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INTERPOLATION BY CONFORMAL MINIMAL SURFACES AND DIRECTED HOLOMORPHIC CURVES

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Dedicated to Franc Forstnerič on the occasion of his sixtieth birthday

Let M be an open Riemann surface and $n \geq 3$ be an integer. We prove that on any closed discrete subset of M one can prescribe the values of a conformal minimal immersion $M \to \mathbb{R}^n$. Our result also ensures jet-interpolation of given finite order, and hence, in particular, one may in addition prescribe the values of the generalized Gauss map. Furthermore, the interpolating immersions can be chosen to be complete, proper into $\mathbb{R}^n$ if the prescription of values is proper, and injective if $n \geq 5$ and the prescription of values is injective. We may also prescribe the flux map of the examples.

We also show analogous results for a large family of directed holomorphic immersions $M \to \mathbb{C}^n$, including null curves.

1. Introduction and main results

The theory of interpolation by holomorphic functions is a central topic in complex analysis which began in the 19th century with the celebrated Weierstrass interpolation theorem [1876]: *on a closed discrete subset of a domain $D \subset \mathbb{C}$, one can prescribe the values of a holomorphic function $D \to \mathbb{C}$.* Much later, Florack [1948] extended the Weierstrass theorem to arbitrary open Riemann surfaces. In this paper we prove an analogue of this classical result for conformal minimal surfaces in Euclidean spaces.

Theorem 1.1 (Weierstrass interpolation theorem for conformal minimal surfaces). *Let Λ be a closed discrete subset of an open Riemann surface M, and let $n \geq 3$ be an integer. Every map $\Lambda \to \mathbb{R}^n$ extends to a conformal minimal immersion $M \to \mathbb{R}^n$.*

Let M be an open Riemann surface and $n \geq 3$ be an integer. By the identity principle it is not possible to prescribe values of a conformal minimal immersion $M \to \mathbb{R}^n$ on a subset that is not closed and discrete; hence the assumptions on Λ in Theorem 1.1 are necessary.

Recall that a conformal immersion $X = (X_1, \dots, X_n) : M \to \mathbb{R}^n$ is minimal if, and only if, X is a harmonic map. If this is the case then, denoting by ∂ the $\mathbb{C}$-linear part of the exterior differential $d = \partial + \bar{\partial}$ on M (here $\bar{\partial}$ denotes the $\mathbb{C}$-antilinear part of d), the 1-form $\partial X = (\partial X_1, \dots, \partial X_n)$ with values in $\mathbb{C}^n$ is holomorphic, has no zeros, and satisfies $\sum_{j=1}^n (\partial X_j)^2 = 0$ everywhere on M. Therefore, ∂X determines the Kodaira-type holomorphic map

$$G_X : M \to \mathbb{CP}^{n-1}, \quad M \ni p \mapsto G_X(p) = [\partial X_1(p) : \cdots : \partial X_n(p)],$$

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which takes values in the complex hyperquadric

$$Q_{n-2} = \{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\} \subset \mathbb{CP}^{n-1}$$

and is known as *the generalized Gauss map of* X. Conversely, every holomorphic map $M \to Q_{n-2} \subset \mathbb{CP}^{n-1}$ is the generalized Gauss map of a conformal minimal immersion $M \to \mathbb{R}^n$; see [Alarcón, Forstnerič and López 2017]. The real part $\Re(\partial X)$ is an exact 1-form on M; the *flux map* (or simply, the *flux*) of X is the group homomorphism $\mathrm{Flux}_X : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ of the first homology group of M with integer coefficients, given by

$$\mathrm{Flux}_X(\gamma) = \int_\gamma \Im(\partial X) = -\mathrm{i} \int_\gamma \partial X, \quad \gamma \in H_1(M; \mathbb{Z}),$$

where $\Im$ denotes the imaginary part and $\mathrm{i} = \sqrt{-1}$.

Conversely, every holomorphic 1-form $\Phi = (\phi_1, \ldots, \phi_n)$ with values in $\mathbb{C}^n$, vanishing nowhere on M, satisfying the nullity condition

$$\sum_{j=1}^{n} (\phi_j)^2 = 0 \quad \text{everywhere on } M, \tag{1-1}$$

and whose real part $\Re(\Phi)$ is exact on M, determines a conformal minimal immersion $X : M \to \mathbb{R}^n$ with $\partial X = \Phi$ by the classical Weierstrass formula

$$X(p) = x_0 + 2\int_{p_0}^{p} \Re(\Phi), \quad p \in M, \tag{1-2}$$

for any fixed base point $p_0 \in M$ and initial condition $X(p_0) = x_0 \in \mathbb{R}^n$. (We refer to [Osserman 1986] for a standard reference on minimal surface theory.) This representation formula has greatly influenced the study of minimal surfaces in $\mathbb{R}^n$ by providing powerful tools coming from complex analysis in one and several variables. In particular, Runge and Mergelyan theorems for open Riemann surfaces (see [Bishop 1958; Runge 1885; Mergelyan 1951]) and, more recently, the modern *Oka theory* (we refer to the monograph [Forstnerič 2017] and to the surveys [Lárusson 2010; Forstnerič and Lárusson 2011; Forstnerič 2013; Kutzschebauch 2014]) have been exploited in order to develop a uniform approximation theory for conformal minimal surfaces in Euclidean spaces which is analogous to the one of holomorphic functions in one complex variable and has found plenty of applications; see [Alarcón and López 2012; 2014; 2015; Alarcón and Forstnerič 2014; Drinovec Drnovšek and Forstnerič 2016; Alarcón, Forstnerič and López 2016a; 2016b; Forstnerič and Lárusson 2016]. In this paper we extend some of the methods invented for developing this approximation theory in order to provide also interpolation on closed discrete subsets of the underlying complex structure.

Theorem 1.1 is a consequence of the following much more general result ensuring not only interpolation but also *jet-interpolation of given finite order*, approximation on holomorphically convex compact subsets, control on the flux, and global properties such as completeness and, under natural assumptions, properness and injectivity. If A is a compact domain in an open Riemann surface, by a *conformal minimal immersion* $A \to \mathbb{R}^n$ *of class* $\mathscr{C}^m(A)$, $m \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, we mean an immersion $A \to \mathbb{R}^n$ of class $\mathscr{C}^m(A)$ whose

restriction to the interior $\mathring{A} = A \setminus bA$ is a conformal minimal immersion; we use the same notation if A is a union of pairwise-disjoint such domains.

Theorem 1.2 (Runge approximation with jet-interpolation for conformal minimal surfaces). *Let M be an open Riemann surface, $\Lambda \subset M$ be a closed discrete subset, and $K \subset M$ be a smoothly bounded compact domain such that $M \setminus K$ has no relatively compact connected components. For each $p \in \Lambda$ let $\Omega_p \subset M$ be a compact neighborhood of p in M, assume that $\Omega_p \cap \Omega_q = \varnothing$ for all $p \neq q \in \Lambda$, and set $\Omega := \bigcup_{p\in\Lambda} \Omega_p$. Also let $X : K \cup \Omega \to \mathbb{R}^n$ ($n \geq 3$) be a conformal minimal immersion of class $\mathscr{C}^1(K \cup \Omega)$ and let $\mathfrak{p} : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ be a group homomorphism satisfying*

$$\mathrm{Flux}_X(\gamma) = \mathfrak{p}(\gamma) \quad \textit{for all closed curves } \gamma \subset K.$$

Then, given $k \in \mathbb{Z}_+$, X may be approximated uniformly on K by complete conformal minimal immersions $\widetilde{X} : M \to \mathbb{R}^n$ enjoying the following properties:

(I) *$\widetilde{X}$ and X have a contact of order k at every point in Λ.*

(II) $\mathrm{Flux}_{\widetilde{X}} = \mathfrak{p}$.

(III) *If the map $X|_\Lambda : \Lambda \to \mathbb{R}^n$ is proper then we can choose $\widetilde{X} : M \to \mathbb{R}^n$ to be proper.*

(IV) *If $n \geq 5$ and the map $X|_\Lambda : \Lambda \to \mathbb{R}^n$ is injective, then we can choose $\widetilde{X} : M \to \mathbb{R}^n$ to be injective.*

Condition (I) in the above theorem is equivalent to $\widetilde{X}|_\Lambda = X|_\Lambda$ and, if $k > 0$, the holomorphic 1-form $\partial(\widetilde{X} - X)$, assuming values in $\mathbb{C}^n$, has a zero of multiplicity (at least) k at all points in Λ; in other words, *the maps $\widetilde{X}$ and X have the same k-jet at every point in Λ* (see Section 2B). This is reminiscent of the generalization of the Weierstrass interpolation theorem provided by [Behnke and Stein 1949] and asserting that on an open Riemann surface one may prescribe values to arbitrary finite order for a holomorphic function at the points in a given closed discrete subset; see [Napier and Ramachandran 2011, Theorem 2.15.1]. In particular, choosing $k = 1$ in Theorem 1.2 we obtain that *on a closed discrete subset of an open Riemann surface M, one can prescribe the values of a conformal minimal immersion $M \to \mathbb{R}^n$ ($n \geq 3$) and of its generalized Gauss map $M \to Q_{n-2} \subset \mathbb{CP}^{n-1}$* (see Corollary 7.1). The case $\Lambda = \varnothing$ in Theorem 1.2 (that is, when one does not take care of the interpolation) was recently proved by Alarcón, Forstnerič, and López [2016a, Theorem 1.2].

Note that the assumptions on $X|_\Lambda$ in assertions (III) and (IV) in Theorem 1.2 are necessary. We also point out that if Λ is infinite then there are injective maps $\Lambda \to \mathbb{R}^n$ which do not extend to a topological embedding $M \to \mathbb{R}^n$; hence, in general, one cannot choose the conformal minimal immersion $\widetilde{X}$ in (IV) to be an *embedding* (i.e., a homeomorphism onto $\widetilde{X}(M)$ endowed with the subspace topology inherited from $\mathbb{R}^n$). On the other hand, since proper injective immersions $M \to \mathbb{R}^n$ are embeddings, we can choose $\widetilde{X}$ in Theorem 1.2 to be a proper conformal minimal embedding provided that $n \geq 5$ and $X|_\Lambda : \Lambda \to \mathbb{R}^n$ is both proper and injective.

Let us now say a word about our methods of proof. Given a holomorphic 1-form θ on M with no zeros (such a θ exists by the Oka–Grauert principle, see [Grauert 1957; 1958; Forstnerič 2017, Theorem 5.3.1]), any holomorphic 1-form $\Phi = (\phi_1, \dots, \phi_n)$ on M with values in $\mathbb{C}^n$ and satisfying the nullity condition

(1-1) can be written in the form $\Phi = f\theta$ where $f : M \to \mathbb{C}^n$ is a holomorphic function taking values in the *null quadric* (also called the *complex light cone*)

$$\mathfrak{A} := \{z=(z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 0\}. \tag{1-3}$$

Therefore, in order to prove Theorem 1.1 one needs to find a holomorphic map

$$f : M \to \mathfrak{A} \setminus \{0\} \subset \mathbb{C}^n$$

such that $\Re(f\theta)$ is an exact real 1-form on M and

$$2\int_{p_0}^{p} \Re(f\theta) = \mathfrak{Z}(p) \quad \text{for all } p \in \Lambda,$$

where $p_0 \in M \setminus \Lambda$ is a fixed base point and $\mathfrak{Z} : \Lambda \to \mathbb{R}^n$ is the given map. Then the formula (1-2) with $x_0 = 0$ and $\Phi = f\theta$ provides a conformal minimal immersion satisfying the conclusion of the theorem. The key in this approach is that the *punctured null quadric*

$$\mathfrak{A}_* := \mathfrak{A} \setminus \{0\} \tag{1-4}$$

is a *complex homogeneous manifold* and hence an *Oka manifold* [Alarcón and Forstnerič 2014, Example 4.4]; thus, there are *many* holomorphic maps $M \to \mathfrak{A}_*$ (see Section 2C for more information).

The proof of Theorem 1.2 is much more involved and elaborate. It requires, in addition to the above, a subtle use of the Runge–Mergelyan theorem with jet-interpolation for holomorphic maps from open Riemann surfaces into Oka manifolds (see Theorem 2.6) to achieve condition (I). Additionally, it requires a conceptually new intrinsic-extrinsic version of the technique from [Jorge and Xavier 1980] to ensure completeness of the interpolating immersions (see Lemma 5.5 and Section 6B), and, in order to guarantee assertion (III), we must extend the recently developed methods in [Alarcón and López 2012; Alarcón and Forstnerič 2014; Alarcón, Forstnerič and López 2016a] for constructing proper minimal surfaces in $\mathbb{R}^n$ with arbitrary complex structure (see Lemma 5.6 and Section 6C). Moreover, in order to prove (IV) we adapt the transversality approach by Abraham [1963] in Theorem 5.3; see [Alarcón and Forstnerič 2014; Alarcón, Drinovec Drnovšek, Forstnerič and López 2015; Alarcón, Forstnerič and López 2016a] for its implementation in minimal surface theory.

The above-described method for constructing conformal minimal surfaces in $\mathbb{R}^n$, based on Oka theory, was introduced in [Alarcón and Forstnerič 2014] and it also works in the more general framework of *directed holomorphic immersions* of open Riemann surfaces into complex Euclidean spaces. Directed immersions have been the focus of interest in a number of classical geometries such as symplectic, contact, Lagrangian, totally real, etc.; we refer for instance to the monograph [Gromov 1986], to [Eliashberg and Mishachev 2002, Chapter 19], and to the introduction of [Alarcón and Forstnerič 2014] for motivation on this subject. Given a (topologically) closed conical complex subvariety $\mathfrak{S}$ of $\mathbb{C}^n$ ($n \geq 3$), a holomorphic immersion $F : M \to \mathbb{C}^n$ of an open Riemann surface M into $\mathbb{C}^n$ is said to be *directed by* $\mathfrak{S}$, or an $\mathfrak{S}$-*immersion*, if its complex derivative F' with respect to any local holomorphic coordinate on M assumes values in

$$\mathfrak{S}_* := \mathfrak{S} \setminus \{0\};$$

see [Alarcón and Forstnerič 2014, Definition 2.1]. If A is a compact domain in an open Riemann surface, or a union of pairwise-disjoint such domains, by an $\mathfrak{S}$-*immersion* $A \to \mathbb{C}^n$ *of class* $\mathscr{A}^m(A)$ ($m \in \mathbb{Z}_+$) we mean an immersion $A \to \mathbb{C}^n$ of class $\mathscr{C}^m(A)$ whose restriction to the interior, $\mathring{A}$, is a (holomorphic) $\mathfrak{S}$-immersion. Among others, general existence, approximation, and desingularization results were proved in [Alarcón and Forstnerič 2014] for certain families of directed holomorphic immersions, including *null curves*, i.e., holomorphic curves in $\mathbb{C}^n$ which are directed by the null quadric $\mathfrak{A} \subset \mathbb{C}^n$; see (1-3). It is well known that the real and imaginary parts of a null curve $M \to \mathbb{C}^n$ are conformal minimal immersions $M \to \mathbb{R}^n$ whose flux map vanishes everywhere on $H_1(M;\mathbb{Z})$; conversely, every conformal minimal immersion $M \to \mathbb{R}^n$ is locally, on every simply connected domain of M, the real part of a null curve $M \to \mathbb{C}^n$; see [Osserman 1986, Chapter 4].

The second main theorem of this paper is an analogue of Theorem 1.2 for a wide family of directed holomorphic curves in $\mathbb{C}^n$ which includes null curves. Given integers $1 \le j \le n$ we denote by $\pi_j : \mathbb{C}^n \to \mathbb{C}$ the coordinate projection $\pi_j(z_1, \dots, z_n) = z_j$.

Theorem 1.3 (Runge approximation with jet-interpolation for directed holomorphic curves). *Let $\mathfrak{S}$ be an irreducible closed conical complex subvariety of $\mathbb{C}^n$ ($n \ge 3$) which is contained in no hyperplane and such that $\mathfrak{S}_* = \mathfrak{S} \setminus \{0\}$ is smooth and an Oka manifold. Let M, Λ, K, and Ω be as in Theorem 1.2 and let $F : K \cup \Omega \to \mathbb{C}^n$ be an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(K \cup \Omega)$. Then, given $k \in \mathbb{N}$, F may be approximated uniformly on K by $\mathfrak{S}$-immersions $\widetilde{F} : M \to \mathbb{C}^n$ such that $\widetilde{F} - F$ has a zero of multiplicity (at least) k at every point in Λ. Moreover, if the map $F|_\Lambda : \Lambda \to \mathbb{C}^n$ is injective, then we can choose $\widetilde{F} : M \to \mathbb{C}^n$ to be injective.*

Furthermore:

(I) *If $\mathfrak{S} \cap \{z_1 = 1\}$ is an Oka manifold and $\pi_1 : \mathfrak{S} \to \mathbb{C}$ admits a local holomorphic section h near $\zeta = 0 \in \mathbb{C}$ with $h(0) \neq 0$, then we may choose $\widetilde{F}$ to be complete.*

(II) *If $\mathfrak{S} \cap \{z_j = 1\}$ is an Oka manifold and $\pi_j : \mathfrak{S} \to \mathbb{C}$ admits a local holomorphic section h_j near $\zeta = 0 \in \mathbb{C}$ with $h_j(0) \neq 0$ for all $j \in \{1, \dots, n\}$, and if the map $F|_\Lambda : \Lambda \to \mathbb{C}^n$ is proper, then we may choose $\widetilde{F} : M \to \mathbb{C}^n$ to be proper.*

In particular, if we are given $\mathfrak{S}$, M, and Λ as in Theorem 1.3 then *every map $\Lambda \to \mathbb{C}^n$ extends to an $\mathfrak{S}$-immersion $M \to \mathbb{C}^n$*. When the subset $\Lambda \subset M$ is empty, the above theorem except for assertion (I) is implied by [Alarcón and Forstnerič 2014, Theorems 7.2 and 8.1]. It is perhaps worth mentioning in this respect that, if $\mathfrak{S}$ is as in assertion (I) and $F|_\Lambda : \Lambda \to \mathbb{C}^n$ is not proper, Theorem 1.3 provides *complete* $\mathfrak{S}$-immersions $M \to \mathbb{C}^n$ which are not proper maps; these seem to be the first known examples of such apart from the case when $\mathfrak{S}$ is the null quadric. Let us emphasize that the particular geometry of $\mathfrak{A}$ allows for the construction of complete null holomorphic curves in $\mathbb{C}^n$ and minimal surfaces in $\mathbb{R}^n$ with a number of different asymptotic behaviors (other than proper in space); see [Alarcón and López 2013a; Alarcón and Forstnerič 2015; Alarcón, Drinovec Drnovšek, Forstnerič and López 2015; ≥ 2019; Alarcón and Castro-Infantes 2018].

Most of the technical parts in the proofs of Theorems 1.2 and 1.3 will be furnished by a general result concerning periods of holomorphic 1-forms with values in a closed conical complex subvariety of $\mathbb{C}^n$

(see Theorem 4.4 for a precise statement). With this at hand, the proofs of Theorems 1.2 and 1.3 are very similar; this is why, with brevity of exposition in mind, we shall spell out in detail the proof of Theorem 1.3 (which is, in some sense, more general) but only briefly sketch the one of Theorem 1.2.

This paper is, to the best of our knowledge, the first contribution to the theory of interpolation by conformal minimal surfaces and directed holomorphic curves in a Euclidean space.

Organization of the paper. In Section 2 we state some notation and the preliminaries which are needed throughout the paper; we also show an observation which is crucial to ensure the jet-interpolation conditions in Theorems 1.2 and 1.3 (see Lemma 2.2). Section 3 is devoted to the proof of several preliminary results on the existence of period-dominating sprays of maps into conical complex subvarieties $\mathfrak{S}_*$ of $\mathbb{C}^n$; we use them in Section 4 to prove the noncritical case of a Mergelyan theorem with jet-interpolation and control on the periods for holomorphic maps into such a $\mathfrak{S}_*$ being Oka (see Lemma 4.2), and the main technical result of the paper (Theorem 4.4). In Section 5 we prove a general position theorem, a completeness lemma, and a properness lemma for $\mathfrak{S}$-immersions, which enable us to complete the proof of Theorem 1.3 in Section 6. Finally, Section 7 is devoted to explaining how the methods in the proof of Theorem 1.3 can be adapted to prove Theorem 1.2.

2. Preliminaries

We define $\mathrm{i} = \sqrt{-1}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathbb{R}_+ = [0, +\infty)$. Given an integer $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $|\cdot|$, $\mathrm{dist}(\cdot, \cdot)$, and $\mathrm{length}(\cdot)$ the Euclidean norm, distance, and length in $\mathbb{K}^n$, respectively. If K is a compact topological space and $f : K \to \mathbb{K}^n$ is a continuous map, we denote by

$$\|f\|_{0,K} := \max\{|f(p)| : p \in K\}$$

the maximum norm of f on K. Likewise, given $x = (x_1, \ldots, x_n)$ in $\mathbb{K}^n$ we define

$$|x|_\infty := \max\{|x_1|, \ldots, |x_n|\} \quad \text{and} \quad \|f\|_{\infty,K} := \max\{|f(p)|_\infty : p \in K\}.$$

If K is a subset of a Riemann surface M, then for any $r \in \mathbb{Z}_+$ we shall denote by $\|f\|_{r,K}$ the standard $\mathscr{C}^r$ norm of a function $f : K \to \mathbb{K}^n$ of class $\mathscr{C}^r(K)$, where the derivatives are measured with respect to a Riemannian metric on M (the precise choice of the metric will not be important).

Given a smooth connected surface S (possibly with nonempty boundary) and a smooth immersion $X : S \to \mathbb{K}^n$, we denote by $\mathrm{dist}_X : S \times S \to \mathbb{R}_+$ the Riemannian distance induced on S by the Euclidean metric of $\mathbb{K}^n$ via X; i.e.,

$$\mathrm{dist}_X(p, q) := \inf\{\mathrm{length}(X(\gamma)) : \gamma \subset S \text{ an arc connecting } p \text{ and } q\}, \quad p, q \in S.$$

Likewise, if $K \subset S$ is a relatively compact subset we define

$$\mathrm{dist}_X(p, K) := \inf\{\mathrm{dist}_X(p, q) : q \in K\}, \quad p \in S.$$

An immersed open surface $X : S \to \mathbb{K}^n$ ($n \geq 3$) is said to be *complete* if the image by X of any proper path $\gamma : [0, 1) \to S$ has infinite Euclidean length; equivalently, if the Riemannian metric on S induced by

dist_X is complete in the classical sense. On the other hand, $X : S \to \mathbb{K}^n$ is said to be *proper* if the image by X of every proper path $\gamma : [0, 1) \to S$ is a divergent path in $\mathbb{K}^n$.

2A. ***Riemann surfaces and spaces of maps.*** Throughout the paper every Riemann surface will be considered connected if the contrary is not indicated.

Let M be an open Riemann surface. Given a subset $A \subset M$ we denote by $\mathcal{O}(A)$ the space of functions $A \to \mathbb{C}$ which are holomorphic on an unspecified open neighborhood of A in M. If A is a smoothly bounded compact domain, or a union of pairwise-disjoint such domains, and $r \in \mathbb{Z}_+$, we denote by $\mathscr{A}^r(A)$ the space of $\mathscr{C}^r$ functions $A \to \mathbb{C}$ which are holomorphic on the interior $\mathring{A} = A \setminus bA$; for simplicity we write $\mathscr{A}(A)$ for $\mathscr{A}^0(A)$. Likewise, we define the spaces $\mathcal{O}(A, Z)$ and $\mathscr{A}^r(A, Z)$ of maps $A \to Z$ to any complex manifold Z. Thus, if $\mathfrak{S}$ is a closed conical complex subvariety of $\mathbb{C}^n$ ($n \geq 3$), by an $\mathfrak{S}$-immersion $A \to \mathbb{C}^n$ of class $\mathscr{A}^r(A)$ we simply mean an immersion of class $\mathscr{A}^r(A)$ whose restriction to $\mathring{A}$ is an $\mathfrak{S}$-immersion. In the same way, a conformal minimal immersion $A \to \mathbb{R}^n$ of class $\mathscr{C}^r(A)$ will be nothing but an immersion of class $\mathscr{C}^r(A)$ whose restriction to $\mathring{A}$ is a conformal minimal immersion.

By a *compact bordered Riemann surface* we mean a compact Riemann surface M with nonempty boundary bM consisting of finitely many pairwise-disjoint smooth Jordan curves. The interior $\mathring{M} = M \setminus bM$ of M is called a *bordered Riemann surface*. It is well known that every compact bordered Riemann surface M is diffeomorphic to a smoothly bounded compact domain in an open Riemann surface $\widetilde{M}$. The spaces $\mathscr{A}^r(M)$ and $\mathscr{A}^r(M, Z)$, for an integer $r \in \mathbb{Z}_+$ and a complex manifold Z, are defined as above.

A compact subset K in an open Riemann surface M is said to be *Runge* (also called *holomorphically convex* or $\mathscr{O}(M)$*-convex*) if every continuous function $K \to \mathbb{C}$, holomorphic in the interior $\mathring{K}$, may be approximated uniformly on K by holomorphic functions on M; by the Runge–Mergelyan theorem [Runge 1885; Mergelyan 1951; Bishop 1958] this is equivalent to $M \setminus K$ having no relatively compact connected components in M. The following particular kind of Runge subsets will play a crucial role in our argumentation.

Definition 2.1. A nonempty compact subset S of an open Riemann surface M is called *admissible* if it is Runge in M and of the form $S = K \cup \Gamma$, where K is the union of finitely many pairwise-disjoint smoothly bounded compact domains in M and $\Gamma := \overline{S \setminus K}$ is a finite union of pairwise-disjoint smooth Jordan arcs and closed Jordan curves meeting K only in their endpoints (or not at all) and such that their intersections with the boundary bK of K are transverse.

If C and C' are oriented arcs in M, and the initial point of C' is the final one of C, we denote by $C * C'$ the product of C and C', i.e., the oriented arc $C \cup C' \subset M$ with initial point the initial point of C and final point the final point of C'.

Every open connected Riemann surface M contains a 1-dimensional embedded CW-complex $C \subset M$ such that there is a strong deformation retraction $\rho_t : M \to M$ ($t \in [0, 1]$); i.e., $\rho_0 = \mathrm{Id}_M$, $\rho_t|_C = \mathrm{Id}|_C$ for all $t \in [0, 1]$, and $\rho_1(M) = C$. It follows that the complement $M \setminus C$ has no relatively compact connected components in M and hence C is Runge. Such a CW-complex $C \subset M$ represents the topology of M and can be obtained, for instance, as the Morse complex of a Morse strongly subharmonic exhaustion function on M. Recall that the first homology group satisfies $H_1(M; \mathbb{Z}) = \mathbb{Z}^l$ for some $l \in \mathbb{Z}_+ \cup \{\infty\}$. It

is not difficult to see that, if M is finitely connected (for instance, if it is a bordered Riemann surface), i.e., if $l \in \mathbb{Z}_+$, then, given a point $p_0 \in M$ there is a CW-complex $C \subset M$ as above which is a bouquet of l circles with base point p_0; i.e., $\{p_0\}$ is the only 0-cell of C, and C has l 1-cells $C_1, \dots, C_l$ which are closed Jordan curves on M that only meet at p_0.

2B. *Jets.* Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds without boundary, $x_0 \in \mathcal{M}$ be a point, and $f, g : \mathcal{M} \to \mathcal{N}$ be smooth maps. The maps f and g have, by definition, a *contact of order* $k \in \mathbb{Z}_+$ at the point x_0 if their Taylor series at this point coincide up to the order k. An equivalence class of maps $\mathcal{M} \to \mathcal{N}$ which have a contact of order k at the point x_0 is called a *k-jet*; see, e.g., [Michor 1980, §1] for a basic reference. Recall that the Taylor series at x_0 of a smooth map $f : \mathcal{M} \to \mathcal{N}$ does not depend on the choice of coordinate charts on $\mathcal{M}$ and $\mathcal{N}$ centered at x_0 and $f(x_0)$ respectively. Therefore, fixing such a pair of coordinates, we can identify the k-jet of f at x_0, which is usually denoted by $j^k_{x_0}(f)$, with the set of derivatives of f at x_0 of order up to and including k; under this identification of jets we have

$$j^0_{x_0}(f) = f(x_0), \quad j^1_{x_0}(f) = \left(f(x_0), \left.\frac{\partial f}{\partial x}\right|_{x_0}\right), \quad j^2_{x_0}(f) = \left(f(x_0), \left.\frac{\partial f}{\partial x}\right|_{x_0}, \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0}\right), \quad \dots$$

Analogously, if $\mathcal{M}$ and $\mathcal{N}$ are complex manifolds then we consider the complex (holomorphic) derivatives with respect to some local holomorphic coordinates. It is clear that the definition of the k-jet of a map at a point is local and hence it can be made for germs of maps at the point. Moreover, if a pair of maps have the same k-jet at a point then, obviously, they also have the same k'-jet at the point for all $k' \in \mathbb{Z}_+,\ k' \le k$.

In particular, if Ω is a neighborhood of a point p in an open Riemann surface M and $f, g : \Omega \to \mathbb{C}^n$ are holomorphic functions, then they have a contact of order $k \in \mathbb{Z}_+$, or the same k-jet, at the point p if, and only if, $f - g$ has a zero of multiplicity (at least) $k+1$ at p; if this is the case then for any distance function $\mathsf{d} : M \times M \to \mathbb{R}_+$ on M (not necessarily conformal) we have

$$|f - g|(q) = O(\mathsf{d}(q, p)^{k+1}) \quad \text{as } q \to p. \tag{2-1}$$

If $f, g : \Omega \to \mathbb{R}^n$ are harmonic maps (as, for instance, conformal minimal immersions), then we say that they have a contact of order $k \in \mathbb{Z}_+$, or the same k-jet, at the point p if, assuming that Ω is simply connected, there are harmonic conjugates $\tilde{f}$ of f and $\tilde{g}$ of g such that the holomorphic functions $f + \mathrm{i}\tilde{f}, g + \mathrm{i}\tilde{g} : \Omega \to \mathbb{C}^n$ have a contact of order k at p; this is equivalent to $f(p) = g(p)$ and, if $k > 0$, the holomorphic 1-form $\partial(f - g)$ has a zero of multiplicity (at least) k at p. Again, if such a pair of maps f and g have the same k-jet at the point $p \in \Omega$ then (2-1) formally holds.

The following observation will be crucial in order to ensure the jet-interpolation in the main results of this paper.

Lemma 2.2. *Let V be a holomorphic vector field in $\mathbb{C}^n$ ($n \in \mathbb{N}$), vanishing at $0 \in \mathbb{C}^n$, and let ϕ_s denote the flow of V for small values of time $s \in \mathbb{C}$. Given an open Riemann surface M, a point $p \in M$, and holomorphic functions $f : M \to \mathbb{C}^n$ and $h : M \to \mathbb{C}$ such that h has a zero of multiplicity $k+1$ at p for some $k \in \mathbb{Z}_+$, then the holomorphic map*

$$q \mapsto \tilde{f}(q) = \phi_{h(q)}(f(q)),$$

which is defined on a neighborhood of p in M, has a contact of order k with f at the point p; that is, f and $\tilde{f}$ have the same k-jet at p.

Proof. The flow ϕ_s of the vector field V at a point $z \in \mathbb{C}^n$ may be expressed as

$$\phi_s(z) = z + sV(z) + O(|s|^2);$$

see, e.g., [Abraham, Marsden and Ratiu 1988, §4.1]. Since h has a zero of multiplicity $k+1$ at p, the conclusion of the lemma follows. □

We shall use the following notation in several places throughout the paper.

Notation 2.3. Let $n \geq 3$ be an integer and $\mathfrak{S}$ be a (topologically) closed conical complex subvariety of $\mathbb{C}^n$; by *conical* we mean that $t\mathfrak{S} = \mathfrak{S}$ for all $t \in \mathbb{C}_* = \mathbb{C} \setminus \{0\}$. We also assume that $\mathfrak{S}$ is contained in no hyperplane of $\mathbb{C}^n$, and $\mathfrak{S}_* := \mathfrak{S} \setminus \{0\}$ is smooth and connected (hence irreducible). We also fix a large integer $N \geq n$ and holomorphic vector fields $V_1, \ldots, V_N$ on $\mathbb{C}^n$ which are tangential to $\mathfrak{S}$ along $\mathfrak{S}$, vanish at $0 \in \mathfrak{S}$, and satisfy

$$\operatorname{span}\{V_1(z), \ldots, V_N(z)\} = T_z\mathfrak{S} \quad \text{for all } z \in \mathfrak{S}_*. \tag{2-2}$$

(Such vector fields exist by Cartan's theorem A [1953].)

$$\text{Let } \phi_s^j \text{ denote the flow of the vector field } V_j \tag{2-3}$$

for $j = 1, \ldots, N$ and small values of the time $s \in \mathbb{C}$.

Remark 2.4. Throughout the paper we shall say that a holomorphic function has a zero of multiplicity $k \in \mathbb{N}$ at a point to mean that the function has a zero of multiplicity *at least* k at the point. When the multiplicity of the zero is exactly k, it will be explicitly mentioned. We will follow the same pattern when claiming that two functions have the same k-jet or a contact of order k at a point.

2C. *Oka manifolds.* We recall the notion of Oka manifold and state some of the properties of such manifolds which will be exploited in our argumentation. A comprehensive treatment of Oka theory can be found in [Forstnerič 2017]; for a briefer introduction to the topic we refer to [Lárusson 2010; Forstnerič and Lárusson 2011; Forstnerič 2013; Kutzschebauch 2014].

Definition 2.5. A complex manifold Z is said to be an *Oka manifold* if every holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^N$ ($N \in \mathbb{N}$) to Z can be approximated uniformly on K by entire maps $\mathbb{C}^N \to Z$.

The central result of Oka theory is that maps $M \to Z$ from a Stein manifold (as, for instance, an open Riemann surface) to an Oka manifold satisfy all forms of the Oka principle; see [Forstnerič 2006]. In this paper we shall use as a fundamental tool the following version of the *Mergelyan theorem with jet-interpolation* which trivially follows from [Forstnerič 2017, Theorems 3.8.1 and 5.4.4]; see also [Forstnerič 2004, Theorem 3.2; Hörmander and Wermer 1968, Theorem 4.1].

Theorem 2.6. *Let Z be an Oka manifold, let M be an open Riemann surface, and let $S = K \cup \Gamma \subset M$ be an admissible subset in the sense of Definition 2.1. Given a finite subset $\Lambda \subset \mathring{K}$ and an integer $k \in \mathbb{Z}_+$, every continuous map $f : S \to Z$ which is holomorphic on $\mathring{K}$ can be approximated uniformly on S by holomorphic maps $M \to Z$ having the same k-jet as f at all points in Λ.*

As we emphasized in the Introduction, the punctured null quadric $\mathfrak{A}_* \subset \mathbb{C}^n$, see (1-3) and (1-4), directing minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$ is an Oka manifold for all $n \geq 3$; see [Alarcón and Forstnerič 2014, Example 4.4 Forstnerič 2017, Example 5.6.2]. Furthermore, for each $j \in \{1, \dots, n\}$ the complex manifold $\mathfrak{A} \cap \{z_j = 1\}$ is an embedded copy of the complex $(n-2)$-sphere

$$\mathbb{C}S^{n-2} = \{w = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1} : w_1^2 + \cdots + w_{n-1}^2 = 1\}.$$

Observe that $\mathbb{C}S^{n-2}$ is homogeneous relative to the complex Lie group $\mathrm{SO}(n-1, \mathbb{C})$, and hence it is an Oka manifold; see [Grauert 1957; Forstnerič 2017, Proposition 5.6.1]. For a more detailed discussion, see [Forstnerič 2017, Example 6.15.7; Alarcón and Forstnerič 2014, Example 7.8]. Moreover, choosing $k \in \{1, \dots, n\}$, $k \neq j$, the map $h = (h_1, \dots, h_n) : \mathbb{C} \to \mathfrak{A}$ given by

$$h_j(\zeta) = \zeta, \quad h_k(\zeta) = \sqrt{1 - \zeta^2}, \quad h_l(\zeta) = \frac{\mathrm{i}}{\sqrt{n-2}} \quad \text{for all } l \neq j, k, \quad \zeta \in \mathbb{C},$$

is a local holomorphic section near $\zeta = 0 \in \mathbb{C}$ of the coordinate projection $\pi_j : \mathfrak{A} \to \mathbb{C}$, $\pi_j(z_1, \dots, z_n) = z_j$, which satisfies $h(0) \neq 0$. Thus, the null quadric $\mathfrak{A} \subset \mathbb{C}^n$ meets the requirements in Theorem 1.3, including the ones in assertions (I) and (II), for all $n \geq 3$.

3. Paths in closed conical complex subvarieties of $\mathbb{C}^n$

We now use Notation 2.3; in particular, $\mathfrak{S} \subset \mathbb{C}^n$ $(n \geq 3)$ denotes a closed conical complex subvariety which is contained in no hyperplane of $\mathbb{C}^n$ and such that $\mathfrak{S}_* = \mathfrak{S} \setminus \{0\}$ is smooth and connected. We need the following:

Definition 3.1. Let Q be a topological space and $n \geq 3$ be an integer. A continuous map $f : Q \to \mathbb{C}^n$ is said to be *flat* if $f(Q) \subset \mathbb{C}z_0 = \{\zeta z_0 : \zeta \in \mathbb{C}\}$ for some $z_0 \in \mathbb{C}^n$, and *nonflat* otherwise. The map f is said to be *nowhere flat* if $f|_A : A \to \mathbb{C}^n$ is nonflat for all open subsets $\varnothing \neq A \subset Q$.

It is easily seen that a continuous map $f : [0, 1] \to \mathfrak{S}_* \subset \mathbb{C}^n$ is nonflat if, and only if,

$$\mathrm{span}\{T_{f(t)}\mathfrak{S} : t \in [0, 1]\} = \mathbb{C}^n.$$

3A. *Paths on I := [0, 1].* We prove a couple of technical results for paths $[0, 1] \to \mathfrak{S}_*$ which pave the way to the construction of period-dominating sprays of holomorphic maps of an open Riemann surface into $\mathfrak{S}_*$ (see Lemma 3.4 in the next subsection).

Lemma 3.2. *Let $f : I \to \mathfrak{S}_*$ and $\vartheta : I \to \mathbb{C}_*$ be continuous maps. Let $\varnothing \neq I' \subset I$ be a closed subinterval and assume that f is nowhere flat on I'. There exist continuous functions $h_1, \dots, h_N : I \to \mathbb{C}$, with*

support on I', and a neighborhood U of $0 \in \mathbb{C}^N$ such that the period map $\mathcal{P} : U \to \mathbb{C}^n$ *given by*

$$\mathcal{P}(\zeta) = \int_0^1 \phi^1_{\zeta_1 h_1(t)} \circ \cdots \circ \phi^N_{\zeta_N h_N(t)}(f(t))\,\vartheta(t)\,dt, \quad \zeta = (\zeta_1, \ldots, \zeta_N) \in U,$$

see (2-3), *is well-defined and has maximal rank equal to n at $\zeta = 0$.*

Proof. We choose continuous functions $h_1, \ldots, h_N : I \to \mathbb{C}$, with support on I', which will be specified later. Then we define for a small neighborhood U of $0 \in \mathbb{C}^N$ a map

$$\Phi : U \times I \to \mathfrak{S}$$

given by

$$\Phi(\zeta, t) := \phi^1_{\zeta_1 h_1(t)} \circ \cdots \circ \phi^N_{\zeta_N h_N(t)}(f(t)), \quad \zeta = (\zeta_1, \ldots, \zeta_N) \in U,\ t \in I.$$

Note that $\Phi(0, t) = f(t)$ for all $t \in I$; recall that each V_j vanishes at 0 for all $j \in \{1, \ldots, N\}$. Thus, since $f(I) \subset \mathfrak{S}_*$ is compact, we may assume that U is small enough so that Φ is well-defined and takes values in $\mathfrak{S}_*$. Furthermore, Φ is holomorphic in the variable ζ and its derivative with respect to ζ_j is

$$\left.\frac{\partial \Phi(\zeta, t)}{\partial \zeta_j}\right|_{\zeta=0} = h_j(t) V_j(f(t)), \quad j = 1, \ldots, N. \tag{3-1}$$

(See (2-2) and (2-3).) Thus, the period map $\mathcal{P} : U \to \mathbb{C}^n$ in the statement of the lemma can be written as

$$\mathcal{P}(\zeta) = \int_0^1 \Phi(\zeta, t)\,\vartheta(t)\,dt, \quad \zeta \in U.$$

Observe that $\mathcal{P}$ is holomorphic and, in view of (3-1),

$$\left.\frac{\partial \mathcal{P}(\zeta)}{\partial \zeta_j}\right|_{\zeta=0} = \int_0^1 h_j(t) V_j(f(t))\,\vartheta(t)\,dt, \quad j = 1, \ldots, N. \tag{3-2}$$

Since f is nowhere flat on I' (see Definition 3.1), (2-2) guarantees the existence of distinct points $t_1, \ldots, t_N \in I'$ such that

$$\operatorname{span}\{V_1(f(t_1)), \ldots, V_N(f(t_N))\} = \mathbb{C}^n. \tag{3-3}$$

Now we specify the values of the function h_j in I' ($j = 1, \ldots, N$); recall that $\operatorname{supp}(h_j) \subset I'$. We choose h_j with support in a small neighborhood $[t_j - \epsilon, t_j + \epsilon]$ of t_j in I', for some $\epsilon > 0$, and such that

$$\int_0^1 h_j(t)\,dt = \int_{t_j-\epsilon}^{t_j+\epsilon} h_j(t)\,dt = 1.$$

Then, for small $\epsilon > 0$, we have

$$\int_0^1 h_j(t) V_j(f(t))\,\vartheta(t)\,dt \approx V_j(f(t_j))\,\vartheta(t_j), \quad j = 1, \ldots, N.$$

Since $\vartheta(t) \neq 0$, (3-3) ensures that the vectors on the right side of the above display span $\mathbb{C}^n$, and hence the same is true for the vectors on the left side provided that $\epsilon > 0$ is chosen sufficiently small. This concludes the proof in view of (3-2). □

Lemma 3.3. *Let $\vartheta : I \to \mathbb{C}_*$ be a continuous map. Given points $u_0, u_1 \in \mathfrak{S}_*$ and $x \in \mathbb{C}^n$, and a domain Ω in $\mathbb{C}^n$ containing 0 and x, there exists a continuous function $g : I \to \mathfrak{S}_*$ which is nowhere flat on a neighborhood of 0 in I and such that*:

(I) $g(0) = u_0$ *and* $g(1) = u_1$.

(II) $\int_0^s g(t)\vartheta(t)\,dt \in \Omega$ *for all* $s \in I$.

(III) $\int_0^1 g(t)\vartheta(t)\,dt = x$.

Proof. Set $I_0 := \left[0, \frac{1}{2}\right]$ and choose any continuous nowhere-flat map $g_0 : I_0 \to \mathfrak{S}_*$ such that

$$g_0(0) = u_0, \qquad \int_0^s g_0(t)\vartheta(t)\,dt \in \Omega \quad \text{for all } s \in I_0. \tag{3-4}$$

Such a map can be constructed as follows. For any $0 < \delta < \frac{1}{2}$ let $f_\delta : I_0 \to [\delta, 1]$ be the continuous map given by $f_\delta(s) = 1 - ((1-\delta)/\delta)s$ for $s \in [0, \delta]$ and $f_\delta(s) = \delta$ for $s \in \left[\delta, \frac{1}{2}\right]$. Choose any continuous nowhere-flat map $\tilde{g}_0 : I_0 \to \mathfrak{S}_*$ with $\tilde{g}_0(0) = u_0$. Then $g_0 := f_\delta \tilde{g}_0 : I_0 \to \mathfrak{S}_*$ satisfies the requirements for any $\delta > 0$ sufficiently small.

Let $\varnothing \neq I' \subset \mathring{I}_0$ be a closed subinterval. Thus, Lemma 3.2 applied to g_0 provides continuous functions $h_1, \dots, h_N : I \to \mathbb{C}$, with support on I', and a neighborhood U of the origin in $\mathbb{C}^N$, such that the period map

$$U \ni \zeta \mapsto \mathcal{P}(\zeta) = \int_0^{1/2} \phi^1_{\zeta_1 h_1(t)} \circ \cdots \circ \phi^N_{\zeta_N h_N(t)}(g_0(t))\vartheta(t)\,dt, \quad \zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N,$$

has maximal rank equal to n at $\zeta = 0$. (See (2-3).) Set

$$\Phi(\zeta, t) := \phi^1_{\zeta_1 h_1(t)} \circ \cdots \circ \phi^N_{\zeta_N h_N(t)}(g_0(t)) \in \mathfrak{S}, \quad \zeta \in U,\ t \in I_0,$$

and observe that $\Phi(0, t) = g_0(t) \in \mathfrak{S}_*$ for all $t \in I_0$. Then, up to shrinking U if necessary, we have:

(a) $\Phi(U \times I_0) \subset \mathfrak{S}_*$ and $\mathcal{P}(U)$ contains a ball in $\mathbb{C}^n$ with radius $\epsilon > 0$ centered at

$$\mathcal{P}(0) = \int_0^{1/2} g_0(t)\vartheta(t)\,dt \in \Omega;$$

see (3-4).

(b) $\Phi(\zeta, t) = g_0(t)$ for all $(\zeta, t) \in U \times \left\{0, \frac{1}{2}\right\}$; recall that $h_j(0) = h_j\left(\frac{1}{2}\right) = 0$ for all $j = 1, \dots, N$.

(c) $\int_0^s \Phi(\zeta, t)\vartheta(t)\,dt \in \Omega$ for all $\zeta \in U$ and $s \in I_0$; see (3-4).

To conclude the proof we adapt the argument in [Alarcón and Forstnerič 2014, Lemma 7.3]. Since the convex hull of $\mathfrak{S}$ is $\mathbb{C}^n$, see [Alarcón and Forstnerič 2014, Lemma 3.1], we may construct a polygonal path $\Gamma \subset \Omega$ connecting $\mathcal{P}(0)$ and x; to be more precise, $\Gamma = \bigcup_{j=1}^m \Gamma_j$ where each Γ_j is a segment of the form $\Gamma_j = w_j + [0, 1]z_j$ for some $w_j \in \mathbb{C}^n$ and $z_j \in \mathfrak{S}_*$, the initial point w_1 of Γ_1 is $\mathcal{P}(0)$, the final point $w_m + z_m$ of Γ_m is x, and the initial point w_j of Γ_j agrees with the final one $w_{j-1} + z_{j-1}$ of Γ_{j-1} for all $j = 2, \dots, m$. Set

$$I_j := \left[\frac{1}{2} + \frac{j-1}{2m}, \frac{1}{2} + \frac{j}{2m}\right], \quad j = 1, \dots, m,$$

and observe that $\bigcup_{j=1}^m I_j = \left[\frac{1}{2}, 1\right]$. For any number $0 < \lambda < 1/(4m)$, set

$$I_j^\lambda := \left[\frac{1}{2} + \frac{j-1}{2m} + \lambda, \frac{1}{2} + \frac{j}{2m} - \lambda\right] \subset I_j, \quad j = 1, \dots, m.$$

Without loss of generality we may assume that $m \in \mathbb{N}$ is large enough so that

$$b_j(\lambda) := \int_{I_j^\lambda} \vartheta(t)\, dt \neq 0 \quad \text{for all } 0 < \lambda < \frac{1}{4m},\ j = 1, \dots, m; \tag{3-5}$$

recall that ϑ has no zeros. Fix a number $0 < \lambda < 1/(4m)$ and set $b_j := b_j(\lambda)$. Pick a constant $\kappa > \max\{|u_0|, |u_1|, |z_1/b_1|, \dots, |z_m/b_m|\}$. Also choose numbers $0 < \tau < \mu < \lambda$, which will be specified later, and consider a continuous map $g_1 : \left[\frac{1}{2}, 1\right] \to \mathfrak{S}_*$ satisfying the following conditions:

(d) $g_1\left(\frac{1}{2}\right) = g_0\left(\frac{1}{2}\right)$ and $g_1(1) = u_1$.

(e) $g_1(t) = z_j/b_j$ for all $t \in I_j^\lambda$.

(f) $|g_1(t)| \leq \kappa$ for all $t \in \left[\frac{1}{2}, 1\right]$.

(g) $|g_1(t)| \leq \tau$ for all $t \in I_j^\tau \setminus I_j^\mu$.

If $\tau > 0$ is chosen sufficiently small, and if μ is close enough to λ, then (e), (f), (g), and (3-5) ensure that:

(h) The image of the map $\left[\frac{1}{2}, 1\right] \ni s \mapsto \mathcal{P}(0) + \int_{1/2}^s g_1(t)\vartheta(t)\, dt$ is close enough to Γ in the Hausdorff distance so that it lies in Ω.

(i) $\left|\mathcal{P}(0) + \int_{1/2}^1 g_1(t)\vartheta(t)\, dt - x\right| < \epsilon$, where $\epsilon > 0$ is the number appearing in (a).

For $\zeta \in U$, let $g^\zeta : I \to \mathfrak{S}_*$ denote the function given by $g^\zeta(t) = \Phi(\zeta, t)$ for $t \in \left[0, \frac{1}{2}\right]$ and $g^\zeta(t) = g_1(t)$ for $t \in \left[\frac{1}{2}, 1\right]$. Properties (a) and (i) guarantee the existence of $\zeta_0 \in U$ such that

$$\int_0^{1/2} g^{\zeta_0}(t)\vartheta(t)\, dt = x - \int_{1/2}^1 g_1(t)\vartheta(t)\, dt,$$

and so $\int_0^1 g^{\zeta_0}(t)\vartheta(t)\, dt = x$. Thus $g := g^{\zeta_0}$ meets (III). By (3-4), (b), and (d), we have that g is continuous and satisfies (I), whereas (c) and (h) ensure (II). □

3B. ***Paths on open Riemann surfaces.*** Let us now state and prove the main result of this section; recall that we are using Notation 2.3.

Lemma 3.4. *Let M be an open Riemann surface and let θ be a holomorphic 1-form vanishing nowhere on M. Let $p_0 \in M$ be a point, $C_1, \dots, C_l$ ($l \in \mathbb{N}$) be a family of oriented Jordan arcs or closed curves in M that only meet at p_0 (i.e., $C_i \cap C_j = \{p_0\}$ for all $i \neq j \in \{1, \dots, l\}$) and such that $C := \bigcup_{i=1}^l C_i$ is Runge in M. Also let $f : C \to \mathfrak{S}_*$ be a continuous map and assume that for each $i \in \{1, \dots, l\}$ there exists a subarc $\widetilde{C}_i \subset C_i$ such that f is nowhere flat on $\widetilde{C}_i$. Then there exist continuous functions $h_{i,1}, \dots, h_{i,N} : C \to \mathbb{C}$, with support on $\widetilde{C}_i$, $i = 1, \dots, l$, and a neighborhood U of $0 \in (\mathbb{C}^N)^l$ such that the* period map *$U \to (\mathbb{C}^n)^l$ whose i-th component $U \to \mathbb{C}^n$ is given by*

$$U \ni \zeta \mapsto \int_{C_i} \phi^1_{\zeta_1^1 h_{1,1}(p)} \circ \cdots \circ \phi^N_{\zeta_N^1 h_{1,N}(p)} \circ \cdots \circ \phi^1_{\zeta_1^l h_{l,1}(p)} \circ \cdots \circ \phi^N_{\zeta_N^l h_{l,N}(p)}(f(p))\theta,$$

see (2-2) *and* (2-3), *where*

$$\zeta = (\zeta^1, \dots, \zeta^l) \in (\mathbb{C}^N)^l, \quad \zeta^i = (\zeta^i_1, \dots, \zeta^i_N) \in \mathbb{C}^N,$$

are holomorphic coordinates, is well-defined and has maximal rank equal to nl at $\zeta = 0$.

Proof. Consider the period map $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_l) : \mathscr{C}(C, \mathbb{C}^n) \to (\mathbb{C}^n)^l$ whose i-th component is defined by

$$\mathscr{C}(C, \mathbb{C}^n) \ni g \mapsto \mathcal{P}_i(g) = \int_{C_i} g\theta, \quad i = 1, \dots, l. \tag{3-6}$$

For each $i = 1, \dots, l$, let $\gamma_i : I = [0, 1] \to C_i$ be a smooth parametrization of C_i such that $\gamma_i(0) = p_0$. If C_i is closed then we choose γ_i with $\gamma_i(1) = p_0$; further, up to changing the orientation of C_i if necessary, we assume that the parametrization γ_i is compatible with the orientation of C_i. Thus,

$$\mathcal{P}_i(g) = \int_0^1 g(\gamma_i(t))\theta(\gamma_i(t), \dot{\gamma}_i(t))\, dt, \quad g \in \mathscr{C}(C, \mathbb{C}^n). \tag{3-7}$$

Let $\varnothing \neq I_i \subset \mathring{I}$ be a closed interval such that $\gamma_i(I_i) \subset \widetilde{C}_i$. Lemma 3.2 applied to I_i, $f \circ \gamma_i$, and $\theta(\gamma_i(\,\cdot\,), \dot{\gamma}_i(\,\cdot\,))$ provides continuous functions $h^i_1, \dots, h^i_N : I \to \mathbb{C}$, supported on I_i, and a neighborhood U_i of $0 \in \mathbb{C}^N$ such that the period map $\mathsf{P}_i : U_i \to \mathbb{C}^n$ given, for any $\zeta^i = (\zeta^i_1, \dots, \zeta^i_{N_i}) \in U_i$, by

$$\mathsf{P}_i(\zeta^i) = \int_0^1 \phi^1_{\zeta^i_1 h^i_1(t)} \circ \cdots \circ \phi^N_{\zeta^i_N h^i_N(t)}(f(\gamma_i(t)))\theta(\gamma_i(t), \dot{\gamma}_i(t))\, dt, \tag{3-8}$$

see (2-2) and (2-3), is well-defined and has maximal rank equal to n at $\zeta^i = 0$. Let U be a ball centered at the origin of $(\mathbb{C}^N)^l$ and contained in $U_1 \times \cdots \times U_l$. For each $i \in \{1, \dots, l\}$ and $j = 1, \dots, N$, we define $h_{i,j} : C \to \mathbb{C}$ by $h_{i,j}(\gamma_i(t)) = h^i_j(t)$ for all $t \in I$, and $h_{i,j}(p) = 0$ for all $p \in C \setminus C_i$. Recall that $h^i_j(0) = 0$ and so $h_{i,j}$ is continuous and $h_{i,j}(p_0) = 0$. Define $\Phi : U \times C \to \mathfrak{S}$ by

$$\Phi(\zeta, p) = \phi^1_{\zeta^1_1 h_{1,1}(p)} \circ \cdots \circ \phi^N_{\zeta^1_N h_{1,N}(p)} \circ \cdots \circ \phi^1_{\zeta^l_1 h_{l,1}(p)} \circ \cdots \circ \phi^N_{\zeta^l_N h_{l,N}(p)}(f(p)),$$

and, up to shrinking U if necessary, assume that $\Phi(U \times C) \subset \mathfrak{S}_*$.

Let $\mathsf{P} : U \to (\mathbb{C}^n)^l$ be the period map whose i-th component $U \to \mathbb{C}^n$, $i = 1, \dots, l$, is given by

$$U \ni \zeta \mapsto \int_{C_i} \Phi(\zeta, \cdot)\theta = \mathsf{P}_i(\zeta^i), \quad \zeta = (\zeta^1, \dots, \zeta^l) \in U;$$

see (3-8) and recall that $h_{i,j}$ vanishes everywhere on $C \setminus C_i$. It follows that P has maximal rank equal to nl at $\zeta = 0$. □

4. Jet-interpolation with approximation

We begin this section with some preparations.

Definition 4.1. Let M be an open Riemann surface. An admissible subset $S = K \cup \Gamma \subset M$ (see Definition 2.1) will be called *simple* if $K \neq \varnothing$, every component of Γ meets K, Γ does not contain closed Jordan curves, and every closed Jordan curve in S meets only one component of K. Further, S will be

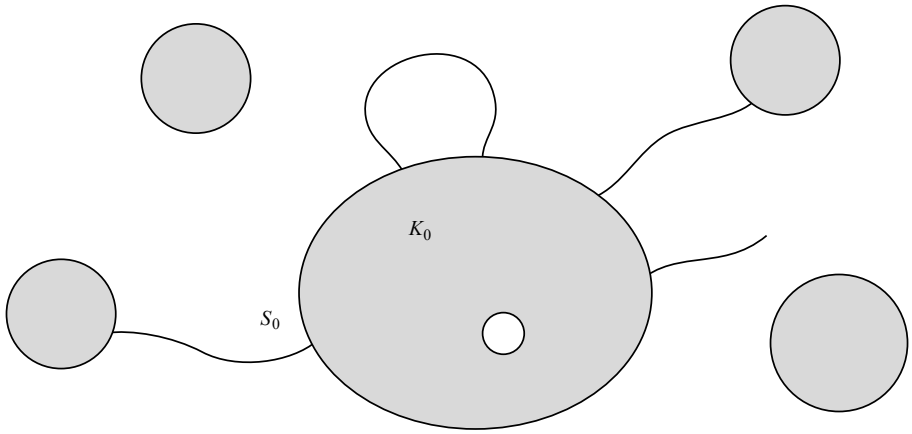

Figure 1. A very simple admissible set.

called *very simple* if it is simple, K has at most one nonsimply connected component K_0, which will be called the *kernel component* of K, and every component of Γ has at least one endpoint in K_0; in this case we denote by S_0 the component of S containing K_0 and call it the *kernel component* of S.

A connected admissible subset $S = K \cup \Gamma$ in an open Riemann surface M is very simple if, and only if, K has $m \in \mathbb{N}$ components $K_0, \dots, K_{m-1}$, where K_i is simply connected for every $i > 0$, and $\Gamma = \Gamma' \cup \Gamma'' \cup \left(\bigcup_{i=1}^{m-1} \gamma_i\right)$ where Γ' consists of components of Γ with both endpoints in K_0, Γ'' consists of components of Γ with an endpoint in K_0 and the other one in $M \setminus K$, and γ_i is a component of Γ connecting K_0 and K_i for all $i = 1, \dots, m-1$. Observe that, in this case, $K_0 \cup \Gamma'$ is a strong deformation retract of S. In general, a very simple admissible subset $S \subset M$ is of the form $S = (K \cup \Gamma) \cup K'$ where $K \cup \Gamma$ is a connected very simple admissible subset and $K' \subset M \setminus (K \cup \Gamma)$ is a (possibly empty) union of finitely many pairwise-disjoint smoothly bounded compact disks. (See Figure 1.)

If $S = K \cup \Gamma \subset M$ is admissible, we denote by $\mathscr{A}(S)$ the space of continuous functions $S \to \mathbb{C}$ which are of class $\mathscr{A}(K)$. Likewise, we define the space $\mathscr{A}(S, Z)$ for maps to any complex manifold Z.

In the remainder of this section we use Notation 2.3.

Lemma 4.2. *Let M be an open Riemann surface and θ be a holomorphic 1-form vanishing nowhere on M. Let $S = K \cup \Gamma \subset M$ be a very simple admissible subset and $L \subset M$ be a smoothly bounded compact domain such that $S \subset \mathring{L}$ and the kernel component S_0 of S is a strong deformation retract of L* (see Definition 4.1). *Denote by $l' \in \mathbb{Z}_+$ the dimension of the first homology group $H_1(S; \mathbb{Z}) = H_1(S_0; \mathbb{Z}) \cong H_1(L; \mathbb{Z})$. Let $K_0, \dots, K_m$, $m \in \mathbb{Z}_+$, denote the components of K contained in S_0, where K_0 is the kernel component of K.*

Let $m' \in \mathbb{Z}_+$, $m' \geq m$, and let $p_0, \dots, p_{m'}$ be distinct points in S such that $p_i \in \mathring{K}_i$ for all $i = 0, \dots, m$ and $p_i \in \mathring{K}_0$ for all $i = m+1, \dots, m'$, and let C_i, $i = 1, \dots, m'$, be pairwise-disjoint oriented Jordan arcs in S with initial point p_0 and final point p_i. Set $l := l' + m'$. Also let C_i, $i = m'+1, \dots, l$, be smooth Jordan curves in S determining a homology basis of S and such that $C_i \cap C_j = \{p_0\}$ for all $i \neq j \in \{1, \dots, l\}$ and $C := \bigcup_{i=1}^{l} C_i$ is Runge in M. (See Figure 2.)

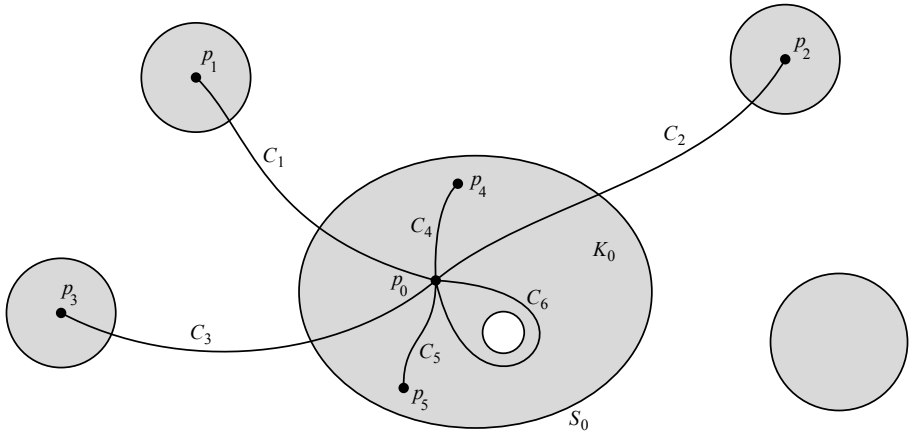

Figure 2. The sets in Lemma 4.2

Given $k \in \mathbb{N}$ and a map $f : S \to \mathfrak{S}_* \subset \mathbb{C}^n$ of class $\mathscr{A}(S)$ which is nonflat on $\mathring{K}_0$ (see Definition 3.1), *the following hold*:

(i) *There exist functions $h_{i,1}, \dots, h_{i,N} : L \to \mathbb{C}$, $i = 1, \dots, l$, of class $\mathscr{A}(L)$ and a neighborhood U of $0 \in (\mathbb{C}^N)^l$ such that*:

(i.1) *$h_{i,j}$ has a zero of multiplicity k at p_r for all $j = 1, \dots, N$ and $r = 1, \dots, m'$.*

(i.2) *Denoting by $\Phi_f : U \times S \to \mathfrak{S}$ the map*

$$\Phi_f(\zeta, p) = \phi^1_{\zeta^1_1 h_{1,1}(p)} \circ \cdots \circ \phi^N_{\zeta^1_N h_{1,N}(p)} \circ \cdots \circ \phi^1_{\zeta^l_1 h_{l,1}(p)} \circ \cdots \circ \phi^N_{\zeta^l_N h_{l,N}(p)}(f(p)),$$

see (2-2) *and* (2-3), *where $\zeta = (\zeta^1, \dots, \zeta^l) \in (\mathbb{C}^N)^l$ and $\zeta^i = (\zeta^i_1, \dots, \zeta^i_N) \in \mathbb{C}^N$, are holomorphic coordinates, the* period map *$U \to (\mathbb{C}^n)^l$ whose i-th component $U \to \mathbb{C}^n$ is given by*

$$U \ni \zeta \mapsto \int_{C_i} \Phi_f(\zeta, \cdot)\theta$$

has maximal rank equal to nl at $\zeta = 0$.

Furthermore, there is a neighborhood V of $g \in \mathscr{A}(S, \mathfrak{S}_*)$ such that the map $V \ni g \mapsto \Phi_g$ can be chosen to depend holomorphically on g.

(ii) *If $\mathfrak{S}_*$ is an Oka manifold, then f may be approximated uniformly on S by maps $\tilde{f} : L \to \mathfrak{S}_*$ of class $\mathscr{A}(L)$ such that*:

(ii.1) *$(\tilde{f} - f)\theta$ is exact on S, equivalently, $\int_{C_r}(\tilde{f} - f)\theta = 0$ for all $r = m' + 1, \dots, l$.*

(ii.2) *$\int_{C_r}(\tilde{f} - f)\theta = 0$ for all $r = 1, \dots, m'$.*

(ii.3) *$\tilde{f} - f$ has a zero of multiplicity k at p_r for all $r = 1, \dots, m'$.*

(ii.4) *No component function of $\tilde{f}$ vanishes everywhere on M.*

Notice that conditions (ii.1) and (ii.2) in the above lemma may be written as a single one in the form

$$\int_{C_r} (\tilde{f} - f)\theta = 0 \quad \text{for all } r = 1, \dots, l.$$

However, we write them separately with the aim of emphasizing that they are useful for different purposes; indeed, (ii.1) concerns the period problem, whereas (ii.2) deals with the problem of interpolation.

Proof. Choose $k \in \mathbb{N}$ and let $f : S \to \mathfrak{S}_*$ be a map of class $\mathscr{A}(S)$ which is nonflat on K_0. Consider the period map $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_l) : \mathscr{C}(C, \mathbb{C}^n) \to (\mathbb{C}^n)^l$ whose i-th component $\mathcal{P}_i : \mathscr{C}(C, \mathbb{C}^n) \to \mathbb{C}^n$ is defined by

$$\mathscr{C}(C, \mathbb{C}^n) \ni g \mapsto \mathcal{P}_i(g) = \int_{C_i} g\theta, \quad i = 1, \dots, l. \tag{4-1}$$

Since S is very simple and f is holomorphic and nonflat on $\mathring{K}_0$, each C_i, $i = 1, \dots, l$, contains a subarc $\widetilde{C}_i \subset \mathring{K}_0 \setminus \{p_0\}$ such that f is nowhere flat on $\widetilde{C}_i$; if $i \in \{m+1, \dots, m'\}$ then we may choose $\widetilde{C}_i \subset C_i \setminus \{p_0, p_i\}$. Thus, Lemma 3.4 applied to the map $f|_C : C \to \mathfrak{S}_*$, the base point p_0, and the curves $C_1, \dots, C_l$ furnishes functions $g_{i,1}, \dots, g_{i,N} : C \to \mathbb{C}$, with support on $\widetilde{C}_i$, $i = 1, \dots, l$, and a neighborhood U of $0 \in (\mathbb{C}^N)^l$ such that the period map $\mathsf{P} : U \to (\mathbb{C}^n)^l$ whose i-th component $\mathsf{P}_i : U \to \mathbb{C}^n$ is given by

$$\mathsf{P}_i(\zeta) := \int_{C_i} \phi^1_{\zeta_1^1 g_{1,1}(p)} \circ \cdots \circ \phi^N_{\zeta_N^1 g_{1,N}(p)} \circ \cdots \circ \phi^1_{\zeta_1^l g_{l,1}(p)} \circ \cdots \circ \phi^N_{\zeta_N^l g_{l,N}(p)} (f(p))\, \theta,$$

see (2-2) and (2-3), is well-defined and has maximal rank equal to nl at $\zeta = 0$. Since $C \subset M$ is Runge, Theorem 2.6 enables us to approximate each $g_{i,j}$ by functions $h_{i,j} \in \mathcal{O}(M) \subset \mathscr{A}(L) \subset \mathscr{A}(S)$ satisfying condition (i.1); recall that every function g_{ij} vanishes on a neighborhood of p_r for all $r = 1, \dots, m'$. Furthermore, if the approximation of $g_{i,j}$ by $h_{i,j}$ is close enough then the period map defined in (i.2) also has maximal rank at $\zeta = 0$. Finally, by varying f locally (keeping the functions $h_{i,j}$ fixed) we obtain a holomorphic family of maps $f \mapsto \Phi_f$ with the desired properties. This proves (i).

Let us now prove assertion (ii), so assume that $\mathfrak{S}_*$ is an Oka manifold. Up to adding to S a smoothly bounded compact disk $D \subset M \setminus S$ and extending f to D as a function of class $\mathscr{A}(S)$ all of whose components are different from the constant 0 on D, we may assume that no component function of f vanishes everywhere on S. Consider the map $\Phi : U \times S \to \mathfrak{S}$ given in (i.2) and, up to shrinking U if necessary, assume that $\Phi(U \times S) \in \mathfrak{S}_*$. Note that the functions $h_{i,j}$ are defined on L but f only on S. By (i), the period map $\mathcal{Q} : U \to (\mathbb{C}^n)^l$ with i-th component

$$\mathcal{Q}_i(\zeta) = \int_{C_i} \Phi(\zeta, \cdot)\theta = \mathcal{P}_i(\Phi(\zeta, \cdot)), \quad \zeta \in U,$$

see (4-1), has maximal rank equal to nl at $\zeta = 0$. It follows that the image by $\mathcal{Q}$ of any open neighborhood of $0 \in U \subset (\mathbb{C}^N)^l$ contains an open ball in $(\mathbb{C}^n)^l$ centered at $\mathcal{Q}(0) = \mathcal{P}(f)$; see (4-1). Since $S \subset M$ is Runge and $\mathfrak{S}_*$ is Oka, Theorem 2.6 allows us to approximate f by holomorphic maps $\hat{f} : M \to \mathfrak{S}_*$ such that

$$\hat{f} - f \text{ has a zero of multiplicity } k \text{ at } p_r \text{ for all } r = 1, \dots, m'. \tag{4-2}$$

Define $\widehat{\Phi} : U \times L \to \mathfrak{S}$ by

$$\widehat{\Phi}(\zeta, p) = \phi^1_{\zeta_1^1 h_{1,1}(p)} \circ \cdots \circ \phi^N_{\zeta_N^1 h_{1,N}(p)} \circ \cdots \circ \phi^1_{\zeta_1^l h_{l,1}(p)} \circ \cdots \circ \phi^N_{\zeta_N^l h_{l,N}(p)}(\hat{f}(p)) \tag{4-3}$$

and, up to shrinking U once again if necessary, assume that $\widehat{\Phi}(U \times L) \subset \mathfrak{S}_*$. Consider now the period map $\widehat{\mathcal{Q}} : U \to (\mathbb{C}^n)^l$ whose i-th component $U \to \mathbb{C}^n$ is given by

$$U \ni \zeta \mapsto \int_{C_i} \widehat{\Phi}(\zeta, \cdot)\,\theta, \quad i = 1, \dots, l.$$

Thus, for any open ball $0 \in W \subset U$, if the approximation of f by $\hat{f}$ is close enough, the range of $\widehat{\mathcal{Q}}(W)$ also contains $\mathcal{P}(f)$. Therefore, there is $\zeta_0 \in W \subset U$ close to $0 \in (\mathbb{C}^N)^l$ such that

$$\tilde{f} := \widehat{\Phi}(\zeta_0, \cdot) : L \to \mathfrak{S}_* \tag{4-4}$$

lies in $\mathscr{A}(L)$ and satisfies (ii.1) and (ii.2); recall that S_0 is a strong deformation retract of L and so the curves C_i, $i = m' + 1, \dots, l$, determine a basis of $H_1(L; \mathbb{Z})$. To finish the proof, Lemma 2.2, (i.1), (4-3), and (4-4) guarantee that $\tilde{f} - \hat{f}$ has a zero of multiplicity (at least) k at p_r for all $r = 1, \dots, m'$. This and (4-2) ensure (ii.3). Finally, if the approximation of f by $\tilde{f}$ on S is close enough, since no component function of f vanishes everywhere on S, no component function of $\tilde{f}$ vanishes everywhere on M, which proves (ii.4) and concludes the proof.

□

We now show the following technical result which will considerably simplify the subsequent proofs.

Proposition 4.3. *Let $n \geq 3$ be an integer and $\mathfrak{S}$ be an irreducible closed conical complex subvariety of $\mathbb{C}^n$ which is not contained in any hyperplane. Let $M = \mathring{M} \cup bM$ be a compact bordered Riemann surface, θ be a holomorphic 1-form vanishing nowhere on M, and $\Lambda \subset \mathring{M}$ be a finite subset. Choose $p_0 \in M \setminus \Lambda$ and, for each $p \in \Lambda$, let $C_p \subset \mathring{M}$ be a smooth Jordan arc with initial point p_0 and final point p such that $C_p \cap C_q = \{p_0\}$ for all $p \neq q \in \Lambda$.*

Let $f : M \to \mathfrak{S}_*$ be a map of class $\mathscr{A}(M)$ which is flat (see Definition 3.1) *and $k \in \mathbb{N}$ be an integer. Then f may be approximated uniformly on M by nonflat maps $\tilde{f} : M \to \mathfrak{S}_*$ of class $\mathscr{A}(M)$ satisfying the following properties*:

(i) *$(\tilde{f} - f)\theta$ is exact on M.*

(ii) *$\int_{C_p} (\tilde{f} - f)\theta = 0$ for all $p \in \Lambda$.*

(iii) *$\tilde{f} - f$ has a zero of multiplicity k at all points $p \in \Lambda$.*

Proof. Without loss of generality we assume that $\Lambda \neq \varnothing$, write $\Lambda = \{p_1, \dots, p_{l'}\}$, and set $C_i := C_{p_i}$, $i = 1, \dots, l'$. Choose $C_{l'+1}, \dots, C_l$ closed Jordan loops in $\mathring{M}$ forming a basis of $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{l-l'}$ such that $C_i \cap C_j = \{p_0\}$ for all $i, j \in \{1, \dots, l\}$, $i \neq j$, and $C := \bigcup_{j=1}^l C_j$ is a Runge subset of M; existence of such loops is ensured by basic topological arguments. Consider smooth parametrizations $\gamma_j : [0, 1] \to C_j$ of the respective curves verifying $\gamma_j(0) = p_0$ and $\gamma_j(1) = p_j$ for $j = 1, \dots, l'$, and $\gamma_j(0) = \gamma_j(1) = p_0$ for $j = l' + 1, \dots, l$.

Since f is flat there exists $z_0 \in \mathfrak{S}_*$ such that $f(M) \subset \mathbb{C}_* z_0$. Observe that $\mathbb{C}z_0$ is a proper complex subvariety of $\mathfrak{S}$. We consider the period map $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_l) : \mathscr{A}(M) \to (\mathbb{C}^n)^l$ defined by

$$\mathscr{A}(M) \ni g \mapsto \mathcal{P}_j(g) := \int_{C_j} g\theta = \int_0^1 g(\gamma_j(t))\theta(\gamma_j(t), \dot{\gamma}_j(t))\, dt, \quad j = 1, \dots, l. \tag{4-5}$$

Note that a map $g \in \mathscr{A}(M)$ meets (i) and (ii) if, and only if, $\mathcal{P}(g) = \mathcal{P}(f)$. So, to finish the proof it suffices to approximate f uniformly on M by nonflat maps $\tilde{f} \in \mathscr{A}(M)$ satisfying the latter condition and also (iii).

Choose a holomorphic vector field V on $\mathbb{C}^n$ which is tangential to $\mathfrak{S}$ along $\mathfrak{S}$, vanishes at 0, and is not everywhere tangential to $\mathbb{C}_* z_0$ along $f(M)$. Let $\phi_s(z)$ denote the flow of V for small values of time $s \in \mathbb{C}$. Choose a nonconstant function $h_1 : M \to \mathbb{C}$ of class $\mathscr{A}(M)$ such that $h_1(p_0) = 0$. Denote by $\mho$ the space of all functions $h : M \to \mathbb{C}$ of class $\mathscr{A}(M)$ having a zero of multiplicity $k \in \mathbb{N}$ at all points $p \in \Lambda$. The following map is well-defined and holomorphic on a small open neighborhood $\mho^*$ of the zero function in $\mho$:

$$\mho^* \ni h \mapsto \mathcal{P}(\phi_{h_1(\cdot)h(\cdot)}(f(\cdot))) \in (\mathbb{C}^n)^l.$$

Each component $\mathcal{P}_j$, $j = 1, \dots, l$, of this map at the point $h = 0$ equals

$$\mathcal{P}_j(\phi_0(f)) = \mathcal{P}_j(f)$$

(recall that V vanishes at $0 \in \mathbb{C}^n$). Since $\mho$ is infinite-dimensional, there is a function $h \in \mho$ arbitrarily close to the function 0 (in particular, we may take $h \in \mho^*$) and nonconstant on M, such that

$$\mathcal{P}(\phi_{h_1(\cdot)h(\cdot)}(f(\cdot))) = \mathcal{P}(f).$$

Set $\tilde{f}(p) = \phi_{h_1(p)h(p)}(f(p))$, $p \in M$. Assume that $\|h\|_{0,M}$ is sufficiently small so that $\tilde{f}$ is well-defined and of class $\mathscr{A}(M)$, $\tilde{f}$ approximates f on M, and $\tilde{f}(p) \in \mathfrak{S}_*$ for all $p \in M$. By the discussion below equation (4-5), $\tilde{f}$ satisfies (i) and (ii). On the other hand, since h has a zero of multiplicity k at every point of Λ and h_1 is not constant, we infer that hh_1 also has a zero of multiplicity (at least) k at all points of Λ. Thus, Lemma 2.2 ensures that $\tilde{f} - f$ satisfies (iii). Finally, since $h_1(p_0) = 0$ and V vanishes at 0, we have $\tilde{f}(p_0) = f(p_0) \in \mathbb{C}_* z_0$, whereas since hh_1 is nonconstant on M and V is not everywhere tangential to $\mathbb{C}_* z_0$ along $f(M)$, there is a point $q \in M$ such that $\tilde{f}(q) \notin \mathbb{C}_* z_0$. This proves that $\tilde{f}$ is nonflat, which concludes the proof. □

The following is the main technical result of this paper.

Theorem 4.4. *Let $n \geq 3$ be an integer and $\mathfrak{S}$ be an irreducible closed conical complex subvariety of $\mathbb{C}^n$ which is not contained in any hyperplane and such that $\mathfrak{S}_* = \mathfrak{S} \setminus \{0\}$ is smooth and an Oka manifold. Let M be an open Riemann surface, θ be a holomorphic 1-form vanishing nowhere on M, $\mathcal{K} \subset M$ be a smoothly bounded Runge compact domain, and $\Lambda \subset M$ be a closed discrete subset. Choose $p_0 \in \mathring{\mathcal{K}} \setminus \Lambda$ and, for each $p \in \Lambda$, let $C_p \subset M$ be an oriented Jordan arc with initial point p_0 and final point p such that $C_p \cap C_q = \{p_0\}$ for all $q \neq p \in \Lambda$ and $C_p \subset \mathcal{K}$ for all $p \in \Lambda \cap \mathcal{K}$. Also, for each $p \in \Lambda$, let $\Omega_p \subset M$ be a compact neighborhood of p in M such that $\Omega_p \cap (\Omega_q \cup C_q) = \varnothing$ for all $q \neq p \in \Lambda$. Set $\Omega := \bigcup_{p \in \Lambda} \Omega_p$.*

Let $f : \mathcal{K} \cup \Omega \to \mathfrak{S}_*$ *be a map of class* $\mathscr{A}(\mathcal{K} \cup \Omega)$, *let* $\mathfrak{q} : H_1(M; \mathbb{Z}) \to \mathbb{C}^n$ *be a group homomorphism, and let* $\mathfrak{Z} : \Lambda \to \mathbb{C}^n$ *be a map, such that*:

(a) $\int_\gamma f\theta = \mathfrak{q}(\gamma)$ *for all closed curves* $\gamma \subset \mathcal{K}$.

(b) $\int_{C_p} f\theta = \mathfrak{Z}(p)$ *for all* $p \in \Lambda \cap \mathcal{K}$.

Then, for any integer $k \in \mathbb{N}$, f *may be approximated uniformly on* $\mathcal{K}$ *by holomorphic maps* $\tilde{f} : M \to \mathfrak{S}_*$ *satisfying the following conditions*:

(A) $\int_\gamma \tilde{f}\theta = \mathfrak{q}(\gamma)$ *for all closed curves* $\gamma \subset M$.

(B) $\tilde{f} - f$ *has a zero of multiplicity* k *at* p *for all* $p \in \Lambda$; *equivalently,* $\tilde{f}$ *and* f *have the same* $(k-1)$*-jet at every point* $p \in \Lambda$.

(C) $\int_{C_p} \tilde{f}\theta = \mathfrak{Z}(p)$ *for all* $p \in \Lambda$.

(D) *No component function of* $\tilde{f}$ *vanishes everywhere on* M.

Proof. Up to slightly enlarging $\mathcal{K}$ if necessary, we may assume without loss of generality that $\Lambda \cap b\mathcal{K} = \varnothing$. Further, up to shrinking the sets Ω_p, we may also assume that, for each $p \in \Lambda$, either $\Omega_p \subset \mathring{\mathcal{K}}$ or $\Omega_p \cap \mathcal{K} = \varnothing$. Finally, by Proposition 4.3 we may assume that $f : \mathcal{K} \to \mathfrak{S}_*$ is nonflat.

Set $M_0 := \mathcal{K}$ and let $\{M_j\}_{j\in\mathbb{N}}$ be a sequence of smoothly bounded Runge compact domains in M such that

$$M_0 \Subset M_1 \Subset M_2 \Subset \cdots \Subset \bigcup_{j\in\mathbb{N}} M_j = M.$$

Assume also that the Euler characteristic $\chi(M_j \setminus \mathring{M}_{j-1})$ of $M_j \setminus \mathring{M}_{j-1}$ is either 0 or -1, and that $\Lambda \cap bM_j = \varnothing$ for all $j \in \mathbb{N}$. Such a sequence can be constructed by basic topological arguments; see, e.g., [Alarcón and López 2013b, Lemma 4.2]. Since Λ is closed and discrete, M_j is compact, and $\Lambda \cap bM_j = \varnothing$ for all $j \in \mathbb{Z}_+$, we know $\Lambda_j := \Lambda \cap M_j = \Lambda \cap \mathring{M}_j$ is either empty or finite. Without loss of generality we assume that $\Lambda_0 \neq \varnothing$ and $\Lambda_j \setminus \Lambda_{j-1} = \Lambda \cap (\mathring{M}_j \setminus M_{j-1}) \neq \varnothing$ for all $j \in \mathbb{N}$, and hence Λ is infinite.

Set $f_0 := f|_{\mathcal{K}}$ and, for each $p \in \Lambda_0 \neq \varnothing$, choose an oriented Jordan arc $C^p \subset \mathring{M}_0$ with initial point p and final point p_0, such that

$$C^p \cap C^q = \{p_0\} \quad \text{for all } p \neq q \in \Lambda_0. \tag{4-6}$$

Such curves trivially exist.

To prove the theorem we shall inductively construct a sequence of maps $f_j : M_j \to \mathfrak{S}_* \subset \mathbb{C}^n$ and a family of oriented Jordan arcs $C^p \subset \mathring{M}_j$, $p \in \Lambda_j \setminus \Lambda_{j-1} \neq \varnothing$, $j \in \mathbb{N}$, with initial point p and final point p_0, satisfying the following properties:

(i_j) $\|f_j - f_{j-1}\|_{0,M_{j-1}} < \epsilon_j$ for a certain constant $\epsilon_j > 0$ which will be specified later.

(ii_j) $\int_\gamma f_j\theta = \mathfrak{q}(\gamma)$ for all closed curves $\gamma \subset M_j$.

(iii_j) $\int_{C^p} f_j\theta = \mathfrak{q}(C_p * C^p) - \mathfrak{Z}(p)$ for all $p \in \Lambda_j$. (Recall that $*$ denotes the product of oriented arcs; see Section 2A.)

(iv_j) $f_j - f$ has a zero of multiplicity k at p for all $p \in \Lambda_j$.

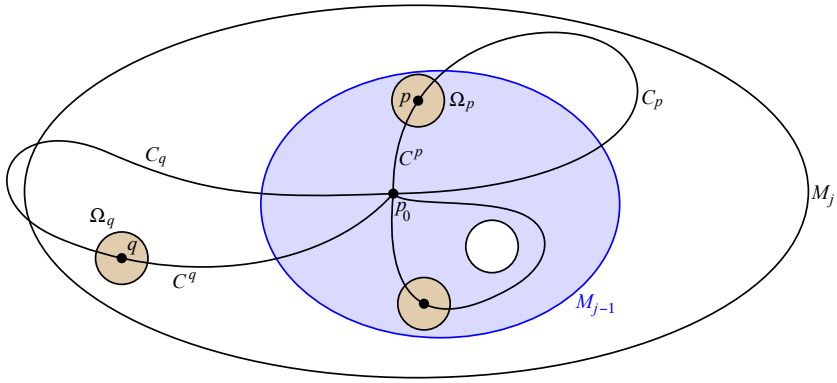

Figure 3. The set $\mathcal{K} \subset M$, the arcs C_p, and the domains Ω_p in Theorem 4.4.

(v_j) $C^p \cap C^q = \{p_0\}$ for all $p \neq q \in \Lambda_j$.

(vi_j) No component function of f_j vanishes everywhere on M_j.

(See Figure 3.) Assume for a moment that we have already constructed such sequence. Then choosing the sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ decreasing to zero fast enough, (i_j) ensures that there is a limit holomorphic map

$$\tilde{f} := \lim_{j\to\infty} f_j : M \to \mathfrak{S}_*$$

which is as close as desired to f uniformly on $\mathcal{K}$, whereas properties (ii_j), (iii_j), (iv_j), (v_j), and (vi_j) guarantee (A), (B), (C), and (D). This would conclude the proof.

The basis of the induction is given by the nonflat map $f_0 = f|_{\mathcal{K}}$ and the already fixed oriented arcs $C^p,\ p \in \Lambda_0$. Condition (i_0) is vacuous, (ii_0) = (a), (iii_0) is implied by (a) and (b), (iv_0) is trivial, and (v_0) = (4-6). For the inductive step, we assume that we already have a map $f_{j-1} : M_{j-1} \to \mathfrak{S}_*$ and arcs $C^p \subset \mathring{M}_{j-1},\ p \in \Lambda_{j-1}$, satisfying properties (ii_{j-1})–(v_{j-1}) for some $j \in \mathbb{N}$, and let us construct a map f_j and arcs C^p for $p \in \Lambda_j \setminus \Lambda_{j-1} = \Lambda \cap (\mathring{M}_j \setminus M_{j-1})$, enjoying conditions ($\mathrm{i}_j$)–($\mathrm{vi}_j$). We distinguish cases depending on the Euler characteristic $\chi(M_j \setminus \mathring{M}_{j-1})$.

Case 1: *the noncritical case. Assume that* $\chi(M_j \setminus \mathring{M}_{j-1}) = 0$. In this case M_{j-1} is a strong deformation retract of M_j. Recall that $\Lambda_j \setminus \Lambda_{j-1}$ is a nonempty finite set. Choose, for each $p \in \Lambda_j \setminus \Lambda_{j-1}$, an oriented Jordan arc $C^p \subset \mathring{M}_j$ with initial point p and final point p_0, so that condition (v_j) holds; such arcs trivially exist. Up to shrinking Ω_p if necessary, we assume without loss of generality that $\Omega_p \subset \mathring{M}_j \setminus M_{j-1}$ for all $p \in \Lambda_j \setminus \Lambda_{j-1}$ and $\Omega_p \cap C^q = \varnothing$ for all $q \in \Lambda_j \setminus \Lambda_{j-1},\ q \neq p$.

Set

$$K := M_{j-1} \cup \Big(\bigcup_{p\in\Lambda_j\setminus\Lambda_{j-1}} \Omega_p \Big), \quad \Gamma := \Big(\bigcup_{p\in\Lambda_j\setminus\Lambda_{j-1}} C^p \Big) \setminus \mathring{K},$$

and, up to slightly modifying the arcs $C^p,\ p \in \Lambda_j \setminus \Lambda_{j-1}$, assume that $S := K \cup \Gamma \subset \mathring{M}_j$ is an admissible subset of M (see Definition 2.1). Notice that S is connected and a strong deformation retract of M_j;

moreover, as admissible set, S is very simple and the kernel component of K is M_{j-1} (see Definition 4.1). Thus, Lemma 3.3 furnishes a map $\varphi : S \to \mathfrak{S}_*$ of class $\mathscr{A}(S)$ such that:

(I) $\varphi = f_{j-1}$ on M_{j-1}.

(II) $\varphi = f$ on $\bigcup_{p\in\Lambda_j\setminus\Lambda_{j-1}} \Omega_p$.

(III) $\int_{C^p} \varphi\theta = \mathfrak{q}(C_p * C^p) - \mathfrak{Z}(p)$ for all $p \in \Lambda_j \setminus \Lambda_{j-1}$.

Now, given $\epsilon_j > 0$, Lemma 4.2(ii) applied to S, M_j, the arcs C^p, $p \in \Lambda_j$, the integer $k \in \mathbb{N}$, and the map φ, provides a map $f_j : M_j \to \mathfrak{S}_*$ of class $\mathscr{A}(M_j)$ satisfying the following conditions:

(IV) $\|f_j - \varphi\|_{0,S} < \epsilon_j$.

(V) $(f_j - \varphi)\theta$ is exact on S.

(VI) $\int_{C^p}(f_j - \varphi)\theta = 0$ for all $p \in \Lambda_j$.

(VII) $f_j - \varphi$ has a zero of multiplicity k at p for all $p \in \Lambda_j$.

(VIII) No component function of f_j vanishes everywhere on M_j.

We claim the map f_j satisfies properties (i_j)–(iv_j); recall that (v_j) is already guaranteed. Indeed, (i_j) follows from (I) and (IV); (ii_j) from (ii_{j-1}), (I), (V), and the fact that M_{j-1} is a strong deformation retract of M_j; (iii_j) from (iii_{j-1}), (I), (III), and (VI); (iv_j) from (iv_{j-1}), (I), (II), and (VII); and (vi_j) = (VIII).

Case 2: *the critical case. Assume that* $\chi(M_j \setminus \mathring{M}_{j-1}) = -1$. Now, the change of topology is described by attaching to M_{j-1} a smooth arc α in $\mathring{M}_j \setminus \mathring{M}_{j-1}$ satisfying M_{j-1} only at its endpoints. Thus, $M_{j-1} \cup \alpha$ is a strong deformation retract of M_j. Further, we may choose α such that $\alpha \cap \Lambda = \varnothing$ and $S := M_{j-1} \cup \alpha$ is an admissible subset of M, which is very simple (see Definition 4.1). Since both endpoints of α lie in bM_{j-1}, there is a closed curve $\beta \subset S$ which contains α as a subarc and is not in the homology of M_{j-1}. Now, Lemma 3.3 furnishes a map $\varphi : S \to \mathfrak{S}_*$ of class $\mathscr{A}(S)$ such that $\varphi = f_{j-1}$ on M_{j-1} and

$$\int_\beta \varphi\theta = \mathfrak{q}(\beta).$$

Choose a smoothly bounded compact domain $L \subset \mathring{M}_j$ such that $S \subset \mathring{L}$, S is a strong deformation retract of L, and $L \cap (\Lambda_j \setminus \Lambda_{j-1}) = \varnothing$. Given $\epsilon_j > 0$, Lemma 4.2(ii) applied to S, L, the arcs C^p, $p \in \Lambda_{j-1}$, the integer $k \in \mathbb{N}$, and the map φ, provides a map $\hat{f} : L \to \mathfrak{S}_*$ of class $\mathscr{A}(L)$ satisfying the following conditions:

(i) $\|\hat{f} - \varphi\|_{0,S} < \frac{1}{2}\epsilon_j$.

(ii) $(\hat{f} - \varphi)\theta$ is exact on S.

(iii) $\int_{C^p}(\hat{f} - \varphi)\theta = 0$ for all $p \in \Lambda_{j-1}$.

(iv) $\hat{f} - \varphi$ has a zero of multiplicity k at p for all $p \in \Lambda_{j-1}$.

Since the Euler characteristic satisfies $\chi(M_j \setminus \mathring{L}) = 0$, this reduces the construction to the noncritical case. This finishes the inductive process and concludes the proof of the theorem. □

To finish this section we prove a Runge–Mergelyan-type theorem with jet-interpolation for holomorphic maps into Oka subvarieties of $\mathbb{C}^n$ in which a component function is preserved provided that it holomorphically extends to the whole source Riemann surface. This will be an important tool to ensure conditions (III) and (IV) in Theorem 1.2 and (I) and (II) in Theorem 1.3.

Lemma 4.5. *Let $n \geq 3$ be an integer and $\mathfrak{S}$ be an irreducible closed conical complex subvariety of $\mathbb{C}^n$ which is not contained in any hyperplane. Assume that $\mathfrak{S}_* = \mathfrak{S} \setminus \{0\}$ is smooth and an Oka manifold, and that $\mathfrak{S} \cap \{z_1 = 1\}$ is also an Oka manifold and the coordinate projection $\pi_1 : \mathfrak{S} \to \mathbb{C}$ onto the z_1-axis admits a local holomorphic section h near $z_1 = 0$ with $h(0) \neq 0$. Let M be an open Riemann surface of finite topology, θ be a holomorphic 1-form vanishing nowhere on M, $S = K \cup \Gamma \subset M$ be a connected very simple admissible Runge subset* (see Definition 4.1) *which is a strong deformation retract of M. Let $\Lambda \subset \mathring{K}_0$ be a finite subset where K_0 is the kernel component of S. Choose $p_0 \in \mathring{K}_0 \setminus \Lambda$ and, for each $p \in \Lambda$, let $C_p \subset \mathring{K}_0$ be an oriented Jordan arc with initial point p_0 and final point p such that $C_p \cap C_q = \{p_0\}$ for all $q \neq p \in \Lambda$.*

Let $f = (f_1, \ldots, f_n) : S \to \mathfrak{S}_*$ be a continuous map, holomorphic on K, such that f_1 extends to a holomorphic map $M \to \mathbb{C}$ which does not vanish everywhere on M. Assume also that $f|_K : K \to \mathfrak{S}_*$ is nonflat. Then, for any integer $k \in \mathbb{Z}_+$, f may be approximated in the $\mathscr{C}^0(S)$-topology by holomorphic maps $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) : M \to \mathfrak{S}_*$ such that:

(i) *$\tilde{f}_1 = f_1$ everywhere on M.*

(ii) *$\tilde{f} - f$ has a zero of multiplicity k at p for all $p \in \Lambda$.*

(iii) *$\int_{C_p} (\tilde{f} - f)\theta = 0$ for all $p \in \Lambda$.*

(iv) *$\int_\gamma (\tilde{f} - f)\theta = 0$ for all closed curves $\gamma \subset S$.*

Proof. We adapt the ideas in [Alarcón and Forstnerič 2014, proof of Theorem 7.7]. Set $\mathfrak{S}' := \mathfrak{S} \cap \{z_1 = 1\}$. By dilations we see that $\mathfrak{S} \setminus \{z_1 = 0\}$ is biholomorphic to $\mathfrak{S}' \times \mathbb{C}_*$ (and hence is Oka), and the projection $\pi_1 : \mathfrak{S}' \to \mathbb{C}$ is a trivial fiber bundle with Oka fiber $\mathfrak{S}'$ except over $0 \in \mathbb{C}$. Write $(f_1, \hat{f}) = (f_1, f_2, \ldots, f_n)$; that is, $\hat{f} := (f_2, \ldots, f_n) : S \to \mathbb{C}^{n-1}$. Since f_1 is holomorphic and nonconstant on M, its zero set $f_1^{-1}(0) = \{a_1, a_2, \ldots\}$ is a closed discrete subset of M. The pullback $f_1^* \pi_1 : E = f^* \mathfrak{S} \to M$ of the projection $\pi_1 : \mathfrak{S} \to \mathbb{C}$ is a trivial holomorphic fiber bundle with fiber $\mathfrak{S}'$ over $M \setminus f_1^{-1}(0)$, but it may be singular over the points $a_j \in f_1^{-1}(0)$. The map $\hat{f} : S \to \mathbb{C}^{n-1}$ satisfies $\hat{f}(x) \in \pi_1^{-1}(f_1(x))$ for all $x \in S$, so $\hat{f}$ corresponds to a section of $E \to M$ over the set S.

Now we need to approximate $\hat{f}$ uniformly by a section $E \to M$ solving the problem of periods and interpolation. (Except for the period and interpolation conditions, a solution is provided by the Oka principle for sections of ramified holomorphic maps with Oka fibers; see [Forstnerič 2003; 2017, §6.14]. We begin by choosing a local holomorphic solution on a small neighborhood of any point $a_j \in M \setminus S$ so that $\hat{f}(a_j) \neq 0$, and we add these neighborhoods to the domain of holomorphicity of $\hat{f}$. Then we need to approximate a holomorphic solution $\hat{f}$ on a smoothly bounded compact set $K \subset M$ by a holomorphic one on a larger domain $L \subset M$ assuming that K is a strong deformation retract of L and $L \setminus K$ does not contain any point a_j. This can be done by applying the Oka principle for maps to the Oka fiber G' of

$\pi : G \to \mathbb{C}$ over $\mathbb{C}_*$. In the critical case we add a smooth Jordan arc α to the domain $K \subset M$ disjoint from the points a_j and such that $K \cup \alpha$ is a strong deformation retract of the next domain. Next, we extend $\hat{f}$ smoothly over α so that the integral $\int_\alpha \hat{f}\theta$ takes the correct value by applying an analogous result of Lemma 3.3 but keeping the first coordinate fixed; this reduces the proof to the noncritical case and concludes the proof of the lemma. □

5. General position, completeness, and properness results

In this section we prove several results that pave the way to the proof of Theorem 1.3 in Section 6. Thus, all the results in this section concern directed holomorphic immersions of open Riemann surfaces into $\mathbb{C}^n$; we point out that the methods of proof easily adapt to give analogous results for conformal minimal immersions into $\mathbb{R}^n$ (see Section 7).

We begin with the following:

Definition 5.1. Let $\mathfrak{S}$ be a closed conical complex subvariety of $\mathbb{C}^n$ $(n \geq 3)$, M be an open Riemann surface, and $S = K \cup \Gamma \subset M$ be an admissible subset (see Definition 2.1). By a *generalized $\mathfrak{S}$-immersion* $S \to \mathbb{C}^n$ we mean a map $F : S \to \mathbb{C}^n$ of class $\mathscr{C}^1(S)$ whose restriction to K is an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(S)$ and the derivative $F'(t)$ with respect to any local real parameter t on Γ belongs to $\mathfrak{S}_*$.

We now prove a Mergelyan-type theorem for generalized $\mathfrak{S}$-immersions which follows from Lemmas 4.2 and 4.5; it will be very useful in the subsequent results.

Proposition 5.2. *Let $\mathfrak{S} \subset \mathbb{C}^n$ be as in Theorem 4.4. Let M be a compact bordered Riemann surface and let $S = K \cup \Gamma \subset \mathring{M}$ be a very simple admissible Runge compact subset such that the kernel component S_0 of S* (see Definition 4.1) *is a strong deformation retract of M. Let $\Lambda \subset \mathring{K}$ be a finite subset and assume that $\Lambda \cap K'$ consists of at most a single point for each component K' of K, $K' \neq K_0$, where K_0 is the kernel component of K. Given an integer $k \in \mathbb{N}$, every generalized $\mathfrak{S}$-immersion $F = (F_1, \ldots, F_n) : S \to \mathbb{C}^n$ which is nonflat on $\mathring{K}_0$ may be approximated in the $\mathscr{C}^1(S)$-topology by $\mathfrak{S}$-immersions $\widetilde{F} = (\widetilde{F}_1, \ldots, \widetilde{F}_n) : M \to \mathbb{C}^n$ such that $\widetilde{F} - F$ has a zero of multiplicity $k \in \mathbb{N}$ at all points $p \in \Lambda$ and $\widetilde{F}$ has no constant component function.*

Furthermore, if $\mathfrak{S} \cap \{z_1 = 1\}$ is an Oka manifold, the coordinate projection $\pi_1 : \mathfrak{S} \to \mathbb{C}$ onto the z_1-axis admits a local holomorphic section h near $z_1 = 0$ with $h(0) \neq 0$, $\Lambda \subset \mathring{K}_0$, and F_1 extends to a nonconstant holomorphic function $M \to \mathbb{C}$, then $\widetilde{F}$ may be chosen with $\widetilde{F}_1 = F_1$.

We point out that an analogous result of the above proposition remains true for arbitrary admissible subsets; we shall not prove the most general statement for simplicity of exposition. Anyway, Proposition 5.2 will suffice for the aim of this paper.

Proof. Let θ be a holomorphic 1-form vanishing nowhere on M. Set $f = dF/\theta : S \to \mathfrak{S}_*$ and observe that f is nonflat on $\mathring{K}_0$ and of class $\mathscr{A}(S)$, and that $f\theta$ is exact on S. Fix a point $p_0 \in \mathring{K}_0 \setminus \Lambda$. If S is not connected then $S \setminus S_0$ consists of finitely many pairwise-disjoint, smoothly bounded compact disks $K_1, \ldots, K_m$. For each $i \in \{1, \ldots, m\}$ choose a smooth Jordan arc $\gamma_i \subset \mathring{M}$ with an endpoint in $(bK_0) \setminus \Gamma$, the other endpoint in bK_i, and otherwise disjoint from S. Choose these arcs so that $S' := S \cup \left(\bigcup_{i=1}^m \gamma_i\right)$ is an

admissible subset of M. It follows that S' is connected, very simple, and a strong deformation retract of M. By Lemma 3.3 we may extend f to a map $f' : S' \to \mathfrak{S}_*$ of class $\mathscr{A}(S')$ such that $F(p_0) + \int_{p_0}^{p} f\theta = F(p)$ for all $p \in S'$. From now on we remove the primes and assume without loss of generality that S is connected.

For each $p \in \Lambda$ choose a smooth Jordan arc $C_p \subset S$ joining p_0 with p such that $C_p \cap C_q = \{p_0\}$ for all $p \neq q \in \Lambda$. By Lemma 4.2(ii) applied to the set $S \subset M$, the map f, the integer $k-1 \geq 0$, and the arcs C_p, $p \in \Lambda$, we may approximate f uniformly on S by a holomorphic map $\tilde{f} : M \to \mathfrak{S}_*$ such that:

(a) $\tilde{f}\theta$ is exact; recall that $f\theta$ is exact on (S and hence on) S_0 and that S_0 is a strong deformation retract of M.

(b) $F(p_0) + \int_{C_p} \tilde{f}\theta = F(p_0) + \int_{C_p} f\theta = F(p)$ for all $p \in \Lambda$.

(c) $\tilde{f} - f$ has a zero of multiplicity $k-1$ at all points $p \in \Lambda$.

(d) No component function of $\tilde{f}$ vanishes everywhere on M.

Then, property (a) ensures that the map $\widetilde{F} : M \to \mathbb{C}^n$ defined by

$$\widetilde{F}(p) := F(p_0) + \int_{p_0}^{p} \tilde{f}\theta, \quad p \in M,$$

is a well-defined $\mathfrak{S}$-immersion and is as close as desired to F in the $\mathscr{C}^1(S)$-topology. Moreover, properties (b) and (c) guarantee that $\widetilde{F} - F$ has a zero of multiplicity k at all points of Λ, whereas (d) ensures that $\widetilde{F}$ has no constant component function. This concludes the first part of the proof.

The second part of the lemma is proved in an analogous way but using Lemma 4.5 instead of Lemma 4.2(ii). Moreover, in order to reduce the proof to the case when S is connected, we need to extend f to a map f' on S' as above such that the first component of f' equals dF_1/θ; this is accomplished by a suitable analogue of Lemma 3.3, we leave the obvious details to the interested reader. □

5A. ***A general position theorem.*** We prove a desingularization result with jet-interpolation for directed immersions of class $\mathscr{A}^1$ on a compact bordered Riemann surface. We use Notation 2.3.

Theorem 5.3. *Let M be a compact bordered Riemann surface and $\Lambda \subset \mathring{M}$ be a finite set. Let $F : M \to \mathbb{C}^n$ ($n \geq 3$) be an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(M)$ such that $F|_\Lambda$ is injective. Then, given $k \in \mathbb{N}$, F may be approximated uniformly on M by a $\mathfrak{S}$-embedding $\widetilde{F} : M \to \mathbb{C}^n$ of class $\mathscr{A}^1(M)$ such that $\widetilde{F} - F$ has a zero of multiplicity k at p for all $p \in \Lambda$.*

Proof. Proposition 4.3 allows us to assume without loss of generality that $F : M \to \mathbb{C}^n$ is nonflat. We assume that M is a smoothly bounded compact domain in an open Riemann surface R. We associate to F the difference map

$$\delta F : M \times M \to \mathbb{C}^n, \quad \delta F(x, y) = F(y) - F(x).$$

Obviously, F is injective if and only if $(\delta F)^{-1}(0) = D_M = \{(x, x) : x \in M\}$.

Since F is an immersion and $F|_\Lambda : \Lambda \to \mathbb{C}^n$ is injective, there is an open neighborhood $U \subset M \times M$ of $D_M \cup (\Lambda \times \Lambda)$ such that $\delta F \neq 0$ everywhere on $\bar{U} \setminus D_M$. To prove the theorem it suffices to find

arbitrarily close to F another $\mathfrak{S}$-immersion $\widetilde{F} : M \to \mathbb{C}^n$ of class $\mathscr{A}^1(M)$ such that $\widetilde{F} - F$ has a zero of multiplicity k at all points of Λ whose difference map $\delta\widetilde{F}|_{M\times M\setminus U}$ is transverse to the origin. Indeed, since $\dim_{\mathbb{C}} M \times M = 2 < n$, this will imply that $\delta\widetilde{F}$ does not assume the value 0 on $M \times M \setminus U$, so $\widetilde{F}(y) \neq \widetilde{F}(x)$ when $(x, y) \in M \times M \setminus U$. On the other hand, if $(x, y) \in \bar{U} \setminus D_M$ then $\widetilde{F}(y) \neq \widetilde{F}(x)$ provided that $\widetilde{F}$ is sufficiently close to F.

To construct such an $\mathfrak{S}$-immersion we will use the standard transversality argument by Abraham [1963]. We need to find a neighborhood $\mathcal{V} \subset \mathbb{C}^N$ of the origin in a complex Euclidean space and a map $H : \mathcal{V} \times M \to \mathbb{C}^n$ of class $\mathscr{A}^1(\mathcal{V} \times M)$ such that:

(a) $H(0, \cdot\,) = F$.

(b) $H - F$ has a zero of multiplicity k at p for all $p \in \Lambda$.

(c) The difference map $\delta H : \mathcal{V} \times M \times M \to \mathbb{C}^n$, defined by

$$\delta H(\zeta, x, y) = H(\zeta, y) - H(\zeta, x), \quad \zeta \in \mathcal{V}, \quad x, y \in M,$$

is a submersive family of maps in the sense that the partial differential

$$d_\zeta \delta H(\zeta, x, y)|_{\zeta=0} : T_0\mathbb{C}^N \cong \mathbb{C}^N \to \mathbb{C}^n$$

is surjective for any $(x, y) \in M \times M \setminus U$.

By openness of the latter condition and compactness of $M \times M \setminus U$ it follows that the partial differential $d_\zeta \delta H$ is surjective for all ζ in a neighborhood $\mathcal{V}' \subset \mathcal{V}$ of the origin in $\mathbb{C}^N$. Hence, the map $\delta H : M \times M \setminus U \to \mathbb{C}^n$ is transverse to any submanifold of $\mathbb{C}^n$, in particular, to the origin $\{0\} \subset \mathbb{C}^n$. The standard argument then shows that for a generic member $H(\zeta, \cdot\,) : M \to \mathbb{C}^n$ of this family, the difference map $\delta H(\zeta, \cdot\,)$ is also transverse to $0 \in \mathbb{C}^n$ on $M \times M \setminus U$. Choosing such a ζ sufficiently close to 0 we then obtain the desired $\mathfrak{S}$-embedding $\widetilde{F} := H(\zeta, \cdot\,)$.

To construct a map H as above we fix a nowhere-vanishing holomorphic 1-form θ on R and write $dF = f\theta$, where $f : M \to \mathfrak{S}_*$ is a map of class $\mathscr{A}^1(M)$. We begin with the following.

Lemma 5.4. *For any point $(p, q) \in M \times M \setminus (D_M \cup (\Lambda \times \Lambda))$ there is a deformation family $H = H^{(p,q)}(\zeta, \cdot\,)$ satisfying conditions* (a) *and* (b) *above, with $\zeta \in \mathbb{C}^n$, such that the differential $d_\zeta \delta H(\zeta, p, q)|_{\zeta=0} : \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism.*

For the proof we adapt the arguments in [Alarcón and Forstnerič 2014, Lemma 6.1] in order to guarantee also the jet-interpolation, i.e., condition (b) of the map H.

Proof. Pick $(p, q) \in M \times M \setminus (D_M \cup (\Lambda \times \Lambda))$. We distinguish cases.

Case 1: *assume that* $\{p, q\} \cap \Lambda \neq \varnothing$. Assume that $p \in \Lambda$ and hence $q \notin \Lambda$; otherwise we reason in a symmetric way. Write $\Lambda = \{p = p_1, \dots, p_{l'}\}$. Pick a point $p_0 \in M \setminus (\Lambda \cup \{q\})$ and choose closed loops $C_j \subset M \setminus \Lambda$, $j = 1, \dots, l''$, forming a basis of $H_1(M, \mathbb{Z}) = \mathbb{Z}^{l''}$, and smooth Jordan arcs $C_{l''+j}$ joining p_0 with p_j, $j = 1, \dots, l'$, such that setting $l := l' + l''$, we have that $C_i \cap C_j = \{p_0\}$ for any $i, j \in \{1, \dots, l\}$ and that $C := \bigcup_{j=1}^{l} C_j$ is a Runge set in M. Also choose another smooth Jordan arc C_q joining p_0 with q and verifying $C \cap C_q = \{p_0\}$. Finally let $\gamma_j : [0, 1] \to C_j$, $j = 1, \dots, l$, and $\gamma : [0, 1] \to C_q$ be smooth

parametrizations of the respective curves verifying $\gamma_j(0) = \gamma_j(1) = p_0$ for $j = 1, \dots, l''$, $\gamma_j(0) = p_0$ and $\gamma_j(1) = p_j$ for $j = l''+1, \dots, l$, and $\gamma(0) = p_0$ and $\gamma(1) = q$.

Since F is nonflat, there exist tangential fields $V_1, \dots, V_n$ on $\mathfrak{S}$, vanishing at 0, and points $x_1, \dots, x_n \in C_q \setminus \{p_0, q\}$ such that, setting $z_i = f(x_i) \in \mathfrak{S}_*$, the vectors $V_1(z_1), \dots, V_n(z_n)$ span $\mathbb{C}^n$. Let $t_i \in (0, 1)$ be such that $\gamma(t_i) = x_i$ and ϕ_t^i be the flow of the vector field V_i for small values of $t \in \mathbb{C}$ in the sense of Notation 2.3. Consider for any $i = 1, \dots, n$ a smooth function $h_i : C \cup C_q \to \mathbb{R}_+ \subset \mathbb{C}$ vanishing on $C \cup \{q\}$; its values on the relative interior of C_q will be specified later. As in the proof of Lemma 4.2, set $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ and consider the map

$$\psi(\zeta, x) = \phi^1_{\zeta_1 h_1(x)} \circ \cdots \circ \phi^n_{\zeta_n h_n(x)}(f(x)) \in \mathfrak{S}, \quad x \in C \cup C_q,$$

which is holomorphic in $\zeta \in \mathbb{C}^n$. Note that $\psi(0, \cdot) = f : M \to \mathfrak{S}_*$ (hence $\psi(\zeta, \cdot)$ does not vanish for ζ in a small neighborhood of the origin) and $\psi(\zeta, x) = f(x)$ for all $x \in C \cup \{q\}$. It follows that

$$\left.\frac{\partial \psi(\zeta, x)}{\partial \zeta_i}\right|_{\zeta=0} = h_i(x) V_i(f(x)), \quad i = 1, \dots, n.$$

We choose h_i with support on a small compact neighborhood of $t_i \in (0, 1)$ in such a way that

$$\int_0^1 h_i(\gamma(t)) V_i(f(\gamma(t))) \theta(\gamma(t), \dot\gamma(t))\, dt \approx V_i(z_i)\, \theta(\gamma(t_i), \dot\gamma(t_i)). \tag{5-1}$$

Assuming that the neighborhoods are sufficiently small, the approximation in (5-1) is close enough so that, since the vectors on the right side above form a basis of $\mathbb{C}^n$, the ones in the left side also do.

Fix a number $\epsilon > 0$. Theorem 2.6 furnishes holomorphic functions $g_i : M \to \mathbb{C}$ such that

$$g_i \text{ has a zero of multiplicity } k-1 \text{ at all points of } \Lambda \tag{5-2}$$

and

$$\sup_{C \cup C_q} |g_i - h_i| < \epsilon, \quad i = 1, \dots, n.$$

Following the arguments in the proof of Lemma 4.2, we define holomorphic maps

$$\Psi(\zeta, x, z) = \phi^1_{\zeta_1 g_1(x)} \circ \cdots \circ \phi^n_{\zeta_n g_n(x)}(z) \in \mathfrak{S},$$
$$\Psi_f(\zeta, x) = \Psi(\zeta, x, f(x)) \in \mathfrak{S}_*,$$

where $x \in M$, $z \in \mathfrak{S}$, and ζ belongs to a sufficiently small neighborhood of the origin in $\mathbb{C}^n$. Observe that $\Psi_f(0, \cdot) = f$. In view of (5-1), if $\epsilon > 0$ is small enough then we have that the vectors

$$\left.\frac{\partial}{\partial \zeta_i}\right|_{\zeta=0} \int_0^1 \Psi_f(\zeta, \gamma(t)) \theta(\gamma(t), \dot\gamma(t))\, dt = \int_0^1 g_i(\gamma(t)) V_i(f(\gamma(t))) \theta(\gamma(t), \dot\gamma(t))\, dt, \tag{5-3}$$

$i = 1, \dots, n$, are close enough to $V_i(z_i)\theta(\gamma(t_i), \dot\gamma(t_i))$ so that they also form a basis of $\mathbb{C}^n$.

To finish the proof it remains to perturb Ψ_f in order to solve the period problem and ensure the jet-interpolation at the points of Λ. From the Taylor expansion of the flow of a vector field it follows that

$$\Psi_f(\zeta, x) = f(x) + \sum_{i=1}^n \zeta_i g_i(x) V_i(f(x)) + O(|\zeta|^2).$$

Since $|g_i| < \epsilon$ on C (recall that $h_i = 0$ on C), the integral of Ψ_f over the curves $C_1, \dots, C_l$ can be estimated by

$$\left|\int_{C_j} (\Psi_f(\zeta, \cdot) - f)\theta\right| = \left|\int_{C_j} \Psi_f(\zeta, \cdot)\theta\right| \le \eta_0 \epsilon |\zeta|, \quad j = 1, \dots, l'' \tag{5-4}$$

(recall that $\int_{C_j} f\theta = \int_{C_j} dF = 0$ for all $j = 1, \dots, l''$, since these curves are closed),

$$\left|\int_{C_j} (\Psi_f(\zeta, \cdot) - f)\theta\right| = \left|F(p_0) - \left(F(p_j) - \int_{C_j} \Psi_f(\zeta, \cdot)\theta\right)\right| \le \eta_0 \epsilon |\zeta|, \quad j = l'' + 1, \dots, l, \tag{5-5}$$

for some constant $\eta_0 > 0$ and sufficiently small $\zeta \in \mathbb{C}^n$. Furthermore, (5-2) guarantees that

$$\Psi_f(\zeta, \cdot) - f \text{ has a zero of multiplicity } k-1 \text{ at all points of } \Lambda \tag{5-6}$$

for ζ in a small neighborhood of the origin (see Lemma 2.2).

Now, Lemma 4.2(i) furnishes holomorphic maps $\Phi(\tilde{\zeta}, x, z)$ and $\Phi_f(\tilde{\zeta}, x) = \Phi(\tilde{\zeta}, x, f(x))$ with the parameter $\tilde{\zeta}$ in a small neighborhood of $0 \in \mathbb{C}^{\widetilde{N}}$ for some large $\widetilde{N} \in \mathbb{N}$ and $x \in M$ such that $\Phi(0, x, z) = z \in \mathfrak{S}$ for all $x \in M$ and

$$\Phi_f(0, \cdot) = \Phi_{\Psi_f(0,\cdot)}(0, \cdot) = f, \tag{5-7}$$

and the differential of the associated period map $\tilde{\zeta} \mapsto \mathcal{P}(\Phi_f(\tilde{\zeta}, \cdot)) \in \mathbb{C}^{ln}$, see (4-1), at the point $\tilde{\zeta} = 0$ has maximal rank equal to ln. The same is true if we allow that f vary locally near the given initial map. Thus, replacing f by $\Psi_f(\zeta, \cdot)$ and considering the map

$$\mathbb{C}^{\widetilde{N}} \times \mathbb{C}^n \times M \ni (\tilde{\zeta}, \zeta, x) \mapsto \Phi(\tilde{\zeta}, x, \Psi_f(\zeta, x)) \in \mathfrak{S}_*$$

defined for $x \in M$ and $(\tilde{\zeta}, \zeta)$ in some sufficiently small neighborhood of $0 \in \mathbb{C}^{\widetilde{N}} \times \mathbb{C}^n$, the implicit function theorem provides a holomorphic map $\tilde{\zeta} = \rho(\zeta)$ near $\zeta = 0 \in \mathbb{C}^n$ with $\rho(0) = 0 \in \mathbb{C}^{\widetilde{N}}$ such that the map defined by $\Phi(\rho(\zeta), x, \Psi_f(\zeta, x))$ satisfies:

(i) $\mathcal{P}(\Phi(\rho(\zeta), \cdot, \Psi_f(\zeta, \cdot))) = \mathcal{P}(\Phi(0, \cdot, \Psi_f(0, \cdot))) = \mathcal{P}(\Psi_f(0, \cdot)) = \mathcal{P}(f)$.

(ii) $\Phi(\rho(\zeta), \cdot, \Psi_f(\zeta, \cdot)) - \Psi_f(\zeta, \cdot)$ has a zero of multiplicity $k-1$ at all points of Λ.

Condition (ii) together with (5-6) ensure that

$$\Phi(\rho(\zeta), \cdot, \Psi_f(\zeta, \cdot)) - f \text{ has a zero of multiplicity } k-1 \text{ at all points of } \Lambda \tag{5-8}$$

for all ζ in a small neighborhood of $0 \in \mathbb{C}^n$. Obviously the map $\rho = (\rho_1, \dots, \rho_n)$ also depends on f. It follows that the integral

$$H_F(\zeta, x) = F(p_0) + \int_{p_0}^{x} \Phi(\rho(\zeta), \cdot, \Psi_f(\zeta, \cdot))\theta \tag{5-9}$$

is independent of the choice of the arc from p_0 to $x \in M$. Moreover,

$$H_F(0, \cdot) = F, \tag{5-10}$$

see (5-7), and $H_F(\zeta,\cdot)$ is an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(M)$ for every $\zeta\in\mathbb{C}^n$ sufficiently close to zero such that

$$H_F(\zeta,\cdot)=F \quad \text{on } \Lambda; \tag{5-11}$$

see (i). In addition, from (5-4) and (5-5) we have

$$|\rho(\zeta)|<\eta_1\epsilon|\zeta|$$

for some $\eta_1>0$. If we call $\widetilde{V}_j$ the vector fields and $\tilde{g}_j$ the functions involved in the construction of the map Φ (see Lemma 4.2), the above estimate gives

$$|\Phi(\rho(\zeta),x,\Psi_f(\zeta,x))-\Psi_f(\zeta,x)|=\Big|\sum\rho_j(\zeta)\tilde{g}_j(x)\widetilde{V}_j(\Psi_f(\zeta,x))+O(|\zeta|^2)\Big|<\eta_2\epsilon|\zeta|$$

for some $\eta_2>0$ and all $x\in M$ and all ζ near the origin in $\mathbb{C}^n$. Clearly, applying this estimate to the arc C_q we have

$$\left|\int_0^1\Phi(\rho(\zeta),\gamma(t),\Psi_f(\zeta,\gamma(t)))\theta(\gamma(t),\dot\gamma(t))-\int_0^1\Psi_f(\zeta,\gamma(t))\theta(\gamma(t),\dot\gamma(t))\right|<\eta_3\epsilon|\zeta|$$

for some $\eta_3>0$. Finally, choosing $\epsilon>0$ small enough, the derivatives

$$\frac{\partial}{\partial\zeta_i}\bigg|_{\zeta=0}\int_0^1\Phi(\rho(\zeta),\gamma(t),\Psi_f(\zeta,\gamma(t)))\,\theta(\gamma(t),\dot\gamma(t))\in\mathbb{C}^n,\quad i=1,\dots,n,$$

are so close to the vectors (5-3) that they also form a basis of $\mathbb{C}^n$. From the definition of H_F, (5-9), and (5-11), we have

$$\begin{aligned}\delta H_F(\zeta,p,q)&=H_F(\zeta,q)-H_F(\zeta,p)\\&=H_F(\zeta,q)-F(p)\\&=F(p_0)-F(p)+\int_0^1\Phi(\rho(\zeta),\gamma(t),\Psi_f(\zeta,\gamma(t)))\theta(\gamma(t),\dot\gamma(t)),\end{aligned}$$

and hence the partial differential

$$\frac{\partial}{\partial\zeta}\bigg|_{\zeta=0}\delta H(\zeta,p,q):\mathbb{C}^n\to\mathbb{C}^n$$

is an isomorphism. This, (5-10), (ii), and (5-11) show that H satisfies the conclusion of the lemma.

Case 2: *assume that* $\{p,q\}\cap\Lambda=\varnothing$. In this case setting $\Lambda':=\Lambda\cup\{p\}$ reduces the proof to Case 1. This proves the lemma. □

The family H_F depending on F given in (5-9) is holomorphically dependent also on F on a neighborhood of a given initial $\mathfrak{S}$-immersion F_0. In particular, if $F(\xi,\cdot):M\to\mathbb{C}^n$ is a family of holomorphic $\mathfrak{S}$-immersions depending holomorphically on $\xi\in\mathbb{C}$ such that $F(\xi,\cdot)-F$ has a zero of multiplicity k at all points $p\in\Lambda$ for any ξ, then $H_{F(\xi,\cdot)}(\zeta,\cdot)$ depends holomorphically on (ζ,ξ). This allows us to compose any finite number of such deformation families by an inductive process. For the case of two families suppose that $H=H_F(\zeta,\cdot)$ and $G=G_F(\xi,\cdot)$ are deformation families with $H_F(0,\cdot)=G_F(0,\cdot)=F$

and such that $H_F(\zeta,\cdot)-F$ and $G_F(\xi,\cdot)-F$ have a zero of multiplicity $k\in\mathbb{N}$ at all points of Λ for all ζ and ξ respectively. Then, we define the composed deformation family by

$$(H\#G)_F(\zeta,\xi,x):=G_{H_F(\zeta,\cdot)}(\xi,x),\quad x\in M.$$

Obviously, we have $(H\#G)_F(0,\xi,\cdot)=G_F(\xi,\cdot)$ and $(H\#G)_F(\zeta,0,\cdot)=H_F(\zeta,\cdot)$, and $H\#G-F$ has a zero of multiplicity k at p for all $p\in\Lambda$. The operation $\#$ is associative but not commutative.

To finish the proof of Theorem 5.3, Lemma 5.4 gives a finite open covering $\{U_i\}_{i=1}^m$ of the compact set $M\times M\setminus U$ and deformation families $H^i=H^i(\zeta^i,\cdot):M\to\mathbb{C}^n$, with $H^i(0,\cdot)=F$, where $\zeta^i=(\zeta^i_1,\dots,\zeta^i_{\eta_i})\in\Omega_i\subset\mathbb{C}^{\eta_i}$ for positive integers $\eta_i\in\mathbb{N}$ and $i=1,\dots,m$. It follows that the difference map $\delta H^i(\zeta^i,p,q)$ is submersive at $\zeta^i=0$ for all $p,q\in U_i$. By taking $\zeta=(\zeta^1,\dots,\zeta^m)\in\mathbb{C}^N$, with $N=\sum_{i=1}^m\eta_i$, and setting

$$H(\zeta,x):=(H^1\#H^2\#\cdots\#H^m)(\zeta^1,\dots,\zeta^m,x),$$

we obtain a deformation family such that $H(0,\cdot)=F$, $H(q,\cdot)-F$ has a zero of multiplicity k at p for all $p\in\Lambda$, and δH is submersive everywhere on $M\times M\setminus U$ for all $\zeta\in\mathbb{C}^N$ sufficiently close to the origin. □

5B. ***A completeness lemma.*** We develop an intrinsic-extrinsic version of the arguments from [Jorge and Xavier 1980] in order to prove the following

Lemma 5.5. *Let $\mathfrak{S}\subset\mathbb{C}^n$ $(n\geq 3)$ be as in Lemma 4.5. Let M be a compact bordered Riemann surface and $K\subset\mathring{M}$ be a smoothly bounded compact domain which is Runge and a strong deformation retract of M. Also let $\Lambda\subset\mathring{K}$ be a finite subset and $p_0\in\mathring{K}\setminus\Lambda$ be a point. Then, given an integer $k\in\mathbb{N}$ and a positive number $\tau>0$, every $\mathfrak{S}$-immersion $F:K\to\mathbb{C}^n$ of class $\mathscr{A}^1(K)$ may be approximated in the $\mathscr{C}^1(K)$-topology by $\mathfrak{S}$-immersions $\widetilde{F}:M\to\mathbb{C}^n$ of class $\mathscr{A}(M)$ satisfying the following conditions*:

(I) *$\widetilde{F}-F$ has a zero of multiplicity k at all points $p\in\Lambda$.*

(II) $\operatorname{dist}_{\widetilde{F}}(p_0,bM)>\tau$.

Proof. Without loss of generality we assume that M is a smoothly bounded compact domain in an open Riemann surface $\widetilde{M}$. By Proposition 5.2 we may assume that F is holomorphic on M and that its first component has no critical points on M. Fix a holomorphic 1-form θ vanishing nowhere on $\widetilde{M}$ and set $dF=f\theta$, where $f=(f_1,\dots,f_n):M\to\mathfrak{S}_*$ is a holomorphic map.

Since K is a strong deformation retract of M, we know $\mathring{M}\setminus K$ consists of a finite family of pairwise-disjoint open annuli. Thus, there exists a finite family of pairwise-disjoint, smoothly bounded, compact disks $L_1,\dots,L_m$ in $\mathring{M}\setminus K$ satisfying the following property: if $\alpha\subset M\setminus\bigcup_{j=1}^m L_j$ is a arc connecting p_0 with bM then

$$\int_\alpha|f_1\theta|>\tau. \tag{5-12}$$

Recall that $f_1\neq 0$. (Such disks can be found as pieces of labyrinths of Jorge–Xavier-type [1980] on the annuli forming $\mathring{M}\setminus K$; see [Alarcón, Fernández and López 2013, proof of Lemma 4.1] for a detailed explanation). Set $L:=\bigcup_{j=1}^m L_j$.

For each $j = 1, \dots, m$, choose a Jordan arc $\gamma_j \subset \mathring{M}$ with an endpoint in K, the other endpoint in L_j, and otherwise disjoint from $K \cup L$, such that $\gamma_i \cap \gamma_j = \varnothing$ for all $i \neq j \in \{1, \dots, m\}$ and the set

$$S := K \cup L \cup \Gamma,$$

where $\Gamma := \bigcup_{j=1}^m \gamma_j$, is an admissible subset of M. It follows that S is a connected very simple admissible subset of M with kernel component K (see Definition 4.1) such that K is a deformation retract of S (hence of M). Take a map $h = (h_1, \dots, h_n) : S \to \mathfrak{S}_*$ of class $\mathscr{A}(S)$ satisfying the following conditions:

(a) $h_1 = f_1|_S$.

(b) $h|_K = f|_K$.

(c) $\left|\int_\alpha h\theta\right| > \tau$ for all arcs $\alpha \subset S$ with initial point p_0 and final point in L.

Existence of such a map is clear; we may for instance choose h close to $0 \in \mathbb{C}^n$ on each component of L and such that $\left|\int_{\gamma_j} h\theta\right|$ is very large for every component γ_j of Γ. Also choose for each $p \in \Lambda$ a smooth Jordan arc $C_p \subset \mathring{K}$ with initial point p_0 and final one p, and assume that $C_p \cap C_q = \{p_0\}$ for all $p \neq q \in \Lambda$. Then, Lemma 4.5 provides a holomorphic map $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n) : M \to \mathfrak{S}_*$ such that:

(i) $\tilde{f}$ approximates h on S.

(ii) $\tilde{f}_1 = f_1$ everywhere on M.

(iii) $\tilde{f} - h$ has a zero of multiplicity $k-1$ at p for all $p \in \Lambda$.

(iv) $\int_{C_p} (\tilde{f} - h)\theta = 0$ for all $p \in \Lambda$.

(v) $\int_\gamma (\tilde{f} - h)\theta = 0$ for all closed curves $\gamma \subset S$.

Since $f\theta = dF$ is exact, properties (b) and (v) and the fact that K is a strong deformation retract of M guarantee that $\tilde{f}\theta$ is exact on M as well. Therefore, the map $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_n) : M \to \mathbb{C}^n$ defined by

$$\widetilde{F}(p) := F(p_0) + \int_{p_0}^p \tilde{f}\theta, \quad p \in M,$$

is well-defined and an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(M)$. We claim that if the approximation in (i) is close enough then $\widetilde{F}$ satisfies the conclusion of the lemma. Indeed, properties (i) and (b) guarantee that $\widetilde{F}$ approximates F as close as desired in the $\mathscr{C}^1(K)$-topology. On the other hand, (iii), (iv), and (b) ensure that $\widetilde{F} - F$ has a zero of multiplicity k at all points of Λ, which proves (I). Finally, in order to check condition (II), let $\alpha \subset M$ be an arc with initial point p_0 and final one in bM. Assume first that $\alpha \cap L \neq \varnothing$ and let $\tilde{\alpha} \subset \alpha$ be a subarc with initial point p_0 and final point q for some $q \in L$. Then we have

$$\text{length}(\widetilde{F}(\alpha)) > \text{length}(\widetilde{F}(\tilde{\alpha})) \geq |\widetilde{F}(q) - \widetilde{F}(p_0)| = \left|\int_{p_0}^q \tilde{f}\theta\right| \overset{\text{(i)}}{\approx} \left|\int_{p_0}^q h\theta\right| \overset{\text{(c)}}{>} \tau.$$

Assume that, on the contrary, $\alpha \cap L = \varnothing$. In this case,

$$\text{length}(\widetilde{F}(\alpha)) = \int_\alpha |\tilde{f}\theta| \geq \int_\alpha |\tilde{f}_1\theta| \overset{\text{(ii)}}{=} \int_\alpha |f_1\theta| \overset{\text{(5-12)}}{>} \tau.$$

This proves (II) and completes the proof. □

5C. ***A properness lemma.*** Recall that given a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ we define $|x|_\infty = \max\{|x_1|, \dots, |x_n|\}$; see Section 2 for notation.

Lemma 5.6. *Let $n \geq 3$ be an integer and $\mathfrak{S}$ be an irreducible closed conical complex subvariety of $\mathbb{C}^n$ which is not contained in any hyperplane. Assume that $\mathfrak{S}_* = \mathfrak{S} \setminus \{0\}$ is smooth and an Oka manifold, and that $\mathfrak{S} \cap \{z_j = 1\}$ is an Oka manifold and the coordinate projection $\pi_j : \mathfrak{S} \to \mathbb{C}$ onto the z_j-axis admits a local holomorphic section h_j near $z_j = 0$ with $h_j(0) \neq 0$ for all $j = 1, \dots, n$. Let M be a compact bordered Riemann surface and $K \subset \mathring{M}$ be a smoothly bounded compact domain which is Runge and a strong deformation retract of M. Also let $\Lambda \subset \mathring{K}$ be a finite subset, $F : K \to \mathbb{C}^n$ be an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(K)$, let $\tau > \rho > 0$ be numbers, and assume that*

$$|F(p)|_\infty > \rho \quad \text{for all } p \in bK. \tag{5-13}$$

Then, given an integer $k \in \mathbb{N}$, F may be approximated in the $\mathscr{C}^1(K)$-topology by $\mathfrak{S}$-immersions $\widetilde{F} : M \to \mathbb{C}^n$ of class $\mathscr{A}^1(M)$ satisfying the following conditions:

(I) *$\widetilde{F} - F$ has a zero of multiplicity k at all points $p \in \Lambda$.*

(II) *$|\widetilde{F}(p)|_\infty > \rho$ for all $p \in M \setminus \mathring{K}$.*

(III) *$|\widetilde{F}(p)|_\infty > \tau$ for all $p \in bM$.*

Proof. Without loss of generality we assume that M is a smoothly bounded compact domain in an open Riemann surface $\widetilde{M}$. By Proposition 5.2 we may assume that $F = (F_1, \dots, F_n)$ is holomorphic on $\widetilde{M}$. Since K is a strong deformation retract of M, we have that $M \setminus \mathring{K}$ consists of finitely many pairwise-disjoint compact annuli. For simplicity of exposition we assume that $\mathcal{A} := M \setminus \mathring{K}$ is connected (and hence a single annulus); the same proof applies in general by working separately on each connected component of $M \setminus \mathring{K}$. We denote by α the boundary component of $\mathcal{A}$ contained in bK and by β the one contained in bM; both α and β are smooth Jordan curves.

From inequality (5-13) there exist an integer $l \geq 3$, subsets $I_1, \dots, I_n$ of $\mathbb{Z}_l$ (where $\mathbb{Z}_l = \{0, 1, \dots, l-1\}$ denotes the additive cyclic group of integers modulus l), and a family of compact connected subarcs $\{\alpha_j : j \in \mathbb{Z}_l\}$ of bK, satisfying the following properties:

(a1) $\bigcup_{j \in \mathbb{Z}_l} \alpha_j = \alpha$.

(a2) α_j and α_{j+1} have a common endpoint p_j and are otherwise disjoint.

(a3) $\bigcup_{a=1}^n I_a = \mathbb{Z}_l$ and $I_a \cap I_b = \varnothing$ for all $a \neq b \in \{1, \dots, n\}$.

(a4) If $j \in I_a$ then $|F_a(p)| > \rho$ for all $p \in \alpha_j$, $a = 1, \dots, n$.

(Possibly $I_a = \varnothing$ for some $a \in \{1, \dots, n\}$.)

Consider for each $j \in \mathbb{Z}_l$ a smooth embedded arc $\gamma_j \subset \mathcal{A}$ with the following properties:

- γ_j joins $\alpha \subset bK$ with $\beta \subset bM$ and intersects them transversely.
- $\gamma_j \cap \alpha = \{p_j\}$.
- $\gamma_j \cap \beta$ consists of a single point, namely, q_j.
- The arcs γ_j, $j \in \mathbb{Z}_l$, are pairwise disjoint.

Consider the admissible set

$$S := K \cup \left(\bigcup_{j \in \mathbb{Z}_l} \gamma_j \right) \subset M$$

and fix a point $x_0 \in \mathring{K} \setminus \Lambda$. Let $z = (z_1, \dots, z_n)$ be the coordinates on $\mathbb{C}^n$ and recall that $\pi_a : \mathbb{C}^n \to \mathbb{C}$ is the a-th coordinate projection $\pi_a(z) = z_a$ for all $a = 1, \dots, n$. Let θ be a holomorphic 1-form vanishing nowhere on $\widetilde{M}$, and let $f : S \to \mathfrak{S}_*$ be a map of class $\mathscr{A}(S)$ such that $f = dF/\theta$ on K and the map $\widetilde{G} : S \to \mathbb{C}^n$ given by

$$\widetilde{G}(p) = F(p_0) + \int_{x_0}^{p} f\theta,$$

which is well-defined since K is a deformation retract of S, satisfies the following conditions:

(b1) $\widetilde{G} = F$ on K and on a neighborhood of p_j for all $j \in \mathbb{Z}_l$.

(b2) If $j \in I_a$ then $|\pi_a(\widetilde{G}(z))| > \rho$ for all $z \in \gamma_{j-1} \cup \gamma_j,\ a = 1, \dots, n$.

(b3) If $j \in I_a$ then $|\pi_a(\widetilde{G}(q_{j-1}))| > \tau$ and $|\pi_a(\widetilde{G}(q_j))| > \tau,\ a = 1, \dots, n$.

Existence of such an f is guaranteed by (a4). Theorem 4.4 provides a map $g : M \to \mathfrak{S}_*$ of class $\mathscr{A}(M)$ such that $g\theta$ is exact on M, and the $\mathfrak{S}$-immersion $G = (G_1, \dots, G_n) : M \to \mathbb{C}^n$ of class $\mathscr{A}^1(M)$ given by $G(p) = F(x_0) + \int_{x_0}^{p} g\theta$ enjoys the following properties:

(c1) G approximates $\widetilde{G}$ in the $\mathscr{C}^1(K)$-topology.

(c2) $G - \widetilde{G}$ has a zero of multiplicity k at all points of Λ.

(c3) If $j \in I_a$ then $|G_a(p)| > \rho$ for all $p \in \gamma_{j-1} \cup \alpha_j \cup \gamma_j,\ a = 1, \dots, n$.

(c4) If $j \in I_a$ then $|G_a(p)| > \tau$ for $p \in \{q_{j-1}, q_j\},\ a = 1, \dots, n$.

Property (c3) follows from (a4) and (b2), whereas (c4) follows from (b3), provided that the approximation of f by g is close enough.

For each $j \in \mathbb{Z}_l$ let $\beta_j \subset \beta$ denote the subarc of β with endpoints q_{j-1} and q_j which does not contain q_i for any $i \in \mathbb{Z}_l \setminus \{j-1, j\}$. It is clear that

$$\bigcup_{j \in \mathbb{Z}_l} \beta_j = \beta. \tag{5-14}$$

Also denote by $D_j \subset \mathcal{A}$ the closed disk bounded by the arcs γ_{j-1}, α_j, γ_j, and β_j; see Figure 4. It follows that

$$\mathcal{A} = \bigcup_{j \in \mathbb{Z}_l} D_j. \tag{5-15}$$

Call $H_0 := G = (H_{0,1}, \dots, H_{0,n})$ and $I_0 := \varnothing$. We shall construct a sequence of $\mathfrak{S}$-immersions $H_b = (H_{b,1}, \dots, H_{b,n}) : M \to \mathbb{C}^n,\ b = 1, \dots, n$, of class $\mathscr{A}^1(M)$ satisfying the following requirements for all $b \in \{1, \dots, n\}$:

(d1$_b$) H_b approximates H_{b-1} in the $\mathscr{C}^1$-topology on $M \setminus \left(\bigcup_{j \in I_b} \mathring{D}_j\right)$.

(d2$_b$) $H_b - H_{b-1}$ has a zero of multiplicity k at all points of Λ.

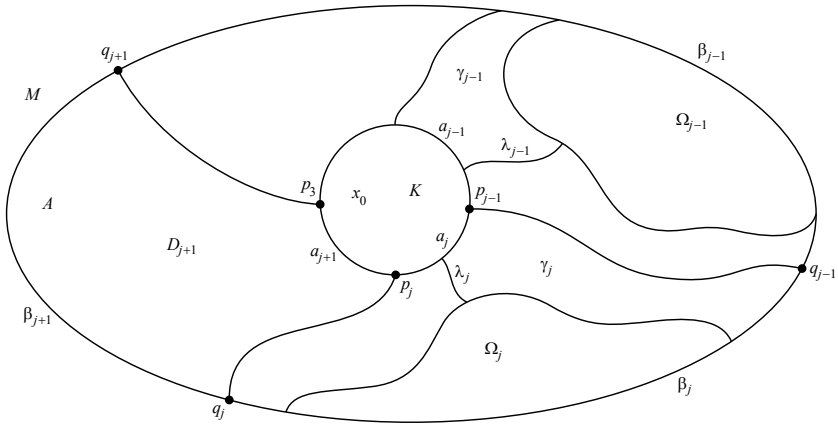

Figure 4. The annulus $\mathcal{A}$.

(d3$_b$) If $j \in \bigcup_{i=1}^{b} I_i$ then $|H_b(p)|_\infty > \rho$ for all $p \in D_j$.

(d4$_b$) If $j \in \bigcup_{i=1}^{b} I_i$ then $|H_b(p)|_\infty > \tau$ for all $p \in \beta_j$.

(d5$_b$) If $j \in I_a$ then $|H_{b,a}(p)| > \rho$ for all $p \in \gamma_{j-1} \cup \alpha_j \cup \gamma_j$, $a = 1, \dots, n$.

(d6$_b$) If $j \in I_a$ then $|H_{b,a}(p)| > \tau$ for $p \in \{q_{j-1}, q_j\}$, $a = 1, \dots, n$.

We claim that the $\mathfrak{S}$-immersion $\widetilde{F} := H_n : M \to \mathbb{C}^n$ satisfies the conclusion of the lemma. Indeed, $\widetilde{F}$ approximates F in the $\mathscr{C}^1(K)$-topology by properties (b1), (c1) and (d1$_1$)–(d1$_n$); condition (I) is guaranteed by (d2), (c2), and (b1); condition (II) by (d3$_n$), (a3), and (5-15); and condition (III) by (d4$_n$), (a3), and (5-14). So, to conclude the proof it suffices to construct the sequence $H_1, \dots, H_n$ satisfying the above properties. We proceed by induction. Assume that we already have $H_0, \dots, H_{b-1}$ for some $b \in \{1, \dots, n\}$ with the desired properties and let us construct H_b. Notice that (d5$_0$) = (c3) and (d6$_0$) = (c4) formally hold. By the continuity of H_{b-1} and conditions (d5$_{b-1}$) and (d6$_{b-1}$), for each $j \in I_b$ there exists a closed disk $\Omega_j \subset D_j \setminus (\gamma_{j-1} \cup \alpha_j \cup \gamma_j)$ such that the following hold:

(i) $\Omega_j \cap \beta_j$ is a compact connected Jordan arc.

(ii) $|H_{b-1,b}(p)| > \rho$ for all $p \in \Upsilon_j := \overline{D_j \setminus \Omega_j}$.

(iii) $|H_{b-1,b}(p)| > \tau$ for all $p \in \overline{\beta_j \setminus \Omega_j}$.

Next, for each $j \in I_b$ choose a smooth embedded arc $\lambda_j \subset \Upsilon_j \setminus (\gamma_{j-1} \cup \gamma_j)$ with an endpoint in α_j and the other one in Ω_j and otherwise disjoint from $b\Upsilon_j$ (see Figure 4). Moreover, choose each λ_j so that the set

$$S_b := \left(M \setminus \bigcup_{j \in I_b} \mathring{\Upsilon}_j\right) \cup \left(\bigcup_{j \in I_b} \lambda_j\right)$$

is admissible. Notice that S_b is connected and very simple in the sense of Definition 4.1.

Set $h = (h_1, \dots, h_n) = dH_{b-1}/\theta$ and let $\tilde{h} = (\tilde{h}_1, \dots, \tilde{h}_n) : S_b \to \mathfrak{S}_*$ be a map of class $\mathscr{A}(S_b)$ such that:

(iv) $\tilde{h} = h$ on $M \setminus \left(\bigcup_{j \in I_b} \mathring{D}_j\right)$.

(v) $\tilde{h}_b = h_b$ on S_b.

(vi) The map $\widetilde{H} : S_b \to \mathbb{C}^n$ given by

$$\widetilde{H}(p) = H_{b-1}(x_0) + \int_{x_0}^{p} \tilde{h}\theta, \quad p \in S_b,$$

satisfies $|\widetilde{H}(p)|_\infty > \tau$ for all $p \in \bigcup_{j\in I_b} \Omega_j$.

To construct such a map $\tilde{h}$ we may for instance choose $\tilde{h} = h$ on $M \setminus \bigcup_{j\in I_b} \mathring{\Upsilon}_j$ and suitably define it on $\bigcup_{j\in I_b} \lambda_j$. Now, Lemma 4.5 furnishes a map $\phi : M \to \mathfrak{S}_*$ of class $\mathscr{A}(M)$ such that $\phi\theta$ is exact on M (take into account that $\tilde{h}\theta = h\theta = dH_{b-1}$ on K and that K is a deformation retract of M) and the map $H_b : M \to \mathbb{C}^n$ given by

$$H_b(p) = H_{b-1}(x_0) + \int_{x_0}^{p} \phi\theta, \quad p \in M,$$

is an $\mathfrak{S}$-immersion of class $\mathscr{A}^1(M)$ enjoying the following properties:

(vii) H_b is as close as desired to $\widetilde{H}$ in the $\mathscr{C}^1$-topology on $M \setminus \big(\bigcup_{j\in I_b} \mathring{D}_j\big)$.

(viii) $H_{b,b} = H_{b-1,b}$ (take into account (v)).

(ix) $H_b - \widetilde{H}$ has a zero of multiplicity k at all points of Λ.

Since $\widetilde{H} = H_{b-1}$ on $M \setminus \big(\bigcup_{j\in I_b} \mathring{D}_j\big) \supset \Lambda \cup \big(\bigcup_{j\in\mathbb{Z}_l} \gamma_{j-1} \cup \alpha_j \cup \gamma_j\big)$, we have $(\mathrm{d1}_b) = $ (vii), $(\mathrm{d2}_b) = $ (ix), and, if the approximation in (vii) is close enough, $(\mathrm{d5}_b)$ and $(\mathrm{d6}_b)$ follow from $(\mathrm{d5}_{b-1})$ and $(\mathrm{d6}_{b-1})$, respectively.

Pick $j \in \bigcup_{i=1}^{b} I_i$ and $p \in D_j$. If $j \notin I_b$ then $(\mathrm{d3}_{b-1})$ and (vii) ensure that $|H_b(p)|_\infty > \rho$. On the other hand, if $j \in I_b$ then (ii) and (viii) ensure that $|H_b(p)| \geq |H_{b,b}(p)| > \rho$ provided that $p \in \Upsilon_j$, whereas (vi) and (vii) guarantee that $|H_b(p)| > \tau > \rho$ provided that $p \in \Omega_j$. This proves $(\mathrm{d3}_b)$.

Finally, choose $j \in \bigcup_{i=1}^{b} I_i$ and $p \in \beta_j$. As above, if $j \notin I_b$ then $(\mathrm{d4}_{b-1})$ and (vii) give that $|H_b(p)|_\infty > \tau$. Likewise, if $j \in I_b$ then (iii) and (viii) ensure that $|H_b(p)| \geq |H_{b,b}(p)| > \tau$ provided that $p \in \beta_j \setminus \Omega_j$, whereas (vi) and (vii) imply that $|H_b(p)| > \tau$ provided that $p \in \beta_j \cap \Omega_j$. This proves $(\mathrm{d4}_b)$ and concludes the proof of the lemma. □

6. Proof of Theorem 1.3

As in the proof of Theorem 4.4 we can assume that $\Lambda \cap bK = \varnothing$ and also that for each $p \in \Lambda$ we have either $\Omega_p \subset \mathring{K}$ or $\Omega_p \cap K = \varnothing$.

Set $M_0 := K$ and let $\{M_j\}_{j\in\mathbb{N}}$ be an exhaustion of M by smoothly bounded Runge compact domains in M such that:

- $M_0 \Subset M_1 \Subset \cdots \Subset \bigcup_{j\in\mathbb{N}} M_j = M$.
- $\chi(M_j \setminus \mathring{M}_{j-1}) \in \{-1, 0\}$ for all $j \in \mathbb{N}$.
- $bM_j \cap \Lambda = \varnothing$ for all $j \in \mathbb{N}$ and so, up to shrinking the sets Ω_p if necessary, we may assume that $\Omega_p \subset \mathring{M}_j$ or $\Omega_p \cap M_j = \varnothing$ for all $p \in \Lambda$.

The existence of such a sequence is guaranteed as in the proof of Theorem 4.4. The set $\Lambda_j := \Lambda \cap M_j = \Lambda \cap \mathring{M}_j$, $j \in \mathbb{Z}_+$, is empty or finite; without loss of generality we may assume that $\Lambda_0 \neq \varnothing$ and $\Lambda_j \setminus \Lambda_{j-1} \neq \varnothing$ for all $j \in \mathbb{N}$, and hence Λ is infinite. Observe that $\Lambda_{j-1} \subsetneq \Lambda_j$ for all $j \in \mathbb{N}$. Fix a sequence $\{\epsilon_j\}_{j\in\mathbb{N}} \searrow 0$ which will be specified later, set

$$F_0 := F|_{M_0} : M_0 \to \mathbb{C}^n$$

and, by Proposition 4.3 and Theorem 5.3, assume without loss of generality that F_0 is nonflat and, if $F|_\Lambda$ is injective, an embedding.

6A. ***Proof of the first part of the theorem.*** For the first part of the theorem we shall construct a sequence $\{F_j\}_{j\in\mathbb{N}}$ of nonflat $\mathfrak{S}$-immersions $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ satisfying:

(i_j) $\|F_j - F_{j-1}\|_{1,M_{j-1}} < \epsilon_j$.

(ii_j) $F_j - F$ has a zero of multiplicity $k \in \mathbb{N}$ at every point $p \in \Lambda_j$.

(iii_j) If $F|_\Lambda$ is injective then F_j is an embedding.

We proceed by induction. The basis is given by the $\mathfrak{S}$-immersion F_0, which clearly satisfies (ii_0) and (iii_0); condition (i_0) is vacuous. For the inductive step assume that we have an $\mathfrak{S}$-immersion $F_{j-1} : M_{j-1} \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_{j-1})$ satisfying (i_{j-1}), (ii_{j-1}), and (iii_{j-1}) for some $j \in \mathbb{N}$, and let us furnish $F_j : M_j \to \mathbb{C}^n$ enjoying the corresponding properties. We distinguish two different cases depending on the Euler characteristic of $M_j \setminus \mathring{M}_{j-1}$.

Noncritical case: *assume that* $\chi(M_j \setminus \mathring{M}_{j-1}) = 0$. It follows that M_{j-1} is a strong deformation retract of M_j, and then Proposition 5.2 applied to the data

$$S = M_{j-1} \cup \Big(\bigcup_{p\in\Lambda_j\setminus\Lambda_{j-1}} \Omega_p \Big), \quad S_0 = M_{j-1}, \quad \Lambda = \Lambda_j, \quad k,$$

and the generalized $\mathfrak{S}$-immersion $S \to \mathbb{C}^n$ agreeing with F_{j-1} on M_{j-1} and with F on $\bigcup_{p\in\Lambda_j\setminus\Lambda_{j-1}} \Omega_p$, provides an $\mathfrak{S}$-immersion $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ that satisfies (i_j) and (ii_j). Finally, if $F|_\Lambda$ is injective then Theorem 5.3 enables us to choose F_j being an embedding; this ensures (iii_j).

Critical case: *assume that* $\chi(M_j \setminus \mathring{M}_{j-1}) = -1$. We then have that the change of topology is described by attaching to M_{j-1} a smooth arc α in $\mathring{M}_j \setminus \mathring{M}_{j-1}$ satisfying M_{j-1} only at its endpoints. Thus, $M_{j-1} \cup \alpha$ is a strong deformation retract of M_j. Further, we may choose α such that $\alpha \cap \Lambda = \varnothing$; and $S := M_{j-1} \cup \alpha$ is an admissible subset of M, which is clearly very simple (see Definition 4.1). We use Lemma 3.3 to extend F_{j-1} to S as a generalized $\mathfrak{S}$-immersion. By Proposition 5.2, we may approximate F_{j-1} in the $\mathscr{C}^1(M_{j-1} \cup \alpha)$-topology by nonflat $\mathfrak{S}$-immersions on a small compact tubular neighborhood $M'_j \Subset M_j$ of $M_{j-1} \cup \alpha$ having a contact of order k with F at all points of Λ_j. Since $\chi(M_j \setminus \mathring{M}'_j) = 0$, this reduces the proof to the previous case and hence concludes the recursive construction of the sequence $\{F_j\}_{j\in\mathbb{N}}$.

Finally, if the number $\epsilon_j > 0$ is chosen sufficiently small at each step in the recursive construction, properties (i_j), (ii_j), and (iii_j) ensure that the sequence $\{F_j\}_{j\in\mathbb{N}}$ converges uniformly on compacta in M to

an $\mathfrak{S}$-immersion

$$\widetilde{F} := \lim_{j\to\infty} F_j : M \to \mathbb{C}^n,$$

which is as close as desired to F uniformly on K, is injective if $F|_\Lambda$ is injective, and is such that $\widetilde{F} - F$ has a zero of multiplicity k at all points of Λ.

6B. ***Proof of assertion*** **(I).** Suppose that the assumptions in assertion (I) hold. Fix a point $p_0 \in \mathring{K} \setminus \Lambda$. We shall now construct a sequence of $\mathfrak{S}$-immersions $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$, $j \in \mathbb{N}$, satisfying conditions (i_j)–(iii_j) above and also:

(iv_j) $\mathrm{dist}_{F_j}(p_0, bM_j) > j$ for all $j \in \mathbb{N}$.

Observe that $F_0 = F|_{M_0}$ satisfies (iv_0) since it is an immersion and $p_0 \in \mathring{K}$. For the inductive step assume that we already have F_{j-1} satisfying (i_j)–(iv_j) for some $j \in \mathbb{N}$ and, reasoning as above, construct an $\mathfrak{S}$-immersion $F'_j : M_j \to \mathbb{C}^n$ satisfying (i_j), (ii_j), and (iii_j). Let $M'_j \subset \mathring{M}_j$ be a smoothly bounded compact domain which is Runge and a strong deformation retract of M_j and contains $M_{j-1} \cup \Lambda_j$ in its relative interior. Then, Lemma 5.5 applied to the data

$$M = M_j, \quad K = M'_j, \quad \Lambda = \Lambda_j, \quad k, \quad \tau = j, \quad \text{and} \quad F = F'_j|_{M'_j},$$

gives an $\mathfrak{S}$-immersion $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ satisfying (ii_j), (iv_j), and also (i_j) provided that the approximation of F'_j by F_j on M'_j is close enough; Theorem 5.3 enables us to assume that F_j also satisfies (iii_j). This closes the induction and concludes the construction of the sequence $\{F_j\}_{j\in\mathbb{N}}$ with the desired properties.

As above, if the number $\epsilon_j > 0$ is chosen sufficiently small at each step in the recursive construction, properties (i_j)–(iii_j) ensure that the sequence $\{F_j\}_{j\in\mathbb{N}}$ converges uniformly on compacta in M to an $\mathfrak{S}$-immersion $\widetilde{F} := \lim_{j\to\infty} F_j : M \to \mathbb{C}^n$ which is as close as desired to F uniformly on K, is injective if F is injective, and is such that $\widetilde{F} - F$ has a zero of multiplicity k at all points of Λ. In addition, property (iv_j) ensures that

$$\lim_{j\to\infty} \mathrm{dist}_{\widetilde{F}}(p_0, bM_j) = +\infty$$

whenever the number $\epsilon_j > 0$ is chosen small enough at each step in the recursive process. This implies that $\widetilde{F}$ is complete and concludes the proof of assertion (I).

6C. ***Proof of assertion*** **(II).** Suppose the assumptions in assertion (II) hold. Observe that $F|_\Lambda : \Lambda \to \mathbb{C}^n$ is a proper map if, and only if, $(F|_\Lambda)^{-1}(C)$ is finite for any compact set $C \subset \mathbb{C}^n$, or, equivalently, if either the closed discrete set Λ is finite or for some (and hence for any) ordering $\Lambda = \{p_1, p_2, p_3, \ldots\}$ of Λ, the sequence $\{F(p_1), F(p_2), F(p_3), \ldots\}$ is divergent in $\mathbb{C}^n$. Since we are assuming that Λ is infinite, there is $j_0 \in \mathbb{N}$ such that

$$F(p) \neq 0 \quad \text{for all } p \in \Lambda \setminus \Lambda_{j_0}. \tag{6-1}$$

In a first step we construct for each $j \in \{0, \ldots, j_0\}$ an $\mathfrak{S}$-immersion $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ satisfying conditions (i_j)–(iii_j) above; we reason as in Section 6A. Now, up to a small deformation of M_{j_0}

if necessary, we may assume without loss of generality that F_{j_0} does not vanish anywhere on bM_{j_0}, and hence there exists $\rho_{j_0} > 0$ such that

$$|F_{j_0}(p)|_\infty > \rho_{j_0} > 0 \quad \text{for all } p \in bM_{j_0}. \tag{6-2}$$

Set

$$\rho_j := \min\{|F(p)|_\infty : p \in \Lambda_j \setminus \Lambda_{j-1}\} \quad \text{for all } j \geq j_0 + 1. \tag{6-3}$$

Recall that $\Lambda_j \setminus \Lambda_{j-1} \neq \varnothing$ for all $j \in \mathbb{N}$. In view of (6-1) and (6-2) we have $\rho_j > 0$ for all $j \geq j_0$. Moreover, since $F|_\Lambda$ is proper,

$$\lim_{j \to +\infty} \rho_j = +\infty. \tag{6-4}$$

In a second step, we shall construct a sequence of $\mathfrak{S}$-immersions $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$, for $j \geq j_0 + 1$, enjoying conditions (i_j)–(iii_j) and also:

(v_j.1) $|F_j(p)|_\infty > \frac{1}{2}\min\{\rho_{j-1}, \rho_j\}$ for all $p \in M_j \setminus \mathring{M}_{j-1}$.

(v_j.2) $|F_j(p)|_\infty > \rho_j$ for all $p \in bM_j$.

We proceed in an inductive way. The basis of the induction is accomplished by F_{j_0}; recall that it satisfies (i_{j_0})–(iii_{j_0}), whereas property (v_{j_0}.1) is vacuous and property (v_{j_0}.2) follows from (6-2). For the inductive step, assume that we already have $F_{j-1} : M_{j-1} \to \mathbb{C}^n$ for some $j \geq j_0 + 1$ satisfying (i_{j-1})–(iii_{j-1}), (v_{j-1}.1), and (v_{j-1}.2) and let us construct an $\mathfrak{S}$-immersion $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ with the corresponding requirements.

By (6-3) and up to a shrinking of the set Ω_p if necessary, we may assume that

$$|F(q)|_\infty > \tfrac{1}{2}\rho_j \quad \text{for all points } q \text{ in } \Omega^j := \bigcup_{p \in \Lambda_j \setminus \Lambda_{j-1}} \Omega_p \neq \varnothing. \tag{6-5}$$

Next, choose a smooth Jordan arc C_p for each $p \in \Lambda_j \setminus \Lambda_{j-1}$ with the initial point in bM_{j-1}, the final point in $b\Omega_p$, and otherwise disjoint from $M_{j-1} \cup \Omega^j$, and such that

$$S' := M_{j-1} \cup \Omega^j \cup \Bigg(\bigcup_{p \in \Lambda_j \setminus \Lambda_{j-1}} C_p \Bigg)$$

is a very simple admissible subset of M_j; in particular $C_p \cap C_q = \varnothing$ if $p \neq q$. If $\chi(M_j \setminus \mathring{M}_{j-1}) = -1$ we then also choose another smooth Jordan arc $\alpha \subset \mathring{M}_j$ with its two endpoints in bM_{j-1} and otherwise disjoint from S' such that $S' \cup \alpha$ is admissible and a strong deformation retract of M_j. If $\chi(M_j \setminus \mathring{M}_{j-1}) = 0$ we set $\alpha := \varnothing$. In any case, the set

$$S := S' \cup \alpha \subset \mathring{M}_j$$

is admissible in M and a strong deformation retract of M_j. Set

$$C := \alpha \cup \Bigg(\bigcup_{p \in \Lambda_j \setminus \Lambda_{j-1}} C_p \Bigg),$$

and observe that $S = (M_{j-1} \cup \Omega^j) \cup C$. Consider a generalized $\mathfrak{S}$-immersion $\widetilde{F}_j : S \to \mathbb{C}^n$ of class $\mathscr{A}^1(S)$ such that:

(A.1) $\widetilde{F}_j|_{M_{j-1}} = F_{j-1}$.

(A.2) $\widetilde{F}_j|_{\Omega^j} = F|_{\Omega^j}$.

(A.3) $|\widetilde{F}_j(q)|_\infty > \frac{1}{2}\min\{\rho_{j-1}, \rho_j\}$ for all $q \in C$.

To ensure (A.3) we use Lemma 3.3; take into account (v_{j-1}.2) and (6-5). Thus, (v_{j-1}.2), (6-5), and (A.3) guarantee that:

(A.4) $|\widetilde{F}_j(p)|_\infty > \frac{1}{2}\min\{\rho_{j-1}, \rho_j\} > 0$ for all $p \in S \setminus \mathring{M}_{j-1} = (bM_{j-1}) \cup \Omega^j \cup C$.

Since $S \subset \mathring{M}_j$ is Runge and a strong deformation retract of M_j, Proposition 5.2 applied to the data

$$M = M_j, \quad S, \quad \Lambda = \Lambda_j, \quad k, \quad \text{and} \quad F = \widetilde{F}_j$$

gives a nonflat $\mathfrak{S}$-immersion $\widehat{F}_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ such that:

(B.1) $\widehat{F}_j$ is as close as desired to $\widetilde{F}_j$ in the $\mathscr{C}^1(S)$-topology.

(B.2) $\widehat{F}_j - \widetilde{F}_j$ has a zero of multiplicity $k \in \mathbb{N}$ at every point $p \in \Lambda_j$.

If the approximation in (B.1) is close enough then, in view of (A.4), there exists a small compact neighborhood N of S in $\mathring{M}_j$, being a smoothly bounded compact domain and a strong deformation retract of M_j, and such that:

(B.3) $|\widehat{F}_j(p)|_\infty > \frac{1}{2}\min\{\rho_{j-1}, \rho_j\} > 0 \quad$ for all $p \in N \setminus \mathring{M}_{j-1}$.

Notice that $\Lambda \cap (M_j \setminus \mathring{N}) = \varnothing$ and hence we may apply Lemma 5.6 to the data

$$M = M_j, \quad K = N, \quad \Lambda = \Lambda_j, \quad F = \widehat{F}_j, \quad \rho = \tfrac{1}{2}\min\{\rho_{j-1}, \rho_j\}, \quad \tau = \rho_j, \quad k,$$

obtaining an $\mathfrak{S}$-immersion $F_j : M_j \to \mathbb{C}^n$ of class $\mathscr{A}^1(M_j)$ such that:

(C.1) F_j is as close as desired to $\widehat{F}_j$ in the $\mathscr{C}^1(N)$-topology.

(C.2) $F_j - \widehat{F}_j$ has a zero of multiplicity k at every point $p \in \Lambda_j \subset \mathring{N}$.

(C.3) $|F_j(p)|_\infty > \frac{1}{2}\min\{\rho_{j-1}, \rho_j\}$ for all $p \in M_j \setminus \mathring{N}$.

(C.4) $|F_j(p)|_\infty > \rho_j$ for all $p \in bM_j$.

We claim that, if the approximations in (B.1) and (C.1) are close enough, the $\mathfrak{S}$-immersion $F_j : M_j \to \mathbb{C}^n$ satisfies properties (i_j)–(iii_j), (v_j.1), and (v_j.2). Indeed, (A.1) ensures (i_j); properties (A.2), (B.2), and (C.2) guarantee (ii_j); condition (v_j.1) follows from (A.4), (B.3), and (C.3); and condition (v_j.2) is implied by (C.4). Finally, if $F|_\Lambda$ is injective then, by Theorem 5.3, we may assume without loss of generality that F_j is an embedding. This closes the inductive step and concludes the recursive construction of the sequence $\{F_j\}_{j \geq j_0+1}$ meeting the desired requirements.

As above, choosing the number $\epsilon_j > 0$ ($j \in \mathbb{N}$) small enough at each step in the construction, properties (i_j)–(iii_j) ensure that the sequence $\{F_j\}_{j\in\mathbb{N}}$ converges uniformly on compact subsets of M to an $\mathfrak{S}$-immersion $\widetilde{F} := \lim_{j\to\infty} F_j : M \to \mathbb{C}^n$ which is as close as desired to F uniformly on K, is injective

if $F|_\Lambda$ is injective, and is such that $\widetilde{F} - F$ has a zero of multiplicity k at all points of Λ. Furthermore, properties (v_j.1) and (6-4) imply that $\widetilde{F}$ is a proper map. Indeed, take a number $R > 0$ and a sequence $\{q_m\}_{m\in\mathbb{N}}$ that diverges on M, and let us check that there is $m_0 \in \mathbb{N}$ such that $|\widetilde{F}(q_m)|_\infty > R$ for all $m \geq m_0$. Indeed, set

$$\varepsilon := \sum_{j\geq 1} \epsilon_j < +\infty$$

and observe that, by properties (i_j),

$$\|\widetilde{F} - F_j\|_{1,M_j} < \varepsilon \quad \text{for all } j \in \mathbb{Z}_+. \tag{6-6}$$

On the other hand, in view of (6-4) there is an integer $j_1 \geq j_0 + 1$ such that

$$\rho_{j-1} > 2(R+\varepsilon) \quad \text{for all } j \geq j_1. \tag{6-7}$$

Now, since the sequence $\{p_m\}_{m\in\mathbb{N}}$ diverges on M and $\{M_j\}_{j\in\mathbb{N}}$ is an exhaustion of M, there is $m_0 \in \mathbb{N}$ such that

$$p_m \in M \setminus M_{j_1} \quad \text{for all } m \geq m_0.$$

Thus, for any $m \geq m_0$ there is an integer $j_m \geq j_1$ such that $q_m \in M_{j_m} \setminus \mathring{M}_{j_m-1}$, and so

$$|\widetilde{F}(q_m)|_\infty \geq |F_{j_m}(q_m)|_\infty - |F_{j_m}(q_m) - \widetilde{F}(q_m)|_\infty \overset{(\mathrm{v}_{j_m}.1),\,(6\text{-}6)}{>} \tfrac{1}{2}\min\{\rho_{j_m-1}, \rho_{j_m}\} - \varepsilon \overset{(6\text{-}7)}{>} R.$$

This proves that $\widetilde{F} : M \to \mathbb{C}^n$ is a proper map and concludes the proof of Theorem 1.3.

7. Sketch of the proof of Theorem 1.2 and the proof of Theorem 1.1

We briefly explain how the arguments in Sections 5 and 6 which enabled us to prove Theorem 1.3 may be adapted in order to guarantee Theorem 1.2; we shall leave the obvious details of the proof to the interested reader. Afterward, we will use Theorem 1.2 to prove Theorem 1.1.

First of all recall that, as pointed out in Section 2C, for any integer $n \geq 3$ the punctured null quadric $\mathfrak{A}_* \subset \mathbb{C}^n$, see (1-3) and (1-4), directing minimal surfaces in $\mathbb{R}^n$ is an Oka manifold and satisfies the assumptions in assertions (I) and (II) in Theorem 1.3. Thus, Theorem 4.4 and Lemma 4.5 hold for $\mathfrak{S} = \mathfrak{A}$.

The first step in the proof of Theorem 1.2 consists of providing an analogue of Proposition 5.2 for *generalized conformal minimal immersions* in the sense of [Alarcón, Forstnerič and López 2016a, Definition 5.2]. In particular we need to show that if we are given M, S, and Λ as in Proposition 5.2 then, for any integer $k \in \mathbb{Z}_+$, every generalized conformal minimal immersion $X : S \to \mathbb{R}^n$ ($n \geq 3$) may be approximated in the $\mathscr{C}^1(S)$-topology by conformal minimal immersions $\widetilde{X} : M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ such that $\widetilde{X}$ and X have a contact of order k at every point in Λ and the flux map $\mathrm{Flux}_{\widetilde{X}}$ equals Flux_X everywhere in the first homology group $H_1(M;\mathbb{Z})$. To do that we reason as in the proof of Proposition 5.2 but working with the map $f := \partial X/\theta : S \to \mathfrak{A}_*$. Since $f\theta$ does not need to be exact (only its real part does) we replace conditions (a) and (b) in the proof of the proposition by the following ones:

- $(\tilde{f} - f)\theta$ is exact on S.
- $X(p_0) + 2\int_{C_p} \Re(\tilde{f}\theta) = X(p_0) + 2\int_{C_p} \Re(f\theta) = X(p)$ for all $p \in \Lambda$.

It can then be easily seen that the conformal minimal immersion $\widetilde{X} : M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ given by

$$\widetilde{X}(p) := X(p) + 2\int_{C_p} \Re(\tilde{f}\theta), \quad p \in M,$$

is well-defined and enjoys the desired properties.

In a second step and following the same spirit, we need to furnish a general position theorem, a completeness lemma, and a properness lemma for conformal minimal immersions of class $\mathscr{C}^1$ on a compact bordered Riemann surface, which are analogues of Theorem 5.3, Lemma 5.5, and Lemma 5.6, respectively. In this case the general position of conformal minimal surfaces is embedded in $\mathbb{R}^n$ for all integers $n \geq 5$; to adapt the proof of Theorem 5.3 to the minimal surfaces framework we combine the argument in [Alarcón, Forstnerič and López 2016a, proof of Theorem 1.1] with the new ideas in Section 5A which allow us to ensure the interpolation condition. Likewise, the analogues of Lemmas 5.5 and 5.6 for conformal minimal surfaces can be proved by adapting the proofs of the cited lemmas in Sections 5B and 5C, respectively; the required modifications follow the pattern described in the previous paragraph: at each step in the proofs we ensure that the real part of the 1-forms is exact and that the periods of the imaginary part agree with the flux map of the initial conformal minimal immersion. Furthermore, obviously, we are allowed to use only the real part in order to ensure the increasing of the intrinsic diameter of the surface to achieve completeness, see Lemma 5.5 (II), and the increasing of the $|\cdot|_\infty$-norm near the boundary to guarantee properness, see Lemma 5.6 (II) and (III). For the former we just replace condition (c) in the proof of Lemma 5.5 (which determines an extrinsic bound) by the following one:

- $|2\int_\alpha \Re(h\theta)| > \tau$ for all arcs $\alpha \subset S$ with initial point p_0 and final point in L.

For the latter, the adaptation is done straightforwardly since all the bounds are of the same nature, namely, extrinsic.

Finally, granted the analogues for conformal minimal surfaces in $\mathbb{R}^n$ of Proposition 5.2, Theorem 5.3, Lemma 5.5, and Lemma 5.6, the proof of Theorem 1.2 follows word for word, up to trivial modifications similar to the ones discussed in the previous paragraphs, the one of Theorem 1.3 in Section 6. It is perhaps worth pointing out that in the noncritical case in the recursive construction (see Section 6A) we now have to extend a conformal minimal immersion of class $\mathscr{C}^1(M_{j-1})$ to a generalized conformal minimal immersion on the admissible set $S = M_{j-1} \cup \alpha \subset \mathring{M}_j$ whose flux map equals $\mathfrak{p}$ for every closed curve in S (here $\mathfrak{p} : H_1(M;\mathbb{Z}) \to \mathbb{R}^n$ denotes the group homomorphism given in the statement of Theorem 1.2, whereas M_{j-1}, α, and M_j are as in Section 6A); this can be easily done as in [Alarcón, Forstnerič and López 2016a, proof of Theorem 1.2]. This concludes the sketch of the proof of Theorem 1.2; as we announced at the very beginning of this section, we leave the details to the interested reader.

To finish the paper we show how Theorem 1.2 can be used in order to prove the following extension to Theorem 1.1 in the Introduction.

Corollary 7.1. *Let M be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Consider also an integer $n \geq 3$ and maps $\mathfrak{Z} : \Lambda \to \mathbb{R}^n$ and*

$$G : \Lambda \to Q_{n-2} = \{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\} \subset \mathbb{CP}^{n-1}.$$

Then there is a conformal minimal immersion $\widetilde{X} : M \to \mathbb{R}^n$ *satisfying* $\widetilde{X}|_\Lambda = \mathfrak{Z}$ *and whose generalized Gauss map* $G_{\widetilde{X}} : M \to \mathbb{CP}^{n-1}$ *equals* G *on* Λ.

Proof. For each $p \in \Lambda$ let Ω_p be a smoothly bounded, simply connected compact neighborhood of p in M, and assume that $\Omega_p \cap \Omega_q = \varnothing$ whenever $p \neq q \in \Lambda$. Set $\Omega := \bigcup_{p\in\Lambda} \Omega_p$ and let $X : \Omega \to \mathbb{R}^n$ be any conformal minimal immersion of class $\mathscr{C}^1(\Omega)$ such that $X|_\Lambda = \mathfrak{Z}$ and the generalized Gauss map satisfies $G_X|_\Lambda = G$. (Such an X always exists; we may for instance choose $X|_{\Omega_p}$ to be a suitable planar disk for each $p \in \Lambda$). Also fix a smoothly bounded simply connected compact domain $K \subset M \setminus \Lambda$, up to shrinking the sets Ω_p if necessary, assume that $K \subset M \setminus \Omega$, and extend X to $\Omega \cup K \to \mathbb{R}^n$ such that $X|_K : K \to \mathbb{R}^n$ is any conformal minimal immersion of class $\mathscr{C}^1(K \cup \Omega)$. Applying Theorem 1.2 to these data, any group homomorphism $H_1(M; \mathbb{Z}) \to \mathbb{R}^n$, and the integer $k = 1$, we obtain a conformal minimal immersion $\widetilde{X} : M \to \mathbb{R}^n$ which has a contact of order 1 with $X|_\Omega$ at every point in Λ. Thus, $\widetilde{X}|_\Lambda = X|_\Lambda = \mathfrak{Z}$ and the generalized Gauss map satisfies $G_{\widetilde{X}}|_\Lambda = [\partial\widetilde{X}]|_\Lambda = [\partial X]|_\Lambda = G_X|_\Lambda = G$. □

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