imaginary line, i.e.,

$$|\delta_7(is)|^2 \leq \tfrac{1}{9} \quad \text{for } s \in \mathbb{R}. \tag{B-2}$$

Note that the function $s \mapsto |\delta_7(is)|^2$ is even. It is therefore enough to prove (B-2) for nonnegative s only. We show that for $t \geq 0$,

$$\left|\delta_7\left(\frac{4t}{t+1}i\right)\right|^2 \leq \tfrac{1}{9} \quad \text{and} \quad |\delta_7((t+4)i)|^2 \leq \tfrac{1}{9}. \tag{B-3}$$

The first estimate above proves (B-2) for $s \in [0, 4)$, while the second one covers the complementary interval $[4, \infty)$. We prove both estimates in (B-3) in the same way and therefore illustrate the proof of the second one only. Note that

$$|\delta_7((t+4)i)|^2 = \frac{Q_1(t)}{Q_2(t)},$$

[2] We use the sectorial formulation of this principle; see, for example, [Titchmarsh 1939, p. 177].

where $Q_j(t) \in \mathbb{Z}[t]$, $\deg Q_j = 32$ and Q_2 has all positive coefficients. Therefore, $|\delta_7((t+4)i)|^2 \le \frac{1}{9}$ is equivalent to $Q_2 - 9Q_1 \ge 0$ and a direct calculation shows that the polynomial $Q_2 - 9Q_1$ has manifestly positive coefficients. □

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