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# A VISCOSITY APPROACH TO THE DIRICHLET PROBLEM FOR DEGENERATE COMPLEX HESSIAN-TYPE EQUATIONS

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A viscosity approach is introduced for the Dirichlet problem associated to complex Hessian-type equations on domains in $\mathbb{C}^n$. The arguments are modeled on the theory of viscosity solutions for real Hessian-type equations developed by Trudinger (1990). As a consequence we solve the Dirichlet problem for the Hessian quotient and special Lagrangian equations. We also establish basic regularity results for the solutions.



## 1. Introduction

Partial differential equations play a pivotal role in modern complex geometric analysis. Their applications typically involve a geometric problem which can be reduced to the solvability of an associated equation. This solvability can be deduced by various methods, yet most of the basic approaches exploit a priori estimates for suitably defined weak solutions. Thus although geometers work in the smooth category, the associated weak theory plays an important role.

One of the most successful such theories is the pluripotential theory associated to the complex Monge–Ampère equation developed by Bedford and Taylor [1976; 1982], Kołodziej [1998], Guedj and Zeriahi [2005] and many others. Roughly speaking, pluripotential theory allows one to define $(i\partial\bar{\partial}u)^k$ as a measure-valued positive closed differential form (i.e., a closed positive current) for any locally bounded plurisubharmonic function, which in turn allows one to deal with nonsmooth weak solutions of Monge–Ampère equations. Unfortunately the pluripotential approach is applicable only for a limited class of nonlinear operators, such as the $m$-Hessian equations — see [Dinew and Kołodziej 2014; Lu 2013].

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Some of the most important examples of nonlinear operators for which pluripotential tools do not seem to apply directly are the complex Hessian quotient operators. These are not only interesting for themselves but also appear in interesting geometrical problems. One such example is the Donaldson equation, which we describe below.

Given a compact Kähler manifold $(X,\omega)$ equipped with another Kähler form $\chi$, one seeks a Kähler form $\tilde{\chi}$ cohomologous to $\chi$ such that

$$\omega\wedge\tilde{\chi}^{n-1}=c\tilde{\chi}^{n}, \tag{1}$$

with the constant $c$ dependent only on the cohomology classes of $\chi$ and $\omega$.

Donaldson [1999] introduced this equation in order to study the properness of the Mabuchi functional. Its parabolic version, known as the $J$-flow, was introduced independently by Donaldson [1999] and Chen [2000] and investigated afterwards by Song and Weinkove [Weinkove 2004; 2006; Song and Weinkove 2008]. It is known that (1) is not always solvable. It was shown in [Song and Weinkove 2008] that a necessary and sufficient condition for the solvability of (1) is that there exists a metric $\chi'$ in $[\chi]$, the Kähler class of $\chi$, satisfying

$$(nc\chi'-(n-1)\omega)\wedge\chi'^{n-2}>0 \tag{2}$$

in the sense of $(n-1,n-1)$-forms. A conjecture of Lejmi and Székelyhidi [2015] predicts that the solvability is linked to positivity of certain integrals which can be viewed as geometric stability conditions. It was also proved that, in general, these positivity conditions are equivalent to the existence of $C$-subsolutions introduced by Székelyhidi [2018]. They are also equivalent to the existence of parabolic $C$-subsolutions for the corresponding flows; see [Phong and Tô 2017]. It would be helpful to study the boundary case when we only have nonnegativity conditions; see [Fang et al. 2014] for Donaldson equation on surfaces. It is expected that in this boundary case the equation admits suitably defined singular solutions which are smooth except on some analytic set. This has been confirmed in complex dimension two in [Fang et al. 2014] but the proof cannot be generalized to higher dimensions. In fact a major part of the problem is to develop the associated theory of weak solutions for the given Hessian quotient equation. An essential problem in applying some version of pluripotential theory for this equation is that one has to define the quotient of two measure-valued operators.

In order to circumvent this difficulty one can look for a possibly different theory of weak solutions. One such approach, known as the viscosity method, was invented long ago in the real setting [Crandall et al. 1992], but was only recently introduced for complex Monge–Ampère equations by Eyssidieux, Guedj and Zeriahi [Eyssidieux et al. 2011], Wang [2012] and Harvey and Lawson [2009].

In the current note we initiate the viscosity theory for general complex nonlinear elliptic PDEs. As the manifold case is much harder we focus only on the local theory; i.e., we deal with functions defined over domains in $\mathbb{C}^n$. Precisely, let $\Omega\subset\mathbb{C}^n$ be a bounded domain. We consider the equation

$$F[u]:=f(\lambda(Hu))=\psi(x,u), \tag{3}$$

where $\lambda(Hu)$ denotes the vector of the eigenvalues of the complex Hessian $Hu$ of the real-valued function $u$ and $\psi:\Omega\times\mathbb{R}\to\mathbb{R}_+$ is a given nonnegative function which is weakly increasing in the second variable. We wish to point out that nonlinear PDEs appear also in geometric problems which are defined over domains in $\mathbb{C}^n$ — see for example [Collins et al. 2017], where a Dirichlet problem for the *special Lagrangian-type equation* is studied. These are the equations defined for a given function $h$ by

$$F[u] := \sum_{i=1}^{n} \arctan \lambda_i = h(z),$$

with $\lambda_i$ denoting the eigenvalues of the Hessian of $u$ at $z$. In the real case, the special Lagrangian equations were introduced by Harvey and Lawson [1982] in the study of *calibrated geometries*. More precisely the graphs of gradients of the solutions correspond to calibrated minimal submanifolds. We show in Section 6 that our method can be applied to solve the Dirichlet problem for the special degenerate Lagrangian-type equation.

In our investigations we heavily rely on the corresponding real theory developed by Trudinger [1990]. It is worth pointing out that the real theory of Hessian and Hessian quotient equations is much better understood thanks to the fundamental results of [Trudinger 1995] and [Chou and Wang 2001]. Some of our results can be seen as complex analogues of the real results that can be found there. In particular we have focused on various comparison principles in Section 3. Our first major result can be summarized as follows (we refer to the next section for the definitions of the objects involved):

**Theorem 1** (comparison principle). *Let $\Gamma$ be the ellipticity cone associated to (3). Assume that the operator $F[u] = f(\lambda(Hu))$ in (3) satisfies*

$$f \in C^0(\overline{\Gamma}), \qquad f > 0 \quad \text{on } \Gamma, \qquad f = 0 \quad \text{on } \partial\Gamma,$$
$$f(\lambda + \mu) \geq f(\lambda) \quad \text{for all } \lambda \in \Gamma,\ \mu \in \Gamma_n.$$

*Assume moreover that either*

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} \lambda_i = \sum_{i=1}^{n} f_i \lambda_i \geq \nu(f) \quad \text{in } \Gamma \qquad \text{and} \qquad \inf_{z \in \Omega} \psi(z, \cdot\,) > 0$$

*for some positive increasing function $\nu$, or*

$$f \text{ is concave and homogeneous.}$$

*Then any bounded subsolution $u$ and supersolution $v$ in $\Omega$ to (3) satisfy*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}\{(u - v)^*, 0\}.$$

We use later on this seemingly technical result to study existence, uniqueness and regularity of the associated Dirichlet problems. One of our main results is the solvability and sharp regularity for viscosity solutions to the Dirichlet problem for a very general class of operators including Hessian quotient-type equations.

**Theorem 2.** *The Dirichlet problem*

$$\begin{cases} F[w] = f(\lambda(Hw)) = \psi(z, w(z)), \\ w = \varphi \quad \text{on } \partial\Omega \end{cases}$$

*admits a continuous solution for any bounded $\Gamma$-pseudoconvex domain $\Omega$. Under natural growth assumptions on $\psi$, the solution is Hölder continuous for any Hölder continuous boundary data $\varphi$.*

Another interesting topic is the comparison between viscosity and pluripotential theory whenever the latter can be reasonably defined. A guiding principle for us is the basic observation made by Eyssidieux, Guedj and Zeriahi [Eyssidieux et al. 2011] that plurisubharmonic functions correspond to viscosity subsolutions to the complex Monge–Ampère equation. We prove several analogous results for general complex nonlinear operators. It has to be stressed that the notion of a supersolution, which does not appear in pluripotential theory, is a very subtle one for nonlinear elliptic PDEs, and several alternative definitions are possible. We in particular compare these and introduce a notion of supersolution that unifies the previously known approaches.

A large part of the note is devoted to complex Hessian quotient equations in domains in $\mathbb{C}^n$. One of our goals in this case was to initiate the construction of the undeveloped pluripotential theory associated to such equations. We rely on connections with the corresponding viscosity theory. Our findings yield in particular that the natural domain of definition of these operators is *strictly smaller* than what standard pluripotential theory would predict. We prove the following theorem:

**Theorem 3.** *Assume that* $0 < \psi \in C^0(\Omega)$ *and* $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *is a viscosity subsolution of*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = \psi(z) \quad \text{in } \Omega.$$

*Then*

$$(dd^c u)^n \geq \psi (dd^c u)^{n-k} \wedge \omega^k$$

*and*

$$(dd^c u)^k \geq \binom{n}{k}^{-1} \psi \omega^k$$

*in the pluripotential sense.*

We guess that this observation, rather obvious in the case of smooth functions, will play an important role in the resolution of the issue caused by the division of measures.

The note is organized as follows: in the next section we collect the basic notions from linear algebra, viscosity and pluripotential theory. Then we investigate the various notions of supersolutions in [Eyssidieux et al. 2011; Lu 2013] and compare them with the complex analogue of Trudinger's supersolutions. Section 3 is devoted to the proof of a very general comparison principle. Then in Section 4 we restrict our attention to operators depending on the eigenvalues of the complex Hessian matrix of the unknown function. We show existence and uniqueness of viscosity solutions under fairly mild conditions. One subsection is devoted to the regularity of these weak solutions. Using classical methods due to Walsh [1968], see also [Bedford and Taylor 1976], we show the optimal Hölder regularity for sufficiently regular data. Section 5 is devoted to comparisons between viscosity and pluripotential subsolutions and supersolutions. Finally in Section 6 we solve the Dirichlet problem for the Lagrangian phase operator.

## 2. Preliminaries

In this section we collect the notation and the basic results and definitions that will be used throughout the note.

**2.1.** ***Linear algebra toolkit.*** We begin by introducing the notion of an admissible cone that will be used throughout the note:

**Definition 4.** A cone $\Gamma$ in $\mathbb{R}^n$ with vertex at the origin is called admissible if:

(1) $\Gamma$ is open and convex, $\Gamma \neq \mathbb{R}^n$.

(2) $\Gamma$ is symmetric; i.e., if $x = (x_1, \dots, x_n) \in \Gamma$ then for any permutation of indices $i = (i_1, \dots, i_n)$, the vector $(x_{i_1}, \dots, x_{i_n})$ also belongs to $\Gamma$.

(3) $\Gamma_n \subset \Gamma$, where $\Gamma_n := \{x \in \mathbb{R}^n \mid x_i > 0,\ i \in 1, \dots, n\}$.

From the very definition it follows that $\Gamma_n$ is an admissible cone. Other examples involve the $\Gamma_k$ cones that we describe below:

Consider the $m$-th elementary symmetric polynomial defined by

$$\sigma_m(x) = \sum_{1 \le j_1 < \cdots < j_m \le n} x_{j_1} x_{j_2} \cdots x_{j_m}.$$

We shall use also the normalized version

$$S_m(x) := \binom{n}{m}^{-1} \sigma_m.$$

**Definition 5.** For any $m = 1, \dots n$, the positive cone $\Gamma_m$ of vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is defined by

$$\Gamma_m = \{x \in \mathbb{R}^n \mid \sigma_1(x) > 0, \dots, \sigma_m(x) > 0\}. \tag{4}$$

It is obvious that these cones are open and symmetric with respect to a permutation of the $x_i$'s. It is a nontrivial but classical fact that $\Gamma_m$ is also convex.

Exploiting the symmetry of $\Gamma$, it is possible to discuss $\Gamma$-positivity for Hermitian matrices:

**Definition 6.** A Hermitian $n \times n$ matrix $A$ is called $\Gamma$-positive (respectively, $\Gamma$-semipositive) if the vector of eigenvalues $\lambda(A) := (\lambda_1(A), \dots, \lambda_n(A))$ belongs to $\Gamma$ (respectively, to the Euclidean closure $\overline{\Gamma}$ of $\Gamma$). The definition is independent of the ordering of the eigenvalues.

Finally one can define, following [Li 2004], the notion of $\Gamma$-admissible and $\Gamma$-subharmonic functions through the following definitions:

**Definition 7.** A $C^2$ function $u$ defined on a domain $\Omega \subset \mathbb{C}$ is called $\Gamma$-admissible if for any $z \in \Omega$ the complex Hessian $Hu(z) := [\partial^2/(\partial z_j\, \partial \bar{z}_k)]_{j,k=1}^n$ is $\Gamma$-positive.

In particular, if $\Gamma$ is an admissible cone, then $\Gamma \subset \Gamma_1$, see [Caffarelli et al. 1985], and hence we have the following corollary:

**Corollary 8.** *Any $\Gamma$-admissible function is subharmonic.*

**Definition 9.** An upper semicontinuous function $v$ defined on a domain $\Omega \subset \mathbb{C}^n$ is called $\Gamma$-subharmonic if near any $z \in \Omega$ it can be written as a decreasing limit of local $\Gamma$-admissible functions.

We refer to [Harvey and Lawson 2009] for a detailed discussion and potential-theoretic properties of general $\Gamma$-subharmonic functions.

**2.2.** ***Viscosity sub- and supersolutions.*** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Consider the equation

$$F[u] := F(x, u, Du, Hu) = 0 \quad \text{on } \Omega, \tag{5}$$

where $Du = (\partial_{z_1} u, \dots, \partial_{z_n} u)$, $Hu = (u_{j\bar{k}})$ is the Hessian matrix of $u$ and $F$ is continuous on $\Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{H}^n$. The operator $F$ is called *degenerate elliptic* at a point $(z, s, p, M)$ if

$$F(z, s, p, M + N) \geq F(z, s, p, M) \quad \text{for all } N \geq 0,\ N \in \mathcal{H}^n, \tag{6}$$

where $\mathcal{H}^n$ is the set of Hermitian matrices of size $n \times n$. We remark that in our case $F(z, s, p, M)$ is not necessarily degenerate elliptic everywhere on $\Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{H}^n$. Motivated by [Trudinger 1990], we pose the following definition:

**Definition 10.** A function $u \in L^\infty(\Omega)$ is a viscosity subsolution of (5) if it is upper semicontinuous in $\Omega$ and for any $z_0 \in \Omega$, and any $C^2$ smooth function $q$ defined in some neighborhood of $z_0$ and satisfying $u \leq q,\ u(z_0) = q(z_0)$, the inequality

$$F[q](z_0) \geq 0 \tag{7}$$

holds. We also say that $F[u] \geq 0$ in the viscosity sense and $q$ is an upper (differential) test for $u$ at $z_0$.

A function $v \in L^\infty(\Omega)$ is a viscosity supersolution of (5) if it is lower semicontinuous and there are no points $z_0 \in \Omega$ and $C^2$ smooth functions defined locally around $z_0$ such that $v \geq q$ in $\Omega,\ v(z_0) = q(z_0)$ and

$$\inf_{N \geq 0} F\big(z_0, q(z_0), Dq(z_0), N + Hq(z_0)\big) > 0. \tag{8}$$

We also say that $F[u] \leq 0$ in the viscosity sense and $q$ is a lower (differential) test for $u$ at $z_0$.

For fixed $(z, s, p) \in \Omega \times \mathbb{R} \times \mathbb{C}^n$, the set of all Hermitian matrices $M$ such that $F$ is degenerate elliptic at $(z, s, p, M)$ is called the *ellipticity set* $\mathcal{A}(z, s, p)$ for the data $(z, s, p)$. Note that the ellipticity set has the property that

$$\mathcal{A}(z, s, p) + \Gamma_n \subset \mathcal{A}(z, s, p),$$

but it may not be a cone. Throughout the note we shall however focus on the situation when the ellipticity set is a cone which is moreover constant for all the possible data sets. We then define the *ellipticity cone* associated to the operator $F$ which is modeled on the notion of a subequation coined by Harvey and Lawson [2009]:

**Definition 11.** An operator $F(z, s, p, M)$ has an ellipticity cone $\Gamma$ if for any $M$ in the ellipticity set the vector $\lambda(M)$ of the eigenvalues of $M$ belongs to the closure $\overline{\Gamma}$ of $\Gamma$. Furthermore $\Gamma$ is the minimal cone with such properties.

Throughout the note we consider only the situation when $\Gamma$ is an admissible cone in the sense of Definition 4. We shall make also the following additional assumption (compare with the condition (2) in Section 4.1):

$$\text{for all } \lambda \in \partial\Gamma,\ \text{ for all } (z,\ s,\ p) \in \Omega \times \mathbb{R} \times \mathbb{C}^n, \quad F(z, s, p, \lambda) \leq 0. \tag{9}$$

This condition arises naturally whenever one seeks solutions to

$$F\big(z, u(z), Du(z), Hu(z)\big) = 0$$

with pointwise Hessian eigenvalues in $\Gamma$ (recall that $F$ increases in the $\Gamma_n$-directions).

It is evident that in Definition 10 the notion of a supersolution is different and substantially more difficult than the notion of a subsolution. The reason for this is that there is no analogue for the role of the positive cone $\Gamma_n$ from the case of subsolutions in the supersolutions' case. As an illustration we recall that while any plurisubharmonic function is a subsolution for $F(u) := \det(H(u)) = 0$, see [Eyssidieux et al. 2011], it is far from true that all supersolutions can be written as the negative of a plurisubharmonic function.

Below we also give another notion of a supersolution that was coined in [Eyssidieux et al. 2011] for the Monge–Ampère equation; see also [Lu 2013] for the case of the $m$-Hessian operator. It can be generalized for all operators admitting an elliptic admissible cone:

**Definition 12.** A lower semicontinuous function $u$ is said to be a supersolution for the operator $F(z,s,p,M)$ with the associated ellipticity cone $\Gamma$ if and only if for any $z_0 \in \Omega$ and every lower differential test $q$ at $z_0$ for which $\lambda(Hq(z_0)) \in \overline{\Gamma}$ one has

$$F\big(z, q(z_0), Dq(z_0), Hq(z_0)\big) \leq 0.$$

Note that in the definition we limit the differential tests only to those for which $\lambda(Hq(z_0)) \in \overline{\Gamma}$.

The next proposition shows that under the assumption (9) the definition above coincides with the one from Definition 10.

**Proposition 13.** *Suppose that the operator $F(z, s, p, M)$ satisfies (9). Then a lower semicontinuous function $u$ defined on a domain $\Omega$ is a supersolution for $F(z, s, p, M) = 0$ in the sense of Definition 12 if and only if it is a supersolution in the sense of Definition 10.*

*Proof.* Suppose first that $u$ is a supersolution in the sense of Definition 12. Fix any $z_0$ in $\Omega$ and $q$ a lower differential test for $u$ at $z_0$. If $\lambda(Hq(z_0)) \in \Gamma$ then

$$F\big(z, q(z_0), Dq(z_0), Hq(z_0)\big) \leq 0;$$

hence taking $N = 0$ in Definition 10 we see that the condition is fulfilled. If $\lambda(Hq(z_0))$ fails to be in $\Gamma$ then there is a positive definite matrix $N$ and a positive number $t$ such that $\lambda(Hq(z_0) + tN) \in \partial\Gamma$. But this implies that $F(z, q(z_0), Dq(z_0), Hq(z_0) + tN) \leq 0$, which fulfills the condition in Definition 10 again.

Suppose now that $u$ is a supersolution in the sense of Definition 10. Again choose $z_0$ in $\Omega$ and $q$ a lower differential test for $u$ at $z_0$. We can assume that $\lambda(Hq(z_0))$ is in $\overline{\Gamma}$, for otherwise such a differential test cannot be applied in Definition 12. But then by ellipticity

$$F\big(z, q(z_0), Dq(z_0), Hq(z_0)\big) \leq F\big(z, q(z_0), Dq(z_0), Hq(z_0) + N\big) \quad \text{for all } N \geq 0,\ N \in \mathcal{H}^n.$$

The infimum over $N$ for the right-hand side is nonpositive by definition, which implies

$$F\big(z, q(z_0), Dq(z_0), Hq(z_0)\big) \leq 0,$$

which was to be proved. ∎

**2.3.** ***Aleksandrov–Bakelman–Pucci maximum principle.*** We now recall a variant of the Aleksandrov–Bakelman–Pucci (ABP) maximum principle following [Jensen 1988]. We first recall the following definition; see [Jensen 1988]:

**Definition 14.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ centered at the origin and $u \in C(\overline{\Omega})$. We define

$$E_\delta = \{x \in \Omega \mid \text{for some } p \in \overline{B(0,\delta)},\ u(z) \le u(x) + p.(z-x) \text{ for all } z \in \Omega\}.$$

Then we have the following lemma due to Jensen [1988], which will be used in the proof of Lemma 21. Recall that a function $u$ is said to be semiconvex if $u + k|z|^2$ is convex for a sufficiently large constant $k$.

**Lemma 15.** *Let $u \in C(\overline{\Omega})$ be semiconvex for some constant $k > 0$. If $u$ has an interior maximum and $\sup_\Omega u - \sup_{\partial\Omega} u = \delta_0 d > 0$, where $d = \operatorname{diam}(\Omega)$, then there is a constant $C = C(n,k) > 0$ such that*

$$|E_\delta| \ge C\delta^n \quad \textit{for all } \delta \in (0, \delta_0). \tag{10}$$

*Proof.* As in [Jensen 1988], by regularization, we can reduce to the case when $u \in C^2(\Omega)$. Now, suppose that $u$ has an interior maximum at $x_0$ and

$$\delta_0 = \frac{\sup_\Omega u - \sup_{\partial\Omega} u}{d} = \frac{u(x_0) - \sup_{\partial\Omega} u}{d},$$

where $d = \operatorname{diam}(\Omega)$.

We now prove that for $\delta < \delta_0$ we have $B(0,\delta) \subset Du(E_\delta)$. Indeed, for any $p \in B(0,\delta)$, consider the hyperplane $\ell_p(x) = h + \langle p, x\rangle$, where $h = \sup_{y\in\Omega}(u(y) - \langle p, y\rangle)$. Then we have $u(x) \le \ell_p(x)$ on $\Omega$ and $u(x_1) = \ell_p(x_1)$ for some $x_1 \in \overline{\Omega}$. If we can prove that $x_1 \in \Omega$, then $Du(x_1) = p$, so $B(0,\delta) \subset Du(E_\delta)$. Suppose by contradiction that $x_1 \in \partial\Omega$, then

$$\begin{aligned}\sup_\Omega u &= u(x_0)\\ &\le \ell_p(x_1) + \langle p, x_0 - x_1\rangle\\ &= u(x_1) + \langle p, x_0 - x_1\rangle \le \sup_{\partial\Omega} u + \delta d < \sup_{\partial\Omega} u + \delta_0 d = \sup_\Omega u,\end{aligned}$$

and hence we get a contradiction.

Next, as we have proved that $B(0,\delta) \subset Du(E_\delta)$, by comparing volumes we infer that

$$c(n)\delta^n \le \int_{E_\delta} |\det(D^2 u)|. \tag{11}$$

Since $u$ is semiconvex with the constant $k > 0$ and $D^2 u \le 0$ in $E_\delta$, we have $|\det(D^2 u)| \le k^n$. It follows that $|E_\delta| \ge c(n)k^{-n}\delta^n$. □

**2.4.** ***Γ-subharmonic functions.*** We have defined Γ-subharmonic functions as limits of admissible ones. Below we present the alternative viscosity and pluripotential points of view:

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Define $\omega = dd^c|z|^2$, where $d := i(\bar\partial + \partial)$ and $d^c := \frac{i}{2\pi}(\bar\partial - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar\partial$. Let $\Gamma \subsetneq \mathbb{R}^n$ be an admissible cone as in Definition 4. We first recall the definition of $k$-subharmonic function:

**Definition 16.** We call a function $u \in C^2(\Omega)$ $k$-subharmonic if for any $z \in \Omega$, the Hessian matrix $(u_{i\bar{j}})$ has eigenvalues forming a vector in the closure of the cone $\Gamma_k$.

Following the ideas of Bedford and Taylor [1982], Błocki [2005] introduced the pluripotential definition of the $k$-subharmonic function.

**Definition 17.** Let $u$ be subharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Then $u$ is called $k$-subharmonic ($k$-sh for short) if for any collection of $C^2$ smooth $k$-sh functions $v_1, \dots, v_{k-1}$, the inequality

$$dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{k-1} \wedge \omega^{n-k} \geq 0$$

holds in the weak sense of currents.

For a general cone $\Gamma$, we have the following definition in the spirit of viscosity theory:

**Definition 18.** An upper semicontinuous function $u$ is called $\Gamma$-subharmonic (respectively, strictly $\Gamma$-subharmonic) if for any $z \in \Omega$, and any upper test function $q$ of $u$ at $z$, we have

$$\lambda(Hq(z)) \in \overline{\Gamma} \quad (\text{respectively, } \lambda(Hq(z)) \in \Gamma).$$

By definition, if $u$ is a $\Gamma$-subharmonic function, it is a $\Gamma$-subsolution in the sense of [Székelyhidi 2018]. In particular, when $\Gamma = \Gamma_k$ for $k = 1, \dots, n$, we have $u$ is a viscosity subsolution of the equation

$$S_k(\lambda(Hu)) = 0,$$

where

$$S_k(\lambda(Hu)) = \frac{(dd^c u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Then it follows from [Eyssidieux et al. 2011; Lu 2013] that $u$ is a $k$-subharmonic function on $\Omega$; hence $u$ is a subharmonic function if $k = 1$ and a plurisubharmonic function if $k = n$.

We also have the following definition generalizing the pseudoconvex domains; see also [Li 2004] for a similar definition for smooth domains:

**Definition 19.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. We say that $\Omega$ is a $\Gamma$-pseudoconvex domain if there is a constant $C_\Omega > 0$ depending only on $\Omega$ so that $-d(z) + C_\Omega d^2(z)$ is $\Gamma$-subharmonic on $\partial\Omega$, where

$$d(z) := \operatorname{dist}(z, \partial\Omega).$$

We recall the following lemma, which was proved in [Li 2004, Theorem 3.1].

**Lemma 20.** *Let $\Omega$ be bounded domain in $\mathbb{C}^n$ with $C^2$ smooth boundary. Let $\rho \in C^2(\overline{\Omega})$ be a defining function of $\Omega$ so that $\lambda(H\rho) \in \Gamma$ on $\partial\Omega$. Then there exists a defining function $\tilde{\rho} \in C^2(\overline{\Omega})$ for $\Omega$ such that $\lambda(H\tilde{\rho}) \in \Gamma$ on $\overline{\Omega}$.*

Finally we wish to recall the survey article [Zeriahi 2013] where the reader may find a thorough discussion of the viscosity theory associated to complex Monge–Ampère-type equations.

## 3. Comparison principles

Comparison principles are basic tools in pluripotential theory — we refer to [Kołodziej 1998; Guedj and Zeriahi 2017] for a thorough discussion of these inequalities. In viscosity theory one compares sub- and supersolutions to the same equation. It is a crucial observation, see [Eyssidieux et al. 2011], that even though supersolutions may fail to have nice pluripotential properties, a version of the comparison principle holds for the complex Monge–Ampère equation. In this section we discuss under what assumptions such comparison principles hold for general operators.

**3.1.** ***A preliminary comparison principle.*** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. In this subsection we prove a comparison principle for viscosity solutions of the equation

$$F[u] := F(x, u, Du, Hu) = 0. \tag{12}$$

It is well known that mere ellipticity is insufficient to guarantee a comparison-type result. Hence we add some natural structural conditions for (12).

First of all we assume that $F$ is decreasing in the $s$-variable, namely

$$\text{for all } r > 0, \quad F(z, s, p, M) - F(z, s + r, p, M) \geq 0. \tag{13}$$

This is a natural assumption in the theory, see [Zeriahi 2013], as it yields an inequality in the "right" direction for the maximum principle.

Next we assume a particular continuity property with respect to the $z$- and $p$-variables:

$$|F(z_1, s, p_1, M) - F(z_2, s, p_2, M)| \leq \alpha_z(|z_1 - z_2|) + \alpha_p(|p_1 - p_2|) \tag{14}$$

for all $z_1, z_2 \in \Omega$, $\sigma \in \mathbb{R}$, $p_1, p_2 \in \mathbb{C}^n$, $M \in \mathcal{H}^n$. Here $\alpha_z$ and $\alpha_p$ are certain moduli of continuity, i.e., increasing functions defined for nonnegative reals which tend to zero as the parameter decreases to zero.

We can now state the following general comparison principle for (12).

**Lemma 21.** *Suppose $u \in L^\infty(\bar{\Omega})$ (respectively, $v \in L^\infty(\bar{\Omega})$) satisfies $F[u] \geq \delta$ (respectively, $F[v] \leq 0$) in $\Omega$ in the viscosity sense for some $\delta > 0$. Then*

$$\sup_\Omega (u - v) \leq \max_{\partial\Omega}\{(u - v)^*, 0\}, \tag{15}$$

*with $^*$ denoting the standard upper semicontinuous regularization.*

*Proof.* The idea comes from [Trudinger 1990]. We use Jensen's approximation [1988] for $u, v$, which is defined by

$$u^\varepsilon(z) = \sup_{z' \in \Omega}\left\{u(z') - \frac{C_0}{\varepsilon}|z' - z|^2\right\}, \quad v_\varepsilon(z) = \inf_{z' \in \Omega}\left\{v(z') + \frac{C_0}{\varepsilon}|z' - z|^2\right\}, \tag{16}$$

where $\varepsilon > 0$ and $C_0 = \max\{\operatorname{osc}_\Omega u, \operatorname{osc}_\Omega v\}$ with $\operatorname{osc}(u) = \sup u_\Omega - \inf_\Omega u$. Then the supremum and infimum in (16) are achieved at points $z^*, z_* \in \Omega$ with $|z - z^*|, |z - z_*| < \varepsilon$ provided that $z \in \Omega_\varepsilon = \{z \in \Omega \mid \operatorname{dist}(z, \partial\Omega) > \varepsilon\}$. It follows from [Caffarelli and Cabré 1995] (see also [Wang 2012] for an adaptation in the complex case) that $u^\varepsilon$ (respectively, $v_\varepsilon$) is Lipschitz and semiconvex (respectively, semiconcave) in $\Omega_\varepsilon$, with

$$|Du^\varepsilon|, |Dv_\varepsilon| \leq \frac{2C_0}{\varepsilon}, \quad Hu^\varepsilon, -Hv_\varepsilon \geq -\frac{2C_0}{\varepsilon^2}\,\mathrm{Id}, \tag{17}$$

whenever these derivatives are well-defined.

Exploiting the definition of viscosity subsolution one can show that $u^\varepsilon$ satisfies

$$F\big(z^*, u^\varepsilon(z), Du^\varepsilon(z), Hu^\varepsilon(z)\big) \geq \delta \tag{18}$$

in the viscosity sense for all $z \in \Omega_\varepsilon$. Indeed, let $q$ be an upper test of $u^\varepsilon$ at $z_0$. Then the function

$$\tilde{q}(z) := q(z + z_0 - z_0^*) + \frac{1}{\varepsilon}|z_0 - z_0^*|^2$$

is an upper test for $u$ at $z_0^*$. Therefore we get (18) as $u$ is a viscosity subsolution. This also implies that

$$F\big(z^*, u^\varepsilon(z), Du^\varepsilon(z), N + Hu^\varepsilon(z)\big) \geq \delta \tag{19}$$

in the viscosity sense for any fixed matrix $N \geq 0$. Since any locally semiconvex (semiconcave) function is twice differentiable almost everywhere by Aleksandrov's theorem, we infer that for almost all $z \in \Omega_\varepsilon$, $F$ is degenerate elliptic at $(z^*, u^\varepsilon(z), Du^\varepsilon(z), Hu^\varepsilon(z))$ and

$$F\big(z^*, u^\varepsilon(z), Du^\varepsilon(z), N + Hu^\varepsilon(z)\big) \geq \delta \tag{20}$$

for all $N \in \mathcal{H}^n$ such that $N \geq 0$.

We assume by contradiction that $\sup_\Omega(u - v) = u(z_0) - v(z_0) = a > 0$ for some $z_0 \in \Omega$. For any $\varepsilon$ sufficiently small, the function $w_\varepsilon := u^\varepsilon - v_\varepsilon$ has a positive maximum on $\Omega_\varepsilon$ at some point $z_\varepsilon \in \Omega_\varepsilon$ such that $z_\varepsilon \to z_0$ as $\varepsilon \to 0$. So we can choose $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, we know $w_\varepsilon := u^\varepsilon - v_\varepsilon$ has a positive maximum on $\Omega_\varepsilon$ at some point $z_\varepsilon \in \Omega$ with $d(z_\varepsilon, \partial\Omega) > \varepsilon_0$. Applying the ABP maximum principle (Lemma 15), for the function $w_\varepsilon$ on $\Omega_{\varepsilon_0}$ and for any $\lambda > 0$ sufficiently small, there exists a set $E_\lambda \subset \Omega_{\varepsilon_0}$ containing $z_\varepsilon$ with $|E_\lambda| \geq c\lambda^n$, where $c$ is $c(n)\varepsilon^{2n}$, such that $|Dw_\varepsilon| \leq \lambda$ and $Hw_\varepsilon \leq 0$ almost everywhere in $E_\lambda$. Since $w_\varepsilon(z_\varepsilon) > 0$, we can choose $\lambda$ small enough such that $w_\varepsilon \geq 0$ in $E_\lambda$. The condition (13) and the fact that $F$ is degenerate elliptic at $(z^*, u^\varepsilon(z), Du^\varepsilon(z), Hu^\varepsilon(z))$ for almost all $z \in E_\lambda$ imply that

$$F\big(z^*, u^\varepsilon(z), Du^\varepsilon(z), N + Hu^\varepsilon(z)\big) \leq F\big(z^*, v_\varepsilon(z), Du^\varepsilon(z), N + Hv^\varepsilon(z)\big). \tag{21}$$

Using (14) and the fact that $|D(u^\varepsilon - v_\varepsilon)| \leq \lambda$, we get

$$F\big(z^*, v_\varepsilon(z), Du^\varepsilon(z), N + Hv^\varepsilon(z)\big) \leq F\big(z^*, v_\varepsilon(z), Dv^\varepsilon(z), N + Hv^\varepsilon(z)\big) + \alpha_p(\lambda).$$

Combining with (14), (20), (21) and $|z^* - z_*| < \varepsilon$ for almost all $z \in E_\lambda$,

$$F\big(z_*, v_\varepsilon(z), Dv_\varepsilon(z), N + Hv_\varepsilon(z)\big) \geq \delta - \alpha_z(\varepsilon) - \alpha_p(\lambda). \tag{22}$$

By taking $\lambda$, and then $\varepsilon$ sufficiently small and using the fact that $v_\varepsilon$ is twice differentiable almost everywhere on $\Omega$, we can find at a fixed point $z_1 \in E_\lambda$ a lower test $q$ of $v$ at $z_1$ such that

$$F\big(z_0, q(z_0), Dq(z_0), N + Hq(z_0)\big) \geq \tfrac{1}{2}\delta \tag{23}$$

for all $N \geq 0$. This contradicts the definition of viscosity supersolution. Therefore we get (15). □

**Remark.** By assuming more properties of $F$, it is possible to obtain $\delta = 0$ in the previous result. This is the case for the Monge–Ampère equation. Otherwise we need to adjust the function $u$ to achieve a strict inequality in order to use Lemma 21.

**3.2.** ***Comparison principle for Hessian-type equations.*** We now consider the Hessian-type equation of the form

$$F[u] = \psi(z, u), \tag{24}$$

where $\psi \in C^0(\Omega \times \mathbb{R})$ and $F[u] = f(\lambda(Hu))$ such that

$$s \mapsto \psi(\,\cdot\,, s) \text{ is weakly increasing,} \tag{25}$$

$$f \in C^0(\overline{\Gamma}), \qquad f > 0 \quad \text{on } \Gamma, \qquad f = 0 \quad \text{on } \partial\Gamma, \tag{26}$$

$$f(\lambda + \mu) \geq f(\lambda) \quad \text{for all } \lambda \in \Gamma,\ \mu \in \Gamma_n. \tag{27}$$

First, *in order to use Lemma 21, we extend $f$ continuously on $\mathbb{R}^n$ by taking $f(\lambda) = 0$ for all $\lambda \in \mathbb{R}^n \setminus \Gamma$.*

For a $\delta$-independent comparison principle, we need more assumptions on $F$. Similarly to [Trudinger 1990], we can assume that the operator $F[u] = f(\lambda(Hu))$ satisfies

$$\sum_{i=1}^n \frac{\partial f}{\partial \lambda_i}\lambda_i = \sum_{i=1}^n f_i\lambda_i \geq \nu(f) \quad \text{in } \Gamma, \qquad \inf_{z\in\Omega} \psi(z, \cdot\,) > 0 \tag{28}$$

for some positive increasing function $\nu$.

This condition is satisfied for example in the case of the complex Hessian equations $F[u] := \sigma_k(\lambda(Hu))$, $k \in \{1, \dots, n\}$.

We also study a *new condition*, namely

$$f \text{ is concave and homogeneous;} \tag{29}$$

i.e., $f(t\lambda) = tf(\lambda)$ for all $t \in \mathbb{R}^+$.

**Theorem 22.** *Let $u, v \in L^\infty(\overline{\Omega})$ be a viscosity subsolution and a viscosity supersolution of (24) in $\Omega$. Assume that either $f$ satisfies either (28) or (29). Then*

$$\sup_\Omega (u - v) \leq \max_{\partial\Omega}\{(u - v)^*, 0\}. \tag{30}$$

*Proof.* Assume first that $f$ satisfies (28). Then following [Trudinger 1990], we set for any $t \in (1, 2)$,

$$u_t(z) = tu(z) - C(t - 1),$$

where $C = \sup_\Omega u$. Therefore we have $u_t(z) \le u(z)$ on $\Omega$ for all $t \in (1, 2)$. Then for any $z_0 \in \Omega$ and an upper test function $q_t(z)$ of $u_t$ at $z_0$, we have $q(z) := t^{-1} q_t(z) - C(t^{-1} - 1)$ is also an upper test for $u$ at $z_0$. Set $\lambda = \lambda[q](z_0)$; then $\lambda[q_t](z_0) = t\lambda$ and $q(z_0) \ge q_t(z_0)$. We also recall that the function $s \mapsto f(s\lambda)$ is increasing on $\mathbb{R}^+$, by (28), and $f(\lambda) \ge \psi(z, u(z_0))$ since $q$ is an upper test for $u$ at $z_0$. It follows that at $z_0$,

$$\begin{aligned} F[q_t] &= f(\lambda[q_t]) = f(t\lambda) \\ &\ge f(\lambda) + (t-1) \sum \lambda_i f_i(t^*\lambda) \\ &\ge \psi(z_0, q(z_0)) + (t-1) \sum \lambda_i f_i(t^*\lambda) \\ &\ge \psi(z_0, q_t(z_0)) + \tfrac{1}{2}(t-1)\nu(\inf_\Omega \psi(z, \inf_\Omega u)) \end{aligned}$$

for $1 \le t^* < t$, sufficiently close to 1. Therefore we have for some $\delta > 0$

$$F[u_t] \ge \psi(z, u_t) + \delta$$

in the viscosity sense in $\Omega$. Thus the inequality (30) follows from Lemma 21.

Next, consider the second case when $f$ is concave and homogeneous. Suppose, without loss of generality, that $0 \in \Omega$. We set

$$u_\tau(z) = u(z) + \tau(|z|^2 - R),$$

where $R = \operatorname{diam}(\Omega)$. Then for any $q_\tau \in C^2(\Omega)$ such that $q_\tau \ge u_\tau$ near $z_0$ and $q_\tau(z_0) = u_\tau(z_0)$, we have $q = q_\tau - \tau(|z|^2 - R) \ge q_\tau$, and $q$ is also an upper test for $u$ at $z_0$. Therefore, we have at $z_0$,

$$F[q_\tau] = 2^d f\left(\frac{\lambda(Hq) + \tau\mathbf{1}}{2}\right) \ge f(\lambda(Hq)) + f(\tau\mathbf{1}) \ge \psi(z_0, q_\tau) + \delta. \tag{31}$$

Therefore $F[u_\tau] \ge \psi + \delta$ in the viscosity sense. Applying Lemma 21 we get (30). □

By definition, we have the following properties of sub- and supersolutions. Their proofs follow in a straightforward way from [Crandall et al. 1992, Proposition 4.3].

**Lemma 23.** (a) *Let $\{u_j\}$ be viscosity subsolutions of (24) in $\Omega$ which are uniformly bounded from above. Then $(\limsup_\Omega u_j)^*$ is also a viscosity subsolution of (24) in $\Omega$.*

(b) *Let $\{v_j\}$ be viscosity supersolutions of (24) in $\Omega$, which are uniformly bounded from below. Then $(\liminf_\Omega v_j)_*$ is also a viscosity supersolution of (24) in $\Omega$.*

Now using Perron's method, see for instance [Crandall et al. 1992], we obtain the next result:

**Lemma 24.** *Suppose $\underline{u}, \bar{u} \in L^\infty(\Omega)$ are a subsolution and a supersolution of (24) on $\Omega$. Suppose that $\underline{u}_*(z) = \bar{u}^*(z)$ on the boundary of $\Omega$. Then the function*

$$u := \sup\{v \in L^\infty(\Omega) \cap \mathrm{USC}(\Omega) \mid v \text{ is a subsolution of (24)}, \ \underline{u} \le v \le \bar{u}\}$$

*satisfies $u \in C^0(\Omega)$ and*

$$F[u] = \psi(x, u) \quad \text{in } \Omega$$

*is the viscosity sense.*

*Proof.* It is straightforward that $u^*$ is a viscosity subsolution of (24). We next prove that $u_*$ is a supersolution of (24). Assume by contradiction that $u_*$ is not a supersolution of (24). Then there exists a point $z_0 \in \Omega$ and a lower differential test $q$ for $u_*$ at $z_0$ such that

$$F[q](z_0) > \psi(z_0, q(z_0)). \tag{32}$$

Set $\tilde{q}(z) = q(z) + b - a|z - z_0|^2$, where $b = \frac{1}{6}ar^2$ with $a, r > 0$ small enough so that $F[\tilde{q}] \geq \psi(x, \tilde{q})$ for all $|z - z_0| \leq r$. Since $u_* \geq q$ for $|z - z_0| \leq r$, we get $u^* \geq u_* > \tilde{q}$ for $\frac{1}{2}r \leq |z - z_0| < r$. Then the function

$$w(z) = \begin{cases} \max\{u^*(z), \tilde{q}(z)\} & \text{if } |z - z_0| \leq r, \\ u^*(z) & \text{otherwise} \end{cases}$$

is a viscosity subsolution of (24). By choosing a sequence $z_n \to z_0$ so that $u(z_n) \to u_*(z_0)$, we have $\tilde{q}(z_n) \to u_*(z_0) + b$. Therefore, for $n$ sufficiently large, we have $w(z_n) > u(z_n)$ and this contradicts the definition of $u$. Thus we have $u_*$ is also a supersolution. Then it follows from Theorem 22 and $\underline{u}_*(z) = \bar{u}^*(z)$ for $z \in \partial\Omega$ that $u^* \leq u_*$ on $\Omega$; hence $u = u_* = u^*$. □

## 4. Dirichlet problems

**4.1.** ***Viscosity solutions in Γ-pseudoconvex domains.*** Let $\Omega \subset \mathbb{C}^n$ be a $C^2$ bounded domain. In this section, we study the Dirichlet problem

$$\begin{cases} F[u] = f(\lambda(Hu)) = \psi(x, u) & \text{on } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{33}$$

where $\varphi \in C^0(\partial\Omega)$ and $\psi \in C^0(\bar{\Omega} \times \mathbb{R})$ such that $\psi > 0$ and

$$s \mapsto \psi(\cdot, s) \text{ is weakly increasing.}$$

Let $\Gamma \subsetneq \mathbb{R}^n$ be an admissible cone. We assume further that $f \in C^0(\overline{\Gamma})$ satisfies:

(1) $f$ is concave and $f(\lambda + \mu) \geq f(\lambda)$ for all $\lambda \in \Gamma$, $\mu \in \Gamma_n$.

(2) $\sup_{\partial\Gamma} f = 0$, and $f > 0$ in $\Gamma$.

(3) $f$ is homogeneous on $\Gamma$.

We remark that the conditions (2) and (3) imply that for any $\lambda \in \Gamma$ we have

$$\lim_{t \to \infty} f(t\lambda) = +\infty. \tag{34}$$

We now can solve (33) in the viscosity sense:

**Theorem 25.** *Let $\Omega$ be a $C^2$ bounded $\Gamma$-pseudoconvex domain in $\mathbb{C}^n$. Then the Dirichlet problem*

$$f(\lambda[u]) = \psi(x, u) \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega$$

*admits a unique admissible solution $u \in C^0(\bar{\Omega})$.*

*In particular, we have an $L^\infty$ bound for $u$ which only depends on $\|\varphi\|_{L^\infty}$ and $\|\psi(x, C)\|_{L^\infty}$ and $\Omega$, where $C$ is a constant depending on $\Omega$.*

*Proof.* By Lemma 20, there is a defining function $\rho \in C^2(\overline{\Omega})$ for $\Omega$ such that $\lambda(H\rho) \in \Gamma$ on $\overline{\Omega}$. The $C^2$-smoothness of the boundary implies the existence of a harmonic function $h$ on $\Omega$ for arbitrary given continuous boundary data $\varphi$. Set

$$\underline{u} = (A_1\rho + h) + A_2\rho,$$

where $A_1 > 0$ is chosen so that $A_1\rho + h$ is admissible and $A_2$ will be chosen later.

By the concavity of $f$ and (34), for $A_2$ sufficiently large we get

$$\begin{aligned} f(\lambda[\underline{u}]) &\geq \tfrac{1}{2} f(2\lambda[A_1\rho + h]) + \tfrac{1}{2} f(2A_2\lambda[\rho]) \\ &\geq \max_{\overline{\Omega}} \psi(x, h) \geq \psi(x, \underline{u}). \end{aligned}$$

Therefore $\underline{u}$ is a subsolution of (33).

Since $h$ is harmonic, for each $z \in \Omega$ there is a Hermitian matrix $N \geq 0$ such that $\lambda(N + H(h)(z)) \in \partial\Gamma$. But then $f(\lambda(N + H(h)(z))) = 0$. Therefore, $\bar{v} := h$ is a supersolution of (33).

Finally, the existence of solution follows from Perron's method. We set

$$u := \sup\{w \text{ is subsolution of (33) on } \Omega,\ \underline{u} \leq w \leq \bar{v}\}.$$

As in the argument from Lemma 24 we have $u^*$ (respectively, $u_*$) is a subsolution (respectively, supersolution) of (33). It follows from the comparison principle (Theorem 22) that

$$u^*(z) - u_*(z) \leq \limsup_{w \to \partial\Omega} (u^* - u_*)^+(w).$$

Since $\underline{u}$ and $\bar{v}$ are continuous and $\underline{u} = \bar{v} = \varphi$ on $\partial\Omega$ we infer that $u^* \leq u_*$ on $\Omega$ and $u^* = u_*$ on $\partial\Omega$. Therefore $u = u^* = u_*$ is a viscosity solution of (33). The uniqueness follows from the comparison principle (Theorem 22). □

As a corollary of Theorem 25, we solve the following Dirichlet problem for Hessian quotient equations:

$$\begin{cases} S_{k,\ell}(\lambda(Hu)) := (S_k/S_\ell)(\lambda(Hu)) = \psi(x, u) & \text{on } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{35}$$

where $\Omega \subset \mathbb{C}^n$ is a smooth bounded $\Gamma_k$-pseudoconvex domain, $1 \leq \ell < k \leq n$, and

$$S_k(\lambda(Hu)) = \frac{(dd^c u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Note that the operator $S_{k,\ell}^{1/(k-l)}$ is concave and homogeneous; see [Spruck 2005].

**Corollary 26.** *The Dirichlet problem* (35) *admits a unique viscosity solution* $u \in C^0(\overline{\Omega})$ *for any continuous data* $\varphi$.

We also remark that a viscosity subsolution is always a $\Gamma$-subharmonic function.

**Lemma 27.** *Any viscosity subsolution of the equation* $f(\lambda(Hu)) = \psi(z, u)$ *is a* $\Gamma$*-subharmonic function. In particular, if* $u$ *is a viscosity subsolution of the equation*

$$S_{k,\ell}(\lambda(Hu)) = \psi(z, u), \tag{36}$$

*then $u$ is $k$-subharmonic.*

*Proof.* Let $z_0 \in \Omega$ and $q \in C^2_{\mathrm{loc}}(\{z_0\})$ such that $u - q$ attains its maximum at $z_0$ and $u(z_0) = q(z_0)$. By definition we have

$$f(\lambda(Hq)(z_0)) > 0.$$

Observe that for any semipositive Hermitian matrix $N$, the function

$$\tilde{q}(z) := q(z) + \langle N(z - z_0), z - z_0\rangle$$

is also an upper test function for $u$ at $z_0$. By the definition of viscosity subsolutions we have

$$f(\lambda(H\tilde{q})(z_0)) > 0. \tag{37}$$

Suppose that $\lambda(Hq)(z_0) \notin \overline{\Gamma}$. Then we can find $N \geq 0$ such that $\lambda(H\tilde{q})(z_0) \in \partial\Gamma$, so $f(\lambda(H\tilde{q})(z_0)) = 0$ by the condition (3) above, and this contradicts (37). Hence we always have $\lambda[q](z_0) \geq 0$, and so $u$ is $\Gamma$-subharmonic. □

**4.2. *Hölder continuity of Hessian-type equations.*** In this subsection, we study the Hölder continuity of the viscosity solution obtained in Section 4.1 to the Dirichlet problem

$$\begin{cases} F[u] = f(\lambda(Hu)) = \psi(x, u) & \text{on } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{38}$$

where $f$, $\varphi$ and $\psi$ satisfy the conditions spelled out in the previous subsection. We prove the following result:

**Theorem 28.** *Let $\Omega$ be a strictly $\Gamma$-pseudoconvex domain. Let $u$ be the viscosity solution of (38). Suppose that $\varphi \in C^{2\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$. If additionally $\psi(z, s)$ satisfies*

(1) $|\psi(z, s)| \leq M_1(s)$ *for some $L^\infty_{\mathrm{loc}}$ function $M_1$,*

(2) $|\psi(z, s) - \psi(w, s)| \leq M_2(s)|z - w|^\alpha$ *for some $L^\infty_{\mathrm{loc}}$ function $M_2$,*

*then $u \in C^\alpha(\overline{\Omega})$.*

**Remark.** Classical examples, see [Bedford and Taylor 1976], show that the claimed regularity cannot be improved. Conditions (1) and (2) can be regarded as weak growth conditions and seem to be optimal. If $\psi$ does not depend on the second variable then these conditions mean that $\psi$ is globally bounded and contained in $C^\alpha$.

*Proof.* The proof relies on the classical idea of Walsh [1968]. A similar argument was used by Bedford and Taylor [1976], who dealt with the complex Monge–Ampère operator. We shall apply a small adjustment in the construction of the local barriers which is due to Charabati [2016].

Suppose for definiteness that $0 \in \Omega$. Assume without loss of generality that the $\Gamma$-subharmonic function $\rho = -\operatorname{dist}(z, \partial\Omega) + C_\Omega \operatorname{dist}(z, \partial\Omega)^2$ satisfies $F(\rho) \geq 2$ (multiply $\rho$ by a constant if necessary and exploit the homogeneity of $F$). Recall that $\rho$ vanishes on $\partial\Omega$. As $\partial\Omega \in C^2$ we know that $\rho \in C^2$ near the boundary. Then it is easy to find a continuation of $\rho$ in the interior of $\Omega$ (still denoted by $\rho$), so that $\rho$ is $\Gamma$-subharmonic and satisfies $F(\rho) \geq 1$.

Fix $\xi \in \partial\Omega$. There is a uniform $C \gg 1$ (dependent on $\Omega$, but independent of $\xi$) such that the function

$$g_\xi(z) := C\rho(z) - |z-\xi|^2$$

is $\Gamma$-sh. In particular $g_\xi \le 0$ in $\bar{\Omega}$.

By definition there is a constant $\widetilde{C}$ such that for any $z \in \partial\Omega$

$$\varphi(z) \ge \varphi(\xi) - \widetilde{C}|z-\xi|^{2\alpha}.$$

Consider the function $h_\xi(z) := -\widetilde{C}(-g_\xi(z))^\alpha$. Then

$$H(h_\xi(z)) \ge \widetilde{C}\alpha(1-\alpha)(-g_\xi(z))^{\alpha-2}H(g_\xi(z)), \tag{39}$$

where $\lambda(H(g_\xi(z))) \in \Gamma$; thus $h_\xi$ is $\Gamma$-subharmonic.

Observe that

$$h_\xi(z) \le -\widetilde{C}|z-\xi|^{2\alpha} \le \varphi(z) - \varphi(\xi).$$

Thus $h_\xi(z) + \varphi(\xi)$ are local boundary barriers constructed following the method of [Charabati 2016] (in [Bedford and Taylor 1976], where the Monge–Ampère case was considered, $h_\xi$ was simply chosen as $-(x_n)^\alpha$ in a suitable coordinate system, but this is not possible in the general case).

At this stage we recall that $u$ is bounded a priori by Theorem 25. Hence we know that for some uniform constant $A$ one has $F[u] \le A$ in the viscosity sense.

From the gathered information, one can produce a global barrier for $u$ in a standard way; see [Bedford and Taylor 1976]. Indeed, consider the function $\tilde{h}(z) := \sup_\xi\{ah_\xi(z) + \varphi(\xi)\}$ for a large but uniform constant $a$. Using the balayage procedure it is easy to show that $F(\tilde{h}(z)) \ge A$ in the viscosity sense once $a$ is taken large enough. Thus $\tilde{h}$ majorizes $u$ by the comparison principle and so is a global barrier for $u$ matching the boundary data given by $\varphi$. By construction $\tilde{h}$ is globally $\alpha$-Hölder continuous.

Note on the other hand that $u$ is subharmonic as $\Gamma \subset \Gamma_1$; thus the harmonic extension $u_\varphi$ of $\varphi$ in $\Omega$ majorizes $u$ from above. Recall that $u_\varphi$ is $\alpha$-Hölder continuous by classical elliptic regularity.

Coupling the information for both the lower and the upper barrier one obtains

$$\text{for all } z \in \bar{\Omega}, \text{ for all } \xi \in \partial\Omega, \quad |u(z) - u(\xi)| \le K|z-\xi|^\alpha. \tag{40}$$

Denote by $K_1$ the quantity $K_2 \operatorname{diam}^2(\Omega)\max\{1, f(\mathbf{1})\} + K$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ is the vector of the eigenvalues of the identity matrix, while $K_2 := \widetilde{C} f(\mathbf{1})^{-1}$ and finally $\widetilde{C}$ is the $\alpha$-Lipschitz constant of $\psi$. Consider for a small vector $\tau \in \mathbb{C}^n$ the function

$$v(z) := u(z+\tau) + K_2|\tau|^\alpha|z|^2 - K_1|\tau|^\alpha$$

defined over $\Omega_\tau := \{z \in \Omega \mid z+\tau \in \Omega\}$.

It is easy to see by using the barriers that if $z+\tau \in \partial\Omega$ or $z \in \partial\Omega$ then

$$v(z) \le u(z) + K|\tau|^\alpha + K_2 \operatorname{diam}^2 \Omega|\tau|^\alpha - K_1|\tau|^\alpha \le u(z).$$

We now claim that $v(z) \le u(z)$ in $\Omega_\tau$. By the previous inequality this holds on $\partial(\Omega_\tau)$. Suppose the claim is false and consider the open subdomain $U$ of $\Omega_\tau$ defined by $U_\tau = \{z \in \Omega_\tau \mid v(z) > u(z)\}$.

We will now prove that $v$ is a subsolution to $F[u] = f(\lambda(Hu)) = \psi(z, u(z))$ in $U$. To this end pick a point $z_0$ and an upper differential test $q$ for $v$ at $z_0$. Observe then that $\tilde{q}(z) := q(z) - K_2|\tau|^\alpha |z|^2 - K_1|\tau|^\alpha$ is then an upper differential test for $u(\tau + \cdot)$ at the point $z_0$. Hence

$$\begin{aligned} F[q(z_0)] &= f\big(\lambda(H\tilde{q}(z_0)) + K_2|\tau|^\alpha \mathbf{1}\big) \\ &\geq f(\lambda(H\tilde{q}(z_0))) + K_2|\tau|^\alpha f(\mathbf{1}) \\ &\geq \psi(z_0 + \tau, u(z_0 + \tau)) + K_2|\tau|^\alpha f(\mathbf{1}), \end{aligned}$$

where we have used the concavity and homogeneity of $f$ in the first inequality and the fact that $\tilde{q}$ is an upper differential test for $u(\tau + \cdot)$ for the second one.

Next

$$\begin{aligned} \psi(z_0 + \tau, u(z_0 + \tau)) + K_2|\tau|^\alpha f(\mathbf{1}) &\geq \psi\big(z_0 + \tau, u(z_0 + \tau) + K_2|\tau|^\alpha |z_0|^2 - K_1|\tau|^\alpha\big) + K_2|\tau|^\alpha f(\mathbf{1}) \\ &= \psi(z_0 + \tau, v(z_0)) + K_2|\tau|^\alpha f(\mathbf{1}) \\ &\geq \psi(z_0 + \tau, u(z_0)) + K_2|\tau|^\alpha f(\mathbf{1}), \end{aligned}$$

where we have exploited twice the monotonicity of $\psi$ with respect to the second variable (and the fact that $z_0 \in U_\tau$).

Exploiting now the Hölder continuity of $\psi$ with respect to the first variable we obtain

$$\psi(z_0 + \tau, u(z_0 + \tau)) + K_2|\tau|^\alpha f(\mathbf{1}) \geq \psi(z_0 + \tau, u(z_0)) + K_2|\tau|^\alpha f(\mathbf{1}) \geq \psi(z_0, u(z_0)).$$

This proves that $F[q(z_0)] \geq \psi(z_0, u(z_0))$ and hence $F[v(z)] \geq \psi(z, v(z))$ in the viscosity sense.

Thus over $U_\tau$, we know $v$ is subsolution and $u$ is a solution, which implies by comparison principle that $v \leq u$ there, a contradiction unless the set $U_\tau$ is empty.

We have thus proven that

$$\text{for all } z \in \Omega_\tau, \quad u(z + \tau) + K_2|\tau|^\alpha |z|^2 - K_1|\tau|^\alpha \leq u(z),$$

which implies the claimed $\alpha$-Hölder continuity. □

## 5. Viscosity vs. pluripotential solutions

Let $\Omega$ be a bounded smooth strictly pseudoconvex domain in $\mathbb{C}^n$. Let $0 < \psi \in C(\bar{\Omega} \times \mathbb{R})$ be a continuous function nondecreasing in the last variable. In this section, we study the relations between viscosity concepts with respect to the inverse $\sigma_k$-equations

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} = \psi(z, u) \quad \text{in } \Omega \tag{41}$$

and pluripotential concepts with respect to the equation

$$(dd^c u)^n = \psi(z, u)(dd^c u)^{n-k} \wedge \omega^k \quad \text{in } \Omega. \tag{42}$$

For the regular case, the following result was shown in [Guan and Sun 2015]:

**Theorem 29** (Guan–Sun). *Let $0 < h \in C^\infty(\bar\Omega)$ and $\varphi \in C^\infty(\partial\Omega)$. Then, there exists a smooth strictly plurisubharmonic function $u$ in $\bar\Omega$ such that*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k}\wedge\omega^k} = h(z) \quad in\ \Omega, \qquad u=\varphi \quad in\ \partial\Omega. \tag{43}$$

Note that the function $u$ in Theorem 29 is a viscosity solution of (41) in the case when $\psi(z,u)=h(z)$. Using Theorem 29, we obtain:

**Proposition 30.** *If $u \in C(\bar\Omega)\cap \mathrm{PSH}(\Omega)$ is a viscosity solution of (41) then there exists a sequence of smooth plurisubharmonic functions $u_j$ in $\Omega$ such that $u_j$ is decreasing to $u$ and the function $(dd^c u_j)^n/((dd^c u_j)^{n-k}\wedge\omega^k)$ converges uniformly to $\psi(z,u)$ as $j\to\infty$. In particular, $u$ is a solution of (42) in the pluripotential sense.*

*Proof.* Let $\varphi_j \in C^\infty(\partial\Omega)$ and $0<\psi_j\in C^\infty(\bar\Omega)$ be sequences of smooth functions such that $\varphi_j \searrow \varphi$ and $\psi_j \nearrow \psi(z,u)$ as $j\to\infty$. Then, by Theorem 29, for any $j=1,2,\dots,$ there exists a smooth strictly plurisubharmonic function $u_j$ in $\bar\Omega$ such that

$$\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k}\wedge\omega^k} = \psi_j(z) \quad \text{in } \Omega, \qquad u_j=\varphi_j \quad \text{in } \partial\Omega. \tag{44}$$

By the comparison principle, we have

$$u_1 \geq u_2 \geq \cdots \geq u_j \geq \cdots \geq u.$$

Let $C > \sup_\Omega |z|^2$. By the homogeneity and the concavity of $S_{n,n-k}^{1/k}$, we have

$$\frac{(dd^c(u_j+\varepsilon|z|^2))^n}{(dd^c(u_j+\varepsilon|z|^2))^{n-k}\wedge\omega^k} \geq \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k}\wedge\omega^k} + \varepsilon^k.$$

Then, by the comparison principle, for any $\varepsilon>0$, there exists $N>0$ such that

$$u_j + \varepsilon(|z|^2 - C) \leq u$$

for any $j>N$. Hence, $u_j$ is decreasing to $u$ as $j\to\infty$. □

Observe that a continuous solution of (42) in the pluripotential sense may not be a viscosity solution of (41). For example, if a continuous plurisubharmonic function $u:\Omega\to\mathbb{R}$ depends only on $n-k-1$ variables then $u$ is a solution of (42) in the pluripotential sense but $u$ is not a viscosity solution of (41). Moreover, by Theorem 34, we know that a viscosity solution of (41) has to satisfy $(dd^c u)^k \geq a\omega^k$ for some $a>0$. The following question is natural:

**Question 31.** *If $u\in \mathrm{PSH}(\Omega)\cap C(\bar\Omega)$ satisfies (42) in the pluripotential sense and*

$$(dd^c u)^k \geq a\omega^k \tag{45}$$

*for some $a>0$, does $u$ satisfy (41) in the viscosity sense?*

At the end of this section, we will give the answer to a special case of this question. Now, we consider the relation between viscosity subsolutions of (41) and *pluripotential subsolutions* of (42). Recall that according to the definition in Section 2.1 for any $n \times n$ complex matrix $A$ and $k \in \{1, \dots, n\}$, $S_k(A)$ denotes the coefficient with respect to $t^{n-k}$ of the polynomial $\binom{n}{k}^{-1} \det(A + t\,\mathrm{Id}_n)$.

Next we prove the following technical result:

**Lemma 32.** *Assume that $A$, $B$ are $n \times n$ complex matrices and $k \in \{1, \dots, n\}$. Then*

$$S_k(AA^*)S_k(BB^*) \geq |S_k(AB^*)|^2.$$

*Proof.* Denote by $a_1, \dots, a_n$ and $b_1, \dots, b_n$, respectively, the row vectors of $A$ and $B$. Then

$$S_k(AA^*) = \left(\binom{n}{k}\right)^{-1} \sum_{\sharp J = k} \det(\langle a_p, a_q \rangle)_{p,q\in J},$$

$$S_k(BB^*) = \left(\binom{n}{k}\right)^{-1} \sum_{\sharp J = k} \det(\langle b_p, b_q \rangle)_{p,q\in J},$$

$$S_k(AB^*) = \left(\binom{n}{k}\right)^{-1} \sum_{\sharp J = k} \det(\langle a_p, b_q \rangle)_{p,q\in J}.$$

We will show that, for any $J = \{p_1, \dots, p_k\}$ with $1 \leq p_1 < \cdots < p_k \leq n$,

$$\det(\langle a_p, a_q \rangle)_{p,q\in J} \cdot \det(\langle b_p, b_q \rangle)_{p,q\in J} \geq |\det(\langle a_p, b_q \rangle)_{p,q\in J}|^2. \tag{46}$$

Indeed, if either $\{a_{p_1}, \dots, a_{p_k}\}$ or $\{b_{p_1}, \dots, b_{p_k}\}$ are linearly dependent then both sides of (46) are equal to 0. Otherwise, exploiting the Gram–Schmidt process, we can assume that $\{a_{p_1}, \dots, a_{p_k}\}$ and $\{b_{p_1}, \dots, b_{p_k}\}$ are orthogonal systems (observe that the quantities in question do not change during the orthogonalization process). Next normalizing the vectors $a_{p_j}$ and $b_{p_j}$, $j = 1, \dots, n$, to unit length, both sides change by the same factor. Hence it suffices to prove the statement for two collections of orthonormal bases.

Under this assumption we have

$$(\langle a_p, a_q \rangle)_{p,q\in J} = (\langle b_p, b_q \rangle)_{p,q\in J} = \mathrm{Id}_k. \tag{47}$$

Let $M = (\langle a_p, b_q \rangle)_{p,q\in J}$. Then $MM^*$ is a semipositive Hermitian matrix, and

$$\mathrm{Tr}(MM^*) = \sum_{l=1}^{k} \sum_{j=1}^{k} |\langle b_{p_j}, a_{p_l} \rangle|^2 = \sum_{j=1}^{k} \left\langle b_{p_j}, \sum_{l=1}^{k} \langle b_{p_j}, a_{p_l} \rangle a_{p_l} \right\rangle \leq \sum_{j=1}^{k} \|b_{p_j}\|^2 = k.$$

Therefore, $|\det(M)| = \sqrt{\det(MM^*)} \leq 1$; hence we obtain (46). Finally, using (46) and the Cauchy–Schwarz inequality, we infer that

$$S_k(AA^*)S_k(BB^*) \geq |S_k(AB^*)|^2,$$

as required. □

For any $n \times n$ Hermitian matrix $A = (a_{j\bar{\ell}})$, we define

$$\omega_A = \sum_{j,\ell=1}^{n} a_{j\bar{\ell}} \frac{i}{\pi} dz_j \wedge d\bar{z}_\ell$$

and

$$\mathcal{B}(A,k) := \left\{ B \in \mathcal{H}_+^n \;\middle|\; \frac{\omega_B^k \wedge \omega_A^{n-k}}{\omega^n} = 1 \right\},$$

where $k = 1, 2 \ldots, n$.

**Theorem 33.** *Let* $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *and* $0 < g \in C(\Omega)$. *Then the following are equivalent*:

(i) $(dd^c u)^n / ((dd^c u)^{n-k} \wedge \omega^k) \geq g^k(z)$ *in the viscosity sense.*

(ii) *For all* $B \in \mathcal{B}(\mathrm{Id}, n-k)$,

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq g^k(z)\omega^n$$

*in the viscosity sense.*

(iii) *For any open set* $U \Subset \Omega$, *there are smooth plurisubharmonic functions* $u_\varepsilon$ *and functions* $0 < g^\varepsilon \in C^\infty(U)$ *such that* $u_\varepsilon$ *are decreasing to* $u$ *and* $g^\varepsilon$ *converge uniformly to* $g$ *as* $\varepsilon \searrow 0$, *and*

$$(dd^c u_\varepsilon) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g^\varepsilon \omega^n \tag{48}$$

*pointwise in* $U$ *for any* $B \in \mathcal{B}(\mathrm{Id}, n-k)$ *and* $A_1, \ldots, A_{k-1} \in \mathcal{B}(B^2, k)$.

(iv) *For any open set* $U \Subset \Omega$, *there are smooth strictly plurisubharmonic functions* $u_\varepsilon$ *and functions* $0 < g^\varepsilon \in C^\infty(U)$ *such that the sequence* $u_\varepsilon$ *is decreasing to* $u$ *and the sequence* $g^\varepsilon$ *converges uniformly to* $g$ *as* $\varepsilon \searrow 0$, *and*

$$\frac{(dd^c u_\varepsilon)^n}{(dd^c u_\varepsilon)^{n-k} \wedge \omega^k} \geq (g^\varepsilon)^k \tag{49}$$

*pointwise in* $U$ *for any* $B \in \mathcal{B}(\mathrm{Id}, n-k)$.

*Proof.* (iv) $\Rightarrow$ (i) is obvious. It remains to show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (ii): Assume that $q \in C^2$ is an upper test for $u$ at $z_0 \in \Omega$. Then $q$ is strictly plurisubharmonic in a neighborhood of $z_0$ and

$$\frac{(dd^c q)^n}{(dd^c q)^{n-k} \wedge \omega^k} \geq g^k$$

at $z_0$.

By using Lemma 32 for $\sqrt{Hq}$ and $(\sqrt{Hq})^{-1}B$, we have

$$\begin{aligned}\frac{(dd^c q)^{n-k} \wedge \omega^k}{(dd^c q)^n} \frac{(dd^c q)^k \wedge \omega_{B^2}^{n-k}}{\omega^n} &= \frac{(dd^c q)^{n-k} \wedge \omega^k}{\omega^n} \frac{(dd^c q)^k \wedge \omega_{B^2}^{n-k}}{(dd^c q)^n} \\ &\geq \left( \frac{\omega_B^{n-k} \wedge \omega^k}{\omega^n} \right)^2\end{aligned}$$

for any $B \in \mathcal{H}_+^n$ (observe that

$$S_{n-k}(CC^*) = \frac{(dd^c q)^k \wedge \omega_{B^2}^{n-k}}{(dd^c q)^n} \quad \text{and} \quad S_{n-k}(\sqrt{Hq}C^*) = \frac{\omega_B^{n-k} \wedge \omega^k}{\omega^n}$$

for $C = (\sqrt{Hq})^{-1}B$).

Then, for any $B \in \mathcal{B}(\mathrm{Id}, n-k)$ we have

$$(dd^c q)^k \wedge \omega_{B^2}^{n-k} \geq g^k \omega^n$$

at $z_0$. Hence

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq g^k \omega^n$$

in the viscosity sense.

(ii) $\Rightarrow$ (iii): Assume that $q \in C^2$ touches $u$ from above at $z_0 \in \Omega$. Then, for any $B \in \mathcal{B}(\mathrm{Id}, n-k)$,

$$(dd^c q)^k \wedge \omega_{B^2}^{n-k} \geq g^k \omega^n$$

at $z_0$. By the same arguments as in [Lu 2013], we have

$$(dd^c q) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g\omega^n$$

for any $B \in \mathcal{B}(\mathrm{Id}, n-k)$, $A_1, \ldots, A_{k-1} \in \mathcal{B}(B^2, k)$. Hence

$$(dd^c u) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g\omega^n \tag{50}$$

in the viscosity sense for any $B \in \mathcal{B}(\mathrm{Id}, n-k)$, $A_1, \ldots, A_{k-1} \in \mathcal{B}(B^2, k)$.

Let $g_j$ be a sequence of smooth functions in $\Omega$ such that $g_j \nearrow g$ as $j \to \infty$. Then

$$(dd^c u) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq g_j\omega^n \tag{51}$$

in the viscosity sense for any $j \in \mathbb{N}$, $B \in \mathcal{B}(\mathrm{Id}, n-k)$ and $A_1, \ldots, A_{k-1} \in \mathcal{B}(B^2, k)$. By the same arguments as in [Eyssidieux et al. 2011, the proof of Proposition 1.5], $u$ satisfies (51) in the sense of positive Radon measures. Using convolution to regularize $u$ and setting $u_\varepsilon = u * \rho_\varepsilon$, we see that $u_\varepsilon$ is smooth strictly plurisubharmonic and

$$(dd^c u_\varepsilon) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq (g_j)_\varepsilon\omega^n$$

pointwise in $\Omega_\varepsilon$. Choosing $g^\varepsilon := (g_{[1/\varepsilon]})_\varepsilon$, we obtain (48).

(iii) $\Rightarrow$ (iv): At $z_0 \in \Omega_\varepsilon$, choosing

$$B = \frac{Hu_\varepsilon(z_0)}{(S_{n-k}(Hu_\varepsilon(z_0)))^{1/(n-k)}}$$

and

$$A_1 = A_2 = \cdots = A_{k-1} = \left(\frac{(dd^c u_\varepsilon(z_0))^k \wedge \omega_{B^2}^{n-k}}{\omega^n}\right)^{-1/k} Hu_\varepsilon(z_0),$$

we get

$$g^\varepsilon \le \left(\frac{(dd^c u_\varepsilon(z_0))^k \wedge \omega_{B^2}^{n-k}}{\omega^n}\right)^{1/k}$$
$$= \left(\frac{(dd^c u_\varepsilon(z_0))^n}{\omega^n}\frac{1}{S_{n-k}(Hu_\varepsilon(z_0))}\right)^{1/k}$$
$$= \left(\frac{(dd^c u_\varepsilon(z_0))^n}{\omega^n}\frac{\omega^n}{(dd^c u_\varepsilon)^{n-k}\wedge\omega^k}\right)^{1/k}$$
$$= \left(\frac{(dd^c u_\varepsilon)^n}{(dd^c u_\varepsilon)^{n-k}\wedge\omega^k}\right)^{1/k}$$

pointwise in $\Omega_\varepsilon$. Then

$$\frac{(dd^c u_\varepsilon)^n}{(dd^c u_\varepsilon)^{n-k}\wedge\omega^k} \ge (g^\varepsilon)^k. \qquad \square$$

As a consequence, our result implies that a viscosity subsolution is a pluripotential subsolution.

**Theorem 34.** *Assume that $\psi(z,s)=\psi(z)$ with $\psi\in C^0(\Omega)$ and $u\in \mathrm{PSH}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ is a viscosity subsolution of (41). Then*

$$(dd^c u)^n \ge \psi (dd^c u)^{n-k}\wedge\omega^k \tag{52}$$

*and*

$$(dd^c u)^k \ge \binom{n}{k}^{-1}\psi\omega^k \tag{53}$$

*in the pluripotential sense. If $u$ is continuous then the conclusion still holds in the case where $\psi$ depends on both variables.*

*Proof.* By Theorem 33, for any open set $U\Subset\Omega$, there are strictly plurisubharmonic functions $u_\varepsilon\in C^\infty(U)$ and functions $0<h^\varepsilon\in C^\infty(U)$ such that $u_\varepsilon$ is decreasing to $u$ and $h^\varepsilon$ converges uniformly to $\psi$ as $\varepsilon\searrow 0$, and

$$\frac{(dd^c u_\varepsilon)^n}{(dd^c u_\varepsilon)^{n-k}\wedge\omega^k} \ge h^\varepsilon \tag{54}$$

pointwise in $U$. Choosing $B=\mathrm{Id}_n$ and letting $\varepsilon\to 0$, we obtain (52).

It also follows from Theorem 33 that we can choose $u_\varepsilon$ and $h^\varepsilon$ so that

$$(dd^c u_\varepsilon)^k\wedge\omega_{B^2}^{n-k} \ge h^\varepsilon\omega^n \tag{55}$$

pointwise in $U$ for any $B\in\mathcal{B}(\mathrm{Id}, n-k)$. Fix $z_0\in U$ and $0<\varepsilon\ll 1$. We can choose complex coordinates so that $Hu_\varepsilon(z_0)=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$, where $0\le\lambda_1\le\cdots\le\lambda_n$. Choosing

$$B=\binom{n}{k}^{1/(n-k)}\mathrm{diag}(0,\dots,\underbrace{0}_{k\text{-th}},1,\dots,1),$$

we get

$$\lambda_1\cdots\lambda_k \ge \binom{n}{k}^{-1}h^\varepsilon.$$

Then

$$(dd^c u_\varepsilon)^k \geq \binom{n}{k}^{-1} h^\varepsilon \omega^k$$

pointwise in $U$. Letting $\varepsilon \to 0$, we obtain (53). □

**Remark.** Note that for strictly positive $\psi$, (53) implies that the *natural* space of functions to consider for the Hessian quotient problem (41) is *not* the space of bounded plurisubharmonic functions but a considerably smaller one.

By assuming some additional conditions, we can also prove that a pluripotential subsolution is a viscosity one.

**Proposition 35.** *Assume that $\psi(z,s) = \psi(z) > 0$ with $\psi \in C^0(\Omega)$ and $u$ is a local bounded plurisubharmonic function in $\Omega$ satisfying*

$$(dd^c u)^k \geq \psi \omega^k$$

*in the pluripotential sense. Then*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \geq \psi$$

*in the viscosity sense.*

*Proof.* By the assumption, for any $A \in \mathcal{H}_+^n$,

$$(dd^c u)^k \wedge \omega_A^{n-k} \geq \psi \omega^k \wedge \omega_A^{n-k} \tag{56}$$

in the pluripotential sense. By [Lu 2013], (56) also holds in the viscosity sense. If $A = B^2$ for some $B \in \mathcal{B}(\mathrm{Id}, n-k)$ then, by using Lemma 32, we have

$$\omega^k \wedge \omega_{B^2}^{n-k} \geq \left( \frac{\omega_B^{n-k} \wedge \omega^k}{\omega^n} \right)^2 \omega^n = \omega^n.$$

Then

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq \psi \omega^n$$

in the viscosity sense, for any $B \in \mathcal{B}(\mathrm{Id}, n-k)$. Applying Theorem 33, we obtain

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \geq \psi$$

in the viscosity sense. □

We now discuss the notion of a *supersolution*. By the same argument as in [Guedj et al. 2017], relying on the idea from [Berman 2013], we obtain the following relation between viscosity supersolutions of (41) and pluripotential supersolutions of (42):

**Proposition 36.** *Let $u \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ be a viscosity supersolution of (41). Then there exists an increasing sequence of strictly plurisubharmonic functions $u_j \in C^\infty(\overline{\Omega})$ such that $u_j$ converges in capacity to $u$ as $j \to \infty$, and*

$$\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} \leq \psi(z, u)$$

*pointwise in* $\Omega$. *In particular*,

$$(dd^c u)^n \le \psi(z,u)(dd^c u)^{n-k} \wedge \omega^k$$

*in the pluripotential sense.*

*If there exists* $a > 0$ *such that* $(dd^c u)^k \ge a\omega^k$ *then* $u_j$ *can be chosen such that*

$$\frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} \ge b$$

*pointwise in* $\Omega$ *for some* $b > 0$.

For the definition of convergence in capacity, we refer to [Guedj and Zeriahi 2017].

*Proof.* Define $\varphi = u|_{\partial\Omega}$ and $g(z) = \psi(z, u(z))$. Then, for any $j \ge 1$, there exists a unique viscosity solution $v_j$ of

$$\begin{cases} (dd^c v_j)^n/((dd^c v_j)^{n-k} \wedge \omega^k) = e^{j(v_j - u)} g(z) & \text{in } \Omega, \\ v_j = \varphi & \text{in } \partial\Omega. \end{cases} \tag{57}$$

Applying the comparison principle to the equation

$$\frac{(dd^c v)^n}{(dd^c v)^{n-k} \wedge \omega^k} = e^{j(v-u)} g(z),$$

we get $u \ge v_j$ and $v_{j+1} \ge v_j$ for any $j \ge 1$.

Note that, by Proposition 30,

$$(dd^c v_j)^n = e^{j(v_j - u)} g(z)(dd^c v_j)^{n-k} \wedge \omega^k$$

in the pluripotential sense. For any $h \in \mathrm{PSH}(\Omega)$ such that $-1 \le h \le 0$, we have

$$\begin{aligned} \varepsilon^n \int_{\{v_j < u - 2\varepsilon\}} (dd^c h)^n &\le \int_{\{v_j < u + \varepsilon h - \varepsilon\}} (dd^c(u + \varepsilon h))^n \\ &\le \int_{\{v_j < u + \varepsilon h - \varepsilon\}} (dd^c v_j)^n \\ &\le \int_{\{v_j < u - \varepsilon\}} e^{j(v_j - u)} g(z)(dd^c v_j)^{n-k} \wedge \omega^k \\ &\le e^{-j\varepsilon} \int_{\{v_1 < u - \varepsilon\}} g(z)(dd^c v_j)^{n-k} \wedge \omega^k \\ &\le C e^{-j\varepsilon}, \end{aligned}$$

where $C > 0$ is independent of $j$. The last inequality holds by the Chern–Levine–Nirenberg inequalities; see [Guedj and Zeriahi 2017]. This implies that $v_j$ converges to $u$ in capacity.

If there exists $a > 0$ such that $(dd^c u)^k \ge a\omega^k$ then, by Proposition 35,

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k} \wedge \omega^k} \ge a$$

in the viscosity sense. Choosing $M \gg 1$ such that $e^{-M}\sup_\Omega g < a$, we get

$$\frac{(dd^c v_j)^n}{(dd^c v_j)^{n-k}\wedge\omega^k} \le a e^{j(v_j-u)+M}.$$

Applying the comparison principle to the equation

$$\frac{(dd^c v)^n}{(dd^c v)^{n-k}\wedge\omega^k} = a e^{j(v-u)},$$

we get $v_j + M/j \ge u$ for any $j \ge 1$. Then

$$\frac{(dd^c v_j)^n}{(dd^c v_j)^{n-k}\wedge\omega^k} = e^{j(v_j-u)}g(z) \ge e^{-M}g(z)$$

for any $j \ge 1$. Hence, by Theorem 34,

$$(dd^c v_j)^k \ge \binom{n}{k}^{-1} e^{-M} g(z) \ge \binom{n}{k}^{-1} e^{-M}\min_{\bar\Omega} g$$

for any $j \ge 1$.

Now, by Proposition 30, for any $j$ we can choose a strictly plurisubharmonic function $u_j \in C^\infty(\bar\Omega)$ such that

$$v_j - \frac{1}{2^j} \le u_j \le v_j - \frac{1}{2^{j+1}}$$

and

$$-\frac{1}{2^j} \le \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k}\wedge\omega^k} - e^{j(v_j-u)}g(z) \le 0.$$

It is easy to see that $u_j$ satisfies the required properties. □

The next result gives the answer to a special case of Question 31:

**Theorem 37.** *Let $u \in \mathrm{PSH}(\Omega)\cap C(\Omega)$ such that*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k}\wedge\omega^k} \le \psi(z,u) \tag{58}$$

*in the viscosity sense and*

$$(dd^c u)^n \ge \psi(z,u)(dd^c u)^{n-k}\wedge\omega^k \tag{59}$$

*in the pluripotential sense. If there exists $a > 0$ such that $(dd^c u)^k \ge a\omega^k$ then $u$ is a viscosity solution of the equation*

$$\frac{(dd^c u)^n}{(dd^c u)^{n-k}\wedge\omega^k} = \psi(z,u). \tag{60}$$

*Proof.* It remains to show that $u$ is a viscosity subsolution of (60) in any smooth strictly pseudoconvex domain $U \Subset \Omega$.

Let $V$ be a smooth strictly pseudoconvex domain such that $U \Subset V \Subset \Omega$. By Proposition 36, there exists an increasing sequence of strictly plurisubharmonic functions $u_j \in C^\infty(\bar{V})$ such that $u_j$ converges in capacity to $u$ as $j \to \infty$, and

$$b \leq \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k} \leq \psi(z, u)$$

pointwise in $V$, where $b > 0$. By Theorem 34, we have $(dd^c u_j)^k \geq \binom{n}{k}^{-1} b\omega^k$. Then, there exists $C > 0$ such that

$$(dd^c u_j)^{n-k} \wedge \omega^k \geq \frac{1}{\psi(z, u)} (dd^c u_j)^n \geq C\omega^n.$$

Define

$$f_j(z) := \frac{(dd^c u_j)^n}{(dd^c u_j)^{n-k} \wedge \omega^k}.$$

Then $f_j(z) \leq \psi(z, u)$ for any $z \in V$, and $(\psi - f_j)(dd^c u_j)^{n-k} \wedge \omega^k \geq C(\psi - f_j)\omega^n$ converges weakly to 0. Hence $f_j$ converges in Lebesgue measure to $\psi$ in $V$ as $j \to \infty$.

Now, by Theorem 33, we have

$$(dd^c u_j) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq (f_j)^{1/k} \omega^n$$

pointwise in $V$ for any $B \in \mathcal{B}(\mathrm{Id}, n-k)$ and $A_1, \ldots, A_{k-1} \in \mathcal{B}(B^2, k)$. Letting $j \to \infty$, we get

$$(dd^c u) \wedge \omega_{A_1} \wedge \cdots \wedge \omega_{A_{k-1}} \wedge \omega_{B^2}^{n-k} \geq \psi^{1/k} \omega^n$$

in the sense of Radon measures. It follows from [Lu 2013] that

$$(dd^c u)^k \wedge \omega_{B^2}^{n-k} \geq \psi^{1/k} \omega^n$$

in the viscosity sense. Using Theorem 33, we get that $u$ is a viscosity subsolution of (60) in $U$. □

## 6. Dirichlet problem for the Lagrangian phase operator

In this section, we prove the existence of a unique viscosity solution to the Dirichlet problem for the Lagrangian phase operator. The existence and uniqueness of the smooth version was obtained recently by Collins, Picard and Wu [Collins et al. 2017]. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Consider the Dirichlet problem

$$\begin{cases} F[u] := \sum_{i=1}^n \arctan \lambda_i = h(z) & \text{on } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{LA}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the complex Hessian $Hu$. We can also write $F[u] = f(\lambda(Hu))$. We assume that $\varphi \in C^0(\partial\Omega)$ and $h : \bar{\Omega} \to \left[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\right)$ is continuous for some $\delta > 0$.

The Lagrangian phase operator $F$ in (LA) arises in geometry and mathematical physics. We refer to [Collins et al. 2015; 2017; Harvey and Lawson 1982; Jacob and Yau 2017; Yuan 2006; Wang and Yuan 2013; 2014] for the details.

Since $h \geq (n-2)\frac{\pi}{2}$, this case is called the *supercritical phase* following [Yuan 2006; Jacob and Yau 2017; Collins et al. 2015; 2017]. Recall first the following properties; see [Yuan 2006; Wang and Yuan 2014; Collins et al. 2017].

**Lemma 38.** *Suppose* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *satisfy* $\sum_i \arctan \lambda_i \geq (n-2)\frac{\pi}{2} + \delta$ *for some* $\delta > 0$. *Then we have*:

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0$ *and* $|\lambda_n| \leq \lambda_{n-1}$.

(2) $\sum_i \lambda_i \geq 0$ *and* $\lambda_n \geq -C(\delta)$.

(3) $\sum \lambda_i^{-1} \leq -\tan(\delta)$ *when* $\lambda_n < 0$.

(4) *For any* $\sigma \in \left((n-2)\frac{\pi}{2}, n\frac{\pi}{2}\right)$, *the set* $\Gamma^\sigma := \left\{\lambda \in \mathbb{R}^n \,\middle|\, \sum_i \arctan \lambda_i > \sigma\right\}$ *is a convex set and* $\partial\Gamma^\sigma$ *is a smooth convex hypersurface.*

It follows from Lemma 38 that the function $f$ can be defined on a cone $\Gamma$ satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. We also remark that if $h \geq (n-1)\frac{\pi}{2}$, then $F$ is concave, while $F$ has concave level sets if $(n-2)\frac{\pi}{2} h \leq (n-1)\frac{\pi}{2}$, but in general $F$ may not be concave; see [Collins et al. 2017]. Therefore we cannot apply Theorem 22 directly. Fortunately, we still have a comparison principle for the Lagrangian operator using Lemma 21.

**Lemma 39.** *Let* $u, v \in L^\infty(\bar{\Omega})$ *be a viscosity subsolution and a viscosity supersolution of the equation* $F[u] = f(\lambda(Hu)) = h$ *on* $\Omega$. *Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}\{(u - v)^*, 0\}. \tag{61}$$

*Proof.* We first define $\varepsilon > 0$ by $\max_{\bar{\Omega}} h = n\frac{\pi}{2} - \varepsilon$. Now for any $0 < \tau \leq \frac{1}{2}\varepsilon$, set $u_\tau = u + \tau|z|^2$. Let $q_\tau$ be any upper test for $u_\tau$ at any point $z_0 \in \Omega$; then $q = q_\tau - \tau|z|^2$ is also an upper test for $u$ at $z_0$. By the definition we have

$$F[q](z_0) = \sum_{i=1}^{n} \arctan \lambda_i(z_0) \geq h(z_0),$$

where $\lambda(z_0) = \lambda(Hq(z_0))$. We also have

$$F[q_\tau](z_0) = \sum_{i=1}^{n} \arctan(\lambda_i(z_0) + \tau). \tag{62}$$

Next, if $F[q](z_0) \geq n\frac{\pi}{2} - \frac{\varepsilon}{2}$, then $F[q](z_0) \geq h(z_0) + \frac{\varepsilon}{2}$; hence

$$F[q_\tau](z_0) \geq h(z_0) + \frac{\varepsilon}{2}. \tag{63}$$

Conversely, if $F[q](z_0) < n\frac{\pi}{2} - \frac{\varepsilon}{2}$, this implies that $\arctan(\lambda_n(z_0)) \leq \frac{\pi}{2} - \frac{\varepsilon}{2n}$. Combining with Lemma 38(2), we get $-C(\delta) \leq \lambda_n(z_0) \leq C(\varepsilon)$. Using the mean value theorem, there exists $\hat{\lambda}_n \in (\lambda_n(z_0), \lambda_n(z_0) + \tau)$ such that

$$\arctan(\lambda_n(z_0) + \tau) - \arctan \lambda_n(z_0) = \frac{1}{1 + \hat{\lambda}_n^2}\tau \geq C(\delta, \varepsilon, \tau) > 0.$$

It follows that

$$F[q_\tau](z_0) \geq F[q](z_0) + C(\delta, \varepsilon, \tau) \geq h(z_0) + C(\delta, \varepsilon, \tau). \tag{64}$$

Combing with (63) yields

$$F[q_\tau](z_0) \geq h(z_0) + C,$$

where $C > 0$ depends only on $\delta, \varepsilon, \tau$. We thus infer that $u_\tau$ satisfies $F[u_\tau] \geq h(z) + C$ in the viscosity sense. Therefore applying Lemma 21 to $u_\tau$ and $v$, then letting $\tau \to 0$, we obtain the desired inequality. □

**Theorem 40.** *Suppose $\Omega$ is a bounded $C^2$ domain. Let $\underline{u}$ be a bounded upper semicontinuous function on $\Omega$ satisfying $F[\underline{u}] \geq h(z)$ in $\Omega$ in the viscosity sense and $\underline{u} = \varphi$ on $\partial\Omega$. Then the Dirichlet equation (LA) admits a unique viscosity solution $u \in C^0(\Omega)$.*

*Proof.* It suffices to find a viscosity supersolution $\bar{u}$ for the equation $F[u] = h(z)$ satisfying $\bar{u} = \varphi$ on $\partial\Omega$. The $C^2$-boundary implies the existence of a harmonic function $\phi$ on $\Omega$ for arbitrary given continuous boundary data $\varphi$. Since $\sum_i \lambda_i(H\phi) = 0$, it follows from Lemma 38 that we have $F[\phi] < (n-2)\frac{\pi}{2} + \delta \leq h$; hence $\phi$ is a supersolution for (LA). The rest of the proof is similar to the one of Theorem 25, by using Lemma 39. □


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