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THE BMO-DIRICHLET PROBLEM FOR ELLIPTIC SYSTEMS IN THE UPPER HALF-SPACE AND QUANTITATIVE CHARACTERIZATIONS OF VMO

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We prove that for any homogeneous, second-order, constant complex coefficient elliptic system L in $\mathbb{R}^n$, the Dirichlet problem in $\mathbb{R}^n_+$ with boundary data in $\mathrm{BMO}(\mathbb{R}^{n-1})$ is well-posed in the class of functions u for which the Littlewood–Paley measure associated with u, namely

$$d\mu_u(x',t) := |\nabla u(x',t)|^2\, t\, dx'\, dt,$$

is a Carleson measure in $\mathbb{R}^n_+$.

In the process we establish a Fatou-type theorem guaranteeing the existence of the pointwise nontangential boundary trace for smooth null-solutions u of such systems satisfying the said Carleson measure condition. In concert, these results imply that the space $\mathrm{BMO}(\mathbb{R}^{n-1})$ can be characterized as the collection of nontangential pointwise traces of smooth null-solutions u to the elliptic system L with the property that μ_u is a Carleson measure in $\mathbb{R}^n_+$.

We also establish a regularity result for the BMO-Dirichlet problem in the upper half-space, to the effect that the nontangential pointwise trace on the boundary of $\mathbb{R}^n_+$ of any given smooth null-solutions u of L in $\mathbb{R}^n_+$ satisfying the above Carleson measure condition actually belongs to Sarason's space $\mathrm{VMO}(\mathbb{R}^{n-1})$ if and only if $\mu_u(T(Q))/|Q| \to 0$ as $|Q| \to 0$, uniformly with respect to the location of the cube $Q \subset \mathbb{R}^{n-1}$ (where $T(Q)$ is the Carleson box associated with Q, and $|Q|$ denotes the Euclidean volume of Q).

Moreover, we are able to establish the well-posedness of the Dirichlet problem in $\mathbb{R}^n_+$ for a system L as above in the case when the boundary data are prescribed in Morrey–Campanato spaces in $\mathbb{R}^{n-1}$. In such a scenario, the solution u is required to satisfy a vanishing Carleson measure condition of fractional order.

By relying on these well-posedness and regularity results we succeed in producing characterizations of the space VMO as the closure in BMO of classes of smooth functions contained in BMO within which uniform continuity may be suitably quantified (such as the class of smooth functions satisfying a Hölder or Lipschitz condition). This improves on Sarason's classical result describing VMO as the closure in BMO of the space of uniformly continuous functions with bounded mean oscillations. In turn, this allows us to show that any Calderón–Zygmund operator T satisfying $T(1) = 0$ extends as a linear and bounded mapping from VMO (modulo constants) into itself. In turn, this is used to describe algebras of singular integral operators on VMO, and to characterize the membership to VMO via the action of various classes of singular integral operators.

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1. Introduction and statement of main theorems

In his ground-breaking article, C. Fefferman [1971] writes “The main idea in proving [that the dual of the Hardy space H^1 is the John–Nirenberg space BMO] is to study the Poisson integral of a function in BMO.” Implicit in this quote is the fact that the Poisson kernel is associated with the Laplace operator, and one of the primary aims of the present paper is to advance this line of work by considering more general systems of partial differential operators than the Laplacian. For example, the key PDE result announced in [Fefferman 1971] states that

$$\begin{array}{c}\text{a measurable function } f \text{ with } \int_{\mathbb{R}^{n-1}} |f(x')|(1+|x'|)^{-n}\,dx' < +\infty\\ \text{belongs to the space } \mathrm{BMO}(\mathbb{R}^{n-1}) \text{ if and only if its Poisson integral}\\ u:\mathbb{R}^n_+\to\mathbb{R}, \text{ with respect to the Laplace operator in } \mathbb{R}^n, \text{ satisfies}\\ \sup_{x'\in\mathbb{R}^{n-1}}\sup_{r>0}\Big\{r^{1-n}\int_{|x'-y'|<r}\int_0^r |(\nabla u)(y',t)|^2\,t\,dt\,dx'\Big\} < +\infty,\end{array} \tag{1-1}$$

and one of the main goals here is to develop machinery that permits us to replace the Laplacian in (1-1) with much more general second-order elliptic systems with complex coefficients. In order to be more specific, we proceed to elaborate on the actual setting adopted in this paper.

Let $M\in\mathbb{N}$ and consider a second-order, homogeneous, $M\times M$ system, with constant complex coefficients, written (with the usual convention of summation over repeated indices in place) as

$$Lu := \big(\partial_r(a^{\alpha\beta}_{rs}\,\partial_s u_\beta)\big)_{1\le\alpha\le M}, \tag{1-2}$$

when acting on a $\mathscr{C}^2$ vector-valued function $u=(u_\beta)_{1\le\beta\le M}$ defined in an open subset of $\mathbb{R}^n$. Assume that L is *strongly elliptic* in the sense that there exists $\kappa_o\in(0,\infty)$ such that

$$\mathrm{Re}[a^{\alpha\beta}_{rs}\xi_r\xi_s\bar\eta_\alpha\eta_\beta] \ge \kappa_o|\xi|^2|\eta|^2 \quad \text{for every } \xi=(\xi_r)_{1\le r\le n}\in\mathbb{R}^n \text{ and } \eta=(\eta_\alpha)_{1\le\alpha\le M}\in\mathbb{C}^M. \tag{1-3}$$

Examples include scalar operators, such as the Laplacian $\Delta=\sum_{j=1}^n\partial_j^2$ or, more generally, operators of the form $\operatorname{div} A\nabla$ with $A=(a_{rs})_{1\le r,s\le n}$ an $n\times n$ matrix with complex entries satisfying the ellipticity condition

$$\inf_{\xi\in S^{n-1}} \mathrm{Re}[a_{rs}\xi_r\xi_s] > 0 \tag{1-4}$$

(where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$), as well as complex versions of the Lamé system of elasticity

$$Lu := \mu\Delta u + (\lambda+\mu)\nabla\operatorname{div} u, \quad u=(u_1,\dots,u_n)\in\mathscr{C}^2. \tag{1-5}$$

Above, the constants $\lambda, \mu \in \mathbb{C}$ (typically called Lamé moduli) are assumed to satisfy

$$\operatorname{Re}\mu > 0 \quad \text{and} \quad \operatorname{Re}(2\mu + \lambda) > 0, \tag{1-6}$$

a condition equivalent to the demand that the Lamé system (1-5) satisfies the Legendre–Hadamard ellipticity condition (1-3). While the Lamé system is symmetric, we stress that the results in this paper require no symmetry for the systems involved.

Returning to the general framework, with every system L as in (1-2)–(1-3) one may associate a Poisson kernel, P^L, which is a $\mathbb{C}^{M\times M}$-valued function defined in $\mathbb{R}^{n-1}$ described in detail in Theorem 2.3. This Poisson kernel played a pivotal role in the treatment of the L^p-Dirichlet boundary value problem for L in the upper half-space in [Martell et al. 2016].

To state our main results, some notation is needed. For a function $\phi : \mathbb{R}^{n-1} \to \mathbb{C}$ set

$$\phi_t(x') := t^{1-n}\phi(x'/t) \quad \text{for every } x' \in \mathbb{R}^{n-1} \text{ and every } t > 0. \tag{1-7}$$

In particular, $P_t^L(x') = t^{1-n} P^L(x'/t)$ for every $x' \in \mathbb{R}^{n-1}$ and $t > 0$. We agree to identify the boundary of the upper half-space

$$\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\} \tag{1-8}$$

with the horizontal hyperplane $\mathbb{R}^{n-1}$ via $(x', 0) \equiv x'$ for any $x' \in \mathbb{R}^{n-1}$. The origin in $\mathbb{R}^{n-1}$ is denoted by $0'$. Having fixed some background parameter $\kappa > 0$, at each point $x' \in \partial\mathbb{R}^n_+$ we define the conical nontangential approach region with vertex at x' as

$$\Gamma_\kappa(x') := \{y = (y', t) \in \mathbb{R}^n_+ : |x' - y'| < \kappa t\}. \tag{1-9}$$

Whenever meaningful, the nontangential pointwise trace of a continuous vector-valued function u defined in $\mathbb{R}^n_+$ is given by

$$\big(u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}\big)(x') := \lim_{\Gamma_\kappa(x') \ni y \to (x',0)} u(y) \quad \text{for } x' \in \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}. \tag{1-10}$$

For each positive integer k denote by $\mathscr{L}^k$ the k-dimensional Lebesgue measure in $\mathbb{R}^k$. A Borel measure μ in $\mathbb{R}^n_+$ is said to be a Carleson measure in $\mathbb{R}^n_+$ provided

$$\|\mu\|_{\mathcal{C}(\mathbb{R}^n_+)} := \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q d\mu(x', t) < \infty, \tag{1-11}$$

where the supremum runs over all cubes Q in $\mathbb{R}^{n-1}$. Here and elsewhere in the paper, by a cube Q in $\mathbb{R}^{n-1}$ we shall understand a cube with sides parallel to the coordinate axes, and its side-length will be denoted by $\ell(Q)$. Also, the $\mathscr{L}^{n-1}$ measure of Q is denoted by $|Q|$ and if $\lambda > 0$ then λQ denotes the cube concentric with Q whose side-length is $\lambda\,\ell(Q)$. Call a Borel measure μ in $\mathbb{R}^n_+$ a vanishing Carleson measure whenever μ is a Carleson measure to begin with and, in addition,

$$\lim_{r \to 0^+} \left(\sup_{Q \subset \mathbb{R}^{n-1},\, \ell(Q) \le r} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q d\mu(x', t) \right) = 0. \tag{1-12}$$

Next, the Littlewood–Paley measure associated with a continuously differentiable function u in $\mathbb{R}^n_+$ is $|\nabla u(x',t)|^2\, t\, dx'\, dt$, and we set

$$\|u\|_{**} := \sup_{Q\subset\mathbb{R}^{n-1}} \left(\frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla u(x',t)|^2\, t\, dx'\, dt \right)^{\frac{1}{2}}. \tag{1-13}$$

In particular, for a continuously differentiable function u in $\mathbb{R}^n_+$ we have

$$\|u\|_{**} < \infty \quad \Longleftrightarrow \quad |\nabla u(x',t)|^2\, t\, dx'\, dt \text{ is a Carleson measure in } \mathbb{R}^n_+. \tag{1-14}$$

We next introduce $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$, the John–Nirenberg space of vector-valued functions of bounded mean oscillations in $\mathbb{R}^{n-1}$, as the collection of $\mathbb{C}^M$-valued functions $f=(f_\alpha)_{1\le\alpha\le M}$ with components in $L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ satisfying

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} := \sup_{Q\subset\mathbb{R}^{n-1}} ⨍_Q |f(x')-f_Q|\, dx' < \infty. \tag{1-15}$$

Above, for every cube Q in $\mathbb{R}^{n-1}$ and every function $h\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ we have abbreviated

$$h_Q := ⨍_Q h(x')\, dx' := \frac{1}{|Q|}\int_Q h(x')\, dx' \in \mathbb{C}^M, \tag{1-16}$$

where the last integration is performed componentwise. To lighten notation, when $M=1$ we simply write $\mathrm{BMO}(\mathbb{R}^{n-1})$ in place of $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C})$. Clearly, for every $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ we have

$$\begin{aligned} \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} &= \|f+C\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} && \text{for all } C\in\mathbb{C}^M,\\ \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} &= \|\tau_{z'}f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} && \text{for all } z'\in\mathbb{R}^{n-1},\\ \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} &= \|\delta_\lambda f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} && \text{for all } \lambda\in(0,\infty), \end{aligned} \tag{1-17}$$

where $\tau_{z'}$ is the operator of translation by z', i.e., $(\tau_{z'}f)(x') := f(x'+z')$ for every $x'\in\mathbb{R}^{n-1}$, and δ_λ is the operator of dilation by λ, i.e., $(\delta_\lambda f)(x') := f(\lambda x')$ for every $x'\in\mathbb{R}^{n-1}$.

We wish to note here that $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}$ is only a seminorm, since for every function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ we have

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} = 0 \quad \Longleftrightarrow \quad f \text{ is constant } \mathscr{L}^{n-1}\text{-a.e. in } \mathbb{R}^{n-1} \text{ (in } \mathbb{C}^M). \tag{1-18}$$

Occasionally, we find it useful to mod out its null-space, in order to render the resulting quotient space Banach. Specifically, for two $\mathbb{C}^M$-valued Lebesgue-measurable functions f,g defined in $\mathbb{R}^{n-1}$ we say that $f\sim g$ provided $f-g$ is constant $\mathscr{L}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$. This is an equivalence relation and we let

$$[f] := \{g:\mathbb{R}^{n-1}\to\mathbb{C}^M : g \text{ measurable and } f\sim g\} \tag{1-19}$$

denote the equivalence class of any given $\mathbb{C}^M$-valued Lebesgue-measurable function f defined in $\mathbb{R}^{n-1}$. If for each $f\in\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ we now set

$$\|[f]\|_{\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1},\mathbb{C}^M)} := \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}, \tag{1-20}$$

then $\|[\cdot]\|_{\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1},\mathbb{C}^M)}$ becomes a genuine norm on the quotient space

$$\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1},\mathbb{C}^M) := \{[f] : f \in \mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)\}. \tag{1-21}$$

In fact, when equipped with the norm (1-20), the space (1-21) is complete (hence Banach).

Moving on, the Sarason space of $\mathbb{C}^M$-valued functions of vanishing mean oscillations in $\mathbb{R}^{n-1}$ is defined by

$$\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M) := \left\{ f \in \mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M) : \lim_{r\to 0^+}\left(\sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r} \fint_Q |f(x')-f_Q|\,dx' \right) = 0 \right\}. \tag{1-22}$$

The space $\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ turns out to be a closed subspace of $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$. In fact, if we let $\mathrm{UC}(\mathbb{R}^{n-1},\mathbb{C}^M)$ denote the space of $\mathbb{C}^M$-valued uniformly continuous functions in $\mathbb{R}^{n-1}$, then

$$\begin{aligned} \mathrm{UC}(\mathbb{R}^{n-1},\mathbb{C}^M) \cap \Bigl(\bigcup_{1\le p\le\infty} L^p(\mathbb{R}^{n-1},\mathbb{C}^M) \Bigr) &\subset \mathrm{UC}(\mathbb{R}^{n-1},\mathbb{C}^M) \cap \mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M) \\ &\subset \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M). \end{aligned} \tag{1-23}$$

To justify the first inclusion, consider $f \in \mathrm{UC}(\mathbb{R}^{n-1},\mathbb{C}^M) \cap L^p(\mathbb{R}^{n-1},\mathbb{C}^M)$ for some $p \in [1,\infty]$. Then there exists $r_0 \in (0,\infty)$ with the property that $|f(x')-f(y')| \le 1$ whenever $x', y' \in \mathbb{R}^{n-1}$ are such that $|x'-y'| \le r_0\sqrt{n-1}$. Suppose now that some arbitrary cube Q in $\mathbb{R}^{n-1}$ has been fixed. If $\ell(Q) \ge r_0$, with the help of Hölder's inequality we estimate

$$\fint_Q |f-f_Q|\,d\mathscr{L}^{n-1} \le 2\fint_Q |f|\,d\mathscr{L}^{n-1} \le \frac{2\|f\|_{L^p(\mathbb{R}^{n-1},\mathbb{C}^M)}}{|Q|^{1/p}} \le \frac{2\|f\|_{L^p(\mathbb{R}^{n-1},\mathbb{C}^M)}}{r_0^{(n-1)/p}}, \tag{1-24}$$

whereas if $\ell(Q) < r_0$ we make use of the uniform continuity of f to estimate

$$\fint_Q |f-f_Q|\,d\mathscr{L}^{n-1} \le \fint_Q\fint_Q |f(x')-f(y')|\,dx'\,dy' \le 1. \tag{1-25}$$

In turn, from (1-24)–(1-25) we then conclude that f belongs to $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$, which establishes the first inclusion in (1-23). The second inclusion in (1-23) is clear from (1-22).

As regards the second inclusion in (1-23), a well-known result of Sarason [1975, Theorem 1, p. 392] implies that, in fact,

$$\begin{gathered} \text{a given function } f \in \mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M) \text{ belongs to } \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M) \\ \text{if and only if there is a sequence } \{f_j\}_{j\in\mathbb{N}} \subset \mathrm{UC}(\mathbb{R}^{n-1},\mathbb{C}^M) \cap \mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M) \\ \text{with } \lim_{j\to\infty} \|f-f_j\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} = 0. \end{gathered} \tag{1-26}$$

We shall refer to this simply by saying that Sarason's VMO space is the closure of UC $\cap$BMO in the space BMO. In relation to (1-23) we wish to note that continuity without uniformity will not preserve the inclusion in (1-23). For example, there exist functions in $\mathscr{C}^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which do not belong to $\mathrm{VMO}(\mathbb{R})$. To see this, consider the mutually disjoint intervals $I_j := [j, j+2/j]$ for each $j \in \mathbb{N}$, $j \ge 3$. Now let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property that, for each $j \in \mathbb{N}$, $j \ge 3$, the graph of $f|_{I_j}$ is the line segment joining the point $(j,-1)$ with $(j+2/j, 1)$ and otherwise the graph of f is made up of curves joining these line segments smoothly within the strip $\mathbb{R}\times[-2,2]$. By design, $f\in\mathscr{C}^\infty(\mathbb{R})\cap L^\infty(\mathbb{R})$. In particular, $f\in\mathrm{BMO}(\mathbb{R})$. However, for each $j\in\mathbb{N}$, $j\geq 3$, we have $f_{I_j}=0$ and

$$⨍_{I_j}|f-f_{I_j}|\,d\mathscr{L}^1=⨍_{I_j}|f|\,d\mathscr{L}^1=\tfrac{1}{2}. \tag{1-27}$$

Since $|I_j|=2/j\to 0$ as $j\to\infty$, from (1-27) and (1-22) it is then clear that $f\notin\mathrm{VMO}(\mathbb{R})$.

Another characterization of $\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ due to Sarason [1975, Theorem 1, p. 392] is as follows:

$$\begin{gathered}\text{a given function } f\in\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M) \text{ belongs to the space } \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)\\ \text{if and only if } \lim_{\mathbb{R}^{n-1}\ni z'\to 0'}\|\tau_{z'}f-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}=0.\end{gathered} \tag{1-28}$$

We are now ready to state our first main result. This concerns the well-posedness of the BMO-Dirichlet problem in the upper half-space for systems L as in (1-2)–(1-3). The existence of a unique solution is established in the class of functions u satisfying a Carleson measure condition, expressed in terms of the finiteness of (1-13). The formulation of our theorem emphasizes the fact that this contains as a "subproblem" the VMO-Dirichlet problem for L in $\mathbb{R}^n_+$ (in which scenario u satisfies a vanishing Carleson measure condition).

Theorem 1.1. *Let L be an $M\times M$ elliptic constant complex coefficient system as in* (1-2)–(1-3). *Then the* BMO-*Dirichlet boundary value problem for L in $\mathbb{R}^n_+$, namely*

$$\begin{cases} u\in\mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M),\\ Lu=0 \text{ in } \mathbb{R}^n_+,\\ |\nabla u(x',t)|^2\,t\,dx'\,dt \text{ is a Carleson measure in } \mathbb{R}^n_+,\\ u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}=f \text{ a.e. in } \mathbb{R}^{n-1}, \quad f\in\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M),\end{cases} \tag{1-29}$$

has a unique solution. Moreover, this unique solution satisfies the following additional properties:

(i) *With P^L denoting the Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3, one has the Poisson integral representation formula*

$$u(x',t)=(P^L_t*f)(x') \quad \text{for all } (x',t)\in\mathbb{R}^n_+. \tag{1-30}$$

(ii) *There exists a constant $C=C(n,L)\in(1,\infty)$ with the property that the solution u of the Dirichlet problem* (1-29) *satisfies the two-sided estimate*

$$C^{-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}\leq\|u\|_{**}\leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{1-31}$$

That is, the size of the solution is comparable to the size of the boundary datum.

(iii) *For each $\varepsilon>0$ the function $u(\cdot,\varepsilon)$ belongs to* $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ *and there exists a constant $C=C(n,L)\in(0,\infty)$ independent of u with the property that the following uniform* BMO *estimate holds*:

$$\sup_{\varepsilon>0}\|u(\cdot,\varepsilon)\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}\leq C\|u\|_{**}. \tag{1-32}$$

Moreover,

$$\lim_{\varepsilon\to 0^+} \|u(\cdot,\varepsilon)-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}=0 \iff \begin{cases} |\nabla u(x',t)|^2\,t\,dx'\,dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}^n_+. \end{cases} \tag{1-33}$$

That is, u *satisfies a vanishing Carleson measure condition in* $\mathbb{R}^n_+$ *if and only if* u *converges to its boundary datum vertically in* $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$.

(iv) *The following regularity results hold*:

$$f\in \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M) \iff \begin{cases} |\nabla u(x',t)|^2\,t\,dx'\,dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}^n_+ \end{cases} \tag{1-34}$$

$$\iff \lim_{\mathbb{R}^n_+\ni z\to 0} \|\tau_z u-u\|_{**}=0, \tag{1-35}$$

where $(\tau_z u)(x):=u(x+z)$ *for each* $x,z\in\mathbb{R}^n_+$.

As a consequence, *the* VMO-*Dirichlet boundary value problem for* L *in* $\mathbb{R}^n_+$, *i.e.*,

$$\begin{cases} u\in\mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M), \\ Lu=0 \text{ in } \mathbb{R}^n_+, \\ |\nabla u(x',t)|^2\,t\,dx'\,dt \text{ is a vanishing Carleson measure in } \mathbb{R}^n_+, \\ u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}=f \text{ a.e. in } \mathbb{R}^{n-1}, \quad f\in \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M), \end{cases} \tag{1-36}$$

is well-posed. Moreover, *its unique solution is given by* (1-30), *satisfies* (1-31)–(1-32), *and*

$$\lim_{\varepsilon\to 0^+} \|u(\cdot,\varepsilon)-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}=0. \tag{1-37}$$

It is reassuring to remark that replacing the boundary datum f by $f+C$ where $C\in\mathbb{C}^M$ in (1-29) changes the solution u into $u+C$ (given that convolution with the Poisson kernel reproduces constants from $\mathbb{C}^M$; see (2-36). As such, the $\widetilde{\mathrm{BMO}}$-Dirichlet problem for L in $\mathbb{R}^n_+$ is also well-posed, if uniqueness of the solution is now understood modulo constants from $\mathbb{C}^M$.

As regards the right-pointing implication in (1-34), for suitable dense subspaces of VMO we are able to precisely quantify the rate at which the Carleson measure $|\nabla u(x',t)|^2\,t\,dx'\,dt$ vanishes in $\mathbb{R}^n_+$. For example, with $\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)$ denoting the homogeneous Hölder space of order $\eta\in(0,1)$ of $\mathbb{C}^M$-valued functions defined in $\mathbb{R}^{n-1}$, it follows from (3-9) in Proposition 3.1, see also (2-19), that

$$\begin{gathered}\text{if } f\in\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M) \text{ with } \eta\in(0,1) \text{ and } u \text{ is as in (1-30), then} \\ \sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r}\Big(\int_0^{\ell(Q)}\fint_Q |\nabla u(x',t)|^2\,t\,dx'\,dt\Big)^{1/2}=O(r^\eta) \text{ as } r\to 0^+,\end{gathered} \tag{1-38}$$

where the multiplicative constant implicit in the big-O condition above depends only on n, L, η, and $\|f\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)}$. The relevance of this result stems from the fact that, for each $\eta\in(0,1)$, the collection of functions from $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ which also belong to $\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)$ makes up a dense subspace of $\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$. The latter density result constitutes one of the main results in this paper, and is formally stated in Theorem 1.5, along with a number of variants and generalizations. Let us also point out here that the decay rate in (1-38) is in agreement with the format of the well-posedness result proved later in Theorem 1.21, in view of (1-164) and (1-160).

The proof of Theorem 1.1 relies on a quantitative Fatou-type theorem, which includes a Poisson integral representation formula along with a characterization of BMO in terms of the traces of solutions to elliptic systems. This is stated next as Theorem 1.2. Among other things, this theorem shows that the conditions stipulated in the first three lines of (1-29) imply that the pointwise nontangential limit considered in the fourth line of (1-29) is always meaningful, and that the boundary datum should necessarily be selected from the space BMO. It also highlights the fact that it is natural to seek a solution of the BMO-Dirichlet problem by taking the convolution of the boundary datum with the Poisson kernel of L in the upper half-space. Finally, Theorem 1.2 is the key ingredient in the proof of uniqueness for the BMO-Dirichlet boundary value problem formulated in (1-29).

Theorem 1.2. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Then there exists a constant $C = C(L,n) \in (1,\infty)$ with the property that*

$$\left.\begin{array}{r} u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\ Lu = 0 \text{ in } \mathbb{R}^n_+ \\ \text{and } \|u\|_{**} < \infty \end{array}\right\} \Longrightarrow \begin{cases} u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} \text{ exists a.e. in } \mathbb{R}^{n-1}, \text{ lies in } \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \\ u(x',t) = \big(P^L_t * (u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+})\big)(x') \text{ for all } (x',t) \in \mathbb{R}^n_+, \\ \text{and } C^{-1}\|u\|_{**} \le \|u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \le C\|u\|_{**}. \end{cases} \tag{1-39}$$

In fact, the following characterization of $\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, *adapted to the system L, holds*:

$$\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \{u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} : u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M),\ Lu = 0 \text{ in } \mathbb{R}^n_+,\ \|u\|_{**} < \infty\}. \tag{1-40}$$

Moreover,

$$\mathrm{LMO}(\mathbb{R}^n_+) := \{u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) : Lu = 0 \text{ in } \mathbb{R}^n_+,\ \|u\|_{**} < \infty\} \tag{1-41}$$

is a linear space on which $\|\cdot\|_{**}$ is a seminorm with null-space $\mathbb{C}^M$, the quotient space $\mathrm{LMO}(\mathbb{R}^n_+)/\mathbb{C}^M$ *becomes complete* (*hence Banach*) *when equipped with $\|\cdot\|_{**}$, and the nontangential pointwise trace operator acting on equivalence classes in the context*

$$\mathrm{LMO}(\mathbb{R}^n_+)/\mathbb{C}^M \ni [u] \mapsto [u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M) \tag{1-42}$$

is a well-defined linear isomorphism between Banach spaces, *where* $[u]$ *in* (1-42) *denotes the equivalence class of u in* $\mathrm{LMO}(\mathbb{R}^n_+)/\mathbb{C}^M$ *and* $[u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}]$ *is interpreted as in* (1-19).

There is a counterpart of the Fatou-type result stated as Theorem 1.2 emphasizing the space VMO in place of BMO. Specifically, we prove the following theorem.

Theorem 1.3. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Then for any function*

$$u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \text{ satisfying } Lu = 0 \text{ in } \mathbb{R}^n_+ \text{ and } \|u\|_{**} < \infty \tag{1-43}$$

one has

$$\left.\begin{array}{r} |\nabla u(x',t)|^2\, t\, dx'\, dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}^n_+ \end{array}\right\} \Longrightarrow u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} \in \mathrm{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M). \tag{1-44}$$

Furthermore, the following characterization of the space $\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$, *adapted to the system* L, *holds*:

$$\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)=\big\{u\big|^{\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}:u\in\mathrm{LMO}(\mathbb{R}^n_+,\mathbb{C}^M)\text{ and }|\nabla u(x',t)|^2\,t\,dx'\,dt \text{ is a vanishing Carleson measure in }\mathbb{R}^n_+\big\}. \tag{1-45}$$

The analogue of Fefferman's theorem, characterizing BMO as in (1-1), in the case of elliptic systems with complex coefficients is the topic of the first item of our next theorem. The second item may be viewed as a characterization of VMO in the spirit of Fefferman's original result.

Theorem 1.4. *Let* L *be an* $M\times M$ *elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider* P^L, *the associated Poisson kernel for* L *in* $\mathbb{R}^n_+$ *from Theorem 2.3. Assume* $f:\mathbb{R}^{n-1}\to\mathbb{C}^M$ *is a Lebesgue-measurable function such that*

$$\int_{\mathbb{R}^{n-1}}\frac{|f(x')|}{1+|x'|^n}\,dx'<\infty. \tag{1-46}$$

Let u *be the Poisson integral of* f *with respect to the system* L, *i.e.*, $u:\mathbb{R}^n_+\to\mathbb{C}^M$ *is given by* $u(x',t):=(P^L_t*f)(x')$ *for each* $(x',t)\in\mathbb{R}^n_+$. *Then the following are true*:

(1) f *belongs to the space* $\mathrm{BMO}(\mathbb{R}^{n-1};\mathbb{C}^M)$ *if and only if* $\|u\|_{**}<\infty$.

(2) f *belongs to the space* $\mathrm{VMO}(\mathbb{R}^{n-1};\mathbb{C}^M)$ *if and only if* $|\nabla u(x',t)|^2\,t\,dx'\,dt$ *is a vanishing Carleson measure in* $\mathbb{R}^n_+$.

In our next result we shall revisit the issue of describing VMO as the closure within BMO of a subspace of functions whose pointwise oscillations vanish as the scale decreases to zero. One such description is contained in (1-26). However, for a variety of purposes (such as the proof of the result recorded in Theorem 1.13 below), the fact that the condition of uniform continuity is of a purely qualitative nature renders the space UC difficult to work with. As such, it is very desirable to replace it, in the context of Sarason's density result recorded in (1-26), by smaller subspaces within which uniform continuity may be suitably quantified. This issue is addressed in Theorem 1.5 below. As a preamble, we introduce some notation. Pick a modulus of continuity, i.e., a function

$$\Upsilon:[0,\infty)\to[0,\infty]\text{ nondecreasing and such that }\lim_{s\to 0^+}\Upsilon(s)=0. \tag{1-47}$$

Given $m\in\mathbb{N}$, consider the space

$$\mathscr{C}^\Upsilon(\mathbb{R}^m):=\big\{f:\mathbb{R}^m\to\mathbb{C}:\text{ there exists }C\in(0,\infty)\text{ such that } |f(a)-f(b)|\le C\,\Upsilon(|a-b|)\text{ for all }a,b\in\mathbb{R}^m\big\} \tag{1-48}$$

and define $\|f\|_{\mathscr{C}^\Upsilon(\mathbb{R}^m)}$ to be the smallest constant C intervening above. In the sequel, the space of $\mathbb{C}^M$-valued functions with components in $\mathscr{C}^\Upsilon(\mathbb{R}^m)$ will be denoted by $\mathscr{C}^\Upsilon(\mathbb{R}^m,\mathbb{C}^M)$. Clearly, $\mathscr{C}^\Upsilon(\mathbb{R}^m)\subseteq\mathrm{UC}(\mathbb{R}^m)$ and, in fact,

$$\mathrm{UC}(\mathbb{R}^m)=\bigcup_{\substack{\Upsilon\text{ modulus of}\\ \text{continuity}}}\mathscr{C}^\Upsilon(\mathbb{R}^m). \tag{1-49}$$

To see the left-to-right inclusion in (1-49), observe that if $f \in \mathrm{UC}(\mathbb{R}^m)$ is arbitrary and we define $\Upsilon_f(s) := \sup\{|f(x) - f(y)| : x, y \in \mathbb{R}^m,\ |x - y| \le s\}$ for each $s \in [0, \infty)$, then Υ_f is a modulus of continuity and $|f(a) - f(b)| \le \Upsilon_f(|a - b|)$ for all $a, b \in \mathbb{R}^m$; hence $f \in \mathscr{C}^{\Upsilon_f}(\mathbb{R}^m)$, as wanted.

Examples of interest are obtained by taking $\eta \in (0, 1]$ and defining $\Upsilon_\eta(s) := s^\eta$ for every $s \ge 0$. Then the space $\mathscr{C}^{\Upsilon_\eta}(\mathbb{R}^m)$ becomes precisely $\dot{\mathscr{C}}^\eta(\mathbb{R}^m)$, the space of functions satisfying a homogeneous Hölder condition of order η in $\mathbb{R}^m$ in the case when $\eta \in (0, 1)$, and becomes $\mathrm{Lip}(\mathbb{R}^m)$, the space of Lipschitz functions in $\mathbb{R}^m$, in the case when $\eta = 1$.

Here is the theorem advertised earlier, which may be regarded as a quantitative description of VMO, improving on Sarason's classical result (1-26).

Theorem 1.5. *Consider the function* $\Upsilon_\# : [0, \infty) \to [0, \infty)$ *given at each* $s \ge 0$ *by*

$$\Upsilon_\#(s) := \min\{1, s\} + \max\{0, \ln s\} = \begin{cases} s & \text{if } s \le 1, \\ 1 + \ln s & \text{if } s > 1. \end{cases} \tag{1-50}$$

Then for every modulus of continuity Υ *with the property that* $\Upsilon_\# \le C\,\Upsilon$ *on* $[0, \infty)$ *for some finite constant* $C > 0$, *the following density result holds for each* $n \in \mathbb{N}$:

$$\begin{gathered} \textit{for every function } f \in \mathrm{VMO}(\mathbb{R}^n) \textit{ there exists a sequence} \\ \{f_j\}_{j\in\mathbb{N}} \subset \mathscr{C}^\Upsilon(\mathbb{R}^n) \cap \mathscr{C}^\infty(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n) \textit{ such that } \|f - f_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \to 0 \textit{ as } j \to \infty. \end{gathered} \tag{1-51}$$

In short, $\mathscr{C}^\Upsilon(\mathbb{R}^n) \cap \mathscr{C}^\infty(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ *is dense in* $\mathrm{VMO}(\mathbb{R}^n)$. *In fact*,

$$\begin{gathered} \textit{the smaller space, consisting of } f \in \mathscr{C}^\Upsilon(\mathbb{R}^n) \cap \mathscr{C}^\infty(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n) \textit{ such that} \\ \partial^\alpha f \in \mathscr{C}^\Upsilon(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \textit{ for every } \alpha \in \mathbb{N}_0^n \textit{ with } |\alpha| \ge 1, \textit{ is also dense in } \mathrm{VMO}(\mathbb{R}^n). \end{gathered} \tag{1-52}$$

The proof of Theorem 1.5 (stated with $n - 1$ in place of n) relies on the fact that, given any $f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, we have, as seen from (1-30) and (1-33)–(1-34),

$$P_t^L * f \to f \text{ in } \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ as } t \to 0^+ \quad \Longleftrightarrow \quad f \in \mathrm{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \tag{1-53}$$

for some (or any) $M \times M$ elliptic system L with constant complex coefficients as in (1-2)–(1-3). A posteriori, once the density result in Theorem 1.5 has been established, we can considerably enlarge the class of approximations to the identity for which a result as in (1-53) holds, as described below.

Theorem 1.6. *Suppose* $\varphi : \mathbb{R}^n \to \mathbb{C}^{M\times M}$ *has the property that there exist* $C \in (0, \infty)$ *and* $\varepsilon \in (0, 1]$ *such that*

$$|\varphi(x)| \le C(1 + |x|)^{-n-\varepsilon} \quad \textit{for every } x \in \mathbb{R}^n \setminus \{0\}, \tag{1-54}$$

and

$$|\varphi(x + h) - \varphi(x)| \le \frac{C|h|^\varepsilon}{|x|^{n+\varepsilon}} \quad \textit{for all } x \in \mathbb{R}^n \setminus \{0\},\ h \in \mathbb{R}^n,\ |h| < |x|/2. \tag{1-55}$$

In addition, assume

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = I_{M\times M} \tag{1-56}$$

(*where* $I_{M\times M}$ *is the* $M\times M$ *identity matrix*). *Then*

$$\text{for each } f\in \mathrm{VMO}(\mathbb{R}^n,\mathbb{C}^M), \text{ it holds that } \varphi_t * f\to f \text{ in } \mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M) \text{ as } t\to 0^+, \tag{1-57}$$

where, *in the present context*, $\varphi_t(x):=t^{-n}\varphi(x/t)$ *for each* $x\in\mathbb{R}^n$ *and each* $t>0$.

As a consequence, *given* $\varphi\in\mathscr{C}^1(\mathbb{R}^n,\mathbb{C}^{M\times M})$ *such that* (1-56) *holds and such that there exists* $C\in(0,\infty)$ *for which*

$$|\varphi(x)|+|(\nabla\varphi)(x)|\le C(1+|x|)^{-n-1} \quad \text{for every } x\in\mathbb{R}^n, \tag{1-58}$$

one has the following real-variable characterization of the membership to VMO:

$$\begin{gathered}\text{for every function } f\in\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M) \text{ there holds}\\ \varphi_t * f\to f \text{ in } \mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M) \text{ as } t\to 0^+ \iff f\in\mathrm{VMO}(\mathbb{R}^n,\mathbb{C}^M).\end{gathered} \tag{1-59}$$

Several density results, of independent interest, are obtained by specializing Theorem 1.5 to moduli of continuity of the form $\Upsilon_\eta(s):=s^\eta$ for $s\ge 0$, with $\eta\in(0,1]$, simply by observing that there exists some finite constant $C_\eta>0$ with the property that $\Upsilon_\#\le C_\eta\Upsilon_\eta$ on $[0,\infty)$. To state these, recall that the inhomogeneous Hölder space of order $\eta\in(0,1)$ in $\mathbb{R}^n$ is defined as

$$\mathscr{C}^\eta(\mathbb{R}^n):=\dot{\mathscr{C}}^\eta(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n). \tag{1-60}$$

Corollary 1.7. *For each* $\eta\in(0,1)$,

$$\begin{gathered}\text{the space consisting of } f\in\dot{\mathscr{C}}^\eta(\mathbb{R}^n)\cap\mathscr{C}^\infty(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n) \text{ such that}\\ \partial^\alpha f\in\mathscr{C}^\eta(\mathbb{R}^n) \text{ for every } \alpha\in\mathbb{N}_0^n \text{ with } |\alpha|\ge 1 \text{ is dense in } \mathrm{VMO}(\mathbb{R}^n).\end{gathered} \tag{1-61}$$

Consequently, *for each* $\eta\in(0,1)$,

$$\dot{\mathscr{C}}^\eta(\mathbb{R}^n)\cap\mathscr{C}^\infty(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n) \text{ is a dense subspace of } \mathrm{VMO}(\mathbb{R}^n). \tag{1-62}$$

In particular, *for each* $\eta\in(0,1)$ *the space* $\dot{\mathscr{C}}^\eta(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n)$ *is dense in* $\mathrm{VMO}(\mathbb{R}^n)$. *Moreover*,

$$\begin{gathered}\text{the space consisting of functions } f\in\mathrm{Lip}(\mathbb{R}^n)\cap\mathscr{C}^\infty(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n) \text{ such that}\\ \partial^\alpha f\in\mathrm{Lip}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n) \text{ for every } \alpha\in\mathbb{N}_0^n \text{ with } |\alpha|\ge 1 \text{ is dense in } \mathrm{VMO}(\mathbb{R}^n).\end{gathered} \tag{1-63}$$

In particular,

$$\mathrm{Lip}(\mathbb{R}^n)\cap\mathscr{C}^\infty(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n) \text{ is a dense subspace of } \mathrm{VMO}(\mathbb{R}^n). \tag{1-64}$$

An interesting feature of Theorem 1.5 is that even though the conclusions are of a purely real-variable nature, its proof makes essential use of the PDE-rooted results established earlier (such as the well-posedness of the BMO-Dirichlet problem for, say, the Laplacian in $\mathbb{R}^n_+$). See Section 5 for details. Theorem 1.5 should be contrasted with the following negative result.

Theorem 1.8. *The space* $\mathrm{UC}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ *is not dense in* $\mathrm{VMO}(\mathbb{R}^n)$.

An example of an unbounded function belonging to $\mathrm{VMO}(\mathbb{R}^n)$ is

$$f(x):=\begin{cases}\ln\ln(1/|x|) & \text{if } |x|\le 1/e,\\ 0 & \text{if } |x|>1/e\end{cases} \qquad \text{for all } x\in\mathbb{R}^n. \tag{1-65}$$

In the context of the main density result presented in Theorem 1.5, the function $\Upsilon_\#$ defined in (1-50) exhibits an optimal behavior both at small and large scales, which cannot be improved, in the following precise sense: *if Υ is a modulus of continuity with the property that*

$$\text{either } \Upsilon(s)/s = o(1) \text{ as } s \to 0^+, \text{ or } \Upsilon(s) = O(1) \text{ as } s \to \infty, \tag{1-66}$$

then

$$\mathscr{C}^\Upsilon(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n) \text{ is not dense in } \mathrm{VMO}(\mathbb{R}^n). \tag{1-67}$$

Indeed, (1-67) is clear when the first eventuality in (1-66) materializes since the space $\mathscr{C}^\Upsilon(\mathbb{R}^n)$ reduces to just constants in this case. Also, in the scenario when the second possibility in (1-66) takes place, $\mathscr{C}^\Upsilon(\mathbb{R}^n)$ becomes a subspace of $\mathrm{UC}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, in which case the desired conclusion follows from Theorem 1.8.

Among other things, the density result stated in Corollary 1.7 permits us to quantify the proximity of a Littlewood–Paley-type measure to the class of vanishing Carleson measures in the upper half-space. This result, of a purely real variable nature, is formally stated in the theorem below.

Theorem 1.9. *Let $\psi \in \mathscr{C}^1(\mathbb{R}^n)$ be a function with the property that there exists $C \in (0, \infty)$ such that*

$$|\psi(x)| \le \frac{C}{(1+|x|)^{n+1}} \quad \text{and} \quad |(\nabla\psi)(x)| \le \frac{C}{(1+|x|)^{n+2}} \quad \text{for every } x \in \mathbb{R}^n, \quad \text{as well as } \int_{\mathbb{R}^n} \psi(x)\,dx = 0. \tag{1-68}$$

For each $x \in \mathbb{R}^n$ and $t > 0$ set $\psi_t(x) := t^{-n}\psi(x/t)$. Then for each function $f \in \mathrm{BMO}(\mathbb{R}^n)$

$$\mu_f(x,t) := |(\psi_t * f)(x)|^2 \,\frac{dx\,dt}{t} \tag{1-69}$$

is a Carleson measure in $\mathbb{R}^{n+1}_+$ satisfying

$$\lim_{r\to 0^+} \Bigg\{ \sup_{\substack{Q\subset\mathbb{R}^n\\ \ell(Q)\le r}} \frac{1}{|Q|} \int_0^{\ell(Q)}\!\!\int_Q |(\psi_t * f)(x)|^2 \,\frac{dx\,dt}{t} \Bigg\} \le C\,\mathrm{dist}(f, \mathrm{VMO}(\mathbb{R}^n))^2, \tag{1-70}$$

where $\mathrm{dist}(f, \mathrm{VMO}(\mathbb{R}^n)) := \inf\{\|f-g\|_{\mathrm{BMO}(\mathbb{R}^n)} : g \in \mathrm{VMO}(\mathbb{R}^n)\}$.

As a corollary,

$$\begin{gathered} \text{if } \psi \in \mathscr{C}^1(\mathbb{R}^n) \text{ is a function satisfying the conditions in (1-68) and } f \in \mathrm{VMO}(\mathbb{R}^n),\\ \text{it follows that } \mu_f(x,t), \text{ defined as in (1-69), is a vanishing Carleson measure in } \mathbb{R}^{n+1}_+. \end{gathered} \tag{1-71}$$

Theorem 1.9 allows us to establish the result stated below, which may be regarded as a quantified version of the equivalence (1-34) in Theorem 1.1.

Theorem 1.10. *Let L be an $M \times M$ elliptic constant complex coefficient system as in* (1-2)–(1-3). *Then there exists a constant $C = C(n, L) \in (0, \infty)$ with the property that for any given $f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ the unique solution u of the* BMO*-Dirichlet boundary value problem* (1-29) *for L in $\mathbb{R}^n_+$ with boundary datum f satisfies*

$$\lim_{r\to 0^+} \Bigg\{ \sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r} \int_0^{\ell(Q)}\!\!⨍_Q |\nabla u(x',t)|^2\, t\,dx'\,dt \Bigg\} \le C\,\mathrm{dist}(f, \mathrm{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M))^2, \tag{1-72}$$

where $\mathrm{dist}(f, \mathrm{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)) := \inf_{g\in\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \|f-g\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}$.

Moving on, if in analogy with (1-21) we also define

$$\widetilde{\mathrm{VMO}}(\mathbb{R}^n) := \{[f] : f \in \mathrm{VMO}(\mathbb{R}^n)\}, \tag{1-73}$$

then $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ becomes a closed subspace of the Banach space $(\mathrm{BMO}(\mathbb{R}^n), \|[\cdot]\|_{\widetilde{\mathrm{BMO}}(\mathbb{R}^n)})$. In particular, $(\widetilde{\mathrm{VMO}}(\mathbb{R}^n), \|[\cdot]\|_{\widetilde{\mathrm{BMO}}(\mathbb{R}^n)})$ is itself a Banach space. Likewise, for each $\eta \in (0,1)$ let us introduce the quotient space[1]

$$\dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim\, := \{[f] : f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n)\} \tag{1-74}$$

and equip it with the norm

$$\|[f]\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim} := \|f\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)} \quad \text{for all } [f] \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim. \tag{1-75}$$

Then $(\dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim, \|[\cdot]\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim})$ becomes a Banach space.

Regarding $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ as a Banach space in the fashion described above, Corollary 1.7 readily implies the following density result.

Corollary 1.11. *For each $\eta \in (0,1)$ the set $(\dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim) \cap \widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ is dense in $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$.*

The quantitative characterizations of the Sarason space provided in Theorem 1.5, Corollary 1.7, and Corollary 1.11 have important consequences as far as the mapping properties of Calderón–Zygmund operators on VMO are concerned. To elaborate on this aspect, we first recall the definition of the latter class of operators.

Definition 1.12. Given $n \in \mathbb{N}$, for each $\gamma \in (0,1]$ denote by $\mathrm{SCZ}(n,\gamma)$ the collection of all linear and continuous mappings $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ which extend to a bounded operator on $L^2(\mathbb{R}^n)$ and have the property that there exist $C', C'' \in (0,\infty)$ such that the Schwartz kernel $K(\cdot,\cdot)$ of T satisfies

$$K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \mathrm{diag}) \tag{1-76}$$

and, for every $x, y \in \mathbb{R}^n$ with $x \neq y$, and each $z \in \mathbb{R}^n$ with $|x-z| < \frac{1}{2}|x-y|$,

$$|K(x,y)| \le \frac{C'}{|x-y|^n} \quad \text{and} \quad |K(x,y) - K(z,y)| \le C'' \frac{|x-z|^\gamma}{|x-y|^{n+\gamma}}. \tag{1-77}$$

Simply call T a semi-Calderón–Zygmund operator in $\mathbb{R}^n$ if $T \in \bigcup_{0<\gamma\le 1} \mathrm{SCZ}(n,\gamma)$.

Also, for each $\gamma \in (0,1]$ introduce $\mathrm{CZ}(n,\gamma) := \{T \in \mathrm{SCZ}(n,\gamma) : T^\top \in \mathrm{SCZ}(n,\gamma)\}$ (where $T^\top : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is the transpose of T, with Schwartz kernel $K^\top(x,y) := K(y,x)$), and refer to the operators in $\bigcup_{0<\gamma\le 1} \mathrm{CZ}(n,\gamma)$ as being Calderón–Zygmund operators in $\mathbb{R}^n$.

Fix a semi-Calderón–Zygmund operator T in $\mathbb{R}^n$. A classical result in harmonic analysis (see, e.g., the proof of [Stein 1993, Theorem 3, p. 114], which readily adapts to the present setting) is the fact that $T^\top$ maps the Hardy space H^1 boundedly into the space of absolutely integrable functions; i.e.,

$$T^\top : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \tag{1-78}$$

[1] Observe that since we are presently dealing with continuous functions, $f \sim g$ means that $f - g$ is everywhere equal to a constant.

is a well-defined, linear, and bounded operator. In particular, this allows us to define $T(1)$ as a functional in $\widetilde{\mathrm{BMO}}(\mathbb{R}^n) = (H^1(\mathbb{R}^n))^*$ acting on any $h \in H^1(\mathbb{R}^n)$ according to

$$\langle T(1), h\rangle := \int_{\mathbb{R}^n} T^\top h \, d\mathscr{L}^n. \tag{1-79}$$

In particular, with the notion of H^1-atom as in (3-37),

$$T(1) = 0 \text{ in } \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \int_{\mathbb{R}^n} T^\top a \, d\mathscr{L}^n = 0 \text{ for each } H^1\text{-atom } a. \tag{1-80}$$

Via interpolation and duality we have

$$\begin{gathered}\text{if } T \text{ is a semi-Calderón–Zygmund operator then } T \text{ is bounded}\\ \text{on } L^p(\mathbb{R}^n) \text{ for each } p \in (2,\infty)\text{; as a consequence, if } T \text{ is a}\\ \text{Calderón–Zygmund operator then } T \text{ is bounded on } L^p(\mathbb{R}^n) \text{ for } p \in (1,\infty).\end{gathered} \tag{1-81}$$

In this vein, we wish to remark that (recall that a function $\Theta : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is said to be positive homogeneous of degree m provided $\Theta(\lambda x) = \lambda^m \Theta(x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$ and each $\lambda \in (0,\infty)$)

$$\begin{gathered}\text{a principal-value convolution-type operator } T_\Theta : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)\text{, given by}\\ T_\Theta f(x) := \lim_{\varepsilon\to 0^+} \textstyle\int_{y\in\mathbb{R}^n\setminus B(x,\varepsilon)} \Theta(x-y) f(y)\,dy \text{ for } f \in \mathscr{S}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n,\\ \text{with a kernel } \Theta \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\}) \text{ which is positive homogenous of degree } -n \text{ and}\\ \text{such that } \textstyle\int_{S^{n-1}} \Theta(\omega)\,d\omega = 0\text{, is a Calderón–Zygmund operator in } \mathbb{R}^n \text{ (in the sense}\\ \text{of Definition 1.12 with } \gamma = 1,\ C' = \|\Theta\|_{L^\infty(S^{n-1})}\text{, and } C'' = \|\nabla\Theta\|_{L^\infty(S^{n-1})})\\ \text{which satisfies } T_\Theta(1) = (T_\Theta)^\top(1) = 0 \text{ in } \widetilde{\mathrm{BMO}}(\mathbb{R}^n)\text{. Moreover, if we define}\\ \widetilde{\Theta}(x) := \Theta(-x) \text{ for each } x \in \mathbb{R}^n \setminus \{0\}\text{, then the transpose of } T_\Theta \text{ acting on } L^p(\mathbb{R}^n)\\ \text{with } 1 < p < \infty \text{ is the operator } T_{\widetilde{\Theta}} \text{ acting on } L^{p'}(\mathbb{R}^n) \text{ where, } 1/p + 1/p' = 1.\end{gathered} \tag{1-82}$$

This is a consequence of the fact that such an operator T_Θ is a multiplier (see, e.g., [Mitrea 2013, Theorem 4.96, pp. 172–173]), i.e., it satisfies $\widehat{T_\Theta \varphi} = m_\Theta \hat{\varphi}$ for each $\varphi \in \mathscr{S}(\mathbb{R}^n)$, where "hat" stands for the Fourier transform. The symbol m_Θ is the Fourier transform of the tempered distribution P.V. Θ, defined as, see [loc. cit., (4.4.2), p. 136],

$$\langle \mathrm{P.V.}\,\Theta, \varphi\rangle := \lim_{\varepsilon\to 0^+} \int_{x\in\mathbb{R}^n,\,|x|>\varepsilon} \Theta(x)\varphi(x)\,dx \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^n); \tag{1-83}$$

i.e.,

$$m_\Theta = \widehat{\mathrm{P.V.}\,\Theta} \quad \text{in } \mathscr{S}'(\mathbb{R}^n). \tag{1-84}$$

From [loc. cit., Theorem 4.71, p. 142] it is known that

$$\begin{aligned} m_\Theta(\xi) &= -\int_{S^{n-1}} \Theta(\omega) \log(i\,\xi\cdot\omega)\,d\omega \\ &= -\int_{S^{n-1}} \Theta(\omega) \Big(\ln\Big|\frac{\xi}{|\xi|}\cdot\omega\Big| + i\,\frac{\pi}{2}\,\mathrm{sgn}(\xi\cdot\omega)\Big)\,d\omega \quad \text{for each } \xi \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{1-85}$$

where the last equality uses the vanishing-moment property of Θ; see [loc. cit., (4.5.15), p. 143]. From this representation it is then apparent (reasoning as in [loc. cit, Step II, pp. 349–350]) that

$$\begin{gathered}\text{the restriction of } m_\Theta \text{ to } \mathbb{R}^n\setminus\{0\} \text{ is a function having the same order of differentiability}\\ \text{as } \Theta\text{, is positive homogeneous of degree zero bounded, satisfies } \overline{m_{\widetilde{\Theta}}}=m_{\overline{\Theta}} \text{ and}\\ \textstyle\int_{S^{n-1}} m_\Theta(\omega)\,d\omega=0, \text{ as well as } m_{\widetilde{\Theta}}(\xi)=m_\Theta(-\xi) \text{ for each } \xi\in\mathbb{R}^n\setminus\{0\}.\end{gathered} \tag{1-86}$$

Let us also note that, starting with (1-85) and making use of [loc. cit, Proposition 13.46, p. 439] it is not difficult to see that

$$\text{for each } p\in(1,\infty] \text{ there exists } C_{n,p}\in[0,\infty) \text{ such that } \|m_\Theta\|_{L^\infty(\mathbb{R}^n)}\le C_{n,p}\|\Theta\|_{L^p(S^{n-1})}. \tag{1-87}$$

In turn, via Parseval's formula these properties imply that T_Θ extends to a linear and bounded operator on $L^2(\mathbb{R}^n)$ which satisfies

$$\widehat{T_\Theta f}=m_\Theta\hat{f} \quad \text{for each } f\in L^2(\mathbb{R}^n). \tag{1-88}$$

In addition, for each $f, g\in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}(T_\Theta f)(x)g(x)\,dx&=(2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{T_\Theta f}(\xi)\hat{g}(-\xi)\,d\xi=(2\pi)^{-n}\int_{\mathbb{R}^n}m_\Theta(\xi)\hat{f}(\xi)\hat{g}(-\xi)\,d\xi\\ &=(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{f}(\xi)\widehat{T_{\widetilde{\Theta}}g}(-\xi)\,d\xi=\int_{\mathbb{R}^n}f(x)(T_{\widetilde{\Theta}}g)(x)\,dx,\end{aligned} \tag{1-89}$$

from which we ultimately conclude that the transpose of T_Θ is $T_{\widetilde{\Theta}}$. Moreover, for each given H^1-atom a, the fact that $T_\Theta a$ belongs to $L^1(\mathbb{R}^n)$, see (1-78), implies that $\widehat{T_\Theta a}$ is a continuous function satisfying $\int_{\mathbb{R}^n} T_\Theta a\, d\mathscr{L}^n=\widehat{T_\Theta a}(0)=\lim_{\xi\to 0} m_\Theta(\xi)\hat{a}(\xi)=0$ since m_Θ is bounded, $\hat{a}$ is continuous (given that $a\in L^1(\mathbb{R}^n)$), and $\hat{a}(0)=\int_{\mathbb{R}^n} a\, d\mathscr{L}^n=0$ thanks to the vanishing-moment property of the atom. In light of (1-80), this shows that $T_\Theta(1)=0$. Finally, in a similar fashion, $(T_\Theta)^\top(1)=0$.

Natural examples of operators of the sort discussed in (1-82) are offered by the Riesz transforms in $\mathbb{R}^n$. These are defined as the family $(R_j)_{1\le j\le n}$ where, for each $j\in\{1,\dots,n\}$ and each $f\in L^p(\mathbb{R}^n)$ with $1\le p<\infty$, we set

$$\begin{aligned}(R_j f)(x)&:=\lim_{\varepsilon\to 0^+}\int_{y\in\mathbb{R}^n\setminus B(x,\varepsilon)}K_j(x-y)f(y)\,dy, \quad x\in\mathbb{R}^n,\\ K_j(z)&:=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\frac{z_j}{|z|^{n+1}} \quad \text{for each } z\in\mathbb{R}^n\setminus\{0\}.\end{aligned} \tag{1-90}$$

These are singular integral operators of convolution type involving odd kernels. A prominent example of a singular integral operator of convolution type involving an even kernel (with vanishing integral on the unit sphere) is offered by the Beurling (or Beurling–Ahlfors) transform in the complex plane

$$(Sf)(z):=-\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\substack{\zeta\in\mathbb{C}\\ |z-\zeta|>\varepsilon}}\frac{f(\zeta)}{(z-\zeta)^2}\,d\mathscr{L}^2(\zeta), \quad z\in\mathbb{C}. \tag{1-91}$$

This has the basic property that

$$S(\partial_{\bar{z}}f)=\partial_z f \quad \text{for each Schwartz function } f\in\mathscr{S}(\mathbb{C}), \tag{1-92}$$

where $\partial_{\bar{z}} := \frac{1}{2}(\partial_x - (1/i)\partial_y)$ and $\partial_z := \frac{1}{2}(\partial_x + (1/i)\partial_y)$ are, respectively, the Cauchy–Riemann operator and its complex conjugate.

To state the result pertaining to the boundedness of semi-Calderón–Zygmund operators on the space of functions of vanishing mean oscillations advertised earlier, recall that the quotient space $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ was defined in (1-73).

Theorem 1.13. *Consider a semi-Calderón–Zygmund operator T in $\mathbb{R}^n$ satisfying $T(1) = 0$. Extend T to a linear and bounded operator $\widetilde{T}$ from $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ into itself by setting* (*with $\langle\cdot,\cdot\rangle$ denoting the $\widetilde{\mathrm{BMO}}$-H^1 duality pairing*; *see item* (iv) *of Proposition 7.6*)

$$\begin{aligned} &\widetilde{T} : \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \to \widetilde{\mathrm{BMO}}(\mathbb{R}^n), \\ &\langle \widetilde{T}[f], g\rangle := \langle [f], T^{\top} g\rangle \quad \text{for all } [f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n), \ \text{for all } g \in H^1(\mathbb{R}^n). \end{aligned} \tag{1-93}$$

Then $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ is an invariant subspace of $\widetilde{T}$. In particular, *its restriction to $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$*,

$$\begin{aligned} &\widetilde{T}|_{\mathrm{VMO}} : \widetilde{\mathrm{VMO}}(\mathbb{R}^n) \to \widetilde{\mathrm{VMO}}(\mathbb{R}^n), \\ &(\widetilde{T}|_{\mathrm{VMO}})[f] := \widetilde{T}[f] \quad \text{for each } [f] \in \widetilde{\mathrm{VMO}}(\mathbb{R}^n), \end{aligned} \tag{1-94}$$

is a well-defined, *linear and bounded operator. Moreover*, *$\widetilde{T}|_{\mathrm{VMO}}$ is compatible with the action of T on Lebesgue spaces in the sense that for each $p \in [2,\infty)$ one has*

$$(\widetilde{T}|_{\mathrm{VMO}})[f] = [Tf] \quad \text{for all } f \in \mathrm{VMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n). \tag{1-95}$$

Example 1. In view of (1-82), Theorem 1.13 applies directly to the Riesz transforms in $\mathbb{R}^n$, as well as the Beurling transform in $\mathbb{C}$. More generally, given any principal-value convolution-type operator T_Θ as in (1-82), its realization as a linear and bounded mapping from the space $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ into itself, via the transposition formula

$$\begin{aligned} &\widetilde{T}_\Theta : \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \to \widetilde{\mathrm{BMO}}(\mathbb{R}^n), \\ &\langle \widetilde{T}_\Theta[f], g\rangle := \langle [f], T_{\widetilde{\Theta}} g\rangle \quad \text{for all } [f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n), \ \text{for all } g \in H^1(\mathbb{R}^n), \end{aligned} \tag{1-96}$$

where $\langle\cdot,\cdot\rangle$ stands for the $\widetilde{\mathrm{BMO}}$-H^1 duality pairing, and $\widetilde{\Theta}(x) := \Theta(-x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$, induces a well-defined, linear and bounded operator

$$\widetilde{T}_\Theta|_{\mathrm{VMO}} : \widetilde{\mathrm{VMO}}(\mathbb{R}^n) \to \widetilde{\mathrm{VMO}}(\mathbb{R}^n). \tag{1-97}$$

Example 2. Recall that, for a given Lipschitz function $A : \mathbb{R} \to \mathbb{C}$, the Calderón commutator of order $m \in \mathbb{N}_0$ is the principal-value singular integral operator C_m on the real line whose kernel is given by

$$K_m(x,y) := \frac{(A(x) - A(y))^m}{(x-y)^{m+1}}, \quad x, y \in \mathbb{R}, \ x \neq y. \tag{1-98}$$

It is then a basic fact that each C_m is a Calderón–Zygmund operator (e.g., C_0 is, up to normalization, the Hilbert transform on the real line). In particular, they all extend to well-defined and bounded linear operators from $L^\infty(\mathbb{R})$ into $\mathrm{BMO}(\mathbb{R})$. Retaining the same notation for the said extensions, a well-known trick (based on integration by parts) then yields the following remarkable recursive identity, see [Meyer 1990, (2.14), p. 266],

$$C_m(1) = C_{m-1}(A') \quad \text{for each } m \in \mathbb{N}. \tag{1-99}$$

In relation to the above family of operators, for each $m \in \mathbb{N}$ let us consider the principal-value singular integral operator T_m on the real line associated with the modified kernel

$$\begin{aligned} k_m(x,y) &:= K_m(x,y) - K_{m-1}(x,y)A'(y) \\ &= \frac{(A(x)-A(y))^{m-1}}{(x-y)^{m+1}}\{A(x)-A(y)-(x-y)A'(y)\}, \quad x,y \in \mathbb{R},\ x \neq y. \end{aligned} \tag{1-100}$$

Since, generally speaking, the function A' is only essentially bounded, the operator T_m is only semi-Calderón–Zygmund (as opposed to C_m which is a genuine Calderón–Zygmund operator). This being said, in contrast with (1-99) we presently have $T_m(1) = C_m(1) - C_{m-1}(A') = 0$. Granted these, Theorem 1.13 applies and gives that

$$\begin{gathered} T_m, \text{ the modified Calderón commutator of order } m \in \mathbb{N} \text{ on the} \\ \text{real line, associated with the kernel } k_m \text{ defined in (1-100),} \\ \text{induces a bounded operator from the space } \widetilde{\mathrm{VMO}}(\mathbb{R}) \text{ into itself.} \end{gathered} \tag{1-101}$$

Example 3. Consider the principal-value Cauchy singular integral operator $\mathcal{C}$ on a curve $\Sigma \subseteq \mathbb{C}$ which is the graph of a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$. That is, $\Sigma := \{z = x + iA(x) : x \in \mathbb{R}\}$ and $\mathcal{C}$ acts on a function $f : \Sigma \to \mathbb{C}$ according to

$$\mathcal{C}f(z) := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\zeta \in \Sigma \setminus B(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Sigma. \tag{1-102}$$

Making the bi-Lipschitz change of variables $\mathbb{R} \ni x \mapsto x + iA(x) \in \Sigma$ and identifying f with the function $g(x) := f(x + iA(x))$ for $x \in \mathbb{R}$, this becomes (after adjusting the truncation; see [Hofmann et al. 2015, Lemma B.1] in this regard) the principal-value singular integral operator on the real line

$$Tg(x) := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{y \in \mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{(1 + iA'(y))g(y)}{y - x + i(A(y) - A(x))}\, dy, \quad x \in \mathbb{R}. \tag{1-103}$$

While the above integral kernel is, generally speaking, lacking smoothness in the y-variable, T is nonetheless a semi-Calderón–Zygmund operator on $\mathbb{R}$, and we claim that $T(1) = 0$. To justify this claim, pick an arbitrary H^1-atom a on the real line and observe that if

$$b : \Sigma \to \mathbb{C} \text{ is defined as } b(x + iA(x)) := \frac{a(x)}{1 + iA'(x)} \text{ for } x \in \mathbb{R}, \tag{1-104}$$

then $\int_\Sigma b(z)\, dz = \int_{\mathbb{R}} a\, d\mathscr{L}^1 = 0$ and

$$\int_{\mathbb{R}} T^\top a\, d\mathscr{L}^1 = -\int_\Sigma (\mathcal{C}b)(z)\, dz = -\int_\Sigma \big((\tfrac{1}{2}I + \mathcal{C})b\big)(z)\, dz = 0. \tag{1-105}$$

The last equality above relies on Cauchy's vanishing formula, see [Mitrea et al. 2017], applied to the function defined in the domain $\Omega \subseteq \mathbb{C}$ lying above the graph Σ by $u(z) := 1/(2\pi i) \int_\Sigma b(\zeta)(\zeta - z)^{-1}\, d\zeta$ for each $z \in \Omega$, which has an integrable nontangential maximal function on $\Sigma = \partial\Omega$ and whose nontangential boundary trace is precisely $\big(\tfrac{1}{2}I + \mathcal{C}\big)b$ at a.e. point on $\Sigma = \partial\Omega$. In view of (1-80), we conclude from (1-105) that, indeed, $T(1) = 0$.

With the knowledge that T is a semi-Calderón–Zygmund operator on $\mathbb{R}$ satisfying $T(1) = 0$, we can apply Theorem 1.13, which gives that

$$\text{the principal-value Cauchy singular integral operator, defined on the real line as in (1-103),} \\ \text{induces a well-defined, linear and bounded operator from the space } \widetilde{\mathrm{VMO}}(\mathbb{R}) \text{ into itself.} \tag{1-106}$$

This result may be further generalized to higher dimensions by considering the principal-value Cauchy–Clifford singular integral operator on a Lipschitz surface as in [Mitrea 1994].

Example 4. Having fixed $n \in \mathbb{N}$, recall the principal-value harmonic double-layer $\mathcal{K}$, defined on a surface $\Sigma \subseteq \mathbb{R}^{n+1}$ which is the graph of a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}$. Specifically,

$$\Sigma := \{X = (x, A(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\},$$

and $\mathcal{K}$ maps a function $f : \Sigma \to \mathbb{C}$ into

$$\mathcal{K}f(X) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int_{Y \in \Sigma \setminus B(X,\varepsilon)} \frac{\langle \nu(Y), Y - X \rangle}{|X - Y|^{n+1}} \, d\sigma(Y), \quad X \in \Sigma, \tag{1-107}$$

where ν and σ, the unit normal and surface measure to Σ, are given by

$$\nu(x, A(x)) = \frac{(\nabla A(x), -1)}{\sqrt{1 + |\nabla A(x)|^2}}, \quad d\sigma(x, A(x)) = \sqrt{1 + |\nabla A(x)|^2}\, dx, \quad x \in \mathbb{R}^n. \tag{1-108}$$

Much as in the case of the Cauchy operator considered earlier, make the bi-Lipschitz change of variables $\mathbb{R}^n \ni x \mapsto (x, A(x)) \in \Sigma$ and identify f with the function $g(x) := f(x, A(x))$ for $x \in \mathbb{R}^n$. This permits us to identify the harmonic double-layer $\mathcal{K}$ with the principal-value singular integral operator in $\mathbb{R}^n$ given by

$$Tg(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int_{y \in \mathbb{R}^n \setminus B(x,\varepsilon)} \frac{A(x) - A(y) - \langle x - y, \nabla A(y) \rangle}{(|x - y|^2 + (A(x) - A(y))^2)^{(n+1)/2}} \, g(y)\, dy, \quad x \in \mathbb{R}^n. \tag{1-109}$$

We remark that the integral kernel above does not, generally speaking, possess any smoothness in the y-variable. Nonetheless, T is bounded on $L^2(\mathbb{R}^n)$, see [Meyer 1990, Théorème 11, p. 320]; hence T is a semi-Calderón–Zygmund operator on $\mathbb{R}^n$. We claim that $T(1) = 0$. To see that this is the case, pick an arbitrary H^1-atom a in $\mathbb{R}^n$ and note that if

$$b : \Sigma \to \mathbb{C} \text{ is defined as } b(x, A(x)) := \frac{a(x)}{\sqrt{1 + |\nabla A(x)|^2}} \text{ for } x \in \mathbb{R}^n, \tag{1-110}$$

then $\int_\Sigma b \, d\sigma = \int_{\mathbb{R}^n} a \, d\mathscr{L}^n = 0$. Also, if we denote by $\mathcal{K}^\top$ the transpose of $\mathcal{K}$ on $L^2(\Sigma)$, then

$$\int_{\mathbb{R}^n} T^\top a \, d\mathscr{L}^n = \int_\Sigma \mathcal{K}^\top b \, d\sigma = \int_\Sigma \big(-\tfrac{1}{2}I + \mathcal{K}^\top\big) b \, d\sigma = 0. \tag{1-111}$$

The last equality above relies on the version of the divergence formula established in [Mitrea et al. 2017], currently used for the vector field defined in the domain $\Omega \subseteq \mathbb{R}^{n+1}$ lying above the surface Σ by

$$\vec{F}(X) := \frac{1}{\omega_n} \int_{\Sigma} \frac{X-Y}{|X-Y|^{n+1}}\, b(Y)\, d\sigma(Y) \quad \text{for all } X \in \Omega, \tag{1-112}$$

which is smooth and divergence-free in Ω, has an integrable nontangential maximal function, and whose nontangential boundary trace $\vec{F}\big|^{\text{n.t.}}_{\partial\Omega}$ satisfies $\nu \cdot (\vec{F}\big|^{\text{n.t.}}_{\partial\Omega}) = \big(-\frac{1}{2}I + K^{\top}\big)b$ at σ-a.e. point on $\Sigma = \partial\Omega$; see [Mitrea et al. 2017] for more details. Having proved (1-111) we then conclude from (1-80) that $T(1) = 0$, as wanted. Given that T is a semi-Calderón–Zygmund operator in $\mathbb{R}^n$ satisfying $T(1) = 0$, from Theorem 1.13 we may then conclude that

the principal-value harmonic double-layer operator, defined in $\mathbb{R}^n$ as in (1-109), induces a well-defined, linear and bounded operator from the space $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ into itself. (1-113)

To close, we mention that similar results are valid for the pull-back from a Lipschitz graph to the Euclidean space of any double-layer potential operator associated with a homogeneous second-order elliptic system.

Moving on, we note that the argument which proves Theorem 1.13 is indicative of a more general principle at play here, to the effect that, regardless of its actual format,

any linear operator which is bounded both on BMO and on a (homogeneous) Hölder space is also bounded on VMO. (1-114)

In relation to (1-114), it is also worth pointing out that the class of operators which are simultaneously bounded on BMO as well as on some common (homogeneous) Hölder space is considerably larger than the class of the semi-Calderón–Zygmund operators considered in Theorem 1.13 since, as opposed to the latter, the former is stable under composition, and hence, in particular, constitutes an algebra. This being said, by additionally hypothesizing a suitable cancellation condition for the transpose, one can identify a (maximal) subfamily of Calderón–Zygmund operators which do make up an algebra. To facilitate stating such a result, for any given Banach space $\mathcal{X}$ we agree to denote by $\mathscr{B}(\mathcal{X})$ the Banach algebra of linear and bounded operators from $\mathcal{X}$ into itself (with respect to the ordinary addition and composition of operators, and ordinary operator norm).

Theorem 1.14. *Fix $n \in \mathbb{N}$ arbitrary. Then the family $\mathscr{A}^{\mathbf{0}}_{\widetilde{\mathrm{CZ}}}$ consisting of all operators $\widetilde{T}|_{\mathrm{VMO}}$ as in* (1-94), *where T is a Calderón–Zygmund operator in $\mathbb{R}^n$ satisfying $T(1) = T^{\top}(1) = 0$, is a subalgebra of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$.*

The family of principal-value convolution-type operators T_Θ associated as in (1-82) with kernels Θ which are actually of class $\mathscr{C}^\infty$ in $\mathbb{R}^n \setminus \{0\}$ also gives rise to an algebra of linear and bounded operators on $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$, of the sort described in our next theorem.

Theorem 1.15. *Fix $n \in \mathbb{N}$ arbitrary. Associate with each complex-valued function*

$\Theta \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$, *positive homogenous of degree $-n$, and with the cancellation property $\int_{S^{n-1}} \Theta(\omega)\, d\omega = 0$* (1-115) *the principal-value convolution-type singular integral operator T_Θ defined as in* (1-82), *and denote by $\widetilde{T}_\Theta$ its realization as a linear and bounded mapping from the space $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ into itself as in* (1-96). *Then, with I denoting the identity operator, the following properties hold*:

(a) *The set*

$$\mathscr{A}_{\widetilde{\mathrm{SIO}}} := \{cI + \widetilde{T}_\Theta|_{\mathrm{VMO}} : \widetilde{\mathrm{VMO}}(\mathbb{R}^n) \to \widetilde{\mathrm{VMO}}(\mathbb{R}^n) : c \in \mathbb{C} \text{ and } \Theta \text{ as in (1-115)}\} \tag{1-116}$$

is a commutative unital subalgebra of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$. In $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$ the following composition law holds: *if $c \in \mathbb{C}$ and the functions $\Theta_1, \dots, \Theta_N, \Theta'_1, \dots, \Theta'_N, \Theta$ are as in* (1-115) *and satisfy*

$$\sum_{j=1}^N m_{\widetilde{\Theta}'_j} m_{\widetilde{\Theta}_j} = c + m_{\widetilde{\Theta}} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{1-117}$$

then

$$\sum_{j=1}^N (\widetilde{T}_{\Theta'_j}|_{\mathrm{VMO}})(\widetilde{T}_{\Theta_j}|_{\mathrm{VMO}}) = cI + \widetilde{T}_\Theta|_{\mathrm{VMO}} \quad \text{in } \mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n)). \tag{1-118}$$

(b) *With the bar denoting the closure in $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$*,

$$\overline{\mathscr{A}}_{\widetilde{\mathrm{SIO}}} = \overline{\mathrm{span}}\{\widetilde{R}_j|_{\mathrm{VMO}}\}_{1 \le j \le n}; \tag{1-119}$$

that is, $\overline{\mathscr{A}}_{\widetilde{\mathrm{SIO}}}$ coincides with the smallest closed subalgebra of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$ containing the Riesz transforms, $\widetilde{R}_j|_{\mathrm{VMO}} \in \mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$ with $1 \le j \le n$.

(c) *Whenever the function Θ is as in* (1-115) *and*

$$c \in \mathbb{C} \setminus \{-m_{\widetilde{\Theta}}(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\}, \tag{1-120}$$

it follows that $cI + \widetilde{T}_\Theta|_{\mathrm{VMO}}$ has an inverse in $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$. More specifically, whenever Θ is as in (1-115) *and c is as in* (1-120), *the operator $cI + \widetilde{T}_\Theta \in \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))$ has an inverse in $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ of the form $c_0 I + \widetilde{T}_{\Theta_0} \in \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))$ for some $c_0 \in \mathbb{C}$ and Θ_0 as in* (1-115), *with the property that $c_0 I + \widetilde{T}_{\Theta_0}|_{\mathrm{VMO}}$ is the inverse of $cI + \widetilde{T}_\Theta|_{\mathrm{VMO}}$ in $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$.*

(d) *Suppose Θ is as in* (1-115) *and c is as in* (1-120). *Then for each $f \in \mathrm{BMO}(\mathbb{R}^n)$ one has*

$$f \in \mathrm{VMO}(\mathbb{R}^n) \quad \Longleftrightarrow \quad (cI + \widetilde{T}_\Theta)[f] \in \widetilde{\mathrm{VMO}}(\mathbb{R}^n). \tag{1-121}$$

More generally, let $N \in \mathbb{N}$ be a given integer and assume $\Theta_1, \dots, \Theta_N$ is a family of functions, each of which as in (1-115). *Also, fix*

$$(c_1, \dots, c_N) \in \mathbb{C}^N \setminus \big\{(-m_{\widetilde{\Theta}_j}(\xi))_{1 \le j \le N} : \xi \in \mathbb{R}^n \setminus \{0\}\big\}. \tag{1-122}$$

Then for each given function $f \in \mathrm{BMO}(\mathbb{R}^n)$ one has

$$f \in \mathrm{VMO}(\mathbb{R}^n) \quad \Longleftrightarrow \quad (c_j I + \widetilde{T}_{\Theta_j})[f] \in \widetilde{\mathrm{VMO}}(\mathbb{R}^n) \text{ for each } j \in \{1, \dots, N\}. \tag{1-123}$$

(e) *Items* (a), (c), *and the first part of* (d), *have natural versions in the case when the functions involved are vector-valued and the kernels of the singular integral operators are matrix-valued. The specifics of this more general setting are as follows. Given a finite-dimensional complex vector space $\mathscr{V}$, consider*

$\mathscr{V}$-valued functions whose scalar components (with respect to some fixed basis of $\mathscr{V}$) are from $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ (or $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$, depending on the context). Also, consider principal-value convolution-type operators T_Θ defined as in (1-82), *associated with kernels Θ as in* (1-115) *taking values in* $\mathrm{Hom}(\mathscr{V},\mathscr{V})$. *In particular, T_Θ may be viewed as a matrix of ordinary scalar, principal-value, convolution-type operators, and extending each individual entry in this matrix as in* (1-96) *then yields a linear and bounded operator $\widetilde{T}_\Theta$ from $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\mathscr{V}$ into itself which leaves the subspace $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)\otimes\mathscr{V}$ invariant.*

The version of item (a) *in this setting is that if one now defines $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$ as in* (1-116), *but with the intervening singular integral operators as just described above and with cI now replaced by $c\in$* $\mathrm{Hom}(\mathscr{V},\mathscr{V})$ *arbitrary, then $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$ becomes a (typically noncommutative) subalgebra of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n)\otimes\mathscr{V})$. Finally, in the case of item* (c) *and the first part of item* (d), *condition* (1-120) *is now replaced by*

$$c+m_{\widetilde{\Theta}}(\xi) \text{ is invertible in } \mathrm{Hom}(\mathscr{V},\mathscr{V}) \text{ for each } \xi\in\mathbb{R}^n\setminus\{0\}. \tag{1-124}$$

Theorem 1.15, whose proof is presented in Section 7, has many consequences of independent interest, which we shall now explore. We begin by stating a version of the first claim in item (d) of Theorem 1.15 for kernels taking values in a finite-dimensional algebra (again, proved in Section 7).

Corollary 1.16. *Let $A=(A,+,\odot,1)$ be a finite-dimensional (complex) unital associative algebra. Fix $n\in\mathbb{N}$ arbitrary and associate with each A-valued function*

$$\begin{gathered}\Theta:\mathbb{R}^n\setminus\{0\}\to A \text{ which is of class } \mathscr{C}^\infty\text{, positive homogenous of}\\ \text{degree } -n\text{, and with the cancellation property } \textstyle\int_{S^{n-1}}\Theta(\omega)\,d\omega=0;\end{gathered} \tag{1-125}$$

consider the principal-value convolution-type operator T_Θ acting on A-valued Schwartz functions $f\in\mathscr{S}(\mathbb{R}^n)\otimes A$ according to $T_\Theta f(x):=\lim_{\varepsilon\to0^+}\int_{y\in\mathbb{R}^n\setminus B(x,\varepsilon)}\Theta(x-y)\odot f(y)\,dy$ for $x\in\mathbb{R}^n$.

Denote by $\widetilde{T}_\Theta$ the realization of the operator T_Θ as a linear and bounded mapping from the space $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes A$ into itself, obtained by extending each scalar component of T_Θ to $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ as in (1-96). *Also, fix some*

$$c\in A \text{ such that } c+m_{\widetilde{\Theta}}(\xi) \text{ is invertible in } A \text{ from the right for each } \xi\in\mathbb{R}^n\setminus\{0\}. \tag{1-126}$$

Then, with I denoting the identity operator, for each $f\in\mathrm{BMO}(\mathbb{R}^n)\otimes A$ one has

$$f\in\mathrm{VMO}(\mathbb{R}^n)\otimes A \iff (cI+\widetilde{T}_\Theta)[f]\in\widetilde{\mathrm{VMO}}(\mathbb{R}^n)\otimes A. \tag{1-127}$$

Historically, the Riesz transforms have been successfully employed in characterizing the regularity of functions in the Euclidean space. For example, it is well known, see, e.g., [García-Cuerva and Rubio de Francia 1985, (4.11), p. 284], that the Hardy space $H^1(\mathbb{R}^n)$ may be described as

$$H^1(\mathbb{R}^n)=\{f\in L^1(\mathbb{R}^n): R_jf\in L^1(\mathbb{R}^n) \text{ for } 1\le j\le n\}. \tag{1-128}$$

Also, if for each $j\in\{1,\dots,n\}$ we denote by $\widetilde{R}_j$ the extension of the j-th Riesz transform, originally acting on $L^2(\mathbb{R}^n)$ as in (1-90), to a bounded operator on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ defined as in (1-96), then the following characterization of the space $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ may be deduced from [Fefferman 1971, Theorem 2, p. 587]:

$$\widetilde{\mathrm{BMO}}(\mathbb{R}^n) = \Big\{ [g_0] + \sum_{j=1}^{n} \widetilde{R}_j [g_j] : g_0, g_1, \dots, g_n \in L^\infty(\mathbb{R}^n) \Big\}. \tag{1-129}$$

In a similar vein, a characterization of the space $\mathrm{VMO}(\mathbb{R})$ as (where H is the Hilbert transform on the real line)

$$\mathrm{VMO}(\mathbb{R}) = \{u + Hv : u, v \in L^\infty(\mathbb{R}) \cap \mathrm{UC}(\mathbb{R})\} \tag{1-130}$$

was given by Sarason [1975, Theorem 1, p. 392]. Let us also mention that regularity results of a geometric flavor involving the Riesz transforms were established in [Mitrea et al. 2016b]. Here is a result along this line of work, providing characterizations of the Sarason space VMO in terms of Riesz and Beurling transforms in the complex plane.

Corollary 1.17. *Work in the two-dimensional setting $\mathbb{R}^2 \equiv \mathbb{C}$ and consider the complex Riesz transform*

$$R_{\mathbb{C}} f(z) := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\zeta \in \mathbb{C} \setminus B(z,\varepsilon)} \frac{z - \zeta}{|z - \zeta|^3} f(\zeta)\, d\mathscr{L}^2(\zeta), \quad z \in \mathbb{C}. \tag{1-131}$$

Denote by $\widetilde{R}_{\mathbb{C}}$ the extension of the complex Riesz transform, originally considered as in (1-131) *on $L^2(\mathbb{C})$, see* (1-82), *to a linear and bounded operator on $\widetilde{\mathrm{BMO}}(\mathbb{C})$, see* (1-96). *Analogously, denote by $\widetilde{S}$ the extension of the Beurling transform defined as in* (1-91) *on $L^2(\mathbb{C})$ to a linear and bounded operator on $\widetilde{\mathrm{BMO}}(\mathbb{C})$. Finally, fix an arbitrary number $c \in \mathbb{C}$ such that $|c| \neq 1$.*

Then for each given function $f \in \mathrm{BMO}(\mathbb{C})$ the following conditions are equivalent:

(i) *f belongs to the Sarason space* $\mathrm{VMO}(\mathbb{C})$.

(ii) $(cI + \widetilde{R}_{\mathbb{C}})[f]$ *belongs to* $\widetilde{\mathrm{VMO}}(\mathbb{C})$.

(iii) $(cI + \widetilde{S})[f]$ *belongs to* $\widetilde{\mathrm{VMO}}(\mathbb{C})$.

The key ingredient in the proof of Corollary 1.17, presented in Section 7, is Theorem 1.13. In turn, the equivalence of (i)–(iv) in Corollary 1.17 may be generalized to higher dimensions using Clifford algebras as a substitute for the field of complex numbers. Specifically, given any $n \in \mathbb{N}$, denote by $(\mathcal{C}\ell_n, +, \odot)$ the (complex) Clifford algebra generated by n anticommuting imaginary units, denoted by $(e_j)_{1 \le j \le n}$. Hence,

$$e_j \odot e_j = -1 \qquad \text{and} \qquad e_j \odot e_k = -e_k \odot e_j \quad \text{whenever } 1 \le j \neq k \le n. \tag{1-132}$$

The Euclidean ambient $\mathbb{R}^n$ embeds canonically into $\mathcal{C}\ell_n$ by identifying $(e_j)_{1 \le j \le n}$ with the standard orthonormal basis in $\mathbb{R}^n$, i.e.,

$$\mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv x := \sum_{j=1}^{n} x_j e_j \in \mathcal{C}\ell_n. \tag{1-133}$$

Under this embedding, (1-132) implies that

$$x \odot x = -|x|^2 \quad \text{for each } x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{1-134}$$

More information on this topic may be found in [Mitrea 1994]. Here is the higher-dimensional version of the portion of Corollary 1.17 dealing with the complex Riesz transform.

Corollary 1.18. *Consider the Clifford–Riesz transform acting on $\mathcal{C}\ell_n$-valued functions f defined in $\mathbb{R}^n$ according to*

$$R_{\mathcal{C}\ell} f(x) := \lim_{\varepsilon\to 0^+} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{y\in\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^{n+1}} \odot f(y)\,dy, \quad x\in\mathbb{R}^n, \tag{1-135}$$

and denote by $\widetilde{R}_{\mathcal{C}\ell}$ its extension to a bounded operator on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\mathcal{C}\ell_n$. Also, consider

$$\begin{gathered} c\in\mathcal{C}\ell_n \text{ such that } c+i\omega \text{ is invertible in } \mathcal{C}\ell_n \text{ from the} \\ \text{right for each vector } \omega\in S^{n-1}\subseteq\mathbb{R}^n\hookrightarrow\mathcal{C}\ell_n. \end{gathered} \tag{1-136}$$

Then for each given function $f\in\mathrm{BMO}(\mathbb{R}^n)\otimes\mathcal{C}\ell_n$ one has

$$f\in\mathrm{VMO}(\mathbb{R}^n)\otimes\mathcal{C}\ell_n \iff (cI+\widetilde{R}_{\mathcal{C}\ell})[f]\in\widetilde{\mathrm{VMO}}(\mathbb{R}^n)\otimes\mathcal{C}\ell_n. \tag{1-137}$$

As discussed in Section 7, the above result is readily implied by Corollary 1.16. We single out another immediate consequence of Theorem 1.15 formulated in terms of scalar-valued functions.

Corollary 1.19. *For each $j\in\{1,\dots,n\}$ denote by $\widetilde{R}_j$ the extension of the j-th Riesz transform, originally acting on $L^2(\mathbb{R}^n)$ as in* (1-90), *to a bounded operator on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ defined as in* (1-96). *Then for each complex-valued function $f\in\mathrm{BMO}(\mathbb{R}^n)$ and each $(c_1,\dots,c_n)\in\mathbb{C}^n\setminus iS^{n-1}$ one has*

$$f\in\mathrm{VMO}(\mathbb{R}^n) \iff (c_jI+\widetilde{R}_j)[f]\in\widetilde{\mathrm{VMO}}(\mathbb{R}^n) \text{ for each } j\in\{1,\dots,n\}. \tag{1-138}$$

In particular, corresponding to the special case when $c_1=\dots=c_n=0$, for each complex-valued function $f\in\mathrm{BMO}(\mathbb{R}^n)$ one has[2]

$$f\in\mathrm{VMO}(\mathbb{R}^n) \iff \widetilde{R}_j[f]\in\widetilde{\mathrm{VMO}}(\mathbb{R}^n) \text{ for each } j\in\{1,\dots,n\}. \tag{1-139}$$

Finally, we note that it is also possible to extend the characterizations of the membership to VMO given in the two-dimensional setting in Corollary 1.17 to higher dimensions and differential forms by introducing suitable higher-dimensional versions of the Beurling and Riesz transforms acting on differential forms. To describe them, we need a some standard notation from differential geometry; see, e.g., [Mitrea et al. 2016a, §2.1]. For each $\ell\in\{0,1,\dots,n\}$ let Λ^ℓ denote the space of differential forms of degree ℓ in $\mathbb{R}^n$, and set $\Lambda:=\bigoplus_{\ell=0}^n\Lambda^\ell$ for the space of differential forms of arbitrary mixed degrees in $\mathbb{R}^n$. The exterior derivative operator d and its formal adjoint δ in $\mathbb{R}^n$ are defined, respectively, as

$$df:=\sum_{j=1}^n dx_j\wedge(\partial_j f), \quad \delta f:=-\sum_{j=1}^n dx_j\vee(\partial_j f) \quad \text{for all } f\in\mathcal{D}'(\mathbb{R}^n)\otimes\Lambda, \tag{1-140}$$

where $\wedge$, $\vee$ stand for the exterior and interior product on Λ, and where the partial derivatives are applied to the individual components of the differential form f. For each $\theta\in\mathbb{C}\setminus\{0\}$ consider then the θ-Beurling

[2] Martell would like to express his gratitude to L. Escauriaza for some conversations pertaining to the one-dimensional case of (1-139).

transform in $\mathbb{R}^n$ defined (on the frequency side) as

$$S_\theta := (\theta\, d\delta - \theta^{-1}\,\delta d)\Delta^{-1} : \mathscr{S}(\mathbb{R}^n)\otimes\Lambda \to \mathscr{S}'(\mathbb{R}^n)\otimes\Lambda. \tag{1-141}$$

In the particular case when $\theta = 1$, this operator appears in [Iwaniec and Martin 2001, (12.71), p. 326]. It is reasonable to think of S_θ above as some kind of generalization of the classical Beurling transform defined in the complex plane in (1-91) due to the following: If for each $\theta \in \mathbb{C}\setminus\{0\}$ we also introduce the first-order differential operators

$$D_\theta := i(\theta\, d - \theta^{-1}\,\delta), \tag{1-142}$$

then $(D_\theta)^2 = d\delta + \delta d = -\Delta$, so each D_θ may be regarded as a square root of the negative Laplacian. Hence, each D_θ is a Dirac-type operator, much like the Cauchy–Riemann operator $\partial_{\bar z}$ and its complex conjugate ∂_z in the complex plane. Moreover, a simple computation (which makes use of the facts that $d^2 = 0$, $\delta^2 = 0$, and $\Delta = -d\delta - \delta d$) shows that

$$S_{\theta_1} D_{\theta_2} = i\ D_{i\theta_1\cdot\theta_2} \quad \text{for each } \theta_1,\theta_2 \in \mathbb{C}\setminus\{0\}, \tag{1-143}$$

which may be viewed as an extension of the classical intertwining properties recorded in (1-92).

An alternative representation of S_θ as an operator on $L^2(\mathbb{R}^n)\otimes\Lambda$, which is visible from (1-140)–(1-141) (upon recalling that the j-th Riesz transform on $L^2(\mathbb{R}^n)$ is the multiplier with symbol $-i\xi_j/|\xi|$), is

$$S_\theta f = -\theta\, R\wedge(R\vee f) + \theta^{-1}\, R\vee(R\wedge f), \quad f \in L^2(\mathbb{R}^n)\otimes\Lambda, \tag{1-144}$$

with the understanding that, in analogy to (1-140),

$$R\wedge f := \sum_{j=1}^n dx_j\wedge(R_j f), \quad R\vee f := -\sum_{j=1}^n dx_j\vee(R_j f) \quad \text{for all } f\in L^2(\mathbb{R}^n)\otimes\Lambda, \tag{1-145}$$

where the Riesz transforms R_j act on the individual components of the differential form f. In particular, if for each $f \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\Lambda$ we also define (with similar conventions as above)

$$\widetilde{R}\wedge[f] := \sum_{j=1}^n dx_j\wedge(\widetilde{R}_j[f]), \quad \widetilde{R}\vee[f] := -\sum_{j=1}^n dx_j\vee(\widetilde{R}_j[f]), \tag{1-146}$$

then Theorem 1.13 permits us to extend the θ-Beurling transform, originally considered as in (1-144), to a linear and bounded operator $\widetilde{S}_\theta$ from $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\Lambda$ into itself given by

$$\widetilde{S}_\theta[f] := -\theta\,\widetilde{R}\wedge(\widetilde{R}\vee[f]) + \theta^{-1}\,\widetilde{R}\vee(\widetilde{R}\wedge[f]), \quad [f]\in\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\Lambda. \tag{1-147}$$

In this vein, let us also introduce the θ-Riesz transforms (once again, on the frequency side) as

$$R_\theta := \frac{D_\theta}{\sqrt{-\Delta}} = i\theta\,\frac{d}{\sqrt{-\Delta}} - i\theta^{-1}\frac{\delta}{\sqrt{-\Delta}} \quad \text{for all } \theta\in\mathbb{C}\setminus\{0\}, \tag{1-148}$$

and note that they induce linear and bounded mappings on $L^2(\mathbb{R}^n)\otimes\Lambda$ according to

$$R_\theta f = -i(\theta\, R\wedge f + \theta^{-1}\, R\vee f), \quad f\in L^2(\mathbb{R}^n)\otimes\Lambda. \tag{1-149}$$

Thanks to Theorem 1.13, the θ-Riesz transforms above may further be extended to linear and bounded operators $\widetilde{R}_\theta$ on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\Lambda$ according to

$$\widetilde{R}_\theta[f] = -i(\theta\,\widetilde{R}\wedge[f] + \theta^{-1}\,\widetilde{R}\vee[f]), \quad [f]\in\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\Lambda. \tag{1-150}$$

In relation to the θ-Beurling transforms in (1-147) and the θ-Riesz transforms in (1-150), we have the following result, akin to the characterization of the membership to VMO in the two-dimensional case given in Corollary 1.17:

Corollary 1.20. *For each $j,k\in\{1,\dots,n\}$ introduce*

$$\Theta_{jk}(x) := \frac{-nx_jx_k+\delta_{jk}|x|^2}{|x|^{n+2}} \quad \text{for all } x\in\mathbb{R}^n\setminus\{0\}, \tag{1-151}$$

and note that

$$\begin{gathered}\Theta_{jk}\in\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\}),\quad \Theta_{kj}=\Theta_{jk},\quad \textstyle\int_{S^{n-1}}\Theta_{jk}(\omega)\,d\omega=0,\quad \text{and}\\ \Theta_{jk}\text{ is even and positive homogeneous of degree } -n \text{ in } \mathbb{R}^n\setminus\{0\}.\end{gathered} \tag{1-152}$$

In particular, these permit introducing the principal-value singular integral operators of convolution type $T_{\Theta_{jk}}$ associated with the Θ_{jk}'s as in (1-82). *Then for each $\theta\in\mathbb{C}\setminus\{0\}$ the operator S_θ is symmetric on $L^2(\mathbb{R}^n)\otimes\Lambda$ and for each $f\in L^2(\mathbb{R}^n)\otimes\Lambda$ one has (with $T_{\Theta_{jk}}$ acting on the differential form f componentwise)*

$$\begin{aligned}S_\theta f = -\frac{\theta}{\omega_{n-1}}\sum_{j,k=1}^n dx_j\wedge(dx_k\vee(T_{\Theta_{jk}}f)) + \frac{\theta^{-1}}{\omega_{n-1}}\sum_{j,k=1}^n dx_j\vee(dx_k\wedge(T_{\Theta_{jk}}f))\\ -\frac{\theta}{n}\sum_{j=1}^n dx_j\wedge(dx_j\vee f)+\frac{\theta^{-1}}{n}\sum_{j=1}^n dx_j\vee(dx_j\wedge f),\end{aligned} \tag{1-153}$$

while for each $[f]\in\widetilde{\mathrm{BMO}}(\mathbb{R}^n)\otimes\Lambda$ one has (with similar conventions as above)

$$\begin{aligned}\widetilde{S}_\theta[f] = -\frac{\theta}{\omega_{n-1}}\sum_{j,k=1}^n dx_j\wedge(dx_k\vee(\widetilde{T}_{\Theta_{jk}}[f])) + \frac{\theta^{-1}}{\omega_{n-1}}\sum_{j,k=1}^n dx_j\vee(dx_k\wedge(\widetilde{T}_{\Theta_{jk}}[f]))\\ -\frac{\theta}{n}\sum_{j=1}^n dx_j\wedge(dx_j\vee[f])+\frac{\theta^{-1}}{n}\sum_{j=1}^n dx_j\vee(dx_j\wedge[f]).\end{aligned} \tag{1-154}$$

Moreover, for each given differential form $f\in\mathrm{BMO}(\mathbb{R}^n)\otimes\Lambda$ the following three conditions are equivalent:

(i) *f belongs to the space $\mathrm{VMO}(\mathbb{R}^n)\otimes\Lambda$.*

(ii) *$(cI+\widetilde{S}_\theta)[f]\in\widetilde{\mathrm{VMO}}(\mathbb{R}^n)\otimes\Lambda$ for some (or every) $\theta\in\mathbb{C}\setminus\{0\}$ and $c\in\mathbb{C}\setminus\{\theta,-\theta^{-1}\}$.*

(iii) *$(cI+\widetilde{R}_\theta)[f]\in\widetilde{\mathrm{VMO}}(\mathbb{R}^n)\otimes\Lambda$ for some (or every) $\theta\in\mathbb{C}\setminus\{0\}$ and $c\in\mathbb{C}\setminus\{\pm1\}$.*

This paper is part of a larger program aimed at treating Dirichlet boundary value problems for $M\times M$ systems with constant complex coefficients as in (1-2)–(1-3) in the upper half-space $\mathbb{R}^n_+$ with boundary datum in various function spaces on $\mathbb{R}^{n-1}$. The space BMO, presently considered, lies at the crossroads of several fundamental scales of function spaces in analysis. For one thing, $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ may be regarded as a natural (rightmost) end-point of the Lebesgue scale $L^p(\mathbb{R}^{n-1},\mathbb{C}^M)$ with $p\in(1,\infty)$. The Dirichlet boundary value problem for elliptic systems L as in (1-2)–(1-3) in the upper half-space with data from the latter scale of spaces has been recently treated in [Martell et al. 2016], where the size of the solution $u:\mathbb{R}^n_+\to\mathbb{C}^M$ is measured using the nontangential maximal operator defined as

$$(\mathcal{N}u)(x'):=(\mathcal{N}_\kappa u)(x'):=\sup\{|u(y)|:y\in\Gamma_\kappa(x')\}\quad\text{for all } x'\in\mathbb{R}^{n-1}. \tag{1-155}$$

In this endeavor, the crux of the matter is the pointwise inequality, see (2-40),

$$\begin{aligned}&(\mathcal{N}u)(x')\le C(\mathcal{M}f)(x') \text{ at each point } x'\in\mathbb{R}^{n-1} \text{ if } u(x',t):=(P^L_t*f)(x') \text{ for}\\ &\text{every } (x',t)\in\mathbb{R}^n_+ \text{ and for some function } f\in L^1(\mathbb{R}^{n-1},1/(1+|x'|^n)\,dx')^M,\end{aligned} \tag{1-156}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator on $\mathbb{R}^{n-1}$; see (2-4).

In fact, estimate (1-156) permitted the treatment in [Martell et al. 2016] of a much larger variety of function lattice spaces. Indeed, one of the main results established in that paper is that the boundedness of the Hardy–Littlewood maximal operator on a Köthe function space $\mathbb{X}$ and on its Köthe dual $\mathbb{X}'$ (both considered in $\mathbb{R}^{n-1}$) is actually equivalent to the well-posedness of the $\mathbb{X}$-Dirichlet and $\mathbb{X}'$-Dirichlet problems in $\mathbb{R}^n_+$ in the class of all second-order, homogeneous, elliptic systems, with constant complex coefficients. As a consequence, in [Martell et al. 2016] the Dirichlet problem for such systems was shown to be well-posed for boundary data in Lebesgue spaces, variable-exponent Lebesgue spaces, Lorentz spaces, and Zygmund spaces, as well as their weighted versions with weights in the Muckenhoupt class.

This being said, the John–Nirenberg space $\mathrm{BMO}(\mathbb{R}^{n-1})$ is not a lattice space (in the sense that a nonnegative measurable function with a pointwise majorant in BMO does not necessarily belong to BMO), so a fresh look at the corresponding Dirichlet problem is warranted. In particular, the nature of the space of solutions (which should be suitably tailored to the specific space of boundary data) now involves a Carleson measure condition in place of the nontangential maximal operator (1-155) which has been extensively used in [Martell et al. 2016].

Another point of view places the John–Nirenberg space $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ as a (leftmost) endpoint for the scale of homogeneous Hölder spaces $\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)$ with $\eta\in(0,1)$ (for pertinent definitions and basic properties regarding this scale see the discussion in the first part of Section 2). Bearing this in mind, it is possible to formulate (a significant portion of) Theorem 1.1 in a manner that reflects the aforementioned feature of BMO. To elaborate on this idea, given $\eta\in[0,1)$ and $p\in[1,\infty)$, for every $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ define

$$\|f\|_*^{(\eta,p)}:=\sup_{Q\subset\mathbb{R}^{n-1}}\ell(Q)^{-\eta}\bigg(⨍_Q|f(x')-f_Q|^p\,dx'\bigg)^{\frac{1}{p}}, \tag{1-157}$$

and introduce the Morrey–Campanato space (which may be regarded as a fractional BMO space, L^p-based, of order η) by setting

$$\mathscr{E}^{\eta,p}(\mathbb{R}^{n-1},\mathbb{C}^M):=\{f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M):\|f\|_*^{(\eta,p)}<\infty\}. \tag{1-158}$$

By the John–Nirenberg inequality it follows that, corresponding to the end-point case $\eta = 0$, we have

$$\mathscr{E}^{0,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \tag{1-159}$$

and it is clear from definitions that, in the regime $\eta > 0$, the vanishing mean oscillation condition (1-22) holds (this time, at a precisely quantified rate of decay) for every function $f \in \mathscr{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$. Going further, for every $u \in \mathscr{C}^1(\mathbb{R}^n_+, \mathbb{C}^M)$ set

$$\|u\|_{**}^{(\eta,p)} := \sup_{Q \subset \mathbb{R}^{n-1}} \ell(Q)^{-\eta} \biggl(⨍_Q \biggl(\int_0^{\ell(Q)} |\nabla u(x',t)|^2 \, t \, dt \biggr)^{\frac{p}{2}} dx' \biggr)^{\frac{1}{p}}. \tag{1-160}$$

The finiteness demand $\|u\|_{**}^{(\eta,p)} < \infty$ may be viewed, compare with (1-14), as a fractional Carleson measure condition (L^p-based, of order η). In particular, it implies that the measure $d\mu(x',t) := |\nabla u(x',t)|^2 \, t \, dt \, dx'$ satisfies the vanishing condition (1-12), with a precisely quantified rate of decay.

Here is the statement of the theorem advertised earlier which deals with the larger, more inclusive context considered above and which complements the end-point case $\eta = 0$ corresponding to the portion of Theorem 1.1 pertaining to the well-posedness of the BMO-Dirichlet boundary value problem.

Theorem 1.21. *Let L be an $M \times M$ elliptic constant complex coefficient system as in* (1-2)–(1-3), *and fix $\eta \in (0,1)$ along with $p, q \in [1,\infty)$. Then the Morrey–Campanato–Dirichlet boundary value problem for L in $\mathbb{R}^n_+$, formulated as*

$$\begin{cases} u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\ Lu = 0 \text{ in } \mathbb{R}^n_+, \\ \|u\|_{**}^{(\eta,q)} < \infty, \\ u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \quad \text{a.e. in } \mathbb{R}^{n-1}, \qquad f \in \mathscr{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M), \end{cases} \tag{1-161}$$

has a unique solution. The solution u of (1-161) *is given by* (1-30) *and there exists a constant $C = C(n, L, \eta, p, q) \in (1,\infty)$ with the property that*

$$C^{-1} \|f\|_*^{(\eta,p)} \le \|u\|_{**}^{(\eta,q)} \le C \|f\|_*^{(\eta,p)}. \tag{1-162}$$

Moreover, u belongs to $\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+, \mathbb{C}^M) = \dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ and, with $C \in (1,\infty)$ as above,

$$C^{-1} \|f\|_*^{(\eta,p)} \le \|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+, \mathbb{C}^M)} \le C \|f\|_*^{(\eta,p)}. \tag{1-163}$$

As a consequence of Theorem 1.21 and its proof, see also (2-2), we obtain that, in fact,

$$\mathscr{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \tag{1-164}$$

as vector spaces, with equivalent norms (the left-to-right inclusion is understood in the sense that if $f \in \mathscr{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ then there exists some $g \in \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$ such that $f = g$ a.e. in $\mathbb{R}^{n-1}$). This offers a new proof (of a PDE flavor) of an old embedding result of N. Meyers [1964]. An inspection of the proof of Theorem 1.21 also reveals that there is a Fatou-type result naturally accompanying the well-posedness result for the boundary value problem (1-161).

We shall now succinctly comment on the literature dealing with Dirichlet boundary value problems for elliptic operators in the upper half-space. As already noted, the nature of these problems strongly depends on the choice of the function space from which the boundary datum f is selected, the specific way in which the size of the solution u is measured, and the very manner in which its boundary trace is considered. To illustrate these distinctions, recall first that there is a vast body of work targeting the case when the solution u is sought in various Sobolev spaces in $\mathbb{R}^n_+$, the boundary datum f belongs to suitable Besov spaces on $\mathbb{R}^{n-1}$, and the boundary trace of u is considered in the sense of Sobolev space theory. Classical references in this regard include [Agmon et al. 1959; 1964, Lions and Magenes 1972; Maz'ya and Shaposhnikova 1985; Taylor 2011a; 2011b; 2011c].

The scenario in which the size of u is measured in terms of the nontangential maximal operator (1-155) and when the trace of u on the boundary of $\mathbb{R}^n_+$ is taken in a nontangential pointwise sense, see (1-10), was treated in [Martell et al. 2016] for the general class of $M \times M$ systems L with constant complex coefficients as in (1-2)–(1-3). This extends classical work carried out in the particular case when $L = \Delta$, where Δ is the Laplacian in $\mathbb{R}^n$, treated in a number of monographs, including [Axler et al. 2001; García-Cuerva and Rubio de Francia 1985; Stein 1970; 1993; Stein and Weiss 1971]. The corresponding higher-order regularity Dirichlet problem in a similar framework was recently considered in [Martell et al. 2014]. See also [Martell et al. 2017] for related work, emphasizing semigroup techniques.

There is also a significant amount of work focused on the classical Dirichlet problem for the Laplacian in the upper half-space with a continuous boundary datum f. In such a case, one seeks a harmonic function $u \in \mathscr{C}^\infty(\mathbb{R}^n_+) \cap \mathscr{C}^0(\overline{\mathbb{R}^n_+})$ satisfying $u|_{\partial\mathbb{R}^n_+} = f$. A remarkable feature, noted in [Helms 1969, p. 42 and p. 158], is that even in the case when the boundary datum f is a bounded continuous function in $\mathbb{R}^{n-1}$, the solution u of this classical Dirichlet problem is not unique. To ensure uniqueness in such a setting one typically specifies the behavior of $u(x', t)$ as $t \to \infty$. A case in point is [Siegel and Talvila 1996], where uniqueness is established in the class of harmonic functions $u \in \mathscr{C}^\infty(\mathbb{R}^n_+) \cap \mathscr{C}^0(\overline{\mathbb{R}^n_+})$ satisfying $u(x) = o(|x| \sec^\gamma \theta)$ as $|x| \to \infty$ (where $\theta := \arccos(x_n/|x|)$ and $\gamma \in \mathbb{R}$ is arbitrary), by proving a Phragmén–Lindelöf principle under the latter growth condition. This builds on the work of [Siegel 1988; Wolf 1941; Yoshida 1996], and others. The works just cited crucially rely on positivity and other various highly specialized properties of the Laplace operator, so the techniques employed there do not adapt to the considerably more general class of elliptic systems considered in the present paper.

In relation to the context just described above, it is instructive to make the following observations. First, the collection of uniformly continuous functions belonging to $\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ is a dense subspace of $\mathrm{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$; see (1-26). Second, in the last part of Theorem 1.1 we have succeeded in proving the well-posedness of the VMO-Dirichlet problem in the class of null-solutions u of a given elliptic system L as in (1-2)–(1-3) which satisfy a vanishing Carleson measure condition. This is a natural condition from the point of view of harmonic analysis which replaces the demand that the solution extends continuously on $\overline{\mathbb{R}^n_+}$, required in the formulation of the classical Dirichlet problem with continuous data.

Apparently, the closest results in the literature to some of the work carried out in this paper are those of E. Fabes, R. Johnson, and U. Neri [Fabes et al. 1976]. Indeed, in their paper they dealt with the BMO-Dirichlet problem for the Laplacian in the upper half-space in the class of harmonic functions satisfying a Carleson measure condition (this being said, we would like to point out that there are certain gaps in some of the key steps of the treatment in that paper, such as the proof of Lemma 1.3 on pp. 161–162,[3] and the proof of estimate (1.5) on page 163[4]). The portion of Theorem 1.1 dealing with (1-29) is a significant generalization of their work, which is thereby extended to a much larger class of systems. Similar attributes are shared by our Theorem 1.21 in relation to the work in [Fabes et al. 1976] dealing with harmonic functions in the upper half-space with traces in Morrey–Campanato spaces. Generalizations of these results appear in [Duong et al. 2014] for the Schrödinger operator of the form $-\Delta + V$ with V being a nonnegative potential belonging to some reverse Hölder class (hence $0 < V < \infty$ a.e.).

We also wish to mention here the work of B. Dahlberg and C. Kenig [1987, Theorem 4.18, p. 463], who have treated the BMO-Dirichlet problem for the Laplacian in bounded Lipschitz domains via layer potentials, building on the earlier work of E. Fabes and U. Neri [1980] who employed harmonic measure techniques. For related work see also [Dindos et al. 2011].

The techniques employed in [Dahlberg and Kenig 1987; Duong et al. 2014; Dindos et al. 2011; Fabes et al. 1976; Fabes and Neri 1980] are largely restricted to scalar equations (as they make essential use of positivity and/or maximum principles). Also, the fact that in [Dahlberg and Kenig 1987; Dindos et al. 2011; Fabes and Neri 1980] the underlying domain is bounded makes the task of proving uniqueness considerably more manageable. In addition, the consideration of PDEs for which the well-posedness of the L^2-Dirichlet problem is known in arbitrary Lipschitz subdomains allows these authors to successfully employ a variety of localization arguments. By way of contrast, most of these key features cease to be effective in the geometric/analytic context considered in this paper. In proving the solvability of the BMO-Dirichlet boundary value problem for an elliptic system L in $\mathbb{R}^n_+$ as formulated in (1-29), our approach makes essential use of the existence and properties of the Poisson kernel associated with L from the work of [Agmon et al. 1959; 1964]. Uniqueness is derived with the help of the Fatou-type result recorded in Theorem 1.2. A considerable amount of effort then goes into establishing the latter theorem, with square-function estimates (see Proposition 3.2), elements of tent-space theory (see Lemma 4.10), interior estimates (see Theorem 2.4), and certain estimates near the boundary from [Maz'ya et al. 2010] for null-solutions of L vanishing on the boundary (see Proposition 2.5), among the tools playing a key role in this regard.

We conclude with a brief overview of the contents of the sections of this paper. Useful background material and auxiliary results are collected in Section 2. The proofs of the existence statements in Theorem 1.1, both for the BMO-Dirichlet problem and the VMO-Dirichlet problem, are carried out in Section 3. Next, Section 4 is reserved for establishing a Fatou result for smooth null-solutions of L satisfying a Carleson measure condition, as well as uniqueness in the BMO-Dirichlet problem, in the upper half-space. Finally, the proofs of Theorems 1.1–1.6, as well as Theorems 1.8–1.10, are given in Section 5, the proof of Theorem 1.21 is contained in Section 6, while the proofs of Theorems 1.13–1.15 and Corollaries 1.16–1.20 are presented in Section 7.

[3]The second equality in the first formula displayed on page 162 is questionable, given that this involves the global gradient in $\mathbb{R}^{n+1}$, which includes the transversal variable t.

[4]Here the authors rely on the implication 3(iii) $\Rightarrow$ 2 from [Fefferman and Stein 1972, pp. 147–148] which is only established under the additional membership to $L^2(\mathbb{R}^n)$.

2. Background material and preliminary results

In this section we collect a number of preliminary results that are useful in the sequel. Throughout, we let $\mathbb{N}$ stand for the collection of all positive integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In this way $\mathbb{N}_0^k$ stands for the set of multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_j \in \mathbb{N}_0$ for $1 \le j \le k$. Also, fix $n \in \mathbb{N}$ with $n \ge 2$. For an arbitrary multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we use the standard notation $\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and we occasionally abbreviate ∂_{x_j} by simply ∂_j for $j \in \{1, \dots, n\}$. The length of the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ is defined as $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We agree to let $\{e_j\}_{1 \le j \le n}$ stand for the standard orthonormal basis in $\mathbb{R}^n$. Occasionally, we canonically identify e_j with a multi-index in $\mathbb{N}_0$ (of length 1). Given an arbitrary set $E \subseteq \mathbb{R}^{n-1}$ we denote by $\mathbf{1}_E$ the characteristic function of E.

Generally speaking, given a metric space (X, d), corresponding to each subset E of X (of cardinality at least 2) and number $\eta > 0$, we associate the homogeneous Hölder space or order η, denoted by $\dot{\mathscr{C}}^\eta(E, \mathbb{C}^M)$, as the collection of functions $w : E \to \mathbb{C}^M$ satisfying

$$\|w\|_{\dot{\mathscr{C}}^\eta(E,\mathbb{C}^M)} := \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|w(x) - w(y)|}{d(x,y)^\eta} < \infty. \tag{2-1}$$

Whenever $E \subseteq F \subseteq X$ (with E having cardinality at least 2) we then have

$$\begin{gathered} \dot{\mathscr{C}}^\eta(E, \mathbb{C}^M) = \dot{\mathscr{C}}^\eta(\overline{E}, \mathbb{C}^M) \text{ isometrically, and} \\ \dot{\mathscr{C}}^\eta(F, \mathbb{C}^M) \ni w \mapsto w|_E \in \dot{\mathscr{C}}^\eta(E, \mathbb{C}^M) \text{ continuously.} \end{gathered} \tag{2-2}$$

Also,

$$\dot{\mathscr{C}}^\eta(E, \mathbb{C}^M) \subseteq \mathrm{UC}(E, \mathbb{C}^M), \tag{2-3}$$

where the latter denotes the space of $\mathbb{C}^M$-valued functions which are uniformly continuous on the set E. Finally, we agree to drop the dependence on the range when $M = 1$, and denote by $\mathrm{Lip}(E)$ the homogeneous Hölder space on E of order $\eta = 1$.

Moving on, we denote by $\mathcal{M}$ the Hardy–Littlewood maximal operator on $\mathbb{R}^{n-1}$ which acts on vector-valued functions with components in $L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ according to

$$(\mathcal{M} f)(x') := \sup_{Q \ni x'} ⨍_Q |f(y')| \, dy' \quad \text{for all } x' \in \mathbb{R}^{n-1}, \tag{2-4}$$

where the supremum runs over all cubes Q in $\mathbb{R}^{n-1}$ containing x'.

We will often work with the weighted Lebesgue space of the form

$$L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^a}\right) := \left\{ f : \mathbb{R}^{n-1} \to \mathbb{C} \text{ Lebesgue-measurable} : \int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1+|x'|^a} \, dx' < \infty \right\}, \tag{2-5}$$

where $a \in (0, \infty)$, and we shall denote by $L^1(\mathbb{R}^{n-1}, dx'/(1+|x'|^a))^M$ the space of $\mathbb{C}^M$-valued functions with components in (2-5). Clearly,

$$L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^a}\right)^M \subset L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad \text{for all } a > 0. \tag{2-6}$$

Next, we record several useful properties of mean oscillations (recall the piece of notation introduced in (1-16)). First we note that if Q and Q' are cubes in $\mathbb{R}^{n-1}$ with the property that $Q' \subseteq Q$, then for any $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and any $p \in [1, \infty)$ we have

$$\left(\fint_{Q'} |f(y') - f_{Q'}|^p\, dy'\right)^{\frac{1}{p}} \le 2\left(\frac{\ell(Q)}{\ell(Q')}\right)^{\frac{n-1}{p}} \left(\fint_{Q} |f(y') - f_{Q}|^p\, dy'\right)^{\frac{1}{p}} \tag{2-7}$$

and

$$\left(\fint_{Q} |f(y') - f_{Q'}|^p\, dy'\right)^{\frac{1}{p}} \le \left[1 + \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\frac{n-1}{p}}\right] \left(\fint_{Q} |f(y') - f_{Q}|^p\, dy'\right)^{\frac{1}{p}}. \tag{2-8}$$

Also,

$$\frac{1}{2}\left(\fint_{Q} |f(y') - f_{Q}|^p\, dy'\right)^{\frac{1}{p}} \le \inf_{c \in \mathbb{C}^M} \left(\fint_{Q} |f(y') - c|^p\, dy'\right)^{\frac{1}{p}} \le \left(\fint_{Q} |f(y') - f_{Q}|^p\, dy'\right)^{\frac{1}{p}}. \tag{2-9}$$

Second, we recall the John–Nirenberg inequality asserting that there exist two-dimensional constants $C_1, C_2 \in (0, \infty)$ with the following significance. Consider an arbitrary cube $Q \subset \mathbb{R}^{n-1}$ along with a function $f \in L^1(Q)$ with the property that

$$N_Q(f) := \sup_{Q' \subseteq Q} \fint_{Q'} |f(y') - f_{Q'}|\, dy' < \infty, \tag{2-10}$$

where the above supremum involves cubes $Q' \subset \mathbb{R}^{n-1}$ contained in Q. Then there holds, see, e.g., [Stein 1993, Corollary 2, p. 154],

$$\mathscr{L}^{n-1}\big(\{y' \in Q : |f(y') - f_Q| > \lambda\}\big) \le C_1\, e^{-\left(\frac{C_2}{N_Q(f)}\right)\lambda}\, |Q| \quad \text{for all } \lambda > 0. \tag{2-11}$$

Third, as a corollary of the John–Nirenberg inequality, we obtain that for every $p \in (0, \infty)$ there exists a constant $C_{n,p} \in (0, \infty)$ with the property that for every cube $Q \subset \mathbb{R}^{n-1}$ and every function $f \in L^1(Q, \mathbb{C}^M)$ we have

$$\left(\fint_{Q} |f(y') - f_{Q}|^p\, dy'\right)^{\frac{1}{p}} \le C_{n,p} \sup_{Q' \subseteq Q} \fint_{Q'} |f(y') - f_{Q'}|\, dy'. \tag{2-12}$$

To proceed, for each $p \in [1, \infty)$, $r \in (0, \infty)$, and $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ define the L^p-based mean oscillations of f at a given scale $r \in (0, \infty)$ as

$$\operatorname{osc}_p(f; r) := \sup_{Q \subset \mathbb{R}^{n-1},\, \ell(Q) \le r} \left(\fint_{Q} |f(x') - f_{Q}|^p\, dx'\right)^{\frac{1}{p}} \in [0, \infty]. \tag{2-13}$$

Some of the main properties of this function are summarized next.

Lemma 2.1. *For each $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ the following properties hold*:

(a) *Fix $p \in [1, \infty)$. Then, as a function of r, the quantity $\operatorname{osc}_p(f; r)$ is nondecreasing in r.*

(b) *For every* $p, q \in [1, \infty)$ *there exists a constant* $C = C(p, q, n) \in (1, \infty)$, *independent of* f, *with the property that*

$$C^{-1}\operatorname{osc}_p(f; r) \leq \operatorname{osc}_q(f; r) \leq C \operatorname{osc}_p(f; r) \quad \text{for every } r \in (0, \infty). \tag{2-14}$$

(c) *The function* f *belongs to* $\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *if and only if* $\operatorname{osc}_p(f; r)$ *as a function in* r *is bounded on* $(0, \infty)$ *for some, or any,* $p \in [1, \infty)$. *Moreover, for each* $p \in [1, \infty)$ *there exists a constant* $C = C(n, p) \in (1, \infty)$, *independent of* f, *with the property that*

$$C^{-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq \sup_{r>0} \operatorname{osc}_p(f; r) = \lim_{r\to\infty} \operatorname{osc}_p(f; r) \leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{2-15}$$

(d) *The function* f *belongs to* $\mathrm{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *if and only if for some, or any exponent* $p \in [1, \infty)$ *one has*

$$\lim_{r\to 0^+} \operatorname{osc}_p(f; r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \operatorname{osc}_p(f; r) < \infty. \tag{2-16}$$

(e) *For every* $\eta \in [0, 1)$ *and* $p \in [1, \infty)$ *we have, recall* (1-157),

$$\operatorname{osc}_p(f; r) \leq r^\eta \|f\|_*^{(\eta,p)} \quad \text{for all } r \in (0, \infty). \tag{2-17}$$

(f) *If* f *belongs to* $\mathscr{C}^\Upsilon(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *for some modulus of continuity* Υ, *recall* (1-47)–(1-48), *then for each* $p \in [1, \infty)$ *one has*

$$\operatorname{osc}_p(f; r) \leq \|f\|_{\mathscr{C}^\Upsilon(\mathbb{R}^{n-1},\mathbb{C}^M)} \Upsilon(\sqrt{n}\, r) \quad \text{for all } r \in (0, \infty). \tag{2-18}$$

In particular, for each $p \in [1, \infty)$ *and* $\eta \in (0, 1)$ *there exists* $C \in (0, \infty)$ *such that for every function* $f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *one has*

$$\operatorname{osc}_p(f; r) \leq C r^\eta \|f\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)} \quad \text{for all } r \in (0, \infty). \tag{2-19}$$

Proof. The claim made in part (a) follows directly from (2-13). The claim in part (b) is a direct consequence of Hölder's inequality and John–Nirenberg's inequality; see (2-12). The latter also implies the claims made in part (c). The claim in part (d) is a consequence of (a)–(c) and (1-22). Estimate (2-17) is immediate from (2-13) and (1-157). Finally, if $f \in \mathscr{C}^\Upsilon(\mathbb{R}^{n-1}, \mathbb{C}^M)$ then for each $p \in [1, \infty)$ and each cube Q in $\mathbb{R}^{n-1}$ Hölder's inequality gives

$$\begin{aligned}\Big(⨍_Q |f(x') - f_Q|^p\, dx'\Big)^{\frac{1}{p}} &\leq \Big(⨍_Q ⨍_Q |f(x') - f(y')|^p\, dy'\, dx'\Big)^{\frac{1}{p}} \\ &\leq \|f\|_{\mathscr{C}^\Upsilon(\mathbb{R}^{n-1},\mathbb{C}^M)} \Upsilon(\sqrt{n}\, \ell(Q)).\end{aligned} \tag{2-20}$$

Then (2-18) follows from (2-20) given that Υ is nondecreasing. □

Next, we discuss the manner in which global integrability properties of a given function are related to the behavior at infinity of its mean oscillation function.

Lemma 2.2. *Fix $\varepsilon > 0$ arbitrary. Then there exists a constant $C_{n,\varepsilon} \in (0,\infty)$ such that for each function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and each cube $Q \subset \mathbb{R}^{n-1}$, with center $x'_Q \in \mathbb{R}^{n-1}$, there holds*

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}}\, dy' \le \frac{C_{n,\varepsilon}}{\ell(Q)^\varepsilon} \int_1^\infty \biggl(⨍_{\lambda Q} |f(y') - f_{\lambda Q}|\, dy' \biggr) \frac{d\lambda}{\lambda^{1+\varepsilon}} \le \frac{C_{n,\varepsilon}}{\ell(Q)^\varepsilon} \int_1^\infty \mathrm{osc}_1(f;\lambda\ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \tag{2-21}$$

As a consequence, for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ one has

$$\int_1^\infty \mathrm{osc}_1(f;\lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} < \infty \quad \Longrightarrow \quad f \in L^1\biggl(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\biggr)^M \tag{2-22}$$

and there exists a constant $C_{n,\varepsilon} \in (0,\infty)$ with the property that

$$\int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1+|x'|^{n-1+\varepsilon}}\, dx' \le C_{n,\varepsilon} \int_1^\infty \mathrm{osc}_1(f;\lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} + C_{n,\varepsilon} ⨍_{Q_0} |f(x')|\, dx', \tag{2-23}$$

where $Q_0 := \bigl(-\frac{1}{2}, \frac{1}{2}\bigr)^{n-1}$ is the cube centered at the origin $0'$ of $\mathbb{R}^{n-1}$ with side-length 1.

In particular, we have

$$\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\biggl(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\biggr)^M \quad \textit{for all } \varepsilon > 0, \tag{2-24}$$

and for each $p \in [1,\infty)$, recall (1-158),

$$\mathscr{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\biggl(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\biggr)^M \quad \textit{for all } \varepsilon > 0,\ \textit{for all } \eta \in [0,\varepsilon), \tag{2-25}$$

while in view of (2-19) *and* (2-22) *we obtain*

$$\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\biggl(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\biggr)^M \quad \textit{for all } \eta \in (0,\varepsilon). \tag{2-26}$$

Proof. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and a cube $Q \subset \mathbb{R}^{n-1}$ with center $x'_Q \in \mathbb{R}^{n-1}$, breaking up the domain of integration allows us to estimate

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}}\, dy' \\
&\quad \le \ell(Q)^{-n+1-\varepsilon} \int_Q |f(y') - f_Q|\, dy' + \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} \frac{|f(y') - f_Q|}{|x'_Q - y'|^{n-1+\varepsilon}}\, dy' \\
&\quad \le \ell(Q)^{-\varepsilon} ⨍_Q |f(y') - f_Q|\, dy' + 2^{2(n-1)+\varepsilon} \ell(Q)^{-\varepsilon} \sum_{k=0}^\infty 2^{-k\varepsilon} ⨍_{2^{k+1}Q} |f(y') - f_Q|\, dy'.
\end{aligned} \tag{2-27}$$

Next, for each $k \in \mathbb{N}_0$ we have

$$\begin{aligned}\fint_{2^{k+1}Q} |f(y')-f_Q|\,dy' &\le \fint_{2^{k+1}Q} |f(y')-f_{2^{k+1}Q}|\,dy' + \sum_{j=0}^{k} |f_{2^jQ}-f_{2^{j+1}Q}| \\ &\le \fint_{2^{k+1}Q} |f(y')-f_{2^{k+1}Q}|\,dy' + 2^{n-1}\sum_{j=0}^{k} \fint_{2^{j+1}Q} |f(y')-f_{2^{j+1}Q}|\,dy'; \qquad (2\text{-}28)\end{aligned}$$

hence,

$$\begin{aligned}&\sum_{k=0}^{\infty} 2^{-k\varepsilon} \fint_{2^{k+1}Q} |f(y')-f_Q|\,dy' \\ &\quad\le \sum_{k=0}^{\infty} 2^{-k\varepsilon} \fint_{2^{k+1}Q} |f(y')-f_{2^{k+1}Q}|\,dy' + 2^{n-1}\sum_{k=0}^{\infty} 2^{-k\varepsilon}\biggl\{\sum_{j=0}^{k} \fint_{2^{j+1}Q} |f(y')-f_{2^{j+1}Q}|\,dy'\biggr\} \\ &\quad= \sum_{k=0}^{\infty} 2^{-k\varepsilon} \fint_{2^{k+1}Q} |f(y')-f_{2^{k+1}Q}|\,dy' + \frac{2^{n-1}}{1-2^{-\varepsilon}} \sum_{j=0}^{\infty} 2^{-j\varepsilon} \fint_{2^{j+1}Q} |f(y')-f_{2^{j+1}Q}|\,dy' \\ &\quad= \Bigl(1+\frac{2^{n-1}}{1-2^{-\varepsilon}}\Bigr) \sum_{k=0}^{\infty} 2^{-k\varepsilon} \fint_{2^{k+1}Q} |f(y')-f_{2^{k+1}Q}|\,dy', \qquad (2\text{-}29)\end{aligned}$$

where the first equality has been obtained by interchanging the sums in k and j. Collectively, (2-27) and (2-29) permit us to conclude that

$$\int_{\mathbb{R}^{n-1}} \frac{|f(y')-f_Q|}{[\ell(Q)+|x'_Q-y'|]^{n-1+\varepsilon}}\,dy' \le 4^{n-1+\varepsilon}\Bigl(1+\frac{2^{n-1}}{1-2^{-\varepsilon}}\Bigr)\ell(Q)^{-\varepsilon}\sum_{k=0}^{\infty} 2^{-k\varepsilon} \fint_{2^kQ} |f(y')-f_{2^kQ}|\,dy'. \tag{2-30}$$

To proceed, observe that (2-7) yields

$$\fint_{2^kQ} |f(y')-f_{2^kQ}|\,dy' \le 2^n \fint_{\lambda Q} |f(y')-f_{\lambda Q}|\,dy' \quad \text{for each } k\in\mathbb{N}_0 \text{ and each } \lambda\in[2^k,2^{k+1}]. \tag{2-31}$$

This, in turn, implies that for each $k \in \mathbb{N}_0$ we have

$$2^{-k\varepsilon} \fint_{2^kQ} |f(y')-f_{2^kQ}|\,dy' \le \frac{2^n\varepsilon}{1-2^{-\varepsilon}} \int_{2^k}^{2^{k+1}} \Bigl(\fint_{\lambda Q} |f(y')-f_{\lambda Q}|\,dy'\Bigr)\frac{d\lambda}{\lambda^{1+\varepsilon}}. \tag{2-32}$$

Availing ourselves of this estimate in (2-30) then establishes the first inequality in (2-21) for the choice

$$C_{n,\varepsilon} := 2^n\,4^{n-1+\varepsilon}\Bigl(1+\frac{2^{n-1}}{1-2^{-\varepsilon}}\Bigr)\cdot\frac{\varepsilon}{1-2^{-\varepsilon}}. \tag{2-33}$$

The second inequality in (2-21) is a direct consequence of this and (2-13). Going further, (2-22)–(2-23) follow from the second inequality in (2-21) with $Q := \bigl(-\frac{1}{2},\frac{1}{2}\bigr)^{n-1}$. In turn, (2-23) together with part (c) in Lemma 2.1 give (2-24), while (2-23) together with part (e) in Lemma 2.1 give (2-25). □

Poisson kernels for elliptic operators in a half-space have a long history; see, e.g., [Agmon et al. 1959; 1964; Solonnikov 1964; 1966]. Here we record the following useful existence and uniqueness result. In its statement (as well as elsewhere in the paper), we make the convention that the convolution between two functions, which are matrix-valued and vector-valued, respectively, takes into account the algebraic multiplication between a matrix and a vector in a natural fashion.

Theorem 2.3. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3). *Then there exists a matrix-valued function $P^L = (P^L_{\alpha\beta})_{1\le\alpha,\beta\le M} : \mathbb{R}^{n-1} \to \mathbb{C}^{M\times M}$ (called the Poisson kernel for L in $\mathbb{R}^n_+$) satisfying the following properties*:

(1) *There exists $C \in (0,\infty)$ such that*

$$|P^L(x')| \le \frac{C}{(1+|x'|^2)^{n/2}} \quad \text{for each } x' \in \mathbb{R}^{n-1}. \tag{2-34}$$

(2) *The function P^L is Lebesgue-measurable and*

$$\int_{\mathbb{R}^{n-1}} P^L(x')\,dx' = I_{M\times M}, \tag{2-35}$$

where $I_{M\times M}$ is the $M\times M$ identity matrix. In particular, for every constant vector $C = (C_\alpha)_{1\le\alpha\le M} \in \mathbb{C}^M$ one has

$$\int_{\mathbb{R}^{n-1}} \sum_{1\le\beta\le M} (P^L_{\alpha\beta})_t(x'-y')C_\beta\,dy' = C_\alpha \quad \text{for all } (x',t) \in \mathbb{R}^n_+. \tag{2-36}$$

(3) *If one sets*

$$K^L(x',t) := P^L_t(x') = t^{1-n}P^L(x'/t) \quad \text{for each } x' \in \mathbb{R}^{n-1} \text{ and } t > 0, \tag{2-37}$$

then the function $K^L = (K^L_{\alpha\beta})_{1\le\alpha,\beta\le M}$ satisfies (in the sense of distributions)

$$LK^L_{\cdot\beta} = 0 \text{ in } \mathbb{R}^n_+ \text{ for each } \beta \in \{1,\dots,M\}, \tag{2-38}$$

where $K^L_{\cdot\beta} := (K^L_{\alpha\beta})_{1\le\alpha\le M}$ is the β-th column in K^L.

Moreover, P^L is unique in the class of $\mathbb{C}^{M\times M}$-valued functions defined in $\mathbb{R}^{n-1}$ and satisfying (1)–(3) *above, and has the following additional properties*:

(4) *One has $P^L \in \mathscr{C}^\infty(\mathbb{R}^{n-1})$ and $K^L \in \mathscr{C}^\infty(\overline{\mathbb{R}^n_+} \setminus B(0,\varepsilon))$ for every $\varepsilon > 0$. Consequently,* (2-38) *holds in a pointwise sense.*

(5) *There holds $K^L(\lambda x) = \lambda^{1-n}K^L(x)$ for all $x \in \mathbb{R}^n_+$ and $\lambda > 0$. In particular, for each multi-index $\alpha \in \mathbb{N}^n_0$ there exists $C_\alpha \in (0,\infty)$ with the property that*

$$|(\partial^\alpha K^L)(x)| \le C_\alpha\,|x|^{1-n-|\alpha|} \quad \text{for all } x \in \overline{\mathbb{R}^n_+} \setminus \{0\}. \tag{2-39}$$

(6) *For each $\kappa > 0$ there exists a finite constant $C_\kappa > 0$ with the property that for each $x' \in \mathbb{R}^{n-1}$,*

$$\sup_{|x'-y'|<\kappa t} |(P_t^L * f)(y')| \leq C_\kappa \, \mathcal{M} f(x') \quad \text{for all } f \in L^1\Big(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n}\, dx'\Big)^M. \tag{2-40}$$

(7) *Fix an arbitrary $\kappa > 0$ and a function*

$$f = (f_\beta)_{1\leq\beta\leq M} \in L^1\Big(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n}\, dx'\Big)^M. \tag{2-41}$$

Then the function $u(x',t) := (P_t^L * f)(x')$ for each $(x',t) \in \mathbb{R}^n_+$ is meaningfully defined via an absolutely convergent integral, satisfies

$$u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad Lu = 0 \text{ in } \mathbb{R}^n_+, \tag{2-42}$$

and, at every Lebesgue point $x_0' \in \mathbb{R}^{n-1}$ of f,

$$\big(u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}\big)(x_0') := \lim_{\substack{(x',t)\to(x_0',0)\\ |x'-x_0'|<\kappa t}} (P_t^L * f)(x') = f(x_0'). \tag{2-43}$$

(8) *The function P^L satisfies the semigroup property*

$$P_{t_1}^L * P_{t_2}^L = P_{t_1+t_2}^L \quad \text{for every } t_1, t_2 > 0. \tag{2-44}$$

Concerning Theorem 2.3, we note that the existence part follows from the classical work of S. Agmon, A. Douglis, and L. Nirenberg [Agmon et al. 1964]. The uniqueness property was recently proved in [Martell et al. 2016], where (2-40), (2-42), (2-43), as well as the semigroup property (2-44), were also established.

Next, we record the following versatile version of interior estimates for higher-order elliptic systems. A proof may be found in [Mitrea 2013, Theorem 11.9, p. 364].

Theorem 2.4. *Assume the system L is as in* (1-2)–(1-3). *Then for each null-solution u of L in a ball $B(x,R)$* (*where $x \in \mathbb{R}^n$ and $R > 0$*), $p \in (0,\infty)$, $\lambda \in (0,1)$, $\ell \in \mathbb{N}_0$, *and* $r \in (0,R)$, *one has*

$$\sup_{z\in B(x,\lambda r)} |\nabla^\ell u(z)| \leq \frac{C}{r^\ell}\Big(\fint_{B(x,r)} |u|^p \, d\mathscr{L}^n\Big)^{\frac{1}{p}}, \tag{2-45}$$

where $C = C(L,p,\ell,\lambda,n) > 0$ is a finite constant.

To proceed we need to introduce some additional terminology. Let

$$W^{1,2}_{\text{bd}}(\mathbb{R}^n_+) := \big\{w \text{ Lebesgue-measurable in } \mathbb{R}^n_+ : w, \nabla w \in L^2(\mathbb{R}^n_+ \cap B(x,r)) \\ \text{for all } x \in \mathbb{R}^n_+, \ \text{for all } r \in (0,\infty)\big\}. \tag{2-46}$$

In the sequel, the space of $\mathbb{C}^M$-valued functions with components in $W^{1,2}_{\text{bd}}(\mathbb{R}^n_+)$ will be denoted by $W^{1,2}_{\text{bd}}(\mathbb{R}^n_+, \mathbb{C}^M)$. Also, (whenever meaningful) the Sobolev trace Tr is defined as

$$(\operatorname{Tr} w)(x') := \lim_{r\to 0^+} \fint_{B((x',0),r)\cap\mathbb{R}^n_+} w \, d\mathscr{L}^n, \quad x' \in \mathbb{R}^{n-1}. \tag{2-47}$$

The following result can be found in [Maz'ya et al. 2010, Corollary 2.4], and it is a consequence of the a priori regularity estimates obtained in [Agmon et al. 1964] and Sobolev embeddings.

Proposition 2.5. *Let L be an $M \times M$ elliptic system as in* (1-2)–(1-3) *and consider a vector-valued function $w \in W^{1,2}_{\mathrm{bd}}(\mathbb{R}^n_+, \mathbb{C}^M)$ such that*

$$\begin{cases} Lw = 0 & \text{in } \mathbb{R}^n_+, \\ \operatorname{Tr} w = 0 & \mathscr{L}^{n-1}\text{-a.e. on } \mathbb{R}^{n-1}. \end{cases} \tag{2-48}$$

Then $w \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$, and for each $z \in \overline{\mathbb{R}^n_+}$ and $\rho > 0$ one has

$$\sup_{\mathbb{R}^n_+ \cap B(z,\rho)} |\nabla w| \leq C\rho^{-1} \sup_{\mathbb{R}^n_+ \cap B(z,2\rho)} |w|, \tag{2-49}$$

where $C \in (0, \infty)$ is a constant independent of the scale ρ, the point z, and the function w.

We will also need an L^p-Fatou-type result obtained in [Martell et al. 2016, Corollary 6.3]. To state it, the reader is invited to recall the nontangential maximal operator from (1-155).

Corollary 2.6. *Assume L is an elliptic $M \times M$ system as in* (1-2)–(1-3). *Then for each $p \in [1, \infty)$,*

$$\left.\begin{array}{r} u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\ Lu = 0 \text{ in } \mathbb{R}^n_+, \\ \mathcal{N}u \in L^p(\mathbb{R}^{n-1}) \end{array}\right\} \implies \begin{cases} u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} \text{ exists a.e. in } \mathbb{R}^{n-1}, \text{ belongs to } L^p(\mathbb{R}^{n-1}, \mathbb{C}^M), \\ \text{and } u(x', t) = (P^L_t * (u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}))(x') \text{ for all } (x', t) \in \mathbb{R}^n_+, \end{cases} \tag{2-50}$$

where P^L is the Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3.

Our last auxiliary result, of a purely real-variable nature, can be found in [Martell et al. 2016, Lemma 3.3].

Lemma 2.7. *Fix $M \in \mathbb{N}$ and let $P = (P_{\alpha\beta})_{1 \leq \alpha,\beta \leq M} : \mathbb{R}^{n-1} \to \mathbb{C}^{M \times M}$ be a Lebesgue-measurable function satisfying, for some $c \in (0, \infty)$,*

$$|P(x')| \leq \frac{c}{(1 + |x'|^2)^{n/2}} \quad \text{for each } x' \in \mathbb{R}^{n-1}. \tag{2-51}$$

Recall that $P_t(x') = t^{1-n} P(x'/t)$ for each $x' \in \mathbb{R}^{n-1}$ and $t \in (0, \infty)$.

Then, for each $t \in (0, \infty)$ fixed, the operator

$$L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n}\, dx'\right)^M \ni f \mapsto P_t * f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n}\, dx'\right)^M \tag{2-52}$$

is well-defined, linear and bounded, with operator norm controlled by $C(t + 1)$. Moreover, for every $\kappa > 0$ there exists a finite constant $C_\kappa > 0$ with the property that for each $x' \in \mathbb{R}^{n-1}$,

$$\sup_{|x'-y'|<\kappa t} |(P_t * f)(y')| \leq C_\kappa\, \mathcal{M} f(x') \quad \text{for all } f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n}\, dx'\right)^M. \tag{2-53}$$

Finally, given any function

$$f = (f_\beta)_{1 \leq \beta \leq M} \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)^M \subset L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M), \tag{2-54}$$

at every Lebesgue point $x_0' \in \mathbb{R}^{n-1}$ of f there holds

$$\lim_{\substack{(x',t)\to(x_0',0)\\ |x'-x_0'|<\kappa t}} (P_t * f)(x') = \left(\int_{\mathbb{R}^{n-1}} P(x')\,dx'\right) f(x_0'), \tag{2-55}$$

and the function

$$\mathbb{R}^n_+ \ni (x',t) \mapsto (P_t * f)(x') \in \mathbb{C}^M \text{ is locally integrable in } \mathbb{R}^n_+. \tag{2-56}$$

3. Proof of the existence statements in Theorem 1.1

This section is devoted to proving Proposition 3.1, dealing with the issue of existence for the BMO-Dirichlet boundary value problem (1-29), the upper estimate in (1-31), and the issue of existence for the VMO-Dirichlet boundary value problem (1-36).

In this regard, we find it useful to adopt a more general point of view, by going beyond the class BMO through the consideration of convolutions of the Poisson kernel with functions f from the weighted Lebesgue space $L^1(\mathbb{R}^{n-1}, dx'/(1+|x'|^n))^M$; recall the inclusion in (2-24). The aforementioned convolutions are then shown to satisfy a variety of Carleson-measure-like conditions, which only require, recall (2-13),

$$\int_1^\infty \operatorname{osc}_1(f;\lambda)\,\frac{d\lambda}{\lambda^2} < \infty. \tag{3-1}$$

Note that this permits the oscillations $\operatorname{osc}_1(f;\lambda)$ of the given function f to grow with the scale λ. In particular, this allows us to simultaneously treat several scales of spaces of interest, including Hölder spaces $\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)$ with $\eta \in (0,1)$, the Morrey–Campanato space $\mathscr{E}^{\eta,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ with $\eta \in (0,1)$ and $p \in [1,\infty)$, as well as the John–Nirenberg space $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$.

An example of a function $f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)$ with $\eta \in (0,1)$ which does not belong to $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ is offered by

$$f(x') := |x'|^\eta \quad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{3-2}$$

Indeed, $\|f\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)} \le 1$ and since $\delta_\lambda f = \lambda^\eta f$, it follows from the last line in (1-17) that necessarily $\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} = \infty$. Incidentally, for f as in (3-2), we have $\operatorname{osc}_1(f;\lambda) = O(\lambda^\eta)$ as $\lambda \to \infty$; hence (3-1) holds in this case.

Here is the formal statement of the result just advertised above.

Proposition 3.1. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and let P^L be the Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Select $f \in L^1(\mathbb{R}^{n-1}, dx'/(1+|x'|^n))^M$ and set*

$$u(x',t) := (P^L_t * f)(x') \quad \text{for all } (x',t) \in \mathbb{R}^n_+. \tag{3-3}$$

Then u is meaningfully defined via an absolutely convergent integral and satisfies

$$u \in \mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M), \quad Lu = 0 \text{ in } \mathbb{R}^n_+, \quad \text{and} \quad u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \text{ a.e. in } \mathbb{R}^{n-1}. \tag{3-4}$$

In addition, u enjoys the following properties:

(a) *For each integer $\ell \geq 1$ there exists a constant $C \in (0,\infty)$ with the property that the following pointwise estimate holds for every $(x',t) \in \mathbb{R}^n_+$*:

$$|(\nabla^\ell u)(x',t)| \leq \frac{C}{t^\ell} \int_1^\infty \operatorname{osc}_1(f;\lambda t)\,\frac{d\lambda}{\lambda^{1+\ell}}. \tag{3-5}$$

In particular, there exists $C \in (0,\infty)$ such that

$$|(\nabla u)(x',t)| \leq \frac{C}{t} \int_1^\infty \operatorname{osc}_1(f;\lambda t)\,\frac{d\lambda}{\lambda^2} \quad \text{for all } (x',t) \in \mathbb{R}^n_+. \tag{3-6}$$

(b) *There exists a constant $C \in (0,\infty)$ such that for every cube Q in $\mathbb{R}^{n-1}$ the following "cube-by-cube" Carleson measure estimates hold*:

$$\begin{aligned}\biggl(\int_0^{\ell(Q)} \fint_Q |(\nabla u)(x',t)|^2\, t\, dx'\, dt\biggr)^{\frac{1}{2}} \\ \leq C \int_1^\infty \biggl(\fint_{\lambda Q} |f(y') - f_{\lambda Q}|\, dy'\biggr) \frac{d\lambda}{\lambda^2} + C \sup_{Q' \subseteq 4Q} \fint_{Q'} |f(y') - f_{Q'}|\, dy'\end{aligned} \tag{3-7}$$

and

$$\biggl(\int_0^{\ell(Q)} \fint_Q |(\nabla u)(x',t)|^2\, t\, dx'\, dt\biggr)^{\frac{1}{2}} \leq C \int_1^\infty \operatorname{osc}_1(f;\lambda\,\ell(Q))\frac{d\lambda}{\lambda^2}. \tag{3-8}$$

(c) *There exists $C \in (0,\infty)$ such that the following local Carleson measure estimate holds for every scale $r \in (0,\infty)$*:

$$\sup_{Q \subset \mathbb{R}^{n-1},\, \ell(Q) \leq r} \biggl(\int_0^{\ell(Q)} \fint_Q |(\nabla u)(x',t)|^2\, t\, dx'\, dt\biggr)^{\frac{1}{2}} \leq C \int_1^\infty \operatorname{osc}_1(f;r\lambda)\,\frac{d\lambda}{\lambda^2}. \tag{3-9}$$

(d) *Whenever f satisfies*

$$\int_1^\infty \operatorname{osc}_1(f;\lambda)\,\frac{d\lambda}{\lambda^2} < \infty, \tag{3-10}$$

the global weighted Carleson measure estimate

$$\sup_{Q \subset \mathbb{R}^{n-1}} \biggl\{ \biggl(\int_1^\infty \operatorname{osc}_1(f;\lambda\,\ell(Q))\,\frac{d\lambda}{\lambda^2}\biggr)^{-1} \biggl(\int_0^{\ell(Q)} \fint_Q |(\nabla u)(x',t)|^2\, t\, dx'\, dt\biggr)^{\frac{1}{2}} \biggr\} \leq C \tag{3-11}$$

holds for some $C \in (0,\infty)$ independent of f.

(e) *There exists a constant $C \in (0,\infty)$ such that the following global Carleson measure estimate holds*:

$$\|u\|_{**} = \sup_{Q \subset \mathbb{R}^{n-1}} \biggl(\int_0^{\ell(Q)} \fint_Q |(\nabla u)(x',t)|^2\, t\, dx'\, dt\biggr)^{\frac{1}{2}} \leq C\, \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{3-12}$$

In particular, thanks to (2-24), *estimate* (3-12) *holds for every $f \in \mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$.*

(f) *Whenever f satisfies*

$$\int_1^\infty \operatorname{osc}_1(f;\lambda)\,\frac{d\lambda}{\lambda^2} < \infty \quad \textit{and} \quad \lim_{r\to 0^+} \operatorname{osc}_1(f;r) = 0, \tag{3-13}$$

the following vanishing Carleson measure condition holds:

$$\lim_{r\to 0^+}\left\{\sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r}\left(\int_0^{\ell(Q)}\fint_Q |(\nabla u)(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}}\right\} = 0. \tag{3-14}$$

In particular, *in the case when $f \in \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ to begin with*, *u has the additional property that*

$$|\nabla u(x',t)|^2\, t\, dx'\, dt \ \textit{is a vanishing Carleson measure in } \mathbb{R}^n_+. \tag{3-15}$$

Ultimately, the proof of Proposition 3.1 relies on square-function estimates. For now, assuming a suitable L^2 bound (implicit in (3-19) below) we may establish some versatile Carleson measure estimates (of local and global nature), as well as vanishing Carleson measure properties for integral operators (modeled upon the gradient of the convolution with the Poisson kernel) acting on function spaces larger than the standard BMO. This is made precise in the following proposition.

Proposition 3.2. *Let $\theta : \mathbb{R}^n_+ \times \mathbb{R}^{n-1} \to \mathbb{C}^{M\times M}$ be a matrix-valued Lebesgue-measurable function*, *with the property that there exist $\varepsilon \in (0,\infty)$ and $C \in (0,\infty)$ such that*

$$|\theta(x',t;y')| \le \frac{C t^\varepsilon}{|(x'-y',t)|^{n-1+\varepsilon}} \quad \textit{for all } (x',t)\in\mathbb{R}^n_+,\ \textit{for all } y'\in\mathbb{R}^{n-1}, \tag{3-16}$$

and the following cancellation condition holds:

$$\int_{\mathbb{R}^{n-1}} \theta(x',t;y')\, dy' = 0 \in \mathbb{C}^{M\times M} \quad \textit{for all } (x',t)\in\mathbb{R}^n_+. \tag{3-17}$$

In relation to the kernel θ, *one may then consider the integral operator Θ acting on arbitrary functions $f \in L^1(\mathbb{R}^{n-1}, dx'/(1+|x'|^{n-1+\varepsilon}))^M$ according to* (*the absolutely convergent integral*)

$$(\Theta f)(x',t) := \int_{\mathbb{R}^{n-1}} \theta(x',t;y')\, f(y')\, dy' \quad \textit{for all } (x',t)\in\mathbb{R}^n_+. \tag{3-18}$$

Then, *under the assumption that the operator*

$$\Theta : L^2(\mathbb{R}^{n-1},\mathbb{C}^M) \to L^2\left(\mathbb{R}^n_+, \frac{dx'\,dt}{t}\right)^M \ \textit{is bounded}, \tag{3-19}$$

the following properties hold:

(a) *There exists a constant $C \in (0,\infty)$ such that for every $f \in L^1(\mathbb{R}^{n-1}, dx'/(1+|x'|^{n-1+\varepsilon}))^M$ and every cube Q in $\mathbb{R}^{n-1}$ the following "cube-by-cube" Carleson measure estimates hold*:

$$\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}} \le C\int_1^\infty\left(\fint_{\lambda Q}|f(y')-f_{\lambda Q}|\,dy'\right)\frac{d\lambda}{\lambda^{1+\varepsilon}} + C\sup_{Q'\subseteq 4Q}\fint_{Q'}|f(y')-f_{Q'}|\,dy' \tag{3-20}$$

and

$$\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}} \le C\int_1^\infty \operatorname{osc}_1(f;\lambda\ell(Q))\frac{d\lambda}{\lambda^{1+\varepsilon}}. \tag{3-21}$$

(b) *There exists* $C\in(0,\infty)$ *such that for every function* $f\in L^1(\mathbb{R}^{n-1},dx'/(1+|x'|^{n-1+\varepsilon}))^M$ *the following local Carleson measure estimate holds for every scale* $r\in(0,\infty)$:

$$\sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r}\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}} \le C\int_1^\infty \operatorname{osc}_1(f;r\lambda)\,\frac{d\lambda}{\lambda^{1+\varepsilon}}. \tag{3-22}$$

(c) *There exists* $C\in(0,\infty)$ *such that for any given function* $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ *with the property that*

$$\int_1^\infty \operatorname{osc}_1(f;\lambda)\,\frac{d\lambda}{\lambda^{1+\varepsilon}}<\infty \tag{3-23}$$

(*which necessarily places* f *into the space* $L^1(\mathbb{R}^{n-1},dx'/(1+|x'|^{n-1+\varepsilon}))^M$ *by* (2-23)) *the following global weighted Carleson measure estimate holds*:

$$\sup_{Q\subset\mathbb{R}^{n-1}}\left\{\left(\int_1^\infty \operatorname{osc}_1(f;\lambda\ell(Q))\,\frac{d\lambda}{\lambda^{1+\varepsilon}}\right)^{-1}\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}}\right\}\le C. \tag{3-24}$$

(d) *There exists a constant* $C\in(0,\infty)$ *such that for every* $f\in\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ *the following global Carleson measure estimate holds*:

$$\sup_{Q\subset\mathbb{R}^{n-1}}\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}}\le C\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{3-25}$$

(e) *Whenever* $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ *is such that*

$$\int_1^\infty \operatorname{osc}_1(f;\lambda)\,\frac{d\lambda}{\lambda^{1+\varepsilon}}<\infty\quad\textit{and}\quad \lim_{r\to0^+}\operatorname{osc}_1(f;r)=0, \tag{3-26}$$

then $f\in L^1(\mathbb{R}^{n-1},dx'/(1+|x'|^{n-1+\varepsilon}))^M$ *and the following vanishing Carleson measure condition holds*:

$$\lim_{r\to0^+}\left\{\sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r}\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}}\right\}=0. \tag{3-27}$$

In particular, (3-27) *holds for every function* $f\in\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$.

Proof. Start by fixing an arbitrary cube Q in $\mathbb{R}^{n-1}$ and denote by x'_Q its center. Given a function $f\in L^1(\mathbb{R}^{n-1},dx'/(1+|x'|^{n-1+\varepsilon}))^M$, use (3-17) in order to write

$$\left(\int_0^{\ell(Q)}\fint_Q |(\Theta f)(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}}=\left(\int_0^{\ell(Q)}\fint_Q |(\Theta(f-f_Q))(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}}\le I+II, \tag{3-28}$$

where

$$I:=\left(\int_0^{\ell(Q)}\fint_Q |\Theta((f-f_Q)\,\mathbf{1}_{4Q})(x',t)|^2\,\frac{dx'\,dt}{t}\right)^{\frac{1}{2}} \tag{3-29}$$

and

$$II := \left(\int_0^{\ell(Q)} \fint_Q |\Theta((f - f_Q)\,\mathbf{1}_{\mathbb{R}^{n-1}\setminus 4Q})(x',t)|^2 \,\frac{dx'\,dt}{t} \right)^{\frac{1}{2}}. \tag{3-30}$$

To estimate I, invoke (3-19), (2-8) with $p = 2$, and (2-12) to estimate

$$\begin{aligned} I &\le \frac{1}{|Q|^{1/2}} \left(\int_{\mathbb{R}^n_+} |\Theta((f - f_Q)\,\mathbf{1}_{4Q})(x',t)|^2 \,\frac{dx'\,dt}{t} \right)^{\frac{1}{2}} \\ &\le C \left(\fint_{4Q} |f(y') - f_Q|^2 \,dy' \right)^{\frac{1}{2}} \le C \left(\fint_{4Q} |f(y') - f_{4Q}|^2 \,dy' \right)^{\frac{1}{2}} \\ &\le C \sup_{Q' \subseteq 4Q} \fint_{Q'} |f(y') - f_{Q'}| \,dy', \end{aligned} \tag{3-31}$$

where $C \in (0,\infty)$ is independent of f and Q. To proceed, observe that there exists a purely dimensional constant $c_n \in (0,\infty)$ (e.g., the choice $c_n := 3/(6 + 2\sqrt{n-1}\,)$ will do) with the property that

$$|x' - y'| \ge c_n(\ell(Q) + |x'_Q - y'|) \quad \text{for each } x' \in Q,\ y' \in \mathbb{R}^{n-1} \setminus 4Q. \tag{3-32}$$

Based on this, (3-18), and (3-16), we may then estimate

$$\begin{aligned} |\Theta((f - f_Q)\,\mathbf{1}_{\mathbb{R}^{n-1}\setminus 4Q})(x',t)| &\le C t^{\varepsilon} \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}} \,dy' \\ &\text{for every point } x' \in Q \text{ and every number } t > 0, \end{aligned} \tag{3-33}$$

for some $C \in (0,\infty)$ depending only on n and the constant appearing in (3-16). In turn, from (3-33) and (2-21) we conclude that

$$\begin{aligned} II &\le C \left(\int_0^{\ell(Q)} \left(\frac{t}{\ell(Q)} \right)^{2\varepsilon} \frac{dt}{t} \right)^{\frac{1}{2}} \cdot \int_1^{\infty} \left(\fint_{\lambda Q} |f(y') - f_{\lambda Q}| \,dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &= C \int_1^{\infty} \left(\fint_{\lambda Q} |f(y') - f_{\lambda Q}| \,dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \end{aligned} \tag{3-34}$$

At this stage, (3-28), (3-31), and (3-34) combine to give (3-20). In turn, (3-21) readily follows from (3-20) and part (a) in Lemma 2.1, which allows us to estimate

$$\begin{aligned} \operatorname{osc}_1(f; 4\ell(Q)) &\le \varepsilon(4^{-\varepsilon} - 5^{-\varepsilon})^{-1} \int_4^5 \operatorname{osc}_1(f; \lambda\,\ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &\le C_\varepsilon \int_1^{\infty} \operatorname{osc}_1(f; \lambda\,\ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \end{aligned} \tag{3-35}$$

In concert with part (a) in Lemma 2.1, estimate (3-21) immediately gives (3-22). Estimate (3-21) also implies the global weighted Carleson measure estimate formulated in (3-24). From (3-22) and part (c) in Lemma 2.1, the global Carleson measure estimate stated in (3-25) follows.

Going further, assume the function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ satisfies the properties listed in (3-26). Then $f \in L^1(\mathbb{R}^{n-1}, dx'/(1 + |x'|^{n-1+\varepsilon}))^M$ by (2-22). Also, thanks to (3-26) and part (a) in Lemma 2.1, Lebesgue's dominated convergence theorem applies and yields

$$\lim_{r\to 0^+}\int_1^\infty \operatorname{osc}_1(f;r\lambda)\,\frac{d\lambda}{\lambda^{1+\varepsilon}}=0. \tag{3-36}$$

Together with (3-22), this ultimately proves the vanishing Carleson measure condition stated in (3-27). Finally, that any function $f\in \mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ actually satisfies the properties listed in (3-26) is clear from (1-22), (2-13), and part (c) in Lemma 2.1. This completes the proof of Proposition 3.2. □

Next the goal is to identify a class of integral kernels θ satisfying (3-16)–(3-17) with the property that the operator Θ associated with θ as in (3-18) enjoys the L^2-boundedness condition formulated in (3-19). We adopt a broader point of view by considering a larger variety of spaces, which turns out to be useful later. To set the stage, let us recall the definition of the Hardy space $H^1(\mathbb{R}^{n-1})$ using $(1,\infty)$-atoms. Specifically, a Lebesgue-measurable function $a:\mathbb{R}^{n-1}\to\mathbb{C}$ is said to be a $(1,\infty)$-atom provided there exists a cube $Q\subset\mathbb{R}^{n-1}$ such that the following localization, normalization, and cancellation properties hold:

$$\operatorname{supp} a\subseteq Q,\quad \|a\|_{L^\infty(\mathbb{R}^{n-1})}\le |Q|^{-1},\quad \int_{\mathbb{R}^{n-1}} a(y')\,dy'=0. \tag{3-37}$$

The space $H^1(\mathbb{R}^{n-1})$ is then defined as the collection of all Lebesgue-measurable functions f defined in $\mathbb{R}^{n-1}$ such that

$$f=\sum_{j=1}^\infty \lambda_j a_j \quad \text{a.e. in } \mathbb{R}^{n-1}, \tag{3-38}$$

with the a_j's being $(1,\infty)$-atoms, and where the sequence $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ satisfies $\sum_{j=1}^\infty|\lambda_j|<\infty$. The norm in $H^1(\mathbb{R}^{n-1})$ is defined as

$$\|f\|_{H^1(\mathbb{R}^{n-1})}:=\inf\sum_{j=1}^\infty|\lambda_j|, \tag{3-39}$$

where the infimum runs over all the atomic decompositions of f as in (3-38). In particular, the series in (3-38) converges in $H^1(\mathbb{R}^{n-1})$. Let us also write $H^1(\mathbb{R}^{n-1},\mathbb{C}^M)$ for the collection of all $\mathbb{C}^M$-valued functions $f=(f_\alpha)_{1\le\alpha\le M}$ with components in $H^1(\mathbb{R}^{n-1})$. In such a scenario, we set

$$\|f\|_{H^1(\mathbb{R}^{n-1},\mathbb{C}^M)}:=\sum_{\alpha=1}^M\|f_\alpha\|_{H^1(\mathbb{R}^{n-1})}. \tag{3-40}$$

Here are the square-function estimates alluded to earlier. For more background and relevant references the reader is referred to the recent exposition in [Hofmann et al. 2017].

Proposition 3.3. *Let θ and Θ be as in* (3-16)–(3-18) *with $\varepsilon=1$ and $M=1$. In addition, assume θ is of class $\mathscr{C}^1$ in the variable $y'\in\mathbb{R}^{n-1}$ and suppose there exists some $C\in(0,\infty)$ with the property that*

$$|\nabla_{y'}\theta(x',t;y')|\le\frac{Ct}{|(x'-y',t)|^{n+1}}\quad \textit{for all } (x',t)\in\mathbb{R}^n_+,\ \textit{for all } y'\in\mathbb{R}^{n-1}. \tag{3-41}$$

Fix a background parameter $\kappa > 0$ and, with the nontangential cone $\Gamma_\kappa(x')$ as in (1-9) *for each $x' \in \mathbb{R}^{n-1}$, define the square function operator S_Θ by setting*

$$(S_\Theta f)(x') := \left(\int_{\Gamma_\kappa(x')} |(\Theta f)(y',t)|^2 \frac{dy'\,dt}{t^n} \right)^{\frac{1}{2}} \quad \textit{for all } x' \in \mathbb{R}^{n-1}. \tag{3-42}$$

Then the following are well-defined, *linear*, *and bounded operators*:

$$\Theta : L^2(\mathbb{R}^{n-1}) \to L^2\left(\mathbb{R}^n_+, \frac{dx'\,dt}{t}\right), \tag{3-43}$$

$$S_\Theta : L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1}) \quad \textit{for all } p \in (1,\infty), \tag{3-44}$$

$$S_\Theta : L^1(\mathbb{R}^{n-1}) \to L^{1,\infty}(\mathbb{R}^{n-1}), \tag{3-45}$$

$$S_\Theta : H^1(\mathbb{R}^{n-1}) \to L^1(\mathbb{R}^{n-1}). \tag{3-46}$$

Proof. We are going to use [Christ 1990, Theorem 20, p. 69]; see also [Christ and Journé 1987]. First observe that (3-16) with $\varepsilon = 1$ presently implies

$$|\theta(x',t;y')| \le C\,\frac{t}{(t+|x'-y'|)^n} \quad \text{for all } x', y' \in \mathbb{R}^{n-1}, \text{ for all } t > 0. \tag{3-47}$$

Second, if x', y', $z' \in \mathbb{R}^{n-1}$ and $t > 0$ are such that $|y'-z'| \le (t+|x'-y'|)/2$, the mean value theorem and (3-41) imply (here and elsewhere, $[a,b]$ denotes the line segment with end-points $a, b \in \mathbb{R}^{n-1}$)

$$\begin{aligned} |\theta(x',t;y') - \theta(x',t;z')| &\le |y'-z'| \sup_{w' \in [y',z']} |\nabla_{y'}\theta(x',t;w')| \\ &\le C|y'-z'| \sup_{w' \in [y',z']} \frac{t}{(t+|x'-w'|)^{n+1}} \\ &\le C\,\frac{|y'-z'|\,t}{(t+|x'-y'|)^{n+1}}. \end{aligned} \tag{3-48}$$

This proves that the family of kernels $\{\theta(x',t;y')\}_{t \in (0,\infty)}$ is a standard family in $\mathbb{R}^{n-1}$ as in [Christ 1990, Definition 19, p. 69]. Third, (3-17) implies that $\Theta 1(x',t) = 0$ for every $(x',t) \in \mathbb{R}^n_+$. We can therefore apply [loc. cit., Theorem 20, p. 69] to conclude that the operator in (3-43) is well-defined, linear and bounded. In particular, the boundedness of the operator in (3-43) implies that there exists a constant $C \in (0,\infty)$ such that for every function $f \in L^2(\mathbb{R}^{n-1})$ there holds

$$\|S_\Theta f\|^2_{L^2(\mathbb{R}^{n-1})} = C_{n,\kappa} \int_{\mathbb{R}^n_+} |(\Theta f)(y',t)|^2 \frac{dy'\,dt}{t} \le C \int_{\mathbb{R}^{n-1}} |f(x')|^2\,dx'. \tag{3-49}$$

Proving the boundedness of the operator in (3-45) comes down to establishing the weak-type-$(1,1)$ estimate for S_Θ. In a first stage, we claim that there exists some constant $C \in (0,\infty)$ with the property that for any cube Q in $\mathbb{R}^{n-1}$ and any function h satisfying

$$h \in L^1(\mathbb{R}^{n-1}), \quad \operatorname{supp} h \subseteq Q, \quad \text{and} \quad \int_{\mathbb{R}^{n-1}} h(y')\,dy' = 0 \tag{3-50}$$

we have

$$|(S_\Theta h)(x')| \le C\,\|h\|_{L^1(\mathbb{R}^{n-1})}\,\frac{\ell(Q)}{|x'-x'_Q|^n} \quad \text{for every } x' \in \mathbb{R}^{n-1}\setminus 2Q, \tag{3-51}$$

where x'_Q is the center of the cube Q. To justify the claim, given $x', z' \in \mathbb{R}^{n-1}$ with $x' \neq z'$ we first use (3-41) combined with natural changes of variables to write

$$\begin{aligned}
\int_{\Gamma_\kappa(x')} |\nabla_{y'}\theta(y',t;z')|^2\,\frac{dy'\,dt}{t^n}
&\le C\int_{|y'-x'|<\kappa t} \frac{t^2}{|(y'-z',t)|^{2(n+1)}}\,\frac{dy'\,dt}{t^n}\\
&= C\int_0^\infty\!\!\int_{|y'|<\kappa} \frac{t^2}{|(y't+(x'-z'),t)|^{2(n+1)}}\,\frac{dy'\,dt}{t}\\
&= C\int_0^\infty\!\!\int_{|y'|<\kappa} \frac{t^{-2n}}{|(y'+(x'-z')/t,1)|^{2(n+1)}}\,\frac{dy'\,dt}{t}\\
&\le C|x'-z'|^{-2n}\int_0^\infty\!\!\int_{|y'|<\kappa} \frac{t^{-2n}}{|(y'+(x'-z')/(t|x'-z'|),1)|^{2(n+1)}}\,\frac{dy'\,dt}{t}\\
&\le C|x'-z'|^{-2n}\sup_{|v'|=1}\int_0^\infty\!\!\int_{|y'|<\kappa} \frac{t^{-2n}}{|y'+v'/t|^{2(n+1)}+1}\,\frac{dy'\,dt}{t}.
\end{aligned} \tag{3-52}$$

Next, fix $v' \in \mathbb{R}^{n-1}$ with $|v'|=1$. If $t<1/(2\kappa)$ and $|y'|<\kappa$ we have

$$t^{-1} = \frac{|v'|}{t} \le \left|y'+\frac{v'}{t}\right| + |y'| < \left|y'+\frac{v'}{t}\right| + \frac{1}{2t} \tag{3-53}$$

and therefore $|y'+v'/t|>1/(2t)$. Thus,

$$\int_0^{\frac{1}{2\kappa}}\!\!\int_{|y'|<\kappa} \frac{t^{-2n}}{|y'+v'/t|^{2(n+1)}+1}\,\frac{dy'\,dt}{t} \le C\int_0^{\frac{1}{2\kappa}}\!\!\int_{|y'|<\kappa} t\,dy'\,dt \le C. \tag{3-54}$$

Also, it is immediate that

$$\int_{\frac{1}{2\kappa}}^\infty \int_{|y'|<\kappa} \frac{t^{-2n}}{|y'+v'/t|^{2(n+1)}+1}\,\frac{dy'\,dt}{t} \le C\int_{\frac{1}{2\kappa}}^\infty \int_{|y'|<\kappa} t^{-2n}\,\frac{dy'\,dt}{t} \le C. \tag{3-55}$$

Combining (3-52), (3-54), and (3-55) we may conclude that

$$\left(\int_{\Gamma_\kappa(x')} |\nabla_{y'}\theta(y',t;z')|^2\,\frac{dy'\,dt}{t^n}\right)^{\frac12} \le \frac{C}{|x'-z'|^n} \quad \text{if } x' \neq z'. \tag{3-56}$$

At this point we return to the proof of (3-51). Fix $x' \in \mathbb{R}^{n-1}\setminus 2Q$ and consider h as in (3-50). Making use of the last property of h recorded in (3-50), the fundamental theorem of calculus, Minkowski's inequality, and (3-56) we may compute

$$
\begin{aligned}
(S_\Theta h)(x') &= \Big(\int_{\Gamma_\kappa(x')} |(\Theta h)(y',t)|^2 \frac{dy'\,dt}{t^n}\Big)^{\frac12}\\
&\le \Big(\int_{\Gamma_\kappa(x')}\Big(\int_{\mathbb{R}^{n-1}} |\theta(y',t;z')-\theta(y',t;x'_Q)|\,|h(z')|\,dz'\Big)^2 \frac{dy'\,dt}{t^n}\Big)^{\frac12}\\
&\le \int_0^1\int_{\mathbb{R}^{n-1}}\Big(\int_{\Gamma_\kappa(x')} |\nabla_{z'}\theta(y',t;x'_Q+s\,(z'-x'_Q))|^2 \frac{dy'\,dt}{t^n}\Big)^{\frac12} |h(z')|\,|z'-x'_Q|\,dz'\,ds\\
&\le C\int_0^1\int_Q \frac{1}{|x'-(x'_Q+s\,(z'-x'_Q))|^n}\,|h(z')|\,|z'-x'_Q|\,dz'\,ds\\
&\le C\,\|h\|_{L^1(\mathbb{R}^{n-1})}\frac{\ell(Q)}{|x'-x'_Q|^n}.
\end{aligned}
\tag{3-57}
$$

For the last inequality in (3-57) we used the observation that for every $s\in(0,1)$ and every $z'\in Q$ one has (keeping in mind that $x'\in\mathbb{R}^{n-1}\setminus 2Q$ and $x'_Q+s\,(z'-x'_Q)\in Q$)

$$
\begin{aligned}
|x'-x'_Q| &\le |x'-(x'_Q+s\,(z'-x'_Q))| + \tfrac12(\sqrt{n-1}\,\ell(Q))\\
&\le |x'-(x'_Q+s\,(z'-x'_Q))| + \sqrt{n-1}\,|x'-(x'_Q+s\,(z'-x'_Q))|\\
&= (1+\sqrt{n-1})|x'-(x'_Q+s\,(z'-x'_Q))|.
\end{aligned}
\tag{3-58}
$$

This finishes the proof of (3-51).

Let us momentarily digress to show that

$$
\int_{\mathbb{R}^{n-1}\setminus Q}\frac{\ell(Q)}{|x'-x'_Q|^n}\,dx' \le \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ}\frac{\ell(Q)}{(\ell(2^kQ)/2)^n}\,dx' \le 2^{2n-1}\sum_{k=0}^\infty 2^{-k} = 4^n. \tag{3-59}
$$

We are now ready to show that S_Θ maps $L^1(\mathbb{R}^{n-1})$ continuously into $L^{1,\infty}(\mathbb{R}^{n-1})$. Following [García-Cuerva and Rubio de Francia 1985, p. 140], given a function $f\in L^1(\mathbb{R}^{n-1})$ and fixed $\lambda>0$, let $\{Q_j\}_j\subset\mathbb{R}^{n-1}$ be the nonoverlapping cubes of the Calderón–Zygmund decomposition of $|f|$ at height λ. That is, the Q_j's are the maximal dyadic cubes for which $|Q_j|^{-1}\int_{Q_j}|f(y')|\,dy'>\lambda$. Set

$$
\Omega_\lambda = \bigcup_{j=1}^\infty Q_j \tag{3-60}
$$

and observe that we have the following properties:

$$
\mathscr{L}^{n-1}(\Omega_\lambda)\le \lambda^{-1}\|f\|_{L^1(\mathbb{R}^{n-1})}, \tag{3-61}
$$

$$
\lambda < \fint_{Q_j}|f(y')|\,dy' \le 2^{n-1}\lambda \quad \text{for all } j\in\mathbb{N}, \tag{3-62}
$$

$$
|f(x')|\le\lambda \quad \text{for } \mathscr{L}^{n-1}\text{-a.e. } x'\in\mathbb{R}^{n-1}\setminus\Omega_\lambda. \tag{3-63}
$$

Finally, split $f = g + b$, where, see [loc. cit., p. 198],

$$g := f\,\mathbf{1}_{\mathbb{R}^{n-1}\setminus\Omega_\lambda} + \sum_{j=1}^{\infty} f_{Q_j}\,\mathbf{1}_{Q_j}, \quad b = \sum_{j=1}^{\infty} b_j, \quad \text{with } b_j := (f - f_{Q_j})\,\mathbf{1}_{Q_j} \text{ for each } j \in \mathbb{N}. \tag{3-64}$$

In particular, (3-60)–(3-64) imply, see [loc. cit., p. 198], that for some constant $C \in (0,\infty)$ independent of f and λ we have

$$\|g\|^2_{L^2(\mathbb{R}^{n-1})} \le 2^{n-1}\lambda\|f\|_{L^1(\mathbb{R}^{n-1})}. \tag{3-65}$$

Making use of (3-64), (3-61), (3-65), (3-49)–(3-51), (3-59) (used with Q_j in place of Q), and bearing in mind that for each $j \in \mathbb{N}$ we have $\operatorname{supp} b_j \subset Q_j$ and $\int_{\mathbb{R}^{n-1}} b_j(y')\,dy' = 0$, we may then estimate

$$\begin{aligned}
&\lambda\mathscr{L}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (S_\Theta f)(x') > \lambda\}) \\
&\quad\le \lambda\mathscr{L}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (S_\Theta g)(x') > \lambda/2\}) + \lambda\mathscr{L}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (S_\Theta b)(x') > \lambda/2\}) \\
&\quad\le \frac{4}{\lambda}\int_{\mathbb{R}^{n-1}} |(S_\Theta g)(x')|^2\,dx' + \lambda\mathscr{L}^{n-1}\Big(\bigcup_{j=1}^{\infty} 2Q_j\Big) + 2\sum_{j=1}^{\infty}\int_{\mathbb{R}^{n-1}\setminus 2Q_j} |(S_\Theta b_j)(x')|\,dx' \\
&\quad\le \frac{C}{\lambda}\int_{\mathbb{R}^{n-1}} |g(x')|^2\,dx' + C\|f\|_{L^1(\mathbb{R}^{n-1})} + C\sum_{j=1}^{\infty}\|b_j\|_{L^1(\mathbb{R}^{n-1})}\int_{\mathbb{R}^{n-1}\setminus 2Q_j}\frac{\ell(Q_j)}{|x'-x'_{Q_j}|^n}\,dx' \\
&\quad\le C\|f\|_{L^1(\mathbb{R}^{n-1})} + C\sum_{j=1}^{\infty}\|b_j\|_{L^1(\mathbb{R}^{n-1})} \\
&\quad\le C\|f\|_{L^1(\mathbb{R}^{n-1})} + C\Big(\sum_{j=1}^{\infty}\int_{Q_j}|f(y')|\,dy'\Big) \\
&\quad\le C\|f\|_{L^1(\mathbb{R}^{n-1})}.
\end{aligned} \tag{3-66}$$

This proves that the operator in (3-45) is well-defined, linear and bounded. The latter and Marcinkiewicz's interpolation theorem imply the boundedness of the operator in (3-44) when $1 < p \le 2$. We may handle the full range $1 < p < \infty$ by invoking [Hofmann et al. 2017, Theorem 1.1, p. 6], applied with $\mathscr{X} := \overline{\mathbb{R}^n_+}$ equipped with the standard Euclidean distance and Lebesgue measure, $E := \mathbb{R}^{n-1}\times\{0\}$, $m = n$, $d = n-1$, $\upsilon = 1$, $\alpha = 1$, $\sigma := \mathscr{L}^{n-1}$, and the integral operator with kernel $t^{-1}\theta(x',t;y')$. The fact that (3-43) holds implies that [loc. cit., (1.25), p. 6] is satisfied. As such, [loc. cit., Theorem 1.1, p. 6] guarantees the validity of [loc. cit., (1.34), p. 7], which, in our current notation, implies that the operator in (3-44) is bounded for every $p \in (1,\infty)$.

Next we consider S_Θ in the context of (3-46). In this regard, we shall first show that there exists some constant $C \in (0,\infty)$ such that for every $(1,\infty)$-atom a one has

$$\|S_\Theta\, a\|_{L^1(\mathbb{R}^{n-1})} \le C. \tag{3-67}$$

To justify (3-67) fix an arbitrary function a satisfying the conditions listed in (3-37). On the one hand, based on Hölder's inequality, (3-49) and the first two properties in (3-37) we may write

$$\int_{2Q} |(S_\Theta\, a)(x')|\,dx' \le C|Q|^{\frac12}\,\|S_\Theta\, a\|_{L^2(\mathbb{R}^{n-1})} \le C|Q|^{\frac12}\,\|a\|_{L^2(\mathbb{R}^{n-1})} \le C \tag{3-68}$$

for some finite constant $C > 0$ independent of a. On the other hand, (3-37) allows us to make use of (3-51) (with a in place of h), which we combine with the second property in (3-37) and (3-59) to obtain

$$\int_{\mathbb{R}^{n-1}\setminus 2Q} |(S_\Theta\, a)(x')|\, dx' \le C\|a\|_{L^1(\mathbb{R}^{n-1})} \int_{\mathbb{R}^{n-1}\setminus 2Q} \frac{\ell(Q)}{|x'-x'_Q|^n}\, dx' \le C, \tag{3-69}$$

with $C \in (0,\infty)$ independent of the atom a. Combining (3-68) and (3-69) then proves that (3-67) holds.

Here is the end-game in the proof of the fact that S_Θ maps $H^1(\mathbb{R}^{n-1})$ boundedly into $L^1(\mathbb{R}^{n-1})$. Let $f \in H^1(\mathbb{R}^{n-1})$ be arbitrary and consider an atomic decomposition $f = \sum_{j=1}^\infty \lambda_j\, a_j$ convergent in $H^1(\mathbb{R}^{n-1})$, where the a_j's are $(1,\infty)$-atoms, which is quasioptimal in the sense that $\sum_{j=1}^\infty |\lambda_j| \approx \|f\|_{H^1(\mathbb{R}^{n-1})}$, where the proportionality constants do not depend on f. In particular, this forces $f = \sum_{j=1}^\infty \lambda_j\, a_j$ in $L^1(\mathbb{R}^{n-1})$ and the weak-type-(1, 1) estimate for S_Θ then implies $S_\Theta f = \sum_{j=1}^\infty \lambda_j\, S_\Theta\, a_j$ in $L^{1,\infty}(\mathbb{R}^{n-1})$. Then the sequence of partial sums associated with the latter series has a subsequence which converges a.e. to $S_\Theta f$. In turn, this allows us to conclude that

$$|(S_\Theta f)(x')| \le \sum_{j=1}^\infty |\lambda_j|\,|(S_\Theta\, a_j)(x')| \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \tag{3-70}$$

In concert, (3-70) and (3-67) then imply

$$\|S_\Theta f\|_{L^1(\mathbb{R}^{n-1})} \le \sum_{j=1}^\infty |\lambda_j|\,\|S_\Theta a_j\|_{L^1(\mathbb{R}^{n-1})} \le C \sum_{j=1}^\infty |\lambda_j| \le C\|f\|_{H^1(\mathbb{R}^{n-1})}, \tag{3-71}$$

as desired, for some constant $C \in (0,\infty)$ independent of f. □

We now have all the ingredients to proceed with the proof of Proposition 3.1.

Proof of Proposition 3.1. Fix an arbitrary $f \in L^1(\mathbb{R}^{n-1}, dx'/(1+|x'|^n))^M$ and define u as in (3-3). Part (7) in Theorem 2.3 then ensures that this function satisfies all properties listed in (3-4).

As in (2-37), write $K^L(x',t) = P_t^L(x')$ for each $(x',t) \in \mathbb{R}^n_+$. To proceed, fix an arbitrary point $(x',t) \in \mathbb{R}^n_+$ and denote by $Q_{x',t}$ the cube in $\mathbb{R}^{n-1}$ centered at x' with side-length t. Making use of (2-36) we obtain

$$\int_{\mathbb{R}^{n-1}} \nabla^\ell[P_t^L(x'-y')]\, dy' = 0 \quad \text{for all } x' \in \mathbb{R}^{n-1},\ \text{ for all } t > 0,\ \text{ for all } \ell \in \mathbb{N}. \tag{3-72}$$

Based on this, for each $\ell \in \mathbb{N}$ we may then write

$$\begin{aligned}(\nabla^\ell u)(x',t) &= \int_{\mathbb{R}^{n-1}} \nabla^\ell[P_t^L(x'-y')] f(y')\, dy' \\ &= \int_{\mathbb{R}^{n-1}} \nabla^\ell[P_t^L(x'-y')][f(y') - f_{Q_{x',t}}]\, dy' \\ &= \int_{\mathbb{R}^{n-1}} (\nabla^\ell K^L)(x'-y',t)[f(y') - f_{Q_{x',t}}]\, dy'. \end{aligned} \tag{3-73}$$

Combining (3-73), (2-39), and (2-21) (with $\varepsilon = \ell$), we may now estimate

$$|(\nabla^\ell u)(x',t)| \le C \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_{Q_{x',t}}|}{|(x'-y',t)|^{n-1+\ell}}\, dy' \le \frac{C}{t^\ell} \int_1^\infty \operatorname{osc}_1(f;\lambda t)\, \frac{d\lambda}{\lambda^{1+\ell}}, \tag{3-74}$$

from which the claims in part (a) of the statement follow.

Moving on, fix an arbitrary $j \in \{1,\dots,n\}$ and, for each $\alpha, \beta \in \{1,\dots,M\}$, set

$$\theta^j_{\alpha\beta}(x',t;y') := t\, \partial_j K^L_{\alpha\beta}(x'-y',t) \quad \text{for all } x', y' \in \mathbb{R}^{n-1}_+, \text{ for all } t > 0. \tag{3-75}$$

In this regard, first observe that (2-39) in Theorem 2.3 implies

$$|\theta^j_{\alpha\beta}(x',t;y')| = t\, |\partial_j K^L_{\alpha\beta}(x'-y',t)| \le C t |(x'-y',t)|^{-n} \tag{3-76}$$

and

$$|\nabla_{y'} \theta^j_{\alpha\beta}(x',t;y')| \le t\, |\nabla^2 K^L_{\alpha\beta}(x'-y',t)| \le C t |(x'-y',t)|^{-n-1}. \tag{3-77}$$

Hence, (3-16) (with $\varepsilon = 1$) and (3-41) hold for $\theta^j_{\alpha\beta}$. Moreover,

$$\int_{\mathbb{R}^{n-1}} \theta^j_{\alpha\beta}(x',t;y')\, dy' = \int_{\mathbb{R}^{n-1}} t\, \partial_j K^L_{\alpha\beta}(x'-y',t)\, dy' = t\, \partial_j \int_{\mathbb{R}^{n-1}} K^L_{\alpha\beta}(y',t)\, dy' = 0 \tag{3-78}$$

since $\partial_j \int_{\mathbb{R}^{n-1}} (P^L_{\alpha\beta})_t(y')\, dy' = 0$ by (3-72). Writing $\Theta^j_{\alpha\beta}$ for the operator associated with the kernel $\theta^j_{\alpha\beta}$ (in place of θ) as in (3-18), it follows from (3-76), (3-77), (3-78), and Proposition 3.3 that each matrix integral operator $\Theta^j := (\Theta^j_{\alpha\beta})_{1 \le \alpha,\beta \le M}$ satisfies all hypotheses in Proposition 3.2, including (3-19). In addition,

$$\begin{aligned}
\left(\int_0^{\ell(Q)} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt \right)^{\frac12} &= \left(\int_0^{\ell(Q)} \fint_Q |t \nabla (P^L_t * f)(x')|^2 \, \frac{dx'\, dt}{t} \right)^{\frac12} \\
&\le \sum_{j=1}^n \left(\int_0^{\ell(Q)} \fint_Q |t\, (\partial_j K^L(\cdot,t) * f)(x')|^2 \, \frac{dx'\, dt}{t} \right)^{\frac12} \\
&= \sum_{j=1}^n \left(\int_0^{\ell(Q)} \fint_Q |(\Theta^j f)(x',t)|^2 \, \frac{dx'\, dt}{t} \right)^{\frac12}.
\end{aligned} \tag{3-79}$$

Granted this, all remaining conclusions in parts (b)–(f) of the statement become direct consequences of Proposition 3.2. □

4. A Fatou result and uniqueness in the BMO-Dirichlet problem

The main result in this section is the following Poisson representation formula and Fatou theorem.

Proposition 4.1. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3). *Assume that*

$$u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad Lu = 0 \textit{ in } \mathbb{R}^n_+, \quad \textit{and} \quad \|u\|_{**} < \infty. \tag{4-1}$$

Then there exists a unique function $f \in L^1(\mathbb{R}^{n-1}, 1/(1+|x'|^n)\,dx')^M$ *such that*

$$u(x',t) = (P_t^L * f)(x') \quad \text{for all } (x',t) \in \mathbb{R}^n_+, \tag{4-2}$$

where P^L *is the Poisson kernel for* L *in* $\mathbb{R}^n_+$ *from Theorem 2.3.*

In fact, $u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}$ *exists at a.e. point in* $\mathbb{R}^{n-1}$, *belongs to* $\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, *and* $f = u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}$. *Moreover, there exists a constant* $C = C(n, L) \in (1, \infty)$ *such that*

$$C^{-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \le \|u\|_{**} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{4-3}$$

Also, as a corollary of Proposition 4.1 we have the following result which, in view of (1-14), implies the uniqueness statements for the BMO-Dirichlet problem and the VMO-Dirichlet problem from Theorem 1.1.

Proposition 4.2. *Let* L *be an* $M \times M$ *elliptic system with constant complex coefficients as in* (1-2)–(1-3). *Assume that*

$$\begin{aligned} &u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad Lu = 0 \text{ in } \mathbb{R}^n_+, \quad \|u\|_{**} < \infty, \\ &u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} \text{ exists and vanishes at a.e. point in } \mathbb{R}^{n-1}. \end{aligned} \tag{4-4}$$

Then necessarily $u \equiv 0$ *in* $\mathbb{R}^n_+$.

The proof of Proposition 4.1 occupies the bulk of this section. To set the stage, we first prove some auxiliary lemmas. The first such lemma contains Bloch-like estimates for smooth null-solutions of L satisfying a Carleson measure condition in the upper half-space. To place things in perspective, recall that a holomorphic function u in the upper half-plane is said to satisfy a Bloch estimate provided

$$\sup_{x\in\mathbb{R},\, y>0} (y|u'(x+iy)|) < \infty. \tag{4-5}$$

Lemma 4.3. *Let* L *be an* $M \times M$ *elliptic system with constant complex coefficients as in* (1-2)–(1-3). *Then for every multi-index* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \ge 1$ *there exists a finite constant* $C = C(n, L, \alpha) > 0$ *with the property that for every function* $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ *satisfying* $Lu = 0$ *in* $\mathbb{R}^n_+$ *and* $\|u\|_{**} < \infty$ *one has*

$$\sup_{(x',t)\in\mathbb{R}^n_+} \{t^{|\alpha|}\,|(\partial^\alpha u)(x',t)|\} \le C\|u\|_{**}. \tag{4-6}$$

In particular, there exists a finite constant $C = C(n, L) > 0$ *with the property that for every function* $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ *such that* $Lu = 0$ *in* $\mathbb{R}^n_+$ *and* $\|u\|_{**} < \infty$ *one has*

$$\sup_{(x',t)\in\mathbb{R}^n_+} t\,|\nabla u(x',t)| \le C\|u\|_{**}. \tag{4-7}$$

Proof. Given a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$, select $j \in \{1, \dots, n\}$ and $\beta \in \mathbb{N}_0^n$ such that $\alpha = \beta + e_j$. Fix $x = (x', t) \in \mathbb{R}^n_+$ and write R_x for the cube in $\mathbb{R}^n$ centered at x with side-length t. Also, let $Q_{x'}$ be the cube in $\mathbb{R}^{n-1}$ centered at x' with side-length t. Since the function $\partial_j u$ is a null-solution of the system L, we may invoke Theorem 2.4 (with $\partial_j u$ in place of u and $p = 2$) in order to conclude

$$|\partial^\beta(\partial_j u)(x',t)| \le \frac{C_\beta}{t^{|\beta|}} \left(\fint_{R_x} |\partial_j u|^2\, d\mathscr{L}^n \right)^{\frac{1}{2}}. \tag{4-8}$$

Hence,

$$t^{|\alpha|}|(\partial^\alpha u)(x',t)| \le Ct\biggl(\fint_{R_x} |\nabla u|^2\, d\mathscr{L}^n\biggr)^{\frac{1}{2}} \le C\biggl(\int_{\frac{t}{2}}^{\frac{3t}{2}} \fint_{Q_{x'}} |\nabla u(y',s)|^2\, s\, dy'\, ds\biggr)^{\frac{1}{2}} \le C\|u\|_{**}, \quad (4\text{-}9)$$

proving (4-6). Estimate (4-7) is a particular case of (4-6). □

We continue by discussing two purely real-variable results. To state the first one, recall the function $\Upsilon_\# : [0,\infty) \to [0,\infty)$ from (1-50). In relation to this, we make two observations. First,

$$\begin{aligned} &\text{for each } \varepsilon \in (0,\infty) \text{ there exists } C_\varepsilon \in (1,\infty) \text{ such that} \\ &C_\varepsilon^{-1}\Upsilon_\#(s) \le \Upsilon_\#(s/\varepsilon) \le C_\varepsilon \Upsilon_\#(s) \text{ for every } s \in [0,\infty). \end{aligned} \quad (4\text{-}10)$$

Second, since for every $\eta \in (0,1]$ there exists a constant $C = C_\eta \in (0,\infty)$ with the property that

$$\Upsilon_\#(s) \le Cs^\eta \quad \text{for all } s \ge 0, \quad (4\text{-}11)$$

we have, see (1-48),

$$\mathscr{C}^{\Upsilon_\#}(\mathbb{R}^{n-1},\mathbb{C}^M) \subset \mathrm{Lip}(\mathbb{R}^{n-1},\mathbb{C}^M) \cap \biggl(\bigcap_{0<\eta<1} \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)\biggr). \quad (4\text{-}12)$$

This is going to be relevant later on, in the proof of Lemma 4.6. For now, here is the first real-variable result advertised above.

Lemma 4.4. *Recall* $\Upsilon_\#$ *from* (1-50) *and let* $u \in \mathscr{C}^1(\mathbb{R}^n_+,\mathbb{C}^M)$ *be such that*

$$C_u := \sup_{(x',t)\in\mathbb{R}^n_+} t|\nabla u(x',t)| < \infty. \quad (4\text{-}13)$$

Then, for every (x',t) *and* (y',t) *in* $\mathbb{R}^n_+$ *one has*

$$|u(x',t) - u(y',t)| \le 2C_u \Upsilon_\#\biggl(\frac{|x'-y'|}{t}\biggr). \quad (4\text{-}14)$$

Proof. The proof follows the argument in [Fabes et al. 1976]. Fix (x',t) and (y',t) in $\mathbb{R}^n_+$. Based on the mean value theorem and (4-13) we may estimate

$$|u(x',t) - u(y',t)| \le \sup_{\xi\in[x',y']} |\nabla u(\xi,t)||x'-y'| \le C_u \frac{|x'-y'|}{t}. \quad (4\text{-}15)$$

Suppose now that $|x'-y'| > t$ and set $r := |x'-y'|$. Applying the fundamental theorem of calculus, (4-15), and (4-13) we obtain

$$\begin{aligned} |u(x',t) - u(y',t)| &\le |u(x',t) - u(x',r)| + |u(x',r) - u(y',r)| + |u(y',r) - u(y',t)| \\ &\le \int_t^r |\partial_n u(x',\lambda)|\, d\lambda + C_u + \int_t^r |\partial_n u(y',\lambda)|\, d\lambda \\ &\le C_u + 2C_u \int_t^r \frac{1}{\lambda}\, d\lambda \le 2C_u\biggl(1 + \ln\frac{|x'-y'|}{t}\biggr). \end{aligned} \quad (4\text{-}16)$$

With this in hand, (4-14) follows from (4-16) (which is valid for $|x'-y'| > t$) and (4-15) used for $|x'-y'| \le t$. □

The second real-variable result mentioned earlier reads as follows.

Lemma 4.5. *Let $\Upsilon_\#$ be the function defined in* (1-50). *Then for every $a > 0$ one has*

$$\Psi(a) := \int_0^\infty \frac{s^{n-1}}{(a+s)^n}\,\Upsilon_\#(s)\,\frac{ds}{s} \le 3\begin{cases} 1+\ln(1/a) & \text{if } a \le 1, \\ (1+\ln a)/a & \text{if } a > 1. \end{cases} \tag{4-17}$$

In particular, $\Psi(a) \le 3\,(1+\log^+ 1/a)$, *where* $\log^+ s := \max\{\ln s, 0\}$ *for every* $s \in (0,\infty)$.

Proof. If $a \ge 1$, we use that $s^{n-2}\,\Upsilon_\#(s)$ is increasing and elementary calculus to obtain

$$\begin{aligned} \Psi(a) &\le a^{-n}\int_0^a s^{n-2}\,\Upsilon_\#(s)\,ds + \int_a^\infty \frac{1+\ln s}{s^2}\,ds \\ &\le a^{-n}\,a^{n-2}\,\Upsilon_\#(a)\,a + a^{-1} + \left[\frac{-1-\ln s}{s}\right]_{s=a}^{s=\infty} \le 3\,\frac{1+\ln a}{a}. \end{aligned} \tag{4-18}$$

On the other hand, if $a < 1$ then

$$\begin{aligned} \Psi(a) &\le \int_0^a \frac{s^{n-1}}{a^n}\,s\,\frac{ds}{s} + \int_a^1 \frac{s^{n-1}}{s^n}\,s\,\frac{ds}{s} + \int_1^\infty \frac{s^{n-1}}{s^n}\,(1+\ln s)\,\frac{ds}{s} \\ &= \frac{1}{n} + \ln\frac{1}{a} + 2 \le 3\Big(1+\ln\frac{1}{a}\Big). \end{aligned} \tag{4-19}$$

Collectively, (4-18)–(4-19) prove the lemma. □

Having dealt with Lemmas 4.4–4.5, in our next two lemmas we study the boundary behavior of the vertical shifts of a smooth null-solution of L which satisfies a Carleson measure condition in the upper half-space.

Lemma 4.6. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Suppose $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfies $Lu = 0$ in $\mathbb{R}^n_+$ and $\|u\|_{**} < \infty$. For each $\varepsilon > 0$, define $u_\varepsilon(x',t) := u(x', t+\varepsilon)$ for every $(x',t) \in \mathbb{R}^n_+$ and $f_\varepsilon(x') := u(x',\varepsilon)$ for every $x' \in \mathbb{R}^{n-1}$. Then there exists a constant $C \in (0,\infty)$ such that for every $\varepsilon > 0$ the following properties are valid*:

(a) *The function u_ε belongs to $\mathscr{C}^\infty(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ and $Lu_\varepsilon = 0$ in $\mathbb{R}^n_+$.*

(b) *One has $\|u_\varepsilon\|_{**} \le C\,\|u\|_{**}$. In fact, for every multi-index $\alpha \in \mathbb{N}_0^n$ there exists a constant $C_\alpha \in (0,\infty)$, independent of u and ε, with the property that $\|\partial^\alpha u_\varepsilon\|_{**} \le C_\alpha \varepsilon^{-|\alpha|}\|u\|_{**}$.*

(c) *For every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$ there exists a constant $C_\alpha \in (0,\infty)$, independent of u, with the property that $\|\partial^\alpha u_\varepsilon\|_{L^\infty(\mathbb{R}^n_+)} \le C_\alpha\,\varepsilon^{-|\alpha|}\,\|u\|_{**}$.*

(d) *The function f_ε belongs to $\mathscr{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \mathscr{C}^{\Upsilon_\#}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, where $\Upsilon_\#$ is as in* (1-50). *In particular*,

$$f_\varepsilon \in \mathrm{Lip}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \Big(\bigcap_{0<\eta<1} \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)\Big); \tag{4-20}$$

hence also $f_\varepsilon \in \mathrm{UC}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and

$$f_\varepsilon \in L^1\Big(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n}\,dx'\Big)^M. \tag{4-21}$$

Moreover,

$$\text{for every } \alpha' \in \mathbb{N}_0^{n-1} \text{ with } |\alpha'| \geq 1 \text{ one has } \partial^{\alpha'} f_\varepsilon \in L^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \mathscr{C}^{\Upsilon_\#}(\mathbb{R}^{n-1}, \mathbb{C}^M). \tag{4-22}$$

(e) *The function* $v_\varepsilon(x', t) := (P_t^L * f_\varepsilon)(x')$ *is well-defined for all* $(x', t) \in \mathbb{R}^n_+$ *via an absolutely convergent integral and*

$$v_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad Lv_\varepsilon = 0 \text{ in } \mathbb{R}^n_+, \quad v_\varepsilon\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = f_\varepsilon \text{ everywhere in } \mathbb{R}^{n-1}. \tag{4-23}$$

(f) *For every* $(y', t) \in \mathbb{R}^n_+$ *one has*

$$|v_\varepsilon(y', t) - f_\varepsilon(y')| + t\,|\nabla v_\varepsilon(y', t)| \leq C\,\|u\|_{**}\,(t/\varepsilon)\,(1 + \log^+(\varepsilon/t)). \tag{4-24}$$

Proof. The claim in part (a) is clear from definitions. To prove the estimate in part (b), fix a cube $Q \subset \mathbb{R}^{n-1}$. Consider first the case $\ell(Q) \geq \varepsilon$, in which scenario a change of variables yields

$$\begin{aligned}\frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q |\nabla u(x', t+\varepsilon)|^2\, t\, dx'\, dt &\leq \frac{1}{|Q|}\int_\varepsilon^{\ell(Q)+\varepsilon}\!\!\int_Q |\nabla u(x', t)|^2\, t\, dx'\, dt \\ &\leq 2^{n-1}\frac{1}{|2Q|}\int_0^{2\ell(Q)}\!\!\int_{2Q} |\nabla u(x', t)|^2\, t\, dx'\, dt \\ &\leq 2^{n-1}\,\|u\|_{**}^2. \end{aligned}\tag{4-25}$$

In the case $\ell(Q) < \varepsilon$, use Lemma 4.3 to conclude that

$$\begin{aligned}\frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q |\nabla u(x', t+\varepsilon)|^2\, t\, dx'\, dt &\leq C^2\,\|u\|_{**}^2\,\frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q \frac{1}{(t+\varepsilon)^2}\, t\, dx'\, dt \\ &\leq C^2\,\|u\|_{**}^2\,\varepsilon^{-2}\int_0^{\ell(Q)} t\, dt \leq \frac{C^2}{2}\|u\|_{**}^2 \end{aligned}\tag{4-26}$$

for some $C \in (0, \infty)$ independent of u and ε. Combining (4-25) and (4-26) and taking the supremum over all cubes Q then proves the first estimate in part (b) for some $C \in (0, \infty)$ independent of u and ε.

To justify the second estimate in part (b), it suffices to consider the case when the multi-index $\alpha \in \mathbb{N}_0^n$ has length $|\alpha| \geq 1$. Assume that this is the case and pick an arbitrary cube $Q \subset \mathbb{R}^{n-1}$. Making use of (4-6) and bearing in mind that $|\alpha| \geq 1$ we may then estimate

$$\begin{aligned}\frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q \big|\nabla[(\partial^\alpha u_\varepsilon)(x', t)]\big|^2\, t\, dx'\, dt &= \frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q \big|\nabla[(\partial^\alpha u)(x', t+\varepsilon)]\big|^2\, t\, dx'\, dt \\ &\leq \frac{C_\alpha\|u\|_{**}^2}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q \frac{1}{(t+\varepsilon)^{2|\alpha|+1}}\, dx'\, dt \\ &\leq C_\alpha\|u\|_{**}^2\int_0^\infty \frac{1}{(t+\varepsilon)^{2|\alpha|+1}}\, dt \leq C_\alpha\|u\|_{**}^2\,\varepsilon^{-2|\alpha|}, \end{aligned}\tag{4-27}$$

from which the desired conclusion readily follows.

Consider next the claim in part (c). Given a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$ we may invoke Lemma 4.3, keeping in mind the conclusions in part (a), in order to conclude that there exists a constant $C_\alpha \in (0, \infty)$, independent of u, such that

$$\sup_{(x',t)\in\mathbb{R}^n_+}|(\partial^\alpha u_\varepsilon)(x',t)|\leq \sup_{(x',t)\in\mathbb{R}^n_+}[(t+\varepsilon)^{|\alpha|}|(\partial^\alpha u)(x',t+\varepsilon)|]\sup_{(x',t)\in\mathbb{R}^n_+}(t+\varepsilon)^{-|\alpha|}\leq C_\alpha\,\|u\|_{**}\,\varepsilon^{-|\alpha|}. \tag{4-28}$$

We now turn to proving the claims in part (d). First, since $f_\varepsilon(y')=u(y',\varepsilon)$ for every $y'\in\mathbb{R}^{n-1}$ we have $f_\varepsilon\in\mathscr{C}^\infty(\mathbb{R}^{n-1},\mathbb{C}^M)$. Second, by using (4-14) (with $t=\varepsilon$), (4-7), and (4-10), for each $x',y'\in\mathbb{R}^{n-1}$ we may estimate

$$|f_\varepsilon(x')-f_\varepsilon(y')|\leq C\,\|u\|_{**}\,\Upsilon_\#\Big(\frac{|x'-y'|}{\varepsilon}\Big)\leq C_{n,L,u,\varepsilon}\,\Upsilon_\#(|x'-y'|). \tag{4-29}$$

This places f_ε in $\mathscr{C}^{\Upsilon_\#}(\mathbb{R}^{n-1},\mathbb{C}^M)$. With this in hand, the conclusions in (4-20) follow with the help of (4-12). As a Hölder function, f_ε also belongs to $L^1(\mathbb{R}^{n-1},1/(1+|x'|^n)\,dx')^M$, see (2-26), proving (4-21).

Next, fix a multi-index $\alpha'\in\mathbb{N}_0^{n-1}$ of length $|\alpha'|\geq 1$. Then $\partial^{(\alpha',0)}u_\varepsilon\in\mathscr{C}^\infty(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ is a null-solution of L in $\mathbb{R}^n_+$ and $\partial^{\alpha'}f_\varepsilon=(\partial^{(\alpha',0)}u_\varepsilon)|_{\partial\mathbb{R}^n_+}$. Now the claim in (4-22) is a consequence of parts (c) and (b), bearing in mind that $\|\partial^{(\alpha',0)}u_\varepsilon\|_{**}<\infty$ (hence, the same type of argument that placed f_ε in $\mathscr{C}^{\Upsilon_\#}(\mathbb{R}^{n-1},\mathbb{C}^M)$ now ensures the membership of $\partial^{\alpha'}f_\varepsilon$ to the latter space).

Moving on, the claim made in part (e) is a consequence of the current part (d) together with part (7) in Theorem 2.3 and the fact that since $f_\varepsilon\in\mathscr{C}^\infty(\mathbb{R}^{n-1},\mathbb{C}^M)$, all points in $\mathbb{R}^{n-1}$ are Lebesgue points for f_ε.

Finally, consider the claim in part (f). Fix $(y',t)\in\mathbb{R}^n_+$. Then the properties of the Poisson kernel recalled in Theorem 2.3, together with Lemmas 4.3, 4.4, and 4.5 permit us to estimate, bearing in mind that $f_\varepsilon=u(\cdot,\varepsilon)$,

$$\begin{aligned}
|v_\varepsilon(y',t)-f_\varepsilon(y')| &\leq \int_{\mathbb{R}^{n-1}}|P^L(z')|\,|f_\varepsilon(y'-tz')-f_\varepsilon(y')|\,dz'\\
&\leq C\,\|u\|_{**}\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|z'|)^n}\,\Upsilon_\#\Big(\frac{t|z'|}{\varepsilon}\Big)\,dz'\\
&\leq C\,\|u\|_{**}\int_0^\infty\frac{r^{n-1}}{(1+r)^n}\,\Upsilon_\#\Big(\frac{tr}{\varepsilon}\Big)\frac{dr}{r}\\
&\leq C\,\|u\|_{**}\,(t/\varepsilon)\int_0^\infty\frac{s^{n-1}}{(t/\varepsilon+s)^n}\,\Upsilon_\#(s)\,\frac{ds}{s}\\
&= C\,\|u\|_{**}\,(t/\varepsilon)\,\Psi(t/\varepsilon)\leq C\,\|u\|_{**}\,(t/\varepsilon)\,(1+\log^+(\varepsilon/t)).
\end{aligned} \tag{4-30}$$

This suits our current purposes.

Consider next the task of estimating ∇v_ε. Using the properties of the Poisson kernel, Theorem 2.3 (recall (2-37) and (2-39)) and Lemmas 4.3, 4.4, 4.5, we write

$$\begin{aligned}
|\nabla v_\varepsilon(x',t)| &= \big|\nabla(P_t^L*(f_\varepsilon(\cdot)-f_\varepsilon(x')))(x')\big|\\
&\leq \int_{\mathbb{R}^{n-1}}|\nabla K^L(x'-y',t)|\,|f_\varepsilon(y')-f_\varepsilon(x')|\,dy'\\
&\leq C\,\|u\|_{**}\int_{\mathbb{R}^{n-1}}\frac{1}{(t+|x'-y'|)^n}\,\Upsilon_\#\Big(\frac{|x'-y'|}{\varepsilon}\Big)\,dy'
\end{aligned}$$

$$\leq C\,\|u\|_{**}\,\varepsilon^{-1}\int_0^\infty \frac{s^{n-1}}{(s+t/\varepsilon)^n}\,\Upsilon_{\#}(s)\,\frac{ds}{s}$$
$$= C\,\|u\|_{**}\,\varepsilon^{-1}\,\Psi(t/\varepsilon) \leq C\,\|u\|_{**}\,\varepsilon^{-1}\,(1+\log^+(\varepsilon/t)). \tag{4-31}$$

Collectively, (4-30) and (4-31) prove (4-24). □

We are now ready to prove that each vertical shift of a smooth null-solution of L which satisfies a Carleson measure condition in the upper half-space has a Poisson integral representation formula.

Lemma 4.7. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Let $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfy $Lu = 0$ in $\mathbb{R}^n_+$ and $\|u\|_{**} < \infty$. For each given $\varepsilon > 0$, define $u_\varepsilon(x', t) := u(x', t+\varepsilon)$ for every $(x', t) \in \mathbb{R}^n_+$.*

Then for every $\varepsilon > 0$ one has $u_\varepsilon \in \mathscr{C}^\infty(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, the restriction $u_\varepsilon|_{\partial\mathbb{R}^n_+}$ belongs to the space $L^1(\mathbb{R}^{n-1}, 1/(1+|x'|^n)\,dx')^M$, and the following Poisson integral representation formula holds:

$$u_\varepsilon(x', t) = (P^L_t * (u_\varepsilon|_{\partial\mathbb{R}^n_+}))(x') \quad \textit{for all } (x', t) \in \mathbb{R}^n_+. \tag{4-32}$$

Proof. For each $\varepsilon > 0$ set $f_\varepsilon := u_\varepsilon|_{\partial\mathbb{R}^n_+}$ and note that by part (d) in Lemma 4.6 we have that f_ε belongs to $L^1(\mathbb{R}^{n-1}, 1/(1+|x'|^n)dx')^M \cap \mathscr{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$. Next, for each $\varepsilon > 0$ define $v_\varepsilon(x', t) := (P^L_t * f_\varepsilon)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. The goal is to show that $w_\varepsilon := v_\varepsilon - u_\varepsilon \equiv 0$ in $\mathbb{R}^n_+$. A key ingredient in this regard is Proposition 2.5.

Notice first that $w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ and $Lw_\varepsilon = 0$ in $\mathbb{R}^n_+$ by parts (a) and (e) in Lemma 4.6. Next, we propose to show that $\operatorname{Tr} w_\varepsilon = 0$, where Tr is as introduced in (2-47). Since by part (a) in Lemma 4.6 we have $\operatorname{Tr} u_\varepsilon = f_\varepsilon$, it remains to prove $\operatorname{Tr} v_\varepsilon = f_\varepsilon$ in $\mathbb{R}^{n-1}$. To this end, given $x' \in \mathbb{R}^{n-1}$, we use part (f) in Lemma 4.6, the fact that $f_\varepsilon(x') = u(x', \varepsilon)$, Lemma 4.3 and Lemma 4.4 (recall that $\Upsilon_{\#}$ is defined in (1-50)) to write

$$\begin{aligned}
&\Big|\fint_{B((x',0),r)\cap\mathbb{R}^n_+} v_\varepsilon\,d\mathscr{L}^n - f_\varepsilon(x')\Big| \\
&\quad\leq \fint_{B((x',0),r)\cap\mathbb{R}^n_+} |v_\varepsilon(y',t) - f_\varepsilon(x')|\,dy'\,dt \\
&\quad\leq \fint_{B((x',0),r)\cap\mathbb{R}^n_+} |v_\varepsilon(y',t) - f_\varepsilon(y')|\,dy'\,dt + \fint_{B((x',0),r)\cap\mathbb{R}^n_+} |f_\varepsilon(y') - f_\varepsilon(x')|\,dy'\,dt \\
&\quad\leq C\,\|u\|_{**}\fint_{B((x',0),r)\cap\mathbb{R}^n_+} (t/\varepsilon)\,(1+\log^+(\varepsilon/t))\,dy'\,dt \\
&\qquad\qquad + C\,\|u\|_{**}\fint_{B((x',0),r)\cap\mathbb{R}^n_+} \Upsilon_{\#}(|x'-y'|/\varepsilon)\,dy'\,dt \\
&\quad\leq C\,\|u\|_{**}\,\frac{r}{\varepsilon}\,(1+\log^+(\varepsilon/r)) + C\,\|u\|_{**}\,\Upsilon_{\#}(r/\varepsilon) \to 0 \quad \text{as } r \to 0^+.
\end{aligned} \tag{4-33}$$

Thus we conclude that $\operatorname{Tr} v_\varepsilon(x') = f_\varepsilon(x')$ for every $x' \in \mathbb{R}^{n-1}$ as desired.

Next we claim that $w_\varepsilon \in W^{1,2}_{\mathrm{bd}}(\mathbb{R}^n_+, \mathbb{C}^M)$; recall the latter space from (2-46). By parts (a) and (c) in Lemma 4.6 we have that $u_\varepsilon \in W^{1,2}_{\mathrm{bd}}(\mathbb{R}^n_+, \mathbb{C}^M)$. For v_ε, fix $R > 0$ arbitrary and rely on (4-24) to estimate

$$\begin{aligned}\|v_\varepsilon\|_{L^2(B(0,R)\cap\mathbb{R}^n_+)} &\le \left(\int_0^R\int_{|x'|\le R} |v_\varepsilon(x',t) - f_\varepsilon(x')|^2\,dx'\,dt\right)^{\frac12} + \left(\int_0^R\int_{|x'|\le R} |f_\varepsilon(x')|^2\,dx'\,dt\right)^{\frac12}\\ &\le C\,\|u\|_{**}\,(R/\varepsilon)\,(1+\log^+(\varepsilon/R))\,R^{\frac{n}{2}} + R^{\frac12}\|f_\varepsilon\|_{L^2(B_{n-1}(0',R))} < \infty,\end{aligned} \tag{4-34}$$

since $f_\varepsilon \in C^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$. Above and elsewhere in the paper we make the convention that

$$B_{n-1}(x', R) \text{ denotes the ball in } \mathbb{R}^{n-1} \text{ centered at } x' \in \mathbb{R}^{n-1} \text{ and of radius } R. \tag{4-35}$$

As regards ∇v_ε, we use (4-24) to write

$$\begin{aligned}\|\nabla v_\varepsilon\|_{L^2(B(0,R)\cap\mathbb{R}^n_+)} &\le C\,R^{\frac{n-1}{2}}\,\|u\|_{**}\,\varepsilon^{-1}\left(\int_0^R (1+\log^+(\varepsilon/t))^2\,dt\right)^{\frac12}\\ &= C\,R^{\frac{n-1}{2}}\,\|u\|_{**}\,\varepsilon^{-1}\left(\int_0^\varepsilon (1+\ln(\varepsilon/t))^2\,dt + \int_\varepsilon^R dt\right)^{\frac12}\\ &\le C\,R^{\frac{n-1}{2}}\,\|u\|_{**}\,\varepsilon^{-1}\left(\varepsilon\int_0^1 (1+\ln(1/s))^2\,ds + R\right)^{\frac12}\\ &\le C\,R^{\frac{n-1}{2}}\,\|u\|_{**}\,\varepsilon^{-1}\,(\varepsilon+R)^{\frac12} < \infty.\end{aligned} \tag{4-36}$$

From (4-34) and (4-36) we conclude that v_ε and, therefore w_ε, belongs to $W^{1,2}_{\mathrm{bd}}(\mathbb{R}^n_+, \mathbb{C}^M)$.

Having established these, we may apply Proposition 2.5 and obtain that for every $z \in \overline{\mathbb{R}^n_+}$ and $\rho > 0$

$$\begin{aligned}&\sup_{\mathbb{R}^n_+\cap B(z,\rho)} |\nabla w_\varepsilon|\\ &\quad\le C\rho^{-1}\sup_{\mathbb{R}^n_+\cap B(z,2\rho)} |w_\varepsilon| = C\rho^{-1}\sup_{\mathbb{R}^n_+\cap B(z,2\rho)} |u_\varepsilon - v_\varepsilon|\\ &\quad\le C\rho^{-1}\sup_{(y',t)\in\mathbb{R}^n_+\cap B(z,2\rho)} |u_\varepsilon(y',t) - f_\varepsilon(y')| + C\rho^{-1}\sup_{(y',t)\in\mathbb{R}^n_+\cap B(z,2\rho)} |v_\varepsilon(y',t) - f_\varepsilon(y')|.\end{aligned} \tag{4-37}$$

Let $(y',t) \in \mathbb{R}^n_+ \cap B(z, 2\rho)$ and note that Lemma 4.3 implies

$$\begin{aligned}|u_\varepsilon(y',t) - f_\varepsilon(y')| &= |u(y', t+\varepsilon) - u(y',\varepsilon)|\\ &\le \int_\varepsilon^{t+\varepsilon} |\partial_n u(y',\lambda)|\,d\lambda\\ &\le C\,\|u\|_{**}\int_\varepsilon^{t+\varepsilon}\frac{1}{\lambda}\,d\lambda\\ &= C\,\|u\|_{**}\ln\frac{t+\varepsilon}{\varepsilon}.\end{aligned} \tag{4-38}$$

Proceeding as in (4-30), Lemma 4.5 implies that for every $t > \varepsilon$ we have

$$|v_\varepsilon(y',t) - f_\varepsilon(y')| \le C\,\|u\|_{**}\,(t/\varepsilon)\,\Psi(t/\varepsilon) \le C\,\|u\|_{**}\,(1+\ln(t/\varepsilon)). \tag{4-39}$$

Returning with (4-38), (4-39) and (4-24) back to (4-37) we obtain that for every $z \in \partial\mathbb{R}^n_+$ and every $\rho > \varepsilon$

$$\begin{aligned}\sup_{\mathbb{R}^n_+ \cap B(z,\rho)} |\nabla w_\varepsilon| &\le C\|u\|_{**}\Big(\rho^{-1}\sup_{0<t<2\rho}\ln\frac{t+\varepsilon}{\varepsilon}+\rho^{-1}\sup_{0<t<\varepsilon}(t/\varepsilon)(1+\log^+(\varepsilon/t))+\rho^{-1}\sup_{\varepsilon<t<2\rho}(1+\ln(t/\varepsilon))\Big)\\ &\le C\|u\|_{**}\Big(\rho^{-1}\ln\frac{2\rho+\varepsilon}{\varepsilon}+\rho^{-1}+\rho^{-1}(1+\ln(2\rho/\varepsilon))\Big). \end{aligned} \tag{4-40}$$

Since the last expression converges to 0 as $\rho \to \infty$ we obtain that $\nabla w_\varepsilon \equiv 0$ in $\mathbb{R}^n_+$. As we have already shown that $w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ this forces w_ε to be constant in $\mathbb{R}^n_+$. In concert with the fact that $\operatorname{Tr} w_\varepsilon = 0$ this ultimately implies $w_\varepsilon \equiv 0$ in $\mathbb{R}^n_+$ as desired. □

Moving on, in Lemmas 4.8–4.12 below we develop essential tools for the proof of Proposition 4.13, where we prove a partial converse to part (e) in Proposition 3.1. Concretely, there we show that if $f \in L^1(\mathbb{R}^{n-1}, 1/(1+|x'|^n)\,dx')^M$ has the property that the Littlewood–Paley measure $|\nabla u(x',t)|^2\,t\,dx'\,dt$ associated with the function u defined as in (3-3) is a Carleson measure in $\mathbb{R}^n_+$ then necessarily f belongs to $\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$.

We begin by introducing some notation. Specifically, consider

$$H^1_a(\mathbb{R}^{n-1}) := \Big\{ g \in L^\infty_{\mathrm{comp}}(\mathbb{R}^{n-1}) : \int_{\mathbb{R}^{n-1}} g\, d\mathscr{L}^{n-1} = 0 \Big\}, \tag{4-41}$$

where $L^\infty_{\mathrm{comp}}(\mathbb{R}^{n-1})$ stands for the space of essentially bounded functions with compact support in $\mathbb{R}^{n-1}$. In particular, since any $g \in H^1_a(\mathbb{R}^{n-1})$ is a scalar multiple of a $(1,\infty)$-atom, recall (3-37), it follows that

$$H^1_a(\mathbb{R}^{n-1}) \text{ is a dense subspace of } H^1(\mathbb{R}^{n-1}). \tag{4-42}$$

In the lemma below we prove a pointwise decay estimate for the vertical maximal operator acting on functions from $H^1_a(\mathbb{R}^{n-1})$. Recall the definition from (1-7).

Lemma 4.8. *Let $\phi = (\phi_{\alpha\beta})_{1\le\alpha,\beta\le M} : \mathbb{R}^{n-1} \to \mathbb{C}^{M\times M}$ be a matrix-valued function with differentiable entries satisfying the property that there exists $C \in (0,\infty)$ such that*

$$|\phi(x')| + |\nabla\phi(x')| \le \frac{C}{1+|x'|^n} \quad \text{for every } x' \in \mathbb{R}^{n-1}. \tag{4-43}$$

Pick a function $g = (g_\alpha)_{1\le\alpha\le M}$ with components in $H^1_a(\mathbb{R}^{n-1})$. Then there exists a constant $C_g \in (0,\infty)$, depending on g, such that

$$\sup_{t>0} |(\phi_t * g)(x')| \le \frac{C_g}{1+|x'|^n} \quad \text{for every } x' \in \mathbb{R}^{n-1}. \tag{4-44}$$

Proof. Take $R = R_g \ge 1$ sufficiently large so that, recall (4-35), $\operatorname{supp} g \subset B_{n-1}(0', R) =: B$. In the case when $x' \in 2B$, for each $t > 0$ we have

$$\begin{aligned}|(\phi_t * g)(x')| &\le \|g\|_{L^\infty(\mathbb{R}^{n-1})}\,\|\phi_t\|_{L^1(\mathbb{R}^{n-1})} = \|g\|_{L^\infty(\mathbb{R}^{n-1})}\,\|\phi\|_{L^1(\mathbb{R}^{n-1})}\\ &\le \frac{\|g\|_{L^\infty(\mathbb{R}^{n-1})}\,\|\phi\|_{L^1(\mathbb{R}^{n-1})}(1+(2R)^n)}{1+|x'|^n}. \end{aligned} \tag{4-45}$$

Corresponding to $x' \notin 2B$, first we use that g has vanishing integral and its support condition to write

$$|(\phi_t * g)(x')| \leq \int_{\mathbb{R}^{n-1}} |\phi_t(x'-y') - \phi_t(x')|\,|g(y')|\,dy' \leq \int_B |\phi_t(x'-y') - \phi_t(x')|\,|g(y')|\,dy'. \tag{4-46}$$

Next, we estimate the integrand in the right-hand side. By recalling (1-7), an application of the mean value theorem combined with (4-43) for each $x' \notin 2B$, $y' \in B$, and $t > 0$, allows us to write

$$|\phi_t(x'-y') - \phi_t(x')| \leq t^{1-n}\,\frac{|y'|}{t} \sup_{\theta\in[0,1]} |\nabla\phi((x'-\theta\,y')/t)| \leq C|y'| \sup_{\theta\in[0,1]} \frac{1}{|x'-\theta\,y'|^n}. \tag{4-47}$$

Moreover, whenever $x' \notin 2B$ and $y' \in B$, for each $\theta \in [0,1]$ we have

$$|x'| \leq |x'-\theta\,y'| + \theta\,|y'| \leq |x'-\theta\,y'| + R \leq |x'-\theta\,y'| + \tfrac{1}{2}|x'|, \tag{4-48}$$

which implies $|x'-\theta\,y'| \geq \frac{1}{2}|x'| \geq \frac{1}{3}(1+|x'|)$. The latter, when used in (4-47) in combination with (4-46), implies

$$|(\phi_t * g)(x')| \leq R\,\frac{C\|g\|_{L^1(\mathbb{R}^{n-1})}}{1+|x'|^n} \quad \text{for all } x' \in \mathbb{R}^{n-1}\setminus(2B), \text{ for all } t > 0. \tag{4-49}$$

Now the desired conclusion follows from (4-45) and (4-49) by taking

$$C_g := \max\{\|g\|_{L^\infty(\mathbb{R}^{n-1})}\,\|\phi\|_{L^1(\mathbb{R}^{n-1})}(1+(2R)^n),\, CR\|g\|_{L^1(\mathbb{R}^{n-1})}\}. \tag{4-50}$$

The proof of the lemma is therefore complete. □

Our next preparatory lemma is needed in the proof of Proposition 4.13.

Lemma 4.9. *Let L be an $M\times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3, as well as K^L, defined in* (2-37)*. Write*

$$\Phi(x') := (\partial_n K^L)(x',1) \quad \textit{for every } x' \in \mathbb{R}^{n-1}, \tag{4-51}$$

and, whenever $0 < a < b < \infty$, also set

$$\Psi_{a,b}(x') := 4\int_a^b (\Phi_t * \Phi_t)(x')\,\frac{dt}{t} \quad \textit{for all } x' \in \mathbb{R}^{n-1}. \tag{4-52}$$

Then, whenever $0 < a < b < \infty$, there holds

$$\Psi_{a,b}(x') = \Phi_{2b}(x') - P^L_{2b}(x') - \Phi_{2a}(x') + P^L_{2a}(x') \quad \textit{for all } x' \in \mathbb{R}^{n-1}. \tag{4-53}$$

Proof. Since ∇K^L is homogeneous of order $-n$ (recall item (5) in Theorem 2.3), for every $(x',t)\in\mathbb{R}^n_+$ we may write

$$\Phi_t(x') = t^{1-n}\Phi(x'/t) = t^{1-n}(\partial_n K^L)(x'/t,1) = t(\partial_n K^L)(x',t) = t\,\partial_t K^L(x',t). \tag{4-54}$$

Consequently, in view of definition (2-37), in the current notation we have

$$\Phi_t(x') = t\,\partial_t[P_t^L(x')] \quad \text{for all } (x',t)\in\mathbb{R}^n_+. \tag{4-55}$$

Fix $h\in\mathscr{C}_0^\infty(\mathbb{R}^{n-1},\mathbb{C}^M)$. Observe that

$$(\Psi_{a,b}*h)(x') = 4\int_a^b\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\Phi_t(x'-z'-y')\,\Phi_t(z')\,h(y')\,dz'\,dy'\,\frac{dt}{t} \tag{4-56}$$

since the triple integral is absolutely convergent in view of the assumptions made on h and (2-39). Set $u(x',t) := (P_t^L*h)(x')$ for each $(x',t)\in\mathbb{R}^n_+$ and in light of (4-55) further write (4-56) in the form

$$\begin{aligned}(\Psi_{a,b}*h)(x') &= 4\int_a^b\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\partial_n K^L(x'-z'-y',t)\,\partial_n K^L(z',t)\,h(y')\,dz'\,dy'\,t\,dt\\ &= 4\int_a^b\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\partial_n K^L(z',t)\,\partial_n K^L(x'-z'-y',t)\,h(y')\,dy'\,dz'\,t\,dt\\ &= 4\int_a^b\int_{\mathbb{R}^{n-1}}\partial_n K^L(z',t)\,\partial_n u(x'-z',t)\,dz'\,t\,dt.\end{aligned} \tag{4-57}$$

Next, for every $(x',t)\in\mathbb{R}^n_+$, define $v(x',t) := (\partial_n u)(x',t)$. By part (7) in Theorem 2.3 we have that $u\in\mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M)$ and $Lu=0$ in $\mathbb{R}^n_+$. In turn, these imply $v\in\mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M)$ and $Lv=0$ in $\mathbb{R}^n_+$.

Moving on, for each $s>0$ set $v_s(x',t) := v(x',t+s)$ for every $(x',t)\in\mathbb{R}^n_+$. Then we have $v_s\in\mathscr{C}^\infty(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ and $Lv_s=0$ in $\mathbb{R}^n_+$. Now recall (1-155). For $\kappa\in(0,\infty)$ arbitrary, if $x'\in\mathbb{R}^{n-1}$ is fixed, Theorem 2.4 allows us to estimate

$$\begin{aligned}|v_s(y',t)| &= |(\partial_n u)(y',t+s)|\\ &\le \frac{C}{s}\fint_{B((y',t+s),\kappa s/\sqrt{1+\kappa^2})}|u|\,d\mathscr{L}^n \le \frac{C}{s}\mathcal{N}u(x') \quad \text{for all } (y',t)\in\Gamma_\kappa(x'),\end{aligned} \tag{4-58}$$

where for the last inequality we have used that $B((y',t+s),\kappa s/\sqrt{1+\kappa^2})\subset\Gamma_\kappa(x')$. Hence, (4-58) combined with (2-40) yields

$$(\mathcal{N}v_s)(x') \le \frac{C}{s}(\mathcal{N}u)(x') \le \frac{C}{s}(\mathcal{M}h)(x') \quad \text{for all } x'\in\mathbb{R}^{n-1}. \tag{4-59}$$

Upon recalling that $h\in\mathscr{C}_0^\infty(\mathbb{R}^{n-1},\mathbb{C}^M)$ and that the Hardy–Littlewood maximal operator is bounded on $L^p(\mathbb{R}^{n-1})$ for $p\in(1,\infty)$, from (4-59) we may infer that $\mathcal{N}v_s\in L^p(\mathbb{R}^{n-1})$ for every $p\in(1,\infty)$. In view of all these, we may apply Corollary 2.6 to v_s and obtain that for each $s \in (0, \infty)$

$$\begin{aligned} v_s(x', t) = (P_t^L * (v_s|_{\partial\mathbb{R}^n_+}))(x') &= \int_{\mathbb{R}^{n-1}} P_t^L(z')\, v_s(x'-z', 0)\, dz' \\ &= \int_{\mathbb{R}^{n-1}} K^L(z', t)\, v_s(x'-z', 0)\, dz' \\ &= \int_{\mathbb{R}^{n-1}} K^L(z', t)\, (\partial_n u)(x'-z', s)\, dz' \quad \text{for all } (x', t) \in \mathbb{R}^n_+. \end{aligned} \tag{4-60}$$

Thus, for every $(x', t) \in \mathbb{R}^n_+$ and every $s > 0$ we have

$$(\partial_n^2 u)(x', t+s) = \partial_n v_s(x', t) = \int_{\mathbb{R}^{n-1}} \partial_n K^L(z', t)\, (\partial_n u)(x'-z', s)\, dz'. \tag{4-61}$$

Applying (4-61) with $s = t$, substituting the resulting equality into (4-57), and making use of (4-55) we obtain

$$\begin{aligned} (\Psi_{a,b} * h)(x') &= 4 \int_a^b (\partial_n^2 u)(x', 2t)\, t\, dt = 4 \left[\tfrac{1}{2} t (\partial_n u)(x', 2t) - \tfrac{1}{4} u(x', 2t)\right]_{t=a}^{t=b} \\ &= [\Phi_{2t} * h(x') - P_{2t}^L * h(x')]_{t=a}^{t=b}. \end{aligned} \tag{4-62}$$

This readily yields

$$(\Psi_{a,b} * h)(x') = (\Phi_{2b} * h)(x') - (P_{2b}^L * h)(x') - (\Phi_{2a} * h)(x') + (P_{2a}^L * h)(x') \tag{4-63}$$

for every $x' \in \mathbb{R}^{n-1}$. Note that (4-63) holds for every $h \in \mathscr{C}_0^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and therefore (4-53) holds for a.e. $x' \in \mathbb{R}^{n-1}$. In addition, by Theorem 2.3 and the fact that $0 < a < b < \infty$, we see that both sides of (4-53) are continuous functions in $\mathbb{R}^{n-1}$. Consequently, the desired equality holds everywhere. The proof of the lemma is complete. □

Given a Lebesgue-measurable function $F : \mathbb{R}^n_+ \to \mathbb{C}$, for every $x' \in \mathbb{R}^{n-1}$ introduce the Lusin area function

$$(\mathcal{A}F)(x') := \left(\int_{\Gamma_\kappa(x')} |F(y', t)|^2 \, \frac{dy'\, dt}{t^n} \right)^{\frac{1}{2}} \tag{4-64}$$

and the Carleson operator

$$(\mathcal{C}F)(x') := \sup_{Q \ni x'} \left(\int_0^{\ell(Q)} \fint_Q |F(y', t)|^2 \, \frac{dy'\, dt}{t} \right)^{\frac{1}{2}}. \tag{4-65}$$

In relation to these operators we recall a result from [Coifman et al. 1985, Theorem 1, p. 313].

Lemma 4.10. *There exists some constant $C \in (0, \infty)$, which depends only on n and κ, with the property that for any Lebesgue-measurable functions $F, G : \mathbb{R}^n_+ \to \mathbb{C}$ there holds*

$$\int_{\mathbb{R}^n_+} |F(x', t)\, G(x', t)| \, \frac{dx'\, dt}{t} \le C \int_{\mathbb{R}^{n-1}} \mathcal{C}F(x')\, \mathcal{A}G(x')\, dx'. \tag{4-66}$$

Strictly speaking, the statement in [Coifman et al. 1985] contains as assumptions the additional requirements $\mathcal{C}F \in L^\infty(\mathbb{R}^{n-1})$ and $\mathcal{A}G \in L^1(\mathbb{R}^{n-1})$. However, these extra assumptions may be eliminated a posteriori via a suitable limiting argument. Specifically, for each $N \in \mathbb{N}$ introduce

$$D_N := \{(x', t) \in \mathbb{R}^n_+ : |(x', t)| < N,\ t > 1/N\} \tag{4-67}$$

and for a generic function $f : \mathbb{R}^n_+ \to \mathbb{C}$ define $f_N : \mathbb{R}^n_+ \to \mathbb{C}$ by setting $f_N(x) := f(x)$ if $x \in D_N$ and $|f(x)| \leq N$ and $f_N(x) := 0$ if either $x \in \mathbb{R}^n \setminus D_N$ or $|f(x)| > N$ for each $x \in \mathbb{R}^n_+$. Then, given $F, G : \mathbb{R}^n_+ \to \mathbb{C}$ arbitrary Lebesgue-measurable functions, for each $N \in \mathbb{N}$ the functions F_N, G_N are Lebesgue-measurable and bounded. It is also immediate from definitions that $\mathcal{C}F_N \in L^\infty(\mathbb{R}^{n-1})$ and $\mathcal{A}G_N \in L^\infty_{\text{comp}}(\mathbb{R}^{n-1}) \subset L^1(\mathbb{R}^{n-1})$. Based on [Coifman et al. 1985, Theorem 1, p. 313] and the monotonicity of the operators $\mathcal{C}$ and $\mathcal{A}$ (with respect to the absolute value of the function to which they are applied) we may write

$$\begin{aligned} \int_{\mathbb{R}^n_+} |F_N(x', t)\, G_N(x', t)| \frac{dx'\, dt}{t} &\leq C \int_{\mathbb{R}^{n-1}} \mathcal{C}F_N(x')\, \mathcal{A}G_N(x')\, dx' \\ &\leq C \int_{\mathbb{R}^{n-1}} \mathcal{C}F(x')\, \mathcal{A}G(x')\, dx'. \end{aligned} \tag{4-68}$$

Now (4-66) follows by taking the limit as $N \to \infty$ of the inequality resulting from (4-68) and applying Lebesgue's monotone convergence theorem.

For further reference we also prove the following companion to Lemma 4.10.

Lemma 4.11. *There exists some constant $C \in (0, \infty)$ (depending only on n and κ) such that for any two Lebesgue-measurable functions $F, G : \mathbb{R}^n_+ \to \mathbb{C}$ one has*

$$\int_{\mathbb{R}^n_+} |F(x', t)\, G(x', t)| \frac{dx'\, dt}{t} \leq C \int_{\mathbb{R}^{n-1}} \mathcal{A}F(x') \mathcal{A}G(x')\, dx'. \tag{4-69}$$

Proof. The idea is to estimate the expression

$$I := \int_{\mathbb{R}^{n-1}} \left(\int_{\Gamma_\kappa(x')} |F(y', t)\, G(y', t)| \frac{dy'\, dt}{t^n} \right) dx' \tag{4-70}$$

in two ways. On the one hand, using Fubini's theorem we may write

$$\begin{aligned} I &= \int_{\mathbb{R}^n_+} |F(y', t)||G(y', t)| \left(\int_{\mathbb{R}^{n-1}} \mathbf{1}_{\Gamma_\kappa(x')}(y', t)\, dx' \right) \frac{dy'\, dt}{t^n} \\ &= C_{\kappa,n} \int_{\mathbb{R}^n_+} |F(y', t)||G(y', t)| \frac{dy'\, dt}{t}. \end{aligned} \tag{4-71}$$

On the other hand, based on Cauchy–Schwarz' inequality we may estimate

$$\begin{aligned} I &\leq \int_{\mathbb{R}^{n-1}} \left(\int_{\Gamma_\kappa(x')} |F(y', t)|^2 \frac{dy'\, dt}{t^n} \right)^{\frac{1}{2}} \left(\int_{\Gamma_\kappa(x')} |G(y', t)|^2 \frac{dy'\, dt}{t^n} \right)^{\frac{1}{2}} dx' \\ &= \int_{\mathbb{R}^{n-1}} \mathcal{A}F(x') \mathcal{A}G(x')\, dx'. \end{aligned} \tag{4-72}$$

Now, (4-69) follows from (4-71) and (4-72). □

To state the final preparatory lemma required in the proof of Proposition 4.13, one more piece of notation is needed. In the sequel, $A^\top$ denotes the transpose of a given matrix A.

Lemma 4.12. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3, as well as K^L as in* (2-37). *Recall Φ from* (4-51) *and for each $x' \in \mathbb{R}^{n-1}$ set $\widetilde{\Phi}(x') := \Phi^\top(-x')$. Furthermore fix $\kappa \in (0, \infty)$ arbitrary and, given a function $f = (f_\beta)_{1 \le \beta \le M} : \mathbb{R}^{n-1} \to \mathbb{C}^M$ with Lebesgue-measurable entries, define for each $x' \in \mathbb{R}^{n-1}$*

$$(S_{\widetilde{\Phi}} f)(x') := \left(\int_{\Gamma_\kappa(x')} |(\widetilde{\Phi}_t * f)(y')|^2 \frac{dy'\,dt}{t^n} \right)^{\frac{1}{2}}$$
$$= \left(\left(\int_{\Gamma_\kappa(x')} \sum_{\beta=1}^{M} |((\widetilde{\Phi}_t)_{\alpha\beta} * f_\beta)(y')|^2 \frac{dy'\,dt}{t^n} \right)^{\frac{1}{2}} \right)_{1 \le \alpha \le M}. \tag{4-73}$$

Then $S_{\widetilde{\Phi}}$ is a bounded operator from $H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)$ into $L^1(\mathbb{R}^{n-1})$.

Proof. For each $\alpha, \beta \in \{1, \dots, M\}$, write $\theta_{\alpha\beta}(x', t; y') := t\, \partial_n K^L_{\beta\alpha}(y' - x', t)$ for every $x', y' \in \mathbb{R}^{n-1}$ and $t > 0$, and denote by $\Theta_{\alpha\beta}$ the integral operator as in (3-18) corresponding to $\theta_{\alpha\beta}$ in place of θ. Notice that (3-76), (3-77) and (3-78) (with $j = n$ and the roles of α and β reversed) allow us to apply Proposition 3.3 and write

$$\|S_{\widetilde{\Phi}} f\|_{L^1(\mathbb{R}^{n-1})} \le \sum_{1 \le \alpha, \beta \le M} \|S_{\Theta_{\alpha\beta}} f_\beta\|_{L^1(\mathbb{R}^{n-1})}$$
$$\le C \sum_{1 \le \beta \le M} \|f_\beta\|_{H^1(\mathbb{R}^{n-1})} = C \|f\|_{H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \tag{4-74}$$

The desired conclusion now follows from (4-74). □

We have seen in Proposition 3.1 part (e) that if $f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ then the Littlewood–Paley measure $|\nabla u(x', t)|^2\, t\, dx'\, dt$ associated with the function u defined as in (3-3) is a Carleson measure in $\mathbb{R}^n_+$; see (1-11). In the proposition below we shall establish the converse implication along with the estimate which naturally accompanies this statement. In the proof, Lemmas 4.8–4.12, as well as the fundamental duality result from [Fefferman and Stein 1972] asserting that

$$(H^1(\mathbb{R}^{n-1}, \mathbb{C}^M))^* = \widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M) \tag{4-75}$$

are going to play a key role.

Proposition 4.13. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3, together with K^L as in* (2-37). *Recall Φ from* (4-51). *Let $f \in L^1(\mathbb{R}^{n-1}, 1/(1 + |x'|^n)\, dx')^M$ and consider the measure in $\mathbb{R}^n_+$ defined by*

$$d\mu(x', t) := |(\Phi_t * f)(x')|^2 \frac{dx'\,dt}{t}. \tag{4-76}$$

Then whenever μ *is a Carleson measure*, *that is*,

$$\|\mu\|_{\mathcal{C}(\mathbb{R}^n_+)} = \sup_{Q\subset\mathbb{R}^{n-1}} \int_0^{\ell(Q)} ⨍_Q |(\Phi_t * f)(x')|^2 \, \frac{dx'\,dt}{t} < \infty, \tag{4-77}$$

one necessarily has $f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *and*

$$\|f\|^2_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \le C\|\mu\|_{\mathcal{C}(\mathbb{R}^n_+)} \tag{4-78}$$

for some constant $C \in (0, \infty)$ *independent of* f.

Proof. Fix a function f as in the hypotheses of the proposition and suppose μ satisfies (4-77). Let $g \in H^1_a(\mathbb{R}^{n-1})$, see (4-41), and for some arbitrary $\alpha_0 \in \{1, \dots, M\}$ define

$$h := (g\,\delta_{\alpha\alpha_0})_{1\le\alpha\le M} \in [H^1_a(\mathbb{R}^{n-1})]^M \subset H^1(\mathbb{R}^{n-1}, \mathbb{C}^M), \tag{4-79}$$

where $\delta_{\alpha\alpha_0}$ denotes the standard Kronecker symbol.

Next, recall the expression of the classical harmonic Poisson kernel (that is, the Poisson kernel associated with the Laplacian Δ)

$$P^{\Delta}(x') := \frac{2}{\omega_{n-1}} \frac{1}{(1+|x'|^2)^{n/2}} \quad \text{for all } x' \in \mathbb{R}^{n-1}, \tag{4-80}$$

where ω_{n-1} stands for the area of the unit sphere in $\mathbb{R}^n$. Then the definition of Φ, (2-39) in Theorem 2.3, and (4-80) imply

$$|\Phi_t(x')| \le C P^{\Delta}_t(x') \quad \text{for all } x' \in \mathbb{R}^{n-1}, \text{ for all } t \in (0, \infty). \tag{4-81}$$

Also, by the semigroup property, (see, e.g., [Stein 1970, (vi), p. 62], or part (8) in Theorem 2.3), for every $\varepsilon \in (0, 1)$ and every $t \in (\varepsilon, \varepsilon^{-1})$ we have

$$P^{\Delta}_t * P^{\Delta}_t = P^{\Delta}_{2t} \le C_\varepsilon\, P^{\Delta}. \tag{4-82}$$

Combining (4-81) and (4-82), for each $\varepsilon \in (0, 1)$ we may write

$$\begin{aligned}
&\int_\varepsilon^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\Phi_t(x'-y'-z')|\,|\Phi_t(z')|\,|f(y')|\,|h(x')| \, dz'\,dy'\,dx' \, \frac{dt}{t} \\
&\qquad \le C \int_\varepsilon^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} P^{\Delta}_t(x'-y'-z')\, P^{\Delta}_t(z')\, |f(y')|\,|g(x')| \, dz'\,dy'\,dx' \, \frac{dt}{t} \\
&\qquad \le C_\varepsilon \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} P^{\Delta}(x'-y')\, |f(y')|\,|g(x')| \, dy'\,dx' \\
&\qquad \le C_\varepsilon \Big(\int_{\mathbb{R}^{n-1}} (1+|x'|^n)\, |g(x')| \, dx' \Big) \Big(\int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1+|y'|^n} \, dx' \Big) < \infty,
\end{aligned} \tag{4-83}$$

where for the last inequality we have used the fact that $1 + |y'| \le (1 + |x'|)\,(1 + |x' - y'|)$ for every $x', y' \in \mathbb{R}^{n-1}$, while the finiteness of the rightmost term in (4-83) follows from our assumptions on f

and g. Thus, recalling the definition of $\Psi_{\varepsilon,\varepsilon^{-1}}$ from (4-52) we have that

$$\int_{\mathbb{R}^{n-1}} \big\langle (\Psi_{\varepsilon,\varepsilon^{-1}} * f)(x'), h(x') \big\rangle \, dx' \\ = \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \big\langle \Phi_t(x'-y'-z')\, \Phi_t(z')\, f(y'), h(x') \big\rangle \, dz'\, dy'\, dx' \, \frac{dt}{t} \tag{4-84}$$

is an absolutely convergent integral. Here and elsewhere we use the notation

$$\langle \lambda, \lambda' \rangle := \sum_{\alpha=1}^{M} \lambda_\alpha \, \lambda'_\alpha, \quad \lambda = (\lambda_\alpha)_{1\le\alpha\le M}, \quad \lambda' = (\lambda'_\alpha)_{1\le\alpha\le M} \in \mathbb{C}^M. \tag{4-85}$$

To continue, we introduce the (matrix-valued) functions

$$\begin{aligned} \widetilde{\Phi}(x') &:= \Phi^\top(-x'), \\ \widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}}(x') &:= \Psi^\top_{\varepsilon,\varepsilon^{-1}}(-x') \\ \widetilde{P}^L(x') &:= (P^L)^\top(-x'), \end{aligned} \tag{4-86}$$

defined for every $x' \in \mathbb{R}^{n-1}$. Then, for every $\varepsilon > 0$, we may write

$$\begin{aligned} \left| \int_{\mathbb{R}^{n-1}} \big\langle f(x'), (\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}} * h)(x') \big\rangle \, dx' \right| &= \left| \int_{\mathbb{R}^{n-1}} \big\langle (\Psi_{\varepsilon,\varepsilon^{-1}} * f)(x'), h(x') \big\rangle \, dx' \right| \\ &= \left| \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \big\langle (\Phi_t * \Phi_t * f)(x'), h(x') \big\rangle \, dx' \, \frac{dt}{t} \right| \\ &= \left| \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \big\langle (\Phi_t * f)(x'), (\widetilde{\Phi}_t * h)(x') \big\rangle \, dx' \, \frac{dt}{t} \right| \\ &= \left| \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \langle F(x',t), H(x',t) \rangle \, dx' \, \frac{dt}{t} \right| \\ &\le \int_{\mathbb{R}^n_+} |\langle F(x',t), H(x',t) \rangle| \, dx' \, \frac{dt}{t}, \end{aligned} \tag{4-87}$$

where $F(x',t) := (\Phi_t * f)(x')$ and $H(x',t) := (\widetilde{\Phi}_t * h)(x')$ for every $(x',t) \in \mathbb{R}^n_+$. Denote by $(F_\alpha)_{1\le\alpha\le M}$ and $(H_\alpha)_{1\le\alpha\le M}$ the scalar components of F and H, respectively. Note that (4-65), the definition of F, and (4-77) imply

$$\|\mathcal{C} F_\alpha\|_{L^\infty(\mathbb{R}^{n-1})} \le \|\mu\|^{\frac{1}{2}}_{\mathcal{C}(\mathbb{R}^n_+)} < \infty \quad \text{for all } \alpha \in \{1, \dots, M\}. \tag{4-88}$$

Also, (4-79), (4-64), and Lemma 4.12 permit us to write

$$\begin{aligned} \|\mathcal{A} H_\alpha\|_{L^1(\mathbb{R}^{n-1})} &\le \|S_{\widetilde{\Phi}} h\|_{L^1(\mathbb{R}^{n-1})} \\ &\le C \|h\|_{H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)} \\ &= C \|g\|_{H^1(\mathbb{R}^{n-1})} \quad \text{for all } \alpha \in \{1, \dots, M\}. \end{aligned} \tag{4-89}$$

Consequently, Lemma 4.10, (4-88), and (4-89) allow us to estimate

$$\begin{aligned}\int_{\mathbb{R}^n_+} |\langle F(x',t), H(x',t)\rangle|\,dx'\,\frac{dt}{t} &\le \sum_{\alpha=1}^{M}\int_{\mathbb{R}^n_+} |F_\alpha(x',t)\,H_\alpha(x',t)|\,dx'\,\frac{dt}{t}\\ &\le C\sum_{\alpha=1}^{M}\int_{\mathbb{R}^{n-1}} \mathcal{C}F_\alpha(x')\,\mathcal{A}H_\alpha(x')\,dx'\\ &\le C\sum_{\alpha=1}^{M}\|\mathcal{C}F_\alpha\|_{L^\infty(\mathbb{R}^{n-1})}\,\|\mathcal{A}H_\alpha\|_{L^1(\mathbb{R}^{n-1})}\\ &= C\,\|\mu\|^{\frac{1}{2}}_{\mathcal{C}(\mathbb{R}^n_+)}\|g\|_{H^1(\mathbb{R}^{n-1})}.\end{aligned}\tag{4-90}$$

At this point we make the claim that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^{n-1}}\big\langle f(x'),(\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}}*h)(x')\big\rangle\,dx' = \int_{\mathbb{R}^{n-1}}\langle f(x'),h(x')\rangle\,dx'.\tag{4-91}$$

The idea is to show that Lebesgue's dominated convergence theorem applies in our setting. With this goal in mind, first observe that by part (5) in Theorem 2.3, for every multi-index $\alpha\in\mathbb{N}_0^{n-1}$, we have

$$|\partial^\alpha\Phi(x')| = |\partial^\alpha\partial_n K^L(x',1)| \le C_\alpha|(x',1)|^{-n-|\alpha|};\tag{4-92}$$

hence $\widetilde{\Phi}$ satisfies the hypotheses of Lemma 4.8. Moreover, by parts (1) and (5) in Theorem 2.3 we also have that $\widetilde{P}^L$ satisfies the hypotheses of Lemma 4.8. Hence, Lemma 4.8 and (4-79) give

$$\sup_{t>0}|(\widetilde{\Phi}_t*h)(x')| + \sup_{t>0}|(\widetilde{P}^L_t*h)(x')| \le \frac{C_h}{1+|x'|^n}\quad\text{for every } x'\in\mathbb{R}^{n-1}.\tag{4-93}$$

In light of (4-53), the latter yields

$$\sup_{0<\varepsilon<1}|(\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}}*h)(x')| \le \frac{C_h}{1+|x'|^n}\quad\text{for every } x'\in\mathbb{R}^{n-1}.\tag{4-94}$$

From this and the fact that $f\in L^1(\mathbb{R}^{n-1},1/(1+|x'|^n)\,dx')^M$ we arrive at the conclusion that

$$\sup_{0<\varepsilon<1}|\langle f,\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}}*h\rangle|\in L^1(\mathbb{R}^{n-1}).\tag{4-95}$$

Next, we focus on the pointwise convergence of the functions under the integral in the left-hand side of (4-91). First, by (2-34), (2-55) in Lemma 2.7, and (2-35) in Theorem 2.3 we obtain

$$\lim_{s\to 0^+}(\widetilde{P}^L_s*h)(x') = \left(\int_{\mathbb{R}^{n-1}}\widetilde{P}^L(y')\,dy'\right)h(x') = h(x')\quad\text{for a.e. } x'\in\mathbb{R}^{n-1}.\tag{4-96}$$

Second, using a suitable change of variables, the properties of h, and Lebesgue's dominated convergence theorem we have

$$\lim_{s\to\infty}(\widetilde{P}^L_s*h)(x') = \lim_{s\to\infty}\int_{\mathbb{R}^{n-1}}\widetilde{P}^L(y')\,h(x'-sy')\,dy' = 0.\tag{4-97}$$

Third, by (3-72) for every $t > 0$ we have

$$\int_{\mathbb{R}^{n-1}} (\partial_n K)(x', t)\, dx' = 0 \quad \text{for all } t > 0, \tag{4-98}$$

which when specialized to $t = 1$ yields

$$\int_{\mathbb{R}^{n-1}} \widetilde{\Phi}(x')\, dx' = \left(\int_{\mathbb{R}^{n-1}} \Phi(-x')\, dx'\right)^{\top} = \left(\int_{\mathbb{R}^{n-1}} \Phi(x')\, dx'\right)^{\top} = 0. \tag{4-99}$$

This, (4-92), and Lemma 2.7 applied to $\widetilde{\Phi}$ then give that

$$\lim_{s\to 0^+} (\widetilde{\Phi}_s * h)(x') = \left(\int_{\mathbb{R}^{n-1}} \widetilde{\Phi}(y')\, dy'\right) h(x') = 0 \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \tag{4-100}$$

Fourth, a suitable change of variables, the properties of h, and Lebesgue's dominated convergence theorem also yield

$$\lim_{s\to\infty} (\widetilde{\Phi}_s * h)(x') = \lim_{s\to\infty} \int_{\mathbb{R}^{n-1}} \widetilde{\Phi}(y')\, h(x' - sy')\, dy' = 0. \tag{4-101}$$

In concert, (4-96), (4-97), (4-100), (4-101), and (4-53) imply the pointwise convergence

$$\lim_{\varepsilon\to 0^+} (\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}} * h)(x') = h(x') \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \tag{4-102}$$

Having dispensed of (4-95) and (4-102), we may apply Lebesgue's dominated convergence theorem to write

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^{n-1}} \big\langle f(x'), (\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}} * h)(x')\big\rangle\, dx' &= \int_{\mathbb{R}^{n-1}} \big\langle f(x'), \lim_{\varepsilon\to 0^+} (\widetilde{\Psi}_{\varepsilon,\varepsilon^{-1}} * h)(x')\big\rangle\, dx' \\ &= \int_{\mathbb{R}^{n-1}} \langle f(x'), h(x')\rangle\, dx', \end{aligned} \tag{4-103}$$

finishing the proof of the claim in (4-91).

From the definition of h, (4-91), (4-87), and (4-90) we may conclude that

$$\left|\int_{\mathbb{R}^{n-1}} f_{\alpha_0}(x')\, g(x')\, dx'\right| \le C\, \|\mu\|^{\frac{1}{2}}_{\mathcal{C}(\mathbb{R}^n_+)} \|g\|_{H^1(\mathbb{R}^{n-1})}. \tag{4-104}$$

In particular, if we define the functional $\Lambda_f^{\alpha_0} : H^1_a(\mathbb{R}^{n-1}) \to \mathbb{C}$ by setting

$$\Lambda_f^{\alpha_0}(g) := \int_{\mathbb{R}^{n-1}} f_{\alpha_0} g\, d\mathscr{L}^{n-1} \quad \text{for every } g \in H^1_a(\mathbb{R}^{n-1}), \tag{4-105}$$

then (4-104) implies $\Lambda_f^{\alpha_0} \in (H^1(\mathbb{R}^{n-1}))^*$; here we also used (4-42). Recalling (4-75), it follows that

$$\begin{aligned} &\text{there exists } b_{\alpha_0} \in \mathrm{BMO}(\mathbb{R}^{n-1}) \text{ such that } \|b_{\alpha_0}\|_{\mathrm{BMO}(\mathbb{R}^{n-1})} \le C\, \|\mu\|^{\frac{1}{2}}_{\mathcal{C}(\mathbb{R}^n_+)} \\ &\text{and } \Lambda_f^{\alpha_0}(g) = \textstyle\int_{\mathbb{R}^{n-1}} b_{\alpha_0}\, g\, d\mathscr{L}^{n-1} \text{ for every function } g \in H^1_a(\mathbb{R}^{n-1}). \end{aligned} \tag{4-106}$$

Thus,

$$\int_{\mathbb{R}^{n-1}} (b_{\alpha_0} - f_{\alpha_0}) g\, d\mathscr{L}^{n-1} = 0 \quad \text{for all } g \in H^1_a(\mathbb{R}^{n-1}). \tag{4-107}$$

Hence, if we set $v_{\alpha_0} := b_{\alpha_0} - f_{\alpha_0}$, then (4-107) implies that

$$v_{\alpha_0} \text{ is constant almost everywhere in } \mathbb{R}^{n-1}. \tag{4-108}$$

Indeed, if the latter were not true, one could find two bounded Lebesgue-measurable sets E^+, E^- in $\mathbb{R}^{n-1}$ such that $0 < |E^{\pm}| < \infty$ and $v_{\alpha_0}(x') \le a < b \le v_{\alpha_0}(y')$ for all $x' \in E^-$, $y' \in E^+$. Then

$$g := \frac{\mathbf{1}_{E^+}}{|E_+|} - \frac{\mathbf{1}_{E^-}}{|E_-|} \quad \text{belongs to } H^1_a(\mathbb{R}^{n-1}) \tag{4-109}$$

and, when used in (4-107), forces

$$0 = \int_{\mathbb{R}^{n-1}} v_{\alpha_0} g \, d\mathscr{L}^{n-1} \ge b - a > 0. \tag{4-110}$$

This contradiction proves (4-108). In summary, we have shown that $b_{\alpha_0} - f_{\alpha_0}$ is constant almost everywhere in $\mathbb{R}^{n-1}$. Upon recalling the first line in (4-106), it follows that $f_{\alpha_0} \in \mathrm{BMO}(\mathbb{R}^{n-1})$ with

$$\|f_{\alpha_0}\|_{\mathrm{BMO}(\mathbb{R}^{n-1})} \le C \|\mu\|^{\frac{1}{2}}_{\mathcal{C}(\mathbb{R}^n_+)}. \tag{4-111}$$

Since $\alpha_0 \in \{1, \dots, M\}$ is arbitrary, we may further conclude that $f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and satisfies (4-78), as wanted. □

In turn, Proposition 4.13 is one of the main ingredients in the proof of the fact that the boundary traces of vertical shifts of a smooth null-solution of L satisfying a Carleson measure condition in the upper half-space belong to BMO, uniformly with respect to the shift.

Lemma 4.14. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Suppose $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfies $Lu = 0$ in $\mathbb{R}^n_+$ and $\|u\|_{**} < \infty$. For each $\varepsilon > 0$, set $u_\varepsilon(x', t) := u(x', t + \varepsilon)$ for every $(x', t) \in \mathbb{R}^n_+$ and $f_\varepsilon := u_\varepsilon|_{\partial\mathbb{R}^n_+}$. Then for each $\varepsilon > 0$ we have $f_\varepsilon \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and*

$$\|f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \le C \|u\|_{**} \tag{4-112}$$

for some $C \in (0, \infty)$ independent of ε.

Proof. We are going to apply Proposition 4.13 to f_ε. Note first that by part (d) in Lemma 4.6 we have $f_\varepsilon \in L^1(\mathbb{R}^{n-1}, 1/(1 + |x'|^n)\, dx')^M \cap \mathscr{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$. Hence we may define μ_ε as in (4-76) associated with f_ε, where we recall that $\Phi(x') = \partial_n K^L(x', 1)$ for every $x' \in \mathbb{R}^{n-1}$ and $K^L(x', t) = t^{1-n} P^L(x'/t)$ for every $x' \in \mathbb{R}^n_+$. Also, Lemma 4.7 and (4-55) imply

$$t\, \partial_t u_\varepsilon(x', t) = t\, \partial_t (P^L_t * f_\varepsilon)(x') = (\Phi_t * f_\varepsilon)(x') \quad \text{for all } (x', t) \in \mathbb{R}^n_+. \tag{4-113}$$

Thus part (b) in Lemma 4.6 yields

$$\begin{aligned} \|\mu_\varepsilon\|_{\mathcal{C}(\mathbb{R}^n_+)} &= \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Phi_t * f_\varepsilon(x')|^2 \, \frac{dx'\, dt}{t} \\ &= \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\partial_t u_\varepsilon(x', t)|^2 \, t \, dx'\, dt \\ &\le \|u_\varepsilon\|^2_{**} \le C \|u\|^2_{**} < \infty. \end{aligned} \tag{4-114}$$

Consequently, we may invoke Proposition 4.13 to conclude that $f_\varepsilon \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and

$$\|f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq C\,\|\mu_\varepsilon\|^{\frac{1}{2}}_{\mathcal{C}(\mathbb{R}^n_+)} \leq C\,\|u\|_{**}, \tag{4-115}$$

as wanted. □

The aim in Lemma 4.15 below is to show that derivatives of the kernel function K^L are multiples of molecules in the sense of Hardy space theory. To make this precise, let us recall the definition of $L^2(\mathbb{R}^{n-1})$-molecules for the Hardy space $H^1(\mathbb{R}^{n-1})$. Specifically, given $\varepsilon > 0$ and a ball $B \subset \mathbb{R}^{n-1}$, a function $m \in L^1(\mathbb{R}^{n-1})$ is said to be an $(L^2(\mathbb{R}^{n-1}), \varepsilon)$-molecule relative to B provided

$$\int_{\mathbb{R}^{n-1}} m(x')\,dx' = 0 \tag{4-116}$$

and

$$\|m\|_{L^2(B)} \leq |B|^{-\frac{1}{2}}, \qquad \|m\|_{L^2(2^k B\setminus 2^{k-1} B)} \leq |2^k\,B|^{-\frac{1}{2}} 2^{-k\,\varepsilon} \quad \text{for all } k \in \mathbb{N}. \tag{4-117}$$

Lemma 4.15. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Then there exists a constant $C \in (0,\infty)$ such that for any fixed $t > 0$, the components of $C\,t\nabla K^L(\cdot\,, t)$ are $(L^2(\mathbb{R}^{n-1}), 1)$-molecules relative to $B_{n-1}(0', t)$. In particular,*

$$\sup_{t>0} \|t\nabla K^L(\cdot\,, t)\|_{H^1(\mathbb{R}^{n-1})} < \infty. \tag{4-118}$$

Consequently, if $f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and the sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ is such that $[f_k] \to [f]$ in the weak- topology on $\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ as $k \to \infty$, i.e.,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n-1}} f_k g\,d\mathscr{L}^{n-1} = \int_{\mathbb{R}^{n-1}} fg\,d\mathscr{L}^{n-1} \quad \textit{for all } g \in H^1(\mathbb{R}^{n-1}, \mathbb{C}^M), \tag{4-119}$$

then for every $(x', t) \in \mathbb{R}^n_+$ fixed one has

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n-1}} t\,\nabla K^L(x'-y', t)\,f_k(y')\,dy' = \int_{\mathbb{R}^{n-1}} t\,\nabla K^L(x'-y', t)\,f(y')\,dy'. \tag{4-120}$$

Proof. Fix $t > 0$, set $B_t := B_{n-1}(0', t)$, and write $m(x') = t\nabla K^L(x', t)$ for every $x' \in \mathbb{R}^{n-1}$. We have already shown in (3-78) that

$$\int_{\mathbb{R}^{n-1}} m(x')\,dx' = 0. \tag{4-121}$$

Also, by part (5) in Theorem 2.3 we have

$$\begin{aligned}\int_{B_t} |m(x')|^2\,dx' &\leq C \int_{|x'|<t} \frac{t^2}{(t+|x'|)^{2n}}\,dx' \leq C \int_{|x'|<t} \frac{t^2}{t^{2n}}\,dx' \\ &= C\,t^{1-n} \leq C_0^2 |B_t|^{-1},\end{aligned} \tag{4-122}$$

and, for every $k \geq 1$,

$$\int_{2^k B_t \setminus 2^{k-1} B_t} |m(x')|^2 \, dx' \leq C \int_{2^{k-1} t < |x'| < 2^k t} \frac{t^2}{(t + |x'|)^{2n}} \, dx' \leq C \int_{2^k B_t} \frac{t^2}{(2^k t)^{2n}} \, dx'$$
$$= C 2^{-2k} (2^k t)^{1-n} \leq C_0^2 \, 2^{-2k} \, |2^k B_t|^{-1} \tag{4-123}$$

for some constant $C_0 \in (0, \infty)$ independent of k, x', and t. All these give that $C_0^{-1} m$ is an $(L^2(\mathbb{R}^{n-1}), 1)$-molecule relative to B_t and (4-118) follows from the molecular characterization of $H^1(\mathbb{R}^{n-1})$; see [Alvarado and Mitrea 2015].

In addition, for each $x' \in \mathbb{R}^{n-1}$ fixed, the function $C_0^{-1} m(x' - \cdot)$ is an $(L^2(\mathbb{R}^{n-1}), 1)$-molecule relative to $B_{n-1}(x', t)$ and therefore belongs to $H^1(\mathbb{R}^{n-1})$. Hence, (4-120) follows from the definition of the weak-* convergence. □

We now have all the ingredients to prove Proposition 4.1:

Proof of Proposition 4.1. Under the notation of Lemma 4.14, from (4-112) we know that the sequence $\{[f_\varepsilon]\}_{0<\varepsilon<1}$ is bounded in the Banach space $\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$. By eventually passing to a subsequence, Alaoglu's theorem guarantees that there is no loss of generality in assuming that $\{[f_\varepsilon]\}_{0<\varepsilon<1}$ converges weakly in $\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ to some $[g] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, where $g \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, satisfying

$$\|[g]\|_{\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**} \tag{4-124}$$

for some finite constant $C > 0$ which does not depend on u. Applying Lemma 4.15, for every $(x', t) \in \mathbb{R}^n_+$ fixed we may conclude with the help of (4-75) that

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \nabla[(P_t^L * f_\varepsilon)(x')] &= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{n-1}} (\nabla K^L)(x' - y', t) f_\varepsilon(y') \, dy' \\
&= \int_{\mathbb{R}^{n-1}} (\nabla K^L)(x' - y', t) g(y') \, dy' \\
&= \nabla[(P_t^L * g)(x')].
\end{aligned} \tag{4-125}$$

On the other hand, by Lemma 4.7 we have

$$\nabla u(x', t + \varepsilon) = \nabla u_\varepsilon(x', t) = \nabla[(P_t^L * f_\varepsilon)(x')] \quad \text{for all } (x', t) \in \mathbb{R}^n_+. \tag{4-126}$$

Together, (4-125) and (4-126) give (keeping in mind part (a) in Lemma 4.6)

$$\nabla u(x', t) = \lim_{\varepsilon \to 0^+} \nabla u(x', t + \varepsilon) = \nabla[(P_t^L * g)(x')] \quad \text{for all } (x', t) \in \mathbb{R}^n_+. \tag{4-127}$$

Consequently, there exists $C \in \mathbb{C}^M$ with the property that

$$u(x', t) = (P_t^L * g)(x') + C \quad \text{for all } (x', t) \in \mathbb{R}^n_+. \tag{4-128}$$

Then $f := g + C \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ satisfies, thanks to (4-124) and (1-20),

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \|g\|_{\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \|[g]\|_{\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}, \tag{4-129}$$

where $C > 0$ is a finite constant which does not depend on u. Moreover, (2-24) ensures that $f \in L^1(\mathbb{R}^{n-1}, 1/(1+|x'|^n)\,dx')^M$, while formula (4-128) becomes, in light of (2-36), precisely (4-2). Granted this, the first conclusion in Proposition 3.1 implies that f is the only function in $L^1(\mathbb{R}^{n-1}, 1/(1+|x'|^n)\,dx')^M$ for which the representation formula (4-2) holds, $u\big|^{\rm n.t.}_{\partial\mathbb{R}^n_+}$ exists at a.e. point in $\mathbb{R}^{n-1}$, and $f = u\big|^{\rm n.t.}_{\partial\mathbb{R}^n_+}$. To conclude the proof of Proposition 4.1 it remains to observe that (4-3) is a consequence of (4-129), (4-2), and (3-12). □

5. Proofs of Theorems 1.1–1.6 and 1.8–1.10

We begin by presenting the proof of the Fatou-type result stated in Theorem 1.2. The main ingredients involved are Propositions 3.1, 4.1, and 4.2.

Proof of Theorem 1.2. The implication in (1-39) is seen directly from Proposition 4.1, which also guarantees the right-to-left inclusion in (1-40). The left-to-right inclusion in (1-40) is a consequence of Proposition 3.1. Going further, it is clear from definitions that $\mathrm{LMO}(\mathbb{R}^n_+)$ is a linear space on which $\|\cdot\|_{**}$ is a seminorm with null-space $\mathbb{C}^M$. The linear mapping in (1-42) is bounded (by the estimate in (1-39)), injective (by Proposition 4.2), and surjective (by Proposition 3.1). Moreover, another reference to the estimate in (1-39) shows that the inverse of the mapping (1-42) is also bounded. Given that $\widetilde{\mathrm{BMO}}(\mathbb{R}^n_+)$ is complete, it follows that the quotient space $\mathrm{LMO}(\mathbb{R}^n_+)/\mathbb{C}^M$ is also complete when equipped with $\|\cdot\|_{**}$. □

Anticipating the proof of Theorem 1.3, below we isolate a key result to the effect that any smooth null-solution of L satisfying a vanishing Carleson measure condition in the upper half-space converges vertically to its nontangential boundary trace in BMO.

Lemma 5.1. *Let L be an $M \times M$ elliptic system with constant complex coefficients as in* (1-2)–(1-3) *and consider P^L, the associated Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3. Suppose $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfies $Lu = 0$ in $\mathbb{R}^n_+$ and $\|u\|_{**} < \infty$ and use Theorem 1.2 to write*

$$f := u\big|^{\rm n.t.}_{\partial\mathbb{R}^n_+} \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M). \tag{5-1}$$

For each number $\varepsilon > 0$, define $u_\varepsilon(x',t) := u(x', t+\varepsilon)$ for every $(x',t) \in \mathbb{R}^n_+$ and consider $f_\varepsilon := u_\varepsilon|_{\partial\mathbb{R}^n_+} \in$ $\mathrm{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ (*see Lemma 4.14*). *Then*

$$\left.\begin{array}{r} |\nabla u(x',t)|^2\, t\, dx'\, dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}^n_+ \end{array}\right\} \implies \lim_{\varepsilon\to 0^+} \|f_\varepsilon - f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} = 0. \tag{5-2}$$

Proof. By Theorem 1.2 we have $u(x',t) = (P^L_t * f)(x')$ for every $(x',t) \in \mathbb{R}^n_+$. Also, Lemma 4.7 implies $u_\varepsilon(x',t) = (P^L_t * f_\varepsilon)(x')$ for every $(x',t) \in \mathbb{R}^n_+$ and each $\varepsilon > 0$. To proceed, for every $(x',t) \in \mathbb{R}^n_+$ set

$$\begin{aligned} v_\varepsilon(x',t) &:= (P^L_t * (f_\varepsilon - f))(x',t) \\ &= (P^L_t * f)(x',t) - (P^L_t * f_\varepsilon)(x',t) \\ &= u_\varepsilon(x',t) - u(x',t). \end{aligned} \tag{5-3}$$

Given that for each parameter $\varepsilon > 0$ the function v_ε satisfies the hypotheses of Theorem 1.2 and almost everywhere $v_\varepsilon\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = f_\varepsilon - f \in \text{BMO}(\mathbb{R}^{n-1})$, it follows that

$$\|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)} \le C\|v_\varepsilon\|_{**} = C\|u_\varepsilon - u\|_{**} \tag{5-4}$$

for every $\varepsilon > 0$. Hence, to complete the proof of (5-2) it suffices to show that

$$\lim_{\varepsilon\to 0^+} \|u_\varepsilon - u\|_{**} = 0. \tag{5-5}$$

To this end, for each $r > 0$ we introduce

$$\|u\|_{**,r} := \sup_{Q\subset\mathbb{R}^{n-1},\,\ell(Q)\le r} \left(\int_0^{\ell(Q)} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}}. \tag{5-6}$$

Note that

$$\|u\|_{**,r} \le \|u\|_{**,s} \le \|u\|_{**} \quad \text{if } r \le s, \tag{5-7}$$

and the fact that $|\nabla u(x',t)|^2\, t\, dx'\, dt$ is a vanishing Carleson measure in $\mathbb{R}^n_+$, recall (1-12), may be rephrased as

$$\|u\|_{**,r} \to 0 \quad \text{as } r \to 0^+. \tag{5-8}$$

We now make the claim that there exists a constant $C = C(n, L) \in (0,\infty)$ such that

$$\|u - u_\varepsilon\|_{**} \le C\big(\|u\|_{**,4\max\{r,\varepsilon\}} + \|u\|_{**}\min\{\varepsilon/r, 1\}\big) \quad \text{for all } r, \varepsilon \in (0,\infty). \tag{5-9}$$

Assume the claim for now and based on (5-9), for every $0 < r < \infty$, we may write

$$\begin{aligned} 0 &\le \limsup_{\varepsilon\to 0^+} \|u - u_\varepsilon\|_{**} \\ &\le C \limsup_{\varepsilon\to 0^+} \|u\|_{**,4\max\{r,\varepsilon\}} + C\|u\|_{**} \limsup_{\varepsilon\to 0^+} [\min\{\varepsilon/r, 1\}] \\ &= C\|u\|_{**,4r}. \end{aligned} \tag{5-10}$$

Recalling now (5-8), we may further take the limit as $r \to 0^+$ in the resulting inequality in (5-10) and conclude that

$$\limsup_{\varepsilon\to 0^+} \|u - u_\varepsilon\|_{**} = 0.$$

This proves (5-5) as wanted.

To finish the proof of the lemma we are left with showing the claim. To do so, we first note that in light of the notation in (5-6), the reasoning in (4-9) (corresponding to $|\alpha| = 1$) yields

$$\begin{aligned} t|\nabla u(x',t)| &\le C\left(\frac{1}{|Q_{x'}|}\int_{\frac{t}{2}}^{\frac{3t}{2}}\int_{Q_{x'}} |\nabla u(y',s)|^2\, s\, dy'\, ds\right)^{\frac{1}{2}} \\ &\le C\|u\|_{**,2t} \end{aligned} \tag{5-11}$$

for each $(x',t) \in \mathbb{R}^n_+$, where $Q_{x'}$ denotes the cube in $\mathbb{R}^{n-1}$ centered at x' with side-length t.

Next, fix a cube $Q \subset \mathbb{R}^{n-1}$ and numbers $r, \varepsilon \in (0,\infty)$ and proceed by analyzing the following possible three cases.

Case 1: $\ell(Q) \le \varepsilon$. Under this assumption, recalling also (5-11) and (5-7), we obtain

$$\begin{aligned}
&\Big(\int_0^{\ell(Q)} \fint_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} \\
&\qquad\le \Big(\int_\varepsilon^{\ell(Q)+\varepsilon} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} + \Big(\int_0^{\ell(Q)} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} \\
&\qquad\le \Big(\int_\varepsilon^{2\varepsilon} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} + \|u\|_{**,\varepsilon} \\
&\qquad\le C\Big(\int_\varepsilon^{2\varepsilon} \frac{\|u\|^2_{**,2t}}{t}\, dt\Big)^{\frac{1}{2}} + \|u\|_{**,4\varepsilon} \\
&\qquad\le C\|u\|_{**,4\varepsilon} \le C\|u\|_{**,4\max\{r,\varepsilon\}}
\end{aligned} \tag{5-12}$$

for some constant $C = C(n, L) \in (0,\infty)$ independent of u, ε, and r.

Case 2: $\varepsilon < \ell(Q) \le r$. Note that in this case $r = \max\{r,\varepsilon\}$ and using again (5-11) and (5-7) we have

$$\begin{aligned}
\Big(\int_0^{\ell(Q)} \fint_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} &\le \Big(\int_\varepsilon^{\ell(Q)+\varepsilon} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} + \|u\|_{**,r} \\
&\le \Big(\int_0^{2\ell(Q)} \fint_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} + \|u\|_{**,\max\{r,\varepsilon\}} \\
&\le C\|u\|_{**,4\max\{r,\varepsilon\}}
\end{aligned} \tag{5-13}$$

for some constant $C = C(n, L) \in (0,\infty)$ independent of u, ε, and r.

Case 3: $\ell(Q) > \max\{r,\varepsilon\}$. In this case, set $\eta := \max\{r,\varepsilon\}$ and write

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_0^{\ell(Q)} \int_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} \\
&\qquad\le \Big(\frac{1}{|Q|}\int_0^{\eta} \int_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} \\
&\qquad\qquad + \Big(\frac{1}{|Q|}\int_\eta^{\ell(Q)} \int_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\Big)^{\frac{1}{2}} \\
&\qquad =: I + II.
\end{aligned} \tag{5-14}$$

To analyze I, let k_0 be the nonnegative integer such that

$$2^{k_0}\,\eta < \ell(Q) \le 2^{k_0+1}\,\eta.$$

Also, consider $\{Q_j\}_{j\in\mathbb{N}}$, the collection of subcubes of Q with pairwise disjoint interiors, satisfying

$$\ell(Q_j) = 2^{-k_0}\,\ell(Q) \text{ for each } j \in \mathbb{N} \quad \text{and} \quad \bigcup_{j\in\mathbb{N}} Q_j = Q.$$

Then $\ell(Q_j) \in (\eta, 2\eta]$ for every $j \in \mathbb{N}$. Bearing this in mind and using the fact that $\varepsilon \leq \eta$, we may then estimate

$$\begin{aligned} I &\leq \left(\frac{1}{|Q|}\int_\varepsilon^{\eta+\varepsilon}\int_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}} + \left(\frac{1}{|Q|}\int_0^{\eta}\int_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}} \\ &\leq 2\left(\frac{1}{|Q|}\int_0^{2\eta}\int_Q |\nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}} \\ &\leq 2\left(\frac{1}{|Q|}\sum_{j\in\mathbb{N}}\int_0^{2\ell(Q_j)}\int_{Q_j} |\nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}} \\ &\leq 2\left(\frac{1}{|Q|}\sum_{j\in\mathbb{N}} \|u\|^2_{**,2\ell(Q_j)}|2Q_j|\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{n+1}{2}}\, \|u\|_{**,4\eta} = 2^{\frac{n+1}{2}}\, \|u\|_{**,4\max\{r,\varepsilon\}}. \end{aligned} \tag{5-15}$$

Up to this point our treatment involved estimating u_ε and u separately, without exploiting any potential cancellations generated by the fact that we are dealing with their difference. However, in the task of estimating II, having the difference $u_\varepsilon - u$ plays a crucial role, as seen next. Given $(x',t) \in \mathbb{R}^n_+$, from Lemma 4.3 we conclude that

$$|\nabla^2 u(x',t)| \leq C\, \|u\|_{**}\, t^{-2}. \tag{5-16}$$

In light of this, the mean value theorem implies that for some $\theta \in (0,1)$ there holds

$$\begin{aligned} |\nabla u_\varepsilon(x',t) - \nabla u(x',t)| = |\nabla u(x',t+\varepsilon) - \nabla u(x',t)| &\leq \varepsilon\, |\nabla^2 u(x',t+\theta\,\varepsilon)| \\ &\leq C\varepsilon \|u\|_{**}\, (t+\theta\,\varepsilon)^{-2} \\ &\leq C\varepsilon \|u\|_{**}\, t^{-2}. \end{aligned} \tag{5-17}$$

Having established (5-17), we may turn to estimating II as follows:

$$\begin{aligned} II &= \left(\frac{1}{|Q|}\int_\eta^{\ell(Q)}\int_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}} \\ &\leq C\varepsilon\, \|u\|_{**} \left(\int_\eta^{\ell(Q)} t^{-3}\, dt\right)^{\frac{1}{2}} \leq C\varepsilon\, \|u\|_{**}\, \eta^{-1} \\ &= C\varepsilon\, \|u\|_{**}\, (\max\{r,\varepsilon\})^{-1} = C\, \|u\|_{**}\, \min\{\varepsilon/r, 1\}. \end{aligned} \tag{5-18}$$

In concert, (5-14), (5-15), and (5-18), allow us to conclude that, under the current assumption $\ell(Q) > \max\{r,\varepsilon\}$, we have

$$\left(\int_0^{\ell(Q)} \fint_Q |\nabla u_\varepsilon(x',t) - \nabla u(x',t)|^2\, t\, dx'\, dt\right)^{\frac{1}{2}} \leq C\big(\|u\|_{**,4\max\{r,\varepsilon\}} + \|u\|_{**}\, \min\{\varepsilon/r, 1\}\big). \tag{5-19}$$

Combining (5-12), (5-13), and (5-19), we obtain that the estimate in (5-19) actually holds for arbitrary cubes Q in $\mathbb{R}^{n-1}$. In turn, the latter yields (5-9) after taking the supremum over all cubes Q in $\mathbb{R}^{n-1}$. With this, the proof of the lemma is completed. □

Having proved the convergence result in Lemma 5.1, we are now prepared to present the proof of Theorem 1.3.

Proof of Theorem 1.3. We start by noticing that since u satisfies the conditions in (1-43), the conclusions in (1-39) hold. Hence if we set $f := u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}$, we have that f exists almost everywhere in $\mathbb{R}^{n-1}$ and belongs to $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$. To proceed in showing that $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, for each $\varepsilon > 0$ define $u_\varepsilon(x', t) := u(x', t+\varepsilon)$ for every $(x', t) \in \mathbb{R}^n_+$, and $f_\varepsilon(x') := u(x', \varepsilon)$ for every $x' \in \mathbb{R}^{n-1}$. Then from Lemma 4.14 and part (d) in Lemma 4.6 we obtain $f_\varepsilon \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \mathscr{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$. In addition, for every $\varepsilon > 0$, based on Lemma 4.3 we obtain

$$\sup_{x' \in \mathbb{R}^{n-1}} |\nabla_{x'} f_\varepsilon(x')| = \sup_{x' \in \mathbb{R}^{n-1}} |\nabla_{x'} u(x', \varepsilon)| \leq C\varepsilon^{-1} \|u\|_{**} < \infty. \tag{5-20}$$

Fix $r \in (0, \infty)$ and let $Q \subset \mathbb{R}^{n-1}$ be a cube in $\mathbb{R}^n_+$ with $\ell(Q) \leq r$. Then using (5-20) we may estimate

$$\fint_Q |f(x') - f_Q|\, dx' \leq \fint_Q |(f - f_\varepsilon)(x') - (f - f_\varepsilon)_Q|\, dx' + \fint_Q |f_\varepsilon(x') - (f_\varepsilon)_Q|\, dx' \tag{5-21}$$

$$\leq \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + \sup_{x' \in \mathbb{R}^{n-1}} |\nabla_{x'} f_\varepsilon(x')|\, \sqrt{n-1}\, \ell(Q) \tag{5-22}$$

$$\leq \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + Cr\varepsilon^{-1}\, \|u\|_{**}. \tag{5-23}$$

Hence,

$$\sup_{Q \subset \mathbb{R}^{n-1},\, \ell(Q) \leq r} \fint_Q |f(x') - f_Q|\, dx' \leq \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + Cr\varepsilon^{-1}\, \|u\|_{**}. \tag{5-24}$$

Letting $r \to 0^+$ first, then sending $\varepsilon \to 0^+$ in (5-24) and recalling that since $|\nabla u(x', t)|^2\, t\, dx'\, dt$ is a vanishing Carleson measure in $\mathbb{R}^n_+$ implication (5-2) holds, we conclude that

$$\lim_{r \to 0^+} \sup_{Q \subset \mathbb{R}^{n-1},\, \ell(Q) \leq r} \fint_Q |f(x') - f_Q|\, dx' = 0. \tag{5-25}$$

Hence, $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, as wanted.

To complete the proof, it remains to establish (1-45). However, the right-to-left inclusion follows from (1-44), while the opposite inclusion is a consequence of Proposition 3.1. □

We continue by presenting the proof of Theorem 1.4.

Proof of Theorem 1.4. Consider first the equivalence in item (1) of Theorem 1.4. The fact that $f \in \text{BMO}(\mathbb{R}^{n-1}; \mathbb{C}^M)$ implies $\|u\|_{**} < \infty$ is part (e) of Proposition 3.1 and (2-24), whereas the converse follows from Proposition 4.1. Regarding the equivalence in item (2) of Theorem 1.4, to see that $f \in \text{VMO}(\mathbb{R}^{n-1}; \mathbb{C}^M)$ implies $|\nabla u(x', t)|^2\, t\, dx'\, dt$ is a vanishing Carleson measure in $\mathbb{R}^n_+$ we use what we just proved in item (1), bearing in mind (1-22), combined with part (f) of Proposition 3.1. The converse follows from (1-44). □

Having dealt with the Fatou-type results from Theorems 1.2 and 1.3, we turn our attention to the proof of Theorem 1.1.

Proof of Theorem 1.1. The fact that the function u defined in (1-30) solves the BMO-Dirichlet boundary value problem (1-29) follows from Proposition 3.1. By Proposition 4.2, this is the only solution of (1-29). Next, the double estimate in (1-31) is a direct consequence of (1-30) and (4-3). The uniform BMO estimate in (1-32) is seen straight from Lemma 4.14.

Moving on, the left-pointing implication in (1-33) follows from Lemma 5.1. For the opposite implication, invoke part (d) in Lemma 4.6 together with (1-32) to conclude that f is the limit in $\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$ of the sequence $\{u(\cdot,\varepsilon)\}_{\varepsilon>0}\subset \mathrm{UC}(\mathbb{R}^{n-1},\mathbb{C}^M)\cap\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$. This places f in $\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$; see (1-26). Having established this, part (f) in Proposition 3.1 gives that $|\nabla u(x',t)|^2\,t\,dx'\,dt$ is a vanishing Carleson measure in $\mathbb{R}^n_+$. Going further, the equivalence in (1-34) is seen from (1-44) and the last part in Proposition 3.1.

As regards the equivalence in (1-35), let us first observe that, as is apparent from its definition in (1-13), the seminorm $\|\cdot\|_{**}$ is invariant to horizontal translations. That is, for every $u\in\mathscr{C}^1(\mathbb{R}^n_+,\mathbb{C}^M)$ we have

$$\|\tau_{(z',0)}u\|_{**}=\|u\|_{**}\quad\text{for every } z'\in\mathbb{R}^{n-1}. \tag{5-26}$$

Given $f\in\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$, the right-pointing implication in (1-34) ensures that

$$|\nabla u(x',t)|^2\,t\,dx'\,dt \text{ is a vanishing Carleson measure in } \mathbb{R}^n_+. \tag{5-27}$$

For each $z=(z',s)\in\mathbb{R}^n_+$ we may write, thanks to (5-26) and the estimate in (1-39),

$$\begin{aligned}\|\tau_z u-u\|_{**}&\le\|\tau_z u-\tau_{(z',0)}u\|_{**}+\|\tau_{(z',0)}u-u\|_{**}\\&=\|\tau_{(0,s)}u-u\|_{**}+\|\tau_{(z',0)}u-u\|_{**}\\&\le C\|u(\cdot,s)-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}+C\|\tau_{z'}f-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}\end{aligned} \tag{5-28}$$

for some constant $C=C(n,L)\in(0,\infty)$. In light of (5-27), the left-pointing implication in (1-33), and (1-28), we then conclude from (5-28) that

$$\lim_{\mathbb{R}^n_+\ni z\to 0}\|\tau_z u-u\|_{**}=0, \tag{5-29}$$

as wanted. Conversely, suppose now that (5-29) holds. Specializing this to the case when $z:=(0',\varepsilon)$ with $\varepsilon>0$ then yields, on account of the estimate in (1-39),

$$\|u(\cdot,\varepsilon)-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}\le C\|\tau_{(0',\varepsilon)}u-u\|_{**}\to 0\quad\text{as }\varepsilon\to 0^+. \tag{5-30}$$

Hence, $\|u(\cdot,\varepsilon)-f\|_{\mathrm{BMO}(\mathbb{R}^{n-1},\mathbb{C}^M)}\to 0$ as $\varepsilon\to 0^+$ which, by virtue of (1-33)–(1-34), implies that $f\in\mathrm{VMO}(\mathbb{R}^{n-1},\mathbb{C}^M)$. This finishes the proofs of the equivalences in part (iv) of the statement.

Finally, all claims about the VMO-Dirichlet boundary value problem (1-36) are direct consequences of what we have proved up to this point. □

Going further, we present the proof of the quantitative characterization of VMO from Theorem 1.5.

Proof of Theorem 1.5. We shall establish all claims stated with $n-1$ in place of n. Fix a modulus of continuity Υ satisfying $\Upsilon_\#\le C\,\Upsilon$ on $[0,\infty)$ for some finite constant $C>0$. This implies that

$$\mathscr{C}^{\Upsilon_\#}(\mathbb{R}^{n-1})\subseteq\mathscr{C}^{\Upsilon}(\mathbb{R}^{n-1}). \tag{5-31}$$

Consider next an arbitrary function $f \in \mathrm{VMO}(\mathbb{R}^{n-1})$ and define $u \in \mathscr{C}^\infty(\mathbb{R}^n_+)$ by setting $u(x',t) := (P_t^\Delta * f)(x')$ for $(x',t) \in \mathbb{R}^n_+$. Then from item (d) in Lemma 4.6, Theorem 1.1 part (iii), and (1-37) we conclude that the sequence of functions $\{f_\varepsilon\}_{\varepsilon>0}$ defined for every $\varepsilon > 0$ by $f_\varepsilon := u(\cdot,\varepsilon)$ in $\mathbb{R}^{n-1}$ satisfies, for each $\varepsilon > 0$,

$$\begin{gathered} f_\varepsilon \in \mathscr{C}^\Upsilon(\mathbb{R}^{n-1}) \cap \mathscr{C}^\infty(\mathbb{R}^{n-1}) \cap \mathrm{BMO}(\mathbb{R}^{n-1}) \text{ and} \\ \partial^{\alpha'} f \in \mathscr{C}^\Upsilon(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1}) \text{ for every } \alpha' \in \mathbb{N}_0^{n-1} \text{ with } |\alpha'| \geq 1, \end{gathered} \tag{5-32}$$

as well as

$$\|f - f_\varepsilon\|_{\mathrm{BMO}(\mathbb{R}^{n-1})} \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{5-33}$$

This establishes (1-51), as well as the stronger claim made in (1-52). □

Going further, we provide the proof of Theorem 1.6.

Proof of Theorem 1.6. First note that condition (1-55) implies that φ is continuous on $\mathbb{R}^n \setminus \{0\}$. As such, φ is a Lebesgue-measurable function $\mathbb{R}^n$ which, in turn, ensures that condition (1-56) is meaningful.

To proceed, observe that if $f \in L^1(\mathbb{R}^n, dx/(1+|x|^{n+\varepsilon}))^M$ then for each $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)||\varphi(x-y)|\,dy &\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n+\varepsilon}} \cdot \frac{(1+|y|)^{n+\varepsilon}}{(1+|x-y|)^{n+\varepsilon}}\,dy \\ &\leq C(1+|x|)^{n+\varepsilon} \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n+\varepsilon}}\,dy < \infty. \end{aligned} \tag{5-34}$$

In light of (2-24) (used here with $n+1$ in place of n), this implies that for every $t > 0$ the convolution $\varphi_t * f$ is well-defined via an absolutely convergent integral whenever the function f belongs to $\mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M)$. In particular, this is the case whenever $f \in \mathrm{VMO}(\mathbb{R}^n, \mathbb{C}^M)$.

Next, fix $t > 0$ and define

$$T_t f := \varphi_t * f \quad \text{for every } f \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M). \tag{5-35}$$

We first claim that there exists some constant $C \in (0,\infty)$ independent of t such that

$$\|T_t f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \quad \text{for all } f \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M). \tag{5-36}$$

To prove this claim, fix $f \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M)$ and an arbitrary cube Q in $\mathbb{R}^n$ with center x_Q; then we have the decomposition

$$f = (f - f_Q)\mathbf{1}_{\lambda Q} + (f - f_Q)\mathbf{1}_{\mathbb{R}^n \setminus \lambda Q} + f_Q, \quad \text{where } \lambda := 2\sqrt{n}. \tag{5-37}$$

Thus, using (1-56) we have

$$(T_t f)(x) = T_t[(f - f_Q)\mathbf{1}_{\lambda Q}](x) + T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n \setminus (\lambda Q)}](x) + f_Q \quad \text{for all } x \in \mathbb{R}^n, \tag{5-38}$$

and if we set

$$c_Q := T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n \setminus (\lambda Q)}](x_Q) + f_Q \in \mathbb{C}^M \tag{5-39}$$

it follows that

$$\begin{aligned}
&⨍_Q |(T_t f)(x) - c_Q|\,dx \\
&\qquad \le ⨍_Q |T_t[(f - f_Q)\mathbf{1}_{\lambda Q}](x)|\,dx + ⨍_Q \big|T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n\setminus\lambda Q}](x) - T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n\setminus\lambda Q}](x_Q)\big|\,dx \\
&\qquad =: I + II.
\end{aligned} \tag{5-40}$$

Since $f \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M)$ we have $(f - f_Q)\mathbf{1}_{\lambda Q} \in L^1(\mathbb{R}^n, \mathbb{C}^M)$. On the other hand, assumption (1-54) implies that T_t is bounded in $L^1(\mathbb{R}^n, \mathbb{C}^M)$ uniformly in t. In concert with (2-8), this permits us to estimate

$$\begin{aligned}
I &= \frac{1}{|Q|}\|T_t[(f - f_Q)\mathbf{1}_{\lambda Q}]\|_{L^1(\mathbb{R}^n,\mathbb{C}^M)} \\
&\le \frac{C}{|Q|}\|(f - f_Q)\mathbf{1}_{\lambda Q}\|_{L^1(\mathbb{R}^n,\mathbb{C}^M)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)}
\end{aligned} \tag{5-41}$$

for some $C \in (0, \infty)$ independent of f, Q, and t. To treat II, first we derive a pointwise estimate. For each $x \in Q$ we have

$$\begin{aligned}
&\big|T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n\setminus\lambda Q}](x) - T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n\setminus\lambda Q}](x_Q)\big| \\
&\qquad\qquad \le t^{-n}\int_{\mathbb{R}^n\setminus\lambda Q} |f(y) - f_Q|\left|\varphi\Big(\frac{x-y}{t}\Big) - \varphi\Big(\frac{x_Q - y}{t}\Big)\right| dy.
\end{aligned} \tag{5-42}$$

Next, pick some arbitrary $x \in Q$ and $y \in \mathbb{R}^n \setminus \lambda Q$; then consider $z := (x_Q - y)/t \in \mathbb{R}^n \setminus \{0\}$ and $h := (x - x_Q)/t \in \mathbb{R}^n$. Since in view of the choice of λ in (5-37) we have

$$|h| \le \frac{\sqrt{n}\,\ell(Q)}{2t} = \frac{\lambda\,\ell(Q)}{4t} \le \frac{|z|}{2}, \tag{5-43}$$

it follows from (1-55) that

$$\begin{aligned}
\left|\varphi\Big(\frac{x-y}{t}\Big) - \varphi\Big(\frac{x_Q - y}{t}\Big)\right| &= |\varphi(z+h) - \varphi(z)| \le \frac{C|h|^\varepsilon}{|z|^{n+\varepsilon}} \\
&\le \frac{C\ell(Q)^\varepsilon t^n}{|y - x_Q|^{n+\varepsilon}} \le \frac{C\ell(Q)^\varepsilon t^n}{(\ell(Q) + |y - x_Q|)^{n+\varepsilon}}.
\end{aligned} \tag{5-44}$$

Combining (5-42)–(5-44) with (2-21) (used here with $n+1$ in place of n) and part (c) in Lemma 2.1, it follows that

$$\begin{aligned}
\big|T_t[(f-f_Q)\mathbf{1}_{\mathbb{R}^n\setminus\lambda Q}](x) - T_t[(f-f_Q)\mathbf{1}_{\mathbb{R}^n\setminus\lambda Q}](x_Q)\big| &\le C\ell(Q)^\varepsilon\int_{\mathbb{R}^n}\frac{|f(y)-f_Q|}{(\ell(Q)+|y-x_Q|)^{n+\varepsilon}}\,dy \\
&\le C\int_1^\infty \operatorname{osc}_1(f;\lambda\,\ell(Q))\,\frac{d\lambda}{\lambda^{1+\varepsilon}} \\
&\le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \quad \text{for all } x \in Q,
\end{aligned} \tag{5-45}$$

where $C \in (0, \infty)$ is independent of f, Q and t. From (5-45) and (5-40) we obtain

$$II \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \tag{5-46}$$

for some $C \in (0,\infty)$ independent of f, Q, and t. In concert, (5-40), (5-41), and (5-46) yield

$$\fint_Q |(T_t f)(x) - c_Q|\,dx \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)}, \tag{5-47}$$

with $c_Q \in \mathbb{C}^M$ as in (5-39). In view of (2-9), this ultimately implies the claim in (5-36).

The second claim we make is that there exists some constant $C \in (0,\infty)$ with the property that for every $t > 0$ and every $\eta \in (0,\varepsilon)$ there holds

$$\|T_t g - g\|_{L^\infty(\mathbb{R}^n,\mathbb{C}^M)} \le Ct^\eta \|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M)} \quad \text{for all } g \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M). \tag{5-48}$$

To prove this claim, fix $t > 0$, $\eta \in (0,\varepsilon)$, $g \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M)$, and, for $x \in \mathbb{R}^n$ arbitrary, estimate

$$\begin{aligned}
|(T_t g)(x) - g(x)| &\le \int_{\mathbb{R}^n} |g(x-y) - g(x)|\,|\varphi_t(y)|\,dy \\
&\le t^\eta \|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M)} \int_{\mathbb{R}^n} \frac{|y|^\eta}{t^\eta}\,|\varphi_t(y)|\,dy \\
&\le t^\eta \|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M)} \int_{\mathbb{R}^n} |z|^\eta\,(1+|z|)^{-n-\varepsilon}\,dz \\
&\le Ct^\eta \|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M)}
\end{aligned} \tag{5-49}$$

for some constant $C = C(\varepsilon,\eta,n,\varphi) \in (0,\infty)$ independent of t and g. The first inequality in (5-49) relies on (1-56), for the third one we have used (1-54) and the change of variables $z = y/t$, while the last one is a consequence of having $\eta \in (0,\varepsilon)$.

Here is the argument involved in the endgame of the proof of Theorem 1.6. Fix $\eta \in (0,\varepsilon)$ and given $f \in \mathrm{VMO}(\mathbb{R}^n,\mathbb{C}^M)$ pick $g \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M) \cap \mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)$. Then for each $t > 0$, we use (5-36) and (5-48) to estimate

$$\begin{aligned}
\|T_t f - f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} &\le \|T_t(f-g)\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} + \|T_t g - g\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} + \|g - f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \\
&\le C\|g-f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} + 2\|T_t g - g\|_{L^\infty(\mathbb{R}^n,\mathbb{C}^M)} \\
&\le C\|g-f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} + Ct^\eta\|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n,\mathbb{C}^M)}.
\end{aligned} \tag{5-50}$$

Thus,

$$\limsup_{t\to 0^+} \|T_t f - f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \le C\|g-f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)}. \tag{5-51}$$

Now (1-57) follows from (5-51) in light of the density result recorded in (1-62).

To prove the very last claim in the statement of Theorem 1.6, let $\varphi \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{C}^{M\times M})$ be a function satisfying (1-58). Then for each $x \in \mathbb{R}^n \setminus \{0\}$ and $h \in \mathbb{R}^n$ with $|h| < |x|/2$ the mean value theorem permits us to estimate

$$\begin{aligned}
|\varphi(x+h) - \varphi(x)| &\le |h| \sup_{\xi\in[x,x+h]} |(\nabla\varphi)(\xi)| \\
&\le C|h| \sup_{\xi\in[x,x+h]} (1+|\xi|)^{-n-1} \le \frac{C|h|}{|x|^{n+1}}.
\end{aligned} \tag{5-52}$$

Hence, both (1-54) and (1-55) hold with $\varepsilon = 1$ in this case, so the left-pointing implication in (1-59) is a consequence of (1-57).

As regards the right-pointing implication in (1-59), let us first observe that from (1-56) and (1-58) we have

$$\int_{\mathbb{R}^n} (\partial_j \varphi)((x-y)/t)\,dy = 0 \quad \text{for all } x \in \mathbb{R}^n, \text{ for all } j \in \{1,\dots,n\}. \tag{5-53}$$

Next, given a function $f \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M)$, fix $x \in \mathbb{R}^n$ and $t > 0$ arbitrary and denote by $Q_{x,t}$ the cube in $\mathbb{R}^n$ centered at x and of side-length t. As usual, abbreviate $f_{Q_{x,t}} := ⨍_{Q_{x,t}} f(y)\,dy$. On account of (5-53), (1-58), (2-21) (used here with $\varepsilon = 1$ and n in place of $n-1$), and (2-15) (used with $p = 1$ and n in place of $n-1$), for each $j \in \{1,\dots,n\}$ we may then estimate

$$\begin{aligned} |\partial_j(\varphi_t * f)(x)| &= t^{-n-1}\left|\int_{\mathbb{R}^n} (\partial_j\varphi)\left(\frac{x-y}{t}\right) f(y)\,dy\right| \\ &= t^{-n-1}\left|\int_{\mathbb{R}^n} (\partial_j\varphi)\left(\frac{x-y}{t}\right) [f(y) - f_{Q_{x,t}}]\,dy\right| \\ &\le C\int_{\mathbb{R}^n} \frac{|f(y) - f_{Q_{x,t}}|}{[t + |x-y|]^{n+1}}\,dy \le C t^{-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)} \end{aligned} \tag{5-54}$$

for some constant $C \in (0,\infty)$ independent of f, x, t. In concert with (5-36), this proves that

$$\varphi_t * f \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^M) \cap \mathrm{Lip}(\mathbb{R}^n, \mathbb{C}^M) \quad \text{for each } t > 0. \tag{5-55}$$

With this in hand, the right-pointing implication in (1-59) readily follows (compare with (1-64)), finishing the proof of Theorem 1.6. □

The proof of the negative result stated in Theorem 1.8 is discussed next.

Proof of Theorem 1.8. From [Bourdaud 2002, Proposition 9, p. 1208] we know that there exists $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that

$$\partial^\alpha f \in \mathrm{BMO}(\mathbb{R}^n) \quad \text{for all } \alpha \in \mathbb{N}_0^n, \tag{5-56}$$

and

$$\inf\{\|f - g\|_{\mathrm{BMO}(\mathbb{R}^n)} : g \in L^\infty(\mathbb{R}^n)\} > 0. \tag{5-57}$$

In concert with [loc. cit., Lemme 6, p. 1211], property (5-56) (used for multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$) gives $f \in \mathrm{UC}(\mathbb{R}^n)$. By once again using (5-56) (with $|\alpha| = 0$), this proves that $f \in \mathrm{UC}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$; hence $f \in \mathrm{VMO}(\mathbb{R}^n)$. On the other hand, (5-57) implies that f does not belong to the closure of $L^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$; hence also f does not belong to the closure of $\mathrm{UC}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. Ultimately, this proves that the space $\mathrm{UC}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is not dense in $\mathrm{VMO}(\mathbb{R}^n)$. □

The penultimate proof in this section is that of Theorem 1.9.

Proof of Theorem 1.9. That for each $f \in \mathrm{BMO}(\mathbb{R}^n)$ the measure μ_f associated with f as in (1-69) satisfies Carleson's condition

$$\|\mu_f\|_{\mathcal{C}(\mathbb{R}^{n+1}_+)} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |(\psi_t * f)(x)|^2 \frac{dx\,dt}{t} \le C\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} \tag{5-58}$$

for some constant $C \in (0,\infty)$ which depends only on the dimension n and the constant in (1-68), is fairly standard. Specifically, having fixed an arbitrary cube $Q \subset \mathbb{R}^n$, take the decomposition $f = f_0 + f_\infty + f_{2Q}$, where $f_0 := (f - f_{2Q})\mathbf{1}_{2Q}$ and $f_\infty := (f - f_{2Q})\mathbf{1}_{\mathbb{R}^n\setminus 2Q}$. On account of the cancellation property of ψ, we may write $\psi_t * f = \psi_t * f_0 + \psi_t * f_\infty$. Then, on the one hand,

$$\begin{aligned}\frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q |(\psi_t * f_0)(x)|^2 \,\frac{dx\,dt}{t} &\le \frac{1}{|Q|}\int_{\mathbb{R}^{n+1}_+} |(\psi_t * f_0)(x)|^2\,\frac{dx\,dt}{t}\\ &\le C|Q|^{-1}\|f_0\|^2_{L^2(\mathbb{R}^n)} \le C\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)},\end{aligned} \tag{5-59}$$

thanks to the square-function estimate (3-43) in Proposition 3.3 (used with n replaced by $n+1$ and the kernel $\theta(x,t;y) := \psi_t(x-y)$ for each $x, y \in \mathbb{R}^n$, $t > 0$), and (2-15). On the other hand, for each $x \in Q$ and $t \in (0, \ell(Q))$ we may estimate

$$\begin{aligned}|(\psi_t * f_\infty)(x)| &\le \int_{\mathbb{R}^n\setminus 2Q} t^{-n}\left|\psi\left(\frac{x-y}{t}\right)\right| |f(y) - f_{2Q}|\,dy\\ &\le Ct\int_{\mathbb{R}^n\setminus 2Q}\frac{|f(y)-f_{2Q}|}{[t+|x-y|]^{n+1}}\,dy \le Ct\int_{\mathbb{R}^n\setminus 2Q}\frac{|f(y)-f_{2Q}|}{|x_Q-y|^{n+1}}\,dy\\ &\le Ct\int_{\mathbb{R}^n}\frac{|f(y)-f_{2Q}|}{[\ell(Q)+|x_Q-y|]^{n+1}}\,dy \le \frac{Ct}{\ell(Q)}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)},\end{aligned} \tag{5-60}$$

by virtue of (2-21) in Lemma 2.2 (used with n replaced by $n+1$ and $\varepsilon = 1$). Combining (5-59) with (5-60) then readily yields (5-58).

Let us next observe that if $g \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n)$ for some $\eta \in (0,1)$ then for each $x \in \mathbb{R}^n$ and $t > 0$ we may estimate, on account of (1-68),

$$\begin{aligned}|(\psi_t * g)(x)| &= \left|\int_{\mathbb{R}^n}\psi_t(y)(g(x-y) - g(x))\,dy\right|\\ &\le \|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\psi_t(y)|\,|y|^\eta\,dy\\ &\le Ct^\eta\|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|y|^\eta}{(1+|y|)^{n+1}}\,dy = Ct^\eta\|g\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)}.\end{aligned} \tag{5-61}$$

Assume now that some function $f \in \mathrm{BMO}(\mathbb{R}^n)$ has been fixed. Pick $\eta \in (0,1)$ and choose $g \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ arbitrary. Then, making use of (5-58) and (5-61), for each cube $Q \subseteq \mathbb{R}^n$ we may bound

$$\begin{aligned}&\frac{1}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q |(\psi_t * f)(x)|^2\,\frac{dx\,dt}{t}\\ &\quad\le \frac{2}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q |(\psi_t * (f-g))(x)|^2\,\frac{dx\,dt}{t} + \frac{2}{|Q|}\int_0^{\ell(Q)}\!\!\int_Q |(\psi_t * g)(x)|^2\,\frac{dx\,dt}{t}\\ &\quad\le C\|f-g\|^2_{\mathrm{BMO}(\mathbb{R}^n)} + C\ell(Q)^{2\eta}\|g\|^2_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n)}.\end{aligned} \tag{5-62}$$

In turn, (5-62) allows us to conclude that

$$\lim_{r\to 0^+}\Bigg\{\sup_{\substack{Q\subset\mathbb{R}^n\\ \ell(Q)\le r}}\frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q|(\psi_t*f)(x)|^2\,\frac{dx\,dt}{t}\Bigg\}\le C\|f-g\|^2_{\mathrm{BMO}(\mathbb{R}^n)},\tag{5-63}$$

which, after taking the infimum over all $g\in\dot{\mathscr{C}}^\eta(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n)$ and bearing in mind the density result in (1-62), yields (1-70). □

We conclude this section by giving the proof of Theorem 1.10.

Proof of Theorem 1.10. Fix $f\in\mathrm{BMO}(\mathbb{R}^n,\mathbb{C}^M)$ and let u be the unique solution u of the BMO-Dirichlet boundary value problem (1-29) for L in $\mathbb{R}^n_+$ with boundary datum f. By (1-30) in Theorem 1.1, we have (with P^L denoting the Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 2.3)

$$u(x',t)=(P^L_t*f)(x')=\int_{\mathbb{R}^{n-1}_+}K^L(x'-y',t)f(y')\,dy'\quad\text{for }(x',t)\in\mathbb{R}^n_+,\tag{5-64}$$

where K^L is as in (2-37). Consider now

$$\psi(z'):=(\psi_1,\dots,\psi_n):=((\partial_jK^L)(z',1))_{1\le j\le n}\quad\text{for each }z'\in\mathbb{R}^{n-1}.\tag{5-65}$$

Then, from item (4) and (2-39) in Theorem 2.3 we deduce that $\psi_j\in\mathscr{C}^\infty(\mathbb{R}^{n-1},\mathbb{C}^{M\times M})$ for each $j\in\{1,\dots,n\}$ and there exists some constant $C\in(0,\infty)$ such that

$$|\psi(z')|\le\frac{C}{(1+|z'|)^n}\quad\text{and}\quad|\nabla\psi(z')|\le\frac{C}{(1+|z'|)^{n+1}}\quad\text{for each }z'\in\mathbb{R}^{n-1}.\tag{5-66}$$

We also claim that

$$\int_{\mathbb{R}^{n-1}}\psi_j(z')\,dz'=0\in\mathbb{C}^{M\times M}\quad\text{for each }j\in\{1,\dots,n\}.\tag{5-67}$$

To see why (5-67) is true, note that based on (5-65) and (2-37) we have

$$\psi_j(z')=\partial_jP^L(z')\quad\text{for all }z'\in\mathbb{R}^{n-1}\text{ and each }j\in\{1,\dots,n-1\},\tag{5-68}$$

while

$$\psi_n(z')=(1-n)P^L(z')-z'\cdot\nabla P^L(z')\quad\text{for all }z'\in\mathbb{R}^{n-1}.\tag{5-69}$$

Now (5-67) follows from (5-68)–(5-69) and (2-35) via integration by parts.

Next, for each $x'\in\mathbb{R}^{n-1}$ and $t>0$ set $\psi_t(x'):=t^{1-n}\psi(x'/t)$. Then from item (5) in Theorem 2.3 it follows that ∇K^L is homogeneous of order $-n$; thus

$$\psi_t(x')=t^{1-n}(\nabla K^L)(x'/t,1)=t(\nabla K^L)(x',t)\quad\text{for each }(x',t)\in\mathbb{R}^{n-1}_+.\tag{5-70}$$

Combining (5-64) and (5-70) yields

$$t(\nabla u)(x',t)=\int_{\mathbb{R}^{n-1}}t(\nabla K^L)(x'-y',t)f(y')\,dy'=(\psi_t*f)(x')\tag{5-71}$$

for each $x' \in \mathbb{R}^{n-1}$ and each $t > 0$. Consequently,

$$|(\psi_t * f)(x')|^2 \, \frac{dx'\,dt}{t} = t|(\nabla u)(x',t)|^2 \, dx'\,dt. \tag{5-72}$$

In light of (5-66)–(5-67) we see that Theorem 1.9 applies componentwise in the current setting (with n replaced by $n-1$) and yields a constant C for which (1-70) holds. The latter becomes (1-72) by invoking (5-72) and finishes the proof of the theorem. □

6. Proof of the well-posedness of the Morrey–Campanato–Dirichlet problem

This section is devoted to presenting the proof of Theorem 1.21. Throughout fix $p, q \in [1,\infty)$. We divide the proof into several steps, the starting point being the following claim:

Step 1: *There exists a constant $C = C(n, L, \eta) \in (0,\infty)$ such that if $f \in \mathscr{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ then the function u given at every point $(x',t) \in \mathbb{R}^n_+$ by $u(x',t) := (P_t^L * f)(x')$ is well-defined (via an absolutely convergent integral) and satisfies $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$, $Lu = 0$ in $\mathbb{R}^n_+$, $u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = f$ a.e. in $\mathbb{R}^{n-1}$, as well as*

$$\sup_{(x',t)\in\mathbb{R}^n_+} [t^{1-\eta}|(\nabla u)(x',t)|] \le C\|f\|_*^{(\eta,p)}. \tag{6-1}$$

The fact that u is well-defined and is a smooth null-solution of L in the upper half-space whose nontangential trace matches f a.e. in $\mathbb{R}^{n-1}$ follows from (2-25) with $\varepsilon = 1$ and item (7) in Theorem 2.3. To proceed, fix an arbitrary point $(x',t) \in \mathbb{R}^n_+$ and, making use of (3-6) and (2-17), estimate

$$|(\nabla u)(x',t)| \le \frac{C}{t}\int_1^\infty \operatorname{osc}_1(f;\lambda t)\,\frac{d\lambda}{\lambda^2} \le \frac{C}{t^{1-\eta}}\|f\|_*^{(\eta,p)}, \tag{6-2}$$

from which (6-1) readily follows.

Step 2: *For every function $u \in \mathscr{C}^1(\mathbb{R}^n_+, \mathbb{C}^M)$ there holds*

$$\|u\|_{**}^{(\eta,q)} \le (2\eta)^{-\frac{1}{2}} \sup_{(x',t)\in\mathbb{R}^n_+} [t^{1-\eta}|(\nabla u)(x',t)|]. \tag{6-3}$$

This is readily seen from (1-160).

Step 3: *There exists a constant $C = C(n, L, \eta, q) \in (0,\infty)$ with the property that for every function $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfying $Lu = 0$ in $\mathbb{R}^n_+$ there holds*

$$\sup_{(x',t)\in\mathbb{R}^n_+} [t^{1-\eta}|(\nabla u)(x',t)|] \le C\|u\|_{**}^{(\eta,q)}. \tag{6-4}$$

For each fixed point $(x',t) \in \mathbb{R}^n_+$ use Theorem 2.4 and repeated applications of Hölder's inequality in order to estimate (recall that $Q_{x',t}$ is the cube in $\mathbb{R}^{n-1}$ centered at x' and of side-length t)

$$\begin{aligned}|(\nabla u)(x',t)| &\le C\fint_{Q_{x',t}\times(\frac{t}{2},\frac{3t}{2})} |(\nabla u)(y',s)|\,dy'\,ds \\ &= C\fint_{Q_{x',t}}\Big(\fint_{(\frac{t}{2},\frac{3t}{2})} |(\nabla u)(y',s)|\,ds\Big)\,dy'\end{aligned}$$

$$
\begin{aligned}
&\leq C\biggl(⨍_{Q_{x',t}}\Bigl(⨍_{(\frac{t}{2},\frac{3t}{2})}|(\nabla u)(y',s)|^2\,ds\Bigr)^{\frac{q}{2}}\,dy'\biggr)^{\frac{1}{q}}\\
&\leq Ct^{-\frac{1}{2}}\biggl(⨍_{Q_{x',t}}\Bigl(⨍_{(\frac{t}{2},\frac{3t}{2})}|(\nabla u)(y',s)|^2 s\,ds\Bigr)^{\frac{q}{2}}\,dy'\biggr)^{\frac{1}{q}}\\
&\leq Ct^{-1}\biggl(⨍_{Q_{x',t}}\Bigl(\int_0^{\frac{3t}{2}}|(\nabla u)(y',s)|^2 s\,ds\Bigr)^{\frac{q}{2}}\,dy'\biggr)^{\frac{1}{q}}\\
&\leq Ct^{-1}\biggl(\frac{1}{\bigl|\frac{3}{2}Q_{x',t}\bigr|}\int_{\frac{3}{2}Q_{x',t}}\Bigl(\int_0^{\ell(\frac{3}{2}Q_{x',t})}|(\nabla u)(y',s)|^2 s\,ds\Bigr)^{\frac{q}{2}}\,dy'\biggr)^{\frac{1}{q}}\\
&\leq Ct^{\eta-1}\|u\|_{**}^{(\eta,q)},
\end{aligned}
\tag{6-5}
$$

where the last inequality is a consequence of (1-160). With this in hand, (6-4) follows.

<u>Step 4</u>: *For every function $u\in\mathscr{C}^1(\mathbb{R}^n_+,\mathbb{C}^M)$ one has*

$$
\sup_{\substack{x,y\in\mathbb{R}^n_+\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\eta}\leq\Bigl(1+\frac{2}{\eta}\Bigr)\sup_{(x',t)\in\mathbb{R}^n_+}\bigl[t^{1-\eta}|(\nabla u)(x',t)|\bigr].
\tag{6-6}
$$

In fact, the opposite inequality holds for smooth null-solutions of L in the upper half-space. Specifically, there exists a constant $C=C(n,L,\eta)\in(0,\infty)$ with the property that for every function $u\in\mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M)$ satisfying $Lu=0$ in $\mathbb{R}^n_+$ there holds

$$
\sup_{(x',t)\in\mathbb{R}^n_+}\bigl[t^{1-\eta}|(\nabla u)(x',t)|\bigr]\leq C\sup_{\substack{x,y\in\mathbb{R}^n_+\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\eta}.
\tag{6-7}
$$

To justify (6-6), abbreviate

$$
C_{u,\eta}:=\sup_{(x',t)\in\mathbb{R}^n_+}\bigl[t^{1-\eta}|(\nabla u)(x',t)|\bigr].
\tag{6-8}
$$

Pick two arbitrary distinct points $x=(x',t)\in\mathbb{R}^n_+$, $y=(y',s)\in\mathbb{R}^n_+$, and set $r:=|x-y|>0$. Then

$$
r^{-\eta}|u(x)-u(y)|\leq I+II+III,
\tag{6-9}
$$

where

$$
\begin{aligned}
I&:=r^{-\eta}|u(x',t)-u(x',t+r)|,\\
II&:=r^{-\eta}|u(x',t+r)-u(y',s+r)|,\\
III&:=r^{-\eta}|u(y',s+r)-u(y',s)|.
\end{aligned}
\tag{6-10}
$$

Then by the fundamental theorem of calculus and the assumption on u,

$$
\begin{aligned}
I&=r^{-\eta}|u(x',t)-u(x',t+r)|=r^{-\eta}\biggl|\int_0^r(\partial_n u)(x',t+\xi)\,d\xi\biggr|\\
&\leq C_{u,\eta}r^{-\eta}\int_0^r(t+\xi)^{\eta-1}\,d\xi\leq C_{u,\eta}r^{-\eta}\int_0^r\xi^{\eta-1}\,d\xi\\
&=C_{u,\eta}r^{-\eta}\eta^{-1}r^\eta=C_{u,\eta}/\eta.
\end{aligned}
\tag{6-11}
$$

Moreover, III may be estimated in a similar manner (with the same bound $C_{u,\eta}/\eta$), while

$$\begin{aligned}
II &= r^{-\eta}|u(x',t+r)-u(y',s+r)| \\
&= r^{-\eta}\left|\int_0^1 \frac{d}{d\theta}[u(\theta(x',t+r)+(1-\theta)(y',s+r))]\,d\theta\right| \\
&= r^{-\eta}\left|\int_0^1 (x'-y',t-s)\cdot(\nabla u)(\theta(x',t+r)+(1-\theta)(y',s+r))\,d\theta\right| \\
&\le C_{u,\eta} r^{-\eta}|x-y|\int_0^1 [\mathrm{dist}(\theta(x',t+r)+(1-\theta)(y',s+r),\partial\mathbb{R}^n_+)]^{\eta-1}\,d\theta \\
&\le C_{u,\eta} r^{-\eta} r\int_0^1 [(1-\theta)s+\theta t+r]^{\eta-1}\,d\theta \le C_{u,\eta} r^{-\eta}\, r\, r^{\eta-1} = C_{u,\eta}.
\end{aligned} \tag{6-12}$$

Now (6-6) follows from (6-9)–(6-12).

Consider next (6-7). Recall (2-1). Fix a point $x=(x',t)\in\mathbb{R}^n_+$ and write R_x for the cube in $\mathbb{R}^n$ centered at x with side-length $t/2$. Using that the function $u(\cdot)-u(x)$ is a null-solution of the system L, we may apply Theorem 2.4 (with $\ell=1$ and $p=1$) to write

$$\begin{aligned}
t|(\nabla u)(x',t)| &\le C \fint_{R_x} |u(y)-u(x)|\,dy \\
&\le C\|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)} \fint_{R_x} |x-y|^\eta\,dy \\
&\le C\|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)}\, t^\eta.
\end{aligned} \tag{6-13}$$

This readily implies (6-7).

<u>Step 5</u>: *There exists a constant $C=C(n,\eta)\in(0,\infty)$ such that for every continuous function $f:\mathbb{R}^{n-1}\to\mathbb{C}^M$ one has*

$$\|f\|_*^{(\eta,p)} \le C \sup_{\substack{x',y'\in\mathbb{R}^{n-1}\\ x'\neq y'}} \frac{|f(x')-f(y')|}{|x'-y'|^\eta}. \tag{6-14}$$

In particular, the inclusion

$$\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M) \hookrightarrow \mathscr{E}^{\eta,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \quad \textit{is continuous.} \tag{6-15}$$

This is a direct consequence of (1-157).

<u>Step 6</u>: *Given $f\in\mathscr{E}^{\eta,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$, the function u defined as in* (1-30) *solves the Dirichlet boundary value problem* (1-161) *and obeys the estimates in* (1-162). *Moreover, $u\in\dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ and* (1-163) *holds as well.*

Fix an arbitrary function $f\in\mathscr{E}^{\eta,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$. From Step 1 we know that u given as in (1-30) is well-defined, $u\in\mathscr{C}^\infty(\mathbb{R}^n_+,\mathbb{C}^M)$, $Lu=0$ in $\mathbb{R}^n_+$, $f=u\big|^{\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ a.e. in $\mathbb{R}^n$, and satisfies (6-1). To proceed, observe that when used in concert, (6-1) and (6-3) imply that

$$\|u\|_{**}^{(\eta,q)} \le C\|f\|_*^{(\eta,p)}. \tag{6-16}$$

Hence, $\|u\|_{**}^{(\eta,q)} < \infty$. On the other hand, combining the results proved in Steps 3 and 4 establishes the membership of u to $\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+, \mathbb{C}^M) = \dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, see (2-2), along with the estimate

$$\|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C\,\|u\|_{**}^{(\eta,q)}. \tag{6-17}$$

Thanks to (6-16)–(6-17) and (2-2), we therefore have $u \in \dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ and

$$\begin{aligned} \|f\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)} &= \|u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)} \\ &= \|u|_{\partial\mathbb{R}^n_+}\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)} \\ &\leq \|u\|_{\dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+},\mathbb{C}^M)} = \|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)} \\ &\leq C\,\|u\|_{**}^{(\eta,q)} \leq C\,\|f\|_{*}^{(\eta,p)}. \end{aligned} \tag{6-18}$$

Using (6-14) and recycling part of the above estimate then yields

$$\|f\|_{*}^{(\eta,p)} \leq C\,\|f\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq C\,\|u\|_{**}^{(\eta,q)}. \tag{6-19}$$

At this stage, all desired properties of u have been established.

<u>Step 7</u>: *Assume that $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\eta(\mathbb{R}^n_+, \mathbb{C}^M)$ for some $\eta \in (0, 1)$ satisfies $Lu = 0$ in $\mathbb{R}^n_+$. Then*

$$u \in \dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M), \quad u|_{\partial\mathbb{R}^n_+} \in \dot{\mathscr{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\biggl(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n}\,dx'\biggr)^M \tag{6-20}$$

and

$$u(x', t) = (P^L_t * (u|_{\partial\mathbb{R}^n_+}))(x') \quad \text{for all } (x', t) \in \mathbb{R}^n_+. \tag{6-21}$$

To justify this, observe that the two memberships listed in (6-20) are direct consequences of (2-2), while the inclusion in (6-20) was proved earlier; see (2-26).

For each fixed $\varepsilon > 0$ consider now the function

$$u_\varepsilon(\cdot) := u(\cdot + \varepsilon e_n) \quad \text{in } \mathbb{R}^n_+, \tag{6-22}$$

which satisfies

$$\begin{gathered} u_\varepsilon \in \mathscr{C}^\infty(\overline{\mathbb{R}^n_+}, \mathbb{C}^M), \quad Lu_\varepsilon = 0 \text{ in } \mathbb{R}^n_+ \quad \text{and} \\ u_\varepsilon \in \dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M) \quad \text{with } \|u_\varepsilon\|_{\dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+},\mathbb{C}^M)} \leq \|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)}. \end{gathered} \tag{6-23}$$

These and (6-7) yield

$$\sup_{x\in\mathbb{R}^n_+} |(\nabla u_\varepsilon)(x)| \leq C(L, \eta, \varepsilon)\|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)}. \tag{6-24}$$

In light of (6-23) (which implies that u_ε is bounded on bounded subsets of $\overline{\mathbb{R}^n_+}$), (6-24) allows us to conclude that

$$u_\varepsilon \in W^{1,2}_{\text{bd}}(\mathbb{R}^n_+, \mathbb{C}^M). \tag{6-25}$$

Going further, set $f_\varepsilon(x') := u(x', \varepsilon)$ for each $x' \in \mathbb{R}^{n-1}$. Then, on the one hand,

$$|f_\varepsilon(x') - f_\varepsilon(y')| = |u(x', \varepsilon) - u(y', \varepsilon)| \leq \|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)}|x' - y'|^\eta \quad \text{for all } x', y' \in \mathbb{R}^{n-1}. \tag{6-26}$$

On the other hand, for all $x', y' \in \mathbb{R}^{n-1}$ we have (with ∇' denoting the gradient in the first $n-1$ variables in $\mathbb{R}^{n-1}$)

$$\begin{aligned} |f_\varepsilon(x') - f_\varepsilon(y')| = |u(x',\varepsilon) - u(y',\varepsilon)| &\le |x'-y'| \sup_{z' \in [x',y']} |(\nabla' u)(z',\varepsilon)| \\ &= |x'-y'| \sup_{z' \in [x',y']} |(\nabla' u_{\varepsilon/2})(z',\varepsilon/2)| \\ &\le |x'-y'|\, C(L,\eta,\varepsilon/2) \|u\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)}, \end{aligned} \tag{6-27}$$

where the last inequality uses (6-24) (written for $u_{\varepsilon/2}$ and for $x = (z', \varepsilon/2)$). A logarithmically convex combination of (6-26)–(6-27) then proves that for every $\theta \in [\eta, 1]$ there exists a finite constant $C(\theta, L, \varepsilon, u) > 0$ such that

$$|f_\varepsilon(x') - f_\varepsilon(y')| \le C(\theta, L, \varepsilon, u) |x'-y'|^\theta \quad \text{for all } x', y' \in \mathbb{R}^{n-1}. \tag{6-28}$$

Hence,

$$f_\varepsilon \in \bigcap_{\eta \le \theta < 1} \dot{\mathscr{C}}^\theta(\mathbb{R}^{n-1}, \mathbb{C}^M). \tag{6-29}$$

Combining (6-29), (6-15), and Step 6 then shows that the function

$$w_\varepsilon(x',t) := (P_t^L * f_\varepsilon)(x') \quad \text{for all } (x',t) \in \mathbb{R}^n_+ \tag{6-30}$$

satisfies

$$w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad L w_\varepsilon = 0 \text{ in } \mathbb{R}^n_+, \quad w_\varepsilon \in \bigcap_{\eta \le \theta < 1} \dot{\mathscr{C}}^\theta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M). \tag{6-31}$$

In addition, from (6-28)–(6-30), Step 5, and Step 1, we see that w_ε has the property that for each $\theta \in [\eta, 1)$ there exists a finite constant $C(\theta, L, \varepsilon, u) > 0$ such that

$$[\operatorname{dist}(x, \partial\mathbb{R}^n_+)]^{1-\theta} |(\nabla w_\varepsilon)(x)| \le C(\theta, L, \varepsilon, u) \quad \text{for all } x \in \mathbb{R}^n_+. \tag{6-32}$$

In particular, choosing $\theta \in \bigl(\max\{\eta, \frac{1}{2}\}, 1\bigr)$, the latter property allows us to estimate for every $R > 0$

$$\begin{aligned} \int_{B(0,R)\cap\mathbb{R}^n_+} |(\nabla w_\varepsilon)(x)|^2\, dx &\le C(\theta, L, \varepsilon, u) \int_{B(0,R)\cap\mathbb{R}^n_+} [\operatorname{dist}(x, \partial\mathbb{R}^n_+)]^{2(\theta-1)}\, dx \\ &= C(\theta, L, \varepsilon, R, u) < +\infty. \end{aligned} \tag{6-33}$$

In concert with the last property in (6-31) (which goes to show that w_ε is bounded on bounded subsets of $\overline{\mathbb{R}^n_+}$), this permits us to conclude that

$$w_\varepsilon \in W^{1,2}_{\mathrm{bd}}(\mathbb{R}^n_+, \mathbb{C}^M). \tag{6-34}$$

From (6-23), (6-25), (6-30), (6-31), and (6-34), we then conclude that the function $v_\varepsilon := u_\varepsilon - w_\varepsilon$ belongs to $\mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ and satisfies

$$v_\varepsilon \in W^{1,2}_{\mathrm{bd}}(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+}, \mathbb{C}^M), \quad L v_\varepsilon = 0 \text{ in } \mathbb{R}^n_+, \quad v_\varepsilon\big|_{\partial\mathbb{R}^n_+} = 0. \tag{6-35}$$

Moreover, the Hölder property gives the growth estimate

$$|v_\varepsilon(x)| \le C(1+|x|^\eta) \quad \text{for all } x \in \mathbb{R}^n_+, \tag{6-36}$$

where $C := \max\{\|v_\varepsilon\|_{\dot{\mathscr{C}}^\eta(\mathbb{R}^n_+,\mathbb{C}^M)}, |v_\varepsilon(0)|\} \in (0,\infty)$.

The estimates near the boundary from Proposition 2.5 then imply (by sending $\rho \to \infty$) that $v_\varepsilon \equiv 0$. This ultimately translates into saying that for each $\varepsilon > 0$ we have

$$u(x', t+\varepsilon) = (P_t^L * f_\varepsilon)(x') \quad \text{for all } (x',t) \in \mathbb{R}^n_+. \tag{6-37}$$

Let us also note that for each $\varepsilon > 0$,

$$\sup_{y' \in \mathbb{R}^{n-1}} |f_\varepsilon(y') - u(y',0)| = \sup_{y' \in \mathbb{R}^{n-1}} |u(y',\varepsilon) - u(y',0)| \le \varepsilon^\eta \|u\|_{\dot{\mathscr{C}}^\eta(\overline{\mathbb{R}^n_+},\mathbb{C}^M)}. \tag{6-38}$$

Hence, $f_\varepsilon \to u|_{\partial\mathbb{R}^n_+}$ as $\varepsilon \to 0^+$, uniformly in $\mathbb{R}^{n-1}$. Since P_t^L is absolutely integrable in $\mathbb{R}^{n-1}$, formula (6-21) then readily follows by passing to limit $\varepsilon \to 0^+$ in (6-37).

Step 8: *Assume that*

$$u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad Lu = 0 \text{ in } \mathbb{R}^n_+, \quad \|u\|_{**}^{(\eta,q)} < \infty, \quad u\big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0. \tag{6-39}$$

Then necessarily $u \equiv 0$ *in* $\mathbb{R}^n_+$.

This is a consequence of Steps 3, 4, and 7.

Step 9: *The end-game in the proof of Theorem 1.21.*

Existence for the Dirichlet boundary value problem (1-161) follows from Step 6. Uniqueness of the Dirichlet boundary value problem (1-161) is seen from Step 8.

7. Calderón–Zygmund operators on VMO

The main goal of this section is to develop the machinery which eventually permits us to prove Theorem 1.13.

We begin by recalling, see, e.g., [Stein 1993, Theorem 1, p. 91], that for each $q \in (0,\infty)$, the Hardy space $H^q(\mathbb{R}^n)$ consists of tempered distributions g in $\mathbb{R}^n$ with the property that their radial maximal function, defined as $(\mathcal{M}_{\text{rad}}\, g)(x) := \sup_{t>0} |(\Phi_t * g)(x)|$ for each $x \in \mathbb{R}^n$ (where Φ is a fixed background Schwartz function in $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \Phi \, d\mathscr{L}^n \neq 0$ and $\Phi_t(x) := t^{-n}\Phi(x/t)$ for each $t > 0$ and $x \in \mathbb{R}^n$), satisfies

$$\|g\|_{H^q(\mathbb{R}^n)} := \|\mathcal{M}_{\text{rad}}\, g\|_{L^q(\mathbb{R}^n)} < +\infty. \tag{7-1}$$

It is then well known that

$$H^q(\mathbb{R}^n) = L^q(\mathbb{R}^n) \quad \text{if } 1 < q < \infty. \tag{7-2}$$

Another classical result in harmonic analysis, see, e.g., [Stein 1993, Theorem 2, p. 107] or [García-Cuerva and Rubio de Francia 1985, Theorem 4.10, p. 283], is the fact that distributions belonging to $H^q(\mathbb{R}^n)$ with $q \in (0,1]$ admit atomic decompositions. To elaborate on this aspect, having fixed $r \in [1,\infty]$, call a

Lebesgue-measurable function $a:\mathbb{R}^n \to \mathbb{C}$ a (q,r)-atom provided there exists a cube $Q \subset \mathbb{R}^n$ such that the following localization, normalization, and cancellation properties hold:

$$\operatorname{supp} a \subseteq Q, \quad \|a\|_{L^r(\mathbb{R}^n)} \le |Q|^{\frac{1}{r}-\frac{1}{q}}, \quad \text{and} \quad \int_{\mathbb{R}^n} x^\alpha a(x)\,dx = 0 \tag{7-3}$$

for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le n(1/q-1)$. Then, given $q \in (0,1]$ and $r \in [1,\infty]$ with $q < r$, any $g \in H^q(\mathbb{R}^n)$ may be written as $g = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $H^q(\mathbb{R}^n)$ for a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfying $\big(\sum_{j\in\mathbb{N}} |\lambda_j|^q\big)^{1/q} \approx \|g\|_{H^q(\mathbb{R}^n)}$ and with each a_j a (q,r)-atom. In particular, this implies that if for each $q \in (0,1]$ and $r \in [1,\infty]$ with $q < r$ we let $H^{q,r}_{\text{fin}}(\mathbb{R}^n)$ stand for the vector space consisting of all finite linear combinations of (q,r)-atoms, then

$$\begin{gathered} H^{q,r}_{\text{fin}}(\mathbb{R}^n) = \Big\{ f \in L^r_{\text{comp}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0 \text{ if } |\alpha| \le n\Big(\frac{1}{q}-1\Big)\Big\}, \\ H^{q,r}_{\text{fin}}(\mathbb{R}^n) \subset H^q(\mathbb{R}^n) \text{ densely}, \quad \text{and} \quad H^{s,r}_{\text{fin}}(\mathbb{R}^n) \subseteq H^{q,r}_{\text{fin}}(\mathbb{R}^n) \text{ if } 0 < s \le q. \end{gathered} \tag{7-4}$$

It turns out that if a given distribution $g \in H^q(\mathbb{R}^n)$ with $0 < q \le 1$ additionally belongs to a Lebesgue space, or another Hardy space, then one may perform an atomic decomposition which converges to g simultaneously in all the said spaces. This is made precise in the lemma below.

Lemma 7.1. *Suppose $0 < p < \infty$, $0 < q \le 1$, $r \in (1,\infty)$ with $r \ge p$, and $0 < s \le \min\{p,q\}$ are given. Then for any $g \in H^q(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ one can find a sequence $\{g_N\}_{N\in\mathbb{N}} \subset H^{s,r}_{\text{fin}}(\mathbb{R}^n)$ which converges to g both in $H^q(\mathbb{R}^n)$ and in $H^p(\mathbb{R}^n)$.*

Proof. Following the suggestion in [Pipher and Verchota 1992, p. 948] (where the treatment in the case $p = 2$ and $q = 1$ is outlined), we revisit the technology used to perform atomic decompositions of distributions in $H^q(\mathbb{R}^n)$ presented in [Torchinsky 1986, pp. 345-348], from which we borrow notation and results; see also the proof of [García-Cuerva and Rubio de Francia 1985, Theorem 4.6, pp. 278-282]. The starting point is the consideration of a function ψ as in [Torchinsky 1986, Lemma 1.7, p. 345]. Among other things,

$$\psi \in \mathscr{C}^\infty_0(\mathbb{R}^n), \qquad \int_{\mathbb{R}^n} x^\alpha \psi(x)\,dx = 0 \quad \text{if } |\alpha| \le n\Big(\frac{1}{s}-1\Big) \text{ and } \psi \text{ is radial.} \tag{7-5}$$

The latter condition implies that $\hat{\psi}$, the Fourier transform of ψ, normalized as in [Mitrea 2013], is also radial. Hence, there exists a real-valued function $\tilde{\psi}$ defined on $[0,\infty)$ such that $\hat{\psi}(x) = \tilde{\psi}(|x|)$ for each $x \in \mathbb{R}^n$. Note that $\tilde{\psi}$ necessarily satisfies

$$\tilde{\psi} \in \mathscr{C}^\infty([0,\infty)), \quad \tilde{\psi}(0) = 0, \quad \text{and} \quad \tilde{\psi} \text{ has rapid decay at infinity.} \tag{7-6}$$

For each $t > 0$ define $\psi_t(x) := t^{-n}\psi(x/t)$ for every $x \in \mathbb{R}^n$.

Fix now an arbitrary distribution $g \in H^q(\mathbb{R}^n)$. From [Torchinsky 1986, Proposition 1.9, p. 346] and the formula at the bottom of page 347 in that paper we know that there exists

$$\begin{gathered} \text{a partition } \{T_{j,k}\}_{j,k} \text{ of } \mathbb{R}^{n+1}_+ \text{ consisting of pairwise disjoint} \\ \text{measurable sets which depend only on } g \end{gathered} \tag{7-7}$$

such that, if P^{Δ} is the Poisson kernel for the Laplacian in $\mathbb{R}^{n+1}$ (see (4-80) with n replaced by $n+1$) and $P_t^{\Delta}(x) := t^{-n} P^{\Delta}(x/t)$ for each $x \in \mathbb{R}^n$ and $t > 0$, then the following properties hold:

(a) For each j, k, the function

$$a_{j,k}(x) := \int_{T_{j,k}} \partial_t (P_t^{\Delta} * g)(y) \psi_t(y-x)\, dy\, dt, \quad x \in \mathbb{R}^n, \tag{7-8}$$

is a multiple of an (s, r)-atom.

(b) Moreover, each $a_{j,k}$ is also a multiple of an (q, r)-atom, and if we write

$$a_{j,k} = \lambda_{j,k} \tilde{a}_{j,k} \text{ for some } \lambda_{j,k} \in \mathbb{C} \text{ and } \tilde{a}_{j,k} \text{ a genuine } (q,r)\text{-atom}, \tag{7-9}$$

then there exists a constant $C > 0$, independent of g, with the property that

$$\Big(\sum_{j,k} |\lambda_{j,k}|^q \Big)^{\frac{1}{q}} \leq C \|g\|_{H^q(\mathbb{R}^n)}. \tag{7-10}$$

(c) One has

$$g = \sum_{j,k} a_{j,k} \quad \text{in } H^q(\mathbb{R}^n). \tag{7-11}$$

If we now set

$$g_N := \sum_{j+k \leq N} a_{j,k} \quad \text{for each } N \in \mathbb{N}, \tag{7-12}$$

then each g_N belongs to $H^{s,r}_{\text{fin}}(\mathbb{R}^n) \subset H^{q,r}_{\text{fin}}(\mathbb{R}^n)$, and (7-11) implies

$$\lim_{N\to\infty} g_N = g \quad \text{in } H^q(\mathbb{R}^n). \tag{7-13}$$

Next, if $0 < p \leq 1$ and $g \in H^q(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$, then running the same argument as in (7-7)–(7-13) (in which we now view g as a distribution in $H^p(\mathbb{R}^n)$) leads to the conclusion that the sequence $\{g_N\}_{N\in\mathbb{N}} \subset H^{s,r}_{\text{fin}}(\mathbb{R}^n)$ constructed earlier in (7-12) also satisfies

$$\lim_{N\to\infty} g_N = g \quad \text{in } H^p(\mathbb{R}^n). \tag{7-14}$$

The lemma is therefore established in the case when $p \in (0, 1]$.

Henceforth, consider the case when $1 < p < \infty$, i.e., assume $g \in H^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$; see (7-2). The goal is to show that, with g_N as in (7-12), we also have

$$\lim_{N\to\infty} g_N = g \quad \text{in } L^p(\mathbb{R}^n). \tag{7-15}$$

This requires some preparation. Since the radial maximal function of g is pointwise dominated by a multiple of the Hardy–Littlewood maximal function of g, see, e.g., [Stein 1993, (16), p. 57], it follows that $\mathcal{M}_{\text{rad}}\, g \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Given that in the current case $q \leq 1 < p$, this forces $\mathcal{M}_{\text{rad}}\, g \in L^1(\mathbb{R}^n)$; hence $g \in H^1(\mathbb{R}^n)$. With this in hand, the same reasoning that has led to (7-13) now gives $\lim_{N\to\infty} g_N = g$

in $H^1(\mathbb{R}^n)$. This further implies $\lim_{N\to\infty} g_N = g$ in $L^1(\mathbb{R}^n)$; hence also (by eventually restricting the index N to a subsequence of $\mathbb{N}$)

$$\lim_{N\to\infty} g_N(x) = g(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{7-16}$$

Consequently, if we set

$$D_N := \bigcup_{j+k\le N} T_{j,k} \quad \text{for each } N \in \mathbb{N}, \tag{7-17}$$

then for each $M, N \in \mathbb{N}$ with $N < M$ we have

$$g_M(x) - g_N(x) = \int_{D_M\setminus D_N} \partial_t(P_t^\Delta * g)(y)\psi_t(y-x)\,dy\,dt, \quad x \in \mathbb{R}^n. \tag{7-18}$$

Hence, if p' is such that $1/p + 1/p' = 1$, for each function $h \in L^{p'}(\mathbb{R}^n)$ and $M, N \in \mathbb{N}$ such that $N < M$ we may write

$$\int_{\mathbb{R}^n} (g_M - g_N)(x)h(x)\,dx = \int_{D_M\setminus D_N} \partial_t(P_t^\Delta * g)(y)(\psi_t * h)(y)\,dy\,dt. \tag{7-19}$$

Next, define

$$\begin{aligned} G(y,t) := t\,\partial_t(P_t^\Delta * g)(y), \quad F(y,t) := (\psi_t * h)(y), \\ \text{and} \quad G_N(y,t) := \mathbf{1}_{D_N}(y,t)\cdot G(y,t) \end{aligned} \tag{7-20}$$

for each $(y,t) \in \mathbb{R}^{n+1}_+$ and $N \in \mathbb{N}$. With the Lusin area function $\mathcal{A}$ defined as in (4-64) (with n replaced by $n+1$), from (7-19), Lemma 4.11 (used with n replaced by $n+1$), and Hölder's inequality we see that

$$\left|\int_{\mathbb{R}^n} (g_M - g_N)(x)h(x)\,dx\right| \le C\|\mathcal{A}F\|_{L^{p'}(\mathbb{R}^n)}\|\mathcal{A}(G_M - G_N)\|_{L^p(\mathbb{R}^n)}. \tag{7-21}$$

We claim that there exists a finite constant $C > 0$, independent of h, such that

$$\|\mathcal{A}F\|_{L^{p'}(\mathbb{R}^n)} \le C\|h\|_{L^{p'}(\mathbb{R}^n)}, \tag{7-22}$$

and that

$$\mathcal{A}(G - G_N) \to 0 \quad \text{in } L^p(\mathbb{R}^n) \text{ as } N \to \infty. \tag{7-23}$$

Granted these, we may then conclude from (7-21) that

$$\begin{aligned} \|g_M - g_N\|_{L^p(\mathbb{R}^n)} &= \sup_{h\in L^{p'}(\mathbb{R}^n),\,\|h\|_{L^{p'}(\mathbb{R}^n)}\le 1} \left|\int_{\mathbb{R}^n} (g_M - g_N)(x)h(x)\,dx\right| \\ &\le C\|\mathcal{A}(G_M - G_N)\|_{L^p(\mathbb{R}^n)} \\ &\le C\|\mathcal{A}(G_M - G)\|_{L^p(\mathbb{R}^n)} + C\|\mathcal{A}(G_N - G)\|_{L^p(\mathbb{R}^n)} \to 0 \quad \text{as } M, N \to \infty; \end{aligned} \tag{7-24}$$

thus, $\{g_N\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\mathbb{R}^n)$. The latter combined with (7-16) yields (7-15).

Turning our attention to (7-22) we first observe that

$$\|\mathcal{A}F\|_{L^{p'}(\mathbb{R}^n)} = C\|S_\Theta h\|_{L^{p'}(\mathbb{R}^n)}, \tag{7-25}$$

where S_Θ is as in (3-42) (with n replaced by $n+1$) corresponding to

$$(\Theta h)(y,t) := \int_{\mathbb{R}^n} \psi_t(y-z)h(z)\,dz \quad \text{for all } (y,t) \in \mathbb{R}^{n+1}_+. \tag{7-26}$$

Since the kernel $\theta(y,t;z) := \psi_t(y-z)$ of the operator Θ satisfies (with $\varepsilon = 1$ and n replaced by $n+1$) (3-16), (3-17), and (3-41), the hypotheses of Proposition 3.3 are satisfied, and (3-44) gives that $\|S_\Theta h\|_{L^{p'}(\mathbb{R}^n)} \le C\|h\|_{L^{p'}(\mathbb{R}^n)}$. The estimate claimed in (7-22) now follows from this and (7-25).

Finally, consider the claim made in (7-23). For starters, observe that

$$0 \le \mathcal{A}G_N \le \mathcal{A}G \quad \text{in } \mathbb{R}^n, \ \text{ for each } N \in \mathbb{N}. \tag{7-27}$$

Also,

$$\|\mathcal{A}G\|_{L^p(\mathbb{R}^n)} = \|S_\Theta g\|_{L^p(\mathbb{R}^n)}, \tag{7-28}$$

where now the operator Θ is taken to be

$$(\Theta g)(y,t) := \int_{\mathbb{R}^n} t\partial_t(P_t^\Delta(y-z))g(z)\,dz \quad \text{for all } (y,t) \in \mathbb{R}^{n+1}_+. \tag{7-29}$$

Since its kernel $\theta(y,t;z) := t\partial_t(P_t^\Delta(y-z))$ once again satisfies (with $\varepsilon = 1$ and n replaced by $n+1$) (3-16), (3-17), and (3-41), Proposition 3.3 applies and (3-44) guarantees that $\|S_\Theta g\|_{L^p(\mathbb{R}^n)} \le C\|g\|_{L^p(\mathbb{R}^n)}$. Together with (7-28), this shows that

$$\mathcal{A}G \in L^p(\mathbb{R}^n). \tag{7-30}$$

In particular, there exists a Lebesgue-measurable set $E \subseteq \mathbb{R}^n$ satisfying

$$\mathscr{L}^n(E) = 0 \quad \text{and} \quad (\mathcal{A}G)(x) < +\infty \text{ for each } x \in \mathbb{R}^n \setminus E. \tag{7-31}$$

For each fixed $x \in \mathbb{R}^n \setminus E$, we have

$$(\mathcal{A}(G-G_N))(x) = \left(\int_{\Gamma_\kappa(x)} \mathbf{1}_{\mathbb{R}^{n+1}_+ \setminus D_N}(y,t)|G(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{7-32}$$

and the fact that $(\mathcal{A}G)(x) < +\infty$ implies that

$$0 \le \mathbf{1}_{\mathbb{R}^{n+1}_+ \setminus D_N}|G| \le |G| \in L^2\left(\Gamma_\kappa(x), \frac{dy\,dt}{t^{n+1}}\right). \tag{7-33}$$

Since, clearly, $\mathbf{1}_{\mathbb{R}^{n+1}_+ \setminus D_N}|G|$ converges pointwise to zero as $N \to \infty$, Lebesgue's dominated convergence theorem applies and gives that $(\mathcal{A}(G-G_N))(x) \to 0$ as $N \to \infty$. With this in hand, one more application of Lebesgue's dominated convergence theorem (bearing in mind (7-30), (7-27), and the fact that $\mathscr{L}^n(E) = 0$) proves (7-23). This completes the proof of Lemma 7.1. □

Having disposed of Lemma 7.1, we now proceed to show that the $\widetilde{\text{BMO}}$-H^1 duality pairing is compatible with integral pairing for dual Lebesgue spaces, as made precise in the next lemma. As a preamble, we recall the specific nature of the duality pairing $\langle\cdot,\cdot\rangle$ between $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. Concretely, [Stein 1993, Theorem 1, p. 142] gives that for each $r\in(1,\infty]$

$$\langle [f],g\rangle=\int_{\mathbb{R}^n} fg\,d\mathscr{L}^n \quad \text{for all } f\in \mathrm{BMO}(\mathbb{R}^n),\ \text{ for all } g\in H^{1,r}_{\mathrm{fin}}(\mathbb{R}^n), \tag{7-34}$$

which further implies that whenever $f\in\mathrm{BMO}(\mathbb{R}^n)$, $g\in H^1(\mathbb{R}^n)$, and $\{g_N\}_{N\in\mathbb{N}}\subseteq H^{1,r}_{\mathrm{fin}}(\mathbb{R}^n)$ is such that $\lim_{N\to\infty} g_N=g$ in $H^1(\mathbb{R}^n)$, then

$$\lim_{N\to\infty}\int_{\mathbb{R}^n} fg_N\,d\mathscr{L}^n \text{ exists and equals } \langle [f],g\rangle. \tag{7-35}$$

As a consequence, whenever $f\in\mathrm{BMO}(\mathbb{R}^n)$, and $g\in H^1(\mathbb{R}^n)$ may be written as $g=\sum_{j\in\mathbb{N}}\lambda_j a_j$ in $H^1(\mathbb{R}^n)$ for a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfying $\sum_{j\in\mathbb{N}}|\lambda_j|<\infty$ and with each a_j a $(1,r)$-atom, we may write

$$\langle [f],g\rangle=\sum_{j=1}^{\infty}\lambda_j\int_{\mathbb{R}^n} fa_j\,d\mathscr{L}^n. \tag{7-36}$$

Lemma 7.2. *Consider $f\in\mathrm{BMO}(\mathbb{R}^n)\cap L^{p'}(\mathbb{R}^n)$ and $g\in H^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, where $p,p'\in(1,\infty)$ are such that $1/p+1/p'=1$. Then, with $\langle\cdot,\cdot\rangle$ denoting the $\widetilde{\mathrm{BMO}}$-H^1 duality bracket, one has*

$$\langle [f],g\rangle=\int_{\mathbb{R}^n} fg\,d\mathscr{L}^n. \tag{7-37}$$

Proof. Having picked $r\in[p,\infty)$, Lemma 7.1 guarantees the existence of a sequence $\{g_N\}_{N\in\mathbb{N}}\subseteq H^{1,r}_{\mathrm{fin}}(\mathbb{R}^n)$ such that $\lim_{N\to\infty} g_N=g$ both in $H^1(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$. Then, thanks to (7-35) and the L^p-$L^{p'}$ duality, we have

$$\langle [f],g\rangle=\lim_{N\to\infty}\int_{\mathbb{R}^n} fg_N\,d\mathscr{L}^n=\int_{\mathbb{R}^n} fg\,d\mathscr{L}^n, \tag{7-38}$$

which establishes (7-37). □

Recall from [García-Cuerva and Rubio de Francia 1985, Theorem 5.30, p. 307] that

$$(H^q(\mathbb{R}^n))^*=\dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim,\qquad \frac{n}{n+1}<q<1,\qquad \eta=n\Big(\frac{1}{q}-1\Big)\in(0,1). \tag{7-39}$$

The manner in which the Hölder–Hardy duality is understood in (7-39) is similar to (7-34)–(7-35). Specifically, with $(\cdot,\cdot)$ denoting the said Hölder–Hardy duality bracket, q,η as in (7-39), and with $r\in(1,\infty]$ fixed, we have

$$([f],g)=\int_{\mathbb{R}^n} fg\,d\mathscr{L}^n \quad \text{for all } f\in\dot{\mathscr{C}}^\eta(\mathbb{R}^n),\ \text{ for all } g\in H^{q,r}_{\mathrm{fin}}(\mathbb{R}^n). \tag{7-40}$$

This further implies that whenever $f\in\dot{\mathscr{C}}^\eta(\mathbb{R}^n)$, $g\in H^q(\mathbb{R}^n)$, and $\{g_N\}_{N\in\mathbb{N}}\subseteq H^{q,r}_{\mathrm{fin}}(\mathbb{R}^n)$ is such that $\lim_{N\to\infty} g_N=g$ in $H^q(\mathbb{R}^n)$, then

$$\lim_{N\to\infty}\int_{\mathbb{R}^n} fg_N\,d\mathscr{L}^n \text{ exists and equals } ([f],g). \tag{7-41}$$

In particular, whenever $f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n)$, and $g \in H^q(\mathbb{R}^n)$ may be written as $g = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $H^q(\mathbb{R}^n)$ for a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfying $\sum_{j\in\mathbb{N}} |\lambda_j|^q < \infty$ and with each a_j a (q, r)-atom, we have

$$([f], g) = \sum_{j=1}^{\infty} \lambda_j \int_{\mathbb{R}^n} f a_j \, d\mathscr{L}^n. \tag{7-42}$$

In a parallel fashion to Lemma 7.2 we have the following compatibility result.

Lemma 7.3. *Suppose* $f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$ *and* $g \in H^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, *where* $p, p' \in (1, \infty)$ *are such that* $1/p + 1/p' = 1$, *while* $q \in (n/(n+1), 1)$ *and* $\eta = n(1/q - 1) \in (0, 1)$. *Then, with* $(\cdot\,,\cdot)$ *denoting the* $\dot{\mathscr{C}}^\eta/_\sim$-$H^q$ *duality bracket, there holds*

$$([f], g) = \int_{\mathbb{R}^n} f g \, d\mathscr{L}^n. \tag{7-43}$$

Proof. Choose some $r \in [p, \infty)$. From Lemma 7.1 we then know that there exists a sequence $\{g_N\}_{N\in\mathbb{N}} \subseteq H^{q,r}_{\text{fin}}(\mathbb{R}^n)$ such that $\lim_{N\to\infty} g_N = g$ both in $H^q(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$. By virtue of (7-41) and the L^p-$L^{p'}$ duality we may then write

$$([f], g) = \lim_{N\to\infty} \int_{\mathbb{R}^n} f g_N \, d\mathscr{L}^n = \int_{\mathbb{R}^n} f g \, d\mathscr{L}^n, \tag{7-44}$$

which proves (7-43). □

There is another compatibility result, discussed in the next lemma, which is going to be relevant for us shortly.

Lemma 7.4. *Suppose* $f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$ *and* $g \in H^q(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ *where* $q \in (n/(n+1), 1)$ *and* $\eta \in (0, 1)$ *are related via* $\eta = n(1/q - 1)$. *Then, with* $(\cdot\,,\cdot)$ *and* $\langle\cdot\,,\cdot\rangle$ *denoting, respectively, the* $\dot{\mathscr{C}}^\eta/_\sim$-$H^q$ *and* $\widetilde{\mathrm{BMO}}$-H^1 *duality brackets, there holds*

$$([f], g) = \langle [f], g \rangle. \tag{7-45}$$

Proof. Fix some $r \in (1, \infty)$ and once again invoke Lemma 7.1 to produce a sequence $\{g_N\}_{N\in\mathbb{N}} \subseteq H^{q,r}_{\text{fin}}(\mathbb{R}^n)$ such that $\lim_{N\to\infty} g_N = g$ both in $H^1(\mathbb{R}^n)$ and in $H^q(\mathbb{R}^n)$. Then

$$([f], g) = \lim_{N\to\infty} \int_{\mathbb{R}^n} f g_N \, d\mathscr{L}^n = \langle [f], g \rangle, \tag{7-46}$$

where the first equality is provided by (7-41) and the second equality is given by (7-35). □

Finally, we record a compatibility result for the Hölder–Hardy duality bracket considered for two choices of the parameters involved in the definitions of these spaces.

Lemma 7.5. *Assume* $f \in \dot{\mathscr{C}}^{\eta_1}(\mathbb{R}^n) \cap \dot{\mathscr{C}}^{\eta_2}(\mathbb{R}^n)$ *and* $g \in H^{q_1}(\mathbb{R}^n) \cap H^{q_2}(\mathbb{R}^n)$, *where* $q_j \in (n/(n+1), 1)$ *and* $\eta_j \in (0, 1)$ *are related via* $\eta_j = n(1/q_j - 1)$ *for* $j = 1, 2$. *Then, if for each* $j = 1, 2$ *one denotes by* $(\cdot\,,\cdot)_j$ *the* $\dot{\mathscr{C}}^{\eta_j}/_\sim$-$H^{q_j}$ *duality bracket, there holds*

$$([f], g)_1 = ([f], g)_2. \tag{7-47}$$

Proof. Pick some $r \in (1, \infty)$ and introduce $q := \min\{q_1, q_2\}$. By once more invoking Lemma 7.1, we can find a sequence $\{g_N\}_{N\in\mathbb{N}} \subseteq H^{q,r}_{\mathrm{fin}}(\mathbb{R}^n)$ such that $\lim_{N\to\infty} g_N = g$ both in $H^{q_1}(\mathbb{R}^n)$ and in $H^{q_2}(\mathbb{R}^n)$. Bearing in mind that each g_N belongs to both $H^{q_1,r}_{\mathrm{fin}}(\mathbb{R}^n)$ and $H^{q_2,r}_{\mathrm{fin}}(\mathbb{R}^n)$, see (7-4), we may write

$$([f], g)_1 = \lim_{N\to\infty} \int_{\mathbb{R}^n} f g_N \, d\mathscr{L}^n = ([f], g)_2, \tag{7-48}$$

where both equalities are implied by (7-40). □

In the proposition below we elaborate on a standard duality procedure according to which one associates a certain bounded mapping on BMO with any given Calderón–Zygmund operator which annihilates constants; see, e.g., [Meyer 1990, Corollaire, p. 239; Stein 1993, p. 156; Fefferman and Stein 1972, Corollary 2, p. 151]. The goal is to prove that the mappings induced by such a Calderón–Zygmund operator on a variety of spaces (Lebesgue, Hardy, BMO, Hölder) are all compatible with one another, and to provide norm estimates in cases of interest. To state this result in precise terms, recall that the class $\mathrm{SCZ}(n, \gamma)$ was introduced in Definition 1.12.

Proposition 7.6. *Fix $n \in \mathbb{N}$, $\gamma \in (0, 1]$, and let $T \in \mathrm{SCZ}(n, \gamma)$ satisfy $T(1) = 0$. Then the following statements are true.*

(i) *For each $p \in [2, \infty)$ the operator T, originally considered on $L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, extends uniquely to a linear and bounded mapping*

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n). \tag{7-49}$$

Moreover, the operators defined as above for any two arbitrary choices of p in $[2, \infty)$ act in a compatible fashion with one another.

(ii) *For each $p' \in (1, 2]$ the operator $T^\top$, originally considered on $L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, extends uniquely to a linear and bounded mapping*

$$T^\top : L^{p'}(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n). \tag{7-50}$$

Moreover, the operators defined as above for any two arbitrary choices of p' in $(1, 2]$ act in a compatible fashion with one another, and whenever $p \in [2, \infty)$ and $p' \in (1, 2]$ are such that $1/p + 1/p' = 1$, the transpose of (7-49) *is precisely* (7-50).

(iii) *The operator* (7-50) *further extends uniquely to a well-defined, linear and bounded mapping in the context of Hardy spaces. Specifically, whenever $n/(n + \gamma) < q \le 1$, there exists a unique linear and bounded operator*

$$T^\top : H^q(\mathbb{R}^n) \to H^q(\mathbb{R}^n), \tag{7-51}$$

which acts in a compatible fashion with (7-50). *Moreover, the operators in* (7-51), *considered for two arbitrary choices of q, are compatible with one another. Also, for each $p \in [2, \infty)$ there exist $\theta \in (0, 1)$ and $c \in (0, \infty)$ depending only on n, γ, q, p such that, with C'' as in* (1-77),

$$\|T^\top\|_{\mathscr{B}(H^q(\mathbb{R}^n))} \le c \|T\|^{1-\theta}_{\mathscr{B}(L^p(\mathbb{R}^n))} \big(C'' + \|T\|_{\mathscr{B}(L^p(\mathbb{R}^n))}\big)^\theta. \tag{7-52}$$

(iv) *The operator*

$$\widetilde{T} : \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \to \widetilde{\mathrm{BMO}}(\mathbb{R}^n), \tag{7-53}$$

defined by setting (*with* $\langle \cdot , \cdot \rangle$ *denoting the* $\widetilde{\mathrm{BMO}}$-H^1 *duality pairing*)

$$\langle \widetilde{T}[f], g\rangle := \langle [f], T^\top g\rangle \quad \textit{for all } [f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n), \ \textit{for all } g \in H^1(\mathbb{R}^n), \tag{7-54}$$

is well-defined, *linear*, *bounded*, *and compatible with* (7-49) *in the sense that for each* $p \in [2, \infty)$ *one has*

$$\widetilde{T}[f] = [Tf] \quad \textit{for all } f \in \mathrm{BMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n). \tag{7-55}$$

Moreover, *for each* $p \in [2, \infty)$ *there exist* $\theta \in (0, 1)$ *and* $c \in (0, \infty)$ *depending only on* n, γ, p *such that*, *with* C'' *as in* (1-77),

$$\|\widetilde{T}\|_{\mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))} \le c\|T\|^{1-\theta}_{\mathscr{B}(L^p(\mathbb{R}^n))}(C'' + \|T\|_{\mathscr{B}(L^p(\mathbb{R}^n))})^\theta. \tag{7-56}$$

(v) *Given any* $\eta \in (0, \gamma)$, *the operator*

$$\widehat{T} : \dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim \ \to \dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim, \tag{7-57}$$

defined by setting, *with* $q := n/(n+\eta) \in (n/(n+\gamma), 1)$ *and* $(\cdot , \cdot)$ *denoting the* $\dot{\mathscr{C}}^\eta/\sim$-$H^q$ *duality pairing*,

$$(\widehat{T}[f], g) := ([f], T^\top g) \quad \textit{for all } [f] \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n)/\sim, \ \textit{for all } g \in H^q(\mathbb{R}^n), \tag{7-58}$$

is well-defined, *linear*, *bounded*, *and compatible with* (7-49) *and* (7-53), *in the sense that for each* $p \in [2, \infty)$ *one has*

$$\widehat{T}[f] = [Tf] \quad \textit{for all } f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \tag{7-59}$$

$$\widehat{T}[f] = \widetilde{T}[f] \quad \textit{for all } f \in \dot{\mathscr{C}}^\eta(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n). \tag{7-60}$$

In addition, *the operators in* (7-57), *considered for two arbitrary choices of* η, *are also compatible with one another.*

Of course, if actually $T \in \mathrm{CZ}(n, \gamma)$ then we may take $p, p' \in (1, \infty)$ arbitrary (retaining condition $1/p + 1/p' = 1$ in the second part of item (ii) though) throughout the statement of Proposition 7.6.

Proof of Proposition 7.6. Working with $T^\top$ which, by design, is a bounded operator on $L^2(\mathbb{R}^n)$ and whose kernel $K^\top \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \mathrm{diag})$ has the property that there exist $C'_K, C''_K \in (0, \infty)$ such that, for every $x, y \in \mathbb{R}^n$ with $x \neq y$ and each $z \in \mathbb{R}^n$ with $|x - z| < \frac{1}{2}|x - y|$,

$$|K^\top(x, y)| \le \frac{C'_K}{|x - y|^n} \quad \text{and} \quad |K^\top(y, x) - K^\top(y, z)| \le C''_K \frac{|x - z|^\gamma}{|x - y|^{n+\gamma}}, \tag{7-61}$$

and relying on the Calderón–Zygmund lemma in the usual fashion, it follows that $T^\top$ induces a well-defined linear and bounded mapping

$$T^\top : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n). \tag{7-62}$$

Hence, via Marcinkiewicz's interpolation theorem, we conclude that $T^\top : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ has a unique extension to a linear and bounded operator from $L^{p'}(\mathbb{R}^n)$ into itself for each $p' \in (1, 2]$. From

[Rudin 1987, Theorem 1.17, p.15] it follows that

$$\begin{aligned}&\text{given } p_1, p_2 \in (1,\infty) \text{ and } f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n), \text{ there exists a}\\&\text{sequence } \{s_j\}_{j\in\mathbb{N}} \text{ of simple functions in } \mathbb{R}^n \text{ which converges}\\&\text{to } f \text{ simultaneously in } L^{p_1}(\mathbb{R}^n) \text{ and in } L^{p_2}(\mathbb{R}^n).\end{aligned} \tag{7-63}$$

In turn, this readily implies that the operators in (7-50), considered for any two arbitrary choices of p' in $(1,2]$, act in a compatible fashion with one another. Consider next $p \in [2,\infty)$ such that $1/p + 1/p' = 1$. Since for each $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we may estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^n} (Tf)g\, d\mathscr{L}^n\right| = \left|\int_{\mathbb{R}^n} f(T^\top g)\, d\mathscr{L}^n\right| &\le \|f\|_{L^p(\mathbb{R}^n)} \|T^\top g\|_{L^{p'}(\mathbb{R}^n)}\\&\le C\|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)},\end{aligned} \tag{7-64}$$

and since, generally speaking,

$$\text{if } h \in L^2(\mathbb{R}^n) \text{ then } \|h\|_{L^p(\mathbb{R}^n)} = \sup_{\substack{g\in L^{p'}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\\ \|g\|_{L^{p'}(\mathbb{R}^n)}\le 1}} \left|\int_{\mathbb{R}^n} hg\, d\mathscr{L}^n\right|, \tag{7-65}$$

we conclude that there exists $C \in (0,\infty)$ such that $\|Tf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$ for every function $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By density it follows that T, originally considered on $L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, extends uniquely to a linear and bounded mapping as in (7-49). By once again appealing to (7-63) we see that the operators in (7-49), considered for any two arbitrary choices of p in $[2,\infty)$, act in a compatible fashion with one another. Finally, granted the continuity properties established above, the identity

$$\int_{\mathbb{R}^n} (Tf)g\, d\mathscr{L}^n = \int_{\mathbb{R}^n} f(T^\top g)\, d\mathscr{L}^n, \quad f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ g \in L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \tag{7-66}$$

further extends by density to

$$\int_{\mathbb{R}^n} (Tf)g\, d\mathscr{L}^n = \int_{\mathbb{R}^n} f(T^\top g)\, d\mathscr{L}^n \quad \text{for all } f \in L^p(\mathbb{R}^n),\ \text{for all } g \in L^{p'}(\mathbb{R}^n), \tag{7-67}$$

where T is as in (7-49) and $T^\top$ is as in (7-50). This finishes the proofs of the claims in items (i)–(ii).

Consider next the claims made in item (iii). Throughout, fix an exponent $p' \in (1,2]$, set $p := p'/(p'-1) \in [2,\infty)$, take $r \in (p',\infty)$, and pick $q \in (n/(n+\gamma), 1]$ arbitrary. Since these choices give $(n+\gamma)/n - 1/p' > 1/q - 1/p'$, it is possible to select

$$\theta \in (0,1) \text{ such that } (n+\gamma)/n - 1/p' > (1/q - 1/p')/\theta. \tag{7-68}$$

We first claim that

$$\begin{aligned}&\text{for each } (q,r)\text{-atom } a \text{ in } \mathbb{R}^n \text{ we have } T^\top a \in H^q(\mathbb{R}^n) \text{ and}\\&\|T^\top a\|_{H^q(\mathbb{R}^n)} \le C := c\|T\|^{1-\theta}_{\mathscr{B}(L^p(\mathbb{R}^n))}\big(C''_K + \|T\|_{\mathscr{B}(L^p(\mathbb{R}^n))}\big)^\theta,\end{aligned} \tag{7-69}$$

where $c \in (0,\infty)$ depends only on n, γ, q, p, and where C''_K is as in (7-61). To see that this is the case, fix some (q,r)-atom a as in (7-3) and observe that, since $a \in L^{p'}(\mathbb{R}^n)$, the function $m := T^\top a$ is meaningfully defined, see (7-50), and satisfies, thanks to (7-50),

$$\|m\|_{L^{p'}(\mathbb{R}^n)} \le \|T^\top\|_{\mathscr{B}(L^{p'}(\mathbb{R}^n))}\|a\|_{L^{p'}(\mathbb{R}^n)} \le \|T\|_{\mathscr{B}(L^{p}(\mathbb{R}^n))}|Q|^{\frac{1}{p'}-\frac{1}{q}}. \tag{7-70}$$

In addition, the vanishing-moment condition of the atom, in concert with the second estimate for the kernel $K^\top$ of $T^\top$ in (7-61) and the size estimate for the atom, yield the decay property

$$|m(x)| \le \frac{c_n C''_K \ell(Q)^\gamma}{|x-x_Q|^{n+\gamma}}|Q|^{1-\frac{1}{q}} \quad \text{for each } x \in \mathbb{R}^n \setminus (2Q), \tag{7-71}$$

where $c_n \in (0,\infty)$ is a purely dimensional constant and C''_K is as in (7-61). Let us also observe that since any (q,r)-atom is a multiple of some $(1,r)$-atom, we have that $a \in H^1(\mathbb{R}^n)$. Granted this, from (1-78) and the fact that $T(1)=0$ we conclude that, see (1-79),

$$m \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} m(x)\,dx = 0. \tag{7-72}$$

In turn, from the estimates recorded in (7-70)–(7-71) one may readily check that if we now introduce $b := (1/q - 1/p')/\theta \in (1/q-1/p',\infty)$ we have

$$\|m\|^{1-\theta}_{L^{p'}(\mathbb{R}^n)}\big\||\cdot - x_Q|^{nb} m\big\|^{\theta}_{L^{p'}(\mathbb{R}^n\setminus 2Q)} \le c\|T\|^{1-\theta}_{\mathscr{B}(L^p(\mathbb{R}^n))}(C''_K)^\theta, \tag{7-73}$$

$$\|m\|^{1-\theta}_{L^{p'}(\mathbb{R}^n)}\big\||\cdot - x_Q|^{nb} m\big\|^{\theta}_{L^{p'}(2Q)} \le c\|T\|_{\mathscr{B}(L^p(\mathbb{R}^n))}, \tag{7-74}$$

where $c \in (0,\infty)$ depends only on n, γ, q, p, and where C''_K is as in (7-61). In the language of [García-Cuerva and Rubio de Francia 1985, Definition 7.13, p. 328], (7-72)–(7-74) amount to saying that m is a (q,p',b)-molecule centered at x_Q. Having established this, we may invoke [loc. cit., Theorem 7.16, p. 330] to conclude that $m \in H^q(\mathbb{R}^n)$ and $\|m\|_{H^q(\mathbb{R}^n)} \le c\|T\|^{1-\theta}_{\mathscr{B}(L^p(\mathbb{R}^n))}(C''_K + \|T\|_{\mathscr{B}(L^p(\mathbb{R}^n))})^\theta$. This proves (7-69).

We next claim that

for each given $g \in L^{p'}(\mathbb{R}^n) \cap H^q(\mathbb{R}^n)$, the function $T^\top g$, originally regarded in $L^{p'}(\mathbb{R}^n)$ by considering the operator $T^\top$ as in (7-50), actually belongs to $H^q(\mathbb{R}^n)$ and satisfies the estimate $\|T^\top g\|_{H^q(\mathbb{R}^n)} \le C\|g\|_{H^q(\mathbb{R}^n)}$ with C of the same format as in (7-69). (7-75)

With this goal in mind, from (7-12)–(7-14) and items (a)–(c) in the proof of Lemma 7.1 we conclude that there exist a constant $c = c_{n,p,q,r} \in (0,\infty)$, along with (q,r)-atoms $\{a_j\}_{j\in\mathbb{N}}$ and numbers $\{\lambda_j\}_{j\in\mathbb{N}}$, such that

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^q\Big)^{\frac{1}{q}} \le c\|g\|_{H^q(\mathbb{R}^n)}, \tag{7-76}$$

and if

$$g_N := \sum_{j=1}^N \lambda_j a_j \quad \text{for each } N \in \mathbb{N} \tag{7-77}$$

then

$$\lim_{N\to\infty} g_N = g \text{ both in } H^q(\mathbb{R}^n) \text{ and in } L^{p'}(\mathbb{R}^n). \tag{7-78}$$

Note that whenever $N, M \in \mathbb{N}$ are such that $N < M$, we may rely on (7-69) to conclude that

$$T^\top g_N,\, T^\top g_M \in H^q(\mathbb{R}^n) \quad \text{and} \quad \|T^\top g_N - T^\top g_M\|_{H^q(\mathbb{R}^n)} \le C\biggl(\sum_{j=N+1}^{M} |\lambda_j|^q\biggr)^{\frac{1}{q}}. \tag{7-79}$$

Given that $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^q$, this proves that the sequence $\{T^\top g_N\}_{N\in\mathbb{N}}$ is Cauchy in $H^q(\mathbb{R}^n)$. Since the latter is a quasi-Banach space, it follows that there exists some $h \in H^q(\mathbb{R}^n)$ such that $\lim_{N\to\infty} T^\top g_N = h$ in $H^q(\mathbb{R}^n)$. On the other hand, from (7-78) and (7-50) we conclude that $\lim_{N\to\infty} T^\top g_N = T^\top g$ in $L^{p'}(\mathbb{R}^n)$. Hence, necessarily, $T^\top g = h$ as distributions in $\mathbb{R}^n$. This goes to show that $T^\top g \in H^q(\mathbb{R}^n)$, and we may also estimate

$$\begin{aligned} \|T^\top g\|_{H^q(\mathbb{R}^n)} &= \|h\|_{H^q(\mathbb{R}^n)} = \lim_{N\to\infty} \|T^\top g_N\|_{H^q(\mathbb{R}^n)} \\ &\le C \limsup_{N\to\infty} \biggl(\sum_{j=1}^{N} |\lambda_j|^q\biggr)^{\frac{1}{q}} = C\biggl(\sum_{j=1}^{\infty} |\lambda_j|^q\biggr)^{\frac{1}{q}} \le C\|g\|_{H^q(\mathbb{R}^n)}, \end{aligned} \tag{7-80}$$

where the constant C has the same format as in (7-69). Above, the second equality uses the fact that $\|\cdot\|_{H^q(\mathbb{R}^n)}$ is a q-norm which defines the topology on $H^q(\mathbb{R}^n)$, the subsequent inequality is a consequence of (7-77), (7-69), and the subadditivity of $\|\cdot\|^q_{H^q(\mathbb{R}^n)}$, while the last inequality comes from (7-76). This finishes the proof of (7-75).

Moving on, consider now an arbitrary $g \in H^q(\mathbb{R}^n)$. Since $L^{p'}(\mathbb{R}^n) \cap H^q(\mathbb{R}^n)$ is dense in $H^q(\mathbb{R}^n)$, there exists a sequence $\{g_j\}_{j\in\mathbb{N}} \subset L^{p'}(\mathbb{R}^n) \cap H^q(\mathbb{R}^n)$ such that $\lim_{j\to\infty} g_j = g$ in $H^q(\mathbb{R}^n)$. From (7-75) it follows that $\{T^\top g_j\}_{j\in\mathbb{N}}$ is Cauchy in $H^q(\mathbb{R}^n)$. Define $T^\top g$ to be the limit of $\{T^\top g_j\}_{j\in\mathbb{N}}$ in $H^q(\mathbb{R}^n)$. By interlacing sequences, it may shown that the limit defining $T^\top g$ does not depend on the actual choice of the sequence $\{g_j\}_{j\in\mathbb{N}}$. In turn, this implies that $T^\top : H^q(\mathbb{R}^n) \to H^q(\mathbb{R}^n)$ is well-defined, linear, and compatible with the action of $T^\top$ on $L^{p'}(\mathbb{R}^n)$. To see that the operator just defined is also bounded, if g and $\{g_j\}_{j\in\mathbb{N}}$ are as before, write

$$\|T^\top g\|_{H^q(\mathbb{R}^n)} = \lim_{j\to\infty} \|T^\top g_j\|_{H^q(\mathbb{R}^n)} \le C \limsup_{j\to\infty} \|g_j\|_{H^q(\mathbb{R}^n)} = C\|g\|_{H^q(\mathbb{R}^n)}, \tag{7-81}$$

where the constant C has the same format as in (7-69). In (7-81), we have used the definition of $T^\top$ on $H^q(\mathbb{R}^n)$, the fact that $\lim_{j\to\infty} g_j = g$ in $H^q(\mathbb{R}^n)$, the estimate in (7-75), and the fact that $\|\cdot\|_{H^q(\mathbb{R}^n)}$ is a q-norm which defines the topology on $H^q(\mathbb{R}^n)$ (in the first and last equalities in (7-81)).

In summary, for each $q \in (n/(n+\gamma), 1]$, we have succeeded in producing a linear and bounded operator $T^\top : H^q(\mathbb{R}^n) \to H^q(\mathbb{R}^n)$ which acts in a compatible fashion with $T^\top$ in (7-50) and which satisfies the estimate in (7-52). It remains to show that these newly produced operators are also compatible with one another as q varies through $(n/(n+\gamma), 1]$. To this end, fix $q_1, q_2 \in (n/(n+\gamma), 1]$ and consider some arbitrary $g \in H^{q_1}(\mathbb{R}^n) \cap H^{q_2}(\mathbb{R}^n)$. Also, fix $p' \in (1, 2]$, choose $r \in (1, \infty)$ with $r \ge p'$, and set $s := \min\{q_1, q_2\}$. Then Lemma 7.1 ensures that there exists some sequence $\{g_N\}_{N\in\mathbb{N}} \subset H^{s,r}_{\mathrm{fin}}(\mathbb{R}^n) \subset L^{p'}(\mathbb{R}^n)$ which converges to g both in $H^{q_1}(\mathbb{R}^n)$ and in $H^{q_2}(\mathbb{R}^n)$. Then, with $T^\top$ considered in the sense of (7-50), the sequence $\{T^\top g_N\}_{N\in\mathbb{N}}$ converges both in $H^{q_1}(\mathbb{R}^n)$ and in $H^{q_2}(\mathbb{R}^n)$. In light of the manner in which the extension to Hardy spaces has been defined earlier, this shows that the operator $T^\top : H^{q_1}(\mathbb{R}^n) \to H^{q_1}(\mathbb{R}^n)$ acting on g, viewed in $H^{q_1}(\mathbb{R}^n)$, agrees with the operator $T^\top : H^{q_2}(\mathbb{R}^n) \to H^{q_2}(\mathbb{R}^n)$ acting on g now viewed as a distribution in $H^{q_2}(\mathbb{R}^n)$. This concludes the justification of the claims made in item (iii).

Going further, the well-definedness, linearity, and boundedness of $T^\top$ in (7-51), together with Fefferman's basic duality result $(H^1(\mathbb{R}^n))^* = \widetilde{\mathrm{BMO}}(\mathbb{R}^n)$, ensure that $\widetilde{T}$ defined as in (7-54) is a well-defined, linear and bounded operator in the context of (7-53). To prove the compatibility condition described in (7-55), fix some $p \in [2, \infty)$ along with an arbitrary function $f \in \mathrm{BMO}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Then, if $p' \in (1, 2]$ is such that $1/p + 1/p' = 1$, for each function $g \in H^1(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$ we may compute

$$\langle \widetilde{T}[f], g\rangle = \langle [f], T^\top g\rangle = \int_{\mathbb{R}^n} f(T^\top g)\, d\mathscr{L}^n = \int_{\mathbb{R}^n} (Tf)g\, d\mathscr{L}^n. \tag{7-82}$$

Above, the first equality is simply (7-54), the second equality is implied by the fact that $T^\top g \in H^1(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$ (see (7-50), (7-51)) and Lemma 7.2, while the last equality is seen from the fact that the adjoint of (7-49) is (7-50). Let us now select a representative $h \in \mathrm{BMO}(\mathbb{R}^n)$ of the equivalence class $\widetilde{T}[f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n)$, and specialize (7-82) to the case when g is a $(1, r)$-atom for some $r \in (1, \infty)$. On account of (7-34), this yields

$$\int_{\mathbb{R}^n} h a\, d\mathscr{L}^n = \int_{\mathbb{R}^n} (Tf)a\, d\mathscr{L}^n \quad \text{for each } (1, r)\text{-atom } a. \tag{7-83}$$

It is not difficult to see that, generally speaking,

$$\begin{gathered}\text{if } q \in (n/(n+1), 1] \text{ and } r, r' \in [1, \infty] \text{ are such that } 1/r + 1/r' = 1 \text{ and } q < r,\\ \text{then a function } \phi \in L^{r'}_{\mathrm{loc}}(\mathbb{R}^n) \text{ satisfying } \textstyle\int_{\mathbb{R}^n} \phi\, a\, d\mathscr{L}^n = 0 \text{ for}\\ \text{each } (q, r)\text{-atom } a \text{ is necessarily constant in } \mathbb{R}^n.\end{gathered} \tag{7-84}$$

This may be seen by considering scalar multiples of (q, r)-atoms of the form

$$a = \mathbf{1}_{B(x,R)}/\mathscr{L}^n(B(x, R)) - \mathbf{1}_{B(0,1)}/\mathscr{L}^n(B(0, 1)), \tag{7-85}$$

with $x \in \mathbb{R}^n$ and $R > 0$ arbitrary, then letting $R \to 0^+$ and invoking Lebesgue's differentiation theorem. In concert, (7-83) and (7-84) then prove that h and Tf differ by a constant. Hence, $\widetilde{T}[f] = [h] = [Tf]$, finishing the proof of (7-55). Finally, the estimate recorded in (7-56) is obtained by noting that (7-54) and the quantitative aspect of the $\widetilde{\mathrm{BMO}}$-H^1 duality yield

$$\|\widetilde{T}\|_{\mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))} \leq c_n \|T^\top\|_{\mathscr{B}(H^1(\mathbb{R}^n))}, \tag{7-86}$$

and then combining this with (7-52) (used here with $q = 1$).

Moving on, from the well-definedness, linearity, and boundedness of $T^\top$ in (7-51), together with the duality result recorded in (7-39) we conclude that $\widehat{T}$ defined in (7-58) is a well-defined, linear and bounded operator in the context of (7-57). Next, the compatibility condition (7-59) is proved much like (7-55), this time making use of Lemma 7.3 instead of Lemma 7.2.

Consider next the compatibility condition in (7-60). With this in mind, select an arbitrary function $f \in \dot{\mathscr{C}}^{\eta}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$. Then for each $g \in H^1(\mathbb{R}^n) \cap H^q(\mathbb{R}^n)$ we have

$$(\widehat{T}[f], g) = ([f], T^\top g) = \langle [f], T^\top g\rangle = \langle \widetilde{T}[f], g\rangle. \tag{7-87}$$

Here, the first equality is based on (7-58), the second equality takes into account the fact that $T^\top g \in H^1(\mathbb{R}^n) \cap H^q(\mathbb{R}^n)$, see (7-51), and uses Lemma 7.4, whereas the last equality is implied by (7-51) and (7-54). Pick a representative $\tilde{h}$ of $\widetilde{T}[f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ along with a representative $\hat{h}$ of $\widehat{T}[f] \in \dot{\mathscr{C}}^{\eta}(\mathbb{R}^n)/\sim$. If we now fix $r \in (1, \infty)$ and specialize the equality of the most extreme sides of (7-87) to the case when g is an arbitrary (q, r)-atom we arrive at the conclusion that

$$\int_{\mathbb{R}^n} \hat{h} a \, d\mathscr{L}^n = \int_{\mathbb{R}^n} \tilde{h} a \, d\mathscr{L}^n \quad \text{for each } (q, r)\text{-atom } a. \tag{7-88}$$

On account of this and (7-84) we may then conclude that the functions $\hat{h}$ and $\tilde{h}$ differ by a constant, which ultimately goes to show that (7-60) holds.

At this stage, it remains to prove that the operators in (7-57) considered for two arbitrary choices of the smoothness parameter are compatible with one another. To this end, pick arbitrary $f \in \dot{\mathscr{C}}^{\eta_1}(\mathbb{R}^n) \cap \dot{\mathscr{C}}^{\eta_2}(\mathbb{R}^n)$ and $g \in H^{q_1}(\mathbb{R}^n) \cap H^{q_2}(\mathbb{R}^n)$, where $q_j \in (n/(n+1), 1)$ and $\eta_j \in (0, 1)$ are related via $\eta_j = n(1/q_j - 1)$ for $j = 1, 2$. For each $j = 1, 2$, we agree to denote the $\dot{\mathscr{C}}^{\eta_j}/\sim$-$H^{q_j}$ duality bracket by $(\cdot, \cdot)_j$. Then

$$(\widehat{T}[f], g)_1 = ([f], T^\top g)_1 = ([f], T^\top g)_2 = (\widehat{T}[f], g)_2, \tag{7-89}$$

where the first and last equalities are based on (7-58), while the middle equality is a consequence of Lemma 7.5. Specializing the coincidence of the most extreme terms in (7-89) to the case when g is a (q, r)-atom for some $r \in (1, \infty)$ and $q := \min\{q_1, q_2\}$ then yields, on account of (7-40),

$$\int_{\mathbb{R}^n} h_1 a \, d\mathscr{L}^n = \int_{\mathbb{R}^n} h_2 a \, d\mathscr{L}^n \quad \text{for each } (q, r)\text{-atom } a, \tag{7-90}$$

where $h_j \in \dot{\mathscr{C}}^{\eta_j}(\mathbb{R}^n)$ is a representative of $\widehat{T}[f] \in \dot{\mathscr{C}}^{\eta_j}(\mathbb{R}^n)/\sim$ for $j = 1, 2$. At this point we invoke (7-84) to conclude that $h_1 - h_2$ is constant in $\mathbb{R}^n$, from which the very last claim in Proposition 7.6 follows. The proof of Proposition 7.6 is therefore complete. □

Having dealt with Proposition 7.6 we are now ready to present the proof of Theorem 1.13.

Proof of Theorem 1.13. Fix $n \in \mathbb{N}$ along with $\gamma \in (0, 1]$ and suppose $T \in \mathrm{SCZ}(n, \gamma)$. Pick $\eta \in (0, \gamma)$ arbitrary. By Proposition 7.6, the operator T extends to a bounded linear mapping $\widetilde{T}$ from $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ into itself and to a bounded linear mapping $\widehat{T}$ from $\dot{\mathscr{C}}^{\eta}(\mathbb{R}^n)/\sim$ into itself. In addition, these extensions are compatible in the sense of (7-60). From these we deduce that $\widetilde{T}$ maps the linear subspace $X := (\dot{\mathscr{C}}^{\eta}(\mathbb{R}^n)/\sim) \cap \widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ of $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ into X. Since $\widetilde{T}$ is continuous on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$, it follows that $\widetilde{T}$ maps the closure of X in $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ linearly and boundedly into itself. Corollary 1.11 tells us that the said closure is simply $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$, so we ultimately conclude that $\widetilde{T}$ maps $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ linearly and boundedly into itself. Keeping in mind that the action of $\widetilde{T}$ in this setting is compatible with that of the original operator T, see (7-55), the desired conclusion follows. □

Theorem 1.13 is the main ingredient in the proof of Theorem 1.14, discussed next.

Proof of Theorem 1.14. According to [Meyer 1990, §9], see also [Meyer 1985, Theorem 5, p. 231],

$$\mathscr{A}^0_{\mathrm{CZ}} := \bigcup_{0<\gamma\leq 1} \{T \in \mathrm{CZ}(n,\gamma) : T(1) = T^{\top}(1) = 0\} \tag{7-91}$$

is the largest subalgebra of $\mathscr{B}(L^2(\mathbb{R}^n))$ consisting of Calderón–Zygmund operators in $\mathbb{R}^n$. Since $\mathscr{A}^0_{\mathrm{CZ}}$ is invariant under transposition, we conclude from Proposition 7.6 and Theorem 1.13 that $\mathscr{A}^0_{\widetilde{\mathrm{CZ}}}$ is indeed a subalgebra of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$. □

Next, we present the proof of Theorem 1.15 which, once again, makes essential use of Theorem 1.13.

Proof of Theorem 1.15. Proposition 7.6 ensures that each principal-value convolution-type operator T_Θ associated as in (1-82) with a function Θ as in (1-115) induces a well-defined linear and bounded mapping $\widetilde{T}_\Theta$ on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$. From Theorem 1.13 we also know that $\widetilde{T}_\Theta|_{\mathrm{VMO}}$, the restriction of $\widetilde{T}_\Theta$ to $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$, is a well-defined linear and bounded operator from the space $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ into itself. Hence, $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$ defined in (1-116) is a subset of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$. Proving that $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$ is actually a commutative subalgebra of $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$ requires some preparations.

Regarding the relationship between a kernel Θ as in (1-115) and its associated symbol m_Θ as in (1-84), two features are particularly significant for us here. First, from (1-86) we know that

$$\begin{gathered}\text{if } \Theta \text{ is as in (1-115), then } m_\Theta \text{ given by (1-84) is positive}\\ \text{homogeneous of degree zero and of class } \mathscr{C}^\infty \text{ in } \mathbb{R}^n\setminus\{0\}.\end{gathered} \tag{7-92}$$

Second, from [Stein 1970, Theorem 6, p. 75], or [Grafakos 2004, Proposition 2.4.7 on p. 128, and Proposition 4.2.3 on p. 267], it follows that

$$\begin{gathered}\text{given any function } m \in \mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\}) \text{ which is positive homogeneous of degree zero,}\\ \text{there exist some unique function } \Theta \text{ as in (1-115) and some unique number } c\in\mathbb{C}\\ \text{such that } m = c + m_\Theta \ \big(\text{actually } c = \textstyle\fint_{S^{n-1}} m(\omega)\,d\omega \in \mathbb{C}\big).\end{gathered} \tag{7-93}$$

Consider next two functions Θ_1, Θ_2 as in (1-115) and associate with them $m_{\Theta_1}, m_{\Theta_2}$ as in (1-84). Since then their product $m_{\Theta_1}m_{\Theta_2}$ belongs to $\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})$, thanks to (7-92), and is positive homogeneous of degree zero (given that both m_{Θ_1} and m_{Θ_2} are), we may invoke (7-93) to conclude that

$$\begin{gathered}\text{there exists a function } \Theta \text{ as in (1-115) with the property that}\\ m_{\Theta_1}m_{\Theta_2} = c + m_\Theta \text{ in } \mathbb{R}^n\setminus\{0\}, \text{ where } c := \textstyle\fint_{S^{n-1}} m_{\Theta_1}(\omega)m_{\Theta_2}(\omega)\,d\omega.\end{gathered} \tag{7-94}$$

If $\mathcal{F}^{-1}_{\xi\to x}$ denotes the inverse Fourier transform (taking functions in the variable ξ into functions in the variable x), then for each $f \in L^2(\mathbb{R}^n)$ we may write

$$\begin{aligned}(T_{\Theta_1}\circ T_{\Theta_2})f(x) &= \mathcal{F}^{-1}_{\xi\to x}[m_{\Theta_1}(\xi)m_{\Theta_2}(\xi)\hat{f}(\xi)]\\ &= \mathcal{F}^{-1}_{\xi\to x}[(c+m_\Theta(\xi))\hat{f}(\xi)] = cf(x) + (T_\Theta f)(x), \quad x\in\mathbb{R}^n.\end{aligned} \tag{7-95}$$

Hence, $T_{\Theta_1} \circ T_{\Theta_2} = cI + T_\Theta$ as operators from the space $L^2(\mathbb{R}^n)$ into itself. Also,

$$\begin{aligned}(T_{\Theta_1} \circ T_{\Theta_2}) f(x) &= \mathcal{F}^{-1}_{\xi\to x}[m_{\Theta_1}(\xi) m_{\Theta_2}(\xi) \hat{f}(\xi)] \\ &= \mathcal{F}^{-1}_{\xi\to x}[m_{\Theta_2}(\xi) m_{\Theta_1}(\xi) \hat{f}(\xi)] = (T_{\Theta_2} \circ T_{\Theta_1}) f(x), \quad x \in \mathbb{R}^n;\end{aligned} \tag{7-96}$$

thus $T_{\Theta_1} \circ T_{\Theta_2} = T_{\Theta_2} \circ T_{\Theta_1}$ on $L^2(\mathbb{R}^n)$. In turn, given that $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, see (7-4), and since $T_{\Theta_1}, T_{\Theta_2}, T_\Theta$ map $H^1(\mathbb{R}^n)$ into itself boundedly and in a compatible fashion with their action on $L^2(\mathbb{R}^n)$, see Proposition 7.6, we may conclude that

$$\begin{gathered} T_{\Theta_1} \circ T_{\Theta_2} = T_{\Theta_2} \circ T_{\Theta_1} \text{ and } T_{\Theta_1} \circ T_{\Theta_2} = cI + T_\Theta \text{ on } H^1(\mathbb{R}^n), \\ \text{whenever } c,\ \Theta \text{ are related to } \Theta_1,\ \Theta_2 \text{ as in (7-94).} \end{gathered} \tag{7-97}$$

Going further, fix Θ_1, Θ_2, Θ as in (1-115). With $\langle \cdot , \cdot \rangle$ denoting the $\widetilde{\mathrm{BMO}}$-H^1 duality bracket, from Proposition 7.6 and (1-82) it follows that $T_{\Theta_1}, T_{\Theta_2}, T_\Theta$ induce linear and bounded operators $\widetilde{T}_{\Theta_1}, \widetilde{T}_{\Theta_2}, \widetilde{T}_\Theta$ from $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ into itself according to

$$\begin{gathered} \langle \widetilde{T}_{\Theta_j}[f], g\rangle = \langle [f], T_{\widetilde{\Theta}_j} g\rangle \text{ for all } f \in \mathrm{BMO}(\mathbb{R}^n), \text{ for all } g \in H^1(\mathbb{R}^n), \text{ for all } j \in \{1,2\}, \\ \text{and } \langle \widetilde{T}_\Theta[f], g\rangle = \langle [f], T_{\widetilde{\Theta}} g\rangle \text{ for all } f \in \mathrm{BMO}(\mathbb{R}^n), \text{ for all } g \in H^1(\mathbb{R}^n), \end{gathered} \tag{7-98}$$

where $\widetilde{\Theta}_j(x) := \Theta_j(-x)$ for $j \in \{1,2\}$, and $\widetilde{\Theta}(x) := \Theta(-x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$. Retaining the symbol I for the identity operator on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$, we claim that these extensions satisfy

$$\begin{gathered} \widetilde{T}_{\Theta_1} \circ \widetilde{T}_{\Theta_2} = \widetilde{T}_{\Theta_2} \circ \widetilde{T}_{\Theta_1} \text{ and } \widetilde{T}_{\Theta_1} \circ \widetilde{T}_{\Theta_2} = cI + \widetilde{T}_\Theta \text{ on } \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \\ \text{provided } m_{\widetilde{\Theta}_1} m_{\widetilde{\Theta}_2} = c + m_{\widetilde{\Theta}} \text{ in } \mathbb{R}^n \setminus \{0\} \text{ for some } c \in \mathbb{C}. \end{gathered} \tag{7-99}$$

Indeed, for each $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $g \in H^1(\mathbb{R}^n)$ based on (7-98) and (7-97) (applied to $\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$ in place of Θ_1, Θ_2) we may write

$$\langle \widetilde{T}_{\Theta_1} \widetilde{T}_{\Theta_2}[f], g\rangle = \langle [f], T_{\widetilde{\Theta}_2} T_{\widetilde{\Theta}_1} g\rangle = \langle [f], T_{\widetilde{\Theta}_1} T_{\widetilde{\Theta}_2} g\rangle = \langle \widetilde{T}_{\Theta_2} \widetilde{T}_{\Theta_1}[f], g\rangle, \tag{7-100}$$

which, in view of the fact that $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$, establishes the first formula in (7-99). As regards the second formula in (7-99), for each $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $g \in H^1(\mathbb{R}^n)$ using (7-98) and (7-97) (applied to $\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$ in place of Θ_1, Θ_2) we may compute

$$\begin{aligned} \langle \widetilde{T}_{\Theta_1} \widetilde{T}_{\Theta_2}[f], g\rangle &= \langle [f], T_{\widetilde{\Theta}_2} T_{\widetilde{\Theta}_1} g\rangle = \langle [f], T_{\widetilde{\Theta}_1} T_{\widetilde{\Theta}_2} g\rangle \\ &= \langle [f], (cI + T_{\widetilde{\Theta}}) g\rangle = \langle (cI + \widetilde{T}_\Theta)[f], g\rangle. \end{aligned} \tag{7-101}$$

The third equality above is provided by the second formula in (7-97), written for $\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$, $\widetilde{\Theta}$ in place of Θ_1, Θ_2, Θ (whose validity is ensured by the assumptions we make on $c \in \mathbb{C}$ and Θ in (7-99)). By once again relying on the fact that $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$, the second formula in (7-99) follows from (7-101). Having established (7-99), we may now conclude (with the help of Theorem 1.13) that $\mathscr{A}_{\widetilde{\mathrm{SIO}}}$ defined as in (1-116) is a commutative unital subalgebra of the algebra of all linear and bounded operators from the space $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ into itself. Also, the fact that if $c \in \mathbb{C}$ and the functions $\Theta_1, \dots, \Theta_N, \Theta'_1, \dots, \Theta'_N, \Theta$ are as in (1-115) and satisfy (1-117) then (1-118) holds is established in a similar fashion to the second formula in (7-99).

Consider next the claim made in item (b). For starters, the right-to-left inclusion in (1-119) is clear from definitions. As regards the opposite inclusion in (1-119), it suffices to show that $\mathscr{A}_{\widetilde{\mathrm{SIO}}} \subseteq \overline{\mathrm{span}}\{\widetilde{R}_j|_{\mathrm{VMO}}\}_{1\le j\le n}$. Since (1-117) holds with $c=-1$, $\Theta=0$, and $\Theta'_j=\Theta_j=K_j$, defined in (1-90), for each $j\in\{1,\dots,n\}$, we conclude from (1-117) that

$$\sum_{j=1}^{n}(\widetilde{R}_j|_{\mathrm{VMO}})^2=-I \quad \text{in } \mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n)). \tag{7-102}$$

In particular, this proves that the identity operator I belongs to the subalgebra spanned by $\{\widetilde{R}_j|_{\mathrm{VMO}}\}_{1\le j\le n}$ in $\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$. Keeping this in mind, formula (1-119) is established as soon as we show that

$$\widetilde{T}_\Theta\in\overline{\mathrm{span}}\{\widetilde{R}_j|_{\mathrm{VMO}}\}_{1\le j\le n} \text{ for each } \Theta \text{ as in (1-115).} \tag{7-103}$$

To this end, fix an arbitrary Θ as in (1-115). To perform a spherical decomposition of $\Theta|_{S^{n-1}}$, we bring in some notation and recall some basic results. Specifically, define the integers

$$H_0:=1,\quad H_1:=n,\quad \text{and}\quad H_\ell:=\binom{n-1+\ell}{\ell}-\binom{n+\ell-3}{\ell-2}\quad \text{if } \ell\ge 2, \tag{7-104}$$

and, for each $\ell\in\mathbb{N}_0$, let $\{\Psi_{i\ell}\}_{1\le i\le H_\ell}$ be an orthonormal basis for the space of spherical harmonics of degree ℓ on the $(n-1)$-dimensional sphere S^{n-1} in $\mathbb{R}^n$. In particular,

$$H_\ell\le(\ell+1)\cdot(\ell+2)\cdots(n+\ell-2)\cdot(n+\ell-1)\le C_n\,\ell^{n-1}\quad \text{for } \ell\ge 2 \tag{7-105}$$

and, if $\Delta_{S^{n-1}}$ denotes the Laplace–Beltrami operator on S^{n-1}, then for each $\ell\in\mathbb{N}_0$ and $1\le i\le H_\ell$,

$$\begin{aligned}&\Delta_{S^{n-1}}\Psi_{i\ell}=-\ell(n+\ell-2)\Psi_{i\ell} && \text{on } S^{n-1},\\ &\Psi_{i\ell}(x/|x|)=P_{i\ell}(x)/|x|^\ell && \text{for every } x\in\mathbb{R}^n\setminus\{0\},\end{aligned} \tag{7-106}$$

for some homogeneous harmonic polynomial $P_{i\ell}$ of degree ℓ in $\mathbb{R}^n$. Also,

$$\{\Psi_{i\ell}\}_{\ell\in\mathbb{N}_0,\,1\le i\le H_\ell} \text{ is an orthonormal basis for } L^2(S^{n-1}); \tag{7-107}$$

hence,

$$\|\Psi_{i\ell}\|_{L^2(S^{n-1})}=1 \text{ for each } \ell\in\mathbb{N}_0 \text{ and } 1\le i\le H_\ell. \tag{7-108}$$

More details on these matters may be found in, e.g., [Stein and Weiss 1971, pp. 137–152; Stein 1993, pp. 68–75]. For further reference let us note here that, having fixed

$$\text{an even integer } d\in\mathbb{N} \text{ with } d>[(n+1)/2], \tag{7-109}$$

Sobolev's embedding theorem then gives that for each $\ell\in\mathbb{N}_0$ and $1\le i\le H_\ell$ we have (with I standing for the identity operator on S^{n-1})

$$\|\Psi_{i\ell}\|_{\mathscr{C}^1(S^{n-1})}\le C_n\|(I-\Delta_{S^{n-1}})^{\frac{d}{2}}\Psi_{i\ell}\|_{L^2(S^{n-1})}\le C_n\ell^d, \tag{7-110}$$

where the last inequality is a consequence of (7-106)–(7-108) and, generally speaking,

$$\|\Psi\|_{\mathscr{C}^1(S^{n-1})}:=\|\Psi\|_{L^\infty(S^{n-1})}+\|\nabla_{\tan}\Psi\|_{L^\infty(S^{n-1})}\quad \text{for all } \Psi\in\mathscr{C}^1(S^{n-1}), \tag{7-111}$$

with $\nabla_{\tan}$ denoting the tangential gradient to S^{n-1}.

At this stage, observe that $\Theta|_{S^{n-1}} \in L^2(S^{n-1})$; hence we may expand

$$\Theta|_{S^{n-1}} = \sum_{\ell=0}^{\infty} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \Psi_{i\ell} \quad \text{in } L^2(S^{n-1}), \tag{7-112}$$

where

$$\lambda_{i\ell} := \int_{S^{n-1}} \Theta(\omega) \Psi_{i\ell}(\omega)\, d\omega \quad \text{for each } \ell \in \mathbb{N}_0 \text{ and } 1 \le i \le H_\ell. \tag{7-113}$$

In relation to (7-113) we claim that $\lambda_{i\ell}$ decays faster than any power of ℓ; i.e.,

$$\begin{gathered} \text{for each } m \in \mathbb{N} \text{ there exists } C_m \in (0,\infty) \text{ such that} \\ |\lambda_{i\ell}| \le C_m (1+\ell)^{-m} \text{ for each } \ell \in \mathbb{N}_0 \text{ and } 1 \le i \le H_\ell. \end{gathered} \tag{7-114}$$

Indeed, if $\ell = 0$, this is immediate from (7-113). In the case when $\ell \in \mathbb{N}$, for each $m \in \mathbb{N}$ and $i \in \{1, \dots, H_\ell\}$ we may estimate

$$\begin{aligned} |\lambda_{i\ell}[-\ell(n+\ell-2)]^m| &= \left| \int_{S^{n-1}} \Theta(\omega)[-\ell(n+\ell-2)]^m \Psi_{i\ell}(\omega)\, d\omega \right| \\ &= \left| \int_{S^{n-1}} \Delta^m_{S^{n-1}}(\Theta|_{S^{n-1}})(\omega) \Psi_{i\ell}(\omega)\, d\omega \right| \\ &\le \|\Delta^m_{S^{n-1}}(\Theta|_{S^{n-1}})\|_{L^2(S^{n-1})} =: C_m < +\infty, \end{aligned} \tag{7-115}$$

thanks to (7-113), the first formula in (7-106), repeated integrations by parts, the Cauchy–Schwarz inequality, and (7-108) (bearing in mind that the finiteness of C_m above is implied by the smoothness of Θ). Now (7-114) readily follows from (7-115).

To proceed, we recall a basic formula and make some notational conventions. Concretely, it is well known, see, e.g., [Stein 1970, Theorem 5, p. 73], that, in general,

$$\begin{gathered} \text{if } P_k \text{ is a harmonic homogeneous polynomial of degree } k \in \mathbb{N} \text{ in } \mathbb{R}^n \text{ then} \\ \mathcal{F}\left(\text{P.V.} \frac{P_k(x)}{|x|^{n+k}} \right)(\xi) = (-i)^k \pi^{\frac{n}{2}} \frac{\Gamma(k/2)}{\Gamma((k+n)/2)} \frac{P_k(\xi)}{|\xi|^k}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \end{gathered} \tag{7-116}$$

Also, for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$ we abbreviate

$$\begin{aligned} R^\alpha &:= R_1^{\alpha_1} \circ \cdots \circ R_n^{\alpha_n} \quad \text{in } \mathscr{B}(L^2(\mathbb{R}^n)), \\ \widetilde{R}^\alpha &:= \widetilde{R}_1^{\alpha_1} \circ \cdots \circ \widetilde{R}_n^{\alpha_n} \quad \text{in } \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n)), \\ (\widetilde{R}|_{\mathrm{VMO}})^\alpha &:= (\widetilde{R}_1|_{\mathrm{VMO}})^{\alpha_1} \circ \cdots \circ (\widetilde{R}_n|_{\mathrm{VMO}})^{\alpha_n} \quad \text{in } \mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n)), \end{aligned} \tag{7-117}$$

and then use these abbreviations to define, for each given polynomial $P(x) = \sum_{|\alpha| \le M} c_\alpha x^\alpha$ in $\mathbb{R}^n$,

$$P(R) := \sum_{|\alpha| \le M} c_\alpha R^\alpha, \quad P(\widetilde{R}) := \sum_{|\alpha| \le M} c_\alpha \widetilde{R}^\alpha, \quad \text{and} \quad P(\widetilde{R}|_{\mathrm{VMO}}) := \sum_{|\alpha| \le M} c_\alpha (\widetilde{R}|_{\mathrm{VMO}})^\alpha. \tag{7-118}$$

For further reference, let us also observe that if $A \in \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))$ is an operator leaving the space $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ invariant then $A|_{\widetilde{\mathrm{VMO}}(\mathbb{R}^n)} \in \mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))$ and

$$\|A|_{\widetilde{\mathrm{VMO}}(\mathbb{R}^n)}\|_{\mathscr{B}(\widetilde{\mathrm{VMO}}(\mathbb{R}^n))} \le \|A\|_{\mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))}. \tag{7-119}$$

Returning to the mainstream discussion, we claim that, with the polynomials $P_{i\ell}$ as in (7-106) and the $\lambda_{i\ell}$'s as in (7-113), we have

$$\pi^{\frac{n}{2}} \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \frac{\Gamma(\ell/2)}{\Gamma((\ell+n)/2)} P_{i\ell}(\widetilde{R}) \to \widetilde{T}_\Theta \quad \text{in } \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n)) \text{ as } N \to \infty. \tag{7-120}$$

Once this is established, we may conclude with the help of (7-117)–(7-119) that the claim in (7-103) holds. This finishes the proof of (1-119), modulo the justification of (7-120).

To facilitate the proof of (7-120), for each $N \in \mathbb{N}$ introduce

$$\Theta_N(x) := \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \frac{P_{i\ell}(x)}{|x|^{n+\ell}} = \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \frac{\lambda_{i\ell}}{|x|^n} \Psi_{i\ell}\Big(\frac{x}{|x|}\Big) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{7-121}$$

Note that (7-113) implies $\lambda_{10} = 0$, given the vanishing-moment of Θ and the fact that $\Psi_{10}|_{S^{n-1}}$ is a constant (as seen from the second line in (7-106), noting that the polynomial P_{10} has degree zero). Then for each $N \in \mathbb{N}$ the function Θ_N is as in (1-115). Bearing this in mind, we may rely on (1-84), (7-121), (7-116), and the fact that each $P_{i\ell}$ is a homogeneous harmonic polynomial of degree ℓ in $\mathbb{R}^n$ to write

$$\begin{aligned} m_{\Theta_N}(\xi) = (\widehat{\mathrm{P.V.}\Theta_N})(\xi) &= \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell}\, \mathcal{F}\Big(\mathrm{P.V.}\, \frac{P_{i\ell}(x)}{|x|^{n+\ell}}\Big)(\xi) \\ &= \pi^{\frac{n}{2}} \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \frac{\Gamma(\ell/2)}{\Gamma((\ell+n)/2)} P_{i\ell}\Big(-i\frac{\xi}{|\xi|}\Big) \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{7-122}$$

for each $N \in \mathbb{N}$. In turn, from (1-88) and (7-122) we see that for each $N \in \mathbb{N}$ and each $f \in L^2(\mathbb{R}^n)$ we have

$$\widehat{T_{\Theta_N} f} = m_\Theta \hat{f} = \pi^{\frac{n}{2}} \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \frac{\Gamma(\ell/2)}{\Gamma((\ell+n)/2)} \widehat{P_{i\ell}(R) f}. \tag{7-123}$$

Thus, for each $N \in \mathbb{N}$,

$$T_{\Theta_N} = \pi^{\frac{n}{2}} \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \frac{\Gamma(\ell/2)}{\Gamma((\ell+n)/2)} P_{i\ell}(R) \quad \text{in } \mathscr{B}(L^2(\mathbb{R}^n)), \tag{7-124}$$

which, with the help of Proposition 7.6, eventually permits us to conclude that

$$\widetilde{T}_{\Theta_N} = \pi^{\frac{n}{2}} \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \frac{\Gamma(\ell/2)}{\Gamma((\ell+n)/2)} P_{i\ell}(\widetilde{R}) \quad \text{in } \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n)) \text{ for each } N \in \mathbb{N}. \tag{7-125}$$

In view of (7-125) and (7-120), the ultimate goal then becomes proving

$$\widetilde{T}_{\Theta_N} \to \widetilde{T}_{\Theta} \quad \text{in } \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n)) \text{ as } N \to \infty. \tag{7-126}$$

With this aim in mind, recall from (7-56) (used with $p = 2$) that there exists $\theta \in (0, 1)$ such that for each $N \in \mathbb{N}$ we have

$$\begin{aligned}
\|\widetilde{T}_{\Theta} - \widetilde{T}_{\Theta_N}\|_{\mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))}
&= \|\widetilde{T}_{\Theta-\Theta_N}\|_{\mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))} \\
&\le C_n \|T_{\Theta-\Theta_N}\|^{1-\theta}_{\mathscr{B}(L^2(\mathbb{R}^n))} \|\nabla\Theta - \nabla\Theta_N\|^{\theta}_{L^\infty(S^{n-1})} + C_n \|T_{\Theta-\Theta_N}\|_{\mathscr{B}(L^2(\mathbb{R}^n))},
\end{aligned} \tag{7-127}$$

where the last inequality uses the current format of the constant C'' from (7-56) given in (1-82). Next, from (1-88) and (1-87) (used with $p = 2$) we deduce that, for each $N \in \mathbb{N}$,

$$\|T_{\Theta-\Theta_N}\|_{\mathscr{B}(L^2(\mathbb{R}^n))} \le C_n \|m_{\Theta-\Theta_N}\|_{L^\infty(\mathbb{R}^n)} \le C_n \|\Theta - \Theta_N\|_{L^2(S^{n-1})}. \tag{7-128}$$

Since (7-121) and (7-112) imply

$$\Theta_N|_{S^{n-1}} = \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \Psi_{i\ell} \to \Theta|_{S^{n-1}} \quad \text{in } L^2(S^{n-1}) \text{ as } N \to \infty, \tag{7-129}$$

it follows that $\|\Theta - \Theta_N\|_{L^2(S^{n-1})} \to 0$ as $N \to \infty$. Granted this, (7-126) becomes a consequence of (7-127) and (7-128) as soon as we establish that

$$\sup_{N \in \mathbb{N}} \|\nabla\Theta_N\|_{L^\infty(S^{n-1})} < +\infty. \tag{7-130}$$

To justify (7-130), fix $N \in \mathbb{N}$ arbitrary and observe that since Θ_N is positive homogeneous of degree $-n$, Euler's formula implies

$$x \cdot (\nabla\Theta_N)(x) = -n\Theta_N(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{7-131}$$

Consequently,

$$\begin{aligned}
\nabla_{\tan}(\Theta_N|_{S^{n-1}})(x) &= (\nabla\Theta_N)(x) - (x \cdot (\nabla\Theta_N)(x))x \\
&= (\nabla\Theta_N)(x) + n\Theta_N(x)x \quad \text{for each } x \in S^{n-1},
\end{aligned} \tag{7-132}$$

which, in light of (7-111), further implies

$$\|\nabla\Theta_N\|_{L^\infty(S^{n-1})} \le n\|\Theta_N\|_{\mathscr{C}^1(S^{n-1})}. \tag{7-133}$$

On the other hand, from (7-121) we know that $\Theta_N = \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} \lambda_{i\ell} \Psi_{i\ell}$ on S^{n-1}; hence for each $m \in \mathbb{N}$ there exists $C_m \in (0, \infty)$ such that

$$\begin{aligned}
\|\nabla\Theta_N\|_{L^\infty(S^{n-1})} &\le n\|\Theta_N\|_{\mathscr{C}^1(S^{n-1})} \\
&\le n \sum_{\ell=0}^{N} \sum_{i=1}^{H_\ell} |\lambda_{i\ell}| \|\Psi_{i\ell}\|_{\mathscr{C}^1(S^{n-1})} \le C_m C_n \sum_{\ell=0}^{N} (1+\ell)^{-m} \ell^d \ell^{n-1},
\end{aligned} \tag{7-134}$$

where the last inequality is based on (7-114), (7-110), and (7-105). Choosing m large enough (depending on n and d) so that the partial sums above converge, we ultimately see that

$$\sup_{N\in\mathbb{N}} \|\nabla\Theta_N\|_{L^\infty(S^{n-1})} \leq C_n C_m \sum_{\ell=0}^{\infty}(1+\ell)^{-m}\ell^d\,\ell^{n-1} < +\infty, \tag{7-135}$$

which establishes (7-130). This finishes the proof of (1-119).

To deal with item (c), assume next that Θ is as in (1-115) and c is as in (1-120). Then

$$m(\xi) := (c + m_{\widetilde{\Theta}}(\xi))^{-1} \quad \text{for each } \xi \in \mathbb{R}^n \setminus \{0\} \tag{7-136}$$

is a well-defined function, which belongs to $\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and is positive homogeneous of degree zero. As such, (7-93) guarantees the existence of a function Θ_0 as in (1-115) with the property that $m = c_0 + m_{\widetilde{\Theta}_0}$, where $c_0 := \fint_{S^{n-1}} m(\omega)\,d\omega \in \mathbb{C}$. We claim that

$$m_{\widetilde{\Theta}}\, m_{\widetilde{\Theta}_0} = (1 - cc_0) + m_{-c\widetilde{\Theta}_0 - c_0\widetilde{\Theta}}. \tag{7-137}$$

This is seen by expanding $m_{-c\widetilde{\Theta}_0 - c_0\widetilde{\Theta}} = -cm_{\widetilde{\Theta}_0} - c_0 m_{\widetilde{\Theta}}$, then replacing $m_{\widetilde{\Theta}_0}$ by $(c + m_{\widetilde{\Theta}})^{-1} - c_0$ throughout. After some simple algebra, (7-137) follows. By virtue of the second formula in (7-99), the identity in (7-137) implies

$$\begin{aligned}\widetilde{T}_\Theta \circ \widetilde{T}_{\Theta_0} &= (1 - cc_0)I + \widetilde{T}_{-c\widetilde{\Theta}_0 - c_0\widetilde{\Theta}} \\ &= (1 - cc_0)I - c\widetilde{T}_{\widetilde{\Theta}_0} - c_0\widetilde{T}_{\widetilde{\Theta}} \quad \text{on } \widetilde{\mathrm{BMO}}(\mathbb{R}^n).\end{aligned} \tag{7-138}$$

The above formula may be recast as

$$(cI + \widetilde{T}_\Theta) \circ (c_0 I + \widetilde{T}_{\Theta_0}) = I \quad \text{on } \widetilde{\mathrm{BMO}}(\mathbb{R}^n). \tag{7-139}$$

In a similar manner we also obtain

$$(c_0 I + \widetilde{T}_{\Theta_0}) \circ (cI + \widetilde{T}_\Theta) = I \quad \text{on } \widetilde{\mathrm{BMO}}(\mathbb{R}^n). \tag{7-140}$$

From (7-139)–(7-140) we conclude that $cI + \widetilde{T}_\Theta$ is invertible as an operator on $\widetilde{\mathrm{BMO}}(\mathbb{R}^n)$, whose inverse is $c_0 I + \widetilde{T}_{\Theta_0} \in \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n))$. Since both operators map $\widetilde{\mathrm{VMO}}(\mathbb{R}^n)$ into itself (see Theorem 1.13), it follows that $c_0 I + \widetilde{T}_{\Theta_0}|_{\mathrm{VMO}} \in \mathscr{A}_{\widetilde{\mathrm{SIO}}}$ is the inverse of $cI + \widetilde{T}_\Theta|_{\mathrm{VMO}}$. This concludes the treatment of item (c).

Moving on, the first claim made in item (d), pertaining to the equivalence stated in (1-121), is seen directly from item (c) (which yields the left-pointing implication), and Theorem 1.13 (which gives the right-pointing implication). Consider next the second claim made in item (d). To set the stage, pick $N \in \mathbb{N}$ and assume $\Theta_1, \dots, \Theta_N$ are as in (1-115), while $c_1, \dots, c_N$ are as in (1-122). If we set

$$Q(\xi) := \sum_{j=1}^{N} |c_j + m_{\widetilde{\Theta}_j}(\xi)|^2 \quad \text{for each } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{7-141}$$

then the present assumptions ensure that Q is a real-valued function which is well-defined, of class $\mathscr{C}^\infty$, positive homogeneous of degree zero, and never zero in $\mathbb{R}^n \setminus \{0\}$. As such, if for each $j \in \{1, \dots, N\}$ we now introduce

$$m_j(\xi) := \frac{\bar{c}_j + m_{\bar{\Theta}_j}(\xi)}{Q(\xi)} = \frac{\overline{c_j + m_{\widetilde{\Theta}_j}(\xi)}}{Q(\xi)} \quad \text{for each } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{7-142}$$

where the second equality is a consequence of one of the formulas in (1-86), then each m_j is a complex-valued function which is well-defined, of class $\mathscr{C}^\infty$, and positive homogeneous of degree zero in $\mathbb{R}^n \setminus \{0\}$. According to (7-93), these properties guarantee the existence of numbers $c'_j \in \mathbb{C}$ and functions Θ'_j as in (1-115) such that

$$m_j = c'_j + m_{\widetilde{\Theta}'_j} \quad \text{in } \mathbb{R}^n \setminus \{0\} \text{ for each } j \in \{1, \dots, N\}. \tag{7-143}$$

Since, by design, $\sum_{j=1}^N m_j(\xi)(c_j + m_{\widetilde{\Theta}_j}(\xi)) = 1$ for each $\xi \in \mathbb{R}^n \setminus \{0\}$, we then conclude that

$$\sum_{j=1}^N (c'_j + m_{\widetilde{\Theta}'_j}(\xi))(c_j + m_{\widetilde{\Theta}_j}(\xi)) = 1 \quad \text{for each } \xi \in \mathbb{R}^n \setminus \{0\} \tag{7-144}$$

or, equivalently,

$$\sum_{j=1}^N m_{\widetilde{\Theta}'_j} m_{\widetilde{\Theta}_j} = c + m_{\widetilde{\Theta}} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{7-145}$$

where

$$c := \left(1 - \sum_{j=1}^N c'_j c_j\right) \in \mathbb{C} \quad \text{and} \quad \Theta := -\sum_{j=1}^N \{c'_j \Theta_j + c_j \Theta'_j\} \text{ is as in (1-115).} \tag{7-146}$$

Similarly to (7-99), from (7-145)–(7-146) we conclude that

$$\sum_{j=1}^N \widetilde{T}_{\Theta'_j} \widetilde{T}_{\Theta_j} = cI + \widetilde{T}_\Theta = \left(1 - \sum_{j=1}^N c'_j c_j\right) I - \sum_{j=1}^N \{c'_j \widetilde{T}_{\Theta_j} + c_j \widetilde{T}_{\Theta'_j}\}, \tag{7-147}$$

which, in turn, implies

$$\sum_{j=1}^N (c'_j I + \widetilde{T}_{\Theta'_j})(c_j I + \widetilde{T}_{\Theta_j}) = I \quad \text{in } \mathscr{B}(\widetilde{\mathrm{BMO}}(\mathbb{R}^n)). \tag{7-148}$$

With this in hand, we may turn to the proof of the equivalence recorded in (1-123) in earnest. The right-pointing implication is clear from Theorem 1.13. As regards the left-pointing implication, assume $f \in \mathrm{BMO}(\mathbb{R}^n)$ is such that there exist $g_1, \dots, g_N \in \mathrm{VMO}(\mathbb{R}^n)$ with the property that

$$[g_j] = (c_j I + \widetilde{T}_{\Theta_j})[f] \quad \text{in } \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \text{ for each } j \in \{1, \dots, N\}. \tag{7-149}$$

Then (7-148) permits us to express $[f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n)$ as

$$[f] = \sum_{j=1}^N (c'_j I + \widetilde{T}_{\Theta'_j})(c_j I + \widetilde{T}_{\Theta_j})[f] = \sum_{j=1}^N (c'_j I + \widetilde{T}_{\Theta'_j})[g_j] \in \widetilde{\mathrm{VMO}}(\mathbb{R}^n), \tag{7-150}$$

where the membership above is provided by Theorem 1.13. Ultimately, from (7-150) we conclude that $f \in \mathrm{VMO}(\mathbb{R}^n)$, finishing the proof of (1-123).

Finally, the proofs of the claims in item (e) closely parallel those in the scalar case, with minor natural adjustments of a purely algebraic nature (designed to accommodate the present matrix-formalism). □

In turn, Theorem 1.15 may be specialized as to yield Corollaries 1.16–1.20 as indicated below.

Proof of Corollary 1.16. The strategy is to devise a suitable dictionary between the algebra formalism, currently used, and the matrix formalism described in item (e) of Theorem 1.15, which is going to yield (1-127) at once. To get started, fix a linear basis $\{e_1, \dots, e_N\}$ in A, regarded as a vector space. Then we have a linear isomorphism

$$A \ni a = \sum_{j=1}^{N} a_j e_j \mapsto a^V := (a_j)_{1\le j\le N} \in \mathbb{C}^N \tag{7-151}$$

identifying algebra elements $a \in A$ with their vector realizations $a^V \in \mathbb{C}^N$. We shall also need to identify each algebra element $a \in A$ with a certain matrix $a^M \in \mathbb{C}^{N\times N}$. To define this matrix realization, consider the family of complex numbers $\lambda_{\ell kj}$, with $1 \le \ell, k, j \le N$, such that

$$e_j \odot e_k = \sum_{\ell=1}^{N} \lambda_{\ell kj} e_\ell \quad \text{for each } j, k \in \{1, \dots, N\}, \tag{7-152}$$

then set

$$a^M := \Big(\sum_{j=1}^{N} \lambda_{\ell kj} a_j\Big)_{1\le \ell,k\le N} \in \mathbb{C}^{N\times N} \quad \text{for all } a = \sum_{j=1}^{N} a_j e_j \in A. \tag{7-153}$$

In relation to these realizations of algebra elements, the following identity holds:

$$a \odot b = c \iff a^M b^V = c^V \quad \text{for all } a, b, c \in A. \tag{7-154}$$

We next claim that

$$\text{if } a \in A \text{ is invertible in } A \text{ from the right then the matrix } a^M \text{ is invertible in } \mathbb{C}^{N\times N}. \tag{7-155}$$

To see this, fix $a \in A$ which has an inverse $a_R^{-1} \in A$ from the right, and pick some arbitrary $(z_1, \dots, z_N) \in \mathbb{C}^N$. Set $c := \sum_{\ell=1}^{N} z_\ell e_\ell \in A$ and consider $b := a_R^{-1} \odot c \in A$. According to (7-154), the fact that $a \odot b = c$ then translates into $a^M b^V = c^V = (z_1, \dots, z_N)$. Since the latter is an arbitrary vector in $\mathbb{C}^N$, this proves that, as a linear map from $\mathbb{C}^N$ into itself, the matrix a^M is surjective, and hence ultimately, invertible.

Consider next an A-valued kernel Θ as in (1-125). Then $\Theta = \sum_{j=1}^{N} \Theta_j e_j$ with each scalar component Θ_j as in (1-115) and, by definition and (7-152),

$$\widetilde{T}_\Theta[f] = \sum_{j,k=1}^{N} \widetilde{T}_{\Theta_j}[f_k] e_j \odot e_k = \sum_{j,k,\ell=1}^{N} \lambda_{\ell kj} \widetilde{T}_{\Theta_j}[f_k] e_\ell \quad \text{for every } f = \sum_{k=1}^{N} f_k e_k \in \mathrm{BMO}(\mathbb{R}^n) \otimes A. \tag{7-156}$$

If we also associate with the A-valued kernel Θ the matrix-valued kernel Θ^M as in (7-153), we may rewrite (7-156) simply as

$$(\widetilde{T}_\Theta[f])^V = \widetilde{T}_{\Theta^M}[f]^V \quad \text{for all } f \in \mathrm{BMO}(\mathbb{R}^n) \otimes A. \tag{7-157}$$

Since (7-154) also gives

$$(c \odot [f])^V = c^M [f]^V \quad \text{for all } c \in A, \ \text{for all } f \in \mathrm{BMO}(\mathbb{R}^n) \otimes A, \tag{7-158}$$

from (7-157)–(7-158) we finally conclude that

$$((cI + \widetilde{T}_\Theta)[f])^V = (c^M + \widetilde{T}_{\Theta^M})[f]^V \quad \text{for all } c \in A, \ \text{for all } f \in \mathrm{BMO}(\mathbb{R}^n) \otimes A. \tag{7-159}$$

It remains to observe that, since $(c + m_{\widetilde{\Theta}}(\xi))^M = c^M + m_{\widetilde{\Theta}^M}(\xi)$ for each $\xi \in \mathbb{R}^n \setminus \{0\}$, from (7-155) we have that

$$\begin{gathered}\text{if } c \text{ is as in (1-126) then for each } \xi \in \mathbb{R}^n \setminus \{0\} \\ \text{the matrix } c^M + m_{\widetilde{\Theta}^M}(\xi) \text{ is invertible in } \mathbb{C}^{N\times N}.\end{gathered} \tag{7-160}$$

Then (7-159)–(7-160) ensure that item (e) of Theorem 1.15 applies (with $\mathscr{V} := \mathbb{C}^N$), which proves (1-127). □

Proof of Corollary 1.17. The complex Riesz transform defined in (1-131) as well as the Beurling transform (1-91) are principal-value convolution operators of the sort discussed in (1-82). Specifically,

$$R_{\mathbb{C}} = T_{\Theta_1} \quad \text{with } \Theta_1(z) := \frac{z}{2\pi|z|^3} \text{ for } z \in \mathbb{C} \setminus \{0\}, \tag{7-161}$$

$$S = T_{\Theta_2} \quad \text{with } \Theta_2(z) := -\frac{1}{\pi z^2} = -\frac{(\bar{z})^2}{\pi|z|^4} \text{ for } z \in \mathbb{C} \setminus \{0\}. \tag{7-162}$$

Their associated symbols are given by, see (7-116),

$$\begin{aligned} m_{\Theta_1}(\xi) &= (\widehat{\mathrm{P.V.}\Theta_1})(\xi) = -i\xi/|\xi| && \text{for } \xi \in \mathbb{C} \setminus \{0\}, \\ m_{\Theta_2}(\xi) &= (\widehat{\mathrm{P.V.}\Theta_2})(\xi) = (\bar{\xi})^2/|\xi|^2 = \bar{\xi}/\xi && \text{for } \xi \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{7-163}$$

Upon observing that for $j \in \{1, 2\}$ we have

$$c \in \mathbb{C} \text{ with } |c| \neq 1 \quad \Longrightarrow \quad c \in \mathbb{C} \setminus \{-m_{\widetilde{\Theta}_j}(\xi) : \xi \in \mathbb{C} \setminus \{0\}\}, \tag{7-164}$$

the first part of item (d) in Theorem 1.15 applies and gives that (i) ⇔ (ii) as well as (i) ⇔ (iii). This finishes the proof of Corollary 1.17. □

Proof of Corollary 1.18. The Clifford–Riesz transform defined in (1-135) is a principal-value convolution operator of form $R_{\mathcal{C}\ell} = T_\Theta$, where the kernel is the Clifford-algebra-valued function, see the convention in (1-133),

$$\Theta : \mathbb{R}^n \setminus \{0\} \to \mathcal{C}\ell_n \quad \text{given by } \Theta(x) := \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x}{|x|^{n+1}} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{7-165}$$

Thanks to (7-116), its associated symbol may be explicitly identified as

$$m_\Theta(\xi) = (\widehat{\mathrm{P.V.}\Theta})(\xi) = -i\xi/|\xi| \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{7-166}$$

In particular, if $c \in \mathcal{C}\ell_n$ is such that $c + i\omega$ is invertible in $\mathcal{C}\ell_n$ from the right for each vector $\omega \in S^{n-1} \subseteq \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$, then

$$c + m_{\widetilde{\Theta}}(\xi) \text{ is invertible in } \mathcal{C}\ell_n \text{ from the right for each } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{7-167}$$

Granted this, Corollary 1.16 applies with $A := \mathcal{C}\ell_n$ and gives the equivalence in (1-137). □

Proof of Corollary 1.19. The equivalence stated in (1-138) is an immediate consequence of (1-123) (used with $N = n$ and $\Theta_j := K_j$, defined in (1-90), for $1 \le j \le n$) upon noting that condition (1-122) presently reads $(c_1, \dots, c_n) \in \mathbb{C}^n \setminus i S^{n-1}$. □

Proof of Corollary 1.20. To recast the operator S_θ in the manner described in (1-153), fix some arbitrary differential form $f \in L^2(\mathbb{R}^n) \otimes \Lambda$ and, starting with (1-144)–(1-145), write (bearing in mind that the j-th Riesz transform on $L^2(\mathbb{R}^n)$ is the multiplier with symbol $-i\xi_j/|\xi|$)

$$\widehat{S_\theta f}(\xi) = -\theta \sum_{j,k=1}^{n} dx_j \wedge \left(dx_k \vee \left(\frac{\xi_j \xi_k}{|\xi|^2} \hat{f}(\xi) \right) \right) + \theta^{-1} \sum_{j,k=1}^{n} dx_j \vee \left(dx_k \wedge \left(\frac{\xi_j \xi_k}{|\xi|^2} \hat{f}(\xi) \right) \right). \tag{7-168}$$

Granted (1-151)–(1-152), we may consider the principal-value distribution P.V. Θ_{jk} associated with Θ_{jk} as in (1-83). Upon recalling, see [Mitrea 2013, Proposition 4.70, p. 141], that for each pair of indices $j, k \in \{1, \dots, n\}$ we have (with $\mathcal{F}$ used as an alternative notation for the Fourier transform hat in $\mathbb{R}^n$, and with δ denoting the standard Dirac distribution in $\mathbb{R}^n$)

$$\frac{\xi_j \xi_k}{|\xi|^2} = \mathcal{F}\Big(\frac{1}{\omega_{n-1}} (\text{P.V. } \Theta_{jk}) + \frac{1}{n} \delta_{jk}\, \delta \Big)(\xi), \tag{7-169}$$

for each $j, k \in \{1, \dots, n\}$ we may express

$$\frac{\xi_j \xi_k}{|\xi|^2} \hat{f}(\xi) = \mathcal{F}\bigg(\Big(\frac{1}{\omega_{n-1}} (\text{P.V. } \Theta_{jk}) + \frac{1}{n} \delta_{jk}\, \delta \Big) * f \bigg)(\xi) = \mathcal{F}\Big(\frac{1}{\omega_{n-1}} T_{\Theta_{jk}} f + \frac{1}{n} \delta_{jk} f \Big)(\xi). \tag{7-170}$$

In turn, from (7-168) and (7-170) we readily conclude that (1-153) holds.

Next, Proposition 7.6 ensures that S_θ, originally considered as in (1-153), further extends to a well-defined linear and bounded operator from the space $H^1(\mathbb{R}^n) \otimes \Lambda$ into itself. Keeping this in mind, for each $[f] \in \widetilde{\text{BMO}}(\mathbb{R}^n) \otimes \Lambda$ and each $g \in H^1(\mathbb{R}^n) \otimes \Lambda$ we may write

$$\begin{aligned}
\langle [f], S_\theta g \rangle = &-\frac{\theta}{\omega_{n-1}} \sum_{j,k=1}^{n} \big\langle [f], dx_k \wedge (dx_j \vee (T_{\widetilde{\Theta}_{jk}} g)) \big\rangle + \frac{\theta^{-1}}{\omega_{n-1}} \sum_{j,k=1}^{n} \big\langle [f], dx_k \vee (dx_j \wedge (T_{\widetilde{\Theta}_{jk}} g)) \big\rangle \\
&- \frac{\theta}{n} \sum_{j=1}^{n} \big\langle [f], dx_j \wedge (dx_j \vee g) \big\rangle + \frac{\theta^{-1}}{n} \sum_{j=1}^{n} \big\langle [f], dx_j \vee (dx_j \wedge g) \big\rangle \\
= &-\frac{\theta}{\omega_{n-1}} \sum_{j,k=1}^{n} \big\langle dx_j \wedge (dx_k \vee (\widetilde{T}_{\Theta_{jk}} [f])), g \big\rangle + \frac{\theta^{-1}}{\omega_{n-1}} \sum_{j,k=1}^{n} \big\langle dx_j \vee (dx_k \wedge (\widetilde{T}_{\Theta_{jk}} [f])), g \big\rangle \\
&- \frac{\theta}{n} \sum_{j=1}^{n} \big\langle dx_j \wedge (dx_j \vee [f]), g \big\rangle + \frac{\theta^{-1}}{n} \sum_{j=1}^{n} \big\langle dx_j \vee (dx_j \wedge [f]), g \big\rangle.
\end{aligned} \tag{7-171}$$

The first equality above uses $\widetilde{\Theta}_{jk} = \Theta_{jk} = \Theta_{kj}$, see (1-152), while the second equality is based on the transposition formula (1-96) and the fact that the interior and exterior product of forms are dual to one another. On the other hand, since for each $[f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \otimes \Lambda$ and $g \in H^1(\mathbb{R}^n) \otimes \Lambda$ we have

$$\begin{aligned}\langle \widetilde{R} \wedge [f], g\rangle &= \Big\langle \sum_{j=1}^{n} dx_j \wedge \widetilde{R}_j[f], g\Big\rangle = \sum_{j=1}^{n} \langle \widetilde{R}_j[f], dx_j \vee g\rangle \\ &= -\sum_{j=1}^{n} \langle [f], dx_j \vee R_j g\rangle = -\langle [f], R \vee g\rangle, \end{aligned} \tag{7-172}$$

and, similarly,

$$\langle \widetilde{R} \vee [f], g\rangle = -\langle [f], R \wedge g\rangle, \tag{7-173}$$

from (1-147) and (7-172)–(7-173) we conclude that

$$\langle \widetilde{S}_\theta[f], g\rangle = \langle [f], S_\theta g\rangle \quad \text{for all } [f] \in \widetilde{\mathrm{BMO}}(\mathbb{R}^n) \otimes \Lambda, \text{ for all } g \in H^1(\mathbb{R}^n) \otimes \Lambda. \tag{7-174}$$

At this stage, by comparing (7-171) with (7-174) and keeping in mind the $\widetilde{\mathrm{BMO}}$-H^1 duality, we conclude that (1-154) holds.

Let us now turn our attention to the equivalences in the last part of the statement of the corollary. As a preamble, for each $\omega = (\omega_1, \dots, \omega_n) \in S^{n-1}$, identified with the differential form of degree one $\omega_1\, dx_1 + \cdots + \omega_n\, dx_n$ in $\mathbb{R}^n$, introduce the operators P_ω, Q_ω acting on an arbitrary differential form $u \in \Lambda$ according to

$$P_\omega u := \omega \wedge (\omega \vee u), \quad Q_\omega u := \omega \vee (\omega \wedge u). \tag{7-175}$$

In the same vein, for each $\theta \in \mathbb{C} \setminus \{0\}$ and $\omega \in S^{n-1}$ let us also set

$$\Omega_{\theta,\omega}\, u := \theta\omega \wedge u + \theta^{-1}\omega \vee u \quad \text{for all } u \in \Lambda. \tag{7-176}$$

Then, with I denoting the identity operator on Λ, for each $\omega \in S^{n-1}$ and $\theta \in \mathbb{C} \setminus \{0\}$ we have, see [Mitrea et al. 2016a, Lemma 2.2, p. 54],

$$\begin{gathered} P_\omega Q_\omega = Q_\omega P_\omega = 0, \quad P_\omega + Q_\omega = I, \\ P_\omega^2 = P_\omega, \quad Q_\omega^2 = Q_\omega, \quad \text{and} \quad \Omega_{\theta,\omega}\Omega_{\theta,\omega} = I. \end{gathered} \tag{7-177}$$

In this notation, it follows from (1-140)–(1-141) that

$$\begin{gathered} S_\theta : L^2(\mathbb{R}^n) \otimes \Lambda \to L^2(\mathbb{R}^n) \otimes \Lambda \text{ acts on each } f \in L^2(\mathbb{R}^n) \otimes \Lambda \\ \text{according to } S_\theta f(x) = \mathcal{F}^{-1}_{\xi\to x}((-\theta P_{\xi/|\xi|} + \theta^{-1} Q_{\xi/|\xi|})\hat{f}(\xi)) \text{ for a.e. } x \in \mathbb{R}^n. \end{gathered} \tag{7-178}$$

Hence, S_θ is a multiplier operator with symbol given by

$$m(\xi) := -\theta P_{\xi/|\xi|} + \theta^{-1} Q_{\xi/|\xi|} \in \mathrm{Hom}(\Lambda, \Lambda) \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{7-179}$$

We now claim that

$$\begin{gathered} \text{if } \theta \in \mathbb{C} \setminus \{0\} \text{ and } c \in \mathbb{C} \setminus \{\theta, -\theta^{-1}\} \text{ then } cI + m(\xi) \text{ is} \\ \text{invertible in } \mathrm{Hom}(\Lambda, \Lambda) \text{ for each } \xi \in \mathbb{R}^n \setminus \{0\}. \end{gathered} \tag{7-180}$$

To see this, assume θ and c are as above and fix some $\xi \in \mathbb{R}^n \setminus \{0\}$ arbitrary. Then, based on (7-177) it is easy to see that $cI + m(\xi) \in \mathrm{Hom}(\Lambda, \Lambda)$ and $cI + \theta^{-1} P_{\xi/|\xi|} - \theta Q_{\xi/|\xi|} \in \mathrm{Hom}(\Lambda, \Lambda)$ commute and their composition is $(c-\theta)(c+\theta^{-1})I$. Hence, (7-180) follows. Granted this, we may then conclude from item (e) of Theorem 1.15 (applied with $\mathscr{V} := \Lambda$) that the equivalence (i) $\Leftrightarrow$ (ii) in the last part of Corollary 1.20 holds.

Likewise, it is visible from (1-149) that

$$\begin{gathered} R_\theta : L^2(\mathbb{R}^n) \otimes \Lambda \to L^2(\mathbb{R}^n) \otimes \Lambda \text{ acts on each } f \in L^2(\mathbb{R}^n) \otimes \Lambda \\ \text{according to } R_\theta f(x) = -\mathcal{F}^{-1}_{\xi \to x}(\Omega_{\theta,\xi/|\xi|} \hat{f}(\xi)) \text{ for a.e. } x \in \mathbb{R}^n; \end{gathered} \tag{7-181}$$

hence R_θ is a multiplier operator with symbol given by

$$m(\xi) := -\Omega_{\theta,\xi/|\xi|} \in \mathrm{Hom}(\Lambda, \Lambda) \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{7-182}$$

Since, thanks to the last formula in (7-177), for each vector $\xi \in \mathbb{R}^n \setminus \{0\}$ we may write

$$(cI - \Omega_{\theta,\xi/|\xi|})(cI + \Omega_{\theta,\xi/|\xi|}) = (c^2 - 1)I,$$

we conclude that

$$\begin{gathered} \text{if } \theta \in \mathbb{C} \setminus \{0\} \text{ and } c \in \mathbb{C} \setminus \{\pm 1\} \text{ then } cI + m(\xi) \\ \text{is invertible in } \mathrm{Hom}(\Lambda, \Lambda) \text{ for each } \xi \in \mathbb{R}^n \setminus \{0\}. \end{gathered} \tag{7-183}$$

As such, item (e) of Theorem 1.15 (once again applied with $\mathscr{V} := \Lambda$) proves the equivalence (i) $\Leftrightarrow$ (iii) in the last part of Corollary 1.20. ☐

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CONVERGENCE OF THE KÄHLER–RICCI ITERATION

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The Ricci iteration is a discrete analogue of the Ricci flow. According to Perelman, the Ricci flow converges to a Kähler–Einstein metric whenever one exists, and it has been conjectured that the Ricci iteration should behave similarly. This article confirms this conjecture. As a special case, this gives a new method of uniformization of the Riemann sphere.

1. Introduction

Let (M, g_1) be a compact Riemannian manifold. A Ricci iteration is a sequence of metrics $\{g_i\}_{i\in\mathbb{N}}$ on M satisfying

$$\operatorname{Ric} g_{i+1} = g_i, \quad i \in \mathbb{N}, \tag{1}$$

where $\operatorname{Ric} g_{i+1}$ denotes the Ricci curvature of g_{i+1}. One may think of (1) as a dynamical system on the space of Riemannian metrics on M. Part of the interest in the Ricci iteration is that, clearly, Einstein metrics are fixed points, and so (1) aims to provide a natural theoretical and numerical approach to uniformization in the challenging case of positive Ricci curvature (different Ricci iterations can be defined in the context of nonpositive curvature, but these are typically easier to understand and will not be discussed here). In essence, the Ricci iteration aims to reduce the Einstein equation to a sequence of prescribed Ricci curvature equations and can be thought of as a discretization of the Ricci flow. Going back to [Rubinstein 2007; 2008c], it has been studied since by a number of authors [Berman 2013; Berman et al. 2016a; Cheltsov et al. 2010; Cheltsov and Shramov 2011a; 2011b; Cheltsov and Wilson 2013; Guedj et al. 2013; Jeffres et al. 2016; Keller 2009; Pulemotov and Rubinstein 2016]; see also the survey [Rubinstein 2014, §6.5]. One of the motivations for considering (1) and not simply repeatedly applying the Ricci tensor (as in [Nadel 1995], see also [Rubinstein 2008a, Remark 4.63]) is the gain of derivatives inherent in (1) as well as monotonicity of certain functionals. Both of these properties will feature below.

Of particular interest has been the study of the Ricci iteration on Kähler manifolds (for the non-Kähler case results are scarce, see [Pulemotov and Rubinstein 2016]). When (M, J, g_1) is Kähler, the Calabi–Yau theorem [Yau 1978] guarantees the existence and uniqueness of the sequence $\{g_i\}_{i\in\mathbb{N}}$ if and only if M is Fano (i.e., has positive first Chern class $c_1(M, \mathrm{J})$) and the Kähler class associated to g_1 is $c_1(M, \mathrm{J})$. Under a rather restrictive technical assumption, one of us showed that g_i converges smoothly to a Kähler–Einstein metric [Rubinstein 2008c, Theorem 3.3] and made the following general conjecture (see Conjecture 3.2 of the same work):

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Conjecture 1.1. *Let* (M, J, g_1) *be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to* g_1 *is* $c_1(M, \mathrm{J})$*. Then the Ricci iteration* (1) *converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric.*

Roughly speaking, (M, g_k) converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric if there exist smooth diffeomorphisms $f_k : M \to M$ such that $f_k^* g_k$ converges smoothly to a Kähler–Einstein metric. As we will see, our methods will actually produce biholomorphisms f_k. For more on Cheeger–Gromov convergence we refer to [Petersen 2016, Chapter 10].

The best result so far on this conjecture is due to Berman et al. [2016a], who replaced the technical assumption of [Rubinstein 2008c, Theorem 3.3] concerning Tian's α-invariant by the weaker assumption of the Mabuchi energy being proper (both of these assumptions imply a Kähler–Einstein metric exists). Therefore, by a classical result of Tian [1997], Conjecture 1.1 holds if M admits no holomorphic vector fields. However, the properness assumption is still too restrictive and fails in general. For example, Conjecture 1.1 is still open even for $M = S^2$, the two-sphere. Furthermore, as recent counterexamples show [Darvas and Rubinstein 2017], certain key theorems in Kähler geometry that one might naively expect to generalize in a straightforward manner from the case of no automorphisms require new tools and ideas when automorphisms are present.

The main result of the present article is the resolution of Conjecture 1.1, and in fact with a stronger convergence:

Theorem 1.2. *Let* (M, J, g_1) *be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to* g_1 *is* $c_1(M, \mathrm{J})$ *and let* $\{g_i\}_{i \in \mathbb{N}}$ *be given by* (1)*. Then there exist holomorphic diffeomorphisms* h_k *such that* $h_k^* g_k$ *converges smoothly to a Kähler–Einstein metric.*

A key ingredient in establishing this result is our use of a Finsler metric structure on the space of Kähler metrics introduced previously by one of us [Darvas 2015]. In this infinite-dimensional geometry, the automorphisms of X act by isometries. We establish the boundedness of the Ricci iteration with respect to this Finsler metric, up to automorphisms of X. This is then shown to imply the key a priori estimates with respect to the stronger C^k norms. In fact, we also show a rather stronger result: discretizations of the Kähler–Ricci flow for *any time step* converge. This is new even for the case of no automorphisms considered in [Rubinstein 2008c; Berman et al. 2016a] and resolves a more general conjecture than Conjecture 1.1; see Theorem 1.6 below.

Uniformization of the two-sphere. As a very special case we obtain the following new method of uniformization. Fix a conformal class of volume V on S^2. As we know, in this class there is a constant curvature metric, the round one. More precisely, let ω_c denote the round form of the constant-c Ricci curvature metric on $M = (S^2, \mathrm{J})$, given locally by

$$\omega_c = \frac{\sqrt{-1}}{c\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Here $V = \int_{S^2} \omega_c = c_1([M])/c = 2/c$. Consequently, $c = \frac{1}{2\pi}$ in the case where we restrict the Euclidean metric of $\mathbb{R}^3$ to the unit sphere.

Let ω be any metric on S^2 with $\int_{S^2}\omega = V = 2/c$. Introduce $u_0 = 0$, and we solve iteratively to find $u_i \in C^\infty(S^2)$ satisfying

$$\Delta_\omega u_i = R_\omega - 2e^{u_{i-1}} \quad \text{and} \quad \int_{S^2} e^{u_i}\,\omega = \frac{2}{c}, \tag{2}$$

so that the scalar curvature of $\omega_i := e^{u_i}\omega$ satisfies $R_{\omega_i} = 2e^{u_{i-1}-u_i}$, or equivalently, $\operatorname{Ric}\omega_i = \omega_{i-1}$. (In two dimensions, $\operatorname{Ric}\omega = \frac{1}{2}R_\omega\omega$, where R_ω is the scalar curvature. If $\omega_1 = e^\phi\omega_0$, then the scalar curvatures of these two metrics satisfy

$$\Delta_{\omega_0}\phi - R_{\omega_0} + R_{\omega_1}e^\phi = 0.$$

We note that the conformal factor is often written $e^{2\phi}$ elsewhere, but this is compensated for here by the fact that $R_\omega = 2K_\omega$, where K_ω is the Gauss curvature.)

Corollary 1.3. *We fix $c > 0$ and let ω be any Kähler form on S^2 with $\int_X \omega = 2/c$. We introduce $\{u_i\} \subset C^\infty(S^2)$ by repeatedly solving the Poisson equation* (2). *Then, there exist Möbius transformations h_i such that $h_i^*(e^{u_i}\omega)$ converges smoothly to the round metric ω_c.*

Discretization of the Ricci flow. One of the original motivations for introducing the Ricci iteration, going back to [Rubinstein 2007; 2008c], is its relation to the Ricci flow. Hamilton's Ricci flow on a Kähler manifold of definite or zero first Chern class is defined as $\{\omega(t)\}_{t\in\mathbb{R}_+}$ satisfying the evolution equation

$$\frac{\partial\omega(t)}{\partial t} = -\operatorname{Ric}\omega(t) + \mu\omega(t), \quad t \in \mathbb{R}_+,$$
$$\omega(0) = \omega,$$

where Ω is a Kähler class satisfying $\mu\Omega = c_1(M, \mathrm{J})$ for $\mu \in \{-1, 0, 1\}$ and $[\omega] = \Omega$ [Hamilton 1982].

The following dynamical system is seen to be a discrete version of this flow [Rubinstein 2008c, Definition 3.1], obtained by a backward Euler discretization with time step τ.

Definition 1.4. Let Ω be a Kähler class satisfying $\mu\Omega = c_1(M, \mathrm{J})$ for $\mu \in \{-1, 0, 1\}$. Given a Kähler form ω with $[\omega] = \Omega$ and a number $\tau > 0$, define the time-τ Ricci iteration to be the sequence of forms $\{\omega_{k\tau}\}_{k\geq 0}$ satisfying the equations

$$\frac{\omega_{k\tau} - \omega_{(k-1)\tau}}{\tau} = -\operatorname{Ric}\omega_{k\tau} + \mu\omega_{k\tau}, \quad k \in \mathbb{N},$$
$$\omega_0 = \omega.$$

Let us assume that $\mu = 1$ from now on; for the cases $\mu \in \{-1, 0\}$ see [Rubinstein 2008c, Theorem 3.3]. Observe that in the case when $\tau = 1$, the time-τ Ricci iteration is precisely the Ricci iteration from (1). Indeed, Conjecture 1.1 is in fact a special case of the following conjecture concerning the time-τ Ricci iteration for any $\tau > 0$ [Rubinstein 2008c, Conjecture 3.2].

Conjecture 1.5. *Let (M, J) be a compact Kähler manifold admitting a Kähler–Einstein metric. Let Ω be a Kähler class such that $\Omega = c_1(M, \mathrm{J})$. Then for any ω with $[\omega] = \Omega$ and for any $\tau > 0$, the time-τ Ricci iteration exists for all $k \in \mathbb{N}$ and converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric.*

The case when $\tau > 1$ is treated in [Rubinstein 2008c, Theorem 3.3]. However, it is the case $\tau \le 1$ that is the most interesting and challenging. The case $\tau = 1$ is perhaps the most interesting due to the simple geometrical interpretation (1), while the cases $\tau < 1$ are interesting due to the connection to the Kähler–Ricci flow. In this regime one may expect the Ricci iteration to converge to the Ricci flow in a certain scaling limit as $\tau \to 0$. The cases $\tau \le 1$ are challenging since the a priori estimates are considerably harder then. While in the regime $\tau > 1$, one has a uniform positive Ricci lower bound along the iteration; this is no longer true when $\tau \le 1$. Thus, there is no a priori control on the diameter or the Poincaré and Sobolev constants. We work around these difficulties by analyzing the Ricci iteration in the metric geometry of the space of Kähler potentials [Darvas 2015].

In this article, we confirm the more general Conjecture 1.5, and treat the iteration for all time steps τ by proving the following result, of which Theorem 1.2 is a special case.

Theorem 1.6. *Let (M, J, g_1) be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to g_1 is $c_1(M, \mathrm{J})$ and let $\{\omega_{k\tau}\}_{k\in\mathbb{N}}$ be the time-τ Ricci iteration given by Definition 1.4. Then there exist holomorphic diffeomorphisms h_k such that $h_k^*\omega_{k\tau}$ converges smoothly to a Kähler–Einstein form.*

2. Energy functionals

Let (M, ω) denote a connected compact closed Kähler manifold. The space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(M) : \omega_\varphi := \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi > 0\} \tag{3}$$

can be identified with $\mathcal{H} \times \mathbb{R}$, where

$$\mathcal{H} = \{\omega_\varphi : \varphi \in C^\infty(M),\ \omega_\varphi > 0\} \tag{4}$$

is the space of all Kähler metrics (or forms) representing the fixed cohomology class $[\omega]$.

From now on let ω be a Kähler form on M, cohomologous to $c_1(M, \mathrm{J})$. The Aubin–Mabuchi functional was introduced by Mabuchi [1986, Theorem 2.3],

$$\mathrm{AM}(\varphi) := \frac{V^{-1}}{n+1} \sum_{j=0}^{n} \int_M \varphi\omega^j \wedge \omega_\varphi^{n-j}, \tag{5}$$

where $V := \int_M \omega_\varphi^n = \int_M \omega_\varphi^n$ is the total volume of the Kähler class. Integration by parts gives the useful estimates

$$\frac{1}{V}\int_M (u-v)\omega_u^n \le \mathrm{AM}(u) - \mathrm{AM}(v) \le \frac{1}{V}\int_M (u-v)\omega_v^n. \tag{6}$$

The subspace

$$\mathcal{H}_0 := \mathrm{AM}^{-1}(0) \cap \mathcal{H}_\omega \tag{7}$$

is isomorphic to $\mathcal{H}$ (4), the space of Kähler metrics.

Let $f_{\omega_\varphi} \in \mathcal{C}^\infty(M)$ denote the unique function (called the Ricci potential of ω_φ) satisfying

$$\sqrt{-1}\,\partial\bar\partial f_{\omega_\varphi} = \operatorname{Ric}\omega_\varphi - \omega_\varphi, \quad \frac{1}{V}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n = 1.$$

The Ding and Mabuchi functionals are given by [Ding 1988; Mabuchi 1986]

$$\begin{aligned} D(\varphi) &:= -\operatorname{AM}(\varphi) - \log \frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n, \\ E(\varphi) &:= \frac{1}{V}\int_X \log \frac{\omega_\varphi^n}{e^{f_\omega}\omega^n}\omega_\varphi^n - \operatorname{AM}(\varphi) + \frac{1}{V}\int_M \varphi\omega_\varphi^n + \frac{1}{V}\int_M f_\omega\omega^n. \end{aligned} \tag{8}$$

Notice that these functionals are invariant under addition of constants to φ; hence they descend to $\mathcal{H}$. Additionally, the critical points of these functionals are exactly the Kähler–Einstein metrics.

For $\varphi \in \mathcal{H}_\omega$ with $\int_M e^{f_\omega - \varphi}\omega^n = V$, Jensen's inequality for the convex weight $t \to t\log t$ yields

$$\operatorname{Ent}(e^{f_\omega-\varphi}\omega^n, \omega_\varphi^n) := \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{e^{f_\omega-\varphi}\omega^n}\omega_\varphi^n = \frac{1}{V}\int_X \frac{\omega_\varphi^n}{e^{f_\omega-\varphi}\omega^n}\log\frac{\omega_\varphi^n}{e^{f_\omega-\varphi}\omega^n}e^{f_\omega-\varphi}\omega^n \geq 0. \tag{9}$$

Thus,

$$E(\omega_\varphi) - \frac{1}{V}\int_M f_\omega\omega^n = \operatorname{Ent}(e^{f_\omega-\varphi}\omega^n, \omega_\varphi^n) - \operatorname{AM}(\varphi) \geq -\operatorname{AM}(\varphi) = D(\omega_\varphi).$$

Moreover, if

$$D(\omega_\varphi) = E(\omega_\varphi) - \frac{1}{V}\int_M f_\omega\omega^n$$

then equality holds in (9). As a result, $\omega_\varphi^n = e^{f_\omega-\varphi}\omega^n = e^{f_{\omega_\varphi}}\omega_\varphi^n$; i.e., ω_φ is Kähler–Einstein. This together with the fact that Kähler–Einstein metrics minimize both D and E allows us to conclude the following result; see also [Rubinstein 2008b, (24)].

Proposition 2.1. *For* $\varphi \in \mathcal{H}_\omega$,

$$D(\omega_\varphi) \leq E(\omega_\varphi) - \frac{1}{V}\int_M f_\omega\omega^n,$$

with equality if and only if $\operatorname{Ric}\omega_\varphi = \omega_\varphi$.

3. The metric completion

All of the functionals introduced in the previous section can be extended to the potential space $\mathcal{E}_1$ introduced by Guedj and Zeriahi [2007], which can be identified with a natural metric completion of $\mathcal{H}$ [Darvas 2015]. The resulting metric theory provides essential tools for proving our main result concerning convergence of the Ricci iteration. We briefly recall this machinery, referring to [Darvas and Rubinstein 2017, §4–5] for more details.

Let

$$\operatorname{PSH}(M,\omega) = \{\varphi \in L^1(M,\omega^n) : \varphi \text{ is upper semicontinuous and } \omega_\varphi \geq 0\}.$$

Following [Guedj and Zeriahi 2007, Definition 1.1] we define the subset of full mass potentials

$$\mathcal{E}(M,\omega) := \left\{\varphi \in \mathrm{PSH}(M,\omega) : \lim_{j\to-\infty} \int_{\{\varphi\le j\}} (\omega + \sqrt{-1}\,\partial\bar\partial \max\{\varphi, j\})^n = 0\right\}.$$

For each $\varphi\in\mathcal{E}(M,\omega)$, define $\omega_\varphi^n := \lim_{j\to-\infty} \mathbf{1}_{\{\varphi>j\}}(\omega+\sqrt{-1}\,\partial\bar\partial \max\{\varphi, j\})^n$. By definition, $\mathbf{1}_{\{\varphi>j\}}(x)$ is equal to 1 if $\varphi(x) > j$ and zero otherwise, and the measure $(\omega + \sqrt{-1}\,\partial\bar\partial \max\{\varphi, j\})^n$ is defined by [Bedford and Taylor 1982] since $\max\{\varphi, j\}$ is bounded. Consequently, $\varphi \in \mathcal{E}(M,\omega)$ if and only if $\int_X \omega_\varphi^n = \int_X \omega^n$, justifying the name of $\mathcal{E}(M,\omega)$.

Next, define a further subset, the space of finite 1-energy potentials

$$\mathcal{E}_1 := \left\{\varphi \in \mathcal{E}(M,\omega) : \int |\varphi|\omega_\varphi^n < \infty\right\}.$$

Consider the following weak Finsler metric on $\mathcal{H}_\omega$ [Darvas 2015]:

$$\|\xi\|_\varphi := V^{-1}\int_M |\xi|\omega_\varphi^n, \quad \xi \in T_\varphi\mathcal{H}_\omega = C^\infty(M). \tag{10}$$

We denote by d_1 the associated pseudometric and recall the result alluded to above, characterizing the d_1-metric completion of $\mathcal{H}_\omega$ [Darvas 2015, Theorems 2 and 3.5]:

Theorem 3.1. *$(\mathcal{H}_\omega, d_1)$ is a metric space whose completion can be identified with $(\mathcal{E}_1, d_1)$, where*

$$d_1(u_0,u_1) := \lim_{k\to\infty} d_1(u_0(k), u_1(k))$$

for any smooth decreasing sequences $\{u_i(k)\}_{k\in\mathbb{N}} \subset \mathcal{H}_\omega$ converging pointwise to $u_i \in \mathcal{E}_1$, $i = 0, 1$.

Also, by [Darvas 2015, Theorem 3], we have the following qualitative estimates for the d_1-metric in terms of analytic quantities:

$$\frac{1}{C}d_1(u,v) \le \int_M |u-v|\omega_u^n + \int_M |u-v|\omega_v^n \le Cd_1(u,v), \quad u,v \in \mathcal{E}_1, \tag{11}$$

where $C > 1$ only depends on ω.

A crucial fact is that the formulas defining the energy functionals discussed in Section 2 actually make sense on the metric completion $\mathcal{E}_1$, and then coincide with the greatest lower semicontinuous extension of the said functionals restricted to $\mathcal{H}_\omega$ [Darvas and Rubinstein 2017, Lemma 5.2, Propositions 5.19 and 5.21]:

Lemma 3.2. (i) AM, $D : \mathcal{H}_\omega \to \mathbb{R}$ *each admit a unique d_1-continuous extension to $\mathcal{E}_1$ and these extensions still satisfy* (5) *and* (8) *respectively.*

(ii) $E : \mathcal{H}_\omega \to \mathbb{R}$ *admits a d_1-lower semicontinuous extension to $\mathcal{E}_1$ and the greatest such extension still satisfies* (8).

Proposition 2.1 was generalized by Berman [2013, Theorem 1.1] to the context of the metric completion (for a proof using the Ricci iteration see [Darvas 2017, Proposition 4.42]):

Theorem 3.3. *Proposition 2.1 holds more generally for all $\varphi \in \mathcal{E}_1$.*

Let $G := \mathrm{Aut}_0(M)$ denote the connected component of the complex Lie group of automorphisms (biholomorphisms) of M. The automorphism group acts on $\mathcal{H}$ by pullback:

$$f.\eta := f^\star \eta, \quad f \in G,\ \eta \in \mathcal{H}. \tag{12}$$

Given the one-to-one correspondence between $\mathcal{H}$ and $\mathcal{H}_0$, recall (7), the group G also acts on $\mathcal{H}_0$. The precise action is described in the next lemma [Darvas and Rubinstein 2017, Lemma 5.8].

Lemma 3.4. *For $\varphi \in \mathcal{H}_0$ and $f \in G$ let $f.\varphi \in \mathcal{H}_0$ be the unique potential such that $f^*\omega_\varphi = \omega_{f.\varphi}$. Then,*

$$f.\varphi = f.0 + \varphi \circ f. \tag{13}$$

Complementing the above, G acts on $\mathcal{H}_0$ by d_1-isometries [Darvas and Rubinstein 2017, Lemma 5.9], which allows us to introduce a natural (pseudo-)metric on the space $\mathcal{H}_0/G$:

$$d_{1,G}(Gu, Gv) = \inf_{g\in G} d_1(u, g.v), \quad u, v \in \mathcal{H}_0. \tag{14}$$

4. Metric convergence of the iteration

We consider the τ-step Ricci iteration equation

$$\frac{\omega_{\psi_{(k+1)\tau}} - \omega_{\psi_{k\tau}}}{\tau} = \omega_{\psi_{(k+1)\tau}} - \mathrm{Ric}\,\omega_{\psi_{(k+1)\tau}}$$

for $\tau \in (0, 1]$. When $\tau = 1$, the iteration simply becomes $\mathrm{Ric}\,\omega_{\psi_{k+1}} = \omega_{\psi_k}$. As explained in [Rubinstein 2008c, (33)], on the level of scalars the iteration can be written in the following manner:

$$\omega^n_{\psi_{(k+1)\tau}} = e^{f_\omega - \frac{1}{\tau}\psi_{k\tau} - \left(1-\frac{1}{\tau}\right)\psi_{(k+1)\tau}} \omega^n, \quad k \in \mathbb{N}, \tag{15}$$

with the natural normalization

$$\frac{1}{V}\int_M e^{f_\omega - \frac{1}{\tau}\psi_{k\tau} - \left(1-\frac{1}{\tau}\right)\psi_{(k+1)\tau}} \omega^n = 1. \tag{16}$$

Since $\tau \in (0, 1]$, note that (15)–(16) has a unique solution $\psi_{(k+1)\tau} \in \mathcal{H}_\omega$, according to [Aubin 1984; Yau 1978].

In our particular case, there will be special emphasis on working in the geodesically complete potential space $\mathcal{H}_0$, and we introduce accordingly

$$\psi'_{k\tau} := \psi_{k\tau} - \mathrm{AM}(\psi_{k\tau}) \in \mathcal{H}_0. \tag{17}$$

First we generalize an inequality of [Rubinstein 2008c] (in the case $\tau = 1$) that provides a comparison of the Ding and Mabuchi energies along the τ-iteration:

Proposition 4.1. *Suppose $\tau \in (0, 1]$ and (M, ω) is a Fano manifold and $\psi_{1\tau} \in \mathcal{H}_\omega$. Then the following estimate holds along the iteration*:

$$E(\omega_{\psi_{(k+1)\tau}}) - \frac{1}{V}\int_M f_\omega \omega^n \le \frac{1}{\tau} D(\omega_{\psi_{k\tau}}) + \left(1 - \frac{1}{\tau}\right) D(\omega_{\psi_{(k+1)\tau}}) \quad \textit{for all } k \in \mathbb{N}. \tag{18}$$

In the argument below (and thereafter) we will suppress the parameter τ from superscripts whenever this will cause no confusion.

Proof. Using (8) and (15),

$$\begin{aligned} E(\omega_{\psi_{k+1}})-\frac{1}{V}\int_M f_\omega\omega^n &= \frac{1}{V}\int_X \log\frac{\omega^n_{\psi_{k+1}}}{e^{f_\omega}\omega^n}\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1}) + \frac{1}{V}\int_M \psi_{k+1}\omega^n_{\psi_{k+1}} \\ &= -\frac{1}{V}\int_M\Big(\frac{1}{\tau}\psi_k+\Big(1-\frac{1}{\tau}\Big)\psi_{k+1}\Big)\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1}) + \frac{1}{V}\int_M \psi_{k+1}\omega^n_{\psi_{k+1}} \\ &= \frac{1}{\tau V}\int_M(\psi_{k+1}-\psi_k)\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1}). \end{aligned}$$

Using this identity, to finish the proof, we notice that it is enough to prove the following two inequalities (and later add them up):

$$\frac{1}{\tau V}\int_M(\psi_{k+1}-\psi_k)\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1}) \le -\frac{1}{\tau}\,\mathrm{AM}(\psi_k) - \Big(1-\frac{1}{\tau}\Big)\mathrm{AM}(\psi_{k+1}), \tag{19}$$

$$0 \le -\frac{1}{\tau}\log\Big(\frac{1}{V}\int_M e^{f_\omega-\psi_k}\omega^n\Big) - \Big(1-\frac{1}{\tau}\Big)\log\Big(\frac{1}{V}\int_M e^{f_\omega-\psi_{k+1}}\omega^n\Big). \tag{20}$$

Notice that, after rearranging terms, (19) is seen to be equivalent to

$$\frac{1}{V}\int_M(\psi_{k+1}-\psi_k)\omega^n_{\psi_{k+1}} \le \mathrm{AM}(\psi_{k+1})-\mathrm{AM}(\psi_k).$$

Thus, (19) follows from (6). To address (20) we prove the following more general claim.

Claim 4.2. *For $\tau \in (0,1]$ and $g,h \in C^\infty(X)$ the following estimate holds*:

$$\Big(\frac{1}{V}\int_M e^{f_\omega-g}\omega^n\Big)^{\frac{1}{\tau}}\Big(\frac{1}{V}\int_M e^{f_\omega-h}\omega^n\Big)^{1-\frac{1}{\tau}} \le \frac{1}{V}\int_M e^{f_\omega-\frac{1}{\tau}g-(1-\frac{1}{\tau})h}\omega^n. \tag{21}$$

By our choice of normalization (16), *this inequality implies* (20).

As (21) is seen to be invariant under adding constants to g and h, we can assume that

$$\frac{1}{V}\int_M e^{f_\omega-h}\omega^n = 1.$$

In particular, we only have to argue that

$$\Big(\frac{1}{V}\int_M e^{-g+h}e^{f_\omega-h}\omega^n\Big)^{\frac{1}{\tau}} \le \frac{1}{V}\int_M (e^{-g+h})^{\frac{1}{\tau}}e^{f_\omega-h}\omega^n.$$

This follows from Jensen's inequality, as the function $f(t)=t^{\frac{1}{\tau}}$ is convex for $t>0$. □

Next we show that in the case a Kähler–Einstein metric exists, the iteration $\{\psi'_k\}_k$ d_1-converges up to pullbacks:

Proposition 4.3. *Let $\tau \in (0,1]$. Suppose a Kähler–Einstein metric exists in $\mathcal{H}$, and let $\{\psi_{k\tau}\}_{k\in\mathbb{N}}$ be the solutions of* (15). *Then there exist $g_k \in G$ such that $g_k.\psi'_{k\tau}$ d_1-converges to a Kähler–Einstein potential.*

Proof. Proposition 4.1 combined with Proposition 2.1 gives

$$D(\omega_{\psi_{k+1}}) \le E(\omega_{\psi_{k+1}}) - \frac{1}{V}\int_M f_\omega \omega^n \le \frac{1}{\tau} D(\omega_{\psi_k}) + \left(1 - \frac{1}{\tau}\right) D(\omega_{\psi_{k+1}}), \quad k \in \mathbb{N}. \tag{22}$$

As a result, $\{D(\omega_{\psi_l})\}_l$ is a decreasing sequence (this is proved in [Rubinstein 2008c, Proposition 4.2(ii)] for $\tau = 1$). We fix a Kähler–Einstein potential

$$\psi_{\mathrm{KE}} \in \mathcal{H}_0.$$

Existence of such a potential implies that both D and E are bounded below [Bando and Mabuchi 1987; Ding and Tian 1992]. Therefore, the (monotone) sequence $\{D(\omega_{\psi_l})\}_l$ converges. Additionally, by (22), $\left\{E(\omega_{\psi_l}) - \frac{1}{V}\int_M f_\omega \omega^n\right\}_l$ converges and

$$\lambda := \lim_l E(\omega_{\psi_l}) - \frac{1}{V}\int_M f_\omega \omega^n = \lim_l D(\omega_{\psi_l}) \in \mathbb{R}.$$

Next we focus on the potentials $\psi'_l \in \mathcal{H}_0$. By [Darvas and Rubinstein 2017, Theorem 2.4], E is G-invariant and

$$E(\psi'_l) \ge C_1 d_{1,G}(0, \psi'_l) - C_2,$$

and so $d_{1,G}(0,\psi'_l) \le C'$. By definition, see (14), there exists $g_l \in G$ such that

$$d_1(\psi_{\mathrm{KE}}, g_l.\psi'_l) \le d_{1,G}(G\psi_{\mathrm{KE}}, G\psi'_l) + \frac{1}{l} \le C' + 1. \tag{23}$$

Remark 4.4. In fact, there exists g_l which achieve the equality $d_1(\psi_{\mathrm{KE}}, g_l.\psi'_l) = d_{1,G}(G\psi_{\mathrm{KE}}, G\psi'_l)$ by [Darvas and Rubinstein 2017, Proposition 6.8] but we do not have to know that for our proof here.

Setting

$$v_l := g_l.\psi'_l,$$

by the G-invariance of E, we obtain that $E(v_l)$ is bounded. On the other hand, a combination of (11) and (23) gives that $\mathrm{AM}(v_l) = 0$ and $\int_M v_l \omega^n_{v_l}$ are bounded as well. Comparing with (4), we see that $\mathrm{Ent}(e^{f_0}\omega^n, \omega^n_{v_l})$ is bounded too.

By (11), d_1-boundedness of potentials implies L^1-boundedness, which in turn implies boundedness of the supremum. As a result, we can apply the compactness result of [Berman et al. 2016a] (see [Darvas and Rubinstein 2017, Theorem 5.6] for a convenient formulation for our context) to conclude that $\{v_l\}_l$ is d_1-precompact.

Next we claim that $d_1(\psi_{\mathrm{KE}}, v_l) \to 0$. If this is not the case, then by possibly choosing a subsequence, we can assume that $d_1(\psi_{\mathrm{KE}}, v_l) > \varepsilon > 0$. By possibly choosing another subsequence, we can assume that $d_1(v_l, u) \to 0$ for some $u \in \mathcal{E}_1$. Lemma 3.2 gives that

$$\lambda = D(u) = E(u) - \frac{1}{V}\int_M f_\omega \omega^n,$$

and in particular u is a Kähler–Einstein potential by Theorem 3.3.

By the Bando–Mabuchi uniqueness theorem [1987] $u = h.\psi_{\mathrm{KE}}$ for some $h \in G$. Combining this with (23), we conclude that

$$d_1(v_{k_l}, \psi_{\mathrm{KE}}) - \frac{1}{k_l} \le d_{1,G}(Gv_l, G\psi_{\mathrm{KE}}) \le d_1(h^{-1}v_l, \psi_{\mathrm{KE}}) = d_1(v_l, h.\psi_{\mathrm{KE}}) = d_1(v_l, u).$$

By choice, the right-hand side converges to zero, and the lim inf of left-hand side is bounded below by $\varepsilon > 0$, giving a contradiction. This implies that $d_1(v_k, \psi_{\mathrm{KE}}) \to 0$, concluding the proof. □

5. A priori estimates and smooth convergence

In this section we prove our main result by strengthening Proposition 4.3.

Theorem 5.1. *Let $\tau \in (0, 1]$. Suppose a Kähler–Einstein metric exists in $\mathcal{H}$, and let $\{\psi_{k\tau}\}_{k\in\mathbb{N}}$ be the solutions of* (15). *Then there exist $g_k \in G$ such that $g_k.\psi'_{k\tau}$ converges smoothly to a Kähler–Einstein potential. In particular, $g_k^*\omega_{\psi_{k\tau}}$ converges smoothly to a Kähler–Einstein metric.*

Proof. By Proposition 4.3 there exists $g_k \in G$ and a Kähler–Einstein potential $\psi_{\mathrm{KE}} \in \mathcal{H}_0$ such that $d_1(g_k.\psi'_k, \psi_{\mathrm{KE}}) \to 0$. We show below that in fact $g_k.\psi'_k \to_{C^\infty} \psi_{\mathrm{KE}}$.

Focusing on the τ-step Ricci iteration recursion, we can write

$$\begin{aligned}(g_{k+1}^{-1} \circ g_k)^* \operatorname{Ric} \omega_{g_{k+1}.\psi'_{k+1}} &= g_k^* \operatorname{Ric} \omega_{\psi'_{k+1}} = g_k^*\left(\frac{1}{\tau}\omega_{\psi'_k} + \left(1 - \frac{1}{\tau}\right)\omega_{\psi'_{k+1}}\right)\\ &= \frac{1}{\tau}\omega_{g_k.\psi'_k} + \left(1 - \frac{1}{\tau}\right)\omega_{g_k.\psi'_{k+1}}\\ &= \frac{1}{\tau}\omega_{g_k.\psi'_k} + \left(1 - \frac{1}{\tau}\right)\omega_{(g_{k+1}^{-1}\circ g_k).g_{k+1}.\psi'_{k+1}}. \end{aligned} \tag{24}$$

Set

$$\begin{aligned}\varphi_k &:= g_k.\psi'_k \in \mathcal{H}_0,\\ f_k &:= g_k^{-1} \circ g_{k-1} \in G.\end{aligned}$$

With this notation, (24) becomes

$$\operatorname{Ric} \omega_{f_{k+1}.\varphi_{k+1}} = \frac{1}{\tau}\omega_{\varphi_k} + \left(1 - \frac{1}{\tau}\right)\omega_{f_{k+1}.\varphi_{k+1}}. \tag{25}$$

Without loss of generality we assume that ω (the reference form) is Kähler–Einstein. Using (25) we can write

$$\sqrt{-1}\,\partial\bar\partial\left(\frac{1}{\tau}\varphi_{k-1} + \left(1 - \frac{1}{\tau}\right) f_k.\varphi_k\right) = \operatorname{Ric} \omega_{f_k.\varphi_k} - \operatorname{Ric}\omega = \sqrt{-1}\,\partial\bar\partial \log(\omega^n/\omega^n_{f_k.\varphi_k}).$$

This implies

$$\frac{1}{\tau}\varphi_{k-1} + \left(1 - \frac{1}{\tau}\right) f_k.\varphi_k + \log(\omega^n_{f_k.\varphi_k}/\omega^n) = B_j \in \mathbb{R}.$$

Since log is a concave function, by Jensen's inequality,

$$\frac{1}{V}\int_M \log(\omega^n_{f_k.\varphi_k}/\omega^n)\omega^n \le \log\frac{1}{V}\int_M \omega^n_{f_k.\varphi_k} = 0.$$

By the triangle inequality, for k sufficiently large,

$$d_1(0, \varphi_{k-1}) \leq d_1(\psi_{\mathrm{KE}}, 0) + 1.$$

Using (11) we conclude that $\int_M \varphi_{k-1}\omega^n \leq C$. These last two estimates combine to give

$$B_j - \left(1 - \frac{1}{\tau}\right)\frac{1}{V}\int_M f_k.\varphi_k\omega^n = \frac{1}{V}\int_M \varphi_{k-1}\omega^n + \frac{1}{V}\int_M \log(\omega^n_{f_k.\varphi_k}/\omega^n)\omega^n \leq C.$$

Since $f_k.\varphi_k \in \mathrm{PSH}(M, \omega)$, it is well known that $\int_M f_k.\varphi_k\omega^n$ and $\sup_M f_k.\varphi_k$ are comparable. As a result,

$$B_j - \left(1 - \frac{1}{\tau}\right)\sup_M f_k.\varphi_k \leq C;$$

hence we can write

$$\omega^n_{f_k.\varphi_k} = e^{B_j - (1-\frac{1}{\tau}) f_k.\varphi_k - \frac{1}{\tau}\varphi_{k-1}}\omega^n \leq e^{C - \frac{1}{\tau}\varphi_{k-1}}\omega^n. \tag{26}$$

Moreover, by Zeriahi's version of the Skoda integrability theorem [Zeriahi 2001] (see [Darvas and Rubinstein 2017, Theorem 5.7] for a formulation that fits our context most), there exists $C > 0$ such that, say,

$$\int_M e^{-\frac{3}{\tau}\varphi_{k-1}}\omega^n \leq C, \quad k \in \mathbb{N}.$$

Combining this estimate with (26), we get that

$$\|\omega^n_{f_k.\varphi_k}/\omega^n\|_{L^3(M,\omega^n)} \leq C.$$

Now Kołodziej's estimate [2005], see also [Błocki 2005], allows us to conclude that the oscillation satisfies $\operatorname{osc} f_k.\varphi_k \leq C$ for some uniform C. Note that for any $u \in \mathcal{H}_0$, it follows from (6) that

$$\inf u \leq \frac{1}{V}\int u\omega^n_u \leq 0 \leq \frac{1}{V}\int u\omega^n \leq \sup u,$$

so u changes signs on M. Thus, since $f_k.\varphi_k \in \mathcal{H}_0$, the oscillation bound implies the uniform bound

$$\|f_k.\varphi_k\|_{L^\infty(M)} \leq C. \tag{27}$$

Consequently, (11) yields

$$d_1(0, f_k.\varphi_k) = d_1(f_k^{-1}.0, \varphi_k) \leq C.$$

Thus,

$$d_1(f_k^{-1}.0, 0) \leq d_1(f_k^{-1}.0, \varphi_k) + d_1(\varphi_k, 0) \leq C'.$$

From Lemma 5.2, proved below, it follows that $\{f_k^{-1}\}_k$ is contained in a bounded set of G. In particular, all derivatives up to order m of f_k^{-1} are bounded by some C_m, independent of k. So, to finish the proof, it suffices to estimate derivatives of

$$h_k := f_k.\varphi_k$$

(since that will imply the same estimates on $f_k^{-1}.h_k = \varphi_k$). From (25) it follows that

$$\operatorname{Ric}\omega_{h_{k+1}} = \operatorname{Ric}\omega_{f_{k+1}.\varphi_{k+1}} \geq \Big(1-\frac{1}{\tau}\Big)\omega_{f_{k+1}.\varphi_{k+1}} = \Big(1-\frac{1}{\tau}\Big)\omega_{h_{k+1}}.$$

Using this, Lemma 5.3 implies $\operatorname{tr}_{\omega_{h_k}}\omega < C$, and using the fact that $\omega^n_{h_k}/\omega^n < C$ by (26) we thus obtain $\operatorname{tr}_\omega \omega_{h_k} < C'$ so $|\Delta_\omega h_k| < C''$, as in [Rubinstein 2008c, p. 1540]. Given the Laplacian bound, the $C^{2,\alpha}$ and higher-order estimates then follow the same way as in [Rubinstein 2008c, Theorem 3.3] (or by applying [Błocki 2012, Theorem 5.1] directly to (26), followed by bootstrapping).

By the Arzelà–Ascoli compactness theorem, $\{\varphi_k\}_k$ is C^k-precompact. From (11) it follows that C^k-convergence implies d_1-convergence. Consequently, any C^k-convergent subsequence of $\{\varphi_k\}_k$ d_1-converges to ψ_{KE}. As a result, $\{\varphi_k\}_k$ C^k-converges to ψ_{KE}, finishing the proof. □

We note that in our arguments above the estimates depend on a positive lower bound to $\tau > 0$. If this could be avoided, then one could hope that these estimates also hold in a scaled limit, as the iteration is expected to converge to the Kähler–Ricci flow.

Lemma 5.2. *Let (X,ω) be a Fano Kähler–Einstein manifold. Let $C > 0$ and suppose that $d_1(g.0,0) \leq C$ for some $g \in G$. Then g is contained in a geodesic ball $B \subset G$ centered at* Id *with radius $R := R(C) > 0$.*

This result is implicit in the arguments of [Darvas and Rubinstein 2017, Proposition 6.8]; see also [Berman et al. 2016b, Lemma 2.7; Darvas and Rubinstein 2017, Claim 7.11].

Proof. By [Darvas and Rubinstein 2017, Propositions 6.2 and 6.9] there exists $k \in \operatorname{Isom}_0(X,\omega)$ and a Hamiltonian vector field $X \in \operatorname{isom}(X,\omega)$ such that $g = k\exp_{\mathrm{Id}} JX$, where $\exp_{\mathrm{Id}}$ is the exponential map of the Lie group G (recall that J is the complex structure of X). It is clear from the definition of the action of G on the level of potentials that $k^{-1}.0 = 0$. Thus we can write

$$C \geq d_1(g.0,0) = d_1(k\exp_{\mathrm{Id}}(JX).0,0) = d_1(\exp_{\mathrm{Id}}(JX).0,k^{-1}.0) = d_1(\exp_{\mathrm{Id}}(JX).0,0).$$

As shown in [Darvas and Rubinstein 2017, Section 7.1], the curve $[0,\infty) \ni t \to \exp_{\mathrm{Id}}(tJX).0 \in \mathcal{H}_0$ is a d_1-geodesic ray, and hence $\|X\|$ is bounded. Since $\operatorname{Isom}_0(X,\omega)$ is compact, we obtain that $g = k\exp_{\mathrm{Id}} JX$ is contained in a geodesic ball $B \subset G$ centered at Id with radius $R := R(C) > 0$. □

For the sake of completeness we recall a version of the Chern–Lu inequality, going back to [Lu 1968], that gives the Laplacian estimate based on a C^0 estimate, elaborated in [Rubinstein 2008c, pp. 1539–1540]; see also [Jeffres et al. 2016, Lemma 7.2]. Since it is stated there in the context of incomplete edge metrics, we state here the simpler smooth version, which follows by setting $D = \varnothing$ in [Jeffres et al. 2016, Lemma 7.2] or [Rubinstein 2014, Corollary 7.8(i)]. Recall that $\operatorname{osc} f := \sup f - \inf f$.

Lemma 5.3. *Let $\varphi \in C^4(M) \cap \mathcal{H}_\omega$. Suppose that* $\operatorname{Ric}\omega_\varphi \geq -C_1\omega - C_2\omega_\varphi$. *Then for some $C = C(M,\omega,C_1,C_2,\operatorname{osc}\varphi) > 0$,*

$$\operatorname{tr}_{\omega_\varphi}\omega \leq C. \tag{28}$$

Proof. Let $f:(M,\omega_\varphi)\to(M,\omega)$ be the identity map. Then consider the Chern–Lu inequality, see, e.g., [Rubinstein 2014, Proposition 7.1],

$$|\partial f|^2\Delta_{\omega_\varphi}\log|\partial f|^2\geq(\operatorname{Ric}\omega_\varphi)^\#\otimes\omega(\partial f,\bar\partial f)-\omega_\varphi{}^\#\otimes\omega_\varphi{}^\#\otimes R_\omega(\partial f,\bar\partial f,\partial f,\bar\partial f),\tag{29}$$

whose meaning (and proof) in local coordinates we now explain. Write

$$\omega_\varphi=\sqrt{-1}g_{i\bar j}(z)dz^i\wedge\overline{dz^j},\quad \omega=\sqrt{-1}h_{i\bar j}(w)dw^i\wedge\overline{dw^j},$$

where we choose *two* holomorphic coordinate charts $(z_1,\dots,z_n)$ and $(w_1,\dots,w_n)$, respectively, centered at the same point $z_0=f(z_0)\in M$ such that the first is normal for ω_φ, while the second is normal for ω. Write $f:z=(z^1,\dots,z^n)\mapsto f(z)=(f^1(z),\dots,f^n(z))$. Then,

$$\partial f=f_i^j\,dz^i|_z\otimes\frac{\partial}{\partial w^j}\bigg|_{f(z)},$$

and the norm of ∂f induced from considering f as the map $f:(M,\omega_\varphi)\to(M,\omega)$ is then $|\partial f|^2=g^{i\bar l}(z)h_{j\bar k}(f(z))f_i^j(z)\overline{f_l^k(z)}$. Thus, at z_0,

$$\begin{aligned}\Delta_\omega|\partial f|^2&=\sum_{p,q}g^{p\bar q}\frac{\partial^2(g^{i\bar l}h_{j\bar k}f_i^j\overline{f_l^k})}{\partial z^p\overline{\partial z^p}}\\&=\sum_p g^{p\bar q}[g^{i\bar l}h_{j\bar k,d\bar e}f_i^j\overline{f_l^k}f_p^d\overline{f_q^e}-h_{j\bar k}g^{i\bar t}g^{s\bar l}g_{s\bar t,p\bar q}f_i^j\overline{f_l^k}+g^{i\bar l}h_{j\bar k}f_{ip}^j\overline{f_{lq}^k}]\\&=-\omega_\varphi^\#\otimes\omega_\varphi^\#\otimes R_\omega(\partial f,\bar\partial f,\partial f,\bar\partial f)+(\operatorname{Ric}\omega_\varphi)^\#\otimes\omega(\partial f,\bar\partial f)+g^{p\bar q}g^{i\bar l}h_{j\bar k}f_{ip}^j\overline{f_{lq}^k}.\end{aligned}\tag{30}$$

Here R_ω denotes the curvature tensor of η (of type $(0,4)$), while $\omega^\#$ denotes the metric g^{-1} on $T^{1,0\star}M$ (i.e., of type $(2,0)$), and similarly $(\operatorname{Ric}\omega)^\#$ denotes the $(2,0)$-type tensor obtained from $\operatorname{Ric}\omega_\varphi$ by raising indices using g. The proof of (29) now follows from (30), the identity $u\Delta_\omega\log u=\Delta_\omega u-u|\partial\log u|^2$, and the Cauchy–Schwarz inequality; see [Rubinstein 2014, p. 102].

We claim that (29) implies

$$\Delta_{\omega_\varphi}(\log\operatorname{tr}_{\omega_\varphi}\omega-(C_2+2C_3+1)\varphi)\geq -C_1-(C_2+2C_3+1)n+\operatorname{tr}_{\omega_\varphi}\omega,\tag{31}$$

where C_3 depends on the curvature of ω. Indeed, the assumption on $\operatorname{Ric}\omega_\varphi$ implies

$$\begin{aligned}(\operatorname{Ric}\omega_\varphi)^\#\otimes\omega(\partial f,\bar\partial f)&=g^{i\bar l}g^{k\bar j}R_{i\bar j}h_{k\bar l}\\&\geq -C_1g^{i\bar l}g^{k\bar j}g_{i\bar j}h_{k\bar l}-C_2g^{i\bar l}g^{k\bar j}h_{i\bar j}h_{k\bar l}\\&\geq -C_1\operatorname{tr}_{\omega_\varphi}\omega-C_2(\operatorname{tr}_{\omega_\varphi}\omega)^2.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}-\omega_\varphi{}^\#\otimes\omega_\varphi{}^\#\otimes R^\omega(\partial f,\bar\partial f,\partial f,\bar\partial f)&=-g^{i\bar j}g^{k\bar l}R^\omega_{i\bar jk\bar l}\\&\geq -C_3g^{i\bar j}g^{k\bar l}(h_{i\bar j}h_{k\bar l}+h_{i\bar l}h_{k\bar j})\geq -2C_3(\operatorname{tr}_{\omega_\varphi}\omega)^2,\end{aligned}$$

where C_3 is an upper bound for the bisectional curvature of ω. Finally, the claim follows since $\mathrm{tr}_{\omega_\varphi}\,\omega = \mathrm{tr}_{\omega_\varphi}(\omega_\varphi - \sqrt{-1}\,\partial\bar\partial\varphi) = n - \Delta_{\omega_\varphi}\varphi$.

Using the inequality now in (31) (at the point where the maximum of $\log \mathrm{tr}_{\omega_\varphi}\,\omega - (C_2 + 2C_3 + 1)\varphi$ is attained), the maximum principle thus gives an estimate on $\mathrm{tr}_{\omega_\varphi}\,\omega$, depending of course also on $\operatorname{osc}\varphi$. $\square$

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CONCENTRATION OF GROUND STATES IN STATIONARY MEAN-FIELD GAMES SYSTEMS

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We provide the existence of classical solutions to stationary mean-field game systems in the whole space $\mathbb{R}^N$, with coercive potential and aggregating local coupling, under general conditions on the Hamiltonian. The only structural assumption we make is on the growth at infinity of the coupling term in terms of the growth of the Hamiltonian. This result is obtained using a variational approach based on the analysis of the nonconvex energy associated to the system. Finally, we show that in the vanishing viscosity limit, mass concentrates around the flattest minima of the potential. We also describe the asymptotic shape of the rescaled solutions in the vanishing viscosity limit, in particular proving the existence of ground states, i.e., classical solutions to mean-field game systems in the whole space without potential, and with aggregating coupling.

1. Introduction

We consider a class of ergodic mean-field games systems set on the whole space $\mathbb{R}^N$ with unbounded decreasing coupling: our problem is, given $\varepsilon > 0$ and $M > 0$, to find a constant $\lambda \in \mathbb{R}$ for which there exists a pair $(u, m) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$, for any $p > 1$, solving

$$\begin{cases} -\varepsilon\Delta u + H(\nabla u) + \lambda = f(m) + V(x), \\ -\varepsilon\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0 \quad \text{on } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m = M. \end{cases} \tag{1-1}$$

The aim of this work is two-fold. Firstly, for any fixed $\varepsilon > 0$, we prove the existence of classical ground states of (1-1). Secondly, we study their behavior in the vanishing viscosity limit $\varepsilon \to 0$.

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The Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$ is strictly convex, $H \in C^2(\mathbb{R}^N \setminus \{0\})$ and it has superlinear growth: we assume that there exist $C_H > 0$, $K > 0$ and $\gamma > 1$ such that, for all $p \in \mathbb{R}^N$,

$$\begin{gathered} C_H|p|^\gamma - K \le H(p) \le C_H|p|^\gamma, \\ \nabla H(p)\cdot p - H(p) \ge K^{-1}|p|^\gamma - K \quad \text{and} \quad |\nabla H(p)| \le K|p|^{\gamma-1}. \end{gathered} \tag{1-2}$$

The coupling term $f : [0, +\infty) \to \mathbb{R}$ is a locally Lipschitz continuous function such that there exist $C_f > 0$ and $K > 0$ for which

$$-C_f m^\alpha - K \le f(m) \le -C_f m^\alpha + K, \tag{1-3}$$

with

$$0 < \alpha < \frac{\gamma}{N(\gamma-1)} = \frac{\gamma'}{N}, \tag{1-4}$$

where $\gamma' = \frac{\gamma}{\gamma-1}$ is the conjugate exponent of γ.

Finally, we assume that the potential V is a locally Hölder continuous function, and that there exist $b > 0$ and a constant $C_V > 0$ such that

$$C_V^{-1}(\max\{|x| - C_V, 0\})^b \le V(x) \le C_V(1 + |x|)^b. \tag{1-5}$$

Note that the requirement of V to be nonnegative is not crucial; we just need it to be bounded from below.

Mean-field games (MFG) is a recent theory that models the behavior of a very large number of indistinguishable rational agents, aiming at minimizing a common cost. The theory was introduced in the seminal works by Lasry and Lions [2006a; 2006b; 2007] and by Huang, Malhamé and Caines [Huang et al. 2006], and has been rapidly growing during the last decade due to its mathematical challenges and several potential applications (from economics and finance, to engineering and models of social systems). In the ergodic MFG setting, the dynamics of a typical agent is given by the controlled stochastic differential equation

$$dX_s = -v_s\,ds + \sqrt{2\varepsilon}\,dB_s, \quad s > 0,$$

where v_s is the control and B_s is a Brownian motion, with initial state given by a random variable X_0. The cost (of long-time average form) is given by

$$\lim_{T\to\infty} \frac{1}{T}\,\mathbb{E}\int_0^T [L(v_s) + V(X_s) + f(m(X_s))]\,ds,$$

where the Lagrangian L is the Legendre transform of H, see (2-1), and $m(x)$ denotes the density of population of small agents at a position $x \in \mathbb{R}^N$. A typical agent minimizes his own cost, and the density of its corresponding distribution law $\mathcal{L}(X_s)$ converges, as $s \to \infty$, to a stationary density μ, which is independent of the initial distribution $\mathcal{L}(X_0)$. In an equilibrium regime, μ coincides with the population density m. This equilibrium is encoded from the PDE viewpoint in (1-1): a solution u of the Hamilton–Jacobi–Bellman (HJB) equation gives an optimal control for the typical agent in feedback form $\nabla H(\nabla u(\cdot))$, and the Kolmogorov equation provides the density m of the agents playing in an optimal way.

The two key points of our setting are the following: Firstly, the cost is monotonically *decreasing* with respect to the population distribution m; namely, agents are attracted toward congested areas. A large part of the MFG literature focuses on the study of systems with competition, namely when the coupling in the cost is monotonically increasing. This assumption is essential if one seeks for uniqueness of equilibria, and it is in general crucial in many existence and regularity arguments; see, e.g., [Gomes et al. 2016]. On the other hand, models with aggregation like (1-1) have been considered in few cases, see [Cesaroni and Cirant 2017; Cirant 2016; 2017; Cirant and Tonon 2018; Gomes et al. 2018].

Secondly, the state of a typical agent here is the *whole euclidean space* $\mathbb{R}^N$. Usually, the analysis of (1-1) is carried out in the periodic setting, in order to avoid boundary issues and the noncompactness of $\mathbb{R}^N$. Few investigations are available in the truly nonperiodic setting: see [Porretta 2017] for time-dependent problems, [Arapostathis et al. 2017] for the case of bounded controls, [Gomes and Pimentel 2016] for some regularity results and [Bardi and Priuli 2014] for the linear-quadratic framework. We observe that the noncompact setting is even more delicate for stationary (ergodic) problems like (1-1): a stable long-time regime of a typical player is ensured if the Brownian motion is compensated by the optimal velocity v_s. In other words, if a force that drives players to bounded states is missing, dissipation eventually leads their distribution to vanish on the whole $\mathbb{R}^N$. This phenomenon is impossible if the state space is compact. The main issue here is that the behavior of the optimal velocity $v_s(\cdot) = \nabla H(\nabla u(\cdot))$ is a priori unknown, and depends in an implicit way on V and the distribution m itself. Note that $V(\cdot)$ represents the spatial preference of a single agent; if it grows as $|x| \to \infty$, it discourages agents from being far away from the origin. At the PDE level, this will compensate the lack of compactness of $\mathbb{R}^N$. Let us mention that even without the coupling term $f(m^\alpha)$, the ergodic control problem in unbounded domains has received considerable attention; see, e.g., [Barles and Meireles 2016; Ichihara 2011; 2015].

In our analysis, we exploit the variational nature of the system (1-1), which has been pointed out already in the first papers on MFG, see [Lasry and Lions 2007], and the more recent work [Mészáros and Silva 2018]. Indeed, solutions to (1-1) can be put in correspondence with critical points of the energy

$$\mathcal{E}(m,w) := \begin{cases} \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V(x)m + F(m)\,dx & \text{if } (m,w) \in \mathcal{K}_{\varepsilon,M}, \\ +\infty & \text{otherwise,} \end{cases} \tag{1-6}$$

where $F(m) = \int_0^m f(n)\,dn$ for $m \geq 0$ and $F(m) = 0$ for $m \leq 0$ and

$$L\left(-\frac{w}{m}\right) := \begin{cases} \sup_{p\in\mathbb{R}^N}\left(-\frac{p\cdot w}{m} - H(p)\right) & \text{if } m > 0, \\ 0 & \text{if } m = 0,\ w = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{1-7}$$

Note that $mL\left(-\frac{\cdot}{m}\right)$ reads as the Legendre transform of $mH(\cdot)$. The constraint set is defined as

$$\mathcal{K}_{\varepsilon,M} := \Big\{(m,w) \in L^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)\times L^1(\mathbb{R}^N) : \\ \varepsilon \int_{\mathbb{R}^N} m(-\Delta\varphi)\,dx = \int_{\mathbb{R}^N} w\cdot\nabla\varphi\,dx \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N),\ \int_{\mathbb{R}^N} m\,dx = M,\ m \geq 0 \text{ a.e.}\Big\}, \tag{1-8}$$

with

$$q = \begin{cases} \frac{N}{N-\gamma'+1}, & \gamma' \leq N, \\ \gamma', & \gamma' > N. \end{cases}$$

Under assumption (1-3) on the coupling term, the energy $\mathcal{E}$ is not convex. Condition (1-4) is necessary for the problem $e_\varepsilon(M) := \min_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m,w)$ to be well-posed. Indeed, consider any

$(m_0, w_0) \in \mathcal{K}_{\varepsilon,M}$ such that m_0 has compact support. An easy computation shows that if $\alpha > \frac{\gamma'}{N}$, then

$$\mathcal{E}(\sigma^{-N} m_0(\sigma^{-1}\cdot), \sigma^{-(N+1)} w_0(\sigma^{-1}\cdot)) \to -\infty$$

as $\sigma \to 0$, so $\mathcal{E}$ is not bounded from below on $\mathcal{K}_{\varepsilon,M}$. We show that (1-4) is indeed sufficient for $e_\varepsilon(M)$ to be finite, and allows us to look for *ground states* of (1-1). This will be accomplished by a study of the Sobolev regularity of the Kolmogorov equation; see in particular Section 2B. Note that the critical case $\alpha = \frac{\gamma'}{N}$ is more delicate, and requires additional analysis. We also mention that another critical exponent is intrinsic in (1-1): if $\alpha > \frac{\gamma'}{N-\gamma'}$, one has to expect nonexistence of solutions; see [Cirant 2016]. We refer to our case as the *subcritical case*, in analogy with the L^2-subcritical regime in nonlinear Schrödinger equations with prescribed mass; see [Cirant 2016, Remark 2.9] for additional comments. The analogy can be made precise in the purely quadratic framework, that is when $H(p) = \frac{1}{2}|p|^2$. Indeed, as observed in [Lasry and Lions 2006a; 2006b], the so-called Hopf–Cole transformation permits us to reduce the number of unknowns in the system. Setting $v^2(x) := m(x) = ce^{-\frac{u(x)}{\varepsilon}}$, with c a normalizing constant, v is a solution to

$$-2\varepsilon^2 \Delta v + (V(x) - \lambda)v = -f(v^2)v,$$

with $\int_{\mathbb{R}^N} v^2(x)\,dx = M$. Then the energy reads $\mathcal{E}(v) = \int_{\mathbb{R}^N} \varepsilon^2 |\nabla v|^2 + \frac{1}{2}V(x)v^2 + \frac{1}{2}F(v^2)\,dx$.

In our approach, to construct solutions to (1-1), we look for minimizers $(m, w) \in \mathcal{K}_{\varepsilon,M}$ of the energy (1-6). These minimizers can be obtained by classical direct methods, by using in particular estimates and compactness in some L^p space for elements (m, w) in $\mathcal{K}_{\varepsilon,M}$ with bounded action, i.e., which satisfy $\int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx \le C$, obtained in Section 2B. Then, the existence of a solution $(u_\varepsilon, \lambda_\varepsilon)$ of the HJB equation in (1-1) is obtained by considering another functional with linearized coupling (around the minimizer) and the associated dual functional in the sense of Fenchel and Rockafellar, as in [Briani and Cardaliaguet 2018]. One has to take care of the interplay between u and m as $|x| \to \infty$. To handle the lack of a priori regularity on the function m, we first regularize the problem, by applying standard regularizing convolution kernels on the coupling (see Section 3). We construct minimizers (m_k, w_k) of the regularized energy and associated solutions (u_k, m_k) of the regularized version of (1-1). Then, in order to come back to the initial problem, we provide some new a priori uniform L^∞ bounds on m_k, which in turn imply a priori uniform bounds on $|\nabla u_k|$ and (local) Hölder regularity of m_k that is uniform in k. This key a priori bound is provided by Theorem 4.1.

Note that we will consider classical solutions to this system (with a slight abuse of terminology), that is, $(u, m) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ for all $p > 1$. The existence result, proved in Section 4, is the following.

Theorem 1.1. *Under the assumptions* (1-2), (1-3), (1-4) *and* (1-5), *for every* $\varepsilon > 0$ *there exists a classical solution* $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$, *for all* $p > 1$, *to* (1-1). *Moreover*, $(m_\varepsilon, -m_\varepsilon \nabla H(\nabla u_\varepsilon))$ *is a minimizer in the set* $\mathcal{K}_{\varepsilon,M}$ *of the energy* (1-6).

We observe (see Remarks 3.5 and 4.2) that Theorem 1.1 holds under more general conditions on H and f, that is, if there exist $C_H, C_f > 0$ and $K > 0$ such that

$$C_H^{-1}|p|^\gamma - K \le H(p) \le C_H(|p|^\gamma + 1), \quad -C_f m^\alpha - K \le f(m) \le C_f^{-1} m^\alpha + K, \tag{1-9}$$

where α satisfies (1-4).

In the second part of the work, in Section 5, we analyze the behavior of the triple $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ coming from a minimizer of $\mathcal{E}$ as $\varepsilon \to 0$, under the assumptions (1-2), (1-3). From the viewpoint of the model, this amounts to removing the Brownian noise from the agents' dynamics. Heuristically, if the diffusion becomes negligible, one should observe aggregation of players (induced by the decreasing monotonicity of coupling in the cost) towards minima of the potential V, which are the preferred sites. Moreover, in the case V has a finite number of minima and polynomial behavior (that is, when (1-13) holds) we specialize the result showing that the limit procedure selects the more stable minima of V, implying, e.g., full convergence in the case that there exists a unique flattest minimum.

In order to bring as much information as possible to the limit, we consider an appropriate rescaling of m, u, namely

$$\bar{m}_\varepsilon(\cdot) = \varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}} m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon), \quad \bar{u}_\varepsilon(\cdot) = \varepsilon^{\frac{N\alpha(\gamma'-1)-\gamma'}{\gamma'-\alpha N}} (u(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon) - u(x_\varepsilon)) \tag{1-10}$$

for all $\varepsilon > 0$. The rescaling is designed so that $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ solves an MFG system where the nonlinearities have the same behavior of the original ones; i.e., $H_\varepsilon \sim |p|^\gamma$ as $p \to \infty$, but the coefficient in front of the Laplacian is equal to 1 for all ε; see (5-19). Moreover, the pair $\bar{u}_\varepsilon, \bar{m}_\varepsilon$ is associated to a minimizer of a rescaled energy $\mathcal{E}_\varepsilon$; see (5-23). It turns out that in this rescaling process, the potential V becomes

$$V_\varepsilon(\cdot) = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot),$$

and vanishes (locally) as $\varepsilon \to 0$. Therefore, as one passes to the limit, the potential cannot compensate anymore for the lack of compactness of $\mathbb{R}^N$, and the convergence of $\bar{m}_\varepsilon$ in $L^1(\mathbb{R}^N)$ has to be proven by other methods. Heuristically, the aggregating force should be strong enough to overcome the dissipation effect, but the clustering point can be hard to predict by lack of spatial preference. This is why we also have to translate in (1-10) by x_ε. We will select x_ε to be the minimum of u_ε: heuristically, since u_ε is the value function, this is the point where most of the players should be located. In order to recover compactness for the sequence $\bar{m}_\varepsilon$, we implement some ideas of the celebrated *concentration-compactness* method [Lions 1984]. This principle states intuitively that if loss of compactness occurs, $\bar{m}_\varepsilon$ splits in (at least) two parts which are going infinitely far away from each other; that is,

$$\bar{m}_\varepsilon \sim \chi_{B_R(0)}\bar{m}_\varepsilon + \chi_{\mathbb{R}^N\setminus B_{2R}(0)}\bar{m}_\varepsilon, \tag{1-11}$$

with $R \to \infty$, $\int \chi_{B_R(0)}\bar{m}_\varepsilon \sim a$ and $\int \chi_{\mathbb{R}^N\setminus B_{2R}(0)}\bar{m}_\varepsilon \sim M - a$ for some $a \in (0, M)$ (a third possibility might happen, but it is easily ruled out here by local estimates). This induces a splitting in the energy $\mathcal{E}$; that is,

$$\inf_{\int m = M} \mathcal{E}_\varepsilon \gtrsim \inf_{\int m = a} \mathcal{E}_\varepsilon + \inf_{\int m = M-a} \mathcal{E}_\varepsilon. \tag{1-12}$$

One then exploits a special feature of $\mathcal{E}_\varepsilon$, which is called subadditivity:

$$\inf_{\int m = M} \mathcal{E}_\varepsilon < \inf_{\int m = a} \mathcal{E}_\varepsilon + \inf_{\int m = M-a} \mathcal{E}_\varepsilon,$$

which makes (1-12) impossible. While subadditivity is easy to prove for $\mathcal{E}_\varepsilon$ (see Lemma 5.5), the splitting (1-12) requires technical work, in particular due to the presence of the term $mL\big(-\frac{w}{m}\big)$ in $\mathcal{E}_\varepsilon$, which becomes increasingly singular as m approaches zero (a simple cut-off as in (1-11) is not useful). The property (1-12) is proven in Theorem 5.6. It relies on the Brezis–Lieb lemma and a perturbation argument. The L^1 convergence of $\bar{m}_\varepsilon$ enables us to obtain the full convergence of $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ to a limit MFG system. By a uniform control of the decay of $\bar{m}_\varepsilon$ as $|x| \to \infty$, which comes from a Lyapunov function built upon $\bar{u}_\varepsilon$, energy arguments and the crucial L^∞ estimate of Theorem 4.1, we are also able to keep track of x_ε. In terms of the nonrescaled density m_ε, x_ε is the point around which most of the mass is located.

The second main result of this work is stated in the following two theorems. The first one is about the concentration of m_ε.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exist sequences $\varepsilon \to 0$ and x_ε such that for all $\eta > 0$ there exist R and ε_0 for which, for all $\varepsilon < \varepsilon_0$,*

$$\int_{|x-x_\varepsilon| \le R\varepsilon^{\gamma'/(\gamma'-\alpha N)}} m_\varepsilon \, dx \ge M - \eta.$$

Moreover, $x_\varepsilon \to \bar{x}$, where $V(\bar{x}) = 0$, i.e., $\bar{x}$ is a minimum of V.

If, in addition, V has the form

$$V(x) = h(x) \prod_{j=1}^{n} |x - x_j|^{b_j}, \qquad C_V^{-1} \le h(x) \le C_V \quad \text{on } \mathbb{R}^N, \tag{1-13}$$

for some $x_j \in \mathbb{R}^N$, and $b_j > 0$ $\big($with $\sum_{j=1}^{n} b_j = b\big)$, then $x_\varepsilon \to x_i$, with $i \in \{j = 1, \dots, n : b_j = \max_k b_k\}$.

Secondly, we describe the asymptotic profile of $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ as $\varepsilon \to 0$. Note that as a byproduct we obtain the existence of solutions to MFG systems without potential.

Theorem 1.3. *Up to subsequences, $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ converges in $C^1_{\text{loc}}(\mathbb{R}^N) \times C_{\text{loc}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, for all $p \ge 1$, to a solution $(\bar{u}, \bar{m})$ of*

$$\begin{cases} -\Delta u + C_H |\nabla u|^\gamma + \lambda = -C_f m^\alpha, \\ -\Delta m - C_H \gamma \operatorname{div}(m |\nabla u|^{\gamma-2} \nabla u) = 0, \\ \int_{\mathbb{R}^N} m = M. \end{cases} \tag{1-14}$$

The function $\bar{u}$ is globally Lipschitz continuous on $\mathbb{R}^N$, and there exist $c_1, c_2 > 0$ such that $0 < \bar{m}(x) \le c_1 e^{-c_2|x|}$.

Finally, if $\bar{w} = -C_H \gamma \bar{m} |\nabla \bar{u}|^{\gamma-2} \nabla \bar{u}$, then

$$\mathcal{E}_0(\bar{m}, \bar{w}) = \min\{\mathcal{E}_0(m, w) : (m, w) \in \mathcal{K}_{1,M},\ m(1 + |y|^b) \in L^1(\mathbb{R}^N)\}, \tag{1-15}$$

where

$$\mathcal{E}_0(m, w) = \int_{\mathbb{R}^N} C_L \frac{|w|^{\gamma'}}{m^{\gamma'-1}} - \frac{1}{\alpha+1} m^{\alpha+1} \, dy. \tag{1-16}$$

We finally observe that by analogous methods, one can prove existence of solutions to more general potential-free MFG systems; see Remark 5.9.

Notation. We will denote a classical solution to the system (1-1) by a triple

$$(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R} \quad \text{for all } p > 1.$$

For any given $p > 1$, we will denote by $p' = \frac{p}{p-1}$ the conjugate exponent of p, and set

$$p^* = \frac{Np}{N-p} \quad \text{if } p < N \qquad \text{and} \qquad p^* = +\infty \quad \text{if } p \geq N.$$

For all $R > 0$, $x \in \mathbb{R}^N$, we define $B_R(x) := \{y \in \mathbb{R}^N : |x - y| < R\}$. We will set $\omega_N := |B_1(0)|$. Finally, $C, C_1, K, K_1, \ldots$ denote (positive) constants we need not specify.

2. Some preliminary regularity results

Let L be the Legendre transform of H, i.e.,

$$L(q) = H^*(q) = \sup_{p \in \mathbb{R}^N} [p \cdot q - H(p)], \quad q \in \mathbb{R}^N. \tag{2-1}$$

The assumptions on H guarantee the following; see, e.g., [Cirant 2014, Proposition 2.1].

Proposition 2.1. *There exist $C_L, C_1, C_2 > 0$ depending on C_H and on γ such that for all $p, q \in \mathbb{R}^N$,*

(i) $L \in C^2(\mathbb{R}^N \setminus \{0\})$ *and it is strictly convex,*

(ii) $0 \leq C_L |q|^{\gamma'} \leq L(q) \leq C_L(|q|^{\gamma'} + 1)$,

(iii) $\nabla L(q) \cdot q - L(q) \geq C_1 |q|^{\gamma'} - C_1^{-1}$,

(iv) $C_1 q|^{\gamma'-1} - C_1^{-1} \leq |\nabla L(q)| \leq C_1^{-1}(|q|^{\gamma'-1} + 1)$,

(v) $C_2 |p|^{\gamma-1} - C_2^{-1} \leq |\nabla H(p)| \leq C_2^{-1}(|p|^{\gamma-1} + 1)$.

We will use the following (standard) result on Hölder functions vanishing at infinity.

Lemma 2.2. *Suppose that $m \geq 0$, $\|m\|_{C^{0,\theta}(\mathbb{R}^N)} \leq c_h$ for some $\theta, c_h > 0$, and $\int_{\mathbb{R}^N} m\, dx < \infty$. Then, $m(x) \to 0$ as $|x| \to \infty$. Moreover, if*

$$\int_{|x| \geq R} m\, dx < \eta$$

for some $\eta, R > 0$, then

$$\max_{|x| \geq R} m(x) \leq C \eta^{\frac{\theta}{\theta + N}}, \tag{2-2}$$

where $C > 0$ depends only on c_h, N.

Proof. By contradiction, suppose that there exists $\delta > 0$ and a sequence $|x_n| \to \infty$ such that $m(x_n) > \delta$ for all n. We may also assume that $|x_{n+1}| \geq |x_n| + 1$ for all n. By the Hölder regularity assumption,

$$m(x) \geq m(x_n) - c_h |x - x_n|^\theta \geq \tfrac{1}{2}\delta,$$

provided that $x \in B_r(x_n)$, and $r^\theta \le \frac{\delta}{2c_h}$. Choose $r = \min\{1, (\frac{\delta}{2c_h})^{\frac{1}{\theta}}\}$, so that $B_r(x_n) \cap B_r(x_m) = \varnothing$ for all $n \neq m$. Then,

$$\int_{\mathbb{R}^N} m\,dx \ge \sum_{n\in\mathbb{N}} \int_{B_r(x_n)} m\,dx \ge \sum_{n\in\mathbb{N}} \tfrac{1}{2}\delta |B_r(0)| = +\infty,$$

which is impossible.

As for the second part, let $M := \max_{|x|\ge R} m(x) = m(\bar{x})$, $|\bar{x}| \ge R$ (note that such a maximum is achieved as a consequence of the first part of the lemma). As before,

$$m(x) \ge m(\bar{x}) - c_h|x-\bar{x}|^\theta \ge \tfrac{1}{2}M$$

for all $x \in B_r(\bar{x})$, where $r = (\frac{M}{2c_h})^{\frac{1}{\theta}}$. Therefore,

$$\eta > \int_{|x|\ge R} m\,dx \ge \tfrac{1}{4}M|B_r(\bar{x})| = \tfrac{1}{4}M|B_1(0)|\left(\frac{M}{2c_h}\right)^{\frac{N}{\theta}},$$

and (2-2) follows. □

We recall the following well-known result, proved in [Brézis and Lieb 1983, Theorem 1].

Theorem 2.3. *Let $f_n \to f$ a.e. in $\mathbb{R}^N$ and assume $\|f_n\|_{L^p(\mathbb{R}^N)} \le C$ for all n and for some $p \in [1, +\infty)$. Then*

$$\lim_n [\|f_n\|^p_{L^p(\mathbb{R}^N)} - \|f_n - f\|^p_{L^p(\mathbb{R}^N)}] = \|f\|^p_{L^p(\mathbb{R}^N)}.$$

From classical elliptic regularity, we have the following result.

Proposition 2.4. *Let $p > 1$ and $m \in L^p(\mathbb{R}^N)$ be such that*

$$\left|\int_{\mathbb{R}^N} m\,\Delta\varphi\,dx\right| \le K\|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)} \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

for some $K > 0$. Then, $m \in W^{1,p}(\mathbb{R}^N)$ and there exists $C > 0$ depending only on p, such that

$$\|\nabla m\|_{L^p(\mathbb{R}^N)} \le CK.$$

Proof. Fix any $R > 1$. Let $\psi \in C_0^\infty(B_2(0))$, $\varphi(Rx) := \psi(x)$ (so, $\varphi \in C_0^\infty(B_{2R}(0))$) and $v(x) := m(Rx)$ on $\mathbb{R}^N$. Then,

$$\begin{aligned}\left|\int_{B_2(0)} v\,\Delta\psi\,dx\right| &= R^{2-N}\left|\int_{B_{2R}(0)} m\,\Delta\varphi\,dy\right| \le KR^{2-N}\left(\int_{B_{2R}(0)} |\nabla\varphi|^{p'}\,dy\right)^{\frac{1}{p'}} \\ &= KR^{1-N+\frac{N}{p}'}\left(\int_{B_2(0)} |\nabla\psi|^{p'}\,dx\right)^{\frac{1}{p'}} \le KR^{1-\frac{N}{p}}\|\psi\|_{W^{1,p'}(B_2(0))}.\end{aligned}$$

Hence, by [Agmon 1959, Theorem 6.1], $v \in W^{1,p}(B_1(0))$ and there exists a constant C, depending on p (but not on R), such that

$$\|\nabla v\|_{L^p(B_1(0))} \le \|v\|_{W^{1,p}(B_1(0))} \le C(KR^{1-\frac{N}{p}} + \|v\|_{L^p(B_2(0))}).$$

Therefore,

$$\left(\int_{B_R(0)} |\nabla m|^p\,dy\right)^{\frac{1}{p}} = R^{\frac{N}{p}-1}\left(\int_{B_1(0)} |\nabla v|^p\,dx\right)^{\frac{1}{p}} \le C\left[K + R^{\frac{N}{p}-1}\left(\int_{B_2(0)} |v|^p\,dx\right)^{\frac{1}{p}}\right]$$
$$= C(K + R^{-1}\|m\|_{L^p(B_{2R}(0))}).$$

Letting $R \to \infty$, we get that $|\nabla m| \in L^p(\mathbb{R}^n)$ and the desired estimate. □

2A. *The Hamilton–Jacobi–Bellman equation on the whole space.* In this section we provide some a priori regularity estimates and existence results for Hamilton–Jacobi–Bellman equations in whole spaces of ergodic type. In particular we will consider families of Hamilton–Jacobi–Bellman equations

$$-\Delta u_n + H_n(\nabla u_n) + \lambda_n = F_n(x) - f_n(x) \quad \text{on } \mathbb{R}^N, \tag{2-3}$$

where $F_n - f_n$ is locally Hölder continuous, $\lambda_n \in \mathbb{R}$ are equibounded in n, that is, $|\lambda_n| \le \lambda$, and $f_n \in L^\infty(\mathbb{R}^N)$, with $\|f_n\|_\infty \le c_f$ for some $c_f > 0$ independent of n. Moreover H_n is for every n a Hamiltonian which satisfies (1-2), with constants γ and C_H independent of n; finally, there exists $C_F \ge 0$ and $b \ge 0$ independent of n such that

$$C_F^{-1}(\max\{|x| - C_F, 0\})^b \le F_n(x) \le C_F(1+|x|)^b \quad \text{for all } n \text{ and all } x \in \mathbb{R}^N. \tag{2-4}$$

Note that, differently from assumption (1-5) for the potential V, the function F_n can also be bounded, if $b = 0$.

Theorem 2.5. *Let $u_n \in C^2(\mathbb{R}^N)$ be a sequence of classical solutions of the HJB equations* (2-3). *Then there exists a constant $K > 0$ depending on $C_H, C_F, c_f, \gamma, N, \lambda$ such that*

$$|\nabla u_n(x)| \le K(1+|x|)^{\frac{b}{\gamma}}, \tag{2-5}$$

where $b \ge 0$ is the growth of F_n appearing in (2-4) *and γ is the growth of H_n appearing in* (1-2).

Proof. Without loss of generality we may consider $H_n(p) = C_H|p|^\gamma$ for all n and p. Indeed, every v_n solves

$$-\Delta u_n + C_H|\nabla u_n|^\gamma + \lambda_n = F_n(x) - f_n(x) + C_H|\nabla u_n|^\gamma - H_n(\nabla u_n) \quad \text{on } \mathbb{R}^N,$$

and since $|C_H|\nabla u_n|^\gamma - H_n(\nabla u_n)| \le C_H$ by (1-2), we can redefine f_n to include $C_H|\nabla u_n|^\gamma - H_n(\nabla u_n)$, which then satisfies the bound $\|f_n\|_\infty \le c_f + C_H$.

We first claim that if $v \in C^2(B_2(0))$ satisfies

$$\left|-\Delta v + C_H|\nabla v|^\gamma\right| \le k \quad \text{on } B_2(0)$$

for some $k > 0$, then we have for any $r \in [1,\infty]$,

$$\|\nabla v\|_{L^r(B_1(0))} \le \widetilde{C}, \tag{2-6}$$

where $\widetilde{C}$ depends only on k, C_H, γ, N, r. If $r \in [1,\infty)$, this is proven in [Lasry and Lions 1989, Theorem A.1]; see also [Cirant 2015, Theorem 19]. The case $r = \infty$ follows by classical elliptic regularity, since if r in (2-6) is large enough, then $-\Delta v$ is bounded in $L^q(B_{\frac{3}{2}}(0))$ for some $q > N$, and the statement follows by Sobolev embeddings.

In view of these considerations, the gradient bound (2-5) easily follows if $b = 0$. For the case $b > 0$, fix $x_0 \in \mathbb{R}^N$, and let $\delta = (1 + |x_0|)^{-\frac{b}{\gamma'}}$. Let

$$v_n(y) := \delta^{\frac{2-\gamma}{\gamma-1}} u_n(x_0 + \delta y) \quad \text{on } \mathbb{R}^N.$$

Then, v_n solves

$$-\Delta v_n + C_H |\nabla v_n|^\gamma = \delta^{\gamma'}(F_n(x_0 + \delta y) - f_n(x_0 + \delta y) - \lambda_n).$$

Since $\delta \le 1$,

$$\delta^{\gamma'} |F_n(x_0 + \delta y) - f_n(x_0 + \delta y) - \lambda_n| \le \frac{C_F (3 + |x_0|)^b + c_f + \lambda}{(1 + |x_0|)^b} \le C_1$$

for all $y \in B_2(0)$ by (2-4) and the bound on f_n.

Therefore, by the first claim,

$$\|\nabla v_n\|_{L^\infty(B_1(0))} \le \widetilde{C}$$

for all n. In particular, choosing $y = 0$,

$$|\nabla u_n(x_0)| = \delta^{-\frac{1}{\gamma-1}} |\nabla v_n(0)| \le \widetilde{C} (1 + |x_0|)^{\frac{b}{\gamma}},$$

and the desired estimate follows. □

Moreover, we prove the following a priori estimates on bounded-from-below solutions to (2-3).

Theorem 2.6. *Let $u_n \in C^2(\mathbb{R}^N)$ be a family of uniformly bounded-from-below classical solutions to* (2-3), *that is, for which there exists $C > 0$ such that $u_n \ge -C$ for every n.*

If $b = 0$ in (2-4), *we moreover assume that there exists $\delta > 0$ and $R > 0$ independent of n such that*

$$F_n(x) - f_n(x) - \lambda_n > \delta > 0 \quad \text{for all } |x| > R. \tag{2-7}$$

Then there exists $C > 0$ such that

$$u_n(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1} \quad \text{for all } n \in \mathbb{N}, x \in \mathbb{R}^N, \tag{2-8}$$

where $b \ge 0$ is the growth power appearing in (2-4) *and γ is the growth power appearing in* (1-2).

Proof. The proof is based on the same argument as in [Barles and Meireles 2016, Proposition 3.4], we sketch it briefly for completeness. Since u_n is bounded from below we can assume $u_n \ge 0$, up to addition of constant C (without changing the equation).

We assume by contradiction that (2-8) does not hold. Then there exist sequences x_l and u_{n_l} such that $|x_l| > 2R$, $|x_l| \to +\infty$, and

$$\frac{u_{n_l}(x_l)}{|x_l|^{1+\frac{b}{\gamma}}} \to 0.$$

Let $a_l = \frac{1}{2}|x_l|$ and we define the function

$$v^l(x) = \frac{1}{a_l^{1+\frac{b}{\gamma}}} u_{n_l}(x_l + a_l x).$$

By Theorem 2.5, we get $|\nabla u_{n_l}(x)| \leq K(1+|x|)^{\frac{b}{\gamma}}$. Therefore, v^l, $|\nabla v^l|$ are uniformly bounded.

Moreover, v^l is a solution to

$$-a_l^{\frac{b}{\gamma}-1} \Delta v^l + H_{n_l}(a_l^{\frac{b}{\gamma}} \nabla v^l) + \lambda_{n_l} = F_{n_l}(x_l + a_l x) - f_{n_l}(x_l + a_l x).$$

In particular, recalling (1-2), we get that v^l is a supersolution to

$$-a_l^{\frac{b}{\gamma}-1-b} \Delta v^l + C_H |\nabla v^l|^\gamma \geq a_l^{-b}(-\lambda_{n_l} + F_{n_l}(x_l + a_l x) - f_{n_l}(x_l + a_l x)).$$

Note that, for every l sufficiently large, by (2-4) and by (2-7) (in the case $b = 0$) the right-hand side above satisfies

$$a_l^{-b}(-\lambda_{n_l} + F_{n_l}(x_l + a_l x) - f_{n_l}(x_l + a_l x)) > 0$$

for x such that $|x| \leq 1$.

Moreover, passing eventually to a subsequence, we get $v^l \to v$ locally uniformly in n and $a_l^{\frac{b}{\gamma}-1-b} \to 0$. So v is a supersolution to $C_H |\nabla v|^\gamma \geq \delta > 0$ in $B(0,1)$ with homogeneous boundary conditions (since $v \geq 0$). By comparison, recalling the explicit formula of the solution to the eikonal equation $|\nabla f|^\gamma = C$ in $B(0,1)$ with homogeneous boundary conditions, we conclude that $v(x) \geq C^{\frac{1}{\gamma}}(1-|x|)$ for all x such that $|x| \leq 1$. Moreover, by uniform convergence, we get that, eventually enlarging C and taking l sufficiently large, $v^l(x) \geq C^{\frac{1}{\gamma}}(1-|x|)$ for all x with $|x| \leq 1$; in particular $v^l(0) \geq C^{\frac{1}{\gamma}}$. Recalling the definition of v^l, we get that $v^l(0) \to 0$, which yields a contradiction. ☐

Define

$$\bar{\lambda}_n := \sup\{\lambda \in \mathbb{R} : \text{(2-3) has a solution } u_n \in C^2(\mathbb{R}^N)\}.$$

Theorem 2.7. *Assume that for every n the function $F_n - f_n$ is bounded from below uniformly in n:*

(i) *$\bar{\lambda}_n < \infty$ for every n, and there exists, for every n, a solution $u_n \in C^2(\mathbb{R}^N)$ to* (2-3) *with $\lambda_n = \bar{\lambda}_n$. Moreover*

$$\bar{\lambda}_n := \sup\{\lambda \in \mathbb{R} : \text{(2-3) } \textit{has a subsolution } u_n \in C^2(\mathbb{R}^N)\}.$$

(ii) *If F_n satisfies* (2-4), *with $b > 0$, then, for every n, the solution u_n to* (2-3) *with $\lambda_n = \bar{\lambda}_n$ is unique up to addition of constants and satisfies* (2-8).

(iii) *If $F_n \equiv 0$, and there exists $\delta > 0$ independent of n such that*

$$\limsup_{|x|\to+\infty} f_n(x) + \bar{\lambda}_n < -\delta < 0, \tag{2-9}$$

then for every n there exists a solution to (2-3) *with $\lambda_n = \bar{\lambda}_n$ which satisfies* (2-8) *with $b = 0$.*

Proof. (i) The proof of this result can be obtained by a straightforward adaptation of the proof of Theorem 2.1 in [Barles and Meireles 2016], using the a priori estimates on the gradient given in Theorem 2.5. Observe that actually in that paper a stronger assumption on the regularity of $F_n - f_n$ is required, in particular local Lipschitz continuity. This assumption is used to derive a priori estimates on the gradient of solutions by using the so-called Bernstein method, see Appendix A in [Barles and Meireles 2016], which depends also on the L^∞ norm of $\nabla(F_n - f_n)$. In our case we can weaken this assumption to just Hölder continuity (so still ensuring classical elliptic regularity) since we are using a priori estimates on the gradient given in Theorem 2.5, which depends only on the L^∞ norm of $F_n - f_n$, and are obtained in [Lasry and Lions 1989] by the so-called integral Bernstein method.

(ii) For the proof we refer to [Ichihara 2011]; see also [Barles and Meireles 2016; Cirant 2014]. In particular in [Ichihara 2011], it is proved that u_n is bounded from below. By looking at the proof, it is easy to check that, due to the uniformity in n of the norms of coefficients, the bound can be taken independent of n, and by Theorem 2.6 we get the estimate on the growth.

(iii) By adapting the argument in [Barles and Meireles 2016, Theorem 2.6], we get that there exists a bounded-from-below solution to (2-3) with $\lambda_n = \bar{\lambda}_n$, with bound uniform in n. Then using Theorem 2.6, we get the estimate on the growth. We give a brief sketch of the proof of the existence of a bounded-from-below solution. For every $R > 0$, we consider the ergodic problem

$$\begin{cases} -\Delta u_n^R + H_n(\nabla u_n^R) + \lambda_n^R = -f, & |x| < R, \\ u_n^R(x) \to +\infty, & |x| \to R. \end{cases} \tag{2-10}$$

Using the result in [Barles et al. 2010], we get that for every $R > 0$ there exists a unique λ_n^R and a unique up to addition of constant solution $u_n^R \in C^2(B_R)$.

First of all we claim that $\lim_R \lambda_n^R = \bar{\lambda}_n$. It is easy to check that if $R' > R$, then $\lambda_n^{R'} \le \lambda_n^R$, and moreover that $\lambda_n^R \ge \bar{\lambda}_n$. So, the sequence λ_n^R is converging as $R \to +\infty$ to some $\lambda_n^\star \ge \bar{\lambda}_n$. Additionally, by the same argument as in Theorem 2.5, we get that for every compact $K \subset \mathbb{R}^N$, there exists a constant $C > 0$ such that $|\nabla u_n^R| \le C$ in K for every R sufficiently large and for all n. Without loss of generality we can assume that $u_n^R(0) = 0$ for every R. So, using the gradient bound, and elliptic regularity, we conclude that u_n^R is bounded in $C^2(K)$ by some constant independent of R. Hence, by the Ascoli–Arzelà theorem, and via a diagonalization procedure, we get that u_n^R converges locally in $\mathbb{R}^N$, with $u_n \in C^2(\mathbb{R}^N)$. Moreover, u_n is a solution to (2-3), with $\lambda = \lambda_n^\star$. Recalling the characterization of $\bar{\lambda}_n$ and the fact that $\lambda_n^\star \ge \bar{\lambda}_n$, we conclude that $\lambda_n^\star = \bar{\lambda}_n$.

Then, we consider $x_n^R \in B_R$ such that

$$u_n^R(x_n^R) = \min_{|x| \le R} u_n^R.$$

Recalling that u_n^R is a solution to (2-10), we get by computing the equation at x_n^R and by recalling that $H_n(0) \le 0$, that

$$\lambda_n^R + f(x_n^R) \ge H_n(0) + \lambda_n^R + f(x_n^R) \ge 0.$$

Using condition (2-9), and recalling that $\lambda_n^R \to \bar{\lambda}_n$, we get that there exists a compact set K (independent of R and of n) and $R_0 > 0$ such that for all $R > R_0$ we have $x_n^R \in K$.

Recalling that $u_n^R(0) = 0$ and $|\nabla u_n^R| \le C$ in K with C independent of n, R, we conclude that $u_n^R(x_R) \ge -C$ for some constant C independent of n, R. But, this implies, since $u_n^R(x) \ge u_n^R(x_n^R)$ for every R, that passing to the limit $u_n(x) \ge -C$, with C independent of n. □

2B. ***A priori estimates for the Kolmogorov equation.*** In this section we provide general a priori estimates for pairs $(m, w) \in (L^1(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)) \times L^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} m(x) = M$ and $-\varepsilon\Delta m + \operatorname{div} w = 0$, where

$$q = \begin{cases} \gamma', & \gamma' \ge N, \\ \frac{N}{N-\gamma'+1}, & \gamma' < N. \end{cases} \tag{2-11}$$

Lemma 2.8. *Let $\beta \le \frac{Nq}{N-q}$ for $q < N$, and $\beta < +\infty$ for $q \ge N$. We define $1 \le r \le \beta$ as follows*:

$$\frac{1}{r} = \frac{1}{\gamma'} + \left(1 - \frac{1}{\gamma'}\right)\frac{1}{\beta}. \tag{2-12}$$

Then, *there exists a constant C*, *depending only on N and β*, *such that*

$$\begin{aligned} \|m\|_{W^{1,r}(\mathbb{R}^N)} &\le C\left(\frac{1}{\varepsilon^{\gamma'}}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + M\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} \\ &\le C\left(\frac{C_L}{\varepsilon^{\gamma'}}\int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx + M\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}}, \end{aligned} \tag{2-13}$$

where $C_L = C_L(C_H, \gamma)$ is the constant appearing in Proposition 2.1.

We now assume that

$$1 < \beta < 1 + \frac{\gamma'}{N}. \tag{2-14}$$

Then, *there exists $\delta > 0$ such that*

$$\|m\|_{L^\beta(\mathbb{R}^N)}^{(1+\delta)\beta} \le C\frac{1}{\varepsilon^{\gamma'}} M^{(1+\delta)\beta-1}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right) \le C C_L \frac{1}{\varepsilon^{\gamma'}} M^{(1+\delta)\beta-1}\int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx, \tag{2-15}$$

where the constant C depends only on γ, *N*, *and β.*

Proof. Since $m \in W^{1,q}(\mathbb{R}^N)$, by Sobolev embedding and interpolation, we get that $m \in L^\beta(\mathbb{R}^N)$. Using $-\varepsilon\Delta m + \operatorname{div} w = 0$, we get for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\varepsilon\int_{\mathbb{R}^N} \nabla m \cdot \nabla\varphi\, dx = \int_{\mathbb{R}^N} w \cdot \nabla\varphi\, dx.$$

Using the Hölder inequality, recalling (2-12), we obtain

$$\begin{aligned} \left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N} w \cdot \nabla\varphi\, dx\right| &\le \int_{\mathbb{R}^N} \frac{1}{\varepsilon}\left|\frac{w}{m}\right| m^{\frac{1}{\gamma'}} m^{1-\frac{1}{\gamma'}} |\nabla\varphi|\, dx \\ &\le \left(\frac{1}{\varepsilon^{\gamma'}}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla\varphi\|_{L^{r'}(\mathbb{R}^N)}. \end{aligned}$$

Therefore, we get that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} \nabla m \cdot \nabla \varphi \, dx\right| \le \left(\frac{1}{\varepsilon \gamma'} \int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla \varphi\|_{r'}.$$

We apply then Proposition 2.4 and we obtain that $m \in W^{1,r}(\mathbb{R}^N)$ and that there exists a constant C, depending only on r, such that

$$\|\nabla m\|_{L^r(\mathbb{R}^N)} \le C\left(\frac{1}{\varepsilon \gamma'} \int_{\mathbb{R}^N} m \left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \tag{2-16}$$

From this inequality, using Proposition 2.1 and recalling that by interpolation, since $\|m\|_{L^1(\mathbb{R}^N)} = M$,

$$\|m\|_{L^r(\mathbb{R}^N)} \le \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} M^{\frac{1}{\gamma'}},$$

we conclude the desired inequality (2-13).

Now we fix η such that

$$\frac{1}{\eta} = \left(\frac{1}{r} - \frac{1}{N}\right)\frac{N}{N+1} + 1 - \frac{N}{N+1} = \frac{N}{N+1}\frac{1}{r}.$$

Note that, by a simple computation using (2-12), we get

$$\frac{1}{\eta} - \frac{1}{\beta} = \frac{N}{N+1}\frac{1}{\beta\gamma'}\left(\beta - 1 - \frac{\gamma'}{N}\right);$$

therefore, by (2-14), we conclude that $\eta > \beta$. By the Gagliardo–Nirenberg inequality, and recalling that $\|m\|_1 = M$, we get

$$\|m\|_{L^\eta(\mathbb{R}^N)} \le C\|\nabla m\|_{L^r(\mathbb{R}^N)}^{\frac{N}{N+1}} M^{\frac{1}{N+1}}. \tag{2-17}$$

Since $\eta > \beta$, by interpolation we get that there exists $\theta > 1$ such that $\|m\|_{L^\beta(\mathbb{R}^N)}^\theta \le \|m\|_{L^\eta(\mathbb{R}^N)} M^{\theta-1}$. Actually

$$\frac{1}{\theta} = \left(1 - \frac{1}{\beta}\right)(N+1)\frac{1}{1 + N\left(1 - \frac{1}{\beta}\right)\left(1 - \frac{1}{\gamma'}\right)}.$$

So, we substitute in (2-17) and (2-16) and we get, elevating both terms to $\gamma' \frac{N+1}{N}$,

$$\|m\|_{L^\beta(\mathbb{R}^N)}^{\theta\gamma'\frac{N+1}{N}} \le C\frac{1}{\varepsilon\gamma'} M^{\gamma'(\theta\frac{N+1}{N} - 1)}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right)\|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{\gamma'}{\gamma}}. \tag{2-18}$$

Now, since $\theta > 1$, by (2-14), we get

$$\theta\gamma'\frac{N+1}{N} - \frac{\gamma'}{\gamma} = \frac{\beta\gamma'}{N(\beta-1)} = \beta + \frac{\beta}{\beta-1}\left[\frac{\gamma'}{N} + 1 - \beta\right] > 0.$$

Therefore we deduce (2-15) from (2-18) with

$$\delta = \frac{1}{\beta-1}\left[\frac{\gamma'}{N} + 1 - \beta\right]. \tag{2-19}$$

This concludes the proof. □

Corollary 2.9. *For every $r < q$, there exists $C > 0$ depending on N, γ' and r such that*

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(C_L \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right)dx + \varepsilon^{\gamma'}M\right). \tag{2-20}$$

Moreover, if $\gamma' > N$ (so $q > N$), then $m \in C^{0,\theta}(\mathbb{R}^N)$ and

$$\|m\|_{C^{0,\theta}(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(C_L \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right)dx + \varepsilon^{\gamma'}M\right). \tag{2-21}$$

Proof. For $q \ge N$ (equivalently $\gamma' \ge N$), we fix $r < q$ and we choose β which satisfies (2-12) for such r. By the Sobolev embedding theorem, $W^{1,r}(\mathbb{R}^N)$ is continuously embedded in $L^\beta(\mathbb{R}^N)$. So, there exists C depending on N and r such that $\|m\|_{L^\beta(\mathbb{R}^N)} \le C\|m\|_{W^{1,r}(\mathbb{R}^N)}$. Using inequality (2-13), we get

$$\|m\|_{L^\beta(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \varepsilon^{\gamma'}M\right).$$

If we substitute again in (2-13) we get

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \varepsilon^{\gamma'}M\right).$$

In particular for $q > N$, we can choose $r > N$ and by the Sobolev embedding theorem we get that there exists $\theta = 1 - \frac{N}{r}$ and a constant $C > 0$ depending on N and r such that

$$\begin{aligned}\|m\|_{C^{0,\theta}(\mathbb{R}^N)} &\le \frac{C}{\varepsilon^{\gamma'}}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \varepsilon^{\gamma'}M\right)\\ &\le \frac{C}{\varepsilon^{\gamma'}}\left(C_L \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right)dx + \varepsilon^{\gamma'}M\right).\end{aligned}$$

For $q < N$, we fix $r < q$, and choose the corresponding β in (2-12), which satisfies $\beta < \frac{N}{N-\gamma'}$. Hence we conclude again using inequality (2-13). □

3. Regularization procedure and existence of approximate solutions for $\varepsilon > 0$

3A. ***The regularized problem.*** We consider the approximation of the system (1-1)

$$\begin{cases} -\varepsilon\Delta u + H(\nabla u) + \lambda = f_k[m](x) + V(x),\\ -\varepsilon\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0,\\ \int_{\mathbb{R}^N} m\,dx = M, \end{cases} \tag{3-1}$$

where

$$f_k[m](x) = f(m \star \chi_k) \star \chi_k(x) = \int_{\mathbb{R}^N} \chi_k(x-y) f\left(\int_{\mathbb{R}^N} m(z)\chi_k(y-z)\,dz\right)dy \tag{3-2}$$

and χ_k, for $k > 0$, is a sequence of standard symmetric mollifiers approximating the unit as $k \to \infty$.

We observe that $f_k[m](x)$ is the L^2-gradient of a C^1 potential $F_k : L^1(\mathbb{R}^N) \to \mathbb{R}$, defined as

$$F_k[m] := \int_{\mathbb{R}^N} F(m \star \chi_k(x))\,dx, \tag{3-3}$$

where $F(m) = \int_0^m f(n)\,dn$ for $m \geq 0$ and $F(m) = 0$ for $m \leq 0$. Note that using Jensen's inequality and (1-3), we get that for all $m \in L^1(\mathbb{R}^N)$ such that $m \geq 0$, and $\int_{\mathbb{R}^N} m(x)\,dx = M$,

$$-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}(x)\,dx - KM \leq F_k[m] \leq -\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} (m \star \chi_k(x))^{\alpha+1}\,dx + KM. \tag{3-4}$$

In order to construct solutions to the system, we follow a variational approach and we associate to (3-1) an energy, as already described in the Introduction. We define the energy

$$\mathcal{E}_k(m,w) := \begin{cases} \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V(x)m\,dx + F_k[m] & \text{if } (m,w) \in \mathcal{K}_{\varepsilon,M}, \\ +\infty & \text{otherwise}, \end{cases} \tag{3-5}$$

where $\mathcal{K}_{\varepsilon,M}$ is defined in (1-8) and L is defined in (1-7). We recall that the exponent q appearing in the definition of $\mathcal{K}_{\varepsilon,M}$ is

$$q = \begin{cases} \frac{N}{N-\gamma'+1}, & \gamma' \leq N, \\ \gamma', & \gamma' > N. \end{cases}$$

Therefore, $q \leq \gamma'$. Observe that, if $q < N$,

$$q^* = \frac{qN}{N-q} = \frac{N}{N-\gamma'},$$

and that $q^* > 1 + \frac{\gamma'}{N} > 1 + \alpha$ by (1-4). If $q = \gamma' \geq N$, then we let $q^* = +\infty$.

3B. ***A priori estimates and energy bounds.*** In this section, we provide bounds from below for the energy $\mathcal{E}_k$, ensuring in particular that the minimum problem is well-defined.

Lemma 3.1. *Let $(m,w) \in \mathcal{K}_{\varepsilon,M}$. Then*

$$\mathcal{E}_k(m,w) \geq -K - C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}, \tag{3-6}$$

where $C, K > 0$ are constants depending only on $N, M, C_L, \gamma, \alpha, M$.

In particular there exists finite

$$e_{k,\varepsilon}(M) = \inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}_k(m,w).$$

Proof. Recalling that $V \geq 0$, using estimate (3-4) and applying (2-15) with $\alpha = \beta - 1$, we get

$$\begin{aligned} \mathcal{E}_k(m,w) &\geq \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right)dx - \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx - KM \\ &\geq C\varepsilon^{\gamma'} M^{1-(1+\delta)(1+\alpha)} \|m\|_{L^{\alpha+1}}^{(1+\alpha)(1+\delta)} - \frac{1}{\alpha+1}\|m\|_{L^{\alpha+1}}^{(1+\alpha)} - KM \\ &\geq -C\delta\varepsilon^{-\frac{\gamma'}{\delta}}\left(\frac{1}{(\delta+1)(\alpha+1)}\right)^{1+\frac{1}{\delta}} - KM, \end{aligned}$$

where C is a constant depending only on $N, M, C_L, \gamma, \alpha$ and

$$\delta = \frac{1}{\alpha}\left[\frac{\gamma'}{N} - \alpha\right]. \tag{3-7}$$

Therefore, substituting in the energy, we get

$$\mathcal{E}_k(m,w) \geq -C\,\frac{(\gamma'-\alpha N)}{\alpha N}\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}\left(\frac{\alpha N}{\gamma'(\alpha+1)}\right)^{\frac{\gamma'}{\gamma'-\alpha N}} - KM,$$

which gives the desired inequality. □

We get also a priori bounds on minimizers and minimizing sequences.

Proposition 3.2. *Let $(m,w)\in\mathcal{K}_{\varepsilon,M}$ be such that $e_{k,\varepsilon}(M)\geq\mathcal{E}_k(m,w)-\eta$ for some positive η. Then*

$$\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}dx \leq C\varepsilon^{-\frac{\gamma' N\alpha}{\gamma'-N\alpha}}+K, \tag{3-8}$$

$$\|m\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^N)} \leq C\varepsilon^{-\frac{\gamma' N\alpha}{\gamma'-N\alpha}}+K \tag{3-9}$$

for some C, K positive constants which depend only on α, N, V, C_L, η.

Proof. First of all we observe that there exists $C\geq 0$ depending on M, C_L, C_V such that

$$e_{k,\varepsilon}(M)\leq C. \tag{3-10}$$

Let $m=ce^{-|x|}$, where c is chosen to have $\int_{\mathbb{R}^n} m\,dx = M$, and $w=\varepsilon\nabla m$, so that $(m,w)\in\mathcal{K}_{\varepsilon,M}$. By assumption (1-5), we get $\int_{\mathbb{R}^n} mV(x)\,dx\leq C$ for some constant $C>0$, by (3-4) we get $F_k[m]\leq KM$ and by the properties of L in Proposition 2.1 we have

$$\int_{\mathbb{R}^n} mL\left(-\frac{w}{m}\right)dx \leq \left(\frac{\varepsilon^{\gamma'}}{c^{\gamma'}}+C_L\right)M.$$

So, in conclusion $e_{k,\varepsilon}(M)\leq\mathcal{E}_k(m,w)\leq C$ as required.

Note that if $(m,w)\in\mathcal{K}_{\varepsilon,M}$, and $e_\varepsilon(M)\geq\mathcal{E}(m,w)-\eta$ for some positive η, then, by (3-4), by the fact that $V\geq 0$, and by the properties of L in Proposition 2.1, we get

$$C+\eta\geq e_\varepsilon(M)+\eta\geq\mathcal{E}_k(m,w)\geq\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}-\frac{C_f}{\alpha+1}m^{\alpha+1}\,dx\; -KM. \tag{3-11}$$

We apply (2-15) with $\alpha=\beta-1$, and we obtain

$$\begin{aligned} C+\eta+KM &\geq \int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}-\frac{C_f}{\alpha+1}m^{\alpha+1}\,dx \\ &\geq C\varepsilon^{\gamma'}M^{1-(1+\delta)(1+\alpha)}\|m\|^{(1+\alpha)(1+\delta)}_{L^{\alpha+1}}-\frac{C_f}{\alpha+1}\|m\|^{(1+\alpha)}_{L^{\alpha+1}}. \end{aligned}$$

Recall that $\delta+1=\frac{\gamma'}{\alpha N}$, which can be computed using (2-19), so

$$\frac{\gamma'}{\delta}=\frac{\gamma' N\alpha}{\gamma'-N\alpha}.$$

Note that if we choose A sufficiently large (depending on δ, M, C_f, C_L), we get

$$C\varepsilon^{\gamma'}M^{1-(1+\delta)(1+\alpha)}(\varepsilon^{-\frac{\gamma'}{\delta}}A)^{1+\delta}-\frac{C_f}{\alpha+1}(\varepsilon^{-\frac{\gamma'}{\delta}}A)\geq C+\eta+KM,$$

from which we conclude that $\|m\|_{L^{\alpha+1}}^{(1+\alpha)} \le \varepsilon^{-\frac{\gamma'}{\delta}} A$, and so estimate (3-9) holds. Estimate (3-8) comes from (3-9) and (3-11). □

3C. ***Existence of a solution.*** We are now in the position to show existence of minimizers of the energy $\mathcal{E}_k$ in the class $\mathcal{K}_{\varepsilon,M}$ for every $\varepsilon, M > 0$.

Proposition 3.3. *For every $\varepsilon > 0$ and $M > 0$, there exists a minimizer $(m_k, w_k) \in \mathcal{K}_{\varepsilon,M}$ of $\mathcal{E}_k$, that is,*

$$\mathcal{E}_k(m_k, w_k) = \inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}_k(m, w).$$

Moreover, for every minimizer $(m_k, w_k) \in \mathcal{K}_{\varepsilon,M}$ of $\mathcal{E}_k$, there holds

$$m_k(1+|x|)^b \in L^1(\mathbb{R}^N), \quad w_k(1+|x|)^{\frac{b}{\gamma}} \in L^1(\mathbb{R}^N), \tag{3-12}$$

and there exist constants $C > 0$ and K, independent of ε and k, such that

$$\int_{\mathbb{R}^N} m_k \left|\frac{w_k}{m_k}\right|^{\gamma'} dx + \int_{R^N} m_k V(x)\,dx + \|m_k\|_{L^{\alpha+1}(\mathbb{R}^N)}^{\alpha+1} \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-N\alpha}} + K. \tag{3-13}$$

Proof. Let $(m_n, w_n) \in \mathcal{K}_{\varepsilon,M}$ be a minimizing sequence, that is, $\mathcal{E}_k(m_n, w_n) \to e_{k,\varepsilon}(M)$. This implies that, choosing n sufficiently large, $\mathcal{E}_k(m_n, w_n) \le e_\varepsilon(M) + 1$. From this and (3-4) we get

$$\begin{aligned}\int_{\mathbb{R}^N} m_n L\left(-\frac{w_n}{m_n}\right) dx + \int_{\mathbb{R}^N} V(x) m_n\,dx &\le \mathcal{E}_k(m_n, w_n) + \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m_n^{\alpha+1}\,dx + KM \\ &\le e_{k,\varepsilon}(M) + 1 + \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m_n^{\alpha+1} + KM. \end{aligned} \tag{3-14}$$

By Proposition 3.2, we get

$$\|m_n\|_{L^{\alpha+1}} + \int_{\mathbb{R}^N} m_n^{1-\gamma'}|w_n|^{\gamma'}\,dx \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K.$$

We conclude also that

$$\int_{\mathbb{R}^N} V(x) m_n(x)\,dx \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K$$

for some $C, K > 0$. These estimates will imply (3-13), after passing to the limit, using Fatou's lemma.

Moreover, by Corollary 2.9, we have that there exists $C_\varepsilon > 0$ depending on ε such that for all $r < q$,

$$\|m_n\|_{W^{1,r}(\mathbb{R}^N)} \le C_\varepsilon.$$

Moreover, due to Sobolev embeddings, we get $\|m_n\|_{L^s(\mathbb{R}^N)} \le C_\varepsilon$ for all $s < q^*$. In addition, by applying the Hölder inequality, we get that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |w_n|^{\frac{\gamma'\alpha+\gamma'}{\gamma'+\alpha}}\,dx \le C\left(\int_{\mathbb{R}^N} m_n^{1-\gamma'}|w_n|^{\gamma'}\,dx\right)^{\frac{\alpha+1}{\gamma'+\alpha}} \|m_n\|_{L^{\alpha+1}(\mathbb{R}^N)}^{\frac{\gamma'-1}{(\alpha+1)(\gamma'+\alpha)}}.$$

By these estimates and Sobolev compact embeddings, we get that eventually extracting a subsequence via a diagonalization procedure, $m_n \to m_k$ weakly in $W^{1,r}(\mathbb{R}^N)$ for all $r < q$ and strongly in $L^s(K)$ for all $1 \le s < q^*$ and for every compact $K \subset \mathbb{R}^N$, and $w_n \to w_k$ weakly in $L^{\frac{\gamma'\alpha+\gamma'}{\gamma'+\alpha}}(\mathbb{R}^N)$. By using the fact that $\int_{\mathbb{R}^N} V(x)m_n(x)\,dx \le C_\varepsilon$ and (1-5), we get that for all $R > 1$,

$$C_\varepsilon \ge \int_{\mathbb{R}^N} m_n(x)V(x)\,dx \ge \int_{|x|>R} m_n(x)V(x)\,dx \ge CR^b \int_{|x|>R} m_n(x)\,dx.$$

So for every $\varepsilon > 0$ fixed and all $\eta > 0$, there exists $R > 0$ for which $\int_{|x|>R} m_n(x)\,dx \le \eta$: up to extracting a subsequence we get that $m_n \to m_k$ in $L^1(\mathbb{R}^N)$, and so $\int_{\mathbb{R}^N} m_k(x)\,dx = M$. By the boundedness of m_n in $L^s(\mathbb{R}^N)$ for all $1 \le s < q^*$, we then have $m_n \to m_k$ strongly in $L^{\alpha+1}(\mathbb{R}^N)$. Finally, observe that from (3-13), using (1-5), we conclude that $m_k(1+|x|^b) \in L^1(\mathbb{R}^N)$. Moreover, we get

$$\int_{\mathbb{R}^N} |w_k|\,dx \le \int_{\mathbb{R}^N} |w_k|(1+|x|)^{\frac{b}{\gamma}}\,dx \le \left(\int_{\mathbb{R}^N} \frac{|w_k|^{\gamma'}}{m_k^{\gamma'-1}}\,dx\right)^{\frac{1}{\gamma'}} \left(\int_{\mathbb{R}^N} m_k(1+|x|)^b\,dx\right)^{\frac{1}{\gamma}},$$

and so $w_k(1+|x|)^{\frac{b}{\gamma}} \in L^1(\mathbb{R}^N)$.

Therefore the convergence is sufficiently strong to ensure that $(m_k, w_k) \in \mathcal{K}_{\varepsilon,M}$. We conclude that (m_k, w_k) is a minimum of the energy, by the lower semicontinuity with respect to weak convergence of the functional $\int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V(x)m\,dx$ and by using the fact that $F_k[m_n] \to F_k[m_k]$, since $m_n \to m_k$ strongly in $L^{\alpha+1}(\mathbb{R}^N)$. □

Using the minimizers we constructed in Proposition 3.3, we prove existence of a classical solution to (3-1).

Proposition 3.4. *There exists a classical solution* (u_k, m_k, λ_k) *to* (3-1) *that satisfies for some constant* $C_{k,\varepsilon} > 0$ *the inequalities*

$$|\nabla u_k(x)| \le C_{k,\varepsilon}(1+|x|^{\frac{b}{\gamma}}), \quad u_k(x) \ge C_{k,\varepsilon}^{-1}(1+|x|^{1+\frac{b}{\gamma}}) - C_{k,\varepsilon}. \tag{3-15}$$

Additionally there exist $C, K > 0$ *not depending on* ε, k *such that*

$$-K - C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} \le \lambda_k \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K. \tag{3-16}$$

Proof. Let (m_k, w_k) be a minimizer of $\mathcal{E}_k$. Define the space of test functions

$$\mathcal{A} = \mathcal{A}_{b,\gamma} := \left\{\psi \in C^2(\mathbb{R}^N) : \limsup_{|x|\to\infty} \frac{|\nabla\psi(x)|}{|x|^{\frac{b}{\gamma}}} < \infty,\ \limsup_{|x|\to\infty} \frac{|\Delta\psi(x)|}{|x|^b} < \infty\right\}. \tag{3-17}$$

Note that we also have, for all $\psi \in \mathcal{A}$,

$$\limsup_{|x|\to\infty} \frac{|\psi(x)|}{|x|^{\frac{b}{\gamma}+1}} < \infty.$$

We claim that

$$-\varepsilon \int_{\mathbb{R}^N} m_k \Delta\psi\,dx = \int_{\mathbb{R}^N} w_k \nabla\psi\,dx \quad \text{for all } \psi \in \mathcal{A}. \tag{3-18}$$

Indeed, consider a radial smooth cutoff function $\chi(x)$ which is identically equal to 1 in $B_1(0)$ and identically zero in $\mathbb{R}^N \setminus B_2(0)$. Set $\chi_R(x) := \chi\big(\frac{x}{R}\big)$; we have $|\nabla \chi_R| \le CR^{-1}$ and $|\Delta \chi_R| \le CR^{-2}$ on $\mathbb{R}^N$ for some positive constant C.

Since the equality $\varepsilon \Delta m_k = \operatorname{div} w_k$ holds in the weak sense on $\mathbb{R}^N$, we may multiply it by $\chi_R \psi$ with $\psi \in \mathcal{A}$ and integrate by parts to obtain

$$-\varepsilon \int_{B_{2R}} m_k(\chi_R \Delta\psi + 2\nabla\psi \cdot \nabla\chi_R + \psi\Delta\chi_R)\, dx = \int_{B_{2R}} w_k \cdot (\chi_R \nabla\psi + \psi \nabla \chi_R)\, dx. \tag{3-19}$$

Note that for some positive C,

$$\int_{\mathbb{R}^N} |w_k \nabla\psi|\, dx \le C \int_{\mathbb{R}^N} |w_k| (1+|x|)^{\frac{b}{\gamma}}\, dx < \infty, \quad \int_{\mathbb{R}^N} m_k |\Delta\psi|\, dx \le C \int_{\mathbb{R}^N} m_k (1+|x|)^b\, dx < \infty$$

by the integrability properties (3-12). Moreover,

$$\begin{aligned} \int_{R \le |x| \le 2R} m_k |\psi| |\Delta \chi_R|\, dx &\le C \int_{R \le |x| \le 2R} m_k \frac{(1+|x|)^{\frac{b}{\gamma}+1}}{R^2}\, dx \\ &\le C_1 \int_{R \le |x| \le 2R} m_k (1+|x|)^{\frac{b}{\gamma}-1}\, dx \to 0 \quad \text{as } R \to \infty, \end{aligned}$$

because $\frac{b}{\gamma} - 1 \le b$. Reasoning in a similar way, we also have that $\int_{R \le |x| \le 2R} m_k \nabla\psi \cdot \nabla\chi_R$ and $\int_{R \le |x| \le 2R} w_k \cdot \psi \nabla \chi_R$ converge to zero as $R \to \infty$. Equality (3-18) then follows by passing to the limit in (3-19).

Therefore, recalling the integrability properties of m_k, w_k obtained in Proposition 3.3, the problem of minimizing $\mathcal{E}_k$ on $\mathcal{K}_{\varepsilon,M}$ is equivalent to minimizing $\mathcal{E}_k$ on $\mathcal{K}$, where

$$\mathcal{K} := \Big\{(w,m) \in (L^1 \cap W^{1,r})(\mathbb{R}^N) \times L^{\frac{\gamma'(\alpha+1)}{\gamma'+\alpha}}(\mathbb{R}^N) : (w,m) \text{ satisfies (3-12), (3-18), } m \ge 0,\ \textstyle\int_{\mathbb{R}^N} m = M\Big\}$$

for some $r < q$. As in [Briani and Cardaliaguet 2018, Proposition 3.1], the convexity of L implies that (m_k, w_k) is also a minimizer of the following convex functional on $\mathcal{K}$:

$$\widetilde{J}(m,w) = \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big) + (V(x) + f_k[m_k])m\, dx.$$

We now aim to prove that

$$\sup\big\{\lambda M : -\varepsilon\Delta\psi + H(\nabla\psi) + \lambda \le V(x) + f_k[m_k] \text{ on } \mathbb{R}^N \text{ for some } \psi \in \mathcal{A}\big\} = \min_{(w,m)\in\mathcal{K}} \widetilde{J}(m,w). \tag{3-20}$$

We proceed as in [Cardaliaguet and Graber 2015, Theorem 3.5]: Setting

$$\mathcal{L}(m,w,\lambda,\psi) := \widetilde{J}(m,w) + \int_{\mathbb{R}^N} \varepsilon m \Delta\psi + w\nabla\psi - \lambda m\, dx + \lambda M,$$

we have

$$\min_{(m,w)\in\mathcal{K}} \widetilde{J}(m,w) = \min_{(m,w)} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \mathcal{L}(m,w,\lambda,\psi),$$

where the minimum in the right-hand side has to be taken over pairs

$$(m, w) \in (L^1 \cap W^{1,r})(\mathbb{R}^N) \times L^{\frac{\gamma'(\alpha+1)}{\gamma'+\alpha}}(\mathbb{R}^N)$$

for some $r < q$, satisfying (3-12). Note that $\mathcal{L}(\cdot\,,\cdot\,,\lambda,\psi)$ is convex, and $\mathcal{L}(m, w,\cdot\,,\cdot)$ is linear. Moreover, since $\mathcal{L}(\cdot\,,\cdot\,,\lambda,\psi)$ is weak-* lower semicontinuous, we can use the min-max theorem, see [Borwein and Vanderwerff 2010, Theorem 2.3.7], to get

$$\begin{aligned}
&\min_{(m,w)} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \mathcal{L}(m, w, \lambda, \psi) \\
&\quad = \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \min_{(m,w)} \mathcal{L}(m, w, \lambda, \psi) \\
&\quad = \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \min_{(m,w)} \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big) + (V(x) + f_k[m_k])m + \varepsilon m\Delta\psi + w\nabla\psi - \lambda m\, dx + \lambda M \\
&\quad = \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \int_{\mathbb{R}^N} \min_{(m,w)\in\mathbb{R}\times\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big) + (V(x) + f_k[m_k])m + \varepsilon m\Delta\psi + w\nabla\psi - \lambda m\, dx + \lambda M,
\end{aligned}$$

where the interchange of the min and the integration is possible by standard results in convex optimization. By computation, $\min_{(m,w)\in\mathbb{R}\times\mathbb{R}^N} mL\big(-\frac{w}{m}\big) + (V(x) + f_k[m_k])m + \varepsilon m\Delta\psi + w\nabla\psi - \lambda m$ is zero whenever $\varepsilon\Delta\psi - H(\nabla\psi) - \lambda + (V(x) + f_k[m_k])$ is positive, and it is $-\infty$ otherwise. Therefore, we have proven (3-20).

By Theorem 2.7(i)–(ii), there exists $u_k \in C^2(\mathbb{R}^N)$ such that

$$-\varepsilon\Delta u_k + H(\nabla u_k) + \lambda_k = V(x) + f_k[m_k] \quad \text{on } \mathbb{R}^N, \tag{3-21}$$

which satisfies

$$|\nabla u_k(x)| \le C_{k,\varepsilon}(1 + |x|)^{\frac{b}{\gamma}}, \quad u_k(x) \ge C_{k,\varepsilon}|x|^{\frac{b}{\gamma}+1} - C_{k,\varepsilon}^{-1}$$

for some $C_{k,\varepsilon} > 0$.

Moreover,

$$\varepsilon|\Delta u_k(x)| \le |H(\nabla u_k(x))| + |\lambda_k| + V(x) - f_k[m_k] \le C_{k,\varepsilon}(1 + |x|)^b \quad \text{on } \mathbb{R}^N,$$

so $u_k \in \mathcal{A}$. Thus, the supremum in the left-hand side of (3-20) is achieved by λ_k, and it holds true that

$$\lambda_k M = \widetilde{J}(m_k, w_k) = \mathcal{E}_k(m_k, w_k) + \int_{\mathbb{R}^N} f_k[m_k]m_k\, dx - F[m_k]. \tag{3-22}$$

This gives in particular (3-16), using Lemma 3.1, estimates (3-10) and recalling Proposition 3.2 and assumptions (1-3), (3-2) and (3-4).

We now use (3-22), (3-21) and (3-18) with $\psi = u_k$ to get

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \Big(L\Big(-\frac{w_k}{m_k}\Big) + V(x) - m_k^\alpha - \lambda_k\Big)m_k\, dx \\
&= \int_{\mathbb{R}^N} \Big(L\Big(-\frac{w_k}{m_k}\Big) - \varepsilon\Delta u_k + H(\nabla u_k)\Big)m_k\, dx \\
&= \int_{\mathbb{R}^N} \Big(L\Big(-\frac{w_k}{m_k}\Big) + H(\nabla u_k) + \nabla u_k \cdot \frac{w_k}{m_k}\Big)m_k\, dx,
\end{aligned}$$

which implies

$$\frac{w_k}{m_k} = -\nabla H(\nabla u_k) \quad \text{on the set } \{m_k > 0\}.$$

Hence, the Kolmogorov equation $\varepsilon \Delta m_k + \operatorname{div}(m_k \nabla H(\nabla u_k)) = 0$ holds in the weak sense, and by elliptic regularity we conclude that (u_k, m_k, λ_k) is a classical solution to (1-1). □

Remark 3.5. Note that if we assume that the local term f satisfies (1-9) instead of (1-3), then the same argument as above applies. In particular there exists a classical solution (u_k, m_k, λ_k) to (3-1) such that

$$|\nabla u_k(x)| \le C_{k,\varepsilon}(1+|x|^{\frac{b}{\gamma}}), \quad u_k(x) \ge C_{k,\varepsilon}^{-1}(1+|x|^{1+\frac{b}{\gamma}}) - C_{k,\varepsilon},$$
$$\int_{\mathbb{R}^N} m_k^{\alpha+1}\,dx, \int_{\mathbb{R}^N} m_k(x)V(x)\,dx \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K.$$

We finally prove that every m_k is bounded from above in $\mathbb{R}^N$ (this is not obvious from Proposition 3.4 unless $\gamma' > N$). Note that the following result does not provide uniform bounds with respect to k. These will be produced in Theorem 4.1 using a much more involved argument.

Proposition 3.6. *Let (u_k, m_k, λ_k) be as in Proposition 3.4. Then, m_k is bounded in $L^\infty(\mathbb{R}^N)$.*

Proof. Let $\phi(x) = u_k(x)^p$, for $p > 1$ to be chosen later. Using the fact that u_k is a classical solution to the HJB equation, we get

$$\begin{aligned}
-\varepsilon\Delta\phi &+ \nabla H(\nabla u_k)\cdot\nabla\phi \\
&= p u_k^{p-1}\left(-\Delta u_k - (p-1)\frac{|\nabla u_k|^2}{u_k} + \nabla H(\nabla u_k)\cdot\nabla u_k\right) \\
&= p u_k^{p-1}\left(-\Delta u_k + H(\nabla u_k) - (p-1)\frac{|\nabla u_k|^2}{u_k} - H(\nabla u_k) + \nabla H(\nabla u_k)\cdot\nabla u_k\right) \\
&= p u_k^{p-1}\left(-(p-1)\frac{|\nabla u_k|^2}{u_k} - H(\nabla u_k) + \nabla H(\nabla u_k)\cdot\nabla u_k - \lambda + f_k[m_k] + V\right). \qquad (3\text{-}23)
\end{aligned}$$

Observe that by (1-2), (1-5), (3-15) and the fact that $f_k[m_k]$ is bounded on $\mathbb{R}^N$, there exist large R and C such that

$$\begin{aligned}
G(x) &= -(p-1)\frac{|\nabla u_k|^2}{u_k} - H(\nabla u_k) + \nabla H(\nabla u_k)\cdot\nabla u_k - \lambda + f_k[m_k] + V(x) \\
&\ge K^{-1}|\nabla u_k|^\gamma - (p-1)\frac{|\nabla u_k|^2}{u_k} - K - \lambda + f_k[m_k] + V(x) \\
&\ge (p-1)|\nabla u_k|^\gamma\left(\frac{1}{K(p-1)} - \frac{|\nabla u_k|^{2-\gamma}}{u_k}\right) - C + C_V^{-1}|x|^b \ge 1 \quad \text{for all } |x| > R.
\end{aligned}$$

Hence, again by (3-15), for all $|x| > R$

$$-\varepsilon\Delta\phi + \nabla H(\nabla u_k)\cdot\nabla\phi \ge c|x|^{(1+\frac{b}{\gamma})(p-1)}.$$

In view of [Metafune et al. 2005, Proposition 2.6], we have $|x|^{(1+\frac{b}{\gamma})(p-1)} m_k \in L^1(\mathbb{R}^N)$. Recall now that $|\nabla H(\nabla u_k)| \le C(1+|x|)^{\frac{b}{\gamma'}}$ by (3-15). Therefore, by choosing p large enough, $|\nabla H(\nabla u_k)|^s m_k \in L^1(\mathbb{R}^N)$ for some $s > N$. We conclude the boundedness of m_k in L^∞ by [Metafune et al. 2005, Theorem 3.5]. □

4. Existence of a solution to the MFG system for $\varepsilon > 0$

Our aim is to pass to the limit $k \to \infty$ for solutions to (3-1).

4A. ***A priori L^∞ bounds.*** We need first a priori L^∞ bounds on m_k that are independent with respect to k. These will be achieved by a blow-up argument, as proposed in [Cirant 2016] for systems set on the flat torus $\mathbb{T}^N$. Here, the unbounded space $\mathbb{R}^N$ and the presence of the unbounded term V make the argument much more involved than the one in that paper. To control the points $x_k \in \mathbb{R}^N$ where $m_k(x_k)$ possibly explodes, some delicate estimates on the decay (in L^1) of its renormalization will be produced.

We provide a more general result, that will be used also in the rescaled framework (Section 5). Let r_k, s_k, t_k be bounded sequences of positive real numbers.

Theorem 4.1. *Let (u_k, λ_k, m_k) be a classical solution to the mean-field game system*

$$\begin{cases} -\Delta u + r_k^\gamma H(r_k^{-1}\nabla u) + \lambda_k = g_k[m] + s_k V(t_k x), \\ -\Delta m - \operatorname{div}(m\, r_k^{\gamma-1} \nabla H(r_k^{-1}\nabla u)) = 0, \\ \int_{\mathbb{R}^N} m\, dx = M, \end{cases}$$

where $g_k : L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ are such that for all $m \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and for all k,

$$\|g_k[m]\|_{L^\infty(\mathbb{R}^N)} \le K(\|m\|^\alpha_{L^\infty(\mathbb{R}^N)} + 1) \tag{4-1}$$

for some $K > 0$. Suppose also that for all k, u_k is bounded from below and m_k is bounded from above on $\mathbb{R}^N$. Then, there exists a constant C independent of k such that

$$\|m_k\|_{L^\infty} \le C.$$

Proof. We argue by contradiction, so we assume that

$$\sup_{\mathbb{R}^N} m_k = L_k \to +\infty.$$

We divide the proof into several steps.

<u>Step 1</u>: rescaling of the solutions. Let

$$\mu_k := L_k^{-\beta}, \quad \beta = \alpha \frac{\gamma - 1}{\gamma} > 0.$$

So, observe that $\mu_k \to 0$ as $k \to 0$. Since u_k is bounded from below, up to adding a suitable constant we can assume that $\min_{\mathbb{R}^N} u_k = 0$. We define the rescaling

$$\begin{cases} v_k(x) = \mu_k^{\frac{2-\gamma}{\gamma-1}} u_k(\mu_k x) + 1, \\ n_k(x) = L_k^{-1} m_k(\mu_k x). \end{cases}$$

Note that $0 \le n_k(x) \le 1$. Moreover, due to (1-4),

$$\int_{\mathbb{R}^N} n_k(x)\,dx = M L_k^{\frac{\alpha N(\gamma-1)}{\gamma}-1} \to 0, \tag{4-2}$$

and $\min v_k = 1$. We define

$$H_k(q) = \mu_k^{\frac{\gamma}{\gamma-1}} r_k^\gamma H(r_k^{-1}\mu_k^{\frac{1}{1-\gamma}} q), \quad \text{so} \quad \nabla H_k(q) = \mu_k r_k^{\gamma-1} \nabla H(r_k^{-1}\mu_k^{\frac{1}{1-\gamma}} q).$$

Recalling (1-2) we have that for all $q \in \mathbb{R}^N$,

$$\begin{gathered} C_H|q|^\gamma - K \le H_k(q) \le C_H(|q|^\gamma + 1), \\ |\nabla H_k(q))| \le C_H|q|^{\gamma-1}, \\ \nabla H_k(q)\cdot q - H_k(q) \ge K^{-1}|q|^\gamma - K. \end{gathered} \tag{4-3}$$

Moreover, we define

$$\tilde{g}_k(x) = \mu_k^{\frac{\gamma}{\gamma-1}} g_k[m_k](\mu_k x).$$

Recalling that $0 \le m_k \le L_k$, by (4-1) we get that for all x and for all k,

$$|\tilde{g}_k(x)| \le \mu_k^{\frac{\gamma}{\gamma-1}} K(L_k^\alpha + 1) \le 2K, \tag{4-4}$$

where we used the fact that $\mu_k = L_k^{-\beta}$ with $\beta = \alpha\frac{\gamma-1}{\gamma}$. Finally, we let

$$\tilde{\lambda}_k = \mu_k^{\frac{\gamma}{\gamma-1}} \lambda_k = \frac{1}{L_k^\alpha}\lambda_k$$

and we observe that

$$|\tilde{\lambda}_k| \le C. \tag{4-5}$$

Finally, let

$$V_k(x) = \mu_k^{\frac{\gamma}{\gamma-1}} s_k V(\mu_k t_k x).$$

By assumption (1-5), we get

$$s_k \mu_k^{\frac{\gamma}{\gamma-1}} C_V^{-1}(\max\{|t_k\mu_k x| - C_V, 0\})^b \le V_k(x) \le C_V(1 + \sigma_k|x|^b), \tag{4-6}$$

where

$$\sigma_k := \mu_k^{\frac{\gamma}{\gamma-1}+b} s_k t_k^b \to 0 \quad \text{as } k \to \infty.$$

In particular we also have the following bound from below for V_k:

$$V_k(x) \ge \frac{C_V^{-1}}{2^b}\sigma_k|x|^b \quad \text{for all } |x| \ge 2C_V(t_k\mu_k)^{-1}. \tag{4-7}$$

An easy computation shows that by rescaling we have that $(v_k, n_k, \tilde{\lambda}_k)$ is a solution to

$$\begin{cases} -\Delta v_k + H_k(\nabla v_k) + \tilde{\lambda}_k = \tilde{g}_k(x) + V_k(x), \\ -\Delta n_k - \operatorname{div}(n_k \nabla H_k(\nabla v_k)) = 0. \end{cases} \tag{4-8}$$

Step 2: a priori bounds on the rescaled solution to the Hamilton–Jacobi equation. We observe that by Theorem 2.5 and (4-6), there exists $C > 0$, independent of k, such that

$$|\nabla v_k(x)| \le C(1 + \sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}}) \quad \text{on } \mathbb{R}^N. \tag{4-9}$$

We recall that we assumed $v_k(\hat{x}_k) = \min v_k = 1$. Since v_k is a classical solution to (4-8), at a minimum point $\hat{x}_k$ we have, by (4-3), (4-4), (4-5) and (4-7),

$$\sigma_k |\hat{x}_k|^b \le C.$$

Therefore, by using this estimate and (4-9), since $|v_k(0)| \le |v_k(\hat{x}_k)| + |\hat{x}_k| \sup_{|y| \le |\hat{x}_k|} |\nabla u_k(y)|$ we obtain

$$|v_k(0)| \le 1 + C(1 + \sigma_k^{\frac{1}{\gamma}} |\hat{x}_k|^{1+\frac{b}{\gamma}}) \le C_1(1 + \sigma_k^{-\frac{1}{b}})$$

and then again by (4-9),

$$|v_k(x)| \le C(1 + \sigma_k^{-\frac{1}{b}} + \sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}+1}) \quad \text{on } \mathbb{R}^N. \tag{4-10}$$

Let χ be a smooth function $\chi : [0, +\infty) \to [0, +\infty)$ such that $\chi \equiv 0$ in $\left(0, \frac{1}{2}\right) \cup \left(\frac{3}{2}, +\infty\right)$, $\chi(1) > 0$ and $|\chi'|, |\chi''| \le 1$. We fix $\tilde{x} \in \mathbb{R}^N$ such that $|\tilde{x}| > 4C_V (t_k \mu_k)^{-1}$, and we set

$$w(x) = \kappa \sigma_k^{\frac{1}{\gamma}} |\tilde{x}|^{1+\frac{b}{\gamma}} \chi\left(\frac{|x|}{|\tilde{x}|}\right),$$

where $\kappa \ge 0$ has to be chosen. We have that $w(x) \le v_k(x)$ for all x such that $|x| \ge \frac{3}{2}|\tilde{x}|$ or $|x| \le \frac{1}{2}|\tilde{x}|$. Moreover, for x such that $\frac{1}{2}|\tilde{x}| \le |x| \le \frac{3}{2}|\tilde{x}|$ we have $|x| > 2C_V (\mu_k t_k)^{-1}$, so using the estimates (4-3), (4-4), (4-5) and (4-7),

$$-\Delta w + H_k(\nabla w) + \tilde{\lambda}_k - \tilde{g}_k(x) - V_k(x) \le \kappa N \sigma_k^{\frac{1}{\gamma}} |\tilde{x}|^{\frac{b}{\gamma}-1} + C_H \kappa^\gamma \sigma_k |\tilde{x}|^b + C - \frac{C_V^{-1}}{2^b} \sigma_k |\tilde{x}|^b.$$

Note that there exist $\kappa > 0$ small and $C_2 > 0$ large, depending only C_V and C_H and not on $|\tilde{x}|$, k, such that the right-hand side of the last expression is negative if

$$\sigma_k |\tilde{x}|^b \ge C_2$$

(this also implies that $t_k \mu_k |\tilde{x}| > 4C_V$, as required). The test function w is then a subsolution of the HJB equation in (4-8); therefore by comparison we get

$$v_k(\tilde{x}) \ge \kappa \chi(1) \sigma_k^{\frac{1}{\gamma}} |\tilde{x}|^{1+\frac{b}{\gamma}}.$$

By the arbitrariness of $\tilde{x}$ we conclude that, for some $C > 0$,

$$v_k(x) \ge C \sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}+1} \quad \text{for all } \sigma_k |x|^b \ge C_2. \tag{4-11}$$

Step 3: estimates on the (approximate) maxima of n_k. We now fix $0 < \delta \ll 1$ and x_k such that $n_k(x_k) = 1 - \delta$. Two possibilities may arise: either $\lim_k \sigma_k |x_k|^b = +\infty$ up to some subsequence, or there exists $C > 0$ such that $\sigma_k |x_k|^b \le C$. We rule out the second possibility by contradiction. Suppose indeed that there exists $C > 0$ such that $\sigma_k |x_k|^b \le C$. By (4-9), $|\nabla v_k| \le C$ on $B_2(x_k)$ for some $C > 0$. Therefore, using the fact that n_k solves the second equation in (4-8), the elliptic estimates in Proposition 2.4, (4-3), the interpolation inequality $\|n\|_q \le \|n\|_1^{\frac{1}{q}} \|n\|_\infty^{1-\frac{1}{q}}$ and the fact that $0 \le n_k \le 1$, we get for all $q > 1$,

$$\|n_k\|_{W^{1,q}(B_1(x_k))} \le C(1 + \|\nabla H_k(\nabla v_k)\|_{L^\infty(B_2(x_k))})\|n_k\|_{L^1(B_2(x_k))}^{\frac{1}{q}} \le C_q \tag{4-12}$$

for some $C_q > 0$ depending on q. This implies, choosing $q > N$, that for all $\theta \in (0, 1)$ there exists C_θ depending on θ (but not on k) such that $\|n_k\|_{C^{0,\theta}(B_1(x_k))} \le C_\theta$. Recalling that $n_k(x_k) = 1 - \delta$, we can fix $r < 1$ such that $n_k(x) \ge \frac{1}{2}$ for all $x \in B_r(x_k)$. It is sufficient to choose $r = C_\theta^{-\frac{1}{\theta}}\left(\frac{1}{2} - \delta\right)^{\frac{1}{\theta}}$. Therefore we have, by (4-2),

$$0 < \tfrac{1}{2}\omega_N r^N \le \int_{B_r(x_k)} n_k(x)\,dx \le \int_{\mathbb{R}^N} n_k(x)\,dx = ML_k^{\frac{\alpha N(\gamma-1)}{\gamma}-1} \to 0.$$

This gives a contradiction. Then we deduce that, up to a subsequence,

$$\lim_k \sigma_k |x_k|^b = +\infty. \tag{4-13}$$

<u>Step 4</u>: construction of a Lyapunov function. Let $\phi(x) = v_k(x)^p$, for $p > 1$ to be chosen later. Using the fact that v_k is a classical solution to (4-8), arguing as in (3-23), we get

$$\begin{aligned} -\Delta\phi + \nabla H_k(\nabla v_k)\cdot\nabla\phi &= p v_k^{p-1}\left(-\Delta v_k - (p-1)\frac{|\nabla v_k|^2}{v_k} + \nabla H_k(\nabla v_k)\cdot\nabla v_k\right) \\ &= p v_k^{p-1}\left(-(p-1)\frac{|\nabla v_k|^2}{v_k} - H_k(\nabla v_k) + \nabla H_k(\nabla v_k)\cdot\nabla v_k - \tilde{\lambda}_k + \tilde{g}_k(x) + V_k(x)\right). \end{aligned}$$

We set

$$G_k(x) = -(p-1)\frac{|\nabla v_k|^2}{v_k} - H_k(\nabla v_k) + \nabla H_k(\nabla v_k)\cdot\nabla v_k - \tilde{\lambda}_k + \tilde{g}_k(x) + V_k(x). \tag{4-14}$$

Using the previous computation and the fact that n_k is a solution to (4-8), we get, by integrating by parts, that

$$0 = \int_{\mathbb{R}^N} n_k(x)\big(-\Delta\phi(x) + \nabla H_k(\nabla v_k(x))\cdot\nabla\phi(x)\big)\,dx = p\int_{\mathbb{R}^N} n_k(x) G_k(x) \phi^{\frac{p-1}{p}}(x)\,dx.$$

Therefore from this, for every $\Lambda > 0$ we get

$$\int_{\{\phi(x)\ge\Lambda^p\}} n_k(x) G_k(x) \phi^{\frac{p-1}{p}}(x)\,dx = -\int_{\{\phi(x)\le\Lambda^p\}} n_k(x) G_k(x)\phi^{\frac{p-1}{p}}(x)\,dx. \tag{4-15}$$

Observe that by (4-3), (4-4), (4-5) and (4-7) we get that for all $t_k\mu_k|x| \ge 2C_V$,

$$\begin{aligned} G_k(x) &\ge K^{-1}|\nabla v_k|^\gamma - (p-1)\frac{|\nabla v_k|^2}{v_k} - K - \tilde{\lambda}_k + \tilde{g}_k(x) + V_k(x) \\ &\ge (p-1)|\nabla v_k|^\gamma\left(\frac{1}{K(p-1)} - \frac{|\nabla v_k|^{2-\gamma}}{v_k}\right) - C + C_V\sigma_k|x|^b. \end{aligned} \tag{4-16}$$

We first claim that by (4-9) and (4-11),

$$\frac{1}{K(p-1)} - \frac{|\nabla v_k|^{2-\gamma}}{v_k} > 0 \quad \text{if } \sigma_k |x|^b \geq C_2,$$

eventually enlarging C_2 in (4-11). Indeed,

$$\frac{|\nabla v_k(x)|^{2-\gamma}}{v_k(x)} \leq C \frac{[1+\sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}}]^{2-\gamma}}{[\sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}}]|x|} \leq \frac{C_H}{p-1} \tag{4-17}$$

whenever $\sigma_k |x|^b$ is large enough. This implies that for all $\sigma_k |x|^b \geq C_2$, by (4-16) we have $G_k(x) \geq -C$. On the other hand, again by the gradient bounds in (4-9) we have that $|\nabla v_k(x)| \leq C(1+C_2)$ on the set $\sigma_k |x|^b \leq C_2$, so (4-16) and $\min v_k = 1$ again guarantee that $G_k(x) \geq -C_3$. In conclusion, there exists $C > 0$ such that

$$G_k(x) \geq -C \quad \text{for all } x \in \mathbb{R}^N.$$

Therefore, going back to (4-15), recalling (4-2), we obtain that

$$\begin{aligned} \int_{\{\phi(x)\geq \Lambda^p\}} n_k(x) G_k(x) \left(\frac{\phi(x)}{\Lambda^p}\right)^{\frac{p-1}{p}} dx &\leq C \int_{\{\phi(x)\leq \Lambda^p\}} n_k(x)\, dx \leq C \int_{\mathbb{R}^N} n_k(x)\, dx \\ &= CM\mu_k^{-N+\frac{\gamma}{\alpha(\gamma-1)}} \to 0 \end{aligned} \tag{4-18}$$

as $k \to \infty$.

Note that by (4-16) and (4-17), if x is such that $G_k(x) \leq 0$, then necessarily $\sigma_k |x|^b \leq C$ for some $C > 0$. Hence, by (4-10), we get that $v_k(x) \leq C_3(1+\sigma_k^{-\frac{1}{b}})$. Therefore if we choose $\Lambda = \Lambda_k = K\sigma_k^{-\frac{1}{b}}$ for a sufficiently large $K > 0$, we get that $G_k(x) > 0$ in the set $\{x : \phi(x) \geq \Lambda^p\}$.

Step 5: integral estimates on n_k. Arguing as in the end of Step 4, we may choose K big enough so that $G_k(x) \geq 1$ in the set $\{x : \phi(x) \geq \Lambda_k^p\}$, where $\Lambda_k = K\sigma_k^{-\frac{1}{b}}$. If k is sufficiently large, by (4-11) and (4-13) it follows that for some $C > 0$,

$$v_k(x) \geq C\sigma_k^{\frac{1}{\gamma}} |x_k|^{1+\frac{b}{\gamma}} \quad \text{in } B_1(x_k), \quad \text{and} \quad B_1(x_k) \subseteq \{x : \phi(x) \geq \Lambda_k^p\}.$$

Therefore, we may conclude that

$$\begin{aligned} \int_{\{\phi(x)\geq \Lambda_k^p\}} n_k(x) G_k(x) \left(\frac{\phi(x)}{\Lambda_k^p}\right)^{\frac{p-1}{p}} dx &\geq C \left(\frac{\sigma_k^{\frac{1}{\gamma}} |x_k|^{1+\frac{b}{\gamma}}}{\sigma_k^{-\frac{1}{b}}}\right)^{p-1} \int_{B_1(x_k)} n_k(x)\, dx \\ &\geq C(\sigma_k^{\frac{1}{\gamma}} |x_k|^{\frac{b}{\gamma}})^{p-1} \int_{B_1(x_k)} n_k(x)\, dx, \end{aligned} \tag{4-19}$$

which together with (4-18) gives

$$\int_{B_1(x_k)} n_k(x)\, dx \leq (\sigma_k^{\frac{1}{\gamma}} |x_k|^{\frac{b}{\gamma}})^{1-p} \tag{4-20}$$

for all k large.

Reasoning as in Step 3, see in particular (4-12), by Proposition 2.4, (4-3), (4-9) and (4-20), we get that for all $q > 1$,

$$\|n_k\|_{W^{1,q}(B_{1/2}(x_k))} \le C(1 + \|\nabla H_k(\nabla v_k)\|_{L^\infty(B_1(x_k))})\|n_k\|_{L^1(B_1(x_k))}^{\frac{1}{q}}$$
$$\le C_4[1 + (\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}})^{\gamma-1}](\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}})^{\frac{1-p}{q}} \le 1,$$

whenever p is such that $\gamma - 1 + \frac{1-p}{q} < 0$ and k is large (recall that we are supposing $\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}} \to +\infty$).

Therefore, we may conclude as in Step 3: choosing $q > N$, for some $\theta \in (0, 1)$ there exists C_θ such that $\|n_k\|_{C^{0,\theta}(B_{1/2}(x_k))} \le C_\theta$. Since $n_k(x_k) = 1-\delta$, we can fix $r < 1$ such that $n_k(x) \ge \frac{1}{2}$ for all $x \in B_r(x_k)$. Finally, by (4-2)

$$0 < \tfrac{1}{2}\omega_N r^N \le \int_{B_r(x_k)} n_k(x)\,dx \le \int_{\mathbb{R}^N} n_k(x)\,dx = ML_k^{\frac{\alpha N(\gamma-1)}{\gamma-1}} \to 0.$$

That gives a contradiction and rules out the possibility that $\sigma_k|x_k|^b \to +\infty$. Therefore, $L_k \to +\infty$ is impossible. ☐

4B. ***Existence of a solution to the MFG system.*** Using the a priori bounds we obtained, we can pass to the limit in k in the MFG system (3-1) to get a solution to (1-1) for every $\varepsilon > 0$.

Proof of Theorem 1.1. First, by Proposition 3.4, the existence for all k of a classical solution (u_k, m_k, λ_k) to (3-1) follows. By (3-16), up to passing to a subsequence we have that $\lambda_k \to \lambda_\varepsilon$.

Note that by Propositions 3.4 and 3.6, u_k and m_k are bounded by below and above respectively, so due to Theorem 4.1 (with $g[m] = f_k[m]$ and $r_k = s_k = t_k = 1$ for all k), we get that there exists $C_\varepsilon > 0$ independent of k (but eventually dependent on $\varepsilon > 0$) such that $\|m_k\|_{L^\infty(\mathbb{R}^N)} \le C_\varepsilon$. Using Theorem 2.5, this implies $|\nabla u_k(x)| \le C_\varepsilon(1 + |x|^{\frac{b}{\gamma}})$ for some C_ε independent of k. We can normalize $u_k(0) = 0$ and using the Ascoli–Arzelà theorem we can extract by a diagonalization procedure a sequence u_k such that $u_k \to u_\varepsilon$ locally uniformly in $\mathbb{R}^N$. Moreover, by using the estimates and the equation we have that actually $u_k \to u_\varepsilon$ locally uniformly in C^1. Note that, denoting by x_k a minimum point of u_k on $\mathbb{R}^N$, we have by the HJB equation that

$$H(0) + \lambda_k - f_k[m_k](x_k) \ge V(x_k).$$

Coercivity (1-5) of V and uniform boundedness of λ_k and $f_k[m_k]$ guarantee that x_k remains bounded, in particular that $u_k \ge -C$ on $\mathbb{R}^N$ by gradient bounds. Theorem 2.6 then applies, and in particular $u_k(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1}$ for all k. This implies, passing to the limit, that

$$u_\varepsilon(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1} \quad \text{on } \mathbb{R}^N. \tag{4-21}$$

By the elliptic estimates in Proposition 2.4, we get that $m_k \to m_\varepsilon$ locally uniformly in $C^{0,\alpha}$ for all $\alpha \in (0, 1)$ and weakly in $W^{1,p}(B_R)$ for every $p > 1$ and $R > 0$. Therefore we get that u_ε is a solution in the viscosity sense of the Hamilton–Jacobi equation, by stability with respect to uniform convergence, and m_ε is a weak solution to the Fokker–Planck equation, by strong convergence of $\nabla u_k \to \nabla u_\varepsilon$. Finally this implies, again by using the regularity of the HJB equation, that $u_k \to u_\varepsilon$ locally uniformly in C^2. Therefore, $u_\varepsilon, m_\varepsilon$ solve in classical sense the system (1-1).

Now we show that $\int_{\mathbb{R}^N} m_\varepsilon(x)\,dx = M$. We have that $m_k \to m_\varepsilon$ locally uniformly in $C^{0,\alpha}$ for every $\alpha \in (0,1)$. Moreover, due to (3-13) and to (1-5), we get that for all $R > 1$,

$$C_\varepsilon \geq \int_{\mathbb{R}^N} m_k(x)V(x)\,dx \geq \int_{|x|>R} m_k(x)V(x)\,dx \geq CR^b \int_{|x|>R} m_k(x)\,dx.$$

This implies $\int_{|x|\leq R} m_k(x)\,dx \geq M - C_\varepsilon R^{-b}$ and then by uniform convergence we get that for every $\varepsilon > 0$, and $\eta > 0$, there exists $R > 0$ such that

$$\int_{|x|\leq R} m_\varepsilon(x)\,dx \geq M - \eta.$$

From this we can conclude that $m_k \to m_\varepsilon$ in $L^1(\mathbb{R}^N)$, that is, $\int_{\mathbb{R}^N} m_\varepsilon(x)\,dx = M$. By the boundedness of m_k in L^∞, it also follows that $m_k \to m_\varepsilon$ in $L^{\alpha+1}(\mathbb{R}^N)$.

Finally, we get that if $w_\varepsilon = -m_\varepsilon \nabla H(\nabla u_\varepsilon)$, then $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$, due to the second equation in (1-1). Moreover, we have that if $m_k \to m$ strongly in $L^{\alpha+1}(\mathbb{R}^N)$, then, due to the Lebesgue dominated convergence theorem and (3-4), $F(m_k \star \chi_k) \to F(m)$ strongly in $L^1(\mathbb{R}^N)$. This implies that the energy $\mathcal{E}_k$ Γ-converges to the energy $\mathcal{E}$, from which we conclude that $(m_\varepsilon, w_\varepsilon)$ is a minimizer of $\mathcal{E}$ in the set $\mathcal{K}_{\varepsilon,M}$. ☐

Remark 4.2. Note that by the very same arguments, recalling Remark 3.5, we have the existence of solutions also in the more general case that condition (1-9) is satisfied.

We conclude proving some estimates on the solution $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ given in Theorem 1.1 that will be useful in the following.

Corollary 4.3. *Let $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ be as in Theorem 1.1. There exist constants $C, C_1, C_2, K, K_1, K_2 > 0$ independent of ε such that*

$$\int_{\mathbb{R}^N} m_\varepsilon |\nabla u_\varepsilon|^\gamma\,dx + \int_{\mathbb{R}^N} m_\varepsilon^{\alpha+1}\,dx + \int_{\mathbb{R}^N} m_\varepsilon(x)V(x)\,dx \leq C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K, \tag{4-22}$$

$$-K_1 - C_1\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} \leq \lambda_\varepsilon \leq K_2 - C_2\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}. \tag{4-23}$$

Proof. We observe that, by the arguments in the proof of Theorem 1.1, $m_k \to m_\varepsilon$ and $|\nabla u_k| \to |\nabla u_\varepsilon|$ almost everywhere, and using the fact that $V(x) \geq 0$, we have that by Fatou's lemma

$$\int_{\mathbb{R}^N} m_\varepsilon(x)|\nabla u_\varepsilon|^\gamma\,dx \leq \liminf_k \int_{\mathbb{R}^N} m_k(x)|\nabla u_k|^\gamma\,dx,$$

$$\int_{\mathbb{R}^N} m_\varepsilon(x)V(x)\,dx \leq \liminf_k \int_{\mathbb{R}^N} m_k(x)V(x)\,dx,$$

$$\int_{\mathbb{R}^N} m_\varepsilon^{\alpha+1}\,dx \leq \liminf_k \int_{\mathbb{R}^N} m_k^{\alpha+1}\,dx.$$

So inequality (3-13) gives immediately (4-22).

Now we prove (4-23). Note that the estimate from below is a direct consequence of (3-16). So, it remains to show that

$$\lambda_\varepsilon \le C_2 - C_2 \varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}.$$

Recalling that formula (3-22) holds and $\int f(m)m - F(m) \le 2KM$ by (1-3), it is sufficient to show that

$$\inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m,w) \le -C_2 \varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + C_2, \tag{4-24}$$

where C_2 is a constant depending only on $N, M, C_L, \gamma, \alpha, V$. We construct a pair $(m,w) \in \mathcal{K}_{\varepsilon,M}$ as follows. First of all we consider a smooth function $\phi : [0,+\infty) \to \mathbb{R}$ which solves the ordinary differential equation

$$\begin{cases} \phi'(r) = -\phi(r)(1+\phi(r)^\alpha)^{\frac{1}{\gamma'}}, \\ \phi(0) = \frac{1}{2}. \end{cases} \tag{$*$}$$

Then, it is easy to check that $0 < \phi(r) \le \frac{1}{2}e^{-r}$. We define $m(x) = A\phi(\tau|x|)$, where A, τ are constants to be fixed, and $w(x) = \varepsilon \nabla m(x)$.

First of all we impose

$$M = \int_{\mathbb{R}^N} m(x)\,dx = \frac{A}{\tau^N}\int_{\mathbb{R}^N} \phi(|y|)\,dy = \frac{A}{\tau^N} C^{-1},$$

recalling that ϕ is exponentially decreasing. So $A = M\tau^N C$, where $C^{-1} = \int_{\mathbb{R}^N} \phi(|y|)\,dy$.

Observe also that

$$\int_{\mathbb{R}^N} m^{\alpha+1}(x)\,dx = M^{\alpha+1}\tau^{\alpha N} C^{\alpha+1}\int_{\mathbb{R}^N} \phi^{\alpha+1}(|y|)\,dy = M^{\alpha+1}\tau^{\alpha N}C^{\alpha+1}C_\alpha, \tag{4-25}$$

where $C_\alpha = \int_{\mathbb{R}^N} \phi^{\alpha+1}(|y|)\,dy$.

We check, recalling the growth condition (1-5), that the following holds:

$$\int_{\mathbb{R}^N} m(x)V(x)\,dx = MC\int_{\mathbb{R}^N} V\Big(\frac{y}{\tau}\Big)\phi(|y|)\,dy = C_1\frac{1}{\tau^b}, \tag{4-26}$$

where K is a constant depending on N, ϕ, C_0.

Moreover, we compute, recalling that ϕ solves the ODE $(*)$,

$$|w|^{\gamma'} = \left|\varepsilon\tau m\Big(1+\frac{1}{M^\alpha C^\alpha \tau^{N\alpha}} m^\alpha\Big)^{\frac{1}{\gamma'}}\right|^{\gamma'} = \varepsilon^{\gamma'}\tau^{\gamma'} m^{\gamma'}\Big(1+\frac{1}{M^\alpha C^\alpha \tau^{N\alpha}}m^\alpha\Big). \tag{4-27}$$

We consider the energy at (m,w)

$$\mathcal{E}(m,w) = \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big) + F(m) + mV(x)\,dx.$$

Observe that by (1-3),

$$F(m) \le -\frac{C_f}{\alpha+1} m^{\alpha+1} + Km.$$

Using Proposition 2.1, computation (4-27) and (4-25), we get

$$\begin{aligned}\int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)+F(m)\,dx &\le \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)\,dx-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx+KM\\ &\le C_L\int_{\mathbb{R}^N} m\frac{|w|^{\gamma'}}{m^{\gamma'}}\,dx+(C_L+K)M-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx\\ &= C_L\varepsilon^{\gamma'}\tau^{\gamma'}\Big(M+\int_{\mathbb{R}^N}\frac{1}{M^\alpha C^\alpha\tau^{N\alpha}}m^{\alpha+1}\,dx\Big)+(C_L+K)M-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\\ &= C_L\varepsilon^{\gamma'}\tau^{\gamma'}M+(C_L+K)M-\Big(\frac{C_f}{\alpha+1}-\frac{\varepsilon^{\gamma'}\tau^{\gamma'-N\alpha}}{M^\alpha C^\alpha}\Big)\int_{\mathbb{R}^N} m^{\alpha+1}\,dx\\ &= (MC_L+MCC_\alpha)\varepsilon^{\gamma'}\tau^{\gamma'}-\frac{C_f}{\alpha+1}M^{\alpha+1}C^{\alpha+1}C_\alpha\tau^{\alpha N}+(C_L+K)M.\end{aligned}$$

We choose now τ such that $\tau=\frac{1}{a}\varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}}$, where a is sufficiently large, in such a way that

$$\int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)\,dx+F(m)\,dx\le -C\,\varepsilon^{-\frac{\gamma' N\alpha}{\gamma'-N\alpha}}+C,$$

where C is a constant depending on α, C_L, M. Substituting this in the energy and recalling (4-26), we get the desired inequality. □

5. Concentration phenomena

In the second part of this work, we are interested in the asymptotic analysis of solutions to (1-1) when $\varepsilon\to 0$.

5A. ***The rescaled problem.*** We consider the rescaling

$$\begin{cases}\tilde m(y):=\varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}}m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y),\\ \tilde u(y):=\varepsilon^{\frac{N\alpha(\gamma'-1)-\gamma'}{\gamma'-\alpha N}}u(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y),\\ \tilde\lambda:=\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}\lambda.\end{cases}\tag{5-1}$$

We introduce the rescaled potential

$$V_\varepsilon(y)=\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y).\tag{5-2}$$

Note that by (1-5), we get

$$C_V^{-1}\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(\max\{|\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y|-C_V,0\})^b\le V_\varepsilon(y)\le C_V\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(1+\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y|)^b.\tag{5-3}$$

The rescaled coupling term is given by

$$f_\varepsilon(\tilde m(y))=\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}f(\varepsilon^{-\frac{N\gamma'}{\gamma'-\alpha N}}m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y)).\tag{5-4}$$

Note that, using (1-3), we obtain

$$-C_fm^\alpha-K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}\le f_\varepsilon(m)\le -C_fm^\alpha+K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}.\tag{5-5}$$

Then we get that

$$\lim_{\varepsilon\to 0} f_\varepsilon(m) = -C_f m^\alpha \quad \text{uniformly in } [0,+\infty). \tag{5-6}$$

Moreover, we define $F_\varepsilon(m) = \int_0^m f_\varepsilon(n)\,dn$ if $m \ge 0$ and $F_\varepsilon(m) = 0$ otherwise, and we get

$$-\frac{C_f}{\alpha+1} m^{\alpha+1} - K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} m \le F_\varepsilon(m) \le -\frac{C_f}{\alpha+1} m^{\alpha+1} + K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} m. \tag{5-7}$$

We define also the rescaled Hamiltonian

$$H_\varepsilon(p) = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} H\big(\varepsilon^{-\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}} p\big). \tag{5-8}$$

By (1-2),

$$\begin{gathered} C_H|p|^\gamma - \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} K \le H_\varepsilon(p) \le C_H|p|^\gamma, \\ |\nabla H_\varepsilon(p)| \le K|p|^{\gamma-1}. \end{gathered} \tag{5-9}$$

So, we get

$$\lim_{\varepsilon\to 0} H_\varepsilon(p) = H_0(p) := C_H|p|^\gamma \quad \text{uniformly in } \mathbb{R}^N. \tag{5-10}$$

Moreover, if we assume that ∇H_ε is locally bounded in $C^{0,\gamma-1}(\mathbb{R}^N)$, then

$$\nabla H_\varepsilon(p) \to \nabla H_0(p) = \frac{C_H}{\gamma}|p|^{\gamma-2}p \quad \text{locally uniformly.}$$

We can define L_ε as in (1-7), with H_ε in place of H and we obtain that condition (5-9) gives that there exists $C_L > 0$ such that

$$C_L|q|^{\gamma'} \le L_\varepsilon(q) \le C_L|q|^{\gamma'} + \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} C_L, \tag{5-11}$$

which in turns gives that

$$L_\varepsilon(q) \to L_0(q) = C_L|q|^{\gamma'} \quad \text{uniformly in } \mathbb{R}^N. \tag{5-12}$$

The rescalings (5-13) lead to the rescaled system

$$\begin{cases} -\Delta\tilde u_\varepsilon + H_\varepsilon(\nabla\tilde u_\varepsilon) + \tilde\lambda_\varepsilon = f_\varepsilon(\tilde m_\varepsilon) + V_\varepsilon(y), \\ -\Delta\tilde m_\varepsilon - \operatorname{div}(\tilde m_\varepsilon \nabla H_\varepsilon(\nabla\tilde u_\varepsilon)) = 0, \\ \int_{\mathbb{R}^N} \tilde m_\varepsilon = M. \end{cases} \tag{5-13}$$

Existence of a triple $(\tilde u_\varepsilon, \tilde m_\varepsilon, \tilde\lambda_\varepsilon)$ solving the previous system is an immediate consequence of Theorem 1.1. We first start by stating some a priori estimates.

Lemma 5.1. *There exist $C, C_1, C_2 > 0$ independent of ε such that the following hold*:

$$-C_1 \le \tilde\lambda_\varepsilon \le -C_2, \tag{5-14}$$

$$\int_{\mathbb{R}^N} \tilde m_\varepsilon |\nabla\tilde u_\varepsilon|^\gamma\,dy + \int_{\mathbb{R}^N} \tilde m_\varepsilon(y) V_\varepsilon(y)\,dy + \|\tilde m_\varepsilon\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^N)} \le C, \tag{5-15}$$

$$\|\tilde m_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C. \tag{5-16}$$

Proof. Estimates (4-23), (4-22) give (5-14), (5-15) by rescaling.

We apply Theorem 4.1 with $g[m](x) = f_\varepsilon(m(x))$, $r_k = \varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}$, $s_k = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}$ and $t_k = \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}$, which are all bounded sequences, and we obtain (5-16). □

Using the a priori bounds on the solutions to (5-13), we want to pass to the limit $\varepsilon \to 0$. The problem is that these estimates are not sufficient to ensure that there is no loss of mass, namely that the limit of $\tilde{m}_\varepsilon$ still has L^1 norm equal to M. Therefore, we need to translate the reference system at a point around which the mass of $\tilde{m}_\varepsilon$ remains positive. This will be done as follows.

Let $y_\varepsilon \in \mathbb{R}^N$ be such that

$$\tilde{u}_\varepsilon(y_\varepsilon) = \min_{\mathbb{R}^N} \tilde{u}_\varepsilon(y), \tag{5-17}$$

note that this point exists due to (4-21).

We will define

$$\begin{aligned} \bar{u}_\varepsilon(y) &= \tilde{u}_\varepsilon(y + y_\varepsilon) - \tilde{u}_\varepsilon(y_\varepsilon), \\ \bar{m}_\varepsilon(y) &= \tilde{m}_\varepsilon(y + y_\varepsilon). \end{aligned} \tag{5-18}$$

Note that $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ is a classical solution to

$$\begin{cases} -\Delta \bar{u}_\varepsilon + H_\varepsilon(\nabla \bar{u}_\varepsilon) + \tilde{\lambda}_\varepsilon = f_\varepsilon(\bar{m}_\varepsilon) + V_\varepsilon(y + y_\varepsilon), \\ -\Delta \bar{m}_\varepsilon - \operatorname{div}(\bar{m}_\varepsilon \nabla H_\varepsilon(\nabla \bar{u}_\varepsilon)) = 0, \\ \int_{\mathbb{R}^N} \bar{m}_\varepsilon = M, \end{cases} \tag{5-19}$$

and in addition $\bar{u}_\varepsilon(0) = 0 = \min_{\mathbb{R}^N} \bar{u}_\varepsilon$.

5B. ***A preliminary convergence result.*** In this section, we provide some preliminary convergence results, where we are not preventing possible loss of mass in the limit. First of all we need some a priori estimates on the solutions to (5-19).

Proposition 5.2. *Let $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ be as in* (5-18). *Then there exists a constant $C > 0$ independent of ε such that the following hold*:

$$\varepsilon^{\frac{(N\alpha+b)\gamma'}{\gamma'-N\alpha}} |y_\varepsilon|^b \le C \quad \textit{and} \quad 0 \le V_\varepsilon(y + y_\varepsilon) \le C(\varepsilon^{\frac{(N\alpha+b)\gamma'}{\gamma'-N\alpha}} |y|^b + 1), \tag{5-20}$$

$$|\nabla \bar{u}_\varepsilon(y)| \le C(1 + |y|)^{\frac{b}{\gamma}} \quad \textit{and} \quad \bar{u}_\varepsilon(y) \ge C|y|^{1+\frac{b}{\gamma}} - C^{-1}, \tag{5-21}$$

$$\int_{B_R(0)} \bar{m}_\varepsilon(y)\, dy \ge C \quad \textit{for all } R \ge 1. \tag{5-22}$$

Finally, if $\bar{w}_\varepsilon = -\bar{m}_\varepsilon \nabla H_\varepsilon(\nabla \bar{u}_\varepsilon)$, then $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ is a minimizer in the set $\mathcal{K}_{1,M}$ of the energy

$$\mathcal{E}_\varepsilon(m, w) = \int_{\mathbb{R}^N} m L_\varepsilon\left(-\frac{w}{m}\right) + V_\varepsilon(y + y_\varepsilon) m + F_\varepsilon(m)\, dy, \tag{5-23}$$

where L_ε and F_ε are defined in Section 5A.

Proof. Since $\bar{u}_\varepsilon$ is a classical solution, we can compute the equation in $y = 0$, obtaining

$$H_\varepsilon(0) + \tilde{\lambda}_\varepsilon \ge f_\varepsilon(\bar{m}_\varepsilon(0)) + V(y_\varepsilon).$$

Using the a priori estimates (5-14), (5-16), (5-9) and the assumptions (5-5), (5-3), this implies

$$\varepsilon^{\frac{(N\alpha+b)\gamma'}{\gamma'-N\alpha}}|y_\varepsilon|^b \le C,$$

and then, again by assumption (5-3), that (5-20) holds.

Using estimates (5-14), (5-16), and (5-20), we conclude by Theorem 2.5 that estimate (5-21) holds.

Again by the equation computed at $y=0$, recalling that $H_\varepsilon(0)\to 0$ and $V_\varepsilon \ge 0$ and estimate (5-14), we deduce that $-f_\varepsilon(\bar m_\varepsilon(0)) \ge -C_2 > 0$. So, by assumption (5-5), we get that there exists $C>0$ independent of ε, such that $\bar m_\varepsilon(0) > C > 0$. Using the estimates (5-21) and (5-16), by Proposition 2.4, we get that there exists a positive constant depending on p such that $\|\bar m_\varepsilon\|_{W^{1,p}(B_2(0))} \le C_p$ for all $p>1$. This, by Sobolev embeddings, gives that $\|\bar m_\varepsilon\|_{C^{0,\alpha}(B_2(0))} \le C_\alpha$ for every $\alpha \in (0,1)$ and for some positive constant depending on α. We choose now $R_0 \in (0,1]$ such that $\bar m_\varepsilon \ge \frac{1}{2}C$ in $B_{R_0}(0)$, using the C^α estimate and the fact that $\bar m_\varepsilon(0) > C > 0$. This implies immediately that $\int_{B_{R_0}(0)} \bar m_\varepsilon(y)\,dy \ge \frac{1}{2}C|B_{R_0}| > 0$. This gives the estimate (5-22), for all radii bigger than R_0.

Finally that $(\bar m_\varepsilon, \bar w_\varepsilon)$ is a minimizer of (5-23) in $\mathcal{K}_{1,M}$ follows from Theorem 1.1, by rescaling. □

We get the first convergence result.

Proposition 5.3. *Let* $(\bar u_\varepsilon, \bar m_\varepsilon, \tilde\lambda_\varepsilon)$ *be the classical solution to* (5-19) *constructed above. Up to subsequences, we get that* $\tilde\lambda_\varepsilon \to \bar\lambda$, *and*

$$\bar u_\varepsilon \to \bar u, \quad \bar m_\varepsilon \to \bar m, \quad \nabla \bar u_\varepsilon \to \nabla \bar u, \quad \nabla H_\varepsilon(\nabla \bar u_\varepsilon) \to \nabla H_0(\nabla \bar u) \tag{5-24}$$

locally uniformly, where $\bar u \ge 0 = \bar u(0)$, *and* $(\bar u, \bar m, \bar\lambda)$ *is a classical solution to*

$$\begin{cases} -\Delta \bar u + H_0(\nabla \bar u) + \bar\lambda = -C_f \bar m^\alpha + g(x), \\ -\Delta \bar m - \operatorname{div}(\bar m \nabla H_0(\nabla \bar u)) = 0 \end{cases} \tag{5-25}$$

for a continuous function g *such that* $0 \le g(x) \le C$ *on* $\mathbb{R}^N$ *for some* $C>0$.

Moreover, there exist $a \in (0, M]$, $C, K, \kappa > 0$ *such that* $\int_{\mathbb{R}^N} \bar m\,dx = a$, *and*

$$\bar u(x) \ge C|x| - C, \qquad |\nabla \bar u| \le K \quad \text{on } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} e^{\kappa|x|}\bar m(x)\,dx < +\infty. \tag{5-26}$$

Proof. First of all observe that, since V is a locally Hölder continuous function, (5-20) implies that, up to a subsequence, $V_\varepsilon(x+y_\varepsilon) \to g(x)$ locally uniformly as $\varepsilon \to 0$, where g is a continuous function such that $0 \le g(x) \le C$ for some $C>0$.

Using the a priori estimate (5-21), and recalling that $\bar u_\varepsilon$ is a classical solution to (5-19), by classical elliptic regularity theory we obtain that $\bar u_\varepsilon$ is locally bounded in $C^{1,\alpha}$ in every compact set, uniformly with respect to ε. So, up to extracting a subsequence via a diagonalization procedure, we get that

$$\bar u_\varepsilon \to \bar u, \quad \nabla \bar u_\varepsilon \to \nabla \bar u, \quad \nabla H_\varepsilon(\nabla \bar u_\varepsilon) \to \nabla H_0(\nabla \bar u)$$

locally uniformly, and $\tilde\lambda_\varepsilon \to \bar\lambda$. Using the estimates (5-21) and (5-16), by Proposition 2.4, and by Sobolev embeddings, for every compact set $K \subset \mathbb{R}^N$, we have that $\|\bar m_\varepsilon\|_{C^{0,\alpha}(K)} \le C_{K,\alpha}$ for every $\alpha \in (0,1)$ and for some positive constant depending on α and K. So, up to extracting a subsequence via a diagonalization procedure, we get that $\bar{m}_\varepsilon \to \bar{m}$ locally uniformly.

So, we can pass to the limit in (5-19) and obtain that $(\bar{u}, \bar{m}, \bar{\lambda})$ is a solution to (5-25), which is classical by elliptic regularity theory.

Using (5-22) and locally uniform convergence, we get that there exists $a \in (0, M]$ such that $\int_{\mathbb{R}^N} \bar{m}\, dy = a$.

Observe that $\bar{u}$ is a solution to

$$-\Delta \bar{u} + H_0(\nabla \bar{u}) + \bar{\lambda} = -C_f \bar{m}^\alpha + g(x).$$

By Theorem 2.5, we get that there exists a constant K depending on $\sup g$ and $-\bar{\lambda}$ such that $|\nabla \bar{u}| \leq K$. Moreover, by construction $\bar{u} \geq 0$.

Since $\bar{m}$ is Hölder continuous and such that $\int_{\mathbb{R}^N} \bar{m}\, dx = a \in (0, M]$, by Lemma 2.2, we get that $\bar{m} \to 0$ as $|x| \to +\infty$. Therefore, we get that $\liminf_{|x| \to +\infty} (-\bar{m}^\alpha(x) + g(x) - \bar{\lambda} - H_0(0)) \geq -\lambda > 0$. So, by Theorem 2.6, recalling that by construction $\bar{u}(0) = 0 \leq \bar{u}(y)$, we get that $\bar{u}$ satisfies

$$\bar{u}(x) \geq C|x| - C \tag{5-27}$$

for some $C > 0$.

To conclude, consider the function $\Phi(x) = e^{\kappa \bar{u}(x)}$. We claim that we can choose $\kappa > 0$ such that there exist $R > 0$ and $\delta > 0$ with

$$-\Delta \Phi + \nabla H_0(\nabla \bar{u}) \cdot \nabla \Phi > \delta \Phi, \quad |x| > R. \tag{5-28}$$

Indeed, since $\bar{u}$ solves the first equation in (5-25), we get

$$-\Delta \Phi + \nabla H_0(\nabla \bar{u}) \cdot \nabla \Phi \geq \kappa(-\bar{\lambda} - \kappa |\nabla \bar{u}|^2 - \bar{m}^\alpha)\Phi.$$

Using (5-27) and $\bar{m} \to 0$ as $|x| \to +\infty$, we obtain the claim. Reasoning as in [Ichihara 2015, Proposition 4.3], or [Metafune et al. 2005, Proposition 2.6], we get that $\int_{\mathbb{R}^N} e^{\kappa \bar{u}} \bar{m}\, dx < +\infty$, which concludes the estimate (5-26). □

Remark 5.4. With estimates (5-26) in force, the pointwise bounds stated in [Metafune et al. 2005, Theorem 6.1] hold; namely there exist positive constants c_1, c_2, such that

$$\bar{m}(x) \leq c_1 e^{-c_2 |x|} \quad \text{on } \mathbb{R}^N.$$

5C. ***Concentration-compactness.*** In this section we show that actually there is no loss of mass when passing to the limit as in Proposition 5.3. In order to do so, we apply a kind of concentration-compactness argument.

First of all we show that the functional $\mathcal{E}_\varepsilon(m, w)$ enjoys the following subadditivity property. Let us set

$$\tilde{e}_\varepsilon(M) = \min_{(m,w) \in \mathcal{K}_M} \mathcal{E}_\varepsilon(m, w).$$

Recalling (3-6), (4-24), and the rescaling (5-1), for every $M > 0$ there exist $C_1(M), C_2(M), K_1, K_2 > 0$ depending on M (and on the other constants of the problem) but not on ε such that there holds

$$-C_1(M) - K_1 \varepsilon^{\frac{N\alpha\gamma'}{\gamma' - N\alpha}} \leq \tilde{e}_\varepsilon(M) \leq -C_2(M) - K_2 \varepsilon^{\frac{N\alpha\gamma'}{\gamma' - N\alpha}}. \tag{5-29}$$

Lemma 5.5. *For all $a \in (0, M)$, there exist $\varepsilon_0 = \varepsilon_0(a)$ and a constant $C = C(a, M) \geq 0$, depending only on a, M and the data (not on ε), such that $C(M, M) = 0 = C(0, M)$, $C(a, M) > 0$ for $0 < a < M$ and*

$$\tilde{e}_\varepsilon(M) \leq \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - C(a, M) \quad \textit{for all } \varepsilon \leq \varepsilon_0. \tag{5-30}$$

Proof. We assume that $a \geq \frac{1}{2}M$ (otherwise it suffices to replace a with $M - a$).

Let $c > 1$ and $B > 0$. For all admissible pairs $(m, w) \in \mathcal{K}_B$ we have, recalling (5-7),

$$\begin{aligned}\tilde{e}_\varepsilon(cB) \leq \mathcal{E}_\varepsilon(cm, cw) &= \int_{\mathbb{R}^N} cmL_\varepsilon\left(-\frac{w}{m}\right) + F_\varepsilon(cm) + cV_\varepsilon(x + y_\varepsilon)m\,dx \\ &= c\mathcal{E}_\varepsilon(m, w) + \int_{\mathbb{R}^N} F_\varepsilon(cm) - cF_\varepsilon(m)\,dx \\ &\leq c\mathcal{E}_\varepsilon(m, w) - \frac{c(c^\alpha - 1)C_f}{\alpha + 1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx + 2KcB\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.\end{aligned} \tag{5-31}$$

Let now (m_n, w_n) be a minimizing sequence of $\mathcal{E}_\varepsilon$ in $\mathcal{K}_B$ such that $\mathcal{E}_\varepsilon(m_n, w_n) \leq \tilde{e}_\varepsilon(B) + \frac{1}{4}C_2(B)$, where $C_2(B)$ is the constant appearing in (5-29), which depends on B and on the data of the problem. Recalling that $V_\varepsilon \geq 0$ and $L_\varepsilon \geq 0$, and using estimate (5-7), we get

$$\tilde{e}_\varepsilon(M) + \tfrac{1}{4}C_2(B) \geq \mathcal{E}_\varepsilon(m_n, w_n) \geq \int_{\mathbb{R}^N} F_\varepsilon(m_n)\,dx \geq -\frac{C_f}{\alpha + 1}\int_{\mathbb{R}^n} m^{\alpha+1}\,dx - KB\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

Using (5-29), we get, for all ε sufficiently small,

$$\frac{C_f}{\alpha + 1}\int_{\mathbb{R}^N} m_n^{\alpha+1}\,dx \geq \tfrac{3}{4}C_2(B) - K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} > \tfrac{1}{2}C_2(B) > 0.$$

So, this estimate in particular holds for a minimizer of $\mathcal{E}_\varepsilon$. Therefore in (5-31) we get, taking (m, w) to be a minimizer of $\mathcal{E}_\varepsilon$ (which exists by Proposition 5.2),

$$\tilde{e}_\varepsilon(cB) < c\tilde{e}_\varepsilon(B) - c(c^\alpha - 1)\tfrac{1}{2}C_2(B) + 2KcB\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}. \tag{5-32}$$

Using (5-32) with $B = a$ and $c = \frac{M}{a}$ we get

$$\tilde{e}_\varepsilon(M) < \frac{M}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

If $a = \frac{1}{2}M$, this permits us to conclude, choosing ε sufficiently small (depending on a). If $a > \frac{1}{2}M$, we use (5-32) with $B = M - a$ and $c = \frac{a}{M-a}$ to get (multiplying everything by $\frac{M-a}{a}$)

$$\begin{aligned}\frac{M-a}{a}\tilde{e}_\varepsilon(a) &< \tilde{e}_\varepsilon(M-a) - \left[\left(\frac{a}{M-a}\right)^\alpha - 1\right]\frac{C_2(M-a)}{2} + 2K(M-a)\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &< \tilde{e}_\varepsilon(M-a) - \left[\left(\frac{a}{M-a}\right)^\alpha - 1\right]\frac{C_2(M-a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \leq \tilde{e}_\varepsilon(M-a) + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.\end{aligned}$$

So putting together the last two inequalities we get

$$\begin{aligned}\tilde{e}_\varepsilon(M) &< \frac{M}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &= \tilde{e}_\varepsilon(a) + \frac{M-a}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &< \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 4KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &\le \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{4}\end{aligned}$$

for ε sufficiently small (depending on a). ☐

Theorem 5.6. *Let $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ be the minimizer of $\mathcal{E}_\varepsilon$ as in Proposition 5.2. Let $\bar{u}, \bar{m}$ as in Proposition 5.3, so that $\bar{m}_\varepsilon \to \bar{m}$, $\bar{w}_\varepsilon \to \bar{w} = -\bar{m}\nabla H_0(\nabla \bar{u})$ locally uniformly, and $\bar{m}$ satisfies the exponential decay* (5-26). *Then,*

$$\int_{\mathbb{R}^N} \bar{m}\,dx = M. \tag{5-33}$$

Proof. Assume by contradiction that $\int_{\mathbb{R}^N} \bar{m}\,dx = a$, with $0 < a < M$. We fix $\varepsilon_0(a)$ as in Lemma 5.5, and we consider from now on $\varepsilon \le \varepsilon_0(a)$. Let $\bar{c} > 0$ be such that $\bar{m} \le \bar{c}e^{-|x|}$ (such $\bar{c}$ exists by Remark 5.4).

For R sufficiently large (to be chosen later), we define

$$\nu_R(x) = \begin{cases} \bar{c}e^{-R}, & |x| \le R, \\ \bar{c}e^{-|x|}, & |x| > R. \end{cases} \tag{5-34}$$

So in particular $\bar{m}(x) \le \nu_R(x)$ for $|x| > R$.

We observe that as $R \to +\infty$

$$\int_{\mathbb{R}^n} \nu_R(x)\,dx = \bar{c}\omega_N e^{-R}R^N + \int_{\mathbb{R}^N\setminus B_R} \bar{c}e^{-|x|}\,dx \le Ce^{-R}R^N \to 0. \tag{5-35}$$

Since $\bar{m}_\varepsilon \to \bar{m}$ and $\nabla H_\varepsilon(\nabla \bar{u}_\varepsilon) \to \nabla H_0(\nabla \bar{u})$ locally uniformly, there exists $\varepsilon_0 = \varepsilon_0(R)$ such that for all $\varepsilon \le \varepsilon_0$,

$$|\bar{m}_\varepsilon - \bar{m}| + |\nabla H_\varepsilon(\nabla \bar{u}_\varepsilon) - \nabla H_0(\nabla \bar{u})| \le \bar{c}e^{-R}, \quad |x| \le R. \tag{5-36}$$

We observe that for all $\varepsilon \le \varepsilon_0$,

$$\bar{m}_\varepsilon - \bar{m} + 2\nu_R \ge \nu_R(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{5-37}$$

Indeed, if $|x| > R$, then $\bar{m}_\varepsilon - \bar{m} + 2\nu_R \ge \bar{m}_\varepsilon + \nu_R \ge \nu_R$, since $\bar{m} \le \nu_R$. On the other hand, if $|x| \le R$, then by (5-36) $\bar{m}_\varepsilon - \bar{m} + 2\nu_R \ge -\bar{c}e^{-R} + 2\bar{c}e^{-R} = \bar{c}e^{-R} = \nu_R$. From (5-37) we deduce that

$$|\bar{m}_\varepsilon - \bar{m}| \le \bar{m}_\varepsilon - \bar{m} + 2\nu_R. \tag{5-38}$$

Moreover, since $\bar{m}_\varepsilon \to \bar{m}$ a.e. by Theorem 2.3, recalling that $\int_{\mathbb{R}^N} \bar{m}_\varepsilon\,dx = M$, $\int_{\mathbb{R}^n} \bar{m} = a$ and using (5-35) and (5-38), we have

$$\int_{\mathbb{R}^N} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx = M - a + 2\int_{\mathbb{R}^N} \nu_R\,dx \to M - a \quad \text{as } R \to +\infty, \tag{5-39}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \bar{m}_\varepsilon^{\alpha+1}\,dx = \int_{\mathbb{R}^N} \bar{m}^{\alpha+1}\,dx + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} |\bar{m}_\varepsilon - \bar{m}|^{\alpha+1}\,dx$$
$$\le \int_{\mathbb{R}^N} \bar{m}^{\alpha+1}\,dx + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R)^{\alpha+1}\,dx. \tag{5-40}$$

We claim that

$$\mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \ge \mathcal{E}_\varepsilon(\bar{m}, \bar{w}) + \mathcal{E}_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R, \bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R) + o_\varepsilon(1) + o_R(1), \tag{5-41}$$

where $o_\varepsilon(1)$ is an error such that $\lim_{\varepsilon\to 0} o_\varepsilon(1) = 0$.

We consider the function $(m, w) \mapsto mL_\varepsilon\big(-\frac{w}{m}\big)$. This is a convex function in (m, w). We compute $\nabla_w\big(mL_\varepsilon\big(-\frac{w}{m}\big)\big) = -\nabla L_\varepsilon\big(-\frac{w}{m}\big)$, so in particular by (5-11) we get

$$C_L\left|\frac{w}{m}\right|^{\gamma'-1} - C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}} \le \left|\nabla_w\left(mL_\varepsilon\left(-\frac{w}{m}\right)\right)\right| \le C_L^{-1}\left|\frac{w}{m}\right|^{\gamma'-1} + C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}. \tag{5-42}$$

Moreover, $\partial_m\big(mL_\varepsilon\big(-\frac{w}{m}\big)\big) = L_\varepsilon\big(-\frac{w}{m}\big) + \frac{w}{m}\cdot\nabla L_\varepsilon\big(-\frac{w}{m}\big)$, therefore, again by (5-11) we get

$$C_L\left|\frac{w}{m}\right|^{\gamma'} - C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}} \le \left|\partial_m\left(mL_\varepsilon\left(-\frac{w}{m}\right)\right)\right| \le C_L^{-1}\left|\frac{w}{m}\right|^{\gamma'} + C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}. \tag{5-43}$$

Note that

$$\int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)\bar{m}_\varepsilon\,dx$$
$$= \int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)\bar{m}\,dx + \int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)(\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx - 2\int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)\nu_R\,dx.$$

Recalling the estimate (5-20) and the definition of ν_R, we have

$$2\int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)\nu_R\,dx \le CR^{b+N}e^{-R}.$$

Hence we obtain

$$\int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)\bar{m}_\varepsilon\,dx$$
$$\ge \int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)\bar{m}\,dx + \int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)(\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx - CR^{b+N}e^{-R}. \tag{5-44}$$

By (5-40) and (5-7) we get

$$\int_{\mathbb{R}^N} F_\varepsilon(\bar{m}_\varepsilon)\,dx \ge -\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} \bar{m}_\varepsilon^{\alpha+1}\,dx - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}$$
$$\ge -\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} \bar{m}^{\alpha+1}\,dx - \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R)^{\alpha+1}\,dx + o_\varepsilon(1)$$
$$\ge \int_{\mathbb{R}^N} F_\varepsilon(\bar{m})\,dx + \int_{\mathbb{R}^N} F_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx + o_\varepsilon(1). \tag{5-45}$$

Finally, we estimate the kinetic terms in the energy. Splitting

$$\int_{\mathbb{R}^N} \bar m_\varepsilon L_\varepsilon\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx = \int_{B_R} \bar m_\varepsilon L_\varepsilon\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx + \int_{\mathbb{R}^N\setminus B_R} \bar m_\varepsilon L\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx,$$

we proceed by estimating separately the two terms.

Estimates in $\mathbb{R}^N \setminus B_R$. First of all, note that by (5-26), (5-9) and (5-11), we get that $L_\varepsilon\big(-\frac{\bar w}{\bar m}\big) = L_\varepsilon(\nabla H_0(\nabla \bar u)) \le C$ for come constant $C > 0$, just depending on the data. Moreover, recalling that $\bar m \le \bar c e^{-|x|}$, we get that, eventually enlarging C,

$$\int_{\mathbb{R}^N\setminus B_R} \bar m L_\varepsilon\Big(-\frac{\bar w}{\bar m}\Big)\,dx \le C\int_{|x|>R} e^{-|x|}\,dx \le CR^N e^{-R}. \tag{5-46}$$

By the convexity of the function $(m,w)\mapsto mL\big(-\frac{w}{m}\big)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R} \bar m_\varepsilon L\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx\\
&\quad\ge \int_{\mathbb{R}^N\setminus B_R} (\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\Big(-\frac{\bar w_\varepsilon - \bar w + 2\nabla \nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\Big)\,dx\\
&\qquad+\int_{\mathbb{R}^N\setminus B_R} \partial_m\bigg((\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\Big(-\frac{\bar w_\varepsilon - \bar w + 2\nabla \nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\Big)\bigg)(\bar m - 2\nu_R)\,dx \qquad (5\text{-}47)\\
&\qquad+\int_{\mathbb{R}^N\setminus B_R} \nabla_w\bigg[(\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\Big(-\frac{\bar w_\varepsilon - \bar w + 2\nabla \nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\Big)\bigg]\cdot(\bar w - 2\nabla\nu_R)\,dx. \qquad (5\text{-}48)
\end{aligned}$$

We recall that $|\bar w| = \bar m|\nabla H_0(\nabla \bar u)| \le C\bar m$ by (5-26) and $|\nabla \nu_R| \le C\nu_R$ by definition. Moreover, by (5-21) and (5-9),

$$|\bar w_\varepsilon| = \bar m_\varepsilon|\nabla H_\varepsilon(\nabla \bar u_\varepsilon)| \le C\bar m_\varepsilon[(1+|x|)^{\frac{b}{\gamma}}]^{\gamma-1} \le C_1\bar m_\varepsilon(1+|x|)^{\frac{b}{\gamma'}}.$$

Using the triangle inequality we get the following, where the constant C can change from line to line:

$$\begin{aligned}
\bigg|\frac{\bar w_\varepsilon - \bar w + 2\nabla \nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\bigg| &\le \frac{\bar m_\varepsilon|\nabla H_\varepsilon(\nabla \bar u_\varepsilon)|}{\bar m_\varepsilon - \bar m + 2\nu_R} + \frac{\bar m|\nabla H_0(\nabla \bar u)|}{\bar m_\varepsilon - \bar m + 2\nu_R} + \frac{C\nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\\
&\le \frac{C\bar m_\varepsilon(1+|x|)^{\frac{b}{\gamma'}}}{\bar m_\varepsilon - \bar m + 2\nu_R} + \frac{C\bar m}{\bar m_\varepsilon - \bar m + 2\nu_R} + \frac{C\nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R} \le C(1+|x|)^{\frac{b}{\gamma'}}
\end{aligned} \tag{5-49}$$

on $\mathbb{R}^N \setminus B_R(0)$, where we used respectively the fact that $\bar m_\varepsilon - \bar m + 2\nu_R \ge \bar m_\varepsilon$, $\bar m \le \nu_R$, and that $\bar m_\varepsilon - \bar m + 2\nu_R \ge \nu_R$.

Now, using (5-43) and (5-49), we can estimate (5-47), and by (5-42) and (5-49) we can estimate (5-48). Indeed, we get

$$\int_{\mathbb{R}^N\setminus B_R}\bigg|\partial_m\bigg((\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\Big(-\frac{\bar w_\varepsilon - \bar w + 2\nabla \nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\Big)\bigg)\bigg|\,|\bar m - 2\nu_R|\,dx \le C\int_{\mathbb{R}^N\setminus B_R}(1+|x|)^b\nu_R(x)\,dx$$

and

$$\int_{\mathbb{R}^N\setminus B_R}\bigg|\nabla_w\bigg[(\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\Big(-\frac{\bar w_\varepsilon - \bar w + 2\nabla \nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\Big)\bigg]\bigg|(|\bar w| + 2|\nabla\nu_R|)\,dx \le C\int_{\mathbb{R}^N\setminus B_R}(1+|x|)^{\frac{b}{\gamma}}\nu_R(x)\,dx,$$

because $\bar{w} \le C\bar{m}$ on $\mathbb{R}^N$. Therefore, we may conclude, possibly enlarging C, that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} \bar{m}_\varepsilon L\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx &\ge \int_{\mathbb{R}^N\setminus B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx - C\int_{\mathbb{R}^N\setminus B_R} (1+|x|)^b \nu_R(x)\,dx \\ &\ge \int_{\mathbb{R}^N\setminus B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx - CR^{N+b}e^{-R}. \qquad (5\text{-}50)\end{aligned}$$

Finally, putting together (5-46) and (5-50), we have, choosing C sufficiently large,

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} \bar{m}_\varepsilon L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx \ge \int_{\mathbb{R}^N\setminus B_R} \bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\,dx \\ + \int_{\mathbb{R}^N\setminus B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx - CR^{N+b}e^{-R}. \qquad (5\text{-}51)\end{aligned}$$

Estimates in B_R. Again by the convexity of the function $(m, w) \mapsto mL\big(-\frac{w}{m}\big)$, we get

$$\begin{aligned}\int_{B_R} \bar{m}_\varepsilon L\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx \ge \int_{B_R} \bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\,dx + \int_{B_R} \partial_m\Big(\bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\Big)(\bar{m}_\varepsilon - \bar{m})\,dx \\ + \int_{B_R} \nabla_w\Big[\bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\Big]\cdot(\bar{w}_\varepsilon - \bar{w})\,dx. \qquad (5\text{-}52)\end{aligned}$$

We now estimate (5-52). We recall that

$$\left|\frac{\bar{w}}{\bar{m}}\right| \le |\nabla H_0(\nabla\bar{u})| \le K$$

and also $|\nabla H_\varepsilon(\nabla\bar{u}_\varepsilon)| \le K$ for all $\varepsilon \le \varepsilon_0(R)$. Then, using these facts and (5-42) and (5-43) and recalling (5-36), we get

$$\int_{B_R} \left|\partial_m\Big(\bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\Big)\right| |\bar{m}_\varepsilon - \bar{m}|\,dx = \int_{B_R} |\partial_m(\bar{m} L_\varepsilon(\nabla H_0(\nabla\bar{u})))|\,|\bar{m}_\varepsilon - \bar{m}|\,dx \le Ce^{-R}R^N$$

and

$$\int_{B_R} |\nabla_w[\bar{m} L_\varepsilon(\nabla H_0(\nabla\bar{u}))]|\,\big(|\nabla H_\varepsilon(\nabla u_\varepsilon)|\,|\bar{m}_\varepsilon - \bar{m}| + |\nabla H_\varepsilon(\nabla\bar{u}_\varepsilon) - \nabla H_0(\nabla\bar{u})|\bar{m}\big)\,dx \le Ce^{-R}R^N.$$

This implies that for all $\varepsilon \le \varepsilon_0(R)$

$$\int_{B_R} \bar{m}_\varepsilon L\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx \ge \int_{B_R} \bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\,dx - Ce^{-R}R^N. \qquad (5\text{-}53)$$

Now we observe that by (5-11),

$$\int_{B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx \le C\int_{B_R} \left[\left|\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\right|^{\gamma'} + 1\right](\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx.$$

By (5-38) we get that $\bar m_\varepsilon-\bar m+2\nu_R\le|\bar m_\varepsilon-\bar m|+2\nu_R\le Ce^{-R}$, eventually enlarging C. Moreover, reasoning as in (5-49), we get

$$\left|\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\right|\le|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)|\frac{|\bar m_\varepsilon-\bar m|}{\bar m_\varepsilon-\bar m+2\nu_R}+\frac{|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)-\nabla H_0(\nabla\bar u)|}{\bar m_\varepsilon-\bar m+2\nu_R}\bar m\le C,$$

where we used that $\nabla\nu_R=0$ for $|x|<R$, that $|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)|\le K$, that by (5-38)

$$\frac{|\bar m_\varepsilon-\bar m|}{\bar m_\varepsilon-\bar m+2\nu_R}\le 1,$$

and that by (5-37) and (5-36)

$$\frac{|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)-\nabla H_0(\nabla\bar u)|}{\bar m_\varepsilon-\bar m+2\nu_R}\le C.$$

So, we conclude that

$$\int_{B_R}(\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\left(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\right)dx\le Ce^{-R}R^N. \tag{5-54}$$

Putting together (5-53) and (5-54) we get, choosing C sufficiently large, for all $\varepsilon\le\varepsilon_0(R)$,

$$\begin{aligned}\int_{B_R}\bar m_\varepsilon L_\varepsilon\left(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\right)dx\ge\int_{B_R}\bar m L_\varepsilon\left(-\frac{\bar w}{\bar m}\right)dx\\ +\int_{B_R}(\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\left(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\right)dx-CR^Ne^{-R}.\end{aligned} \tag{5-55}$$

Therefore, summing up (5-55), (5-51), (5-44) and (5-45), we conclude for all $\varepsilon\le\varepsilon_0(R)$,

$$\mathcal{E}_\varepsilon(\bar m_\varepsilon,\bar w_\varepsilon)\ge\mathcal{E}_\varepsilon(\bar m,\bar w)+\mathcal{E}_\varepsilon(\bar m_\varepsilon-\bar m+2\nu_R,\bar w_\varepsilon-\bar w+2\nabla\nu_R)+o_\varepsilon(1)-CR^{b+N}e^{-R}. \tag{5-56}$$

Let now

$$c_R=\frac{M-a}{M-a+2\int_{\mathbb{R}^N}\nu_R\,dx}.$$

We have $c_R\to1$ as $R\to+\infty$ and $c_R<1$. In particular, $(c_R(\bar m_\varepsilon-\bar m+2\nu_R),c_R(\bar w_\varepsilon-\bar w+2\nabla\nu_R))\in\mathcal{K}_{M-a}$. By the same computation as in (5-31), we get

$$\begin{aligned}&c_R\mathcal{E}_\varepsilon(\bar m_\varepsilon-\bar m+2\nu_R,\bar w_\varepsilon-\bar w+2\nabla\nu_R)\\ &=\mathcal{E}_\varepsilon(c_R(\bar m_\varepsilon-\bar m+2\nu_R),c_R(\bar w_\varepsilon-\bar w+2\nabla\nu_R))+\int_{\mathbb{R}^N}c_RF_\varepsilon(\bar m_\varepsilon-\bar m+2\nu_R)-F_\varepsilon(c_R(\bar m_\varepsilon-\bar m+2\nu_R))\,dx\\ &\ge\mathcal{E}_\varepsilon(c_R(\bar m_\varepsilon-\bar m+2\nu_R),c_R(\bar w_\varepsilon-\bar w+2\nabla\nu_R))\\ &\qquad+c_R\frac{c_R^\alpha-1}{\alpha+1}C_f\int_{\mathbb{R}^N}(\bar m_\varepsilon-\bar m+2\nu_R)^{\alpha+1}\,dx-2K\left(M-a+2\int_{\mathbb{R}^N}\nu_R\,dx\right)\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.\end{aligned} \tag{5-57}$$

Observe that by (5-15) there exists C independent of ε such that

$$0\le\int_{\mathbb{R}^N}(\bar m_\varepsilon-\bar m+2\nu_R)^{\alpha+1}\,dx\le(\|\bar m_\varepsilon\|_{\alpha+1}+\|\bar m\|_{\alpha+1}+\|2\nu_R\|_{\alpha+1})^{\alpha+1}\le C.$$

Therefore, (5-57) reads (recalling that $c_R < 1$ and enlarging the constants C, K)

$$\begin{aligned} c_R\mathcal{E}_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R, \bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R) &\\ \geq \mathcal{E}_\varepsilon(c_R(\bar{m}_\varepsilon - \bar{m} + 2\nu_R), c_R(\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R)) &+ c_R\frac{c_R^\alpha - 1}{\alpha+1}C - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ \geq \tilde{e}_\varepsilon(M-a) + c_R\frac{c_R^\alpha - 1}{\alpha+1}C - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}. \end{aligned}$$

Using this inequality, and using the fact that $\mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) = \tilde{e}_\varepsilon(M)$ and that $\mathcal{E}_\varepsilon(\bar{m}, \bar{w}) \geq \tilde{e}_\varepsilon(a)$, we obtain from (5-56)

$$\begin{aligned} \tilde{e}_\varepsilon(M) \geq \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) + (1-c_R)\mathcal{E}_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R, \bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R) \\ + Cc_R\frac{c_R^\alpha - 1}{\alpha+1} - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} + o_\varepsilon(1) - CR^{b+N}e^{-R}. \end{aligned}$$

Moreover by (5-29) we get that there exists $K = K(M-a) > 0$ such that

$$\mathcal{E}_\varepsilon(\bar{m}_\varepsilon - \bar{m} + 2\nu_R, \bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R) \geq -K;$$

therefore the previous inequality gives

$$\tilde{e}_\varepsilon(M) \geq \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - (1-c_R)K + Cc_R\frac{c_R^\alpha - 1}{\alpha+1} + o_\varepsilon(1) - CR^{b+N}e^{-R}. \tag{5-58}$$

By Lemma 5.5, we get

$$\tilde{e}_\varepsilon(M) \leq \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - C(a, M),$$

where $C(a, M) > 0$ for $a < M$ and $C(M, M) = 0$. This implies in particular that

$$0 > -C(a, M) \geq -(1-c_R)K + Cc_R\frac{c_R^\alpha - 1}{\alpha+1} + o_\varepsilon(1) - CR^{b+N}e^{-R}.$$

Recalling that $c_R \to 1$ as $R \to +\infty$, this gives a contradiction, choosing R sufficiently large and $\varepsilon < \varepsilon_0(R)$. □

An immediate corollary of the previous theorem is the following convergence result.

Corollary 5.7. *Let $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ and $(\bar{u}, \bar{m}, \bar{\lambda})$ be as in Proposition 5.3. Then,*

$$\bar{m}_\varepsilon \to \bar{m} \quad \text{in } L^1(\mathbb{R}^N) \text{ and } L^{\alpha+1}(\mathbb{R}^N). \tag{5-59}$$

Finally for all $\eta > 0$, there exist $R > 0$ and ε_0 such that for all $\varepsilon \leq \varepsilon_0$,

$$\int_{B(0,R)} \bar{m}_\varepsilon \, dx \geq M - \eta. \tag{5-60}$$

Proof. By Proposition 5.3 we get that $\bar{m}_\varepsilon \to \bar{m}$ almost everywhere, and by Theorem 5.6, $\int_{\mathbb{R}^N} \bar{m}_\varepsilon = M = \int_{\mathbb{R}^N} \bar{m}$. This implies the convergence in $L^1(\mathbb{R}^N)$. Indeed, by Fatou's lemma

$$2M \leq \liminf_\varepsilon \int_{\mathbb{R}^N} \bar{m}_\varepsilon + \bar{m} - |\bar{m}_\varepsilon - \bar{m}| \, dx \leq 2M - \limsup_\varepsilon \int_{\mathbb{R}^N} |\bar{m}_\varepsilon - \bar{m}| \, dx.$$

Moreover, recalling (5-16), we get

$$\|\bar{m}_\varepsilon - \bar{m}\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^N)} \le \|\bar{m}_\varepsilon - \bar{m}\|_{L^1(\mathbb{R}^N)}(\|\bar{m}\|_{L^\infty(\mathbb{R}^N)} + \|\bar{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}) \to 0.$$

Finally observe that for all R, by Remark 5.4,

$$\int_{B_R(0)} \bar{m}_\varepsilon \, dy \ge \int_{B_R(0)} \bar{m} \, dy - \int_{B_R(0)} |\bar{m}_\varepsilon - \bar{m}| \, dy \ge M - CR^{N-1}e^{-R} - \int_{\mathbb{R}^N} |\bar{m}_\varepsilon - \bar{m}| \, dy.$$

So, using the L^1 convergence we conclude the desired estimate. □

5D. *Existence of ground states.* In this subsection we aim at proving that as ε goes to zero, $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ converges to a solution of the limiting MFG system (1-14), without potential terms. In particular, we will prove Theorem 1.3.

We first need a Γ-convergence-type result, proved in the following lemma.

Lemma 5.8. *Let $(m_\varepsilon, w_\varepsilon), (m, w) \in \mathcal{K}_{1,M}$ be such that $m_\varepsilon \to m$ in $L^1 \cap L^{\alpha+1}(\mathbb{R}^N)$ and $w_\varepsilon \rightharpoonup w$ weakly in $L^q(\mathbb{R}^N)$ for some $q > 1$. Then*

$$\liminf_\varepsilon \mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) \ge \mathcal{E}_0(m, w), \tag{5-61}$$

where $\mathcal{E}_0$ is defined in (1-16).

Let $(m, w) \in \mathcal{K}_{1,M}$ be such that $m(1 + |y|^b) \in L^1(\mathbb{R}^N)$. Then

$$\lim_\varepsilon \mathcal{E}_\varepsilon(m(\cdot - y_\varepsilon), w(\cdot - y_\varepsilon)) \le \mathcal{E}_0(m, w). \tag{5-62}$$

Proof. We recall that $L_\varepsilon(q) \to C_L|q|^{\gamma'}$ uniformly in $\mathbb{R}^N$ by (5-11) and $F_\varepsilon(m) \to -\frac{1}{\alpha+1}m^{\alpha+1}$ uniformly in $[0, +\infty)$ by (5-7). Moreover we observe that the energy $\mathcal{E}_0$ is lower semicontinuous with respect to weak L^q convergence of w and strong $L^{\alpha+1} \cap L^1$ convergence of m. Since $V \ge 0$, we get

$$\begin{aligned}
\liminf_\varepsilon \mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) &\ge \liminf_\varepsilon \int_{\mathbb{R}^N} m_\varepsilon L_\varepsilon\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) + F_\varepsilon(m_\varepsilon) \, dx \\
&\ge \liminf_\varepsilon \int_{\mathbb{R}^N} C_L m_\varepsilon^{1-\gamma'}|w_\varepsilon|^{\gamma'} - \frac{C_f}{\alpha+1} m_\varepsilon^{\alpha+1} \, dx \\
&\ge \int_{\mathbb{R}^N} C_L m^{1-\gamma'}|w|^{\gamma'} - \frac{C_f}{\alpha+1} m^{\alpha+1} \, dx = \mathcal{E}_0(m, w).
\end{aligned}$$

Now we observe that for all m such that $m(1 + |y|^b) \in L^1(\mathbb{R}^N)$, using (5-3), we get

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} m(y + y_\varepsilon) V_\varepsilon(y + y_\varepsilon) \, dy \le \lim_\varepsilon C_V \varepsilon^{\frac{N\alpha\gamma'}{\gamma' - \alpha N}} \int_{\mathbb{R}^N} (1 + |y|)^b m(y) \, dy = 0. \tag{5-63}$$

Therefore, recalling again the uniform convergence of $L_\varepsilon(q) \to C_L|q|^{\gamma'}$ and $F_\varepsilon(m) \to -\frac{1}{\alpha+1}m^{\alpha+1}$, we conclude (noting that if we translate m, w of y_ε the energy $\mathcal{E}_0$ remains the same)

$$\lim_\varepsilon \mathcal{E}_\varepsilon(m(\cdot - y_\varepsilon), w(\cdot - y_\varepsilon)) = \mathcal{E}_0(m, w) + \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} m(y + y_\varepsilon) V_\varepsilon(y + y_\varepsilon) \, dy \le \mathcal{E}_0(m, w). \quad □$$

Proof of Theorem 1.3. We first show that $(\bar{u}, \bar{m})$ obtained in Proposition 5.3 are associated to minimizers of an appropriate energy, without potential term, so that (1-15) holds.

Note that $(\bar{m}, \bar{w}) \in \mathcal{K}_{1,M}$, where $\bar{w} = -\bar{m}\nabla H_0(\nabla \bar{u})$, due to Proposition 5.3 and Theorem 5.6 and $\bar{m}(1+|y|^b) \in L^1(\mathbb{R}^N)$ by the exponential decay (5-26). Moreover $\bar{m}_\varepsilon \to \bar{m}$ in $L^1 \cap L^{\alpha+1}$ by Corollary 5.7 and

$$\bar{w}_\varepsilon = -\bar{m}_\varepsilon \nabla H_\varepsilon(\nabla \bar{u}_\varepsilon) \to \bar{w} = -\bar{m}\nabla H_0(\nabla \bar{u})$$

locally uniformly (by Proposition 5.3) and weakly in $L^{\frac{\gamma'(\alpha+1)}{\gamma'+\alpha}}$ by the same argument as in the proof of Proposition 3.3.

Let now $(m, w) \in \mathcal{K}_{1,M}$ be such that $m(1+|y|^b) \in L^1(\mathbb{R}^N)$. Using the minimality of $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$, (5-61) and (5-62), we conclude that

$$\mathcal{E}_0(m, w) \geq \lim_\varepsilon \mathcal{E}_\varepsilon(m(\cdot - y_\varepsilon), w(\cdot - y_\varepsilon)) \geq \lim_\varepsilon \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \geq \mathcal{E}_0(\bar{m}, \bar{w}).$$

This implies (1-15).

To obtain the first part of the theorem, that is, the existence of a solution to (1-14), we need to prove that the function g appearing in Proposition 5.3 is actually zero on $\mathbb{R}^N$. To do that, we derive a better estimate on the term $V_\varepsilon(y + y_\varepsilon)$; in particular we show that $V_\varepsilon(y + y_\varepsilon) \to 0$ locally uniformly in $\mathbb{R}^N$.

By the minimality of $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ and $(\bar{m}, \bar{w})$, (5-11), (5-7) and (5-63) we get

$$\begin{aligned}\mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) &\leq \mathcal{E}_\varepsilon(\bar{m}(\cdot + y_\varepsilon), \bar{w}(\cdot + y_\varepsilon)) \\ &\leq \mathcal{E}_0(\bar{m}, \bar{w}) + \int_{\mathbb{R}^N} \bar{m}(y + y_\varepsilon)V_\varepsilon(y + y_\varepsilon)\, dy + C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \leq \mathcal{E}_0(\bar{m}_\varepsilon, \bar{w}_\varepsilon) + C_1\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.\end{aligned}$$

Again using (5-7) and (5-11) we get

$$\mathcal{E}_0(\bar{m}_\varepsilon, \bar{w}_\varepsilon) + C_1\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \leq \int_{\mathbb{R}^N} \bar{m}_\varepsilon L_\varepsilon\left(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\right) + F_\varepsilon(\bar{m}_\varepsilon)\, dy + C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} M + C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

So, putting together the last two inequalities, we conclude that

$$\int_{\mathbb{R}^N} \bar{m}_\varepsilon V_\varepsilon(y + y_\varepsilon)\, dy \leq C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}. \tag{5-64}$$

Recalling (5-2), this implies that for all $R > 0$, we get

$$C_V^{-1}(\max\{\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y_\varepsilon| - \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}R - C_V, 0\})^b \int_{B(0,R)} \bar{m}_\varepsilon\, dy \leq C.$$

Using (5-60), we conclude that there exists $C > 0$ such that

$$\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y_\varepsilon| \leq C. \tag{5-65}$$

In turn this gives, recalling again (5-2), that

$$0 \leq V_\varepsilon(y + y_\varepsilon) \leq C_V\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(1 + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y| + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y_\varepsilon|)^b \leq C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(1 + |y|)^b,$$

which implies that $V_\varepsilon(y + y_\varepsilon) \to 0$ locally uniformly. ☐

Remark 5.9. If H and f satisfy the growth conditions (1-2) and (1-3), arguing as before one has that there exists a classical solution to the potential-free version of (1-1),

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(m), \\ -\Delta m - \operatorname{div}(\nabla H(\nabla u)m) = 0, \\ \int_{\mathbb{R}^N} m = M. \end{cases} \tag{5-66}$$

In addition, $(m, -\nabla H(\nabla u)m)$ is a minimizer of

$$(m, w) \mapsto \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big) + F(m)\, dx$$

among $(m, w) \in \mathcal{K}_{1,M}$, $m(1 + |y|^b) \in L^1(\mathbb{R}^N)$. This can be done as follows: Start with a sequence $(u_\delta, m_\delta, \lambda_\delta)$ solving

$$\begin{cases} -\Delta u_\delta + H(\nabla u_\delta) + \lambda_\delta = f(m_\delta) + \delta|x|^b, \\ -\Delta m_\delta - \operatorname{div}(\nabla H(\nabla u_\delta)m_\delta) = 0, \\ \int_{\mathbb{R}^N} m_\delta = M, \end{cases} \tag{5-67}$$

with $\delta = \delta_n \to 0$. Such a sequence exists by Theorem 1.1. The problem of passing to the limit in (5-67) to obtain (5-66) is the same as passing to the limit in (5-13), and it is even simpler: in (5-13), one has to be careful as the Hamiltonian H_ε and the coupling f_ε vary as $\varepsilon \to 0$ (still, they converge uniformly), while in (5-67) they are fixed, and only the potential is vanishing. We observe that $b > 0$ could be chosen arbitrarily; the perturbation $\delta|x|^b$ always disappears in the limit. Still, the limit m, u somehow retains a memory of b in terms of energy properties: m minimizes an energy among competitors satisfying $m(1 + |y|^b) \in L^1(\mathbb{R}^N)$.

Remark 5.10. We stress that uniqueness of solutions for (1-14) does not hold in general; for example, a triple (u, m, λ) solving the system may be translated in space to obtain a full family of solutions. On the other hand, a more subtle issue is the uniqueness of m in the second equation (with ∇u fixed); that is, if (u, m_1, λ) and (u, m_2, λ) are solutions, then $m_1 \equiv m_2$. This property is intimately related to the ergodic behavior of the optimal trajectory $dX_s = -\nabla H_0(\nabla u(X_s))\, ds + \sqrt{2\varepsilon}\, dB_s$; see, for example, [Cirant 2014]. It is well known that uniqueness for the Kolmogorov equation is guaranteed by the existence of a so-called Lyapunov function; in our cases, it can be checked that u itself (or increasing functions of u, as in (5-28)) acts as a Lyapunov function, so uniqueness of m and ergodicity hold for (1-14) and (1-1).

5E. ***Concentration of mass.*** The last problem we address is the localization of the point y_ε, to conclude the proof of Theorem 1.2. Rewriting (5-60) in view of (5-1) and (5-18), we get that for all $\eta > 0$ there exist R, ε_0 such that for all $\varepsilon \le \varepsilon_0$,

$$\int_{B(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon, \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} R)} m(x)\, dx \ge M - \eta, \tag{5-68}$$

where m is the classical solution to (1-1) given in Theorem 1.1, and

$$\bar{m}_\varepsilon(y) = \varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}} m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon).$$

By (5-65), we know that, up to subsequences, $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon \to \bar{x}$. Our aim is to locate this point, which is the point where mass concentrates. We need a preliminary lemma stating the existence of suitable competitors that will be used in the sequel.

Lemma 5.11. *For all $\varepsilon \le \varepsilon_0$, there exists $(\hat{m}_\varepsilon, \hat{w}_\varepsilon) \in \mathcal{K}_{1,M}$ that minimizes*

$$(m, w) \mapsto \int_{\mathbb{R}^N} m L_\varepsilon\left(-\frac{w}{m}\right) + F_\varepsilon(m)\, dy \tag{5-69}$$

among $(m, w) \in \mathcal{K}_{1,M}$, $m(1+|y|^b) \in L^1(\mathbb{R}^N)$. Moreover, for some positive constants c_1, c_2 independent of ε,

$$\hat{m}_\varepsilon(y) \le c_1 e^{-c_2|y|} \quad \text{on } \mathbb{R}^N. \tag{5-70}$$

Proof. The existence of $(\hat{m}_\varepsilon, \hat{w}_\varepsilon)$ is stated in Remark 5.9, together with a solution $(\hat{u}_\varepsilon, \hat{m}_\varepsilon, \hat{\lambda}_\varepsilon)$ to the associated MFG system, as the optimality conditions; see (5-71) below. To obtain the uniform exponential decay, we can argue by Lyapunov functions as in Proposition 5.3; here, we have to be careful, since the argument in Proposition 5.3 mainly requires

$$f_\varepsilon(\hat{m}_\varepsilon) - \hat{\lambda}_\varepsilon - H_\varepsilon(0) \ge -\tfrac{1}{2}\hat{\lambda}_\varepsilon > 0$$

outside some fixed ball $B_r(0)$. This claim can be proved as follows: First, $-\hat{\lambda}_\varepsilon$ is bounded away from zero for ε small. Indeed,

$$\hat{\lambda}_\varepsilon M = \int_{\mathbb{R}^N} \hat{m}_\varepsilon L_\varepsilon\left(-\frac{\hat{w}_\varepsilon}{\hat{m}_\varepsilon}\right) + f_\varepsilon(\hat{m}_\varepsilon)\hat{m}_\varepsilon\, dy \le \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) + o_\varepsilon(1) \le -C.$$

The inequality follows by the minimality of $(\hat{m}_\varepsilon, \hat{w}_\varepsilon)$ and $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$, and (rescaled) (4-24).

We now prove that $\hat{m}_\varepsilon$ decays as $|x| \to \infty$ uniformly in ε. Note that $\hat{w}_\varepsilon = -\nabla H_\varepsilon(\nabla \hat{u}_\varepsilon)\hat{m}_\varepsilon$, where $(\hat{u}_\varepsilon, \hat{m}_\varepsilon, \hat{\lambda}_\varepsilon)$ solves

$$\begin{cases} -\Delta \hat{u}_\varepsilon + H_\varepsilon(\nabla \hat{u}_\varepsilon) + \lambda = f_\varepsilon(\hat{m}_\varepsilon), \\ -\Delta \hat{m}_\varepsilon - \operatorname{div}(\nabla H_\varepsilon(\nabla \hat{u}_\varepsilon)\hat{m}_\varepsilon) = 0, \\ \int_{\mathbb{R}^N} \hat{m}_\varepsilon = M. \end{cases} \tag{5-71}$$

We derive local estimates for $\hat{u}_\varepsilon$ and $\hat{m}_\varepsilon$. We shift the x-variable so that $\hat{u}_\varepsilon(0) = 0 = \min_{\mathbb{R}^N} \hat{u}_\varepsilon$ for all ε. Choose $p > N$ such that

$$\alpha < \frac{\gamma'}{p} < \frac{\gamma'}{N}.$$

If one considers the HJB equation solved by $\hat{u}_\varepsilon$, recalling (5-5) and (5-9), Theorem 2.5 gives the existence of $C > 0$ such that

$$\|\nabla \hat{u}_\varepsilon\|_{L^\infty(B_{2R}(x_0))} \le K(\|\hat{m}_\varepsilon\|^\alpha_{L^\infty(B_{4R}(x_0))} + 1)^{\frac{1}{\gamma}}.$$

Note that $C > 0$ does not depend on ε and x_0. Turning to the Kolmogorov equation, again by (5-9) and Proposition 2.4,

$$\|\hat{m}_\varepsilon\|_{W^{1,p}(B_R(x_0))} \le C(\|\nabla \hat{u}_\varepsilon\|^{\gamma-1}_{L^\infty(B_{2R}(x_0))} + 1)\|m_\varepsilon\|_{L^p(B_{2R}(x_0))}.$$

By the previous L^∞ estimate on ∇u_ε and interpolation of the L^p norm of m between L^1 and L^∞ we get

$$\|\hat{m}_\varepsilon\|_{W^{1,p}(B_R(x_0))} \leq C(\|\hat{m}_\varepsilon\|_{L^\infty(B_{4R}(x_0))}^{\frac{\alpha}{\gamma'}} + 1)\|\hat{m}_\varepsilon\|_{L^1(B_{4R}(x_0))}^{\frac{1}{p}} \|\hat{m}_\varepsilon\|_{L^\infty(B_{4R}(x_0))}^{1-\frac{1}{p}}.$$

Recall that $\|\hat{m}_\varepsilon\|_{L^1(B_{4R}(x_0))} \leq M$; then, since $p > N$, by Sobolev embeddings we obtain that for some $\beta > 0$,

$$\|\hat{m}_\varepsilon\|_{C^{0,\beta}(B_R(x_0))} \leq C(\|\hat{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}^{\frac{\alpha}{\gamma'}} + 1)\|\hat{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{1}{p}}. \tag{5-72}$$

First, since C does not depend on x_0, this yields $\|\hat{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq C$, by the choice of $p < \frac{\gamma'}{\alpha}$. Secondly, plugging this estimate back into (5-72), we conclude

$$\|\hat{m}_\varepsilon\|_{C^{0,\beta}(\mathbb{R}^N)} \leq C.$$

Then, using these estimates, we get that up to subsequences, $\hat{\lambda}_\varepsilon \to \hat{\lambda}$, $\hat{u}_\varepsilon \to \hat{u}$ locally uniformly in C^1, and $\hat{m}_\varepsilon \to \hat{m}$ locally uniformly, where $(\hat{u}, \hat{m}, \hat{\lambda})$ is a solution to (5-25) with $g \equiv 0$. Arguing exactly as in Proposition 5.3, we get that $\tilde{u}$, $\tilde{m}$ satisfy the estimates (5-26) (eventually modifying the constants). Moreover,

$$\int_{\mathbb{R}^N} \hat{m}\, dx = a \in (0, M].$$

Observe now that Lemma 5.5 and Theorem 5.6 hold also for the energy (5-69), since it coincides with the energy $\mathcal{E}_\varepsilon$ without the potential term $\int_{\mathbb{R}^N} V_\varepsilon m\, dx$. Therefore we can apply Theorem 5.6 to $\hat{m}$ to conclude that actually $\int_{\mathbb{R}^N} \hat{m}\, dx = M$. So, by Corollary 5.7, we obtain that for all $\eta > 0$, there exist $R > 0$ and ε_0 such that for all $\varepsilon \leq \varepsilon_0$,

$$\int_{B(0,R)} \hat{m}_\varepsilon\, dx \geq M - \eta. \tag{5-73}$$

By (5-72) and (5-73), using Lemma 2.2, we get

$$f_\varepsilon(\hat{m}_\varepsilon) \geq \tfrac{1}{4}\hat{\lambda}_\varepsilon$$

outside a ball $B_r(0)$. Since $H_\varepsilon(0) \to 0$, the claim

$$f_\varepsilon(\hat{m}_\varepsilon) - \hat{\lambda}_\varepsilon - H_\varepsilon(0) \geq -\tfrac{1}{2}\hat{\lambda}_\varepsilon > 0 \tag{5-74}$$

outside a ball $B_r(0)$ follows. As previously mentioned, we may now proceed and conclude as in Proposition 5.3; basically, (5-74) implies that $x \mapsto e^{k\hat{u}_\varepsilon(x)}$ acts as a Lyapunov function for $\hat{m}_\varepsilon$ for some small $k > 0$, giving

$$c \int_{\mathbb{R}^N} e^{k|x|-k_1}\hat{m}_\varepsilon \leq \int_{\mathbb{R}^N} e^{k\hat{u}_\varepsilon}\hat{m}_\varepsilon \leq C$$

for all ε small, which easily implies the pointwise exponential decay (5-70) of $\hat{m}_\varepsilon$ by the Hölder regularity of $\hat{m}_\varepsilon$ itself. □

For general potentials, the point where mass concentrates is a minimum for V.

Proposition 5.12. *Up to subsequences,* $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon \to \bar{x}$, *where* $V(\bar{x}) = 0$, *i.e.,* $\bar{x}$ *is a minimum of* V.

Proof. Fix a generic $z \in \mathbb{R}^N$ and observe that $(\hat{m}_\varepsilon(\cdot+z), \hat{w}_\varepsilon(\cdot+z))$ is still a minimizer of $\int mL_\varepsilon\bigl(-\frac{w}{m}\bigr)+F_\varepsilon(m)$. By the minimality of $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ and of $(\hat{m}_\varepsilon(\cdot+z), \hat{w}_\varepsilon(\cdot+z))$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{m}_\varepsilon L_\varepsilon\Bigl(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Bigr) + F_\varepsilon(\bar{m}_\varepsilon)\,dy + \int_{\mathbb{R}^N} \bar{m}_\varepsilon(y)V_\varepsilon(y+y_\varepsilon)\,dy \\ &= \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \le \mathcal{E}_\varepsilon(\hat{m}_\varepsilon(\cdot+z), \hat{w}_\varepsilon(\cdot+z)) \\ &\le \int_{\mathbb{R}^N} \bar{m}_\varepsilon L_\varepsilon\Bigl(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Bigr) + F_\varepsilon(\bar{m}_\varepsilon) + \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y+z)V_\varepsilon(y+y_\varepsilon)\,dy.\end{aligned}$$

In particular this gives

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{m}_\varepsilon(y)V_\varepsilon(y+y_\varepsilon)\,dy &\le \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y+z)V_\varepsilon(y+y_\varepsilon)\,dy \\ &= \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y)V_\varepsilon(y+y_\varepsilon-z)\,dy \quad \text{for all } z \in \mathbb{R}^N. \qquad (5\text{-}75)\end{aligned}$$

Recalling the rescaling of V_ε and of $\bar{m}_\varepsilon$ in (5-1), this is equivalent to

$$\int_{\mathbb{R}^N} m(x)V(x)\,dx \le \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y)V\bigl(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon - \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}z\bigr)\,dy \quad \text{for all } z \in \mathbb{R}^N, \qquad (5\text{-}76)$$

where m is the classical solution to (1-1) given in Theorem 1.1 such that

$$\bar{m}_\varepsilon(y) = \varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}} m\bigl(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon\bigr).$$

By (5-65), we get that up to passing to a subsequence, $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^N$. Then by (5-68), we get

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} m(x)V(x)\,dx \ge \liminf_{\varepsilon\to 0} \int_{B(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon, \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}R)} m(x)V(x)\,dx \ge (M-\eta)V(\bar{x}). \qquad (5\text{-}77)$$

We fix $\bar{z}$ such that $V(\bar{z}) = 0$ and we choose in (5-76) $z = y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}}\bar{z}$. We have, by the Lebesgue convergence theorem and (5-70),

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y)V\bigl(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \bar{z}\bigr)\,dy \le \limsup_{\varepsilon\to 0} c_1 \int_{\mathbb{R}^N} e^{-c_2|y|}V\bigl(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \bar{z}\bigr)\,dy = 0. \qquad (5\text{-}78)$$

By (5-77), (5-78) and (5-76), we conclude $V(\bar{x}) = 0$. □

If we assume that the potential V has a finite number of minima and polynomial behavior, that is, it satisfies assumption (1-13), then we get that at the limit $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon$ selects at the limit the more stable minima of V, as we will show in the next proposition.

Proposition 5.13. *Assume that V satisfies assumption* (1-13). *Then, up to subsequences, there holds*

$$\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon \to x_i \quad \textit{as } \varepsilon \to 0,$$

where $i \in \{j = 1, \dots, n : b_j = \max_k b_k\}$.

Proof. By Proposition 5.12, we know that up to subsequences, $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon \to x_\iota$ for some $\iota = 1, \dots n$. It remains to prove that $b_\iota = \max_i b_i$. Assume by contradiction that it is not true, and then $b_\iota < \max_i b_i$.

We compute for $j \in 1, \dots n$, recalling the uniform exponential decay of $\hat{m}_\varepsilon$ given in (5-70),

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{m}_\varepsilon(y + y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j) V_\varepsilon(y + y_\varepsilon)\, dy \\ &= \int_{\mathbb{R}^n} \hat{m}_\varepsilon(y) V_\varepsilon(y + \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j)\, dy \\ &\le C_V \varepsilon^{\frac{\gamma' N\alpha}{\gamma'-N\alpha}} \int_{\mathbb{R}^n} \hat{m}_\varepsilon(y) \varepsilon^{\frac{b_j\gamma'}{\gamma'-N\alpha}} |y|^{b_j} \prod_{i\ne j} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y - x_i + x_j|^{b_i}\, dy \\ &\le C\varepsilon^{\frac{\gamma'(N\alpha+b_j)}{\gamma'-N\alpha}} \int_{\mathbb{R}^n} \hat{m}_\varepsilon(y)|y|^{b_j} \prod_{i\ne j} |y - x_i + x_j|^{b_i}\, dy \le C\varepsilon^{\frac{\gamma'(N\alpha+b_j)}{\gamma'-N\alpha}}. \end{aligned} \tag{5-79}$$

Note in particular that we can choose in the previous inequality $b_j = \max_i b_i$.

We get from (5-75) applied to $z = y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j$, where j is such that $b_j = \max_i b_i$, and from (5-79) the following improvement of (5-64):

$$\int_{B(0,R)} \bar{m}_\varepsilon V_\varepsilon(y + y_\varepsilon)\, dy \le \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y + y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j) V_\varepsilon(y + y_\varepsilon)\, dy \le C\varepsilon^{\frac{(N\alpha+\max b_j)\gamma'}{\gamma'-N\alpha}} \tag{5-80}$$

for all $R \ge 0$. We choose $R > 0$ sufficiently large such that $\int_{B(0,R)} \bar{m}_\varepsilon\, dy \ge \frac{1}{2}M$. Recalling the rescaling of V, (5-80) implies

$$C\varepsilon^{\frac{\max b_j \gamma'}{\gamma'-N\alpha}} \ge \tfrac{1}{2} M C_V^{-1} \min_{y\in B(0,R)} \prod_{j=1}^{n} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_j|^{b_j}. \tag{5-81}$$

Note that for ε sufficiently small $|\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_j| \ge \delta > 0$ for all $i \ne \iota$ and all $y \in B(0, R)$. So, by (5-81) we get that there exists $C > 0$ for which

$$\min_{y\in B(0,R)} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_\iota|^{b_\iota} \le C\varepsilon^{\frac{\max b_j \gamma'}{\gamma'-N\alpha}}$$

and then

$$|\hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota} = \min_{y\in B(0,R)} |y + y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota} \le C\varepsilon^{\frac{(\max b_j - b_\iota)\gamma'}{\gamma'-N\alpha}} \to 0 \tag{5-82}$$

for some $\hat{y}_\varepsilon \in B(y_\varepsilon, R)$. Let $z_\varepsilon = \hat{y}_\varepsilon - y_\varepsilon \in B(0, R)$. Up to subsequences we can assume that $z_\varepsilon \to \bar{z} \in B(0, R)$.

We use now (5-80), recalling assumption (1-13), and we get

$$\begin{aligned} C\varepsilon^{\frac{\max b_j \gamma'}{\gamma'-N\alpha}} &\ge C_V^{-1} \int_{B(0,R)} \bar{m}_\varepsilon(y) \prod_{j=1}^{n} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_j|^{b_j}\, dy \\ &\ge c_1 \varepsilon^{\frac{b_\iota\gamma'}{\gamma'-N\alpha}} \int_{B(0,R)} \bar{m}_\varepsilon(y)|y - z_\varepsilon + \hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota}\, dy. \end{aligned}$$

In particular this implies

$$\lim_{\varepsilon\to 0}\int_{B(0,R)} \bar{m}_\varepsilon(y)|y - z_\varepsilon + \hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota}\, dy = 0. \tag{5-83}$$

Recalling that $\bar{m}_\varepsilon \to \bar{m}$ locally uniformly, see (5-24), that $\hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota \to 0$ by (5-82), and that $z_\varepsilon \to \bar{z}$, we get

$$\lim_{\varepsilon\to 0}\int_{B(0,R)} \bar{m}_\varepsilon(y)|y - z_\varepsilon + \hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota}\, dy = \int_{B(0,R)} \bar{m}(y)|y-\bar{z}|^{b_\iota}\, dy > 0.$$

This gives a contradiction with (5-83). □

As a consequence of the previous results, we can conclude with the following.

Proof of Theorem 1.2. Setting $x_\varepsilon = \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon$, it suffices to recall (5-68) and Propositions 5.12, 5.13. □

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GENERALIZED CRYSTALLINE EVOLUTIONS AS LIMITS OF FLOWS WITH SMOOTH ANISOTROPIES

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We prove existence and uniqueness of weak solutions to anisotropic and crystalline mean curvature flows, obtained as a limit of the viscosity solutions to flows with smooth anisotropies.

1. Introduction

In this note we deal with anisotropic, and possibly crystalline, mean curvature flows, that is, flows of sets $t \mapsto E(t)$ governed by the law

$$V(x,t) = -\psi(\nu^{E(t)}(x))(\kappa_\phi^{E(t)}(x) + g(x,t)), \tag{1-1}$$

where

- $V(x,t)$ stands for the outer normal velocity of the boundary $\partial E(t)$ at x,
- ϕ is a given norm on $\mathbb{R}^N$ representing the *surface tension*,
- $\kappa_\phi^{E(t)}$ is the *anisotropic mean curvature* of $\partial E(t)$ associated with the anisotropy ϕ,
- ψ is a norm evaluated at the outer unit normal $\nu^{E(t)}$ to $\partial E(t)$, and g is a forcing term.

The factor ψ plays the role of a *mobility*.

We refer to [Chambolle et al. 2017a] for the motivations to study this flow, which originate in problems from phase transitions and materials science; see for instance [Taylor 1978; Gurtin 1993]. Its mathematical well-posedness is established in the smooth setting, that is, when ϕ, ψ, g and the initial set are sufficiently smooth and ϕ satisfies suitable ellipticity conditions. However, it is also well known that in dimensions $N \geq 3$ singularities may form in finite time even in the smooth case and for regular initial sets. When this occurs, the strong formulation of (1-1) ceases to be meaningful and thus needs to be replaced by weaker notions of global-in-time solution.

Among the different weak approaches that have been proposed in the literature for the classical mean curvature flow (and for several other "regular" flows) here we recall the so-called *level-set formulation* [Osher and Sethian 1988; Evans and Spruck 1991; 1992a; Chen et al. 1991; Giga 2006] and the *flat flow formulation*, proposed by Almgren, Taylor and Wang [Almgren et al. 1993] and based on the *minimizing movements* variational scheme (referred to as the ATW scheme).

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However, when the anisotropy ϕ in (1-1) is nondifferentiable or crystalline, the lack of smoothness of the involved differential operators makes it much harder to pursue the aforementioned approaches. In fact, in the crystalline case the problem of finding a suitable weak formulation of (1-1) in dimension $N \geq 3$ leading to a unique global-in-time solution for general initial sets has remained open until the very recent works [Chambolle et al. 2017a; 2017b; Giga and Požár 2016; 2018].

We refer also to [Giga et al. 1998; Caselles and Chambolle 2006; Bellettini et al. 2006] for previous results holding for special classes of initial data, and to [Giga et al. 2014] for a well-posedness result dealing with a very specific anisotropy. The two-dimensional case is somewhat easier and has been essentially settled in [Giga and Giga 2001] (when g is constant) by developing a crystalline version of the viscosity approach for the level-set equation; see also [Taylor 1978; Almgren and Taylor 1995; Angenent and Gurtin 1989; Giga and Giga 1998; Giga and Gurtin 1996] for relevant former work. We also mention the recent papers [Chambolle and Novaga 2015; Mercier et al. 2016], where short time existence and uniqueness of strong solutions for initial "regular" sets (in a suitable sense) is shown.

Let us now briefly describe the most recent progress on the problem. In [Chambolle et al. 2017b], the first global-in-time existence and uniqueness result for the level-set flow associated to (1-1), valid in all dimensions, for arbitrary (possibly unbounded) initial sets, and for general (including crystalline) anisotropies ϕ was established, but under the particular choice $\psi = \phi$ (and $g = 0$). The main contribution of that work is the observation that the variant of the ATW scheme proposed in [Chambolle 2004; Caselles and Chambolle 2006] converges to solutions that satisfy a new stronger *distributional* formulation of the problem in terms of distance functions. Such a formulation is only reminiscent of, but not quite the same as, the distance formulation studied in [Soner 1993], see also [Barles et al. 1993; Ambrosio and Soner 1996; Caselles and Chambolle 2006; Ambrosio 2000], and because of its distributional character it enables the use of parabolic PDE's arguments in order to establish a comparison result yielding uniqueness.

In [Chambolle et al. 2017a], we first observe that the methods of [Chambolle et al. 2017b] can be pushed to treat bounded spatially Lipschitz continuous forcing terms g and more general mobilities ψ, which are "regular" with respect to the anisotropy ϕ. More precisely, a norm ψ is said to be ϕ-regular if the associated ψ-Wulff shape W^ψ satisfies a uniform inner ϕ-Wulff shape condition at all points of its boundary. Such a condition implies that the ϕ-curvature κ_ϕ of ∂W^ψ is bounded above and it enables us to show that a distributional formulation in the spirit of [Chambolle et al. 2017b] still holds true. Next, owing to the simple observation that the ϕ-regular mobilities are dense, we succeed in extending the notion of solution to general mobilities by an approximation procedure. More precisely, by establishing delicate stability estimates on the ATW scheme, we show that if ψ is any norm and $\psi_n \to \psi$, with ψ_n a ϕ-regular mobility for every n, then the corresponding distributional level-set solutions u^{ψ_n}, with the given initial datum u^0, admit a unique limit u^ψ (independent of the choice of the approximating ψ_n), which we may therefore regard as the unique solution to the level-set flow with mobility ψ and initial datum u^0. As a byproduct of this analysis, we also settle the problem of the uniqueness (up to fattening) of flat flows for general mobilities. Once again, our results hold in all dimensions, for arbitrary (possibly unbounded) initial sets and general, possibly crystalline, anisotropies ϕ.

By completely different methods, in [Giga and Požár 2016], and more recently in [Giga and Požár 2018], the authors succeed in extending the viscosity approach of [Giga and Giga 2001] to the case $N = 3$ and to the general case $N \geq 3$, respectively. In fact, as in [Giga and Giga 2001], they are able to deal with very general equations of the form

$$V = f(\nu^E, -\kappa_\phi^E),$$

with f continuous and nondecreasing with respect to the second variable, but without spatial dependence, establishing existence and uniqueness for the corresponding level-set formulation. Important achievements in their work are the definition of a crystalline curvature for sets with appropriate regularity, a comparison result for such a curvature, and an approximation result showing that any compact set is arbitrarily close to sets with well-defined curvature. They can also deduce stability results, [Giga and Požár 2016, Theorem 8.9; 2018, Theorem 1.5], which ensure in particular that the viscosity solution of the nonsmooth problem can be built as the limit of a sequence of classical viscosity solutions of the problem with smooth regularized anisotropies. As our approach in the current paper is based on the same idea, a by-product is that when both are defined, their evolutions and ours coincide (Remarks 3.8 and 3.9). On the other hand, their method currently works only for purely crystalline anisotropies ϕ, bounded initial sets, and constant forcing terms.

As said, we propose here an approach different from our previous work [Chambolle et al. 2017a]: Following [Giga and Požár 2016; 2018], we derive existence, uniqueness and some properties of anisotropic and crystalline flows directly from the corresponding properties of smooth (i.e., with smooth anisotropies) flows, appropriately defined as viscosity solutions of a geometric PDE. This leads to a more direct and easier proof of the well-posedness of (1-1) for general mobilities and anisotropies, relying on purely viscosity methods. On the other hand, our new estimates are too weak to provide information about the uniqueness of flat flows, shown in [Chambolle et al. 2017a].

Let us describe the new approach in more detail. The starting point is the observation that when the anisotropy is smooth, the distributional formulation of [Chambolle et al. 2017a; 2017b] is equivalent to the classical viscosity formulation; see Section 2B. Next, in Section 2C we show that if $\phi_n \to \phi$, with ϕ_n smooth, and if $\psi_n \to \psi$, with ψ_n ϕ_n-regular "uniformly" with respect to n (see the statement of Theorem 2.8 below for the precise meaning), then the corresponding viscosity (and thus distributional) level-set solutions u_n converge locally uniformly to the unique distributional level-set flow with anisotropy ϕ and (ϕ-regular) mobility ψ. This leads to a new proof of the existence of distributional level-set solutions for ϕ-regular mobilities, without using the ATW scheme as in [Chambolle et al. 2017a].

In Sections 3A and 3B we establish the crucial stability estimates of the flow with respect to changing ϕ-regular mobilities. This is achieved once again by exploiting the viscosity formulation in order to prove first the estimates in the case of smooth anisotropies and to conclude by approximation.

Finally, in Section 3C we prove the main existence and uniqueness result for the level-set formulation of (1-1), in the case of general anisotropies and mobilities. In this last step we proceed essentially as in [Chambolle et al. 2017a]: we approximate any mobility ψ by a sequence ϕ-regular mobilities ψ_n and show, by means of the stability estimates of the previous sections, that the corresponding solutions admit a unique limit.

2. Distributional mean curvature flows

Given a norm η on $\mathbb{R}^N$ (a convex, even, one-homogeneous real-valued function with $\eta(\nu) > 0$ if $\nu \neq 0$), we define a polar norm η° by $\eta^\circ(\xi) := \sup_{\eta(\nu)\le 1} \nu \cdot \xi$ and an associated anisotropic perimeter P_η as

$$P_\eta(E) := \sup\left\{ \int_E \operatorname{div} \zeta \, dx : \zeta \in C_c^1(\mathbb{R}^N; \mathbb{R}^N),\ \eta^\circ(\zeta) \le 1 \right\}.$$

As is well known, $(\eta^\circ)^\circ = \eta$ so that when the set E is smooth enough one has

$$P_\eta(E) = \int_{\partial E} \eta(\nu^E)\, d\mathcal{H}^{N-1},$$

which is the perimeter of E weighted by the surface tension $\eta(\nu)$.

We will make repeated use of the identities

$$\begin{aligned} \partial\eta(\nu) &= \{\xi : \eta^\circ(\xi) \le 1 \text{ and } \xi \cdot \nu \ge \eta(\nu)\} \\ &= \{\xi : \eta^\circ(\xi) = 1 \text{ and } \xi \cdot \nu = \eta(\nu)\} \end{aligned} \tag{2-1}$$

(and the symmetric statement for η°) for $\nu \neq 0$, where $\partial\eta(\nu)$ denotes the subdifferential of η at ν. Moreover, $\partial\eta(0) = \{\xi : \eta^\circ(\xi) \le 1\}$, while $\partial\eta^\circ(0) = \{\xi : \eta(\xi) \le 1\}$. For $R > 0$ we define

$$W^\eta(x, R) := \{y : \eta^\circ(y - x) \le R\}.$$

Such a set is called the *Wulff shape* (of radius R and center x) associated with the norm η and represents the unique (up to translations) solution of the anisotropic isoperimetric problem

$$\min\{P_\eta(E) : |E| = |W^\eta(0, R)|\};$$

see for instance [Fonseca and Müller 1991].

We denote by $\operatorname{dist}^\eta(\,\cdot\,, E)$ the distance from E induced by the norm η; that is, for any $x \in \mathbb{R}^N$,

$$\operatorname{dist}^\eta(x, E) := \inf_{y\in E} \eta(x - y) \tag{2-2}$$

if $E \neq \varnothing$ and $\operatorname{dist}^\eta(x, \varnothing) := +\infty$. Moreover, we denote by d_E^η the signed distance from E induced by η, i.e.,

$$d_E^\eta(x) := \operatorname{dist}^\eta(x, E) - \operatorname{dist}^\eta(x, E^c).$$

so that $\operatorname{dist}^\eta(x, E) = d_E^\eta(x)^+$ and $\operatorname{dist}^\eta(x, E^c) = d_E^\eta(x)^-$, where we adopt the standard notation $t^+ := t \vee 0$ and $t^- := (-t)^+$. Note that by (2-1) we have $\eta(\nabla d_E^{\eta^\circ}) = \eta^\circ(\nabla d_E^\eta) = 1$ a.e. in $\mathbb{R}^N \setminus \partial E$.

Finally we recall that a sequence of closed sets E_n in $\mathbb{R}^m$ converges to a closed set E in the *Kuratowski sense* if the following conditions are satisfied:

(i) if $x_n \in E_n$, any limit point of $\{x_n\}$ belongs to E,

(ii) any $x \in E$ is the limit of a sequence $\{x_n\}$, with $x_n \in E_n$,

and we write

$$E_n \xrightarrow{\mathcal{K}} E.$$

Since $E_n \xrightarrow{\mathcal{K}} E$ if and only if (for any norm η) $\mathrm{dist}^\eta(\,\cdot\,, E_n) \to \mathrm{dist}^\eta(\,\cdot\,, E)$ locally uniformly in $\mathbb{R}^m$, by the Ascoli–Arzelà theorem any sequence of closed sets admits a converging subsequence in the Kuratowski sense.

2A. *The weak formulation of the crystalline flow.* In this section we recall the weak formulation of the crystalline mean curvature flow introduced in [Chambolle et al. 2017a; 2017b].

In what follows, we will consider forcing terms $g : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ satisfying the following two hypotheses:

(H1) $g \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.

(H2) There exists $L > 0$ such that $g(\,\cdot\,, t)$ is L-Lipschitz continuous with respect to the metric ψ° for a.e. $t > 0$. Here ψ is the norm representing the mobility in (1-1).

Remark 2.1. Assumption (H1) can be in fact weakened and replaced by

(H1′) for every $T > 0$, we have $g \in L^\infty(\mathbb{R}^N \times (0, T))$.

Indeed under the weaker assumption (H1′), all the arguments and the estimates presented throughout the paper continue to work in any time interval $(0, T)$, with some of the constants involved possibly depending on T. In the same way, if one restricts our study to the evolution of sets with compact boundary, then one could assume that g is only locally bounded in space. We assume (H1) instead of (H1′) only to simplify the presentation.

Let ϕ, ψ be two (possibly crystalline) norms representing the anisotropy and the mobility in (1-1), respectively. We recall the following distributional formulation of (1-1).

Definition 2.2 [Chambolle et al. 2017a]. Let $E^0 \subset \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$ and for each $t \geq 0$ define $E(t) := \{x \in \mathbb{R}^N : (x, t) \in E\}$. We say that E is a *superflow* of (1-1) with initial datum E^0 if:

(a) (initial condition) $E(0) \subseteq E^0$.

(b) (left continuity) $E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$.

(c) (extinction time) If $E(t) = \varnothing$ for $t \geq 0$, then $E(s) = \varnothing$ for all $s > t$.

(d) (differential inequality) Set $T^* := \inf\{t > 0 : E(s) = \varnothing \text{ for } s \geq t\}$, and

$$d(x, t) := \mathrm{dist}^{\psi^\circ}(x, E(t)) \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then there exists $M > 0$ such that the inequality

$$\partial_t d \geq \operatorname{div} z + g - Md \tag{2-3}$$

holds in the distributional sense in $\mathbb{R}^N \times (0, T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0, T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, and

$$(\operatorname{div} z)^+ \in L^\infty(\{(x, t) \in \mathbb{R}^N \times (0, T^*) : d(x, t) \geq \delta\}) \quad \text{for every } \delta \in (0, 1).$$

We say that A, an open set in $\mathbb{R}^N \times [0, +\infty)$, is a subflow of (1-1) with initial datum E^0 if A^c is a superflow of (1-1) with g replaced by $-g$ and with initial datum $(\mathring{E}^0)^c$.

Finally, we say that E, a closed set in $\mathbb{R}^N \times [0, +\infty)$, is a solution of (1-1) with initial datum E^0 if it is a superflow and if $\mathring{E}$ is a subflow, both with initial datum E^0.

It is shown in [Chambolle et al. 2017a] (see also [Chambolle et al. 2017b] for a simpler equation), using quite standard parabolic comparison arguments, that such evolutions satisfy a comparison principle:

Theorem 2.3 [Chambolle et al. 2017a, Theorem 2.7]. *Let E be a superflow with initial datum E^0 and F be a subflow with initial datum F^0 in the sense of Definition 2.2. Assume that* $\operatorname{dist}^{\psi^\circ}(E^0, \mathbb{R}^N \setminus F^0) =: \Delta > 0$. *Then,*

$$\operatorname{dist}^{\psi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta e^{-Mt} \quad \textit{for all } t \geq 0,$$

where $M > 0$ is as in (2-3) *for both E and F.*

We now recall the corresponding notion of sub- and supersolution to the level-set flow associated with (1-1); see again [Chambolle et al. 2017a].

Definition 2.4 (level-set subsolutions and supersolutions). Let u^0 be a uniformly continuous function on $\mathbb{R}^N$. We will say that a lower semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *level-set supersolution* corresponding to (1-1), with initial datum u^0, if $u(\,\cdot\,, 0) \geq u^0$ and if for a.e. $\lambda \in \mathbb{R}$, the closed sublevel set $\{(x, t) : u(x, t) \leq \lambda\}$ is a superflow of (1-1) in the sense of Definition 2.2, with initial datum $\{u_0 \leq \lambda\}$.

We will say that an upper-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *level-set subsolution* corresponding to (1-1), with initial datum u^0, if $-u$ is a superlevel-set flow in the previous sense, with initial datum $-u_0$ and with g replaced by $-g$.

Finally, we will say that a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *solution* to the level-set flow corresponding to (1-1) if it is both a level-set subsolution and supersolution.

As shown in [Chambolle et al. 2017a], Theorem 2.3 easily yields that almost all closed sublevels of a solution of the level-set flows are solutions of (1-1) in the sense of Definition 2.2. Moreover, the following comparison principle between level-set subsolutions and supersolutions holds true.

Theorem 2.5 [Chambolle et al. 2017a, Theorem 2.8]. *Let u^0, v^0 be uniformly continuous functions on $\mathbb{R}^N$ and let u, v be respectively a level-set subsolution with initial datum u^0 and a level-set supersolution with initial datum v^0, in the sense of Definition 2.4. If $u^0 \leq v^0$, then $u \leq v$.*

For smooth anisotropies, solutions to the level-set flow and (minus the characteristic function of) solutions of the geometric flow in the sense of Definition 2.2 are in fact viscosity solutions of the (degenerate) parabolic equation (2-4) below. This classical fact will be shown and exploited to some extent to nonsmooth anisotropies in the next two sections.

2B. ***Viscosity solutions.*** We show here that in the smooth cases, the notion of solution in Definition 2.2 coincides with the definition of standard viscosity solutions for geometric motions, as for instance in [Barles and Souganidis 1998]. This property will be helpful to establish estimates using standard approaches for viscosity solutions.

Lemma 2.6. *Assume that ϕ, ψ, $\psi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, and that g is continuous. Let E be a superflow in the sense of Definition 2.2. Then, $-\chi_E$ is a viscosity supersolution of*

$$u_t = \psi(\nabla u)(\operatorname{div} \nabla\phi(\nabla u) + g) \tag{2-4}$$

in $\mathbb{R}^N \times (0, T^*]$, where T^* is the possible extinction time of E.

Conversely, a viscosity supersolution $-\chi_E$ of (2-4) defines a superflow in the sense of Definition 2.2.

Proof. A similar statement (in a simpler context) is proved in [Chambolle et al. 2017b, Appendix], while it is proved in [Chambolle et al. 2017a] that a superflow defines a viscosity supersolution. We therefore here focus on the converse: Given an evolving set $E(t)$ such that $-\chi_E$ is a viscosity supersolution of (2-4), we show that $E(t)$ is a superflow in the sense of Definition 2.2, with the constant M in (2-3) equal to the Lipschitz constant L of $g(\,\cdot\,, t)$ appearing in the assumption (H2).

Step 1: left continuity and extinction time. Let $T^* \in [0, +\infty]$ be the (first) extinction time of E, and assume without loss of generality $T^* > 0$. Let $d(x, t) := \operatorname{dist}^{\psi^\circ}(x, E(t))$. We fix $\delta > 0$ and we set $A = (\mathbb{R}^N \times [0, T^*)) \setminus E$ and $A^\delta = A \cap \{d > \delta\}$. Let (x, t) with $d(x, t) = R > \delta > 0$. Then $W^\psi(x, R - \varepsilon) \cap E(t) = \varnothing$ for any $\varepsilon > 0$ (small). There exists a constant C (depending on ϕ, ψ) such that, letting

$$W(s) = \mathbb{R}^N \setminus W^\psi\Big(x, R - \varepsilon - \Big(\frac{C}{R} + \|g\|_\infty\Big)s\Big),$$

$-\chi_{W(s)}$ is a viscosity subsolution of (2-6) for $s \le R^2/(2(C + R\|g\|_\infty))$ and $\varepsilon \le R/4$. By standard comparison results [Barles et al. 1993], it follows that $E(t + s) \subset W(s)$ for such times s, so that $d(x, t + s) \ge R - \varepsilon - (C/R + \|g\|_\infty)s$. Hence, letting $\varepsilon \to 0$, we find that

$$d(x, t + s) \ge d(x, t) - \Big(\frac{C}{\delta} + \|g\|_\infty\Big)s \quad \text{if } (x, t) \in A^\delta. \tag{2-5}$$

In particular, it follows that $\partial_t d$ is bounded from below in such sets and hence is a measure. By (2-5) and the fact that E is closed we deduce that the left continuity (b) of Definition 2.2 holds for $E(t)$. Moreover, the same argument shows that if $t > T^*$ then $d(x, t) = +\infty$, showing also point (c).

Step 2: the distance function is a viscosity supersolution. We now show that the function $d(x, t)$ is a viscosity supersolution of

$$u_t = \psi(\nabla u)(D^2\phi(\nabla u) : D^2 u + g - Lu). \tag{2-6}$$

In fact, this is essentially classical [Soner 1993]; however the proof in this reference needs to be adapted to deal with the forcing term. An elementary proof is as follows: Let η be a smooth test function and assume $(\bar{x}, \bar{t})$ is a contact point, where $\eta(\bar{x}, \bar{t}) = d(\bar{x}, \bar{t})$ and $\eta \le d$. If the common value of η, d at $(\bar{x}, \bar{t})$ is zero then it is also a contact point of $1 - \chi_E$ and η, so that

$$\partial_t \eta(\bar{x}, \bar{t}) \ge \psi(\nabla\eta(\bar{x}, \bar{t}))\big(D^2\phi(\nabla\eta(\bar{x}, \bar{t})) : D^2\eta(\bar{x}, \bar{t}) + g(\bar{x}, \bar{t}) - L\eta(\bar{x}, \bar{t})\big) \tag{2-7}$$

obviously holds, by definition (recalling (2-4) and that $\eta(\bar{x}, \bar{t}) = 0$). Hence we consider the case where $R = d(\bar{x}, \bar{t}) > 0$. Let $\bar{y} \in \partial E(\bar{t})$ such that $R = \psi^\circ(\bar{x} - \bar{y})$. We let

$$\eta'(y, t) := \eta(y + \bar{x} - \bar{y}, t) - R \le d(y + \bar{x} - \bar{y}, t) - R \le d(y, t)$$

since d is 1-Lipschitz in the ψ° norm. In particular, in a neighborhood of $(\bar{y}, \bar{t})$ we have $\eta'(y,t) \le 1 - \chi_{E(t)}(y)$. On the other hand, $\eta'(\bar{y}, \bar{t}) = 0 = d(\bar{y}, \bar{t}) = 1 - \chi_{E(\bar{t})}(\bar{y})$. Hence, by (2-4)

$$\partial_t \eta(\bar{x}, \bar{t}) = \partial_t \eta'(\bar{y}, \bar{t}) \ge \psi(\nabla \eta'(\bar{y}, \bar{t}))\big(D^2\phi(\nabla\eta'(\bar{y}, \bar{t})) : D^2\eta'(\bar{y}, \bar{t}) + g(\bar{y}, \bar{t})\big).$$

Since $g(\bar{y}, \bar{t}) \ge g(\bar{x}, \bar{t}) - L\eta(\bar{x}, \bar{t})$, (2-7) follows.

Step 3: differential inequality. A classical remark is that d^2, as an infimum of the uniformly semiconcave functions $\psi^\circ(\cdot - y)^2$, $y \in E(t)$, is semiconcave; hence in A^δ one has $D^2 d \le C/\delta\, I$ in the sense of measures for some constant C depending only on ψ°. In particular, $\operatorname{div} \nabla\phi(\nabla d) = D^2\phi(\nabla d) : D^2 d \le C/\delta$ in A^δ in the sense of measures.

We proceed as in [Chambolle et al. 2017b]: For $n \ge 1$, let $d_n(x,t) := \min_s (d(x, t-s) + ns^2)$, which is semiconcave and converges to d as $n \to \infty$. Moreover, one can easily check that $d_n(\cdot, t) \to d(\cdot, t)$ locally uniformly if t is a continuity point of d. Let $B \subset A^\delta$ be an open ball (where in particular d is bounded from above and it is bounded from below by δ) and observe that d_n is still a supersolution of (2-6), provided $g(x,t)$ is replaced with $g(x,t) - \omega_n$ for some $\omega_n \to 0$ as $n \to +\infty$. Since d_n, which is semiconcave, has a second-order jet a.e. in B, (2-6) holds for d_n a.e. in B. Reasoning as in [Chambolle et al. 2017b, Appendix], we deduce that

$$\partial_t d_n \ge \psi(\nabla d_n)(\operatorname{div} z_n + g - \omega_n - L d_n) \tag{2-8}$$

in the distributional sense (or as measures) in B, where $z_n := \nabla\phi(\nabla d_n)$. It remains to send $n \to \infty$: Clearly, $\partial_t d_n \to \partial_t d$ in the distributional sense. Consider (x,t) a point where $\nabla d(x,t)$ and $\nabla d_n(x,t)$ exist for all n. First, if $d(x, t-s) + ns^2$ attains the minimum at s_n, one has for any $p \in \partial^+ d(x, t - s_n)$ (the spatial supergradient of the semiconcave function $d(\cdot, t - s_n)$) that

$$\begin{aligned} d_n(x+h, t) &\le d(x+h, t-s_n) + ns_n^2 \\ &\le d(x, t-s_n) + p\cdot h + \frac{C}{\delta}|h|^2 + ns_n^2 = d_n(x,t) + p \cdot h + \frac{C}{\delta}|h|^2, \end{aligned}$$

showing that $p \in \partial^+ d_n(x,t) = \{\nabla d_n(x,t)\}$. We deduce that $d(\cdot, t - s_n)$ is differentiable at x, with gradient $\nabla d_n(x,t)$, and in particular that $\psi(\nabla d_n(x,t)) = 1$.

Assume now that in addition d is continuous at t. Then $d_n(\cdot, t) \to d(\cdot, t)$ uniformly in $B \cap (\mathbb{R}^N \times \{t\})$, and using the (uniform) semiconcavity of these functions, one also deduces that $\nabla d_n(x,t) \to \nabla d(x,t)$ a.e.; hence, $z_n(x,t) = \nabla\phi(\nabla d_n(x,t))$ converges to $z(x,t) = \nabla\phi(\nabla d(x,t))$ a.e. Hence we may send n to ∞ in (2-8) to find that

$$\partial_t d \ge \operatorname{div} z + g - Ld$$

in the distributional sense in B, with $z = \nabla\phi(\nabla d)$ a.e.

This shows the lemma. □

2C. ***The level-set formulation.*** Let u^0 be a bounded, uniformly continuous function on $\mathbb{R}^N$. Then, it is well known [Chen et al. 1991] that if $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$ and ψ, g are continuous, then there exists a unique viscosity solution u of (2-4) with initial datum u^0. Moreover, for all $\lambda \in \mathbb{R}$, we know $-\chi_{\{u<\lambda\}}$

is a viscosity supersolution and $-\chi_{\{u\le\lambda\}}$ is a viscosity subsolution of the same equation. If in addition $\psi, \psi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, it follows from Lemma 2.6 that $E_\lambda(t) := \{u(\cdot, t) \le \lambda\}$ is a superflow in the sense of Definition 2.2, while $A_\lambda(t) := \{u(\cdot, t) < \lambda\}$ is a subflow.[1]

In what follows we will say that a given norm η is *smooth and elliptic* if both η and η° belong to $C^2(\mathbb{R}^N \setminus \{0\})$.

We now consider sequences ϕ_n, ψ_n of smooth and elliptic anisotropies/mobilities converging to ϕ, ψ. We also consider $g_n(x, t)$ a smooth forcing term, which converges to $g(x, t)$ weakly-$*$ in $L^\infty(\mathbb{R}^N \times [0, +\infty))$. We assume also that g_n is uniformly spatially Lipschitz continuous and we denote by L, M the (uniform) Lipschitz constants of g_n with respect to ψ_n° and ϕ_n°, respectively. Given u_n, the corresponding unique viscosity solution of (2-4) (with ψ_n, ϕ_n, g_n instead of ψ, ϕ, g) with initial datum u^0, we want to study the possible limits of u_n. If the limiting anisotropies and forcing term are still smooth enough, it is well known that the limiting u is the unique viscosity solution of the corresponding limit problem. If not, we will show that the limit is still unique. We recall, see [Chambolle et al. 2017a], the following:

Definition 2.7. We will say that a norm ψ is ϕ-regular if the associated Wulff shape $W^\psi(0, 1)$ satisfies a uniform interior ϕ-Wulff shape condition, that is, if there exists $\varepsilon_0 > 0$ with the following property: for every $x \in \partial W^\psi(0, 1)$ there exists $y \in W^\psi(0, 1)$ such that $W^\phi(y, \varepsilon_0) \subseteq W^\psi(0, 1)$ and $x \in \partial W^\phi(y, \varepsilon_0)$.

Notice that it is equivalent to saying that $W^\psi(0, 1)$ is the sum of a convex set and $W^\phi(0, \varepsilon_0)$, or equivalently that $\psi(\nu) = \psi_0(\nu) + \varepsilon_0\phi(\nu)$ for some convex function ψ_0.

We now show the following result.

Theorem 2.8. *Let $(\psi_n)_n$, $(\phi_n)_n$ and $(g_n)_n$ be as above, and, in addition, assume that the mobilities $(\psi_n)_n$ are uniformly ϕ_n-regular, meaning that $\varepsilon_0 > 0$ in Definition 2.7 does not depend on n. Let u_n be the level-set solutions to* (1-1) *in the sense of Definition 2.4, with initial datum u^0, anisotropy $(\psi_n)_n$, mobility $(\phi_n)_n$ and forcing term $(g_n)_n$. Then, the u_n converge locally uniformly to the unique level-set solution u to* (1-1) *in the sense of Definition 2.4, with initial datum u^0, anisotropy ψ, mobility ϕ and forcing term g.*

Proof. A first observation is that the functions u_n remain uniformly continuous in space and time on $\mathbb{R}^N \times [0, T]$ for all $T > 0$, with a modulus depending only on the modulus of continuity ω of u^0 and the Lipschitz constant M. Indeed, by Proposition 3.4 below it follows that for any $\lambda < \lambda'$

$$\operatorname{dist}^{\phi_n^\circ}\big(\{u_n(\cdot, t) \le \lambda\}, \{u_n(\cdot, t) \ge \lambda'\}\big) \ge \Delta e^{-\beta M t},$$

where $\Delta := \omega^{-1}(\lambda' - \lambda) \ge \operatorname{dist}^{\phi^\circ}(\{u^0 \le \lambda\}, \{u^0 \ge \lambda'\}) > 0$, and $\beta > 0$ depends (for large n) only on ϕ and ψ; see (3-16). Therefore, $u_n(\cdot, t)$ is uniformly continuous with modulus of continuity with respect to the norm ϕ_n° given by $\omega(e^{\beta M t}\,\cdot)$. As for the equicontinuity in time, we set $\omega_T(s) := \omega(e^{\beta M T} s)$ and we start by observing that for any $x \in \mathbb{R}^N$, $\varepsilon > 0$, $t \in (0, T]$, and $n \in N$ we have

$$W^{\phi_n}(x, \omega_T^{-1}(\varepsilon)) \subseteq \{y : u_n(y, t) > u_n(x, t) - \varepsilon\}.$$

[1]In the case of "fattening", also $\overline{\{u < \lambda\}}$ is a superflow, and the interior of $\{u \le \lambda\}$ a subflow.

Therefore, by standard comparison results we have $u_n(x,t') > u_n(x,t) - \varepsilon$ provided that $0 < t' - t < \tau$, where τ is the extinction time for $W^{\phi_n}(x, \omega_T^{-1}(\varepsilon))$ under the evolution (1-1). Analogously, one shows that $u_n(x,t') < u_n(x,t) + \varepsilon$ if $0 < t' - t < \tau$. Since τ is bounded away from zero by a quantity independent of n (depending only on ε, $\sup_n \|g_n\|_\infty$ and, for n large, on ϕ and ψ); see for instance [Chambolle et al. 2017a, Remark 4.6]. This establishes the equicontinuity in time.

Hence, up to a subsequence (not relabeled), we may assume that u_n converges locally uniformly to some u. In view of Theorem 2.5, it is enough to show that u is a solution in the sense of Definition 2.4, that is, that for a.e. $\lambda \in \mathbb{R}$ the set $E_\lambda := \{u \le \lambda\}$ is a superflow in the sense of Definition 2.2 and $A_\lambda := \{u < \lambda\}$ a subflow.

We prove the assertion for E_λ. We first notice that since $u_n \to u$ locally uniformly, the Kuratowski limit superior of the sets $E_n := \{u_n \le \lambda\}$ as $n \to \infty$ is contained in E_λ.

By Lemma 2.6, the sets E_n are superflows in the sense of Definition 2.2. We consider $d_n(x,t) := \mathrm{dist}^{\psi_n^\circ}(x, E_n(t))$ and $d(x,t) := \mathrm{dist}^{\psi^\circ}(x, E_\lambda(t))$, the corresponding distance functions, which are finite up to some time $T_n^*, T^* \in (0, +\infty]$ respectively, where T^* is defined according with Definition 2.2. Notice that T^* is increasing with respect to λ, and that if λ is a continuity point, then we have $T_n^* \to T^*$, as $n \to \infty$.

Recalling (2-5), one can deduce that for $s > 0$ small,

$$\frac{d_n(t+s) - d_n(t)}{s} \ge -\frac{C}{d_n(t)} - \|g\|_\infty,$$

where the constant C does not depend on n, as it is essentially the maximal speed, without forcing, of the Wulff shape $W^{\psi_n} := W^{\psi_n}(0,1)$, which is bounded by $(\max_\xi \psi_n) \times (\max_{\partial W^{\psi_n}} \kappa_{\phi_n})$. The curvature κ_{ϕ_n} of ∂W^{ψ_n} is in $[0, (N-1)/\varepsilon_0]$, thanks to the assumption that $\psi_n' := \psi_n - \varepsilon_0 \phi_n$ is convex, which yields that $W^{\psi_n} = W^{\psi_n'} + \varepsilon_0 W^{\phi_n}$. We deduce $\partial_t d_n \ge -C/d_n - \|g\|_\infty$, which yields that there is an increasing function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\Theta(d_n(t+s)) \ge \Theta(d_n(t)) - \|g\|_\infty s \quad \text{for all } t, s > 0. \tag{2-9}$$

Actually Θ is explicitly given by

$$\Theta(d) = d - \frac{C}{\|g\|_\infty} \log\left(1 + \frac{\|g\|_\infty}{C} d\right)$$

for $d \ge 0$. Notice that, for small $d \ge 0$, we have $\Theta(d) \approx \|g\|_\infty d^2/(2C)$, while $\Theta(d) \approx d$ for large d.

It follows from (2-9) (see for instance details in the proof of [Chambolle et al. 2017b, Proposition 4.4], which is an adaptation of Helly's selection theorem) that one can find an at most countable set $\mathcal{N} \subset (0, T^*)$ such that for all $t \notin \mathcal{N}$, $d_n(\cdot, t) \to d(\cdot, t)$ locally uniformly. If $B \Subset (\mathbb{R}^N \times (0, T^*)) \setminus E_\lambda$, one has $B \cap E_n = \varnothing$ for n large enough and

$$\partial_t d_n \ge \operatorname{div} z_n + g_n - L d_n$$

in the distributional sense in B, thanks to (2-3) and Lemma 2.6. Here, $z_n = \nabla \phi_n(\nabla d_n)$. Notice that the z_n are (for n large) well-defined and bounded in $L^\infty(\mathbb{R}^N \times (0,T))$ for any $T < T^*$. In the limit, we find that (2-3) holds for d, with z the weak-$*$ (local in time) limit of $(z_n)_n$ (or rather, in fact, a subsequence). It remains to show that $z \in \partial\phi(\nabla d)$ a.e. in B. An important observation is that, using again the ϕ_n-regularity of ψ_n, one can show that $\operatorname{div} \nabla\phi_n(\nabla d_n) \le (N-1)/(\varepsilon_0 d_n)$; hence it is bounded in $\{d_n > \delta\}$. In particular, in the limit, $(\operatorname{div} z)^+ \chi_{\{d>\delta\}} \in L^\infty(\mathbb{R}^N \times (0, T^*))$.

To show $z \in \partial\phi(\nabla d)$ a.e. in B, we establish that $z \cdot \nabla d \ge \phi(\nabla d)$ a.e. in B. The proof here is as in [Chambolle et al. 2017b]. There exists δ such that for all n large enough, $d_n \ge \delta$ in B; hence $\operatorname{div} z_n \le (N-1)/(\varepsilon_0 \delta)$. Let $\eta \in C_c^\infty(B; \mathbb{R}_+)$; then

$$\int_B \phi(\nabla d)\eta \, dx \, dt \le \liminf_n \int_B \phi_n(\nabla d_n)\eta \, dx \, dt = \liminf_n \int_B (z_n \cdot \nabla d_n)\eta \, dx \, dt.$$

On the other hand,

$$\int_B (z_n \cdot \nabla d_n)\eta \, dx \, dt = \int_B (z_n \cdot \nabla d)\eta \, dx \, dt + \int_B (z_n \cdot \nabla(d_n - d))\eta \, dx \, dt, \tag{2-10}$$

and $\lim_n \int_B (z_n \cdot \nabla d)\eta \, dx \, dt = \int_B (z \cdot \nabla d)\eta \, dx \, dt$ since $z_n \overset{*}{\rightharpoonup} z$.

It remains to prove that the second addend in the right hand side of (2-10) tends to zero as $n \to +\infty$. Set

$$m_n(t) = \min_{x:(x,t)\in \bar{B}} (d_n(x,t) - d(x,t)), \qquad M_n(t) = \max_{x:(x,t)\in \bar{B}} (d_n(x,t) - d(x,t)).$$

Then $M_n(t) - m_n(t) \to 0$ for all $t \notin \mathcal{N}$. One has

$$\begin{aligned} \int_B (z_n \cdot \nabla(d_n - d))\eta \, dx \, dt &= \int_B (z_n \cdot \nabla(d_n - d - m_n(t)))\eta \, dx \, dt \\ &= -\int_B (d_n - d - m_n)\eta \operatorname{div} z_n \, dx \, dt - \int_B (d_n - d - m_n) z_n \cdot \nabla\eta \, dx \, dt. \end{aligned}$$

The last integral goes to zero as $n \to \infty$. Since $(d_n - d - m_n(t))\eta \ge 0$ we have

$$-\int_B (d_n - d - m_n)\eta \operatorname{div} z_n \, dx \, dt \ge -\frac{N-1}{\varepsilon_0 \delta} \int_B (d_n - d - m_n)\eta \, dx \, dt \xrightarrow{n\to\infty} 0.$$

Using instead $d_n - d - M_n$, we show the reverse inequality, and we deduce

$$\int_B \phi(\nabla d)\eta \, dx \, dt \le \int_B (z \cdot \nabla d)\eta \, dx \, dt,$$

which concludes the proof. □

3. Existence by approximation

3A. ***A useful estimate: comparison with different forcing terms.*** We prove in this section and the following a series of comparison results, which will then be combined together to deduce a global comparison result for flows with possibly different mobilities. In this section, we shall assume that the surface tensions ϕ, ψ are smooth and elliptic, so that we can work in the classical viscosity setting. In the limit, our main estimate will also hold for crystalline flows in the sense of Definition 2.2.

We start by recalling standard comparison results for flows with constant velocities; however, we pay special attention to the particular metrics in which our velocities are expressed. We first consider the equation

$$u_t = \psi(\nabla u) g(x,t). \tag{3-1}$$

The following is a slight variant of the well-known result [Barles 2013, Theorem 8.1]:

Lemma 3.1. *Consider $u^0 : \mathbb{R}^N \to \mathbb{R}$, bounded and Λ-Lipschitz continuous with respect to a norm η, smooth and elliptic, and let $\beta > 0$ be such that*

$$\psi \le \beta \eta^\circ. \tag{3-2}$$

Assume g is bounded, continuous and M-Lipschitz in space in the norm η. Let $u(x,t)$ be a viscosity solution of (3-1) *with initial datum u_0. Then for all $t \ge 0$, the function $u(\,\cdot\,,t)$ is $\Lambda e^{\beta M t}$-Lipschitz continuous in the norm η.*

Proof. We start by observing that by classical results the solution u is uniformly continuous locally in time; see for instance [Giga et al. 1991]. The rest of the proof is an adaptation of the argument in [Barles 2013, proof of Theorem 8.1]. Let $\delta > 0$ be given, and let C be a smooth function such that

$$C' - \beta M C \ge \beta M \delta > 0, \tag{3-3}$$

with $C(0) = \Lambda$. Set

$$\sigma := \sup_{\substack{x,y\in\mathbb{R}^N \\ t\in[0,T]}} u(x,t) - u(y,t) - C(t)\eta(x-y).$$

We claim that $\sigma = 0$. Using this claim, we have

$$u(x,t) - u(y,t) \le (\Lambda e^{\beta M t} + \delta(e^{\beta M t} - 1))\eta(x-y)$$

for all $x, y, t \le T$, and sending $\delta \to 0$ we conclude the proof of the lemma.

We are left to prove the claim that $\sigma = 0$. Arguing by contradiction, assume that $\sigma > 0$. Consider a maximum point $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$ in $\mathbb{R}^{2N} \times [0,T]^2$ for the function

$$\varphi(x,y,s,t) = u(x,t) - u(y,s) - C(t)\eta(x-y) - \frac{|t-s|^2}{2a} - b\frac{|x|^2 + |y|^2}{2},$$

where $a, b > 0$ are small parameters (notice that $\varphi(x,y,0,0) \le 0$). For b small enough, $\varphi(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \ge \sigma/2 > 0$, and then by standard arguments (using in particular that $|\bar{x}|, |\bar{y}| \le c/\sqrt{b}$, and that for fixed b, both $\bar{t}$ and $\bar{s}$ converge, up to a subsequence, to the same positive value as $a \to 0$, see for instance [Barles 2013, Lemma 5.2]) we may assume $0 < \bar{t}, \bar{s} \le T$, so that

$$C'(\bar{t})\eta(\bar{x}-\bar{y}) + \frac{\bar{t}-\bar{s}}{a} \le \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) + b\bar{x}) g(\bar{x},\bar{t}),$$

$$\frac{\bar{t}-\bar{s}}{a} \ge \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) - b\bar{y}) g(\bar{y},\bar{s}).$$

Evaluating the difference and recalling (3-3) we obtain

$$\beta M(C(\bar{t}) + \delta)\eta(\bar{x}-\bar{y}) \le \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) + b\bar{x}) g(\bar{x},\bar{t}) - \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) - b\bar{y}) g(\bar{y},\bar{s}).$$

For fixed $b > 0$, we can then let $a \to 0$ and denote by $\tilde{t} \in (0, T]$ the common limit (along a subsequence) of $\bar{t}$ and $\bar{s}$ as $a \to 0$, and by $\tilde{x}$ and $\tilde{y}$ the limits (along a subsequence) of $\bar{x}$ and $\bar{y}$, respectively. Thus, using (3-2), we obtain

$$\begin{aligned}
&M(C(\tilde{t})+\delta)\eta(\tilde{x}-\tilde{y})\\
&\le \frac{1}{\beta}\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})g(\tilde{x},\tilde{t})-\frac{1}{\beta}\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})-b\tilde{y})g(\tilde{y},\tilde{t})\\
&\le \frac{1}{\beta}\big(\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})-\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})-b\tilde{y})\big)g(\tilde{y},\tilde{t})+\eta^\circ(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+bx)M\eta(\tilde{x}-\tilde{y}).
\end{aligned}$$

We deduce

$$C(\tilde{t})+\delta \le \eta^\circ(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})+\frac{\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})-\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})-b\tilde{y})}{\beta M\eta(\tilde{x}-\tilde{y})}\|g\|_\infty,$$

and sending $b \to 0$ (and observing that $\eta(\tilde{x}-\tilde{y}) \not\to 0$ as $\sigma > 0$ and u is uniformly continuous), we find that if $\hat{t}$ is a limit point of $\tilde{t}$, then $C(\hat{t})+\delta \le C(\hat{t})$, which gives a contradiction. Hence one must have $\sigma = 0$. □

In the next lemma we show that if $E^0 \subset F^0$ are initial sets and $-\chi_E$, $-\chi_F$ are viscosity solutions of (3-1), starting from $-\chi_{E^0}$ and $-\chi_{F^0}$, respectively, then $\operatorname{dist}^\eta(\partial E(t), \partial F(t)) \ge \operatorname{dist}^\eta(\partial E^0, \partial F^0)e^{-\beta M t}$.

A splitting strategy will then extend this result to the solutions of (2-4).

Lemma 3.2. *Let η be a smooth and elliptic norm satisfying* (3-2). *Let g_1, g_2 be two admissible forcing terms satisfying assumptions* (H1), (H2) *of Section 2A, and both M-Lipschitz in the η norm. Assume*

$$g_2 - g_1 \le c < +\infty \quad \text{in } \mathbb{R}^N \times [0, +\infty). \tag{3-4}$$

Let $E^0 \subset F^0$ be two closed sets with $\operatorname{dist}^\eta(E^0, \mathbb{R}^N \setminus F^0) := \Delta > 0$. *Assume that $-\chi_{E(t)}$ is a viscosity supersolution of $u_t = \psi(\nabla u)g_1(x,t)$ starting from $-\chi_{E^0}$, and $-\chi_{F(t)}$ is a subsolution of $v_t = \psi(\nabla v)g_2(x,t)$ starting from $-\chi_{F^0}$. Then at any time $t \ge 0$,*

$$\operatorname{dist}^\eta(E(t), \mathbb{R}^N \setminus F(t)) \ge \Delta e^{-\beta M t} - c\frac{1-e^{-\beta M t}}{M}. \tag{3-5}$$

Proof. With Lemma 3.1 at hand, this is a straightforward application of standard comparison principles. We consider first $u_0(x) := -\Delta \vee (2\Delta \wedge d^\eta_E(x))$ and $v_0(x) := -2\Delta \vee (\Delta \wedge d^\eta_F(x))$, so that $v_0 + \Delta \le u_0$. These functions are both 1-Lipschitz in the norm η. We then consider the viscosity solutions u of $u_t = \psi(\nabla u)g_1(x,t)$ starting from u_0, and v of $v_t = \psi(\nabla v)g_2(x,t)$, starting from v_0. By standard comparison results, $E(t) \subseteq \{u(t) \le 0\}$ and $F(t) \supseteq \{v(t) \le 0\}$ for all $t \ge 0$.

Thanks to Lemma 3.1, $u(\,\cdot\,,t)$, $v(\,\cdot\,,t)$ are $e^{\beta M t}$-Lipschitz. Let now

$$w(\,\cdot\,,t) = v(\,\cdot\,,t) + \Delta - c\frac{e^{\beta M t}-1}{M}.$$

Then at $t = 0$, we have $w(\,\cdot\,,0) = v_0 + \Delta \le u_0$. We show that w is a subsolution of $u_t = \psi(\nabla u)g_1(x,t)$, so that $w \le u$. Indeed, if φ is a smooth test function and $(\bar{x}, \bar{t})$ is a point of maximum of $w - \varphi$, then it is a point of maximum of $v-(\varphi-\Delta+c\beta(e^{\beta Mt}-1)/M)$ so that, using (3-4) and the fact that v is a subsolution, we get

$$\partial_t\varphi(\bar{x},\bar{t})+c\beta e^{\beta M\bar{t}}\le\psi(\nabla\varphi(\bar{x},\bar{t}))g_2(\bar{x},\bar{t})\le\psi(\nabla\varphi(\bar{x},\bar{t}))g_1(\bar{x},\bar{t})+c\psi(\nabla\varphi(\bar{x},\bar{t})).$$

Since $\bar{x}$ is a contact point of the smooth function $\varphi(\,\cdot\,,\bar{t})$ and the $e^{\beta M\bar{t}}$-Lipschitz function $w(\,\cdot\,,\bar{t})$ (in the η norm), we have $\eta^\circ(\nabla\varphi)\le e^{\beta M\bar{t}}$ at $(\bar{x},\bar{t})$. By (3-2), $c\psi(\nabla\varphi(\bar{x},\bar{t}))\le c\beta e^{\beta M\bar{t}}$, whence

$$\partial_t\varphi\le\psi(\nabla\varphi)g_1$$

and this shows that w is a subsolution of this equation, and hence that $w\le u$. Therefore, for all x,t, $v(x,t)\le u(x,t)-\Delta+c(e^{\beta Mt}-1)/M$. Thus, for $t\ge 0$ and $x,y\in\mathbb{R}^N$, recalling that v is $e^{\beta M\bar{t}}$-Lipschitz,

$$v(y,t)\le u(x,t)-e^{\beta Mt}\left(\Delta e^{-\beta Mt}-c\frac{1-e^{-\beta Mt}}{M}-\eta(x-y)\right).$$

It follows that if $\operatorname{dist}^\eta(y,E(t))\le\Delta e^{-\beta Mt}-c(1-e^{-\beta Mt})/M$, then $v(y,t)\le 0$, and hence $y\in F(t)$, which shows the lemma. □

3B. *Comparison for different mobilities.* In this section we provide the crucial stability estimates with respect to varying mobilities, not necessarily smooth and elliptic.

3B1. *A comparison result with a constant forcing term.* In this subsection we shall assume that ϕ,ψ_1,ψ_2 are smooth and elliptic, and that

$$(1-\delta)\psi_2(\xi)\le\psi_1(\xi)\le(1+\delta)\psi_2(\xi)\quad\text{for all }\xi\in\mathbb{R}^N,\tag{3-6}$$

for some (small) $\delta>0$. We first show the following:

Lemma 3.3. *There exists a constant $c_0>0$ depending only on N such that the following holds: Let $\Delta>0$, and let E be a superflow for the equation $V=-\psi_1(\nu)\kappa_\phi$ and F be a subflow for the equation $V=-\psi_2(\nu)(\kappa_\phi-c_0\delta/\Delta)$, with $\operatorname{dist}^{\phi^\circ}(E(0),\mathbb{R}^N\setminus F(0))=\Delta$. Then for all t until extinction of E or F^c, we have $\operatorname{dist}^{\phi^\circ}(E(t),\mathbb{R}^N\setminus F(t))\ge\Delta$.*

Proof. We first assume that $\partial E(t)$, $\partial F(t)$ are bounded for all t.

We shall use the fact that $u(x,t)=-\chi_E(x,t)$ is a viscosity supersolution of

$$\partial_t u=\psi_1(\nabla u)\operatorname{div}\nabla\phi(\nabla u),\tag{3-7}$$

while $v(x,t)=-\chi_F(x,t)$ is a viscosity subsolution of (see Lemma 2.6)

$$\partial_t v=\psi_2(\nabla v)\Big(\operatorname{div}\nabla\phi(\nabla v)-c_0\frac{\delta}{\Delta}\Big).\tag{3-8}$$

A first remark is that since the equations are translationally invariant, we also have

$$u'(x,t)=\inf_{\phi^\circ(z)\le\Delta/4}u(x+z,t)$$

is a supersolution of (3-7), and similarly,

$$v'(x,t)=\sup_{\phi^\circ(z)\le\Delta/4} v(x+z,t)$$

is a subsolution of (3-8). Note that $u'=-\chi_{E'}$ and $v'=-\chi_{F'}$, with the tubes E', F' defined by

$$E'(t)=E(t)+W^\phi\big(0,\tfrac14\Delta\big),$$
$$\mathbb{R}^N\setminus F'(t)=(\mathbb{R}^N\setminus F(t))+W^\phi\big(0,\tfrac14\Delta\big)$$

until their respective extinction times. We denote by t^* the minimum extinction time of these sets. In particular,

$$\mathrm{dist}^{\phi^\circ}(E'(0),\mathbb{R}^N\setminus F'(0))=\tfrac12\Delta.$$

Using [Chambolle et al. 2017a, Lemma 2.6], there is a time t_0 such that for $t\le t_0$,

$$\mathrm{dist}^{\phi^\circ}(E'(t),\mathbb{R}^N\setminus F'(t))\ge\tfrac14\Delta.$$

Let $\varepsilon>0$, and consider a point $(\bar x,\bar t,\bar y,\bar s)$ (depending on ε) which attains

$$M_\varepsilon=\min_{\substack{x,\,y\in\mathbb{R}^N\\ 0\le s,\,t<t_0}}\frac1\varepsilon(1+u'(x,t)-v'(y,s))+\frac{\phi^\circ(x-y)^2}{2}+\frac{(t-s)^2}{2\varepsilon}+\frac{\varepsilon}{t_0-t}+\frac{\varepsilon}{t_0-s}. \tag{3-9}$$

Observe that for every fixed $x\in E'(0)$, $y\notin F'(0)$ and $s=t=0$, this quantity is less than

$$\frac{\phi^\circ(x-y)^2}{2}+2\frac{\varepsilon}{t_0}$$

and in particular, $M_\varepsilon\le\Delta^2/8+2\varepsilon/t_0$. If ε is small enough, one must have $1+u'(\bar x,\bar t)-v'(\bar y,\bar s)=0$, that is, $\bar x\in E'(\bar t)$ and $\bar y\notin F'(\bar s)$; hence

$$\phi^\circ(\bar x-\bar y)=\mathrm{dist}^{\phi^\circ}(E'(\bar t),\mathbb{R}^N\setminus F'(\bar s)).$$

If both $\bar t,\bar s>0$, then from [Crandall et al. 1992, Theorem 3.2] (with $\varepsilon=1$ in their notation), there exist $(N+1)\times(N+1)$ symmetric matrices

$$\widetilde X=\begin{pmatrix}X&\zeta\\ \zeta^T&\zeta_0\end{pmatrix},\quad \widetilde Y=\begin{pmatrix}Y&\eta\\ \eta^T&\eta_0\end{pmatrix} \tag{3-10}$$

such that

$$\begin{aligned}\Big(\frac{\bar s-\bar t}{\varepsilon}-\frac{\varepsilon}{(t_0-\bar t)^2},\nabla\phi^\circ(\bar y-\bar x),\widetilde X\Big)&\in\overline{\mathcal{P}^{2,-}}\frac{u'}{\varepsilon}(\bar x,\bar t),\\ \Big(\frac{\bar s-\bar t}{\varepsilon}+\frac{\varepsilon}{(t_0-\bar s)^2},\nabla\phi^\circ(\bar y-\bar x),\widetilde Y\Big)&\in\overline{\mathcal{P}^{2,+}}\frac{v'}{\varepsilon}(\bar y,\bar s),\end{aligned} \tag{3-11}$$

and such that

$$-(1+\|A\|)\mathrm{Id}\le\begin{pmatrix}-\widetilde X&0\\ 0&\widetilde Y\end{pmatrix}\le A+A^2, \tag{3-12}$$

where in (3-11) we used the standard notation for the (closed) parabolic second-order sub/superjets, see [Crandall et al. 1992], and

$$A = \begin{pmatrix} D^2\phi^\circ(\bar{x}-\bar{y}) & 0 & -D^2\phi^\circ(\bar{x}-\bar{y}) & 0 \\ 0 & 1/\varepsilon - 2\varepsilon/(t_0-\bar{t})^3 & 0 & -1/\varepsilon \\ -D^2\phi^\circ(\bar{x}-\bar{y}) & 0 & D^2\phi^\circ(\bar{x}-\bar{y}) & 0 \\ 0 & -1/\varepsilon & 0 & 1/\varepsilon - 2\varepsilon/(t_0-\bar{s})^3 \end{pmatrix}.$$

In particular, for all $\xi \in \mathbb{R}^N$, letting $\tilde{\xi} = (\xi, 0, \xi, 0) \in \mathbb{R}^{2N+2}$, from (3-12) and (3-10) we get

$$-\xi^T X \xi + \xi^T Y \xi \le \tilde{\xi}^T A \tilde{\xi} + \tilde{\xi}^T A^2 \tilde{\xi} = 0,$$

which gives the inequality

$$X \ge Y. \tag{3-13}$$

Recall that u'/ε is a supersolution and v'/ε is a subsolution. Thanks to (3-11), letting $p = \nabla\phi^\circ(\bar{y}-\bar{x})$ and $a = (\bar{s}-\bar{t})/\varepsilon$, one has

$$a - \frac{\varepsilon}{(t_0-\bar{t})^2} \ge \psi_1(p) D^2\phi(p) : X,$$
$$a + \frac{\varepsilon}{(t_0-\bar{s})^2} \le \psi_2(p)\Big(D^2\phi(p) : Y - c_0\frac{\delta}{\Delta}\Big),$$

yielding

$$0 < \frac{\varepsilon}{(t_0-\bar{t})^2} + \frac{\varepsilon}{(t_0-\bar{s})^2} \le \psi_2(p)\Big(D^2\phi(p) : Y - c_0\frac{\delta}{\Delta}\Big) - \psi_1(p) D^2\phi(p) : X. \tag{3-14}$$

Now, we observe that as $E'(\bar{t}) = E(\bar{t}) + W^\phi(0, \Delta/4)$ and (necessarily) $\bar{x} \in \partial E'(\bar{t})$, we find that (p, X) is also a subjet of $-\chi_{W^\phi(x',\Delta/4)}$ for some $x' \in E(\bar{t})$ with $\phi^\circ(\bar{x}-x') = \Delta/4$. In particular, it follows that $D^2\phi(p) : X \le 4(N-1)/\Delta$. In the same way, $D^2\phi(p) : Y \ge -4(N-1)/\Delta$ and using (3-13), we obtain

$$-4\,\frac{N-1}{\Delta} \le D^2\phi(p) : Y \le D^2\phi(p) : X \le 4\,\frac{N-1}{\Delta}. \tag{3-15}$$

Thanks to (3-6) and (3-15),

$$\begin{aligned} -\psi_1(p) D^2\phi(p) : X &\le -\psi_2(p) D^2\phi(p) : X + \delta\psi_2(p)|D^2\phi(p) : X| \\ &\le -\psi_2(p) D^2\phi(p) : X + 4(N-1)\frac{\delta}{\Delta}\psi_2(p), \end{aligned}$$

so that (3-14) and (3-13) yield

$$\begin{aligned} 0 &< \psi_2(p)\Big(D^2\phi(p) : Y - c_0\frac{\delta}{\Delta}\Big) - \psi_1(p) D^2\phi(p) : X \\ &= \psi_2(p)\Big(D^2\phi(p) : (Y-X) - c_0\frac{\delta}{\Delta}\Big) + (\psi_1(p) - \psi_2(p)) D^2\phi(p) : X \\ &\le \psi_2(p)\Big(D^2\phi(p) : (Y-X) - (c_0 - 4(N-1))\frac{\delta}{\Delta}\Big) \le 0 \end{aligned}$$

as soon as $c_0 \ge 4(N-1)$, yielding a contradiction.

We deduce that at least one of $\bar{t}$ or $\bar{s}$ is zero; without loss of generality let us assume $\bar{s} = 0$. For any $t < t_0$, thanks to (3-9) (choosing $s = t$), if ε is small enough one has

$$\tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(t), \mathbb{R}^N \setminus F'(t))^2 + 2\frac{\varepsilon}{t_0 - t} \geq \tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(\bar{t}), \mathbb{R}^N \setminus F'(0))^2 + \frac{\bar{t}^2}{2\varepsilon} + \frac{\varepsilon}{t_0 - \bar{t}} + \frac{\varepsilon}{t_0},$$

from which we see, in particular, that $\bar{t} \to 0$ as $\varepsilon \to 0$. Hence, in the limit $\varepsilon \to 0$, using also that E is closed, see [Chambolle et al. 2017a, Remark 2.3] for more details, we deduce

$$\begin{aligned}\tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(t), \mathbb{R}^N \setminus F'(t))^2 &\geq \liminf_{\bar{t}\to 0} \tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(\bar{t}), \mathbb{R}^N \setminus F'(0))^2 \\ &\geq \tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(0), \mathbb{R}^N \setminus F'(0))^2 = \tfrac{1}{8}\Delta^2,\end{aligned}$$

which shows the thesis of the lemma, until $t = t_0$ (thanks to the continuity property (b)). Starting again from t_0, we have proven the lemma for bounded sets (or sets with bounded boundary).

If $\partial E(0)$ or $\partial F(0)$ is unbounded, we proceed as follows: We first consider, for $\varepsilon > 0$, the sets

$$\begin{aligned}E_0^\varepsilon &:= E(0) + W^\phi(0, \varepsilon),\\ F_0^\varepsilon &:= \mathbb{R}^N \setminus \big((\mathbb{R}^N \setminus F(0)) + W^\phi(0, \varepsilon)\big),\end{aligned}$$

which satisfy $\operatorname{dist}^{\phi^\circ}(E_0^\varepsilon, \mathbb{R}^N \setminus F_0^\varepsilon) \geq \Delta - 2\varepsilon$.

Then, for $R > 0$, we consider the initial sets $E_0^{\varepsilon,R} = E_0^\varepsilon \cap B_R$ and $F_0^{\varepsilon,R} = F_0^\varepsilon \cap (B_R + W^\phi(0, \Delta))$. The result holds for the evolutions starting from these two sets, with the distance $\Delta - 2\varepsilon$. Hence in the limit $R \to \infty$, it must hold for the (viscosity) evolutions starting from E_0^ε and F_0^ε (which are unique for almost all ε).

By standard comparison results for discontinuous viscosity solutions [Barles 1994; Barles and Souganidis 1998; Barles et al. 1993], it then follows that the superflow E (which is also a viscosity superflow) is contained in the evolution starting from E_0^ε, while F contains the evolution starting from F_0^ε (the ε-regularization has been introduced to avoid issues due to the possible nonuniqueness of viscosity solutions).

We deduce that $\operatorname{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta - 2\varepsilon$ for all t, until extinction. Since this is true for any $\varepsilon > 0$, the lemma is proven. □

3B2. *Comparison with a nonconstant forcing term.* In this section we prove the crucial stability estimate for motions corresponding to different but close mobilities. We start with the following:

Proposition 3.4. *Assume that ϕ, ψ_1, ψ_2 are smooth and elliptic, that ψ_1, ψ_2 satisfy* (3-6), *and that $\beta > 0$ is such that*

$$\psi_2(\xi) \leq \beta\phi(\xi) \quad \textit{for all } \xi \in \mathbb{R}^N. \tag{3-16}$$

Let $E_0 \subset F_0$ be a closed and an open set, respectively, such that $\operatorname{dist}^{\phi^\circ}(E_0, \mathbb{R}^N \setminus F_0) =: \Delta > 0$, *and let E, F be a closed and an open "tube" in $\mathbb{R}^N \times [0, \infty)$, respectively, with $E(0) = E_0$, $F(0) = F_0$, such that $-\chi_E$ is a supersolution of*

$$u_t = \psi_1(\nabla u)(\operatorname{div} \nabla\phi(\nabla u) + g), \tag{3-17}$$

and $-\chi_F$ *is a subsolution of*

$$u_t = \psi_2(\nabla u)(\operatorname{div} \nabla\phi(\nabla u) + g). \tag{3-18}$$

Then,

$$\operatorname{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta e^{-\beta M t} - \delta \frac{2c_0/\Delta + \|g\|_\infty}{M}(1 - e^{-\beta M t}) \tag{3-19}$$

as long as this quantity is larger than $\Delta/2$, *where* c_0 *is as in Lemma 3.3 and* M *is the Lipschitz constant of* g *with respect to* ϕ°.

Proof. In order to obtain the estimate, we combine the results of Lemmas 3.3 and 3.2 (with $\eta = \phi^\circ$), together with a splitting result which follows from [Barles and Souganidis 1991]; see Example 1 of that paper, as well as [Barles 2006].

As before, we may need to slightly perturb the initial sets, considering rather $E_0^s = E_0 + W^\phi(0, s)$ and $F_0^s = \mathbb{R}^N \setminus (\mathbb{R}^N \setminus F_0 + W^\phi(0, s))$ for a small s (which eventually will go to 0).

Given $s > 0$ small, we start with building, for $\varepsilon > 0$ given, the motions $u^\varepsilon(x, t)$, $v^\varepsilon(x, t)$ defined as follows: We let $u^\varepsilon(x, 0) = -\chi_{E_0^s}$ and define recursively u^ε for $j \geq 0$ as a viscosity solution of

$$u_t^\varepsilon = \begin{cases} 2\psi_1(\nabla u^\varepsilon) \operatorname{div} \nabla\phi(\nabla v^\varepsilon), & 2j\varepsilon < t \leq 2j\varepsilon + \varepsilon, \\ 2\psi_1(\nabla u^\varepsilon) \displaystyle⨍_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds, & 2j\varepsilon + \varepsilon < t \leq 2(j+1)\varepsilon. \end{cases}$$

(In the case of nonuniqueness, we select for instance the smallest (super)solution, corresponding to the largest set $E^\varepsilon(t) = \{u^\varepsilon = -1\}$.) Similarly, we let $v^\varepsilon(x, 0) = -\chi_{F_0^s}$ and let $v^\varepsilon(x, t)$ be the largest (sub)solution of

$$v_t^\varepsilon = \begin{cases} 2\psi_2(\nabla v^\varepsilon)\Big(\operatorname{div} \nabla\phi(\nabla v^\varepsilon) - 2c_0 \dfrac{\delta}{\Delta}\Big), & 2j\varepsilon < t \leq 2j\varepsilon + \varepsilon, \\ 2\psi_2(\nabla v^\varepsilon)\Big(\displaystyle⨍_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds + 2c_0 \dfrac{\delta}{\Delta}\Big), & 2j\varepsilon + \varepsilon < t \leq 2(j+1)\varepsilon, \end{cases}$$

where c_0 is as in Lemma 3.3. Thanks to [Barles and Souganidis 1991; Barles 2006], as $\varepsilon \to 0$ these functions converge to the viscosity solutions of (3-17) and (3-18), respectively, starting from $-\chi_{E_0^\varepsilon}$ and $-\chi_{F_0^\varepsilon}$, provided these solutions are uniquely defined, which is known to be true for almost all ε (in fact all but a countable set of values), in which case it is also known that they are (negative of) characteristic functions.

We now show that we can estimate the distance between the corresponding geometric evolutions, using Lemmas 3.3 and 3.2.

Let δ be as in (3-6). A first observation is that, for $j \geq 0$, if we consider on the interval $[2j\varepsilon+\varepsilon, 2(j+1)\varepsilon]$ the smallest solution $\tilde{u}^\varepsilon(x, t)$ of

$$\tilde{u}_t^\varepsilon = 2\psi_2(\nabla\tilde{u}^\varepsilon)\Big(⨍_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds - \delta\|g\|_\infty\Big), \quad \tilde{u}^\varepsilon(\,\cdot\,, 2j\varepsilon + \varepsilon) = u^\varepsilon(\,\cdot\,, 2j\varepsilon + \varepsilon),$$

then, since for any $p \in \mathbb{R}^N$,

$$\psi_1(p) ⨍_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds \geq \psi_2(p) ⨍_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds - \delta\psi_2(p)\|g\|_\infty,$$

one has $\tilde{u}^\varepsilon(x,t) \le u^\varepsilon(x,t)$ for $2j\varepsilon+\varepsilon \le t \le 2(j+1)\varepsilon$, and thus $E^\varepsilon(t) \subseteq \{\tilde{u}^\varepsilon(\cdot,t)=-1\}$. Hence, Lemma 3.2 yields that for $2j\varepsilon+\varepsilon \le t \le 2(j+1)\varepsilon$,

$$\begin{aligned}\mathrm{dist}^{\phi^\circ}(E^\varepsilon(t), \mathbb{R}^N\setminus F^\varepsilon(t)) &\ge \mathrm{dist}^{\phi^\circ}\big(\{\tilde{u}^\varepsilon(\cdot,t)=-1\}, \mathbb{R}^N\setminus F^\varepsilon(t)\}\big)\\ &\ge \Big(\mathrm{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon+\varepsilon), \mathbb{R}^N\setminus F^\varepsilon(2j\varepsilon+\varepsilon)) - \frac{c}{M}\Big)e^{-2\beta M(t-2j\varepsilon-\varepsilon)} + \frac{c}{M}\end{aligned}$$

for $c=-\delta(2c_0/\Delta+\|g\|_\infty)$. Note that here we use the fact that the mobility $2\psi_2$ satisfies $2\psi_2\le 2\beta\phi$; see (3-16).

On the other hand, Lemma 3.3 yields that for all $j\ge 0$ and $2j\varepsilon\le t\le 2j\varepsilon+\varepsilon$,

$$\mathrm{dist}^{\phi^\circ}(E^\varepsilon(t), \mathbb{R}^N\setminus F^\varepsilon(t)) \ge \mathrm{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon), \mathbb{R}^N\setminus F^\varepsilon(2j\varepsilon))$$

as long as $\mathrm{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon), \mathbb{R}^N\setminus F^\varepsilon(2j\varepsilon)) \ge \Delta/2$.

In particular, setting $d_j = \mathrm{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon), \mathbb{R}^N\setminus F^\varepsilon(2j\varepsilon))$, one obtains by induction that

$$d_{j+1} \ge \Big(d_j - \frac{c}{M}\Big)e^{-2\beta M\varepsilon} + \frac{c}{M} \ge \Big(d_0 - \frac{c}{M}\Big)e^{-2\beta M(j+1)\varepsilon} + \frac{c}{M},$$

as long as $d_j\ge\Delta/2$. In the limit, we find that, letting $E^s(t)=\{u(\cdot,t)=-1\}$ and $F^s(t)=\{v(\cdot,t)=-1\}$ and recalling that $\mathrm{dist}^{\phi^\circ}(E^s_0, \mathbb{R}^N\setminus F^s_0)\ge \Delta-2s$,

$$\mathrm{dist}^{\phi^\circ}(E^s(t), \mathbb{R}^N\setminus F^s(t)) \ge (\Delta-2s)e^{-\beta Mt} - \delta\frac{2c_0/\Delta+\|g\|_\infty}{M}(1-e^{-\beta Mt})$$

as long as this quantity is larger than $\Delta/2$.

By comparison, it is clear that $E\subset E^s$ and $F^s\subset F$; hence (letting eventually $s\to 0$), we deduce that (3-19) holds as long as the right-hand side is larger than $\Delta/2$. □

We are now ready to state and prove the main result of the section.

Theorem 3.5. *Let ψ_1, ψ_2 and ϕ satisfy* (3-6) *and* (3-16). *Assume also that ψ_1, ψ_2 are ϕ-regular in the sense of Definition 2.7. Let the forcing term $g(x,t)$ be continuous, bounded, and spatially M-Lipschitz continuous with respect to the distance ϕ°, and denote by E a superflow for $V=-\psi_1(\nu)(\kappa_\phi+g)$ and by F a subflow for $V=-\psi_2(\nu)(\kappa_\phi+g)$, both in the sense of Definition 2.2. Finally, assume that* $\mathrm{dist}^{\phi^\circ}(E(0), \mathbb{R}^N\setminus F(0))\ge\Delta>0$. *Then for all t,*

$$\mathrm{dist}^{\phi^\circ}(E(t), \mathbb{R}^N\setminus F(t)) \ge \Delta e^{-\beta Mt} - \delta\frac{2c_0/\Delta+\|g\|_\infty}{M}(1-e^{-M\beta t}) \tag{3-20}$$

as long as this quantity is larger than $\Delta/2$.

Proof. Consider smooth, elliptic approximations of ψ_i $(i=1,2)$, ϕ, denoted by ψ_i^n, ϕ^n, such that (3-6)–(3-16) hold also for ψ_i^n, ϕ^n (with slightly larger constants δ and β that, with a small abuse of notation, will not be relabeled) and with $\psi_i^n-\varepsilon\phi^n$ convex $(i=1,2)$, that is, ψ_i^n are uniformly ϕ^n-regular (see the statement of Theorem 2.8).

Consider as before, for $s>0$ small, the initial sets $E_0^s := E_0 + W^{\phi^n}(0,s)$ and $F_0^s := \mathbb{R}^N\setminus[(\mathbb{R}^N\setminus F_0) + W^{\phi^n}(0,s)]$. As in Theorem 2.8 we can build subflows A_n^s and superflows B_n^s for the evolution $V = -\psi_1^n(\nu)(\kappa_{\phi^n}+g)$, both starting from E_0^s, such that $A_n^s\subset B_n^s$, and a subflow $A_n'^s$ and superflow $B_n'^s$ for the evolution $V = -\psi_2^n(\nu)(\kappa_{\phi^n} + g)$, both starting from F_0^s, such that $A_n'^s \subset B_n'^s$. Thanks to Lemma 2.6, $-\chi_{B_n^s}$ is a viscosity supersolution and $-\chi_{A_n'^s}$ is a viscosity subsolution, so that we can apply Proposition 3.4 and estimate their $(\phi^n)^\circ$-distance according to (3-19).

Again thanks to Theorem 2.8, $\mathbb{R}^N \setminus A_n^s$ converges in the Kuratowski sense as $n \to \infty$ to the complement of a subflow, which contains E thanks to Theorem 2.3, and analogously $B_n'^s$ converges to a superflow contained in F. We deduce (3-20), letting $s \to 0$. □

3C. ***Existence and uniqueness by approximation.*** We recall that the existence theory for level-set flows (in the sense of Definition 2.4) that we have so far works only for ϕ-regular mobilities. The goal of this section is to extend the existence theory to general mobilities. To this aim, we consider the following notion of *solution via approximation*:

Definition 3.6 (level-set flows via approximation). Let ψ be a mobility, g an admissible forcing term and u^0 a uniformly continuous function on $\mathbb{R}^N$.

We will say that a continuous function $u^\psi : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *solution via approximation* to the level-set flow corresponding to (1-1), with initial datum u^0, if $u^\psi(\cdot, 0) = u^0$ and if there exists a sequence $\{\psi_n\}$ of ϕ-regular mobilities such that $\psi_n \to \psi$ and, denoting by u^{ψ_n} the unique solution to (1-1) (in the sense of Definition 2.4) with mobility ψ_n and initial datum u^0, we have $u^{\psi_n} \to u^\psi$ locally uniformly in $\mathbb{R}^N \times [0, +\infty)$.

The next theorem is the main result of this section: it shows that for any mobility ψ, a solution via approximation u^ψ in the sense of the previous definition always exists; such a solution is also unique in that it is independent of the choice of the approximating sequence of ϕ-regular mobilities $\{\psi_n\}$. In particular, in the case of a ϕ-regular mobility, the notion of solution via approximation is consistent with that of Definition 2.4.

Theorem 3.7. *Let ψ, g, and u^0 be as in Definition 3.6. Then, there exists a unique solution u^ψ in the sense of Definition 3.6 with initial datum u^0.*

Proof. We have to prove that for any sequence $\{\psi_n\}$ of ϕ-regular mobilities such that $\psi_n \to \psi$, the corresponding solutions u^{ψ_n} to (1-1) with initial datum u^0 converge to some function u locally uniformly in $\mathbb{R}^N \times [0, +\infty)$. We split the proof of the theorem into two steps.

Step 1. Let β be as in (3-16). Let $T_0 > 0$ be defined by $e^{-2\beta M T_0} = \frac{3}{4}$, where as usual M is the spatial Lipschitz constant of the forcing term g with respect to the distance induced by ϕ°. We claim that for every $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\|u^{\psi_n} - u^{\psi_m}\|_{L^\infty(\mathbb{R}^N \times [0,T_0])} \le \varepsilon \quad \text{for all } n, m \ge \bar{n}. \tag{3-21}$$

To this aim, we observe that since $\psi_n \to \psi$, for n large enough

$$\psi_n(\xi) \le 2\beta\phi(\xi) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{3-22}$$

and there exists $\delta_j \to 0$ such that

$$(1 - \delta_j)\psi_n \le \psi_m \le (1 + \delta_j)\psi_n \quad \text{for all } m, n \ge j. \tag{3-23}$$

Set $E_\lambda^{\psi_n}(t) := \{u^{\psi_n}(\,\cdot\,,t) \le \lambda\}$, $F_\lambda^{\psi_n}(t) := \{u^{\psi_n}(\,\cdot\,,t) < \lambda\}$ and recall that $E_\lambda^{\psi_n}$ is a superflow, while $F_\lambda^{\psi_n}$ is a subflow in the sense of Definition 2.2.

Let ω be a modulus of continuity for u^0 with respect to ϕ° and recall that for any $\lambda \in \mathbb{R}$

$$\mathrm{dist}^{\phi^\circ}(E_\lambda^{\psi_m}(0), \mathbb{R}^N \setminus F_{\lambda+\varepsilon}^{\psi_n}(0)) = \mathrm{dist}^{\phi^\circ}(\{u^0 \le \lambda\}, \{u^0 \ge \lambda+\varepsilon\}) \ge \omega^{-1}(\varepsilon).$$

By (3-22), (3-23) and Theorem 3.5, for all $n, m \ge j$ we have

$$\mathrm{dist}^{\phi^\circ}(E_\lambda^{\psi_m}(t), \mathbb{R}^N \setminus F_{\lambda+\varepsilon}^{\psi_n}(t)) \ge \omega^{-1}(\varepsilon)e^{-2\beta Mt} - \delta_j \frac{2c_0/\omega^{-1}(\varepsilon) + \|g\|_\infty}{M}(1 - e^{-2\beta Mt}),$$

as long as the right-hand side is larger than $\omega^{-1}(\varepsilon)/2$, that is, for all $t \in [0, T_0]$, provided j is large enough. In particular, for n, m large enough $E_\lambda^{\psi_m}(t) \subset F_{\lambda+\varepsilon}^{\psi_n}(t)$ for all $t \in [0, T_0]$, which yields

$$u^{\psi_n}(\,\cdot\,,t) \le u^{\psi_m}(\,\cdot\,,t) + \varepsilon \quad \text{for all } t \in [0, T_0].$$

By switching the roles of n and m we deduce (3-21).

Step 2. First arguing as in the proof of Theorem 2.8 and using (3-22) we see that $\omega(e^{2\beta Mt}\,\cdot)$ is a spatial modulus of continuity for $u^{\psi_n}(\,\cdot\,,t)$ for all n. Observe that from (3-21) it follows that for n, m large enough we have

$$E_\lambda^{\psi_m}(T_0) \subseteq E_{\lambda+\varepsilon}^{\psi_n}(T_0),$$

which in turn implies

$$\mathrm{dist}^{\phi^\circ}(E_\lambda^{\psi_m}(T_0), \mathbb{R}^N \setminus F_{\lambda+2\varepsilon}^{\psi_n}(T_0)) \ge \mathrm{dist}^{\phi^\circ}(E_{\lambda+\varepsilon}^{\psi_n}(T_0), \mathbb{R}^N \setminus F_{\lambda+2\varepsilon}^{\psi_n}(T_0)) \ge \omega^{-1}(e^{2\beta MT_0}\varepsilon).$$

We can now argue as in Step 1 to conclude that, for n, m large enough,

$$\|u^{\psi_n} - u^{\psi_m}\|_{L^\infty(\mathbb{R}^N \times [T_0, 2T_0])} \le 2\varepsilon.$$

Therefore, by an easy iteration argument we can show that, for every given $T > 0$, the sequence $\{u^{\psi_n}\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^N \times [0, T])$. □

We conclude by recalling the following remarks, referring to [Chambolle et al. 2017a] for the details.

Remark 3.8 (stability). As a byproduct of the previous theorem, and a standard diagonalization argument, we have the following stability property for solutions to (1-1): Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of mobilities and ϕ_n a sequence of anisotropies such that $\psi_n \to \psi$ and $\phi_n \to \phi$ as $n \to +\infty$. Then u^{ψ_n} converge to u^ψ locally uniformly in $\mathbb{R}^N \times [0, +\infty)$ as $h \to 0$ (where u^{ψ_n} is the solution to (1-1) with ψ replaced by ψ_n and ϕ replaced by ϕ_n).

Remark 3.9 (comparison with the Giga–Požár solution). When ϕ is purely crystalline and $g \equiv c$ for some $c \in \mathbb{R}$, the unique level-set solution in the sense of Definition 3.6 coincides with the viscosity solution constructed in [Giga and Požár 2016; 2018].

We also recall that when g is constant, (1-1) admits a phase-field approximation by means of the anisotropic Allen–Cahn equation; see [Chambolle et al. 2017a, Remark 6.2] for the details.

In the next theorem we recall the main properties of the level-set solutions introduced in Definition 3.6. In the statement of the theorem, we will say that a uniformly continuous initial function u^0 is *well-prepared* at $\lambda \in \mathbb{R}$ if the following two conditions hold:

(a) If $H \subset \mathbb{R}^N$ is a closed set such that $\mathrm{dist}(H, \{u_0 \geq \lambda\}) > 0$, then there exists $\lambda' < \lambda$ such that $H \subseteq \{u_0 < \lambda'\}$.

(b) If $A \subset \mathbb{R}^N$ is an open set such that $\mathrm{dist}(\{u_0 \leq \lambda\}, \mathbb{R}^N \setminus A) > 0$, then there exists $\lambda' > \lambda$ such that $\{u_0 \leq \lambda'\} \subset A$.

Here $\mathrm{dist}(\,\cdot\,,\cdot\,)$ denotes the distance function with respect to a given norm. Clearly, the properties stated in (a) and (b) above do not depend on the choice of such a norm.

Remark 3.10. Note that the above assumption of well-preparedness is automatically satisfied if the set $\{u_0 \leq \lambda\}$ is bounded.

Theorem 3.11 (properties of the level-set flow). *Let u^ψ be a solution in the sense of Definition 3.6, with initial datum u^0. The following properties hold true*:

(i) (*nonfattening of level sets*) *There exists a countable set $N \subset \mathbb{R}$ such that for all $t \in [0, +\infty)$, $\lambda \notin N$,*

$$\begin{aligned} \{(x,t) : u^\psi(x,t) < \lambda\} &= \mathrm{Int}(\{(x,t) : u^\psi(x,t) \leq \lambda\}), \\ \overline{\{(x,t) : u^\psi(x,t) < \lambda\}} &= \{(x,t) : u^\psi(x,t) \leq \lambda\}. \end{aligned} \tag{3-24}$$

(ii) (*distributional formulation when ψ is ϕ-regular*) *If ψ is ϕ-regular, then u^ψ coincides with the distributional solution in the sense of Definition 2.4.*

(iii) (*comparison*) *Assume that $u^0 \leq v^0$ and denote the corresponding level-set flows by u^ψ and v^ψ, respectively. Then $u^\psi \leq v^\psi$.*

(iv) (*geometricity*) *Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and uniformly continuous. Then u^ψ is a solution with initial datum u^0 if and only if $f \circ u^\psi$ is a solution with initial datum $f \circ u^0$.*

(v) (*independence of the initial level-set function*) *Assume that u^0 and v^0 are well-prepared at λ. If $\{u^0 < \lambda\} = \{v^0 < \lambda\}$, then $\{u^\psi(\,\cdot\,, t) < \lambda\} = \{v^\psi(\,\cdot\,, t) < \lambda\}$ for all $t > 0$. Analogously, if $\{u^0 \leq \lambda\} = \{v^0 \leq \lambda\}$, then $\{u^\psi(\,\cdot\,, t) \leq \lambda\} = \{v^\psi(\,\cdot\,, t) \leq \lambda\}$ for all $t > 0$.*

For the proof we refer to [Chambolle et al. 2017a, Theorem 5.9].

We conclude with a remark about conditions that prevent the occurrence of fattening.

Remark 3.12 (star-shaped sets, convex sets and graphs). It is well-known [Soner 1993, Section 9] that for the motion without forcing, strictly star-shaped sets do not develop fattening so that, in particular, their evolution is unique. The proof of this fact, given for instance in [Soner 1993] for the mean curvature flow, works also for solutions in the sense of Definition 2.2 when the mobility ψ is ϕ-regular, and in turn, by approximation, also for the *generalized motion* associated to level-set solutions in the sense of Definition 3.6, when ψ is general. Uniqueness also holds for motions with a time-dependent forcing $g(t)$ [Bellettini et al. 2009, Theorem 5] as long as the set remains strictly star-shaped. This remark obviously applies to initial convex sets, which, in addition, remain convex for all times, as was shown in [Bellettini et al. 2006; 2009; Caselles and Chambolle 2006] with a spatially constant forcing term.[2] The case of unbounded initial convex sets was not considered in these references but can be easily addressed by approximation (and uniqueness still holds with the same proof).

In the same way, if the initial set $E_0 = \{x_N \leq v^0(x_1, \dots, x_{N-1})\}$ is the subgraph of a uniformly continuous function v^0, and the forcing term does not depend on x_N, then one can show that fattening does not develop and $E(t)$ is still the subgraph of a uniformly continuous function for all $t > 0$, as in the classical case [Ecker and Huisken 1989; Evans and Spruck 1992b]; see also [Giga and Giga 1998] for the two-dimensional crystalline case.

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GLOBAL WEAK SOLUTIONS OF THE TEICHMÜLLER HARMONIC MAP FLOW INTO GENERAL TARGETS

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We analyse finite-time singularities of the Teichmüller harmonic map flow — a natural gradient flow of the harmonic map energy — and find a canonical way of flowing beyond them in order to construct global solutions in full generality. Moreover, we prove a no-loss-of-topology result at finite time, which completes the proof that this flow decomposes an arbitrary map into a collection of branched minimal immersions connected by curves.

1. Introduction

The Teichmüller harmonic map flow is a gradient flow of the harmonic map energy that evolves a given map $u_0 : M \to N$ from a closed oriented surface M of arbitrary genus $\gamma \geq 0$ into a closed target manifold N of arbitrary dimension, and simultaneously evolves the domain metric on M within the class of constant curvature metrics. It tries to evolve u_0 to a branched minimal immersion — a critical point of the energy functional in this situation — but in general there is no such immersion homotopic to u_0, so something more complicated must occur.

The development of the theory so far has suggested that the flow should instead decompose u_0 into a *collection* of branched minimal immersions from lower-genus surfaces. This paper provides the remaining part of the jigsaw in order to prove this in full generality, by analysing the finite-time singularities that may occur, finding a canonical way of flowing beyond them, and analysing their fine structure in order to prove that no topology is lost except by the creation of additional branched minimal immersions and connecting curves. The resulting global generalised solution will have at most finitely many singular times, together, possibly, with singular behaviour at infinite time that was analysed in [Rupflin and Topping 2016; Rupflin et al. 2013; Huxol et al. 2016].

Consider the harmonic map energy

$$E(u, g) = \frac{1}{2} \int_M |du|_g^2 \, dv_g$$

acting on a sufficiently regular map $u : M \to (N, g_N)$, and a metric g in the space $\mathcal{M}_c$ of constant (Gauss-) curvature -1, 0 or 1 (depending on the genus) metrics on M with fixed unit area in the case that the curvature is 0. Critical points are weakly conformal harmonic maps $u : (M, g) \to (N, g_N)$, which are then

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branched minimal immersions [Gulliver et al. 1973] (allowing constant maps in addition). The gradient flow, introduced in [Rupflin and Topping 2016], can be written with respect to a fixed parameter $\eta > 0$ as

$$\frac{\partial u}{\partial t} = \tau_g(u); \qquad \frac{\partial g}{\partial t} = \tfrac{1}{4}\eta^2 \operatorname{Re}(P_g(\Phi(u, g))), \tag{1-1}$$

where $\tau_g(u) = \operatorname{tr}_g(\nabla_g du)$ denotes the tension field of u, P_g represents the L^2-orthogonal projection from the space of quadratic differentials on (M, g) onto the space $\mathcal{H}(M, g)$ of *holomorphic* quadratic differentials, and $\Phi(u, g)$ is the Hopf differential. The flow decreases the energy $E(t) := E(u(t), g(t))$ according to

$$\begin{aligned}\frac{dE}{dt} &= -\int_M \big[|\tau_g(u)|^2 + \big(\tfrac{1}{4}\eta\big)^2 |\operatorname{Re}(P_g(\Phi(u, g)))|^2\big] \\ &= -\|\partial_t u\|_{L^2}^2 - \frac{1}{\eta^2}\|\partial_t g\|_{L^2}^2 \\ &= -\|\tau_g(u)\|_{L^2}^2 - \tfrac{1}{32}\eta^2 \|P_g(\Phi(u, g))\|_{L^2}^2,\end{aligned} \tag{1-2}$$

where we use that $\|P_g(\Phi(u, g))\|_{L^2}^2 = 2\| \operatorname{Re}(P_g(\Phi(u, g))\|_{L^2}^2$. We refer to [Rupflin and Topping 2016] for further details.

When the genus γ of M is zero, there are no nonvanishing holomorphic quadratic differentials, so g remains fixed, and we recover the harmonic map flow [Eells and Sampson 1964], which has been studied in detail for two-dimensional domains; see [Struwe 1985; Topping 2004, Theorem 1.6] and the references therein. In the case that $\gamma = 1$, this flow can be shown to be equivalent to a flow of Ding, Li and Liu [Ding et al. 2006], as pointed out in [Rupflin and Topping 2016], and analysed in [Ding et al. 2006; Huxol et al. 2016].

1.1. ***Construction of a global flow.*** In both cases $\gamma = 0$ and $\gamma = 1$, one obtains global weak solutions starting with any initial map $u_0 \in H^1(M, N)$ and any initial metric $g_0 \in \mathcal{M}_c$ [Struwe 1985; Ding et al. 2006]. For $\gamma \geq 2$ it was shown in [Rupflin 2014] that a weak solution exists on a time interval $[0, T)$ for some $T \in (0, \infty]$, and if $T < \infty$ then the domain must degenerate in the sense that the injectivity radius of (M, g) must approach zero as $t \uparrow T$. In all these cases the flow will be smooth away from finitely many times and, as time increases to a singular time, the map u splits off one or more (but finitely many) nonconstant harmonic 2-spheres, which will then automatically be branched minimal spheres (see, e.g., [Eells and Lemaire 1978, (10.6)] for this latter fact) as bubbling occurs. At each such singular time τ, the continuation of this weak solution is constructed by taking a (unique) limit $(u(\tau), g(\tau)) \in H^1(M, N) \times \mathcal{M}_c$ as $t \uparrow \tau$ and continuing the flow past the singular time by restarting with $(u(\tau), g(\tau))$ as new initial data. This process gives a unique flow within the class of weak solutions with nonincreasing energy. It was shown in [Ding and Tian 1995; Topping 2004, Theorem 1.6] that for the harmonic map flow, and in particular for the case $\gamma = 0$ above, we have no loss of energy and precise control on the bubble scales at these singular times. A very similar argument establishes the same properties for all genera γ, and the case $\gamma \geq 2$ even follows directly from Proposition 3.3 below, which we need for other reasons. The upshot of this singularity analysis is that the flow map before a singular time can be reconstructed from the flow map after the singular time together with the branched minimal spheres representing the bubbles.

Whenever a *global* weak solution of (1-1) exists, i.e., when $T=\infty$ for $\gamma \geq 2$, and in all cases for $\gamma = 0, 1$, it was shown in [Rupflin and Topping 2016; Rupflin et al. 2013; Huxol et al. 2016], see also [Ding et al. 2006; Struwe 1985], that either the flow subconverges to a branched minimal immersion, or it subconverges to a *collection* of branched minimal immersions. This collection may consist partly of bubbles, and it may include a limit branched minimal immersion parametrised over the original domain, but in general, for $\gamma \geq 2$, the domain can split into a collection of lower-genus closed surfaces, and the map converges to a branched minimal immersion on some or all of these lower-genus surfaces. The way the domain surface can split into lower-genus surfaces is described by the classical Deligne–Mumford-type description of how hyperbolic surfaces can degenerate; see, e.g., [Rupflin and Topping 2018a, Theorem A.4]. In particular, when the domain splits, the length of the shortest closed geodesic in the domain will shrink to zero and so-called collar regions around such shrinking geodesics, described by the collar lemma of Keen and Randol [Randol 1979], see, e.g., [Rupflin and Topping 2018a, Lemma A.1], will degenerate. In all cases, if one is careful to capture all bubbles, including those disappearing down any degenerating collars, it was shown in [Huxol et al. 2016] that *all* energy in the limit is accounted for by branched minimal immersions from closed surfaces. The upshot of this asymptotic analysis is that when a global weak solution exists, for a domain of arbitrary genus, the map $u(t)$ can be reconstructed from the branched minimal immersions we find, connected together with curves. (See [Huxol et al. 2016] for precise statements.)

The theory above leaves open the possibility of the flow stopping in finite time in the case $\gamma \geq 2$ if it happens that the injectivity radius of the domain converges to zero, i.e., we have collar degeneration as above but in *finite* time. We showed in [Rupflin and Topping 2018b] that the flow exists and is smooth for all time in the case that the target (N, g_N) has nonpositive sectional curvature, mirroring the seminal work [Eells and Sampson 1964] (although the asymptotic behaviour is more elaborate in our situation, with infinite time singularities reflecting the more complicated structure of the space of critical points). However, in the case of general targets, the theory above has the major omission that the existence time T for $\gamma \geq 2$ could be finite, and by such time we cannot expect the flow to have decomposed $u(t)$ into branched minimal immersions. The existence of solutions of variants of Teichmüller harmonic map flow that degenerate in finite time is proved in [Robertson and Rupflin 2018].

In this paper we show how the flow can be continued in a canonical fashion when this domain degeneration occurs, with the continuation being a finite collection of new flows. By repeating this process a finite number of times, we arrive at a global solution that is smooth except at finitely many singular times. Moreover, our analysis of the collar degeneration singularity allows us to account for all "lost topology" at the singular time in terms of branched minimal spheres, some of which may be conventional bubbles, together with connecting curves, despite the tension field diverging to infinity in general. Combined with the earlier work described above, a consequence is that the flow realises the following:

Any smooth map $u_0 : M \to (N, g_N)$ is decomposed by the flow (1-1) *into a finite collection of branched minimal immersions $v_i : \Sigma_i \to (N, g_N)$ from closed Riemann surfaces $\{\Sigma_i\}$ of total genus no more than γ. The original M can be reconstructed from the surfaces $\{\Sigma_i\}$ by removing a finite collection of pairs of tiny discs in $\coprod_i \Sigma_i$ and gluing in cylinders. The map u_0*

is homotopic to the corresponding combination of the $\{v_i\}$ *together with connecting curves on the glued-in cylinders.*

For other situations in which maps are decomposed into collections of minimal objects, see [Meeks et al. 1982; Hass and Scott 1988], for example.

In order to make a continuation of the flow, we require the following basic description of the convergence of the flow as we approach a finite-time singularity. This can be applied to a weak solution (including bubbling) by restricting to a short time interval just prior to a time when the injectivity radius drops to zero, thus avoiding the bubbling and allowing us to consider a smooth flow for simplicity. A far more refined description will be required later in order to ensure that the continuation after the singularity properly reflects the flow just before.

Theorem 1.1. *Let* M *be any closed oriented surface of genus* $\gamma \geq 2$ *and let* (N, g_N) *be any smooth closed Riemannian manifold. Let* (u, g) *be a smooth solution of* (1-1) *defined on a time interval* $[0, T)$ *with* $T < \infty$ *that is maximal in the sense that*

$$\liminf_{t \uparrow T} \operatorname{inj}_{g(t)}(M) = 0. \tag{1-3}$$

Then the following properties hold:

(1) *The "pinching set"* $F \subset M$ *defined by*

$$F := \big\{ p \in M : \liminf_{t \uparrow T} \operatorname{inj}_{g(t)}(p) = 0 \big\} \tag{1-4}$$

is nonempty and closed, and its complement $\mathcal{U} := M \setminus F$ *is nonempty and supports a complete hyperbolic metric* h *with finite volume and cusp ends, so that* $(\mathcal{U}, h)$ *is conformally equivalent to a finite disjoint union of closed Riemann surfaces* M_i *with finitely many punctures and genus strictly less than that of* M, *and so that*

$$g(t) \to h \quad \text{smoothly locally on } \mathcal{U} \text{ as } t \uparrow T.$$

(2) *The "bubble" set*

$$S := \big\{ p \in \mathcal{U} : \text{there exists } \varepsilon > 0 \text{ such that } \limsup_{t \uparrow T} E(u(t), g(t), V) \geq \varepsilon \quad \text{for all neighbourhoods } V \subset M \text{ of } p \big\} \tag{1-5}$$

is a finite set, and there exists a smooth map $\bar{u} : \mathcal{U} \setminus S \to N$, *with* $\bar{u} \in H^1(\mathcal{U}, h, N)$, *such that*

$$u(t) \to \bar{u} \quad \text{as } t \uparrow T$$

smoothly locally in $\mathcal{U} \setminus S$ *and weakly locally in* H^1 *on* $\mathcal{U}$. *Moreover*, $\bar{u}$ *extends to a collection of maps* $u_i \in H^1(M_i, N)$.

The convergence of the metric $g(t)$ here should be contrasted with the convergence of a sequence $g(t_n)$, with $t_n \uparrow T$, that could be deduced from the differential geometric form of Deligne–Mumford compactness; see, e.g., [Rupflin and Topping 2018a, Theorem A.4]. Our convergence is uniform in time, and does not require modification by diffeomorphisms.

This theorem already tells us enough to be able to define the continuation of the flow beyond time T. We simply take each closed Riemann surface M_i, equip it with a conformal metric g_i in the corresponding space $\mathcal{M}_c$ of metrics of constant curvature, and restart the flow on each M_i separately with u_i as the initial map. The choice of g_i is uniquely determined when the genus of M_i is at least 1, but on the sphere it is initially defined only up to pull-back by Möbius maps. In this case, we must find a way of making a canonical choice of g_i in order to obtain a canonical choice of continuation. We do this by returning to the limit metric h, which induces a smooth conformal complete hyperbolic metric of finite area on the sphere with punctures, and choose the metric g_i to be the limit g_∞ of the rescaled Ricci flow on the sphere that starts with the metric h, as given by the following theorem which follows immediately from a combination of [Topping 2012, Theorem 1.2] (see also the simplifications arising from [Topping and Yin 2017]) and [Hamilton 1988; Chow 1991] (see also [Giesen and Topping 2011]). Note that by Gauss–Bonnet, the volume of the metric h must be $2\pi(n-2)$, where n is the number of punctures.

Theorem 1.2. *Suppose $\{p_1, \dots, p_n\} \subset S^2$ is a finite set of points and h is a complete conformal hyperbolic metric on $S^2 \setminus \{p_1, \dots, p_n\}$. Then there exists a unique smooth Ricci flow $g(t)$ on S^2, $t \in (0, T)$, $T = \frac{1}{4}(n-2)$, i.e., a smooth complete solution of $\partial g/\partial t = -2Kg$ with curvature uniformly bounded below and such that $g(t) \to h$ smoothly locally on $S^2 \setminus \{p_1, \dots, p_n\}$ as $t \downarrow 0$. (Here K is the Gauss curvature.) Moreover, there exists a smooth conformal metric g_∞ on S^2 of constant Gauss curvature 1 such that $g(t)/(2(T-t)) \to g_\infty$ smoothly as $t \uparrow T$.*

Theorem 1.1, with the aid of Theorem 1.2, thus establishes that our flow can be continued canonically beyond the singular time T as a finite collection of flows. The construction does not require us to stop prior to the singular time T and perform surgery. Instead, we flow right to the singular time, and the surgery we do consists of nothing more than adding points to fill in punctures in the domain (the analogue of adding an arbitrary cap in a traditional surgery argument).

1.2. *No loss of information at finite-time collar degenerations.* At this stage we have however not yet established a very strong connection between the flow prior to a collar degeneration singularity and the flows after the singularity. We need to relate the topology of M to the topology of the surfaces M_i, and to relate the topology of the map $u(t)$ prior to the singularity to the flow maps afterwards, and most of this paper will be devoted to achieving this. The former issue is dealt with by the following:

Proposition 1.3. *In the setting of Theorem 1.1, the injectivity radius converges* ***uniformly*** *to a continuous limit:*

$$\operatorname{inj}_{g(t)}(x) \to \begin{cases} \operatorname{inj}_h(x) & \text{for } x \in \mathcal{U}, \\ 0 & \text{for } x \in F = M \setminus \mathcal{U} \end{cases} \tag{1-6}$$

as $t \uparrow T$. Moreover, the set F from (1-4) consists of $k \in \{1, \dots, 3(\gamma-1)\}$ components $\{F_j\}$, and the total number of punctures in Theorem 1.1 is $2k$.

Furthermore, there exist $\delta_0 \in (0, \operatorname{arsinh}(1))$ and $t_0 \in [0, T)$ such that for every $t \in [t_0, T)$ there are exactly k simple closed geodesics $\sigma_j(t) \subset (M, g(t))$ with length $\ell_j(t) = L_{g(t)}(\sigma_j(t)) < 2\delta_0$ and the lengths of these geodesics decay according to

$$\ell_j(t) \le C(T-t)(E(t) - E(T)) \to 0 \quad \text{as } t \uparrow T \tag{1-7}$$

for some $C = C(\eta, \gamma)$. In addition, for every $\delta \in (0, \delta_0]$ and $t \in [t_0, T)$ the set δ-thin$(M, g(t))$ consists of the union of the (possibly empty) disjoint cylindrical "subcollar" regions $\mathcal{C}_j = \mathcal{C}_j(t, \delta)$ around $\sigma_j(t)$ which are isometric to

$$(-X_j, X_j) \times S^1 \text{ equipped with the metric } \rho_j^2(s)(ds^2 + d\theta^2) \tag{1-8}$$

where

$$X_j = X_j(t, \delta) = \frac{2\pi}{\ell_j(t)} \arccos\left(\frac{\sinh(\ell_j(t)/2)}{\sinh \delta}\right) \quad \text{if } 2\delta \geq \ell_j(t), \qquad \text{while } X_j = 0 \quad \text{if } 2\delta < \ell_j(t) \tag{1-9}$$

and

$$\rho_j(s) = \rho_{\ell_j(t)}(s) = \frac{\ell_j(t)}{2\pi \cos(\ell_j(t)s/(2\pi))},$$

and for all t sufficiently large (depending in particular on δ) we have $F_j \subset \mathcal{C}_j(t, \delta)$.

The subcollars $\mathcal{C}_j$ are subsets of collar neighbourhoods of the collapsing simple closed geodesics described by the collar lemma; see, e.g., [Rupflin and Topping 2018a, Lemma A.1]. If $\delta(t) \downarrow 0$ sufficiently slowly so that $\delta(t)^{-1}(T - t)(E(t) - E(T)) \to 0$ as $t \uparrow T$, then $X_j(t, \delta(t)) \to \infty$ as $t \uparrow T$.

This proposition gives us a topological description of how M can be reconstructed from the M_i. We remove $2k$ small discs from the M_i at the punctures described in Theorem 1.1, and glue in cylinders corresponding to the k collar regions from the proposition. (We see that there will be $2k$ punctures.)

The proposition also demonstrates what we must establish in order to relate the flow map before the singularity to the flow maps after the singularity. The continuation of the flow is given in terms of the smooth local limit $\bar{u}$ on $\mathcal{U} \setminus S$ from Theorem 1.1. Therefore we can potentially lose parts of the map at the points S and parts of the map "at infinity" in $\mathcal{U}$. As we shall see in part (1) of Theorem 1.4, the loss of energy at points in S is entirely accounted for in terms of bubbles, i.e., maps $\omega_i : S^2 \to N$ that are harmonic and nonconstant and are thus themselves branched minimal spheres.

On the other hand, we have to be concerned about parts of the map that are lost at infinity in $\mathcal{U}$. By Proposition 1.3, we must specifically be concerned with the restriction of the flow map $u(t)$ to the collar regions $\mathcal{C}_j$. If we view these collar regions conformally as the cylinder (1-8) with the flat metric $g_0 = ds^2 + d\theta^2$, then any fixed-length portion of an end of these cylinders will have injectivity radius $\mathrm{inj}_{g(t)}$ bounded below by a positive number, uniformly as $t \uparrow T$, and thus by Proposition 1.3, it will remain in a compact subset of $\mathcal{U}$ and the map there will be captured in the limit $\bar{u}$. However, this says nothing about what happens away from the ends of the cylinders, and we have to be concerned because the map there need not become harmonic since the tension field is a priori unbounded in L^2. Nevertheless, part (3) of Theorem 1.4 will show that near enough the centre of these cylinders — essentially on the $[T-t]^{1/2}$-thin part of $(M, g(t))$ — we will be able to describe the map as a collection of bubbles connected together by curves.

This leaves the worry that a little outside this thin region (for example where the injectivity radius is of the order of $[T-t]^{1/2-\varepsilon}$) we might accumulate "unstructured" energy that is lost down the collars in the limit, and does not represent any branched minimal immersion or curve, but instead represents some arbitrary map. Again, this is ruled out in the following Theorem 1.4, part (2), where we show that all lost energy lives not just on the $[T-t]^{1/2}$-thin part but even on the $[T-t]$-thin part.

Theorem 1.4. *In the setting of Theorem 1.1, we can extract a finite collection of branched minimal spheres at the singular time in order to obtain no loss of energy/topology in the following sense. There exists a sequence $t_n \uparrow T$ such that*

$$\big[\|\tau_g(u)(t_n)\|_{L^2(M,g(t_n))} + \|P_g(\Phi(u,g))(t_n)\|_{L^2(M,g(t_n))}\big]\cdot (T-t_n)^{1/2} \to 0, \tag{1-10}$$

and so that:

(1) *At each $x \in S$, finitely many bubbles, i.e., nonconstant harmonic maps $S^2 \to (N, g_N)$, develop as $t_n \uparrow T$. All of these bubbles develop at scales of order $o((T-t_n)^{1/2})$ and they account for all of the energy that is lost near $x \in S$, as is made precise in part* (2) *of Proposition 3.3. In particular, if $\omega_1, \dots, \omega_m$ is the complete list of bubbles developing at points in S then*

$$\begin{aligned} E_{\text{thick}} &:= \lim_{\delta\downarrow 0}\lim_{t\uparrow T} E(u(t), g(t), \delta\text{-thick}(\mathcal{U}, h)) \\ &= E(\bar{u}, h, \mathcal{U}) + \sum_{l=1}^{m} E(\omega_l) = \sum_i E(u_i, M_i) + \sum_{l=1}^{m} E(\omega_l). \end{aligned} \tag{1-11}$$

(2) *All the energy*

$$E_{\text{thin}} := E(T) - E_{\text{thick}},$$

*$E(T) := \lim_{t\uparrow T} E(t)$, lost down the collars concentrates on the $[T-t]$-*thin *part in the sense that*

$$E_{\text{thin}} = \lim_{t\uparrow T} E\big(u(t), g(t), [T-t]\text{-thin}(M, g(t))\big). \tag{1-12}$$

In fact, we have the more refined information that

$$E_{\text{thin}} = \lim_{K\to\infty} \liminf_{t\uparrow T} E\big(u(t), g(t), [K(T-t)(E(t)-E(T))]\text{-thin}(M, g(t))\big). \tag{1-13}$$

(3) *The restriction of the maps $u(t_n)$ to the $(T-t_n)^{1/2}$-*thin *part of the degenerating subcollars $\mathcal{C}_j$ from Proposition 1.3 has tension $\|\tau_{g_0}(u(t_n))\|_{L^2} \to 0$ as $n\to\infty$ with respect to $g_0 = ds^2 + d\theta^2$ and hence can be assumed to converge to a full bubble branch as explained in Proposition 1.5 below.*

In the following proposition from [Huxol et al. 2016], we write $a_n \ll b_n$, for sequences a_n and b_n if $a_n < b_n$ for each n and $b_n - a_n \to \infty$ as $n \to \infty$.

Proposition 1.5 [Huxol et al. 2016, Theorem 1.9 and Definition 1.10]. *For any sequence of maps $u_n : [-\widehat{X}_n, \widehat{X}_n] \times S^1 \to N$, $\widehat{X}_n \to \infty$, for which the tension with respect to the flat metric $g_0 = ds^2 + d\theta^2$ satisfies $\|\tau_{g_0}(u_n)\|_{L^2} \to 0$, there exists a subsequence that converges to a full bubble branch in the following sense*:

There exist a finite number of sequences s_n^m (for $m \in \{0, \dots, \bar{m}\}$, $\bar{m} \in \mathbb{N}$) with

$$-\widehat{X}_n =: s_n^0 \ll s_n^1 \ll \cdots \ll s_n^{\bar{m}} := \widehat{X}_n$$

such that:

(1) *The* ***connecting cylinders*** $(s_n^{m-1}+\lambda, s_n^m-\lambda)\times S^1$, λ *large*, *are mapped near curves in the sense that*

$$\lim_{\lambda\to\infty}\limsup_{n\to\infty}\sup_{s\in(s_n^{m-1}+\lambda,s_n^m-\lambda)}\operatorname{osc}(u_n;\{s\}\times S^1)=0 \tag{1-14}$$

for each $m\in\{1,\dots,\bar m\}$.

(2) *For each* $m\in\{1,\dots,\bar m-1\}$ (*if nonempty*) *the translated maps* $u_n^m(s,\theta):=u_n(s+s_n^m,\theta)$ *converge weakly in* H^1 *locally on* $(-\infty,\infty)\times S^1$ *to a harmonic map* ω^m *and strongly in* $H^1_{\mathrm{loc}}((-\infty,\infty)\times S^1)$ *away from a finite number of points at which bubbles can be extracted in a way that each bubble is counted no more than once*, *and so that in this convergence of* u_n^m *to a bubble branch there is no loss of energy on compact subsets of* $(-\infty,\infty)\times S^1$. *Since* $(-\infty,\infty)\times S^1$ *is conformally equivalent to the sphere with two points removed*, ω^m *extends to a harmonic map from* S^2. *This map can then be considered as a bubble* (*in particular a branched minimal immersion*) *if it is nonconstant. If it is constant*, *then there must be a nonzero number of bubbles developing within.* (See Theorem 1.5 of [Huxol et al. 2016] for details.)

Remark 1.6. Proposition 3.3 will give a more general version of part (1) of Theorem 1.4, establishing the no-loss-of-energy property and control on the bubble scales also at finite-time singularities as considered in [Rupflin 2014] at which the metrics do not degenerate. As mentioned earlier, the analogue of this result when the underlying surface is $M=S^2$ can already be found in [Topping 2004, Theorem 1.6] since (1-1) is then just the harmonic map flow. That theorem also elaborates on the sense in which the finite collection of bubbles develop, and the strategy of its proof broadly carries over to our situation here.

The key point of Theorem 1.4 is that the degenerating collars, and indeed the whole surface, can be divided up into two regions: First, the cylinders making up $[T-t]$-thin$(M,g(t))$ (and even those making up $[T-t]^{1/2}$-thin) are sufficiently collapsed that when we rescale, the evolving map u can be seen to have very small tension and can thus be represented in terms of branched minimal spheres. Second, on the remaining $[T-t]$-thick part, the limiting energy is fully accounted for by the energy of the limits u_i and the energy of the bubbles. This latter assertion is not a priori so clear since one might have a part of the flow map drifting down the collar, always living in a region such as where the injectivity radius is of the order of, e.g., $[T-t_n]^{1/3}$. Such a part of the map would have no reason to look harmonic in any way, and might carry some nontrivial topology. This "unstructured" energy could in principle drift down the collar not because energy was flowing around the domain, but because the injectivity radius itself is evolving.

The key to ruling out this latter bad behaviour is the following theorem, which gives a more precise description of the convergence of the metric than the one given in Theorem 1.1 and which asserts essentially that by time $t\in[0,T)$, the metric $g(t)$ has settled down to its limit h on the $[T-t]$-thick part. As we shall see, this represents an instance of a more general theory from [Rupflin and Topping 2018a] describing the convergence of a general "horizontal curve" of hyperbolic metrics.

Theorem 1.7. *In the setting of Theorem 1.1*, *there exists* $\overline{K}<\infty$ *depending on* η *and the genus of* M (*and determined in Lemma 2.2*) *such that the following holds true*:

The "pinching set" $F\subset M$ *defined in* (1-4) *can be characterised as*

$$F=\bigcap_{t<T}\{p\in M:\operatorname{inj}_{g(t)}(p)<\delta_K(t)\} \tag{1-15}$$

for any $K \geq \overline{K}$, where $\delta_K(t) := K(T-t)(E(t)-E(T)) \downarrow 0$ as $t \uparrow T$ and $E(T) := \lim_{t\uparrow T} E(t)$. Equivalently, we have

$$\mathcal{U} := M \setminus F = \bigcup_{t<T} [\delta_K(t)]\text{-thick}(M, g(t)). \tag{1-16}$$

In addition to the claims on $\mathcal{U}$, h and the convergence $g(t) \to h$ made in Theorem 1.1, for any $K \geq \overline{K}$, $t_0 \in [0, T)$ and $t \in [t_0, T)$, we have that for every $l \in \mathbb{N}$

$$\|g(t) - h\|_{C^l([\delta_K(t_0)]\text{-thick}(M,g(t_0)),g(t_0))} + \|g(t) - h\|_{C^l([\delta_K(t_0)]\text{-thick}(\mathcal{U},h),h)} \leq CK^{-1/2}\left[\frac{\delta(t)}{\delta(t_0)}\right]^{1/2}, \tag{1-17}$$

where we abbreviate $\delta(t) = \delta_1(t)$ and where C depends only on l, the genus of M and η. Furthermore, for $K > 0$ sufficiently large (depending on η, the genus of M and an upper bound E_0 for the initial energy) and for all $t_0 \in [0, T)$ — or for arbitrary $K > 0$ and $t_0 \in [0, T)$ sufficiently large — we have

$$\sup_{t\in[t_0,T)} \|g(t) - h\|_{C^l([K(T-t_0)]\text{-thick}(M,g(t_0)),g(t_0))} + \sup_{t\in[t_0,T)} \|g(t) - h\|_{C^l([K(T-t_0)]\text{-thick}(\mathcal{U},h),h)} \leq C\frac{(E(t_0) - E(T))^{1/2}}{K^{1/2}} \to 0 \tag{1-18}$$

as $t_0 \uparrow T$, where C depends on l, the genus of M and η.

Remark 1.8. Although we do not require it here, one should be able to improve the smooth local convergence $u(t) \to \bar{u}$ of Theorem 1.1 to quantitative control on the size of $u(t) - \bar{u}$ over, say, the $[T-t]^{1/2}$-thick part of $(M, g(t))$, away from S, with respect to an appropriate weighted norm.

In summary we obtain that the flow (1-1) decomposes any smooth map $u_0 : M \to N$ into a collection of branched minimal immersions $v_i : \Sigma_i \to N$ through global solutions that are smooth away from finitely many times as follows: As discussed earlier, at each singular time t_m for which $\mathrm{inj}_{g(t)}(M) \not\to 0$ as $t \uparrow t_m$, all of the lost energy is accounted for in terms of bubbles $\omega_m^j : S^2 \to (N, g_N)$, which we add to the collection of minimal immersions v_i (adding that same number of copies of S^2 to the collection of domains Σ_i). At singular times for which $\mathrm{inj}_{g(t)}(M) \to 0$, the results discussed above apply and we add both the bubbles developing at the singular points $S \subset \mathcal{U}$ and those that are disappearing down one of the degenerating collars to the set of minimal immersions v_i (again adding the corresponding number of S^2's to the collection of the Σ_i's) and continue the flow on the closed lower-genus surfaces M_i as described above.

If the genus of any of the closed surfaces M_i is 0 or 1, then its continuation will be a weak solution that flows forever afterwards according to the theory of the harmonic map flow [Struwe 1985] or the theory in [Ding et al. 2006]. If the genus of any of the surfaces M_i is larger than 1, then the subsequent flow might develop a further finite-time singularity at which a collar degenerates, in which case we repeat the process above to continue the flow further still. Each time we restart the flow after a singularity caused by the degeneration of one or more collars, the genus of the surfaces underlying the continued flows will decrease, so repeating the process finitely many times gives us a global weak solution as desired. As the energy is conformally invariant, the resulting global solution has nonincreasing energy and the total number of singular times t_m is a priori bounded in terms of the genus, the initial energy and the target (N, g_N).

We can relate the domain(s) and map(s) before a singular time t_m to the flow(s) after the singular time as explained above and can thus reconstruct the initial map and the initial domain in terms of the map(s) and domain(s) at any time $t \in (t_m, t_{m+1})$ and the collection of all of the bubbles v_i obtained at the singular times $t_1 < \cdots < t_m$ as well as connecting curves on cylinders.

We can then apply the asymptotic analysis as discussed above, principally from [Huxol et al. 2016], to each of the obtained global flows, eventually adding also the bubbles developing at infinite time as well as the limiting maps $u_j^\infty : M_j^\infty \to N$ obtained at infinite time, which are branched minimal immersions defined on surfaces of total genus no more than γ, to the collection of the (Σ_i, v_i). This gives the decomposition of the initial map into branched minimal immersions $v_i : \Sigma_i \to N$ described earlier on.

This paper is organised as follows. In Section 2 we carry out the analysis of the metric component of the flow, proving part (1) of Theorem 1.1 as well as Theorem 1.7 and Proposition 1.3. The resulting control on the evolution of the metric then allows us to analyse the map component in the subsequent Section 3. In Section 3.2 we focus on the properties of the map on the nondegenerate part of the surface, stating and proving Proposition 3.3, which yields both part (2) of Theorem 1.1 as well as part (1) of Theorem 1.4. Parts (2) and (3) of Theorem 1.4 are then proven in Section 3.3 where we analyse the map component on the degenerating part of the surface.

2. Analysis of the metric component

In this section we analyse the metric component of the flow, proving first part (1) of Theorem 1.1, then Theorem 1.7, and finally Proposition 1.3. This analysis is based on the theory of general horizontal curves we developed in [Rupflin and Topping 2018a], henceforth abbreviated [RT2018a], some of which we recall here.

A *horizontal* curve of metrics on a smooth closed oriented surface M of genus at least 2 is a smooth one-parameter family $g(t)$ of hyperbolic metrics on M for t within some interval $I \subset \mathbb{R}$ so that for each $t \in I$, there exists a holomorphic quadratic differential $\Psi(t)$ such that $\partial_t g = \mathrm{Re}(\Psi)$. This makes $g(t)$ move orthogonally to modifications by diffeomorphisms, as described in [RT2018a].

An important property of horizontal curves is that we can bound the C^l norm of their velocity, $l \in \mathbb{N}$, in terms of a much weaker norm of $\partial_t g$ and the injectivity radius. In fact, [RT2018a, Lemma 2.6] gives that for any $x \in M$ and $l \in \mathbb{N}$

$$|\partial_t g(t)|_{C^l(g(t))}(x) \le C[\mathrm{inj}_{g(t)}(x)]^{-1/2} \|\partial_t g(t)\|_{L^2(M,g(t))}, \tag{2-1}$$

with C depending only on the genus of M and l, where $|\Omega|_{C^l(g)}(x) := \sum_{k=0}^{l} |\nabla_g^k \Omega|_g(x)$, with ∇_g the Levi-Civita connection, or its extension.

We furthermore recall that for every point $x \in M$ the map $t \mapsto \mathrm{inj}_{g(t)}(x)$ is locally Lipschitz on the interval I over which g is defined, see [RT2018a, Lemma 2.1], and that

$$\left| \frac{d}{dt} [\mathrm{inj}_{g(t)}(x)]^{1/2} \right| \le K_0 \|\partial_t g(t)\|_{L^2(M,g(t))} \tag{2-2}$$

holds true for a constant $K_0 < \infty$ that depends only on the genus of M, see [RT2018a, Lemma 2.2].

These estimates play an important role in the proof of the following convergence result for finite-length horizontal curves, proven in [RT2018a], which we will use to analyse the metric component of the flow. In order to state this result, we introduce some more notation: If $g(t)$ is defined for t in some interval $[0, T)$, then we let

$$\mathcal{L}(s) := \int_s^T \|\partial_t g(t)\|_{L^2(M,g(t))}\, dt \in [0, \infty]$$

denote the length of the restriction of g to the interval $[s, T)$. Given a tensor Ω defined in a neighbourhood of some $K \subset M$, we write

$$\|\Omega\|_{C^l(K,g)} := \sup_K |\Omega|_{C^l(g)}. \tag{2-3}$$

Theorem 2.1 [RT2018a, Theorem 1.2]. *Let M be a closed oriented surface of genus $\gamma \geq 2$, and suppose $g(t)$ is a smooth horizontal curve in $\mathcal{M}_{-1}$ for $t \in [0, T)$, with finite length $\mathcal{L}(0) < \infty$. Then there exist a nonempty open subset $\mathcal{U} \subset M$, whose complement has $k \in \{0, \dots, 3(\gamma - 1)\}$ components, and a complete hyperbolic metric h on $\mathcal{U}$ for which $(\mathcal{U}, h)$ is of finite volume and is conformally a finite disjoint union of closed Riemann surfaces (of genus strictly less than that of M if $\mathcal{U}$ is not the whole of M) with $2k$ punctures, such that*

$$g(t) \to h$$

smoothly locally on $\mathcal{U}$. Moreover, defining $\mathcal{I} : M \to [0, \infty)$ by

$$\mathcal{I}(x) = \begin{cases} \mathrm{inj}_h(x) & \text{on } \mathcal{U}, \\ 0 & \text{on } F = M \setminus \mathcal{U}, \end{cases}$$

we have $\mathrm{inj}_{g(t)} \to \mathcal{I}$ uniformly on M as $t \uparrow T$, and indeed that

$$\|[\mathrm{inj}_{g(t)}]^{1/2} - \mathcal{I}^{1/2}\|_{C^0} \leq K_0 \mathcal{L}(t) \to 0 \quad \text{as } t \uparrow T, \tag{2-4}$$

where K_0 is chosen as in (2-2) and depends only on γ. Furthermore, for any $l \in \mathbb{N}$ and $\delta > 0$, if we take $t_0 \in [0, T)$ sufficiently large so that

$$(2K_0 \mathcal{L}(t_0))^2 < \delta, \quad \text{where } K_0 \text{ is the constant obtained in (2-2)}, \tag{2-5}$$

then $\delta\text{-thick}(M, g(s)) \subset \mathcal{U}$ for every $s \in [t_0, T)$, and we have for every $t \in [t_0, T)$

$$\|g(t) - h\|_{C^l(\delta\text{-thick}(\mathcal{U},h),h)} + \|g(t) - h\|_{C^l(\delta\text{-thick}(M,g(s)),g(s))} \leq C\delta^{-1/2}\mathcal{L}(t), \tag{2-6}$$

where C depends only on l and γ.

We first apply this result to prove the properties of the metric component claimed in our basic convergence result, i.e., part (1) of Theorem 1.1 To this end we first note that for any smooth solution (u, g) of (1-1) defined on $[0, T)$, $T < \infty$, the metric component is by definition a smooth horizontal curve. Furthermore, its length is finite as

$$\begin{aligned} \mathcal{L}(t)^2 = \left(\int_t^T \|\partial_t g(t)\|_{L^2(M,g(t))}\, dt \right)^2 &\leq (T - t) \int_t^T \|\partial_t g(t)\|^2_{L^2(M,g(t))}\, dt \\ &\leq \eta^2 (T - t)(E(t) - E(T)), \end{aligned} \tag{2-7}$$

by (1-2), where we abbreviate $E(t) := E(u(t), g(t))$ and $E(T) := \lim_{s\uparrow T} E(s)$. In particular, defining

$$\bar{K} := 5K_0^2\eta^2 \tag{2-8}$$

to depend only on η and γ, and defining

$$\delta_K(t) := K(T-t)(E(t)-E(T)), \tag{2-9}$$

which we will be considering for $K \geq \bar{K}$ and $t \in [0, T)$, we have

$$[K_0\mathcal{L}(t)]^2 \leq \tfrac{1}{5}\delta_{\bar{K}}(t) \tag{2-10}$$

for all $t \in [0, T)$. We may thus analyse the metric component g of any solution of (1-1) with the above Theorem 2.1.

In the setting of Theorem 1.1, the assumption (1-3) that the metric component degenerates as t approaches T combined with the uniform convergence of the injectivity radius furthermore guarantees that the pinching set F must be nonempty.

Part (1) of Theorem 1.1 concerning the local convergence of $g(t)$ to a limit h and the properties of $h, \mathcal{U}$ and F is thus a direct consequence of Theorem 2.1 and the fact that $\mathcal{L}(0) < \infty$.

To prove the refined properties of the metric component stated in Theorem 1.7 and Proposition 1.3 we shall use the following lemma, where $\bar{K}$ will be chosen as in (2-8) above.

Lemma 2.2. *Let (u, g) be a smooth solution of* (1-1) *on* $[0, T)$, $T < \infty$, *on a surface M of genus $\gamma \geq 2$. Then there exists a constant $\bar{K}$ depending only on η and γ so that the following holds true. If we define $\delta_K(t)$ as in* (2-9) *then for every $t_0 \in [0, T)$ the assumption* (2-5) *of Theorem 2.1 is satisfied for t_0 and any $\delta > 0$ with $\delta \geq \delta_{\bar{K}}(t_0)$ and thus estimate* (2-6) *holds true for any $t_0 \in [0, T)$, $s, t \in [t_0, T)$, and any $\delta > 0$ with $\delta \geq \delta_{\bar{K}}(t_0)$. Furthermore*:

(1) *For every $K \geq \bar{K}$ the pinching set F defined in* (1-4) *can be characterised by* (1-15).

(2) *The metrics $(g(t))_{t\in[t_0,T)}$ are uniformly equivalent and their injectivity radii are of comparable size at points $x \in \delta_{\bar{K}}(t_0)$*-thick$(M, g(t_0))$ *in the sense that for every $s, t \in [t_0, T)$*

$$g(s)(x) \leq C_1 \cdot g(t)(x) \quad \text{and} \quad C_1^{-1} \cdot h(x) \leq g(t)(x) \leq C_1 \cdot h(x) \tag{2-11}$$

and

$$\operatorname{inj}_{g(s)}(x) \leq C_2 \cdot \operatorname{inj}_{g(t)}(x), \tag{2-12}$$

where $C_1 \geq 1$ depends only on the genus of M, while $C_2 \geq 1$ is a universal constant.

(3) *For every $K \geq \bar{K}$, every $x \in \delta_K(t_0)$*-thick$(M, g(t_0))$, *every $s, t \in [t_0, T)$ and every $l \in \mathbb{N}$ we have*

$$|\partial_t g(t)|_{C^l(g(s))}(x) \leq C\delta_K(t_0)^{-1/2}\|\partial_t g(t)\|_{L^2(M,g(t))}, \tag{2-13}$$

where C depends only on l and the genus of M.

Proof of Lemma 2.2. We first remark that the claims are trivially true if $\delta_K(t_0) = 0$ and hence $g|_{[t_0,T)}$ is constant in time, so we may assume without loss of generality that $\delta_K(t_0) > 0$.

Define $\bar{K}$ as in (2-8). Then (2-10) implies

$$(2K_0\mathcal{L}(t_0))^2 \le \tfrac{4}{5}\delta_{\bar{K}}(t_0) < \delta_{\bar{K}}(t_0), \tag{2-14}$$

and so (2-5) is satisfied for $\delta \ge \delta_{\bar{K}}(t_0)$ as claimed in the lemma.

To prove part (1) of the lemma we combine (2-4) with (2-10) to obtain that $\operatorname{inj}_{g(t)}(p) \le (K_0\mathcal{L}(t))^2 < \delta_{\bar{K}}(t)$ for every $p \in F$ and every $t \in [0, T)$ and thus that

$$F \subset \bigcap_{t\in[0,T)} \delta_K(t)\text{-thin}(M, g(t)) \quad \text{for any } K \ge \bar{K}.$$

As the reverse inclusion is trivially satisfied this establishes the characterisation (1-15) of the pinching set for each $K \ge \bar{K}$.

The proofs of parts (2) and (3) of the lemma are now based on estimates on the velocity and the injectivity radius that were derived in [RT2018a] for general horizontal curves under the same hypothesis that (2-5) holds true: Lemma 3.2 and Remark 3.5 of [RT2018a] establish that (2-11) and (2-12) hold true for arbitrary horizontal curves, times $s, t \in [t_0, T)$ and points $x \in \delta\text{-thick}(M, g(t_0))$ provided t_0 and δ are such that (2-5) is satisfied. Combined with (2-14) this immediately yields part (2) of the lemma. Finally, (2-13), and hence part (3) of the lemma, follows immediately from [RT2018a, Lemma 3.2], with δ there equal to $\delta_K(t_0)$ here, because the hypotheses of that lemma are implied by (2-14). □

Parts (2) and (3) of Lemma 2.2 will be used in the next section for the fine analysis of the map component, but before that we complete the proofs of Theorem 1.7 and Proposition 1.3.

Proof of Theorem 1.7. We let $\bar{K}$ be the constant obtained in Lemma 2.2, i.e., given by (2-8), and set as usual $\delta_K(t) = K(T-t)(E(t) - E(T))$. For this choice of $\bar{K}$ the characterisation (1-15) of the pinching set F has already been proven in Lemma 2.2 and from this lemma we furthermore know that (2-5) holds true for any t_0 and any $\delta \ge \delta_{\bar{K}}(t_0)$ and thus in particular for $\delta = \delta_K(t_0)$, $K \ge \bar{K}$. Hence (1-17) follows from the corresponding estimate (2-6) of Theorem 2.1 and the bound (2-7) on $\mathcal{L}(t)$.

It remains to prove (1-18). For this we observe that for $K > 0$ sufficiently large and for all $t_0 \in [0, T)$ — or for arbitrary $K > 0$ and $t_0 \in [0, T)$ sufficiently large — we can be sure that $\bar{K}(E(t_0) - E(T)) \le K$ and hence by (2-10) that (2-5) is satisfied for t_0 and $\delta = K(T - t_0)$. This allows us to apply estimate (2-6) of Theorem 2.1 also for such values of δ, which then gives

$$\sup_{t\in[t_0,T)} \|g(t) - h\|_{C^l([K(T-t_0)]\text{-thick}(M,g(t_0)),g(t_0))} \le C\frac{\mathcal{L}(t_0)}{K^{1/2}(T-t_0)^{1/2}} \le C\frac{(E(t_0) - E(T))^{1/2}}{K^{1/2}} \to 0 \tag{2-15}$$

as $t_0 \uparrow T$, using (2-7), as well as

$$\sup_{t\in[t_0,T)} \|g(t) - h\|_{C^l([K(T-t_0)]\text{-thick}(\mathcal{U},h),h)} \le C\frac{(E(t_0) - E(T))^{1/2}}{K^{1/2}} \to 0, \tag{2-16}$$

where C depends only on l, η and the genus of M. □

Proof of Proposition 1.3. The uniform convergence of the injectivity radius follows from Theorem 2.1 as $(g(t))_{t\in[0,T)}$, $T < \infty$, is a horizontal curve of finite length.

We furthermore recall that standard results from the theory of hyperbolic surfaces give that for any $\delta < \operatorname{arsinh}(1)$ the δ-thin part of a hyperbolic surface is given by the union of disjoint subcollar regions around the simple closed geodesics of length $\ell < 2\delta$ as described in the proposition; refer to the appendix of [Rupflin and Topping 2018b] for further details.

For $K \geq K_0$, with K_0 as in (2-2), we define closed sets

$$F_K(t) := \{p : \operatorname{inj}_{g(t)}(p) \leq (K\mathcal{L}(t))^2\}$$

for $t \in [0, T)$. It follows from the slightly stronger result [RT2018a, Lemma 3.1] that the sets $F_K(t)$ are nested, becoming only smaller as t increases, and that the pinching set F can be written as

$$F = \bigcap_{t\in[0,T)} F_K(t). \tag{2-17}$$

It is useful for us to appeal to this fact for some $K > K_0$, and we choose $K = 2K_0$.

Thus for t_0 sufficiently large, chosen in particular so that $(2K_0\mathcal{L}(t_0))^2 < \operatorname{arsinh}(1)$, the pinching set F has the same number $k \in \{1, \dots, 3(\gamma - 1)\}$ of connected components as the sets $F_{2K_0}(t)$, $t \in [t_0, T)$, with the connected components of $F_{2K_0}(t)$ being disjoint closed subcollars around geodesics $\sigma_j(t)$ of length $\ell_j(t) \leq 2(2K_0\mathcal{L}(t))^2 \leq C(T - t)(E(t) - E(T))$ whose interior is as described in the proposition. In particular given any $\delta \in (0, \operatorname{arsinh}(1))$ and $t \in [t_0, T)$ sufficiently large (depending in particular on δ), we know that the connected components F_j of the pinching set are contained in the corresponding subcollar $\mathcal{C}_j(t, \delta)$, as claimed in the proposition.

It thus remains to show that there exists a number $\delta_0 \in (0, \operatorname{arsinh}(1))$ such that any simple closed geodesic in $(M, g(t))$, $t \in [t_0, T)$, that does not coincide with one of the $\sigma_j(t)$ must have length at least $2\delta_0$. To this end we observe that the characterisation (2-17), this time with $K = K_0$, gives

$$\Omega := (2K_0\mathcal{L}(t_0))^2\text{-thick}(M, g(t_0)) \subset M \setminus F_{K_0}(t_0) \subset \mathcal{U}$$

and since Ω is closed, it is a compact subset of M and hence also of $\mathcal{U}$. Therefore over Ω the injectivity radius $\operatorname{inj}_{g(t)}(\cdot)$, $t \in [t_0, T)$, is bounded uniformly from below by some constant $\delta_0 \in (0, \operatorname{arsinh}(1))$ thanks to (1-6). Consequently, any simple closed geodesic in $(M, g(t))$ that enters Ω must have length at least $2\delta_0$.

The only alternative is that the simple closed geodesic in $(M, g(t))$ is fully contained in one of the k cylinders $\mathcal{C}_j(t_0, (2K_0\mathcal{L}(t_0))^2)$, in which case it must be homotopic to $\sigma_j(t)$ (up to change of orientation) and hence coincide with $\sigma_j(t)$. □

3. Analysis of the map component

The challenges of analysing the map component are of a different nature depending on whether we consider a region where the metric has already settled down or a region in a collar that will ultimately degenerate. Roughly speaking, on the nondegenerate part of the surface we control the metric but cannot hope to bound the tension, while on the degenerating part of the surface the metric is not controlled but the tension tends to zero when computed with respect to the flat metric in collar coordinates along a sequence of times $t_n \uparrow T$ as considered in Theorem 1.4. We will analyse the map component separately on these two different regions, with the analysis on the nondegenerate part, and hence the proofs of part (2) of Theorem 1.1 and of part (1) of Theorem 1.4, carried out in Section 3.2. Parts (2) and (3) of Theorem 1.4, which concern the part of the map that is lost on degenerating collars, are then proven in Section 3.3. In both of these sections we use a local energy estimate that is derived in Section 3.1.

3.1. ***Local energy estimates.*** The goal of this section is to prove the following lemma on the evolution of cut-off energies

$$E_\varphi(t) := \frac{1}{2}\int \varphi^2 |du(t)|^2_{g(t)}\, dv_{g(t)} \tag{3-1}$$

for functions $\varphi \in C^\infty(M, [0, 1])$.

Lemma 3.1 (Local energy estimate). *Let (u, g) be a smooth solution of* (1-1) *on a closed surface of genus at least* 2, *and for an interval* $[0, T)$, $T < \infty$, *and let $\varphi \in C^\infty(M, [0, 1])$ be such that there exists $t_0 \in [0, T)$ and $K \geq \overline{K}$ for $\overline{K}$ the constant obtained in Lemma 2.2, so that*

$$\operatorname{supp}(\varphi) \subset \delta_K(t_0)\text{-thick}(M, g(t_0)), \tag{3-2}$$

where as usual $\delta_K(t) := K(T-t)(E(t)-E(T))$.

Then the limit $\lim_{t\uparrow T} E_\varphi(t)$ exists and (assuming the flow is not constant in time on $[t_0, T)$) for any $t_0 \leq t < s < T$ we have

$$\begin{aligned}|E_\varphi(t) - E_\varphi(s)| &\leq E(t) - E(s) + C[\delta_K(t_0)^{-1/2} + \|d\varphi\|_{L^\infty(M,g(t_0))}](s-t)^{1/2}(E(t)-E(s))^{1/2}\\ &\leq E(t) - E(T) + C[\delta_K(t_0)^{-1/2} + \|d\varphi\|_{L^\infty(M,g(t_0))}](T-t)^{1/2}(E(t)-E(T))^{1/2},\end{aligned} \tag{3-3}$$

where C depends only on the coupling constant η, the genus of M and an upper bound E_0 for the initial energy.

A first step in the proof of Lemma 3.1 is to show the following analogue of well-known local energy estimates for harmonic map flow as found, e.g., in [Topping 2004, Section 2].

Lemma 3.2. *Let (u, g) be a (smooth) solution of* (1-1) *on* $[0, T)$ *and let $\varphi \in C^\infty(M, [0, 1])$ be any given function. Then the evolution of the cut-off energy $E_\varphi(t)$ defined in* (3-1) *is controlled by*

$$\begin{aligned}&\left|\frac{d}{dt}E_\varphi + \int \varphi^2 |\tau_g(u)|^2\, dv_g\right|\\ &\qquad \leq 2\sqrt{2}E(u,g)^{1/2}\|d\varphi\|_{L^\infty(M,g)}\left(\int \varphi^2|\tau_g(u)|^2\, dv_g\right)^{1/2} + \|\partial_t g\|_{L^\infty(\operatorname{supp}(\varphi),g)}E_\varphi.\end{aligned} \tag{3-4}$$

Proof. The equation of the map component can be described by

$$\partial_t u - \Delta_g u = A_g(u)(du, du) = g^{ij}A(u)(\partial_{x_i}u, \partial_{x_j}u) \perp T_u N \tag{3-5}$$

if we view (N, g_N) as a submanifold of Euclidean space using Nash's embedding theorem and denote by A the second fundamental form of $N \hookrightarrow \mathbb{R}^K$. We multiply this equation with $\varphi^2\, \partial_t u$ and integrate over

(M, g) to obtain

$$0=\int \varphi^2|\partial_t u|^2\,dv_g+\int \langle\partial_t\,du,du\rangle_g\varphi^2 dv_g+\partial_t u\langle du,d(\varphi^2)\rangle_g\,dv_g.$$

We now recall that $\partial_t g$ is given as the real part of a quadratic differential and thus has zero trace, which implies $(d/dt)dv_g=0$. As φ is independent of time while $\partial_t u=\tau_g(u)$, we thus obtain

$$\begin{aligned}\Big|\frac{d}{dt}E_\varphi+\int \varphi^2|\tau_g(u)|^2\,dv_g\Big| &\le \frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int |du|^2_{g(t+\varepsilon)}\varphi^2\,dv_g+2\int \varphi|d\varphi|\cdot|\tau_g(u)|\cdot|du|\,dv_g\\ &\le \|\partial_t g\|_{L^\infty(\mathrm{supp}(\varphi),g)}E_\varphi+2\|d\varphi\|_{L^\infty(M,g)}(2E(u,g))^{1/2}\cdot\Big(\int \varphi^2|\tau_g(u)|^2\,dv_g\Big)^{1/2} \qquad (3\text{-}6)\end{aligned}$$

as claimed. ☐

Based on this lemma, as well as the control on the metric on the $\delta_K(t_0)$-thick part of the domain obtained in Lemma 2.2, we can now prove our main energy estimate.

Proof of Lemma 3.1. Given $K\ge\bar{K}$, with $\bar{K}$ as in Lemma 2.2, we set as usual $\delta_K(t):=K(T-t)(E(t)-E(T))$ and consider a cut-off function φ as in the lemma for which (3-2) is satisfied for some t_0.

This assumption on the support of φ allows us to bound any C^l norm of $\partial_t g$ on $\mathrm{supp}(\varphi)$ using estimate (2-13) of Lemma 2.2, which implies in particular that

$$\|\partial_t g(t)\|_{L^\infty(\mathrm{supp}\,\varphi,g(t))}\le C[\delta_K(t_0)]^{-1/2}\|\partial_t g(t)\|_{L^2(M,g(t))}\quad \text{for any } t\in[t_0,T) \qquad (3\text{-}7)$$

holds true with a constant C that depends only on the genus.

Furthermore, the equivalence (2-11) of the metrics on $\delta_{\bar{K}}(t_0)$-thick$(M, g(t_0))$, and thus in particular on $\mathrm{supp}(\varphi)$, obtained in the same lemma, allows us to bound

$$\|d\varphi\|_{L^\infty(M,g(t))}\le\sqrt{C_1}\|d\varphi\|_{L^\infty(M,g(t_0))}\quad \text{for } t\in[t_0,T).$$

The local energy estimate (3-4) of Lemma 3.2 thus reduces to

$$\begin{aligned}\Big|\frac{d}{dt}E_\varphi\Big| &\le\|\tau_g(u)\|^2_{L^2(M,g)}+C\|d\varphi\|_{L^\infty(M,g(t_0))}\|\tau_g(u)\|_{L^2(M,g)}+C\delta_K(t_0)^{-1/2}\|\partial_t g\|_{L^2(M,g)}\\ &\le\Big(-\frac{dE}{dt}\Big)+C\cdot[\|d\varphi\|_{L^\infty(M,g(t_0))}+\delta_K(t_0)^{-1/2}]\Big(-\frac{dE}{dt}\Big)^{1/2} \qquad (3\text{-}8)\end{aligned}$$

for $t\in[t_0,T)$, where the constant C now depends not only on the genus but also on the coupling constant and an upper bound E_0 on $E(0)\ge E(t)$ and where we used the evolution equation (1-2) of the total energy in the second step.

Integrating (3-8) over $[t,s]\subset[t_0,T)$ yields the claim of Lemma 3.1. ☐

Lemma 3.1 will allow us to determine both the part of the degenerating collars where energy can be lost as well as the scale at which energy concentrates around points in the singular set $S\subset\mathcal{U}$. This will then allow us to capture two collections of bubbles, one developing at the bubble points S, but also an additional collection of bubbles that are disappearing down the collars. By further analysing what can happen between these bubbles, this will allow us to prove Theorem 1.4.

This bubbling analysis will be carried out along a sequence of times t_n for which (1-10) holds. The existence of such $t_n \uparrow T$ follows by a standard argument: Integrating (1-2) in time implies that

$$\int_0^T \|\tau_g(u)\|_{L^2}^2\,dt \quad \text{and} \quad \int_0^T \|P_g(\Phi)\|_{L^2}^2\,dt$$

are bounded in terms of an upper bound E_0 for the initial energy, and (for the second integral) the coupling constant η. Here we suppress the dependence of Φ on u and g, of the L^2 measure on g, and of u and g on t. These bounds imply that whenever a smooth function $f : [0, T) \to [0, \infty)$ has infinite integral, there exists a sequence of times $t_n \uparrow T$ such that

$$[\|\tau_g(u)\|_{L^2}^2 + \|P_g(\Phi)\|_{L^2}^2](t_n) < f(t_n).$$

In particular, we may always choose some sequence $t_n \uparrow T$ so that (1-10) holds true for t_n (and thus also for any subsequence that we take later).

3.2. ***Analysis of the map component on the nondegenerate part of the surface.*** On compact subsets of $\mathcal{U}$ we can control the metric component using Lemma 2.2 and may thus think of the evolution equation (1-1) for the map component as a solution of a flow that is akin to the classical harmonic map flow albeit with a (well-controlled) time-dependent metric. This will allow us to adapt well-known techniques from the theory of the harmonic map flow, in particular from [Struwe 1985; Topping 2004], to analyse the solution on this part of the domain in detail: We prove that as $t \uparrow T$, energy concentrates only at finitely many points S away from which the maps converge in C^l, for each $l \in \mathbb{N}$, and that, along a subsequence of times $t_n \uparrow T$ as in (1-10), we can extract a finite number of bubbles at each point in S which account for all the energy that is lost near these point. This last part is equivalent to proving that no energy is lost on so-called neck-regions around the bubbles (not to be confused with collar regions around the degenerating geodesics). This fine analysis of the map component on the thick part of the surface applies not only in the case of a finite time degeneration as considered in the present paper but (as a by-product of the following proposition) also gives refined information at singular times as considered in [Rupflin 2014], across which the metric remains controlled.

Proposition 3.3 (cf. [Topping 2004]). *Let (u, g) be any smooth solution of* (1-1) *for $t \in [0, T)$ on a surface of genus at least* 2. *Let F be the* (*possibly empty*) *set given by* (1-4) *and let S be defined as in* (1-5). *Then S is a finite set and*:

(1) *$u(t)$ converges smoothly locally on $M \setminus (F \cup S)$ and weakly locally in H^1 on $M \setminus F$ as $t \uparrow T$, to a limit that we denote by $u(T)$.*

(2) *We have no loss of energy at points in S, and the scales of bubbles developing at the points of S, along a subsequence of times $t_n \uparrow T$ as in* (1-10), *are small compared with $(T - t_n)^{1/2}$. Indeed, if $\omega_1, \ldots, \omega_{m'}$ are the bubbles developing at $x \in S$ then for every $\nu > 0$*

$$\begin{aligned}\lim_{r\downarrow 0}\lim_{t\uparrow T} E(u(t), g(t), B_h(x, r)) &= \lim_{r\downarrow 0}\lim_{t\uparrow T} E(u(t), g(t), B_{g(t)}(x, r)) \\ &= \lim_{t\uparrow T} E\big(u(t), g(t), B_{g(t)}(x, \nu(T-t)^{1/2})\big) = \sum_{l=1}^{m'} E(\omega_l). \qquad (3\text{-}9)\end{aligned}$$

In particular, if $\omega_1, \dots, \omega_{m''}$ is the complete list of bubbles developing at points in S and if $\Omega \Subset \mathcal{U}$ is chosen large enough so that S is contained in the interior of Ω then

$$\lim_{t\uparrow T} E(u(t), g(t), \Omega) = E(\bar{u}, h, \Omega) + \sum_{l=1}^{m''} E(\omega_l). \tag{3-10}$$

In the setting of Theorem 1.1, i.e., in the case that $\mathrm{inj}_{g(t)}(M) \to 0$ as $t \uparrow T$, part (1) of the proposition yields the convergence of the maps $u(t)$ on $\mathcal{U}$ and on $\mathcal{U} \setminus S$ claimed in part (2) of Theorem 1.1. As the resulting limiting maps can be extended across the punctures to H^1 maps from M_i (since their energy is bounded) and as the properties of the metric component claimed in part (1) of Theorem 1.1 have already been proven in Section 2, this then completes the proof of Theorem 1.1, modulo the proof of Proposition 3.3.

The second part of Proposition 3.3 implies part (1) of Theorem 1.4: the first part of (1-11) follows from (3-10) since δ-thick$(\mathcal{U}, h)$ is compact for every $\delta > 0$, while the second part of (1-11) is due to the conformal invariance of the energy.

For the proof of Proposition 3.3 we shall use the following standard ε-regularity result.

Proposition 3.4. *There exist constants $\varepsilon_0 > 0$ and $C \in \mathbb{R}$ depending only on the target manifold so that the following holds true. Let $u : B_{g_H}(x, r) \to N$ be any smooth map from a ball of radius $r \in (0, 1]$ in the hyperbolic plane (H, g_H) with energy*

$$E(u, g_H, B_{g_H}(x, r)) \leq \varepsilon_0.$$

Then

$$\int \varphi^2 [|\nabla_{g_H} du|^2_{g_H} + |du|^4_{g_H}]\, dv_{g_H} \leq C \|d\varphi\|^2_{L^\infty(H,g_H)} E(u, g_H, B_{g_H}(x, r)) + C \int \varphi^2 |\tau_{g_H}(u)|^2\, dv_{g_H} \tag{3-11}$$

holds true for every function $\varphi \in C_c^\infty(B_{g_H}(x, r), [0, 1])$.

Note that the Hessian term $|\nabla_{g_H} du|^2_{g_H}$ is not referring to the intrinsic Hessian. That term is instead the sum of the corresponding terms for each component of u viewed as a map into Euclidean space, and depends on the isometric embedding of N that we chose. This term can be controlled in terms of the integral of $\varphi^2 |\Delta_g u|^2$ and lower-order terms simply using integration by parts. This leading-order term can be rewritten using (3-5) and the resulting quartic term in du controlled with the Sobolev inequality. The details of a very similar argument can be found in [Rupflin 2008, Proposition 2.4].

Proof of Proposition 3.3. Part (1) of the proposition represents the analogue of Lemma 3.10′ of [Struwe 1985] and we shall use properties of horizontal curves from Lemma 2.2 to control the evolution of the metric; see also [Rupflin 2014] for a related proof in the nondegenerate case.

In the following we shall use several times that for any compact subset $\Omega \subset \mathcal{U} = M \setminus F$ there exists a number $t_0 = t_0(\Omega) \in [0, T)$ so that

$$\Omega \subset \delta_{2\overline{K}}(t_0)\text{-thick}(M, g(t_0)), \tag{3-12}$$

where $\delta_K(t) = K(T - t)(E(t) - E(T))$ and $\overline{K}$ are as in Lemma 2.2. Indeed, for solutions of (1-1) which degenerate as described in (1-3), this is a consequence of the uniform convergence of the injectivity radius obtained in Proposition 1.3, while otherwise $\mathrm{inj}_{g(t)}(M)$ is bounded away from zero uniformly so (3-12) is trivially satisfied for t_0 sufficiently close to T. As a consequence of (3-12) also

$$\mathrm{inj}_{g(t_0)}(x) \geq \delta_{\overline{K}}(t_0) \quad \text{for all } x \in M \text{ with } \mathrm{dist}_{g(t_0)}(x, \Omega) \leq \delta_{\overline{K}}(t_0), \tag{3-13}$$

which allows us to apply Lemma 2.2 to control the evolution of the metric as well as Lemma 3.1 to bound the cut-energy on this neighbourhood of Ω.

We first apply this idea to prove that for any point $p \in \mathcal{U}$ for which

$$\limsup_{t \uparrow T} E(u(t), g(t), V) \geq \varepsilon_0 \quad \text{for every neighbourhood } V \subset M \text{ of } p, \tag{3-14}$$

where $\varepsilon_0 > 0$ is the constant obtained in Proposition 3.4, we also have

$$\liminf_{t \uparrow T} E(u(t), g(t), W) \geq \varepsilon_0 \quad \text{for every neighbourhood } W \subset M \text{ of } p. \tag{3-15}$$

In particular, the set $\widetilde{S}$ of points in $\mathcal{U}$ for which (3-14) holds is a finite set and we will later see that it agrees with the singular set S defined in (1-5).

To show (3-15) for a given $p \in \widetilde{S}$ we let $t_0 \in [0, T)$ be large enough so that (3-12) holds for $\Omega = \{p\}$. Given any neighbourhood W of p we then choose $r \in (0, \delta_{\overline{K}}(t_0))$ small enough so that $B_{g(t_0)}(p, r) \subset W$ and select a cut-off function $\varphi \in C_c^\infty(B_{g(t_0)}(p, r), [0, 1])$ with $\varphi \equiv 1$ in a neighbourhood V of p.

Lemma 3.1 implies that the limit $\lim_{t \uparrow T} E_\varphi(t)$ of the cut-off energy defined in (3-1) exists and thus that, by (3-14),

$$\liminf_{t \uparrow T} E(u(t), g(t), W) \geq \lim_{t \uparrow T} E_\varphi(t) \geq \limsup_{t \uparrow T} E(u(t), g(t), V) \geq \varepsilon_0$$

as claimed. Having thus established that there is only a finite subset $\widetilde{S}$ of points in $\mathcal{U}$ for which (3-14) holds, we now want to prove that $u(t)$ converges smoothly on every compact subset V of $\mathcal{U} \setminus \widetilde{S}$ as $t \uparrow T$.

Given such a compact subset V of $\mathcal{U} \setminus \widetilde{S}$ we may choose $r_0 \in (0, 1)$ small enough that

$$E(u(t), g(t), B_{g(t_0)}(p, r_0)) < \varepsilon_0 \quad \text{for all } t \in [0, T), \text{ and all } p \in V. \tag{3-16}$$

Then choosing $t_0 \in [0, T)$ so that (3-12) holds true for $\Omega = V$ and reducing r_0 if necessary to ensure that $r_0 < \delta_{\overline{K}}(t_0)$, we know from (3-13) that we can apply both Lemmas 2.2 and 3.1 on balls $B_{g(t_0)}(p, r)$, $r \leq r_0$, $p \in V$, as they are contained in $\delta_{\overline{K}}(t_0)$-thick$(M, g(t_0))$.

We first note that (2-11) from Lemma 2.2 guarantees that for every $t \in [t_0, T)$

$$B_{g(t_0)}\left(p, \frac{r_0}{C_1}\right) \subset B_{g(t)}\left(p, \frac{r_0}{\sqrt{C_1}}\right) \subset B_{g(t_0)}(p, r_0). \tag{3-17}$$

We furthermore note that $r_0/\sqrt{C_1}$ cannot be larger than $\mathrm{inj}_{g(t)}(p)$ for any $t \in [t_0, T)$ as otherwise $B_{g(t)}(p, r_0/\sqrt{C_1})$, and thus also $B_{g(t_0)}(p, r_0)$, would need to contain a curve σ starting and ending in p that is not contractible in M, which would contradict the fact that $r_0 < \mathrm{inj}_{g(t_0)}(p)$.

Hence $B_{g(t)}(p, r_0/\sqrt{C_1})$ is isometric to a ball in the hyperbolic plane and so the smallness of the energy $E(u(t), g(t), B_{g(t)}(p, r_0/\sqrt{C_1})) < \varepsilon_0$ obtained from (3-16) and (3-17) allows us to apply Proposition 3.4

for any $\varphi \in C_c^\infty(B_{g(t_0)}(p, r_0/C_1), [0,1])$ and any time $t \in [t_0, T)$. This will be crucial in the proof of the following:

Claim. For any $p \in V$ and $\varphi \in C_c^\infty(B_{g(t_0)}(p, r_0/C_1), [0,1])$ (with $r_0 > 0$ chosen as above) we have

$$\sup_{t\in[t_0,T)} \int \varphi^2 |\partial_t u|^2 \, dv_g < \infty. \tag{3-18}$$

In particular there exists a neighbourhood W of V so that

$$\sup_{t\in[t_0,T)} \|u(t)\|_{H^2(W,g(t))} < \infty.$$

Proof of Claim. To prove the first part of the claim, we differentiate (3-5) in time, test with $\varphi^2\, \partial_t u$ and use that $(d/dt) dv_g = 0$ to write

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \varphi^2|\partial_t u|^2\,dv_g + \int \varphi^2 |d\,\partial_t u|^2\,dv_g \\
&\quad = -\int \langle d\,\partial_t u, d(\varphi^2)\rangle_g \cdot \partial_t u\,dv_g + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int \Delta_{g(t+\varepsilon)} u\cdot\varphi^2\,\partial_t u\,dv_{g(t+\varepsilon)} \\
&\qquad\qquad + \int \partial_t(A_g(u)(du,du))\cdot\varphi^2\,\partial_t u\,dv_g \\
&\quad \le \frac{1}{8}\int \varphi^2|d\,\partial_t u|^2\,dv_g + C\|d\varphi\|^2_{L^\infty(M,g)}\cdot\|\partial_t u\|^2_{L^2(M,g)} - \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int \langle du, d(\varphi^2\,\partial_t u)\rangle_{g(t+\varepsilon)}\,dv_g \\
&\qquad\qquad + C\|\partial_t g\|^2_{L^\infty(\mathrm{supp}(\varphi),g)} E(u,g) + C\int |\partial_t u|^2 |du|_g^2 \varphi^2\,dv_g \\
&\quad \le \frac{1}{4}\int \varphi^2|d\,\partial_t u|^2\,dv_g + C\|d\varphi\|^2_{L^\infty(M,g)}\|\partial_t u\|^2_{L^2(M,g)} + C\|\partial_t g\|^2_{L^\infty(\mathrm{supp}(\varphi),g)} \\
&\qquad\qquad + \widehat{C}\int |\partial_t u|^2|du|_g^2\varphi^2\,dv_g,
\end{aligned} \tag{3-19}$$

where C and $\widehat{C}$ depend only on a bound E_0 on the initial energy and the target manifold, and the value of $\widehat{C}$ is fixed in what follows.

To estimate the last term in (3-19) we first apply Proposition 3.4 to get

$$\int \varphi^2|\partial_t u|^2|du|_g^2\,dv_g \le C\Big(\int \varphi^2|\partial_t u|^4 dv_g\Big)^{1/2}\cdot\Big[\int \varphi^2|\partial_t u|^2\,dv_g + C\|d\varphi\|^2_{L^\infty(M,g)}\Big]^{1/2}.$$

We then recall that $\mathrm{supp}(\varphi)$ is contained in the ball $B_{g(t)}(p, r_0/\sqrt{C_1})$ for every $t \in [t_0, T)$ and that $r_0/\sqrt{C_1} \le \min(\mathrm{inj}_{g(t)}(p), 1)$. We may thus view $(\mathrm{supp}(\varphi), g(t))$ as a subset of the unit ball in the hyperbolic plane and apply the Sobolev embedding theorem to estimate the first factor in the above inequality by

$$\begin{aligned}
\Big(\int \varphi^2|\partial_t u|^4\,dv_g\Big)^{1/2} &= \|\varphi|\partial_t u|^2\|_{L^2} \le C\|d(\varphi|\partial_t u|^2)\|_{L^1} \\
&\le C\|\partial_t u\|_{L^2(M,g)}\Big(\int |d\,\partial_t u|^2\varphi^2\,dv_g\Big)^{1/2} + C\|d\varphi\|_{L^\infty(M,g)}\|\partial_t u\|^2_{L^2(M,g)}.
\end{aligned} \tag{3-20}$$

Combined, this allows us to estimate the final term in (3-19) by

$$\widehat{C}\int |\partial_t u|^2 |du|^2 \varphi^2\, dv_g \le \frac{1}{4}\int |d\,\partial_t u|^2\varphi^2\, dv_g + C\|\partial_t u\|^2_{L^2(M,g)}\biggl[\int \varphi^2|\partial_t u|^2\, dv_g + C\|d\varphi\|^2_{L^\infty(M,g)}\biggr]$$

and thus to reduce (3-19) to

$$\frac{d}{dt}\int \varphi^2|\partial_t u|^2\, dv_g + \int \varphi^2 |d\,\partial_t u|^2\, dv_g \le C\|\partial_t u\|^2_{L^2(M,g)}\cdot\biggl[\|d\varphi\|^2_{L^\infty(M,g)} + \int \varphi^2|\partial_t u|^2\, dv_g\biggr] + C\|\partial_t g\|^2_{L^\infty(\mathrm{supp}(\varphi),g)}. \tag{3-21}$$

Since $\partial_t g$ is controlled on $\mathrm{supp}(\varphi)\subset \delta_{\overline{K}}(t_0)$-thick$(M, g(t_0))$ by the estimate (2-13) of Lemma 2.2, while estimate (2-11) from the same lemma implies $\|d\varphi\|_{L^\infty(M,g(t))} \le \sqrt{C_1}\|d\varphi\|_{L^\infty(M,g(t_0))}$, we thus conclude that

$$\begin{aligned}\frac{d}{dt}&\int \varphi^2|\partial_t u|^2\, dv_g\\ &\le C\|d\varphi\|^2_{L^\infty(M,g(t_0))}\|\partial_t u\|^2_{L^2(M,g)} + C\|\partial_t u\|^2_{L^2(M,g)}\int \varphi^2|\partial_t u|^2\, dv_g + C\delta_{\overline{K}}(t_0)^{-1}\|\partial_t g\|^2_{L^2(M,g)}\\ &\le C\Bigl(-\frac{dE}{dt}\Bigr)\int \varphi^2|\partial_t u|^2\, dv_g + C\Bigl(-\frac{dE}{dt}\Bigr)\cdot[\|d\varphi\|^2_{L^\infty(M,g(t_0))} + \delta_{\overline{K}}(t_0)^{-1}],\end{aligned} \tag{3-22}$$

by (1-2), where C now depends also on the genus of M and η. Hence (3-18) follows using Gronwall's lemma. The second part of the claim is now an immediate consequence of (3-18) and Proposition 3.4. □

Based on the claim we have just proven, we can now establish convergence of $u(t)$ in $C^l(V)$ for every $l\in\mathbb{N}$ by well-known arguments: First of all, we may reduce the neighbourhood W of V if necessary to ensure that $W\subset\delta_{\overline{K}}(t_0)$-thick$(M, g(t_0))$; compare (3-12) and (3-13). We then apply the Sobolev embedding theorem to obtain that

$$\sup_{t\in[t_0,T)}\|du(t)\|_{L^p(W,g(t))} < \infty \quad \text{for every } 1\le p<\infty.$$

The control on the metrics $g(t)$, $t\in[t_0,T)$, obtained in Lemma 2.2 thus allows us to view (3-5) as a uniformly parabolic equation on the fixed surface $(W, g(t_0))$, for times t in this interval $[t_0,T)$, whose right-hand side is in L^p for every $p<\infty$. Standard parabolic theory combined with the fact that u is by assumption smooth away from T, implies that u is in the parabolic Sobolev space $W^{2,1;p}(\widetilde{W}\times[t_0,T))$ for every $p<\infty$ for a slightly smaller neighbourhood $\widetilde{W}$ of V. In particular u is Hölder continuous with exponent α for every $\alpha<1$ on $\widetilde{W}\times[0,T)$.

Taking covariant derivatives $\nabla^l_{g(t)}$ of (3-5) allows us to repeat the above argument and obtain that $(x,t)\mapsto(\nabla^l_{g(t)}u)(x,t)$ is Hölder continuous on $V\times[t_0,T)$ for every $l\in\mathbb{N}$. As the metrics converge smoothly to h on V, this allows us to conclude that also $u(t)\to\bar{u}$ in $C^l(V,h)$ for every $l\in\mathbb{N}$, for some $\bar{u}$.

Since the obtained convergence implies in particular that the set S defined in (1-5), as used in Proposition 3.3, agrees with the set $\widetilde{S}$ of points satisfying (3-14) considered here, this completes the proof of part (1) of Proposition 3.3.

For the proof of part (2) of the proposition we closely follow the arguments of [Topping 2004, Section 2].

Let $p \in S$. As above we choose $t_0 < T$ so that (3-12) holds true for $\Omega = \{p\}$, which we recall allows us to apply Lemmas 2.2 and 3.1 on balls $B_{g(t_0)}(p, r_0)$, $r_0 \in (0, \delta_{\overline{K}}(t_0))$ since (3-13) ensures that such balls are contained in $\delta_{\overline{K}}(t_0)$-thick$(M, g(t_0))$. We fix such a radius r_0 which is small enough so that $B_{g(t_0)}(p, r_0)$ contains no other element of the singular set S.

Given any fixed cut-off function $\psi \in C_c^\infty([0,1),[0,1])$ with $\psi \equiv 1$ on $\left[0, \frac{1}{2}\right]$ and with $\|\psi'\|_{L^\infty} \leq 4$, we set

$$\varphi_r(x) := \psi\left(\frac{\operatorname{dist}_{g(t_0)}(p,x)^2}{r^2}\right), \quad 0 < r < r_0,$$

and note that $\|d\varphi_r\|_{L^\infty(M, g(t_0))} \leq C/r$. As $\operatorname{supp}(\varphi_r) \subset B_{g(t_0)}(p, r_0)$ we can apply Lemma 3.1 to control the associated cut-off energies $E_r(t) := E_{\varphi_r}(t)$ defined in (3-1) and obtain in particular that $\lim_{t\uparrow T} E_r(t)$ exists for every $r \in (0, r_0)$. Combined with the local C^l convergence of $u(t) \to \bar{u}$ on $\mathcal{U} \setminus S$ and the convergence of the metrics obtained in part (1) of Theorem 1.1 this implies that

$$\widehat{E}_p := \lim_{t\uparrow T} E_r(t) - \frac{1}{2}\int \varphi_r^2 |d\bar{u}|_h^2 \, dv_h \tag{3-23}$$

is independent of $r \in (0, r_0)$.

Let now $\nu > 0$. For $t \in [t_0, T)$ sufficiently close to T so that $\nu(T-t)^{1/2} < r_0$ we can apply Lemma 3.1 to $s \mapsto E_{\nu(T-t)^{1/2}}(s)$, $s \in [t_0, T)$, in order to obtain the second inequality of

$$\begin{aligned}
&|E_{\nu(T-t)^{1/2}}(t) - \widehat{E}_p| \\
&\leq \left|E_{\nu(T-t)^{1/2}}(t) - \lim_{s\uparrow T} E_{\nu(T-t)^{1/2}}(s)\right| + \frac{1}{2}\int \varphi^2_{\nu(T-t)^{1/2}} |d\bar{u}|_h^2 \, dv_h \\
&\leq E(t) - E(T) + C[\nu^{-1} + \delta_{\overline{K}}^{-1/2}(t_0)\cdot(T-t)^{1/2}]\cdot(E(t)-E(T))^{1/2} + E(\bar{u}, h, B_{g(t_0)}(p, \nu(T-t)^{1/2})).
\end{aligned} \tag{3-24}$$

We furthermore note that $B_{g(t_0)}(p, \nu(T-t)^{1/2}) \subset B_h(p, \sqrt{C_1}\nu(T-t)^{1/2})$, compare with (2-11) of Lemma 2.2, and thus that the last term in (3-24) tends to zero as $t \uparrow T$. Passing to the limit $t \uparrow T$ in (3-24) we thus obtain that also

$$\lim_{t\uparrow T} E_{\nu(T-t)^{1/2}}(t) = \widehat{E}_p \quad \text{for every } \nu > 0.$$

Combined with the equivalence (2-11) of the metrics obtained in Lemma 2.2 we therefore get that for any $\nu > 0$

$$\begin{aligned}
\lim_{r\downarrow 0}\lim_{t\uparrow T} E\big(u(t), g(t), B_{g(t)}(p,r)\big) &\leq \lim_{r\downarrow 0}\lim_{t\uparrow T} E\big(u(t), g(t), B_{g(t_0)}(p, \sqrt{C_1}r)\big) \leq \lim_{r\downarrow 0}\lim_{t\uparrow T} E_{2\sqrt{C_1}r}(t) \\
&= \widehat{E}_p = \lim_{t\uparrow T} E_{\nu C_1^{-1/2}(T-t)^{1/2}}(t) \\
&\leq \liminf_{t\uparrow T} E\big(u(t), g(t), B_{g(t_0)}(p, \nu C_1^{-1/2}(T-t)^{1/2})\big) \\
&\leq \liminf_{t\uparrow T} E\big(u(t), g(t), B_{g(t)}(p, \nu(T-t)^{1/2})\big).
\end{aligned} \tag{3-25}$$

As the "reverse" inequality

$$\limsup_{t\uparrow T} E\big(u(t), g(t), B_{g(t)}(p, \nu(T-t)^{1/2})\big) \le \lim_{r\downarrow 0}\lim_{t\uparrow T} E\big(u(t), g(t), B_{g(t)}(p, r)\big)$$

is trivially true, this proves the second equality in (3-9), including the existence of the limits taken, while the first inequality of (3-9) follows directly from the equivalence (2-11) of the metrics $g(t)$ and h obtained in Lemma 2.2.

To establish the final inequality of (3-9) we closely follow [Topping 2004, Section 2]. Given a sequence of times $t_n \uparrow T$ as in (1-10) and a point $p \in S$, we pick local isothermal coordinates centred at p for each of the $g(t_n)$ by identifying $B_{g(t_n)}(p, r_0)$ with the corresponding ball centred at zero of the Poincaré hyperbolic disc, viewed conformally as the unit disc centred at the origin in $\mathbb{R}^2$, and rescale to obtain a sequence of maps

$$u_n(x) := u(r_n x, t_n), \quad r_n := (T - t_n)^{1/2},$$

for which $\|\tau(u_n)\|_{L^2(\mathcal{K})} \to 0$ for every $\mathcal{K} \Subset \mathbb{R}^2$.

Since (3-25) implies $E(u_n, B(0,\Lambda)\setminus B(0,\lambda)) \to 0$ for any $0<\lambda<\Lambda$, a subsequence of the maps u_n converges strongly in H^1 away from 0 to a constant map, while bubbles $\{\omega_j\}_{j=1}^{m'}$ develop near the origin at scales $\hat\lambda_n^j \to 0,\ n\to\infty$.

The scales at which the bubbles ω_j develop in the original sequence are thus $\lambda_n^j = r_n\hat\lambda_n^j = o((T-t_n)^{1/2})$ and the "no-loss-of-energy" result for bubble tree convergence of almost harmonic maps of [Ding and Tian 1995] ensures that all the energy of the u_n is captured by these bubbles, i.e., for every $\Lambda > 0$ we have

$$\lim_{n\to\infty} E(u_n, B_\Lambda(0)) = \sum_{l=1}^{m'} E(\omega_l).$$

Taking the limit $\Lambda \downarrow 0$, and bearing in mind that all but the final equality of (3-9) has already been established, we find that for every $p \in S$, we have

$$\lim_{r\downarrow 0}\lim_{t\uparrow T} E(u(t), g(t), B_{g(t)}(p, r)) = \sum_{l=1}^{m'} E(\omega_l), \tag{3-26}$$

completing the proof of (3-9).

Finally, given any compact subset $\Omega \Subset \mathcal{U}$ which is large enough for S to be contained in the interior of Ω, we can combine (3-9) with the strong H^1_{loc} convergence of $u(t) \to \bar u$ on $\mathcal{U}\setminus S$ and the convergence of the metrics to obtain that indeed

$$\lim_{t\uparrow T} E(u(t), g(t), \Omega) = E(\bar u, h, \Omega) + \sum_{l=1}^{m''} E(\omega_l), \tag{3-27}$$

where $\{\omega_l\}_{l=1}^{m''}$ is the set of all bubbles developing at points in S along a sequence of times t_n as considered in the proposition. □

3.3. ***All energy lost down collars is represented by bubbles.*** At this point we have a good description of the convergence of $u(t)$ and $g(t)$ locally on $\mathcal{U} = M \setminus F$, with Proposition 3.3 completing the proof of Theorem 1.1 and establishing part (1) of Theorem 1.4. In this section we prove parts (2) and (3) of Theorem 1.4, which show that near the centre of degenerating collars, the map looks like a collection of bubbles, while on larger scales that are nevertheless vanishing scales, where we have no way of showing that the map is becoming harmonic, no energy can be lost.

Proof of part (2) *of Theorem 1.4.* As a next step we now prove part (2) of Theorem 1.4, which can be seen as quantifying the size of the part of $\mathcal{U}$ on which the energy has almost reached its limit. As we can only apply the local energy estimate from Lemma 3.1 on regions with sufficiently large injectivity radius, we will obtain the existence of a limit of the energy on the $[T-t]$-thin part by proving that the limit on the $[T-t]$-thick part exists and agrees with E_{thick} and then appealing to the existence of a limit of the total energy $E(t)$.

As above it will be more convenient to work not with energies over given sets, but with cut-off energies E_φ as defined in (3-1). To this end we let $\delta_K(t) = K(T-t)(E(t)-E(T))$, $K \geq \overline{K}$, be as in Lemma 2.2 and recall that the characterisation of the pinching set (1-15) implies in particular that for every $t_0 \in [0, T)$

$$\operatorname{inj}_{g(t_0)}(M) < \delta_K(t_0)$$

and thus that

$$A_{K,t_0} := \{x \in M : \operatorname{inj}_{g(t_0)}(x) \leq \delta_K(t_0)\}$$

is nonempty. We will always assume that $t_0 \in [0, T)$ is sufficiently large, depending in particular on K, so that $\delta_K(t_0)\cdot(\pi e) < \operatorname{arsinh}(1)$. In this way, not only can we be sure that every point in A_{K,t_0} has injectivity radius less than $\operatorname{arsinh}(1)$, and is thus lying within some collar region around a geodesic of length less than $2\operatorname{arsinh}(1)$, we can also be sure that the 1-fattening of A_{K,t_0}, i.e., $\{p \in M : \operatorname{dist}_{g(t_0)}(p, A_{K,t_0}) < 1\}$, must lie within $\delta_{e\pi K}(t_0)$-thin$(M, g(t_0))$, and hence also lie within a union of such (pairwise disjoint) collars, since by [RT2018a, Lemma A.3] if $x \in A_{K,t_0}$, and $y \in B_{g(t_0)}(x, 1)$ lies in the same collar, then $\operatorname{inj}_{g(t_0)}(y) \leq \operatorname{inj}_{g(t_0)}(x)\cdot(\pi e) \leq \delta_K(t_0)\cdot(\pi e) < \operatorname{arsinh}(1)$, so we cannot escape this collar within a distance 1 of x. In particular, the function $x \mapsto \operatorname{dist}_{g(t_0)}(x, A_{K,t_0})$ is smooth on the 1-fattening of A_{K,t_0}.

Given any smooth cut-off function $\phi : \mathbb{R} \to [0, 1]$ such that $\phi(x) = 0$ for $x \leq 0$, $\phi(x) = 1$ for $x \geq 1$ and $|\phi'| \leq 2$, we can thus define the induced *smooth* cut-off $\varphi_{K,t_0} : M \to [0, 1]$ by

$$\varphi_{K,t_0}(x) := \phi(\operatorname{dist}_{g(t_0)}(x, A_{K,t_0})). \tag{3-28}$$

It is immediately apparent that

$$\varphi_{K,t_0} \equiv 0 \quad \text{on } \delta_K(t_0)\text{-thin}(M, g(t_0)), \tag{3-29}$$

and that the support of φ_{K,t_0} lies within $\delta_K(t_0)$-thick$(M, g(t_0))$ and hence φ_{K,t_0} has compact support within $\mathcal{U}$ owing to (1-16). This will shortly allow us to apply Lemma 3.1 to the corresponding local energy $E_{K,t_0}(t) := E_{\varphi_{K,t_0}}(t)$ that serves as a substitute for the energy of $u(t)$ over $\delta_K(t_0)$-thick$(M, g(t_0))$.

We also claim that

$$\varphi_{K,t_0} \equiv 1 \quad \text{on } \delta_{e\pi K}(t_0)\text{-thick}(M, g(t_0)). \tag{3-30}$$

Indeed, the only way this could fail would be if we could find a point in the 1-fattening of A_{K,t_0} that lies in $\delta_{e\pi K}(t_0)$-thick$(M, g(t_0))$, which we ruled out above.

By (3-29), we see that $E_{K,t_0}(t) \le E\big(u(t), g(t), \delta_K(t_0)\text{-thick}(M, g(t_0))\big)$, and so

$$\lim_{K\to\infty} \limsup_{t\uparrow T} E_{K,t}(t) \le \lim_{K\to\infty} \limsup_{t\uparrow T} E\big(u(t), g(t), \delta_K(t)\text{-thick}(M, g(t))\big). \tag{3-31}$$

On the other hand, by (3-30), we see that $E\big(u(t), g(t), \delta_{e\pi K}(t_0)\text{-thick}(M, g(t_0))\big) \le E_{K,t_0}(t)$, and hence we have the converse inequality

$$\lim_{K\to\infty} \limsup_{t\uparrow T} E\big(u(t), g(t), \delta_K(t)\text{-thick}(M, g(t))\big) \le \lim_{K\to\infty} \limsup_{t\uparrow T} E_{K,t}(t), \tag{3-32}$$

i.e., we have equality in (3-31) and (3-32). Therefore to prove (1-13), it suffices to show that

$$E_{\text{thick}} = \lim_{K\to\infty} \limsup_{t\uparrow T} E_{K,t}(t). \tag{3-33}$$

We claim first that

$$E_{\text{thick}} = \limsup_{t_0\uparrow T} \lim_{t\uparrow T} E_{K,t_0}(t), \tag{3-34}$$

where the existence of $\lim_{t\uparrow T} E_{K,t_0}(t)$ is guaranteed by Lemma 3.1. To see (3-34), first recall that for K, t_0 as above, the support of φ_{K,t_0} is compact within $\mathcal{U}$, and is thus contained within δ-thick$(\mathcal{U}, h)$ for sufficiently small $\delta > 0$. By reducing δ further, we may assume that all bubble points in S lie within the interior of δ-thick$(\mathcal{U}, h)$. Therefore we have $E(u(t), g(t), \delta\text{-thick}(\mathcal{U}, h)) \ge E_{K,t_0}(t)$, and taking the limits $t \uparrow T$, $\delta \downarrow 0$ and $t_0 \uparrow T$ in that order, we find that $E_{\text{thick}} \ge \limsup_{t_0\uparrow T} \lim_{t\uparrow T} E_{K,t_0}(t)$. To see the converse inequality, we observe that by (3-30), for any $\delta > 0$ and $t_0 < T$ sufficiently large (depending on δ, K etc.) we have $\varphi_{K,t_0} \equiv 1$ on δ-thick$(M, g(t_0))$, and so $E(u(t), g(t), \delta\text{-thick}(\mathcal{U}, h)) \le E_{K,t_0}(t)$. This time we take limits in the order $t \uparrow T$, $t_0 \uparrow T$ and then $\delta \downarrow 0$ to give $E_{\text{thick}} \le \limsup_{t_0\uparrow T} \lim_{t\uparrow T} E_{K,t_0}(t)$, and hence (3-34).

Thus (1-13) would follow if we could prove that as $K \to \infty$ we have

$$\limsup_{t_0\uparrow T} |E_{K,t_0}(t_0) - \lim_{t\uparrow T} E_{K,t_0}(t)| \to 0. \tag{3-35}$$

But this follows from Lemma 3.1, which implies that for $t_0 \in [0, T)$ as large as considered above, and every $t \in [t_0, T)$, we have

$$|E_{K,t_0}(t) - E_{K,t_0}(t_0)| \le E(t_0) - E(T) + \frac{C}{K^{1/2}} + C(T - t_0)^{1/2}(E(t_0) - E(T))^{1/2}, \tag{3-36}$$

with C depending only on the genus of M, η and an upper bound on the initial energy, which thus yields (3-35) after taking the limits $t \uparrow T$, $t_0 \uparrow T$ and $K \to \infty$, in that order.

Now that (1-13) has been proved, we verify that (1-12) follows as a result. In particular, we verify that the limit taken in (1-12) exists. However large we take $K > 0$, for sufficiently large $t < T$ we have $T - t \ge \delta_K(t)$, and hence

$$E\big(u(t), g(t), [T-t]\text{-thin}(M, g(t))\big) \ge E\big(u(t), g(t), \delta_K(t)\text{-thin}(M, g(t))\big).$$

Taking a lim inf as $t \uparrow T$ and then the limit $K \to \infty$, and using (1-13) we find that

$$\liminf_{t\uparrow T} E\big(u(t), g(t), [T-t]\text{-thin}(M, g(t))\big) \geq E_{\text{thin}}. \tag{3-37}$$

To obtain the converse inequality, observe that given any $\delta > 0$, for sufficiently large $t < T$ we have $\delta\text{-thin}(\mathcal{U}, h) \supset [T-t]\text{-thin}(M, g(t))$, see (1-6), and therefore

$$E\big(u(t), g(t), \delta\text{-thin}(\mathcal{U}, h)\big) \geq E\big(u(t), g(t), [T-t]\text{-thin}(M, g(t))\big).$$

Provided $\delta > 0$ is sufficiently small (so that the singular set S is in the interior of $\delta\text{-thick}(\mathcal{U}, h)$), we can then take a limit as $t \uparrow T$, followed by a limit as $\delta \downarrow 0$, to give

$$E_{\text{thin}} \geq \limsup_{t\uparrow T} E\big(u(t), g(t), [T-t]\text{-thin}(M, g(t))\big),$$

which when combined with (3-37) completes the proof of (1-12) and hence of part (2) of the theorem. □

While part (2) of Theorem 1.4 gives good control on where energy can concentrate on the degenerating part of the surface, we currently have no control of what parts of the map are lost down the degenerating parts of the collar at the singular time T. This is addressed by part (3), which we shall now prove.

Proof of part (3) *of Theorem 1.4.* Proposition 1.3 tells us that the length $\ell(t_n)$ of the central geodesic of each degenerating collar is controlled like $\ell(t_n) = o(T - t_n)$ and hence that the $[T-t_n]$-thin part of such a collar, where all of the lost energy lives, is represented by longer and longer cylinders $\widetilde{\mathcal{C}}_n := \mathcal{C}(t_n, \delta_n) = (-\widetilde{X}_n, \widetilde{X}_n) \times S^1$, $\delta_n = T - t_n$, equipped with the corresponding collar metrics $g = \rho^2 g_0$.

We can indeed consider the maps on the larger subcollars $\widehat{\mathcal{C}}_n = (-\widehat{X}_n, \widehat{X}_n)$ which correspond to the $[T-t_n]^{1/2}$-thin parts of the collar, where we note that $1 \ll \widetilde{X}_n \ll \widehat{X}_n \ll X(\ell_n)$; compare with (1-9).

We recall from [Rupflin and Topping 2018b, (A.9)] that $\rho(y) \leq \text{inj}_{g(t)}(y)$ as y varies within each collar. Therefore, throughout $\widehat{\mathcal{C}}_n$ we have $\rho \leq (T - t_n)^{1/2}$. By the scaling of the tension field, if we switch from the hyperbolic metric $g_n = g(t_n)$ to the flat cylinder metric $g_0 = ds^2 + d\theta^2$ on each such subcollar, then we can estimate the tension of $u_n := u(t_n)$ according to

$$\|\tau_{g_0}(u_n)\|_{L^2(\widehat{\mathcal{C}}_n, g_0)} \leq \big(\sup_{\widehat{\mathcal{C}}_n} \rho\big) \|\tau_{g_n}(u_n)\|_{L^2(\widehat{\mathcal{C}}_n, g_n)} \leq (T - t_n)^{1/2} \|\tau_{g_n}(u_n)\|_{L^2(M, g_n)} \to 0 \tag{3-38}$$

by (1-10).

We can thus view the u_n's as almost-harmonic maps from longer and longer cylinders $(\widehat{\mathcal{C}}_n, g_0)$ and apply Proposition 1.5 to pass to a subsequence that converges to a full bubble branch.

It is this estimate (3-38) and the precise information on the degenerate region where energy can concentrate obtained in part (2) of Theorem 1.4 that allows us to represent the maps on these parts in terms of branched minimal immersions and curves. We stress that we would not be able to perform this analysis on the whole collar.

We also remark that in our situation we obtain the additional information that any bubble obtained in the convergence to a full bubble branch described in Proposition 1.5 will be contained in the $[T-t_n]$-thin part of the surface, as we already know that no energy can be lost on $\{p : \text{inj}_{g(t)}(p) \in [(T-t), (T-t)^{1/2}]\}$. □

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A RIGOROUS DERIVATION FROM THE KINETIC CUCKER–SMALE MODEL TO THE PRESSURELESS EULER SYSTEM WITH NONLOCAL ALIGNMENT

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We consider the kinetic Cucker–Smale model with local alignment as a mesoscopic description for the flocking dynamics. The local alignment was first proposed by Karper, Mellet and Trivisa (2014), as a singular limit of a normalized nonsymmetric alignment introduced by Motsch and Tadmor (2011). The existence of weak solutions to this model was obtained by Karper, Mellet and Trivisa (2014), and in the same paper they showed the time-asymptotic flocking behavior. Our main contribution is to provide a rigorous derivation from a mesoscopic to a macroscopic description for the Cucker–Smale flocking models. More precisely, we prove the hydrodynamic limit of the kinetic Cucker–Smale model with local alignment towards the pressureless Euler system with nonlocal alignment, under a regime of strong local alignment. Based on the relative entropy method, a main difficulty in our analysis comes from the fact that the entropy of the limit system has no strict convexity in terms of density variable. To overcome this, we combine relative entropy quantities with the 2-Wasserstein distance.

1. Introduction

This article is mainly devoted to providing a rigorous justification of the hydrodynamic limit of the kinetic Cucker–Smale model to the pressureless Euler system with nonlocal alignment force. Cucker and Smale [2007] introduced an agent-based model capturing a flocking phenomenon observed within complex systems, such as flocks of birds, schools of fish and swarms of insects. The Cucker–Smale (CS) model has received extensive attention in the mathematical community, as well as physics, biology, engineering and social science, etc.; see for instance [Carlen et al. 2015; Cañizo et al. 2011; Carrillo et al. 2010; Duan et al. 2010; Fornasier et al. 2011; Ha et al. 2014c; 2017; Ha and Tadmor 2008; Poyato and Soler 2017; Zavlanos et al. 2011]. Motsch and Tadmor [2011] proposed a modified Cucker–Smale model by replacing the original CS alignment by a normalized nonsymmetric alignment. Karper, Mellet, and Trivisa [Karper et al. 2014] proposed a new kinetic flocking model as a combination of the CS alignment and a local alignment interaction, where the latter was obtained as a singular limit of the nonsymmetric alignment introduced by Motsch and Tadmor.

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We consider the kinetic flocking model without Brownian noise, proposed by Karper, Mellet and Trivisa [Karper et al. 2013] on $\mathbb{T}^d \times \mathbb{R}^d$:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) + \nabla_v \cdot ((u-v)f) = 0, \\ &L[f](t,x,v) = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \psi(x-y) f(t,y,w)(w-v)\, dw\, dy, \\ &u(t,x) = \frac{\int_{\mathbb{R}^d} vf\, dv}{\int_{\mathbb{R}^d} f\, dv}, \quad \|f(0)\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = 1. \end{aligned} \tag{1-1}$$

Here $\psi : \mathbb{T}^d \to \mathbb{R}^d$ is a Lipschitz communication weight that is positive and symmetric, i.e., $\psi(x-y) = \psi(y-x)$. The term $\nabla_v \cdot (L[f]f)$ describes a nonlocal alignment due to the original Cucker–Smale flocking mechanism, while the last term $\nabla_v \cdot ((u-v)f)$ describes a local alignment interaction, because of the averaged local velocity u. The global existence of weak solutions to (1-1) was proved in [Karper et al. 2013]. The flocking behaviors of (1-1), however, have not been studied so far. We here provide its time-asymptotic behavior.

As a mesoscopic description, the kinetic model (1-1) is posed in $(t,x,v) \in \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$, i.e., in $2d+1$ dimensions. This feature provides an accurate description for a significant number of particles. However, its numerical test is very costly with respect to an associated macroscopic description. Hence, it is very important to find a suitable parameter regime on which the complexity of (1-1) is reduced.

The main goal of this article is to show a singular limit of (1-1) in a regime of strong local alignment:

$$\begin{aligned} &\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nabla_v \cdot (L[f^\varepsilon]f^\varepsilon) + \frac{1}{\varepsilon}\nabla_v \cdot ((u^\varepsilon - v)f^\varepsilon) = 0, \\ &L[f^\varepsilon](t,x,v) = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \psi(x-y) f^\varepsilon(t,y,w)(w-v)\, dw\, dy, \\ &u^\varepsilon = \frac{\int_{\mathbb{R}^d} vf^\varepsilon\, dv}{\int_{\mathbb{R}^d} f^\varepsilon\, dv}, \\ &f^\varepsilon|_{t=0} = f_0^\varepsilon, \quad \|f_0^\varepsilon\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = 1. \end{aligned} \tag{1-2}$$

As $\varepsilon \to 0$, it is expected that the solution f^ε of (1-2) converges, in some weak sense, to a monokinetic distribution

$$\delta_{v=u(t,x)} \otimes \rho(t,x); \tag{1-3}$$

see Remark 1.1. Here, $\delta_{v=u(t,x)}$ denotes a Dirac mass in v centered on $u(t,x)$. Also, as we shall explain later, at least formally ρ and u should solve the associated limit system given by the pressureless Euler system with nonlocal flocking dissipation:

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0, \\ &\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\, dy, \\ &\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad \|\rho_0\|_{L^1(\mathbb{T}^d)} = 1. \end{aligned} \tag{1-4}$$

The main difficulty in the justification of this limit comes from the singularity of the monokinetic distribution. To the best of our knowledge, there is no general method to handle the hydrodynamic limit from some kinetic equations to the pressureless Euler systems, no matter what regime is considered. Indeed, there are few results on this kinds of limit; see [Jabin and Rey 2017; Kang 2018; Kang and Vasseur 2015] (see also [Jabin 2000] for a general treatment of similar regimes that lead to the Dirac formation and pressureless gases equations).

Remark 1.1. In this paper we will use the symbol $\otimes$ in two different contexts: if μ is a measure on a complete metric space X, and $\{\nu_x\}_{x\in X}$ is a family of measures on a complete metric space Y, then $\nu_x \otimes \mu$ denotes the measure on $X \times Y$ defined as

$$\int_{X\times Y} \varphi\, d[\nu_x \otimes \mu] = \int_X \left(\int_Y \varphi(x, y)\, d\nu_x(y) \right) d\mu(x) \quad \text{for all } \varphi \in C_c(X \times Y).$$

When ν_x is independent of x (that is, $\nu_x = \nu$ for all x), we use the more standard notation $\mu \otimes \nu$ (instead of $\nu \otimes \mu$, as done before) to denote the product measure:

$$\int_{X\times Y} \varphi\, d[\mu \otimes \nu] = \int_X \left(\int_Y \varphi(x, y)\, d\nu(y) \right) d\mu(x) \quad \text{for all } \varphi \in C_c(X \times Y).$$

Finally, if $a, b \in \mathbb{R}^d$ are vectors, then $a \otimes b$ denotes the $(d \times d)$-matrix with entries

$$(a \otimes b)_{ij} = a_i b_j \quad \text{for all } i, j = 1, \dots, d.$$

The meaning will always be clear from the context.

It is worth mentioning that the pressureless Euler system without the nonlocal alignment has been used for the formation of large-scale structures in astrophysics and the aggregation of sticky particles [Silk et al. 1983; Zeldovich 1970]. For more theoretical studies on the pressureless gases, we for example refer to [Bouchut 1994; Bouchut and James 1999; Boudin 2000; Brenier and Grenier 1998; Huang and Wang 2001; Poupaud and Rascle 1997; Weinan et al. 1996].

The macroscopic flocking model (1-4) or its variants have been formally derived under a monokinetic ansatz (1-3), and studied in various topics; see for example [Do et al. 2018; Ha et al. 2014a; 2014b; 2015; Tadmor and Tan 2014]. In [Ha et al. 2014b], the authors showed the global well-posedness of (1-4) with suitably smooth and small initial data, and the time-asymptotic flocking behavior. In [Ha et al. 2015], the authors dealt with a moving boundary problem of (1-4) with compactly supported initial density. We also refer to [Ha et al. 2014a] for a reformulation of (1-4) into hyperbolic conservation laws with damping in one dimension.

In [Karper et al. 2015], the authors showed the hydrodynamic limit of the kinetic flocking model (1-1) with Brownian motion, that is, a Vlasov–Fokker–Planck-type equation, under the regime of strong local alignment and strong Brownian motion:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nabla_v \cdot (L[f^\varepsilon] f^\varepsilon) + \frac{1}{\varepsilon} \nabla_v \cdot ((u^\varepsilon - v) f^\varepsilon) - \frac{1}{\varepsilon} \Delta_v f^\varepsilon = 0. \tag{1-5}$$

In this case, as $\varepsilon \to 0$, f^ε converges to a smooth local equilibrium given by a local Maxwellian, contrary to (1-3). There, the authors used the relative entropy method, heavily relying on a strict convexity of the entropy of the isothermal Euler system (as a limit system of (1-5)):

$$\begin{aligned}&\partial_t \rho + \nabla\cdot(\rho u) = 0,\\ &\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\rho = \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\,dy.\end{aligned}$$

The relative entropy method based on a strict convex entropy has been successfully used to prove the hydrodynamic limit of Vlasov–Fokker–Planck-type equations; we refer to [Berthelin and Vasseur 2005; Carrillo et al. 2016; Goudon et al. 2004; Mellet and Vasseur 2008; Vasseur 2008].

On the other hand, the pressureless Euler system (1-4) has a convex entropy given by

$$\eta(\rho,\rho u) = \rho\,\tfrac{1}{2}(|u|^2), \tag{1-6}$$

which is not strictly convex with respect to ρ. For this reason, the associated relative entropy (1-6) is not enough to control the convergence of the nonlocal alignment term (compare with [Kang and Vasseur 2015], where the nonlocal alignment is not present). To overcome this difficulty, we first estimate an L^2-distance of characteristics generated by vector fields u^ε and u that controls the 2-Wasserstein distance of densities, and then combine the estimates of the relative entropy and the L^2-distance of characteristics.

As a related work on (1-5), we refer to [Carrillo et al. 2016], where the authors studied the flocking behavior and hydrodynamic limit of a coupled system of (1-5) and fluid equations via drag force.

The rest of this paper is organized as follows. In Section 2, we mention different scales of Cucker–Smale models from a microscopic level to a macroscopic level, and then specify some known existence results on the two descriptions (1-1) and (1-4). In Section 3, we present our main theorem on the hydrodynamic limit, and collect some useful results on the relative entropy method and the optimal transportation theory that are used in the proof of the main theorem. In Section 4, we present some structural hypotheses to guarantee the hydrodynamic limit in a general setting. Then we apply the general result to our systems by verifying the hypotheses in Section 5. In the Appendix, we provide the proof of the long time-asymptotic flocking dynamics and the existence of monokinetic solutions for the kinetic model (1-1).

2. Various scales of Cucker–Smale models

We first present various scales of Cucker–Smale models, from a microscopic level to a macroscopic level. Then we state some known results on global existence of weak solutions to the kinetic description (1-1), and local existence of smooth solutions to the limit system (1-4). Those results are crucially used in the proof of the main theorem. Finally, in Theorem 2.2, we present the time-asymptotic flocking behavior of the kinetic model (1-1).

Variants of Cucker–Smale models. We briefly present the kinetic CS model and its variants. Cucker and Smale [2007] proposed a mathematical model to explain the flocking phenomenon:

$$\begin{aligned}&\frac{dx_i}{dt} = v_i, \quad i=1,\dots,N,\\ &\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N}\psi(x_j-x_i)(v_j-v_i),\end{aligned} \tag{2-1}$$

where $x_i, v_i \in \mathbb{R}^d$ denote the spatial position and velocity of the i-th particle for an ensemble of N self-propelled particles. The kernel $\psi(|x_j - x_i|)$ is a communication weight given by

$$\psi(x_j - x_i) = \frac{\lambda}{(1 + |x_j - x_i|^2)^\beta}, \quad \beta \geq 0,\ \lambda > 0. \tag{2-2}$$

The system (2-1) with (2-2) was used as an analytical description of the Vicsek model [Vicsek et al. 1995] without resorting to the first principle of physics.

When the number of particles is sufficiently large, the ensemble of particles can be described by the one-particle density function $f = f(t, x, v)$ at the spatial-velocity position $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ at time t. Then, the evolution of f is governed by the following Vlasov-type equation:

$$\begin{gathered} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f] f) = 0, \\ L[f](t, x, v) = \int_{\mathbb{R}^{2d}} \psi(x - y) f(t, y, w)(w - v)\, dw\, dy. \end{gathered} \tag{2-3}$$

This was first introduced by Ha and Tadmor [2008] using the BBGKY hierarchy from the particle CS model (2-1). A rigorous mean-field limit was given in [Ha and Liu 2009].

Motsch and Tadmor [2011] recognized a drawback of the CS model (2-1), which is due to the normalization factor $1/N$. More precisely, when a small group of agents are located far away from a much larger group of agents, the internal dynamics of the small group is almost halted since the total number of agents is relatively very large. To solve this issue, they replaced the nonlocal alignment $L[f]$ by a normalized nonsymmetric alignment operator:

$$\bar{L}[f](t, x, v) := \frac{\int_{\mathbb{R}^{2d}} K^r(x - y) f(t, y, w)(w - v)\, dw\, dy}{\int_{\mathbb{R}^{2d}} K^r(x - y) f(t, y, w)\, dw\, dy},$$

where the kernel K^r is a communication weight and r denotes the radius of influence of K^r.

In [Karper et al. 2014], the authors considered the case when the communication weight is extremely concentrated near each agent, so that the alignment term $\bar{L}[f]$ corresponds to a short-range interaction. More precisely, they rigorously justified the singular limit $r \to 0$, i.e., as K^r converges to the Dirac distribution δ_0, in which case $\bar{L}[f]$ converges to a local alignment term:

$$\bar{L}[f](t, x, v) \to \frac{\int_{\mathbb{R}^d} f(t, x, w)(w - v)\, dw}{\int_{\mathbb{R}^d} f(t, x, w)\, dw} = u(t, x) - v,$$

where $u(t, x)$ denotes the averaged local velocity defined as

$$u(t, x) = \frac{\int_{\mathbb{R}^d} v f(t, x, v)\, dv}{\int_{\mathbb{R}^d} f(t, x, v)\, dv}.$$

Hence, their new model became (1-1), which consists of two kinds of alignment force: a nonlocal alignment due to the original CS model, plus a local alignment.

***Existence of weak solutions to* (1-2).** In [Karper et al. 2013], the authors showed the existence of weak solutions to the kinetic Cucker–Smale model with local alignment, noise, self-propulsion, and friction:

$$
\begin{aligned}
\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(L[f]f) + \nabla_v\cdot((u-v)f) &= \sigma\Delta_v f - \nabla_v\cdot((a-b|v|^2)vf),\\
L[f] &= \int_{\mathbb{R}^{2d}} \psi(x-y) f(t,y,w)(w-v)\,dw\,dy,
\end{aligned}
\tag{2-4}
$$

where the kernel ψ is the same as (1-2) and a, b, and σ are nonnegative constants. By their result applied with $a=b=\sigma=0$ inside the periodic domain $\mathbb{T}^d$, we obtain existence of solutions for (1-2). To precisely state such an existence result, we need to define a (mathematical) entropy $\mathcal{F}(f^\varepsilon)$ and kinetic dissipations $\mathcal{D}_1(f^\varepsilon)$, $\mathcal{D}_2(f^\varepsilon)$ for (1-2):

$$
\begin{aligned}
\mathcal{F}(f^\varepsilon) &:= \int_{\mathbb{R}^d} \frac{|v|^2}{2} f^\varepsilon\,dv,\\
\mathcal{D}_1(f^\varepsilon) &:= \int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon |u^\varepsilon - v|^2\,dv\,dx,\\
\mathcal{D}_2(f^\varepsilon) &:= \frac{1}{2}\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(x,v) f^\varepsilon(y,w)|v-w|^2\,dx\,dy\,dv\,dw.
\end{aligned}
\tag{2-5}
$$

Proposition 2.1. *For any $\varepsilon>0$, assume that f_0^ε satisfies*

$$
f_0^\varepsilon \ge 0,\qquad f_0^\varepsilon \in L^1\cap L^\infty(\mathbb{R}^{2d}),\qquad |v|^2 f_0^\varepsilon \in L^1(\mathbb{R}^{2d}). \tag{2-6}
$$

Then there exists a weak solution $f^\varepsilon \ge 0$ of (1-2) *such that*

$$
\begin{aligned}
f^\varepsilon &\in C(0,T;L^1(\mathbb{R}^{2d})) \cap L^\infty((0,T)\times\mathbb{R}^{2d}),\\
|v|^2 f^\varepsilon &\in L^\infty(0,T;L^1(\mathbb{R}^{2d})),
\end{aligned}
\tag{2-7}
$$

and (1-2) *holds in the sense of distributions, that is, for any $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^{2d})$, the weak formulation holds*:

$$
\int_0^t\int_{\mathbb{R}^{2d}} f^\varepsilon\Big(\partial_t\varphi + v\cdot\nabla_x\varphi + L[f^\varepsilon]\cdot\nabla_v\varphi + \frac{1}{\varepsilon}(u^\varepsilon - v)\cdot\nabla_v\varphi\Big)\,dv\,dx\,ds + \int_{\mathbb{R}^{2d}} f_0^\varepsilon\varphi(0,\cdot)\,dv\,dx = 0. \tag{2-8}
$$

Moreover, *f^ε preserves the total mass and satisfies the entropy inequality*

$$
\int_{\mathbb{T}^d}\mathcal{F}(f^\varepsilon)(t)\,dx + \frac{1}{\varepsilon}\int_0^t \mathcal{D}_1(f^\varepsilon)(s)\,ds + \int_0^t \mathcal{D}_2(f^\varepsilon)(s)\,ds \le \int_{\mathbb{T}^d}\mathcal{F}(f_0^\varepsilon)\,dx. \tag{2-9}
$$

The entropy inequality (2-9) is crucially used in the proof of Theorem 3.1.

***Flocking behavior of the kinetic model* (1-1).** We now present the time-asymptotic flocking behavior of solutions to the kinetic model (1-1). For that, we define the following two Lyapunov functionals:

$$
\begin{aligned}
\mathcal{E}_1(t) &:= \int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)|u(t,x)-v|^2\,dv\,dx,\\
\mathcal{E}_2(t) &:= \int_{\mathbb{T}^{2d}} \rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy,
\end{aligned}
$$

where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$. We remark that $\mathcal{E}_1$ measures a local alignment, and $\mathcal{E}_2$ measures alignment of the averaged local velocities. Then, for the flocking estimate, we combine the two functionals as follows:

$$\mathcal{E}(t) := \mathcal{E}_1(t) + \tfrac{1}{2}\mathcal{E}_2(t). \tag{2-10}$$

Theorem 2.2. *Let f be a solution to* (1-1). *Then, we have the time-asymptotic flocking estimate*

$$\mathcal{E}(t) \le \mathcal{E}(0)\exp\left(-2\min\{1,\psi_m\}t\right), \quad t > 0, \tag{2-11}$$

where ψ_m is the minimum communication weight:

$$\psi_m := \min_{x,y\in\mathbb{T}^d} \psi(x-y) > 0.$$

In addition, if u is uniformly Lipschitz continuous on a time interval $[0,T]$, *namely*

$$\ell_T := \sup_{t\in[0,T]} \|\nabla_x u\|_{L^\infty(\mathbb{T}^d)} < \infty,$$

then

$$\mathcal{E}_1(t) \le \mathcal{E}_1(0)e^{2(\ell_T-1)} \quad \textit{for all } t\in[0,T]. \tag{2-12}$$

Proof. We postpone the proof to the Appendix. □

Remark 2.3. As an interesting consequence of (2-12) one obtains that, for smooth solutions, $\mathcal{E}_1(0) = 0$ implies that $\mathcal{E}_1(t) = 0$ for all $t \in [0,T]$. In other words, monokinetic initial conditions remain monokinetic as long as the velocity field is Lipschitz. One can note that monokinetic solutions to (1-1) simply correspond to solutions of the pressureless Euler system (1-4); hence the short time existence of Lipschitz solutions is guaranteed by Proposition 2.4 and Remark 2.5 below.

***Formal derivation of the hydrodynamic Cucker–Smale system* (1-4).** We consider the hydrodynamic variables $\rho^\varepsilon := \int_{\mathbb{R}^d} f^\varepsilon\,dv$ and $\rho^\varepsilon u^\varepsilon := \int_{\mathbb{R}^d} v f^\varepsilon\,dv$.

First of all, integrating (1-2) with respect to v, we get the continuity equation

$$\partial_t\rho^\varepsilon + \nabla_x\cdot(\rho^\varepsilon u^\varepsilon) = 0.$$

Multiplying (1-2) by v, and then integrating it with respect to v, we have

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \nabla_x\cdot\left(\int_{\mathbb{R}^d} v\otimes v f^\varepsilon\,dv\right) = \int_{\mathbb{T}^d}\psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)(u^\varepsilon(t,y)-u^\varepsilon(t,x))\,dy,$$

where we used

$$u^\varepsilon = \frac{\int_{\mathbb{R}^d} v f^\varepsilon\,dv}{\int_{\mathbb{R}^d} f^\varepsilon\,dv}.$$

Then, we rewrite the system for ρ^ε and u^ε as

$$\begin{aligned} &\partial_t\rho^\varepsilon + \nabla_x\cdot(\rho^\varepsilon u^\varepsilon) = 0,\\ &\partial_t(\rho^\varepsilon u^\varepsilon) + \nabla_x\cdot(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon + P^\varepsilon) = \int_{\mathbb{T}^d}\psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)(u^\varepsilon(t,y)-u^\varepsilon(t,x))\,dy, \end{aligned} \tag{2-13}$$

where P^ε is the stress tensor given by

$$P^\varepsilon := \int_{\mathbb{R}^d} (v-u^\varepsilon)\otimes(v-u^\varepsilon) f^\varepsilon \, dv.$$

If we take $\varepsilon \to 0$ in (1-2), the local alignment term $\nabla_v \cdot ((u^\varepsilon - v) f^\varepsilon)$ converges to 0. Hence, if $\rho^\varepsilon \to \rho$ and $\rho^\varepsilon u^\varepsilon \to \rho u$ for some limiting functions ρ and u, we have that $f^\varepsilon \to \delta_{v=u} \otimes \rho$ (in some suitable sense). Hence, the stress tensor P^ε should vanish in the limit, since

$$\int_{\mathbb{R}^d} (v-u)\otimes(v-u)\delta_{v=u}\rho \, dv = 0.$$

Therefore, at least formally, the limit quantities ρ and u satisfy the pressureless Euler system with nonlocal alignment:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) &= \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\, dy. \end{aligned}$$

***Existence of classical solutions to* (1-4).** We present here the local existence of classical solutions to the pressureless Euler system (1-4).

Proposition 2.4. *Assume that*

$$\rho_0 > 0 \quad \text{in } \mathbb{T}^d \qquad \text{and} \qquad (\rho_0, u_0) \in H^s(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d) \quad \text{for } s > \tfrac{1}{2}d + 1. \tag{2-14}$$

Then, there exists $T_* > 0$ such that (1-4) *has a unique classical solution* (ρ, u) *satisfying*

$$\begin{aligned} &\rho \in C^0([0,T_*]; H^s(\mathbb{T}^d)) \cap C^1([0,T_*]; H^{s-1}(\mathbb{T}^d)), \\ &u \in C^0([0,T_*]; H^{s+1}(\mathbb{T}^d)) \cap C^1([0,T_*]; H^s(\mathbb{T}^d)). \end{aligned} \tag{2-15}$$

Remark 2.5. Since $s > \frac{1}{2}d + 1$, by the Sobolev inequality it follows that $(\rho, u) \in C^1([0, T_*] \times \mathbb{T}^d)$.

Proposition 2.4 has been proven in [Ha et al. 2014b]. There, the authors obtained also a global well-posedness of classical solutions, provided an initial datum is suitably smooth and small.

3. Main result and preliminaries

We first present our main result on the hydrodynamic limit of (1-2). We next present useful results on the relative entropy method and the optimal transportation theory, which are used as main tools in the next section.

Main result. For the hydrodynamic limit, we consider a well-prepared initial data f_0^ε satisfying (2-6) and

($\mathcal{A}1$) $\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} (f_0^\varepsilon \frac{1}{2}|v|^2 - \rho_0 \frac{1}{2}|u_0|^2)\, dv\, dx = \mathcal{O}(\varepsilon)$,

($\mathcal{A}2$) $\|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{T}^d)} = \mathcal{O}(\varepsilon)$,

($\mathcal{A}3$) $\|u_0^\varepsilon - u_0\|_{L^\infty(\mathbb{T}^d)} = \mathcal{O}(\varepsilon)$.

We now specify our main result on the hydrodynamic limit.

Theorem 3.1. *Assume that the initial data f_0^ε and (ρ_0, u_0) satisfy* (2-6), (2-14), *and* $(\mathcal{A}1)$–$(\mathcal{A}3)$. *Let f^ε be a weak solution to* (1-2) *satisfying* (2-9), *and (ρ, u) be a local-in-time smooth solution to* (1-4) *satisfying* (2-15) *up to the time T_*. Then, there exists a positive constant C_* (depending on T_*) such that, for all $t \le T_*$,*

$$\int_{\mathbb{T}^d} \rho^\varepsilon(t)|(u^\varepsilon - u)|^2(t)\,dx + W_2^2(\rho^\varepsilon(t), \rho(t)) \le C_*\varepsilon, \tag{3-1}$$

where $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon\,dv$, $\rho^\varepsilon u^\varepsilon = \int_{\mathbb{R}^d} v f^\varepsilon\,dv$, and W_2 denotes the 2-Wasserstein distance.

Therefore, we have

$$f^\varepsilon \rightharpoonup \delta_{v=u(t,x)} \otimes \rho(t,x) \quad \text{in } \mathcal{M}((0,T_*) \times \mathbb{T}^d \times \mathbb{R}^d), \tag{3-2}$$

where $\mathcal{M}((0,T_*) \times \mathbb{T}^d \times \mathbb{R}^d)$ is the space of nonnegative Radon measures on $(0,T_*) \times \mathbb{T}^d \times \mathbb{R}^d$.

The proof of this result is postponed to Section 5. In the next subsections we collect some preliminary facts that will be used later in the proof.

Relative entropy method. First of all, we rewrite the limit system (1-4) in an abstract form using the notation

$$P = \rho u, \quad U = \begin{pmatrix} \rho \\ P \end{pmatrix}, \quad A(U) = \begin{pmatrix} P^T \\ (P \otimes P)/\rho \end{pmatrix},$$

$$F(U) = \begin{pmatrix} 0 \\ \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\,dy \end{pmatrix}.$$

Then we can rewrite (1-4) as the balance law

$$\partial_t U + \operatorname{div}_x A(U) = F(U). \tag{3-3}$$

We consider the relative entropy and relative flux

$$\begin{aligned} \eta(V \mid U) &= \eta(V) - \eta(U) - D\eta(U)\cdot(V-U), \\ A(V \mid U) &= A(V) - A(U) - DA(U)\cdot(V-U), \end{aligned} \tag{3-4}$$

where $DA(U)\cdot(V-U)$ is a matrix defined as

$$(DA(U)\cdot(V-U))_{ij} = \sum_{k=1}^{d+1} \partial_{U_k} A_{ij}(U)(V_k - U_k), \quad 1 \le i \le d+1,\ 1 \le j \le d.$$

By the theory of conservation laws, the system (3-3) has a convex entropy $\eta(U) = \rho\,\frac{1}{2}|u|^2$ with entropy flux G given by the identity

$$\partial_{U_i} G_j(U) = \sum_{k=1}^{d+1} \partial_{U_k}\eta(U)\,\partial_{U_i} A_{kj}(U), \quad 1 \le i \le d+1,\ 1 \le j \le d.$$

Since $\eta(U) = |P|^2/(2\rho)$, and

$$D\eta(U) = \begin{pmatrix} D_\rho \eta \\ D_P \eta \end{pmatrix} = \begin{pmatrix} -|P|^2/(2\rho^2) \\ P/\rho \end{pmatrix} = \begin{pmatrix} -|u|^2/2 \\ u \end{pmatrix}, \tag{3-5}$$

for given $V = \binom{q}{qw}$, $U = \binom{\rho}{\rho u}$, we have

$$\begin{aligned}\eta(V \mid U) &= \tfrac{1}{2}q|w|^2 - \tfrac{1}{2}\rho|u|^2 + \tfrac{1}{2}|u|^2(q-\rho) - u(qw - \rho u)\\ &= \tfrac{1}{2}q|u-w|^2.\end{aligned} \tag{3-6}$$

The next proposition provides a cornerstone to verify the hydrodynamic limit through the relative entropy method. For its proof, we refer to the proof of Proposition 4.2 in [Karper et al. 2015]; see also [Vasseur 2008].

Proposition 3.2. *Let U be a strong solution to balance law* (3-3) *and V be any smooth function. Then, the following holds*:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} \eta(V \mid U)\,dx = \frac{d}{dt}\int_{\mathbb{T}^d}\eta(V)\,dx &- \int_{\mathbb{T}^d} \nabla_x\big(D\eta(U)\big) : A(V \mid U)\,dx\\ &- \int_{\mathbb{T}^d} D\eta(U)\cdot[\partial_t V + \operatorname{div}_x A(V) - F(V)]\,dx\\ &- \int_{\mathbb{T}^d}[D^2\eta(U)F(U)(V-U) + D\eta(U)F(V)]\,dx.\end{aligned}$$

Wasserstein distance and representation formulae for solutions of the continuity equation. For $p \geq 1$, the p-Wasserstein distance between two probability measures μ_1 and μ_2 on $\mathbb{R}^d$ is defined by

$$W_p^p(\mu_1,\mu_2) := \inf_{\nu\in\Lambda(\mu_1,\mu_2)} \int_{\mathbb{R}^{2d}} |x-y|^2\,d\nu(x,y),$$

where $\Lambda(\mu_1,\mu_2)$ denotes the set of all probability measures ν on $\mathbb{R}^{2d}$ with marginals μ_1 and μ_2, i.e,

$$\pi_{1\#}\nu = \mu_1, \quad \pi_{2\#}\nu = \mu_2,$$

where $\pi_1 : (x,y) \mapsto x$ and $\pi_2 : (x,y) \mapsto y$ are the natural projections from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$, and $\pi_{\#}\nu$ denotes the push forward of ν through a map π, i.e., $\pi_{\#}\nu(B) := \nu(\pi^{-1}(B))$ for any Borel set B. This same definition can be extended to measures on the torus $\mathbb{T}^d$ with the understanding that $|x-y|$ denotes the distance on the torus.

To make a connection between the L^2-distance of velocities and the 2-Wasserstein distance of densities (see Lemma 5.2), we will use two different representation formulas for solutions to the continuity equation

$$\partial_t\mu_t + \operatorname{div}_x(u_t\mu_t) = 0. \tag{3-7}$$

Let us recall that, if the velocity field $u_t : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz with respect to x, uniformly in t, then for any x there exists a global-in-time unique characteristic X generated by u_t starting from x,

$$\dot{X}(t,x) = u_t(X(t,x)), \quad X(0,x) = x,$$

and the solution μ_t of (3-7) is the push forward of the initial data μ_0 through $X(t)$, i.e.,

$$\mu_t = X(t)_{\#}\mu_0; \tag{3-8}$$

e.g., see [Ambrosio et al. 2005, Proposition 8.1.8]. On the other hand, if the velocity field u_t is not Lipschitz with respect to x, the uniqueness of the characteristics is not guaranteed anymore. Still, a probabilistic representation formula for solutions to (3-7) holds (recall that a curve of probability measures in $\mathbb{R}^d$ is called narrowly continuous if it is continuous in the duality with continuous bounded functions):

Proposition 3.3. *For a given $T > 0$, let $\mu_t : [0, T] \to \mathcal{P}(\mathbb{R}^d)$ be a narrowly continuous solution of* (3-7) *for a Borel vector field u_t satisfying*

$$\int_0^T \int_{\mathbb{R}^d} |u_t(x)|^p \, d\mu_t(x)\, dt < \infty \quad \text{for some } p > 1.$$

Let Γ_T denote the space of continuous curves from $[0, T]$ into $\mathbb{R}^d$. Then, there exists a probability measure η on $\Gamma_T \times \mathbb{R}^d$ satisfying the following properties:

(i) *η is concentrated on the set of pairs (γ, x) such that γ is an absolutely continuous curve solving the ODE*

$$\dot\gamma(t) = u_t(\gamma(t)) \quad \text{for a.e. } t \in (0, T), \text{ with } \gamma(0) = x.$$

(ii) *μ_t satisfies*

$$\int_{\mathbb{R}^d} \varphi(x)\, d\mu_t(x) = \int_{\Gamma_T \times \mathbb{R}^d} \varphi(\gamma(t))\, d\eta(\gamma, x) \quad \text{for all } \varphi \in C_b^0(\mathbb{R}^d),\ t \in [0, T].$$

Again, this result readily extends on the torus.

Note that, in the case when u_t is Lipschitz, there exists a unique curve γ solving the ODE and starting from x (i.e., $\gamma = X(\cdot\,, x)$), so the measure η is given by the formula

$$d\eta(\gamma, x) = \delta_{\gamma = X(\cdot\,, x)} \otimes d\mu_0(x).$$

We refer to [Ambrosio et al. 2005, Theorem 8.2.1] for more details and a proof.

Useful inequality. We here present a standard inequality that is used in the proof of Lemma 5.2, for the convenience of the reader:

Lemma 3.4. *Let $\rho_1, \rho_2 : \mathbb{T}^d \to \mathbb{R}$ be two probability densities. Then*

$$W_2^2(\rho_1, \rho_2) \le \tfrac{1}{8} d \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)}.$$

Proof. The idea is simple: to estimate the transportation cost from ρ_1 to ρ_2 it suffices to consider a transport plan that keeps at rest all the mass in common between ρ_1 and ρ_2 (namely $\min\{\rho_1, \rho_2\}$) and sends $\rho_1 - \min\{\rho_1, \rho_2\}$ onto $\rho_2 - \min\{\rho_1, \rho_2\}$ in an arbitrary way. For instance, assuming without loss of generality that $\rho_1 \ne \rho_2$ (otherwise the result is trivial), we set

$$m := \|\rho_1 - \min\{\rho_1, \rho_2\}\|_{L^1(\mathbb{T}^d)} = \|\rho_2 - \min\{\rho_1, \rho_2\}\|_{L^1(\mathbb{T}^d)} = \tfrac{1}{2}\|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} > 0.$$

Then, a possible choice of transport plan between ρ_1 and ρ_2 is given by

$$\pi(dx, dy) := \delta_{x=y}(dy) \otimes \min\{\rho_1(x), \rho_2(x)\}dx \\ + \frac{1}{m}[\rho_1(x) - \min\{\rho_1(x), \rho_2(x)\}]dx \otimes [\rho_2(y) - \min\{\rho_1(y), \rho_2(y)\}]dy.$$

Since the diameter of $\mathbb{T}^d$ is bounded by $\frac{1}{2}\sqrt{d}$, we deduce that the W_2^2-cost to transport $\rho_1 - \min\{\rho_1, \rho_2\}$ onto $\rho_2 - \min\{\rho_1, \rho_2\}$ is at most

$$\int_{\mathbb{T}^{2d}} |x-y|^2\, d\pi(x,y) = \frac{1}{m}\int_{\mathbb{T}^{2d}} |x-y|^2(\rho_1(x)-\min\{\rho_1(x),\rho_2(x)\})(\rho_2(y)-\min\{\rho_1(y),\rho_2(y)\})\,dx\,dy$$
$$\le \tfrac{1}{4}d\|\rho_1-\min\{\rho_1,\rho_2\}\|_{L^1(\mathbb{T}^d)} = \tfrac{1}{8}d\|\rho_1-\rho_2\|_{L^1(\mathbb{T}^d)},$$

as desired. □

4. Structural lemma

In a general system, we first present some structural hypotheses to provide a Gronwall-type inequality on the relative entropy that is also controlled by 2-Wasserstein distance.

Hypotheses. Let f^ε be a solution to a given kinetic equation KE_ε scaled with $\varepsilon > 0$ corresponding to initial data f_0^ε. Let U^ε and U_0^ε consist of hydrodynamic variables of f^ε and f_0^ε respectively.

Let U be a solution to a balance law (as a limit system of KE_ε):

$$\partial_t U + \operatorname{div}_x A(U) = F(U), \quad U|_{t=0} = U_0.$$

($\mathcal{H}1$) The kinetic equation KE_ε has a kinetic entropy $\mathcal{F}$ such that $\int \mathcal{F}(f^\varepsilon)(t)\,dx \ge 0$ and

$$\int \mathcal{F}(f^\varepsilon)(t)\,dx + \frac{1}{\varepsilon}\int_0^t D_1(f^\varepsilon)(s)\,ds + \int_0^t D_2(f^\varepsilon)(s)\,ds \le \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx,$$

where $D_1, D_2 \ge 0$ are some dissipations.

($\mathcal{H}2$) There exists a constant $C > 0$ (independent of ε) such that

$$\int \eta(U_0^\varepsilon \mid U_0)\,dx \le C\varepsilon, \quad \int (\mathcal{F}(f_0^\varepsilon) - \eta(U_0^\varepsilon))\,dx \le C\varepsilon, \quad \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx \le C.$$

($\mathcal{H}3$) The balance law has a convex entropy η, and the following minimization property holds:

$$\eta(U^\varepsilon) \le \mathcal{F}(f^\varepsilon).$$

($\mathcal{H}4$) There exists a constant $C > 0$ (independent of ε) such that

$$\left|\int \nabla_x(D\eta(U)) : A(U^\varepsilon \mid U)\,dx\right| \le C\int \eta(U^\varepsilon \mid U)\,dx.$$

($\mathcal{H}5$) There exists a constant $C > 0$ (independent of ε) such that

$$\left|\int D\eta(U)\cdot[\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon)]\,dx\right| \le C D_1(f^\varepsilon).$$

($\mathcal{H}6$) Let ρ^ε be the hydrodynamic variable of f^ε as the local mass, and ρ be the corresponding variable for the balance law. Then,

$$-\int [D^2\eta(U)F(U)(U^\varepsilon - U) + D\eta(U)F(U^\varepsilon)]\,dx \le D_2(f^\varepsilon) + CW_2^2(\rho^\varepsilon,\rho) + C\int \eta(U^\varepsilon \mid U)\,dx.$$

($\mathcal{H}7$) There exists a constant $C > 0$ (independent of ε) such that

$$W_2^2(\rho^\varepsilon, \rho)(t) \le C\int_0^t \int \eta(U^\varepsilon \mid U)\, dx\, ds + C\varepsilon.$$

Remark 4.1. (1) The hypotheses ($\mathcal{H}1$)–($\mathcal{H}5$) provide a basic structure in applying the relative entropy method to hydrodynamic limits as in previous results, for example, [Kang and Vasseur 2015; Karper et al. 2015; Mellet and Vasseur 2008]. On the other hand, the hypotheses ($\mathcal{H}6$)–($\mathcal{H}7$) provide a crucial connection between the relative entropy and Wasserstein distance.

(2) The (kinetic) entropy inequality ($\mathcal{H}1$) plays an important role in controlling the dissipations D_1, D_2 in ($\mathcal{H}5$) and ($\mathcal{H}6$).

(3) ($\mathcal{H}2$) is related to a kind of well-prepared initial data.

Lemma 4.2. *Assume the hypotheses* ($\mathcal{H}1$)–($\mathcal{H}7$). *Then, for a given* $T > 0$, *there exists a constant* $C > 0$ *such that*

$$\int \eta(U^\varepsilon \mid U)(t)\, dx + W_2^2(\rho^\varepsilon, \rho)(t) \le C\varepsilon, \quad t \le T.$$

Proof. First of all, using Proposition 3.2, we have

$$\int_{\mathbb{T}^d} \eta(U^\varepsilon \mid U)(t)\, dx \le I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned}
I_1 &:= \int_{\mathbb{T}^d} \eta(U_0^\varepsilon \mid U_0)\, dx,\\
I_2 &:= \int_{\mathbb{T}^d} (\eta(U^\varepsilon)(t) - \eta(U_0^\varepsilon))\, dx,\\
I_3 &:= -\int_0^t \int_{\mathbb{T}^d} \nabla_x(D\eta(U)) : A(U^\varepsilon \mid U)\, dx\, ds,\\
I_4 &:= -\int_0^t \int_{\mathbb{T}^d} D\eta(U)\cdot[\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon)]\, dx\, ds,\\
I_5 &:= -\int_0^t \int_{\mathbb{T}^d} [D^2\eta(U)F(U)(U^\varepsilon - U) + D\eta(U)F(U^\varepsilon)]\, dx\, ds.
\end{aligned}$$

It follows from ($\mathcal{H}2$) that $I_1 \le C\varepsilon$.

We decompose I_2 as

$$I_2 = \underbrace{\int_{\mathbb{T}^d} (\eta(U^\varepsilon)(t) - \mathcal{F}(f^\varepsilon)(t))\, dx}_{=:I_2^1} + \underbrace{\int_{\mathbb{T}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon))\, dx}_{=:I_2^2} + \underbrace{\int_{\mathbb{T}^d} (\mathcal{F}(f_0^\varepsilon) - \eta(U_0^\varepsilon))\, dx}_{=:I_2^3}. \tag{4-1}$$

First, $I_2^1 \le 0$ by ($\mathcal{H}3$).

Since ($\mathcal{H}1$) yields

$$I_2^2 \le -\int_0^t D_2(f^\varepsilon)\, ds,$$

it follows from ($\mathcal{H}6$) that

$$I_2^2 + I_5 \le C\int_0^t W_2^2(\rho^\varepsilon, \rho)\, ds + C\int_0^t \int_{\mathbb{T}^d} \eta(U^\varepsilon \mid U)\, dx\, ds.$$

By ($\mathcal{H}2$), $I_2^3 \le C\varepsilon$.

It follows from ($\mathcal{H}4$) that

$$I_3 \le C\int_0^t \int_{\mathbb{T}^d} \eta(U^\varepsilon \mid U)\, dx\, ds.$$

Since ($\mathcal{H}1$) and ($\mathcal{H}2$) imply

$$\int_0^t D_1(f^\varepsilon)(s)\, ds \le C\varepsilon,$$

we have $I_4 \le C\varepsilon$.

Therefore, we have

$$\int \eta(U^\varepsilon \mid U)(t)\, dx \le C\varepsilon + C\int_0^t \left[\int \eta(U^\varepsilon \mid U)(s)\, dx\, ds + W_2^2(\rho^\varepsilon, \rho)\right] ds.$$

Hence, combining it with ($\mathcal{H}7$), and using Gronwall's inequality, we have the desired result. □

5. Proof of Theorem 3.1

The main part of the proof consists in proving the estimate (3-1).

***Proof of* (3-1).** This will be done by verifying the hypotheses ($\mathcal{H}1$)–($\mathcal{H}7$), and then completed by Lemma 4.2.

Verification of ($\mathcal{H}1$). ($\mathcal{H}1$) is satisfied thanks to Lemma 5.1 below. There we show that one can replace the nonlocal dissipation $\mathcal{D}_2$ in the kinetic entropy inequality (2-9) by another dissipation $\widetilde{\mathcal{D}}_2$ defined in terms of the hydrodynamic variables ρ^ε and u^ε.

Lemma 5.1. *For any $\varepsilon > 0$, assume that f_0^ε satisfies*

$$f_0^\varepsilon \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d), \quad |v|^2 f_0^\varepsilon \in L^1(\mathbb{T}^d \times \mathbb{R}^d).$$

Then the weak solution f^ε in Proposition 2.1 also satisfies

$$\int_{\mathbb{T}^d} \mathcal{F}(f^\varepsilon)(t)\, dx + \frac{1}{\varepsilon}\int_0^t \mathcal{D}_1(f^\varepsilon)(s)\, ds + \int_0^t \widetilde{\mathcal{D}}_2(f^\varepsilon)(s)\, ds \le \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\, dx, \tag{5-1}$$

where $\mathcal{F}$ and $\mathcal{D}_1$ are as in (2-5), *and*

$$\widetilde{\mathcal{D}}_2(f^\varepsilon) := \frac{1}{2}\int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)|u^\varepsilon(t,x) - u^\varepsilon(t,y)|^2\, dx\, dy.$$

Proof. Recalling (2-9), it is enough to show $\widetilde{\mathcal{D}}_2(f^\varepsilon) \le \mathcal{D}_2(f^\varepsilon)$. We first rewrite $\widetilde{\mathcal{D}}_2(f^\varepsilon)$ in terms of the mesoscopic variables as follows: using $\psi(x-y) = \psi(y-x)$, we have

$$
\begin{aligned}
\widetilde{\mathcal{D}}_2(f^\varepsilon) &= \frac{1}{2}\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot(u^\varepsilon(t,x)-u^\varepsilon(t,y))\,dv\,dw\,dx\,dy \\
&= \int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot u^\varepsilon(t,x)\,dv\,dw\,dx\,dy \\
&= \underbrace{\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot v\,dv\,dw\,dx\,dy}_{=:\mathcal{I}_1} \\
&\qquad + \underbrace{\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot(u^\varepsilon(t,x)-v)\,dv\,dw\,dx\,dy}_{=:\mathcal{I}_2}.
\end{aligned}
$$

First, we have

$$
\mathcal{I}_1 = \frac{1}{2}\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)|v-w|^2\,dx\,dy\,dv\,dw = \mathcal{D}_2(f^\varepsilon).
$$

We next claim $\mathcal{I}_2 \le 0$.

Indeed, since

$$
\rho^\varepsilon|u^\varepsilon|^2 = \frac{\left(\int_{\mathbb{R}^d} v f^\varepsilon\,dv\right)^2}{\int_{\mathbb{R}^d} f^\varepsilon\,dv} \le \int_{\mathbb{R}^d} |v|^2 f^\varepsilon\,dv, \tag{5-2}
$$

we have

$$
\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)|v|^2\,dv\,dw\,dx\,dy \ge \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,y)\rho^\varepsilon(t,x)|u^\varepsilon(t,x)|^2\,dx\,dy.
$$

Then, since

$$
\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w) u^\varepsilon(t,x)\cdot w\,dv\,dw\,dx\,dy = \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y) u^\varepsilon(t,x)\cdot u^\varepsilon(t,y)\,dx\,dy,
$$

$$
\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w) u^\varepsilon(t,x)\cdot v\,dv\,dw\,dx\,dy = \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)|u^\varepsilon(t,x)|^2\,dx\,dy,
$$

$$
\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w) v\cdot w\,dv\,dw\,dx\,dy = \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y) u^\varepsilon(t,x)\cdot u^\varepsilon(t,y)\,dx\,dy,
$$

we conclude that $\mathcal{I}_2 \le 0$, as desired. □

Verification of $(\mathcal{H}2)$. We show that the assumptions $(\mathcal{A}1)$–$(\mathcal{A}3)$ for initial data imply $(\mathcal{H}2)$. Using (3-6) and assumption $(\mathcal{A}3)$, we have

$$\int_{\mathbb{T}^d} \eta(U_0^\varepsilon \mid U_0)\,dx = \frac{1}{2}\int_{\mathbb{T}^d} \rho_0^\varepsilon |u_0^\varepsilon - u_0|^2\,dx \le C\varepsilon^2 \int_{\mathbb{T}^d} \rho_0^\varepsilon\,dx \le C\varepsilon^2.$$

Since it follows from $(\mathcal{A}1)$–$(\mathcal{A}3)$ that

$$\int_{\mathbb{T}^d} (\mathcal{F}(f_0^\varepsilon) - \eta(U_0))\,dx = \mathcal{O}(\varepsilon),$$

and

$$\begin{aligned}\int_{\mathbb{T}^d} (\eta(U_0) - \eta(U_0^\varepsilon))\,dx &= \frac{1}{2}\int_{\mathbb{R}^d} (\rho_0|u_0|^2 - \rho_0^\varepsilon |u_0^\varepsilon|^2) \\ &\le \frac{1}{2}\int_{\mathbb{T}^d} |\rho_0 - \rho_0^\varepsilon|\,|u_0|^2 + \frac{1}{2}\int_{\mathbb{T}^d} \rho_0^\varepsilon \big||u_0^\varepsilon|^2 - |u_0|^2\big| = \mathcal{O}(\varepsilon),\end{aligned}$$

we have

$$\int_{\mathbb{T}^d} (\mathcal{F}(f_0^\varepsilon) - \eta(U_0^\varepsilon))\,dx = \mathcal{O}(\varepsilon).$$

It is obvious that $(\mathcal{A}1)$ implies

$$\int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx \le C.$$

Verification of $(\mathcal{H}3)$. It follows from (5-2) that

$$\eta(U^\varepsilon) = \rho^\varepsilon \tfrac{1}{2}|u^\varepsilon|^2 \le \int_{\mathbb{R}^d} \tfrac{1}{2}|v|^2 f^\varepsilon\,dv = \mathcal{F}(f^\varepsilon). \tag{5-3}$$

Verification of $(\mathcal{H}4)$. Since

$$A(U) = \begin{pmatrix} P^T \\ (P\otimes P)/\rho \end{pmatrix},$$

we have

$$\begin{aligned} DA(U)\cdot(U^\varepsilon - U) &= D_\rho A(U)(\rho^\varepsilon - \rho) + D_{P_i}A(U)(P_i^\varepsilon - P_i) \\ &= \begin{pmatrix} (P^\varepsilon - P)^T \\ -((\rho^\varepsilon - \rho)/\rho^2)P\otimes P + (1/\rho)P\otimes(P^\varepsilon - P) + (1/\rho)(P^\varepsilon - P)\otimes P \end{pmatrix},\end{aligned}$$

which yields

$$\begin{aligned} &A(U^\varepsilon \mid U) \\ &= \begin{pmatrix} 0 \\ (1/\rho^\varepsilon)P^\varepsilon\otimes P^\varepsilon - (1/\rho)P\otimes P + ((\rho^\varepsilon-\rho)/\rho^2)P\otimes P - (1/\rho)P\otimes(P^\varepsilon - P) - (1/\rho)(P^\varepsilon - P)\otimes P \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \rho^\varepsilon (u^\varepsilon - u)\otimes(u^\varepsilon - u) \end{pmatrix}.\end{aligned}$$

Therefore, using (3-5) and (3-6), we have

$$\begin{aligned} \left|\int \nabla_x(D\eta(U)) : A(U^\varepsilon \mid U)\,dx\right| &= \left|\int_0^t\int_{\mathbb{T}^d} \rho^\varepsilon(u^\varepsilon - u)\otimes(u^\varepsilon - u) : \nabla_x u\,dx\,ds\right| \\ &\le C\|\nabla_x u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)} \int_0^t\int_{\mathbb{T}^d} \eta(U^\varepsilon \mid U)\,dx\,ds.\end{aligned}$$

Verification of ($\mathcal{H}5$). For a weak solution f^ε to (1-2), it follows from (2-13) that $U^\varepsilon = \binom{\rho^\varepsilon}{P^\varepsilon}$ solves the system

$$\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon) = \operatorname{div}_x \begin{pmatrix} 0 \\ -\int_{\mathbb{R}^d} (v - u^\varepsilon) \otimes (v - u^\varepsilon) f^\varepsilon \, dv \end{pmatrix}, \tag{5-4}$$

where the equality holds in the sense of distributions; see (2-8). Therefore, we have

$$\begin{aligned} &\left| \int D\eta(U) \cdot [\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon)] \, dx \right| \\ &\qquad = \left| \int_{\mathbb{T}^d} \nabla_x u : \left(\int_{\mathbb{R}^d} (v - u^\varepsilon) \otimes (v - u^\varepsilon) f^\varepsilon \, dv \right) dx \right| \\ &\qquad \le C \|\nabla_x u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v - u^\varepsilon|^2 f^\varepsilon \, dv \, dx = C \|\nabla_x u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)} \mathcal{D}_1(f^\varepsilon). \end{aligned}$$

Verification of ($\mathcal{H}6$). From the proof of Proposition 4.2 in [Karper et al. 2015], we see

$$-\int_{\mathbb{T}^d} [D^2\eta(U) F(U)(U^\varepsilon - U) + D\eta(U) F(U^\varepsilon)] \, dx = K_1 + K_2 + K_3,$$

where

$$\begin{aligned} K_1 &:= -\frac{1}{2} \int_{\mathbb{T}^{2d}} \psi(x-y) \rho^\varepsilon(x) \rho^\varepsilon(t, y) \big| (u^\varepsilon(x) - u(x)) - (u^\varepsilon(y) - u(y)) \big|^2 \, dx \, dy, \\ K_2 &:= \frac{1}{2} \int_{\mathbb{T}^{2d}} \psi(x-y) \rho^\varepsilon(x) \rho^\varepsilon(y) \, |u^\varepsilon(x) - u^\varepsilon(y)|^2 \, dx \, dy, \\ K_3 &:= \int_{\mathbb{T}^{2d}} \psi(x-y) \rho^\varepsilon(x) (\rho^\varepsilon(y) - \rho(y)) (u(y) - u(x)) (u^\varepsilon(x) - u(x)) \, dx \, dy. \end{aligned}$$

Notice that $K_1 \le 0$, and $K_2 = \widetilde{\mathcal{D}}_2(f^\varepsilon)$ where $\widetilde{\mathcal{D}}_2(f^\varepsilon)$ is in Lemma 5.1.

To estimate K_3, we separate it into two parts:

$$\begin{aligned} K_3 = \int_{\mathbb{T}^d} &\left(\int_{\mathbb{T}^d} \psi(x-y) u(y) (\rho^\varepsilon(y) - \rho(y)) \, dy \right) \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \, dx \\ &- \int_{\mathbb{T}^d} \left(\int_{\mathbb{T}^d} \psi(x-y) (\rho^\varepsilon(y) - \rho(y)) \, dy \right) u(x) \rho^\varepsilon(x) (u^\varepsilon(x) - u(x)) \, dx. \end{aligned}$$

Since ψ and u are Lipschitz, we use the Kantorovich–Rubinstein theorem, see [Villani 2009, Theorem 5.10 and Particular Case 5.16], to estimate

$$\begin{aligned} K_3 \le W_1(\rho^\varepsilon, \rho) \big(\sup_{x \in \mathbb{T}^d} \|\psi(x - \cdot) u\|_{L^\infty(0,T_*; W^{1,\infty}(\mathbb{T}^d))} + \|\psi\|_{L^\infty(0,T_*; W^{1,\infty}(\mathbb{T}^d))} \|u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)} \big) \\ \times \int_{\mathbb{T}^d} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)| \, dx. \end{aligned}$$

Therefore, since $W_1(\rho^\varepsilon, \rho) \le W_2(\rho^\varepsilon, \rho)$, we obtain

$$K_3 \le C \left(W_2^2(\rho^\varepsilon, \rho) + \int_{\mathbb{T}^d} \rho^\varepsilon(x) |u^\varepsilon(x) - u(x)|^2 \, dx \right).$$

Hence we have verified ($\mathcal{H}6$).

Verification of ($\mathcal{H}7$). This will be shown by Lemma 5.2 below. We first derive some estimates for the characteristics generated by the velocity fields u^ε and u.

For the velocity u in the limit system (1-4), let X be a characteristic generated by it, that is,

$$\dot{X}(t,x) = u(t, X(t,x)), \quad X(0,x) = x. \tag{5-5}$$

Then, thanks to the smoothness of u, it follows from (3-8) that

$$X(t)_\# \rho_0(x)\,dx = \rho(t,x)\,dx.$$

On the other hand, since u^ε is not Lipschitz with respect to x, we use a probabilistic representation for ρ^ε as a solution of the continuity equation in (3-3). More precisely, (5-3) and (2-9) imply

$$\int_{\mathbb{T}^d} |u^\varepsilon(t)|^2 \rho^\varepsilon(t)\,dx \le \int_{\mathbb{T}^d} \mathcal{F}(f^\varepsilon)(t)\,dx \le \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx < \infty,$$

so it follows from Proposition 3.3 that there exists a probability measure η^ε in $\Gamma_{T_*} \times \mathbb{T}^d$ that is concentrated on the set of pairs (γ, x) such that γ is a solution of the ODE

$$\dot{\gamma}(t) = u^\varepsilon(\gamma(t)), \quad \gamma(0) = x, \tag{5-6}$$

and

$$\int_{\mathbb{T}^d} \varphi(x)\rho^\varepsilon(t,x)\,dx = \int_{\Gamma_{T_*}\times\mathbb{T}^d} \varphi(\gamma(t))\,d\eta^\varepsilon(\gamma,x) \quad \text{for all } \varphi \in C^0(\mathbb{T}^d),\ t \in [0, T_*]. \tag{5-7}$$

In particular, this says that the time marginal of the measure η^ε at time 0 is given by $\rho^\varepsilon(0) = \rho_0^\varepsilon$. Hence, by the disintegration theorem of measures, see for instance [Ambrosio et al. 2005, Theorem 5.3.1] and the comments at the end of Section 8.2 in [Ambrosio et al. 2005], we can write

$$d\eta^\varepsilon(\gamma, x) = \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx,$$

where $\{\eta_x^\varepsilon\}_{x\in\mathbb{T}^d}$ is a family of probability measures on Γ_{T^*} concentrated on solutions of (5-6).

For the flow X in (5-5), we also consider the densities $\tilde{\rho}^\varepsilon(t)$ defined as

$$\tilde{\rho}^\varepsilon(t,x)\,dx = X(t)_\# \rho_0^\varepsilon(x)\,dx. \tag{5-8}$$

Note that, since

$$\begin{aligned} \|\rho(t) - \tilde{\rho}^\varepsilon(t)\|_{L^1(\mathbb{T}^d)} &= \sup_{\|\varphi\|_\infty \le 1} \int_{\mathbb{T}^d} \varphi(x)[\rho(t,x) - \tilde{\rho}^\varepsilon(t,x)]\,dx \\ &= \sup_{\|\varphi\|_\infty \le 1} \int_{\mathbb{T}^d} \varphi(X(t,x))[\rho_0(x) - \rho_0^\varepsilon(x)]\,dx \le \|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{T}^d)}, \end{aligned}$$

we have

$$\|\rho(t) - \tilde{\rho}^\varepsilon(t)\|_{L^1(\mathbb{T}^d)} \le \|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{T}^d)}. \tag{5-9}$$

We now consider the measure ν^ε on $\Gamma_{T_*} \times \Gamma_{T_*} \times \mathbb{T}^d$ defined as

$$d\nu^\varepsilon(\gamma, \sigma, x) = \eta_x^\varepsilon(d\gamma) \otimes \delta_{X(\,\cdot\,,x)}(d\sigma) \otimes \rho_0^\varepsilon(x)\,dx.$$

If we consider the evaluation map

$$E_t : \Gamma_{T_*} \times \Gamma_{T_*} \times \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{T}^d, \quad E_t(\gamma, \sigma, x) = (\gamma(t), \sigma(t)),$$

it follows that the measure $\pi_t^\varepsilon := (E_t)_\# \nu^\varepsilon$ on $\mathbb{T}^d \times \mathbb{T}^d$ has marginals $\rho^\varepsilon(t,x)\,dx$ and $\tilde{\rho}^\varepsilon(t,y)\,dy$ for all $t \geq 0$. Therefore, we have

$$\begin{aligned}\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - X(t,x)|^2 \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx &= \int_{\Gamma_{T_*}\times\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-\sigma(t)|^2\,d\nu^\varepsilon(\gamma,\sigma,x) \\ &= \int_{\mathbb{T}^{2d}} |x-y|^2\,d\pi_t^\varepsilon(x,y) \\ &\geq W_2^2(\rho^\varepsilon(t), \tilde{\rho}^\varepsilon(t)). \end{aligned} \tag{5-10}$$

We now use the above results to prove the following lemma.

Lemma 5.2. *Under the same assumptions as in Theorem 3.1, we have*

$$W_2^2(\rho^\varepsilon(t), \rho(t)) \leq Ce^{T_*}\int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(s,x) - u(s,x)|^2 \rho^\varepsilon(s,x)\,dx\,ds + \mathcal{O}(\varepsilon), \quad t \leq T_*. \tag{5-11}$$

Proof. Let $\tilde{\rho}^\varepsilon$ be defined as in (5-8). We begin by observing that, thanks to Lemma 3.4, (5-9), and assumption ($\mathcal{A}2$), it follows that

$$W_2^2(\tilde{\rho}^\varepsilon(t), \rho(t)) \leq \mathcal{O}(\varepsilon).$$

Hence, to prove (5-11), it is enough to bound $W_2^2(\rho^\varepsilon(t), \tilde{\rho}^\varepsilon(t))$.

To this aim, we try to get a Gronwall-type inequality on

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - X(t,x)|^2 \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx.$$

Since

$$\dot{\gamma}(t) - \dot{X}(t,x) = (u^\varepsilon(\gamma(t)) - u(\gamma(t))) + (u(\gamma(t)) - u(X(t,x)))$$

by (5-5) and (5-6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx &\leq \int_{\Gamma_{T_*}\times\mathbb{T}^d} |u^\varepsilon(\gamma(t))-u(\gamma(t))|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \\ &\quad + \int_{\Gamma_{T_*}\times\mathbb{T}^d} |u(\gamma(t))-u(X(t,x))|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \\ &\quad + 2\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx. \end{aligned}$$

Notice that, thanks to (5-7),

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |u^\varepsilon(\gamma(t)) - u(\gamma(t))|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx = \int_{\mathbb{T}^d} |u^\varepsilon(t,x) - u(t,x)|^2 \rho^\varepsilon(t,x)\,dx.$$

Moreover, since

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |u(\gamma(t))-u(X(t,x))|^2\, d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \\ \le \|u\|_{L^\infty(0,T_*;W^{1,\infty}(\mathbb{T}^d))}\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-X(t,x)|^2\, d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx,$$

we have

$$\frac{d}{dt}\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-X(t,x)|^2\, d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \\ \le C\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-X(t,x)|^2\, d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx + \int_{\mathbb{T}^d}|u^\varepsilon(t,x)-u(t,x)|^2\rho^\varepsilon(t,x)\,dx.$$

Therefore, using Gronwall's inequality together with $\gamma(0)=X(0,x)=x$ for η_x^ε-a.e. γ, we obtain

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t)-X(t,x)|^2\, d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \le Ce^{T_*}\int_0^t\int_{\mathbb{T}^d}|u^\varepsilon(s,x)-u(s,x)|^2\rho^\varepsilon(s,x)\,dx\,ds, \quad t\le T_*.$$

Hence, using (5-10) we get the desired control on $W_2^2(\rho^\varepsilon(t),\tilde\rho^\varepsilon(t))$, which concludes the proof. □

***Proof of* (3-2).** Here we use the estimate (3-1) to show the convergence (3-2).

First, since (5-1) and ($\mathcal{A}$1) imply

$$\int_0^t \mathcal{D}_1(f^\varepsilon)(s)\,ds \le C\varepsilon,$$

using (3-1), we have

$$\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon|v-u|^2\,dx\,dv\,ds \le 2\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon(|v-u^\varepsilon|^2+|u^\varepsilon-u|^2)\,dx\,dv\,ds \le C(1+T_*)\varepsilon. \tag{5-12}$$

Then, for any $\varphi\in C_c^1((0,T_*)\times\mathbb{T}^d\times\mathbb{R}^d)$,

$$\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(s,x,v)f^\varepsilon\,dx\,dv\,ds - \int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(s,x,v)\rho\,\delta_u(dv)\,dx\,ds \\ = \int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(s,x,v)f^\varepsilon\,dx\,dv\,ds - \int_0^{T_*}\int_{\mathbb{T}^d}\varphi(s,x,u)\rho\,dx\,ds \\ = \underbrace{\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon(\varphi(s,x,v)-\varphi(s,x,u))\,dx\,dv\,ds}_{=:I_1^\varepsilon} + \underbrace{\int_0^{T_*}\int_{\mathbb{T}^d}\varphi(s,x,u)(\rho^\varepsilon-\rho)\,dx\,ds}_{=:I_2^\varepsilon}.$$

Using (5-12), we have

$$I_1^\varepsilon \le \|\nabla_v\varphi\|_\infty\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon|v-u|\,dx\,dv\,ds \\ = \|\nabla_v\varphi\|_\infty\biggl(\int_0^{T_*}\int_{|v-u|\le\sqrt{\varepsilon}} f^\varepsilon|v-u|\,dx\,dv\,ds + \int_0^{T_*}\int_{|v-u|>\sqrt{\varepsilon}} f^\varepsilon|v-u|\,dx\,dv\,ds\biggr)$$

$$
\begin{aligned}
&\le \|\nabla_v \varphi\|_\infty \left(\sqrt{\varepsilon} T_* + \frac{1}{\sqrt{\varepsilon}} \int_0^{T_*} \int_{|v-u|>\sqrt{\varepsilon}} f^\varepsilon |v-u|^2 \, dv \, dx \, ds \right) \\
&\le C(1+T_*)\sqrt{\varepsilon}.
\end{aligned}
$$

Since $W_1(\rho^\varepsilon, \rho) \le W_2(\rho^\varepsilon, \rho) \to 0$ by (3-1), we also have $I_2^\varepsilon \to 0$ as $\varepsilon \to 0$.

Hence, this completes the proof of (3-2).

Appendix: Proof of Theorem 2.2

We first estimate $(d/dt)\mathcal{E}_1$ as follows:

$$
\frac{d}{dt}\mathcal{E}_1 = 2\int_{\mathbb{T}^d \times \mathbb{R}^d} f(u-v)\,\partial_t u \, dv \, dx + \int_{\mathbb{T}^d \times \mathbb{R}^d} \partial_t f |u-v|^2 \, dv \, dx := I_1 + I_2.
$$

First of all, by the definition of u, we have $\int f(u-v)\,dv = 0$; hence $I_1 = 0$.

Concerning I_2, it follows from (1-1) that

$$
\begin{aligned}
I_2 &= \int_{\mathbb{T}^d \times \mathbb{R}^d} |u-v|^2 \big(-\nabla_x \cdot (vf) - \nabla_v \cdot (L[f]f) - \nabla_v \cdot ((u-v)f) \big) \, dv \, dx \\
&= \underbrace{2\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x u (u-v) \cdot vf \, dv \, dx}_{=:I_{21}} \underbrace{-2\int_{\mathbb{T}^d \times \mathbb{R}^d} (u-v) \cdot L[f] f \, dv \, dx}_{=:I_{22}} \underbrace{-2\int_{\mathbb{T}^d \times \mathbb{R}^d} |u-v|^2 f \, dv \, dx}_{=-2\mathcal{E}_1}.
\end{aligned}
$$

Then, we use the stress tensor $P = \int_{\mathbb{R}^d} (v-u) \otimes (v-u) f \, dv$ to rewrite I_{21} as

$$
I_{21} = 2\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x u (u-v) \cdot (v-u) f \, dv \, dx = 2\int_{\mathbb{T}^d} (\nabla_x \cdot P) \cdot u \, dx.
$$

Thanks to the estimate on $\mathcal{I}_2$ in the proof of Lemma 5.1, we see that

$$
I_{22} = -2\int_{\mathbb{T}^{2d} \times \mathbb{R}^{2d}} \psi(x-y) f(t,x,v) f(t,y,w) (u(t,x)-v) \cdot (w-v) \, dv \, dw \, dx \, dy \le 0.
$$

Therefore, we have

$$
\frac{d}{dt}\mathcal{E}_1 \le 2\int_{\mathbb{T}^d} (\nabla_x \cdot P) \cdot u \, dx - 2\mathcal{E}_1. \tag{A-1}
$$

We next estimate $(d/dt)\mathcal{E}_2$ as follows:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_2 &= 2\int_{\mathbb{T}^{2d}} \partial_t \rho(t,x) \rho(t,y) |u(t,x)-u(t,y)|^2 \, dx \, dy \\
&\qquad + 2\int_{\mathbb{T}^{2d}} \rho(t,x)\rho(t,y)(u(t,x)-u(t,y))\partial_t(u(t,x)-u(t,y)) \, dx \, dy \\
&:= J_1 + J_2.
\end{aligned}
$$

Since it follows from (2-13) with $\varepsilon = 1$ that

$$
\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\rho\, \partial_t u + \rho u \cdot \nabla_x u + \nabla_x \cdot P = \int_{\mathbb{R}^d} L[f] f \, dv,
\end{aligned}
$$

we obtain (recall that $\|\rho\|_{L^1(\mathbb{T}^d)} = 1$)

$$J_1 = -2\int_{\mathbb{T}^{2d}} \nabla_x \cdot (\rho u)(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy$$
$$= 4\int_{\mathbb{T}^d} \rho u\cdot \nabla_x u \cdot u\,dx - 4\int_{\mathbb{T}^d} \rho u\cdot \nabla_x u\,dx \cdot \int_{\mathbb{T}^d} \rho u\,dx,$$

and

$$J_2 = 4\int_{\mathbb{T}^{2d}} \rho(t,y)u(t,x)\rho(t,x)\partial_t u(t,x)\,dx\,dy - 4\int_{\mathbb{T}^{2d}} \rho(t,y)u(t,y)\rho(t,x)\partial_t u(t,x)\,dx\,dy$$
$$= -4\int_{\mathbb{T}^d} \rho u\cdot\nabla_x u\cdot u\,dx - 4\int_{\mathbb{T}^d} \nabla_x\cdot P\cdot u\,dx + 4\underbrace{\int_{\mathbb{T}^d\times\mathbb{R}^d} u\cdot L[f]f\,dx\,dv}_{:=J_{21}} + 4\int_{\mathbb{T}^d}\rho u\cdot\nabla_x u\,dx\cdot\int_{\mathbb{T}^d}\rho u\,dx$$
$$+ 4\underbrace{\int_{\mathbb{T}^d}\nabla_x\cdot P\,dx}_{=0}\cdot\int_{\mathbb{T}^d}\rho u\,dx - 4\underbrace{\int_{\mathbb{T}^d\times\mathbb{R}^d} L[f]f\,dx\,dv}_{:=J_{22}}\cdot\int_{\mathbb{T}^d}\rho u\,dx.$$

Now, we compute the above terms J_{21} and J_{22} as follows:

$$J_{21} = \int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y)f(t,x,v)f(t,y,w)(w-v)\cdot u(t,x)\,dv\,dw\,dx\,dy$$
$$= \int_{\mathbb{T}^{2d}} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\cdot u(t,x)\,dx\,dy$$
$$= -\frac{1}{2}\int_{\mathbb{T}^{2d}} \psi(x-y)\rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy,$$

$$J_{22} = \int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y)f(t,x,v)f(t,y,w)(w-v)\,dv\,dw\,dx\,dy = 0.$$

Therefore, we have

$$\frac{d}{dt}\mathcal{E}_2 = -4\int_{\mathbb{T}^d}\nabla_x\cdot P\cdot u\,dx - 2\int_{\mathbb{T}^{2d}}\psi(x-y)\rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy.$$

Recalling (A-1), proves that

$$\frac{d}{dt}\mathcal{E} \le -2\mathcal{E}_1 - \int_{\mathbb{T}^{2d}}\psi(x-y)\rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy$$
$$\le -2\mathcal{E}_1 - \psi_m\mathcal{E}_2 \le -2\min\{1,\psi_m\}\mathcal{E},$$

which completes the proof of (2-11).

To show the second bound (2-12), note that if $\ell_T := \sup_{t\in[0,T]}\|\nabla_x u\|_{L^\infty(\mathbb{T}^d)} < \infty$ then (A-1) yields

$$\frac{d}{dt}\mathcal{E}_1(t) \le -2\int_{\mathbb{T}^d}\nabla_x u : P\,dx - 2\mathcal{E}_1 \le 2\ell_T\int_{\mathbb{T}^d\times\mathbb{R}^d}|u-v|^2 f\,dv\,dx - 2\mathcal{E}_1(t) = 2(\ell_T-1)\mathcal{E}_1(t),$$

which proves (2-12). □

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