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CONVERGENCE OF THE KÄHLER–RICCI ITERATION

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The Ricci iteration is a discrete analogue of the Ricci flow. According to Perelman, the Ricci flow converges to a Kähler–Einstein metric whenever one exists, and it has been conjectured that the Ricci iteration should behave similarly. This article confirms this conjecture. As a special case, this gives a new method of uniformization of the Riemann sphere.

1. Introduction

Let (M, g_1) be a compact Riemannian manifold. A Ricci iteration is a sequence of metrics $\{g_i\}_{i\in\mathbb{N}}$ on M satisfying

$$\operatorname{Ric} g_{i+1} = g_i, \quad i \in \mathbb{N}, \tag{1}$$

where $\operatorname{Ric} g_{i+1}$ denotes the Ricci curvature of g_{i+1}. One may think of (1) as a dynamical system on the space of Riemannian metrics on M. Part of the interest in the Ricci iteration is that, clearly, Einstein metrics are fixed points, and so (1) aims to provide a natural theoretical and numerical approach to uniformization in the challenging case of positive Ricci curvature (different Ricci iterations can be defined in the context of nonpositive curvature, but these are typically easier to understand and will not be discussed here). In essence, the Ricci iteration aims to reduce the Einstein equation to a sequence of prescribed Ricci curvature equations and can be thought of as a discretization of the Ricci flow. Going back to [Rubinstein 2007; 2008c], it has been studied since by a number of authors [Berman 2013; Berman et al. 2016a; Cheltsov et al. 2010; Cheltsov and Shramov 2011a; 2011b; Cheltsov and Wilson 2013; Guedj et al. 2013; Jeffres et al. 2016; Keller 2009; Pulemotov and Rubinstein 2016]; see also the survey [Rubinstein 2014, §6.5]. One of the motivations for considering (1) and not simply repeatedly applying the Ricci tensor (as in [Nadel 1995], see also [Rubinstein 2008a, Remark 4.63]) is the gain of derivatives inherent in (1) as well as monotonicity of certain functionals. Both of these properties will feature below.

Of particular interest has been the study of the Ricci iteration on Kähler manifolds (for the non-Kähler case results are scarce, see [Pulemotov and Rubinstein 2016]). When (M, J, g_1) is Kähler, the Calabi–Yau theorem [Yau 1978] guarantees the existence and uniqueness of the sequence $\{g_i\}_{i\in\mathbb{N}}$ if and only if M is Fano (i.e., has positive first Chern class $c_1(M, \mathrm{J})$) and the Kähler class associated to g_1 is $c_1(M, \mathrm{J})$. Under a rather restrictive technical assumption, one of us showed that g_i converges smoothly to a Kähler–Einstein metric [Rubinstein 2008c, Theorem 3.3] and made the following general conjecture (see Conjecture 3.2 of the same work):

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Conjecture 1.1. *Let (M, J, g_1) be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to g_1 is $c_1(M, \mathrm{J})$. Then the Ricci iteration (1) converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric.*

Roughly speaking, (M, g_k) converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric if there exist smooth diffeomorphisms $f_k : M \to M$ such that $f_k^* g_k$ converges smoothly to a Kähler–Einstein metric. As we will see, our methods will actually produce biholomorphisms f_k. For more on Cheeger–Gromov convergence we refer to [Petersen 2016, Chapter 10].

The best result so far on this conjecture is due to Berman et al. [2016a], who replaced the technical assumption of [Rubinstein 2008c, Theorem 3.3] concerning Tian's α-invariant by the weaker assumption of the Mabuchi energy being proper (both of these assumptions imply a Kähler–Einstein metric exists). Therefore, by a classical result of Tian [1997], Conjecture 1.1 holds if M admits no holomorphic vector fields. However, the properness assumption is still too restrictive and fails in general. For example, Conjecture 1.1 is still open even for $M = S^2$, the two-sphere. Furthermore, as recent counterexamples show [Darvas and Rubinstein 2017], certain key theorems in Kähler geometry that one might naively expect to generalize in a straightforward manner from the case of no automorphisms require new tools and ideas when automorphisms are present.

The main result of the present article is the resolution of Conjecture 1.1, and in fact with a stronger convergence:

Theorem 1.2. *Let (M, J, g_1) be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to g_1 is $c_1(M, \mathrm{J})$ and let $\{g_i\}_{i \in \mathbb{N}}$ be given by (1). Then there exist holomorphic diffeomorphisms h_k such that $h_k^* g_k$ converges smoothly to a Kähler–Einstein metric.*

A key ingredient in establishing this result is our use of a Finsler metric structure on the space of Kähler metrics introduced previously by one of us [Darvas 2015]. In this infinite-dimensional geometry, the automorphisms of X act by isometries. We establish the boundedness of the Ricci iteration with respect to this Finsler metric, up to automorphisms of X. This is then shown to imply the key a priori estimates with respect to the stronger C^k norms. In fact, we also show a rather stronger result: discretizations of the Kähler–Ricci flow for *any time step* converge. This is new even for the case of no automorphisms considered in [Rubinstein 2008c; Berman et al. 2016a] and resolves a more general conjecture than Conjecture 1.1; see Theorem 1.6 below.

Uniformization of the two-sphere. As a very special case we obtain the following new method of uniformization. Fix a conformal class of volume V on S^2. As we know, in this class there is a constant curvature metric, the round one. More precisely, let ω_c denote the round form of the constant-c Ricci curvature metric on $M = (S^2, \mathrm{J})$, given locally by

$$\omega_c = \frac{\sqrt{-1}}{c\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Here $V = \int_{S^2} \omega_c = c_1([M])/c = 2/c$. Consequently, $c = \frac{1}{2\pi}$ in the case where we restrict the Euclidean metric of $\mathbb{R}^3$ to the unit sphere.

Let ω be any metric on S^2 with $\int_{S^2}\omega = V = 2/c$. Introduce $u_0 = 0$, and we solve iteratively to find $u_i \in C^\infty(S^2)$ satisfying

$$\Delta_\omega u_i = R_\omega - 2e^{u_{i-1}} \quad \text{and} \quad \int_{S^2} e^{u_i}\omega = \frac{2}{c}, \tag{2}$$

so that the scalar curvature of $\omega_i := e^{u_i}\omega$ satisfies $R_{\omega_i} = 2e^{u_{i-1}-u_i}$, or equivalently, $\operatorname{Ric}\omega_i = \omega_{i-1}$. (In two dimensions, $\operatorname{Ric}\omega = \frac{1}{2}R_\omega\omega$, where R_ω is the scalar curvature. If $\omega_1 = e^\phi\omega_0$, then the scalar curvatures of these two metrics satisfy

$$\Delta_{\omega_0}\phi - R_{\omega_0} + R_{\omega_1}e^\phi = 0.$$

We note that the conformal factor is often written $e^{2\phi}$ elsewhere, but this is compensated for here by the fact that $R_\omega = 2K_\omega$, where K_ω is the Gauss curvature.)

Corollary 1.3. *We fix $c > 0$ and let ω be any Kähler form on S^2 with $\int_X \omega = 2/c$. We introduce $\{u_i\} \subset C^\infty(S^2)$ by repeatedly solving the Poisson equation (2). Then, there exist Möbius transformations h_i such that $h_i^*(e^{u_i}\omega)$ converges smoothly to the round metric ω_c.*

Discretization of the Ricci flow. One of the original motivations for introducing the Ricci iteration, going back to [Rubinstein 2007; 2008c], is its relation to the Ricci flow. Hamilton's Ricci flow on a Kähler manifold of definite or zero first Chern class is defined as $\{\omega(t)\}_{t\in\mathbb{R}_+}$ satisfying the evolution equation

$$\begin{aligned}\frac{\partial\omega(t)}{\partial t} &= -\operatorname{Ric}\omega(t) + \mu\omega(t), \quad t \in \mathbb{R}_+,\\ \omega(0) &= \omega,\end{aligned}$$

where Ω is a Kähler class satisfying $\mu\Omega = c_1(M, \mathrm{J})$ for $\mu \in \{-1, 0, 1\}$ and $[\omega] = \Omega$ [Hamilton 1982].

The following dynamical system is seen to be a discrete version of this flow [Rubinstein 2008c, Definition 3.1], obtained by a backward Euler discretization with time step τ.

Definition 1.4. Let Ω be a Kähler class satisfying $\mu\Omega = c_1(M, \mathrm{J})$ for $\mu \in \{-1, 0, 1\}$. Given a Kähler form ω with $[\omega] = \Omega$ and a number $\tau > 0$, define the time-τ Ricci iteration to be the sequence of forms $\{\omega_{k\tau}\}_{k\geq 0}$ satisfying the equations

$$\begin{aligned}\frac{\omega_{k\tau} - \omega_{(k-1)\tau}}{\tau} &= -\operatorname{Ric}\omega_{k\tau} + \mu\omega_{k\tau}, \quad k \in \mathbb{N},\\ \omega_0 &= \omega.\end{aligned}$$

Let us assume that $\mu = 1$ from now on; for the cases $\mu \in \{-1, 0\}$ see [Rubinstein 2008c, Theorem 3.3]. Observe that in the case when $\tau = 1$, the time-τ Ricci iteration is precisely the Ricci iteration from (1). Indeed, Conjecture 1.1 is in fact a special case of the following conjecture concerning the time-τ Ricci iteration for any $\tau > 0$ [Rubinstein 2008c, Conjecture 3.2].

Conjecture 1.5. *Let (M, J) be a compact Kähler manifold admitting a Kähler–Einstein metric. Let Ω be a Kähler class such that $\Omega = c_1(M, \mathrm{J})$. Then for any ω with $[\omega] = \Omega$ and for any $\tau > 0$, the time-τ Ricci iteration exists for all $k \in \mathbb{N}$ and converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric.*

The case when $\tau > 1$ is treated in [Rubinstein 2008c, Theorem 3.3]. However, it is the case $\tau \le 1$ that is the most interesting and challenging. The case $\tau = 1$ is perhaps the most interesting due to the simple geometrical interpretation (1), while the cases $\tau < 1$ are interesting due to the connection to the Kähler–Ricci flow. In this regime one may expect the Ricci iteration to converge to the Ricci flow in a certain scaling limit as $\tau \to 0$. The cases $\tau \le 1$ are challenging since the a priori estimates are considerably harder then. While in the regime $\tau > 1$, one has a uniform positive Ricci lower bound along the iteration; this is no longer true when $\tau \le 1$. Thus, there is no a priori control on the diameter or the Poincaré and Sobolev constants. We work around these difficulties by analyzing the Ricci iteration in the metric geometry of the space of Kähler potentials [Darvas 2015].

In this article, we confirm the more general Conjecture 1.5, and treat the iteration for all time steps τ by proving the following result, of which Theorem 1.2 is a special case.

Theorem 1.6. *Let (M, J, g_1) be a compact Kähler manifold admitting a Kähler–Einstein metric. Suppose the Kähler class associated to g_1 is $c_1(M, \mathrm{J})$ and let $\{\omega_{k\tau}\}_{k\in\mathbb{N}}$ be the time-τ Ricci iteration given by Definition 1.4. Then there exist holomorphic diffeomorphisms h_k such that $h_k^*\omega_{k\tau}$ converges smoothly to a Kähler–Einstein form.*

2. Energy functionals

Let (M, ω) denote a connected compact closed Kähler manifold. The space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(M) : \omega_\varphi := \omega + \sqrt{-1}\,\partial\bar\partial\varphi > 0\} \tag{3}$$

can be identified with $\mathcal{H} \times \mathbb{R}$, where

$$\mathcal{H} = \{\omega_\varphi : \varphi \in C^\infty(M),\ \omega_\varphi > 0\} \tag{4}$$

is the space of all Kähler metrics (or forms) representing the fixed cohomology class $[\omega]$.

From now on let ω be a Kähler form on M, cohomologous to $c_1(M, \mathrm{J})$. The Aubin–Mabuchi functional was introduced by Mabuchi [1986, Theorem 2.3],

$$\mathrm{AM}(\varphi) := \frac{V^{-1}}{n+1} \sum_{j=0}^{n} \int_M \varphi\omega^j \wedge \omega_\varphi^{n-j}, \tag{5}$$

where $V := \int_M \omega_\varphi^n = \int_M \omega_\varphi^n$ is the total volume of the Kähler class. Integration by parts gives the useful estimates

$$\frac{1}{V}\int_M (u-v)\omega_u^n \le \mathrm{AM}(u) - \mathrm{AM}(v) \le \frac{1}{V}\int_M (u-v)\omega_v^n. \tag{6}$$

The subspace

$$\mathcal{H}_0 := \mathrm{AM}^{-1}(0) \cap \mathcal{H}_\omega \tag{7}$$

is isomorphic to $\mathcal{H}$ (4), the space of Kähler metrics.

Let $f_{\omega_\varphi} \in C^\infty(M)$ denote the unique function (called the Ricci potential of ω_φ) satisfying

$$\sqrt{-1}\,\partial\bar\partial f_{\omega_\varphi} = \mathrm{Ric}\,\omega_\varphi - \omega_\varphi, \qquad \frac{1}{V}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n = 1.$$

The Ding and Mabuchi functionals are given by [Ding 1988; Mabuchi 1986]

$$\begin{aligned} D(\varphi) &:= -\mathrm{AM}(\varphi) - \log\frac{1}{V}\int_M e^{f_\omega-\varphi}\omega^n, \\ E(\varphi) &:= \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{e^{f_\omega}\omega^n}\omega_\varphi^n - \mathrm{AM}(\varphi) + \frac{1}{V}\int_M \varphi\omega_\varphi^n + \frac{1}{V}\int_M f_\omega\omega^n. \end{aligned} \tag{8}$$

Notice that these functionals are invariant under addition of constants to φ; hence they descend to $\mathcal{H}$. Additionally, the critical points of these functionals are exactly the Kähler–Einstein metrics.

For $\varphi \in \mathcal{H}_\omega$ with $\int_M e^{f_\omega-\varphi}\omega^n = V$, Jensen's inequality for the convex weight $t \to t\log t$ yields

$$\mathrm{Ent}(e^{f_\omega-\varphi}\omega^n, \omega_\varphi^n) := \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{e^{f_\omega-\varphi}\omega^n}\omega_\varphi^n = \frac{1}{V}\int_X \frac{\omega_\varphi^n}{e^{f_\omega-\varphi}\omega^n}\log\frac{\omega_\varphi^n}{e^{f_\omega-\varphi}\omega^n}e^{f_\omega-\varphi}\omega^n \ge 0. \tag{9}$$

Thus,

$$E(\omega_\varphi) - \frac{1}{V}\int_M f_\omega\omega^n = \mathrm{Ent}(e^{f_\omega-\varphi}\omega^n, \omega_\varphi^n) - \mathrm{AM}(\varphi) \ge -\mathrm{AM}(\varphi) = D(\omega_\varphi).$$

Moreover, if

$$D(\omega_\varphi) = E(\omega_\varphi) - \frac{1}{V}\int_M f_\omega\omega^n$$

then equality holds in (9). As a result, $\omega_\varphi^n = e^{f_\omega-\varphi}\omega^n = e^{f_{\omega_\varphi}}\omega_\varphi^n$; i.e., ω_φ is Kähler–Einstein. This together with the fact that Kähler–Einstein metrics minimize both D and E allows us to conclude the following result; see also [Rubinstein 2008b, (24)].

Proposition 2.1. *For* $\varphi \in \mathcal{H}_\omega$,

$$D(\omega_\varphi) \le E(\omega_\varphi) - \frac{1}{V}\int_M f_\omega\omega^n,$$

with equality if and only if $\mathrm{Ric}\,\omega_\varphi = \omega_\varphi$.

3. The metric completion

All of the functionals introduced in the previous section can be extended to the potential space $\mathcal{E}_1$ introduced by Guedj and Zeriahi [2007], which can be identified with a natural metric completion of $\mathcal{H}$ [Darvas 2015]. The resulting metric theory provides essential tools for proving our main result concerning convergence of the Ricci iteration. We briefly recall this machinery, referring to [Darvas and Rubinstein 2017, §4–5] for more details.

Let

$$\mathrm{PSH}(M,\omega) = \{\varphi \in L^1(M,\omega^n) : \varphi \text{ is upper semicontinuous and } \omega_\varphi \ge 0\}.$$

Following [Guedj and Zeriahi 2007, Definition 1.1] we define the subset of full mass potentials

$$\mathcal{E}(M,\omega) := \left\{\varphi \in \mathrm{PSH}(M,\omega) : \lim_{j\to-\infty} \int_{\{\varphi\le j\}} (\omega + \sqrt{-1}\,\partial\bar\partial \max\{\varphi, j\})^n = 0\right\}.$$

For each $\varphi \in \mathcal{E}(M,\omega)$, define $\omega_\varphi^n := \lim_{j\to-\infty} \mathbf{1}_{\{\varphi>j\}}(\omega+\sqrt{-1}\,\partial\bar\partial \max\{\varphi, j\})^n$. By definition, $\mathbf{1}_{\{\varphi>j\}}(x)$ is equal to 1 if $\varphi(x) > j$ and zero otherwise, and the measure $(\omega + \sqrt{-1}\,\partial\bar\partial \max\{\varphi, j\})^n$ is defined by [Bedford and Taylor 1982] since $\max\{\varphi, j\}$ is bounded. Consequently, $\varphi \in \mathcal{E}(M,\omega)$ if and only if $\int_X \omega_\varphi^n = \int_X \omega^n$, justifying the name of $\mathcal{E}(M,\omega)$.

Next, define a further subset, the space of finite 1-energy potentials

$$\mathcal{E}_1 := \left\{\varphi \in \mathcal{E}(M,\omega) : \int |\varphi|\omega_\varphi^n < \infty\right\}.$$

Consider the following weak Finsler metric on $\mathcal{H}_\omega$ [Darvas 2015]:

$$\|\xi\|_\varphi := V^{-1}\int_M |\xi|\omega_\varphi^n, \quad \xi \in T_\varphi\mathcal{H}_\omega = C^\infty(M). \tag{10}$$

We denote by d_1 the associated pseudometric and recall the result alluded to above, characterizing the d_1-metric completion of $\mathcal{H}_\omega$ [Darvas 2015, Theorems 2 and 3.5]:

Theorem 3.1. *$(\mathcal{H}_\omega, d_1)$ is a metric space whose completion can be identified with $(\mathcal{E}_1, d_1)$, where*

$$d_1(u_0,u_1) := \lim_{k\to\infty} d_1(u_0(k), u_1(k))$$

for any smooth decreasing sequences $\{u_i(k)\}_{k\in\mathbb{N}} \subset \mathcal{H}_\omega$ converging pointwise to $u_i \in \mathcal{E}_1$, $i = 0, 1$.

Also, by [Darvas 2015, Theorem 3], we have the following qualitative estimates for the d_1-metric in terms of analytic quantities:

$$\frac{1}{C}d_1(u,v) \le \int_M |u-v|\omega_u^n + \int_M |u-v|\omega_v^n \le Cd_1(u,v), \quad u,v \in \mathcal{E}_1, \tag{11}$$

where $C > 1$ only depends on ω.

A crucial fact is that the formulas defining the energy functionals discussed in Section 2 actually make sense on the metric completion $\mathcal{E}_1$, and then coincide with the greatest lower semicontinuous extension of the said functionals restricted to $\mathcal{H}_\omega$ [Darvas and Rubinstein 2017, Lemma 5.2, Propositions 5.19 and 5.21]:

Lemma 3.2. (i) AM, $D : \mathcal{H}_\omega \to \mathbb{R}$ *each admit a unique d_1-continuous extension to $\mathcal{E}_1$ and these extensions still satisfy (5) and (8) respectively.*

(ii) *$E : \mathcal{H}_\omega \to \mathbb{R}$ admits a d_1-lower semicontinuous extension to $\mathcal{E}_1$ and the greatest such extension still satisfies (8).*

Proposition 2.1 was generalized by Berman [2013, Theorem 1.1] to the context of the metric completion (for a proof using the Ricci iteration see [Darvas 2017, Proposition 4.42]):

Theorem 3.3. *Proposition 2.1 holds more generally for all $\varphi \in \mathcal{E}_1$.*

Let $G := \mathrm{Aut}_0(M)$ denote the connected component of the complex Lie group of automorphisms (biholomorphisms) of M. The automorphism group acts on $\mathcal{H}$ by pullback:

$$f.\eta := f^\star \eta, \quad f \in G,\ \eta \in \mathcal{H}. \tag{12}$$

Given the one-to-one correspondence between $\mathcal{H}$ and $\mathcal{H}_0$, recall (7), the group G also acts on $\mathcal{H}_0$. The precise action is described in the next lemma [Darvas and Rubinstein 2017, Lemma 5.8].

Lemma 3.4. *For $\varphi \in \mathcal{H}_0$ and $f \in G$ let $f.\varphi \in \mathcal{H}_0$ be the unique potential such that $f^*\omega_\varphi = \omega_{f.\varphi}$. Then,*

$$f.\varphi = f.0 + \varphi \circ f. \tag{13}$$

Complementing the above, G acts on $\mathcal{H}_0$ by d_1-isometries [Darvas and Rubinstein 2017, Lemma 5.9], which allows us to introduce a natural (pseudo-)metric on the space $\mathcal{H}_0/G$:

$$d_{1,G}(Gu, Gv) = \inf_{g \in G} d_1(u, g.v), \quad u, v \in \mathcal{H}_0. \tag{14}$$

4. Metric convergence of the iteration

We consider the τ-step Ricci iteration equation

$$\frac{\omega_{\psi_{(k+1)\tau}} - \omega_{\psi_{k\tau}}}{\tau} = \omega_{\psi_{(k+1)\tau}} - \mathrm{Ric}\,\omega_{\psi_{(k+1)\tau}}$$

for $\tau \in (0, 1]$. When $\tau = 1$, the iteration simply becomes $\mathrm{Ric}\,\omega_{\psi_{k+1}} = \omega_{\psi_k}$. As explained in [Rubinstein 2008c, (33)], on the level of scalars the iteration can be written in the following manner:

$$\omega^n_{\psi_{(k+1)\tau}} = e^{f_\omega - \frac{1}{\tau}\psi_{k\tau} - (1-\frac{1}{\tau})\psi_{(k+1)\tau}}\omega^n, \quad k \in \mathbb{N}, \tag{15}$$

with the natural normalization

$$\frac{1}{V}\int_M e^{f_\omega - \frac{1}{\tau}\psi_{k\tau} - (1-\frac{1}{\tau})\psi_{(k+1)\tau}}\omega^n = 1. \tag{16}$$

Since $\tau \in (0, 1]$, note that (15)–(16) has a unique solution $\psi_{(k+1)\tau} \in \mathcal{H}_\omega$, according to [Aubin 1984; Yau 1978].

In our particular case, there will be special emphasis on working in the geodesically complete potential space $\mathcal{H}_0$, and we introduce accordingly

$$\psi'_{k\tau} := \psi_{k\tau} - \mathrm{AM}(\psi_{k\tau}) \in \mathcal{H}_0. \tag{17}$$

First we generalize an inequality of [Rubinstein 2008c] (in the case $\tau = 1$) that provides a comparison of the Ding and Mabuchi energies along the τ-iteration:

Proposition 4.1. *Suppose $\tau \in (0, 1]$ and (M, ω) is a Fano manifold and $\psi_{1\tau} \in \mathcal{H}_\omega$. Then the following estimate holds along the iteration*:

$$E(\omega_{\psi_{(k+1)\tau}}) - \frac{1}{V}\int_M f_\omega \omega^n \le \frac{1}{\tau} D(\omega_{\psi_{k\tau}}) + \left(1 - \frac{1}{\tau}\right) D(\omega_{\psi(k+1)\tau}) \quad \textit{for all } k \in \mathbb{N}. \tag{18}$$

In the argument below (and thereafter) we will suppress the parameter τ from superscripts whenever this will cause no confusion.

Proof. Using (8) and (15),

$$\begin{aligned}
E(\omega_{\psi_{k+1}})-\frac{1}{V}\int_M f_\omega\omega^n &= \frac{1}{V}\int_X \log\frac{\omega^n_{\psi_{k+1}}}{e^{f_\omega}\omega^n}\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1})+\frac{1}{V}\int_M \psi_{k+1}\omega^n_{\psi_{k+1}}\\
&= -\frac{1}{V}\int_M\left(\frac{1}{\tau}\psi_k+\left(1-\frac{1}{\tau}\right)\psi_{k+1}\right)\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1})+\frac{1}{V}\int_M \psi_{k+1}\omega^n_{\psi_{k+1}}\\
&= \frac{1}{\tau V}\int_M(\psi_{k+1}-\psi_k)\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1}).
\end{aligned}$$

Using this identity, to finish the proof, we notice that it is enough to prove the following two inequalities (and later add them up):

$$\frac{1}{\tau V}\int_M(\psi_{k+1}-\psi_k)\omega^n_{\psi_{k+1}} - \mathrm{AM}(\psi_{k+1}) \le -\frac{1}{\tau}\mathrm{AM}(\psi_k)-\left(1-\frac{1}{\tau}\right)\mathrm{AM}(\psi_{k+1}), \tag{19}$$

$$0 \le -\frac{1}{\tau}\log\left(\frac{1}{V}\int_M e^{f_\omega-\psi_k}\omega^n\right)-\left(1-\frac{1}{\tau}\right)\log\left(\frac{1}{V}\int_M e^{f_\omega-\psi_{k+1}}\omega^n\right). \tag{20}$$

Notice that, after rearranging terms, (19) is seen to be equivalent to

$$\frac{1}{V}\int_M(\psi_{k+1}-\psi_k)\omega^n_{\psi_{k+1}} \le \mathrm{AM}(\psi_{k+1})-\mathrm{AM}(\psi_k).$$

Thus, (19) follows from (6). To address (20) we prove the following more general claim.

Claim 4.2. *For $\tau\in(0,1]$ and $g,h\in C^\infty(X)$ the following estimate holds*:

$$\left(\frac{1}{V}\int_M e^{f_\omega-g}\omega^n\right)^{\frac{1}{\tau}}\left(\frac{1}{V}\int_M e^{f_\omega-h}\omega^n\right)^{1-\frac{1}{\tau}} \le \frac{1}{V}\int_M e^{f_\omega-\frac{1}{\tau}g-(1-\frac{1}{\tau})h}\omega^n. \tag{21}$$

By our choice of normalization (16), this inequality implies (20).

As (21) is seen to be invariant under adding constants to g and h, we can assume that

$$\frac{1}{V}\int_M e^{f_\omega-h}\omega^n=1.$$

In particular, we only have to argue that

$$\left(\frac{1}{V}\int_M e^{-g+h}e^{f_\omega-h}\omega^n\right)^{\frac{1}{\tau}} \le \frac{1}{V}\int_M (e^{-g+h})^{\frac{1}{\tau}}e^{f_\omega-h}\omega^n.$$

This follows from Jensen's inequality, as the function $f(t)=t^{\frac{1}{\tau}}$ is convex for $t>0$. □

Next we show that in the case a Kähler–Einstein metric exists, the iteration $\{\psi_k'\}_k$ d_1-converges up to pullbacks:

Proposition 4.3. *Let $\tau \in (0, 1]$. Suppose a Kähler–Einstein metric exists in $\mathcal{H}$, and let $\{\psi_{k\tau}\}_{k\in\mathbb{N}}$ be the solutions of (15). Then there exist $g_k \in G$ such that $g_k.\psi'_{k\tau}$ d_1-converges to a Kähler–Einstein potential.*

Proof. Proposition 4.1 combined with Proposition 2.1 gives

$$D(\omega_{\psi_{k+1}}) \le E(\omega_{\psi_{k+1}}) - \frac{1}{V}\int_M f_\omega \omega^n \le \frac{1}{\tau} D(\omega_{\psi_k}) + \left(1 - \frac{1}{\tau}\right) D(\omega_{\psi_{k+1}}), \quad k \in \mathbb{N}. \tag{22}$$

As a result, $\{D(\omega_{\psi_l})\}_l$ is a decreasing sequence (this is proved in [Rubinstein 2008c, Proposition 4.2(ii)] for $\tau = 1$). We fix a Kähler–Einstein potential

$$\psi_{\mathrm{KE}} \in \mathcal{H}_0.$$

Existence of such a potential implies that both D and E are bounded below [Bando and Mabuchi 1987; Ding and Tian 1992]. Therefore, the (monotone) sequence $\{D(\omega_{\psi_l})\}_l$ converges. Additionally, by (22), $\{E(\omega_{\psi_l}) - \frac{1}{V}\int_M f_\omega \omega^n\}_l$ converges and

$$\lambda := \lim_l E(\omega_{\psi_l}) - \frac{1}{V}\int_M f_\omega \omega^n = \lim_l D(\omega_{\psi_l}) \in \mathbb{R}.$$

Next we focus on the potentials $\psi'_l \in \mathcal{H}_0$. By [Darvas and Rubinstein 2017, Theorem 2.4], E is G-invariant and

$$E(\psi'_l) \ge C_1 d_{1,G}(0, \psi'_l) - C_2,$$

and so $d_{1,G}(0, \psi'_l) \le C'$. By definition, see (14), there exists $g_l \in G$ such that

$$d_1(\psi_{\mathrm{KE}}, g_l.\psi'_l) \le d_{1,G}(G\psi_{\mathrm{KE}}, G\psi'_l) + \frac{1}{l} \le C' + 1. \tag{23}$$

Remark 4.4. In fact, there exists g_l which achieve the equality $d_1(\psi_{\mathrm{KE}}, g_l.\psi'_l) = d_{1,G}(G\psi_{\mathrm{KE}}, G\psi'_l)$ by [Darvas and Rubinstein 2017, Proposition 6.8] but we do not have to know that for our proof here.

Setting

$$v_l := g_l.\psi'_l,$$

by the G-invariance of E, we obtain that $E(v_l)$ is bounded. On the other hand, a combination of (11) and (23) gives that $\mathrm{AM}(v_l) = 0$ and $\int_M v_l \omega^n_{v_l}$ are bounded as well. Comparing with (4), we see that $\mathrm{Ent}(e^{f_0}\omega^n, \omega^n_{v_l})$ is bounded too.

By (11), d_1-boundedness of potentials implies L^1-boundedness, which in turn implies boundedness of the supremum. As a result, we can apply the compactness result of [Berman et al. 2016a] (see [Darvas and Rubinstein 2017, Theorem 5.6] for a convenient formulation for our context) to conclude that $\{v_l\}_l$ is d_1-precompact.

Next we claim that $d_1(\psi_{\mathrm{KE}}, v_l) \to 0$. If this is not the case, then by possibly choosing a subsequence, we can assume that $d_1(\psi_{\mathrm{KE}}, v_l) > \varepsilon > 0$. By possibly choosing another subsequence, we can assume that $d_1(v_l, u) \to 0$ for some $u \in \mathcal{E}_1$. Lemma 3.2 gives that

$$\lambda = D(u) = E(u) - \frac{1}{V}\int_M f_\omega \omega^n,$$

and in particular u is a Kähler–Einstein potential by Theorem 3.3.

By the Bando–Mabuchi uniqueness theorem [1987] $u = h.\psi_{\mathrm{KE}}$ for some $h \in G$. Combining this with (23), we conclude that

$$d_1(v_{k_l}, \psi_{\mathrm{KE}}) - \frac{1}{k_l} \le d_{1,G}(Gv_l, G\psi_{\mathrm{KE}}) \le d_1(h^{-1}v_l, \psi_{\mathrm{KE}}) = d_1(v_l, h.\psi_{\mathrm{KE}}) = d_1(v_l, u).$$

By choice, the right-hand side converges to zero, and the lim inf of left-hand side is bounded below by $\varepsilon > 0$, giving a contradiction. This implies that $d_1(v_k, \psi_{\mathrm{KE}}) \to 0$, concluding the proof. □

5. A priori estimates and smooth convergence

In this section we prove our main result by strengthening Proposition 4.3.

Theorem 5.1. *Let $\tau \in (0, 1]$. Suppose a Kähler–Einstein metric exists in $\mathcal{H}$, and let $\{\psi_{k\tau}\}_{k\in\mathbb{N}}$ be the solutions of (15). Then there exist $g_k \in G$ such that $g_k.\psi'_{k\tau}$ converges smoothly to a Kähler–Einstein potential. In particular, $g_k^*\omega_{\psi_{k\tau}}$ converges smoothly to a Kähler–Einstein metric.*

Proof. By Proposition 4.3 there exists $g_k \in G$ and a Kähler–Einstein potential $\psi_{\mathrm{KE}} \in \mathcal{H}_0$ such that $d_1(g_k.\psi'_k, \psi_{\mathrm{KE}}) \to 0$. We show below that in fact $g_k.\psi'_k \to_{C^\infty} \psi_{\mathrm{KE}}$.

Focusing on the τ-step Ricci iteration recursion, we can write

$$\begin{aligned}(g_{k+1}^{-1} \circ g_k)^* \operatorname{Ric} \omega_{g_{k+1}.\psi'_{k+1}} = g_k^* \operatorname{Ric} \omega_{\psi'_{k+1}} &= g_k^*\Big(\frac{1}{\tau}\omega_{\psi'_k} + \Big(1 - \frac{1}{\tau}\Big)\omega_{\psi'_{k+1}}\Big) \\ &= \frac{1}{\tau}\omega_{g_k.\psi'_k} + \Big(1 - \frac{1}{\tau}\Big)\omega_{g_k.\psi'_{k+1}} \\ &= \frac{1}{\tau}\omega_{g_k.\psi'_k} + \Big(1 - \frac{1}{\tau}\Big)\omega_{(g_{k+1}^{-1}\circ g_k).g_{k+1}.\psi'_{k+1}}. \end{aligned} \tag{24}$$

Set

$$\begin{aligned} \varphi_k &:= g_k.\psi'_k \in \mathcal{H}_0, \\ f_k &:= g_k^{-1} \circ g_{k-1} \in G. \end{aligned}$$

With this notation, (24) becomes

$$\operatorname{Ric} \omega_{f_{k+1}.\varphi_{k+1}} = \frac{1}{\tau}\omega_{\varphi_k} + \Big(1 - \frac{1}{\tau}\Big)\omega_{f_{k+1}.\varphi_{k+1}}. \tag{25}$$

Without loss of generality we assume that ω (the reference form) is Kähler–Einstein. Using (25) we can write

$$\sqrt{-1}\,\partial\bar\partial\Big(\frac{1}{\tau}\varphi_{k-1} + \Big(1 - \frac{1}{\tau}\Big) f_k.\varphi_k\Big) = \operatorname{Ric}\omega_{f_k.\varphi_k} - \operatorname{Ric}\omega = \sqrt{-1}\,\partial\bar\partial \log(\omega^n/\omega^n_{f_k.\varphi_k}).$$

This implies

$$\frac{1}{\tau}\varphi_{k-1} + \Big(1 - \frac{1}{\tau}\Big) f_k.\varphi_k + \log(\omega^n_{f_k.\varphi_k}/\omega^n) = B_j \in \mathbb{R}.$$

Since log is a concave function, by Jensen's inequality,

$$\frac{1}{V}\int_M \log(\omega^n_{f_k.\varphi_k}/\omega^n)\omega^n \le \log\frac{1}{V}\int_M \omega^n_{f_k.\varphi_k} = 0.$$

By the triangle inequality, for k sufficiently large,

$$d_1(0, \varphi_{k-1}) \le d_1(\psi_{\mathrm{KE}}, 0) + 1.$$

Using (11) we conclude that $\int_M \varphi_{k-1}\omega^n \le C$. These last two estimates combine to give

$$B_j - \left(1 - \frac{1}{\tau}\right)\frac{1}{V}\int_M f_k.\varphi_k \omega^n = \frac{1}{V}\int_M \varphi_{k-1}\omega^n + \frac{1}{V}\int_M \log(\omega^n_{f_k.\varphi_k}/\omega^n)\omega^n \le C.$$

Since $f_k.\varphi_k \in \mathrm{PSH}(M,\omega)$, it is well known that $\int_M f_k.\varphi_k\omega^n$ and $\sup_M f_k.\varphi_k$ are comparable. As a result,

$$B_j - \left(1 - \frac{1}{\tau}\right)\sup_M f_k.\varphi_k \le C;$$

hence we can write

$$\omega^n_{f_k.\varphi_k} = e^{B_j - (1-\frac{1}{\tau}) f_k.\varphi_k - \frac{1}{\tau}\varphi_{k-1}}\omega^n \le e^{C - \frac{1}{\tau}\varphi_{k-1}}\omega^n. \tag{26}$$

Moreover, by Zeriahi's version of the Skoda integrability theorem [Zeriahi 2001] (see [Darvas and Rubinstein 2017, Theorem 5.7] for a formulation that fits our context most), there exists $C > 0$ such that, say,

$$\int_M e^{-\frac{3}{\tau}\varphi_{k-1}}\omega^n \le C, \quad k \in \mathbb{N}.$$

Combining this estimate with (26), we get that

$$\|\omega^n_{f_k.\varphi_k}/\omega^n\|_{L^3(M,\omega^n)} \le C.$$

Now Kołodziej's estimate [2005], see also [Błocki 2005], allows us to conclude that the oscillation satisfies $\operatorname{osc} f_k.\varphi_k \le C$ for some uniform C. Note that for any $u \in \mathcal{H}_0$, it follows from (6) that

$$\inf u \le \frac{1}{V}\int u\omega_u^n \le 0 \le \frac{1}{V}\int u\omega^n \le \sup u,$$

so u changes signs on M. Thus, since $f_k.\varphi_k \in \mathcal{H}_0$, the oscillation bound implies the uniform bound

$$\|f_k.\varphi_k\|_{L^\infty(M)} \le C. \tag{27}$$

Consequently, (11) yields

$$d_1(0, f_k.\varphi_k) = d_1(f_k^{-1}.0, \varphi_k) \le C.$$

Thus,

$$d_1(f_k^{-1}.0, 0) \le d_1(f_k^{-1}.0, \varphi_k) + d_1(\varphi_k, 0) \le C'.$$

From Lemma 5.2, proved below, it follows that $\{f_k^{-1}\}_k$ is contained in a bounded set of G. In particular, all derivatives up to order m of f_k^{-1} are bounded by some C_m, independent of k. So, to finish the proof, it suffices to estimate derivatives of

$$h_k := f_k.\varphi_k$$

(since that will imply the same estimates on $f_k^{-1}.h_k = \varphi_k$). From (25) it follows that

$$\operatorname{Ric}\omega_{h_{k+1}} = \operatorname{Ric}\omega_{f_{k+1}.\varphi_{k+1}} \geq \left(1-\frac{1}{\tau}\right)\omega_{f_{k+1}.\varphi_{k+1}} = \left(1-\frac{1}{\tau}\right)\omega_{h_{k+1}}.$$

Using this, Lemma 5.3 implies $\operatorname{tr}_{\omega_{h_k}}\omega < C$, and using the fact that $\omega_{h_k}^n/\omega^n < C$ by (26) we thus obtain $\operatorname{tr}_\omega \omega_{h_k} < C'$ so $|\Delta_\omega h_k| < C''$, as in [Rubinstein 2008c, p. 1540]. Given the Laplacian bound, the $C^{2,\alpha}$ and higher-order estimates then follow the same way as in [Rubinstein 2008c, Theorem 3.3] (or by applying [Błocki 2012, Theorem 5.1] directly to (26), followed by bootstrapping).

By the Arzelà–Ascoli compactness theorem, $\{\varphi_k\}_k$ is C^k-precompact. From (11) it follows that C^k-convergence implies d_1-convergence. Consequently, any C^k-convergent subsequence of $\{\varphi_k\}_k$ d_1-converges to ψ_{KE}. As a result, $\{\varphi_k\}_k$ C^k-converges to ψ_{KE}, finishing the proof. □

We note that in our arguments above the estimates depend on a positive lower bound to $\tau > 0$. If this could be avoided, then one could hope that these estimates also hold in a scaled limit, as the iteration is expected to converge to the Kähler–Ricci flow.

Lemma 5.2. *Let (X,ω) be a Fano Kähler–Einstein manifold. Let $C > 0$ and suppose that $d_1(g.0,0) \leq C$ for some $g \in G$. Then g is contained in a geodesic ball $B \subset G$ centered at* Id *with radius $R := R(C) > 0$.*

This result is implicit in the arguments of [Darvas and Rubinstein 2017, Proposition 6.8]; see also [Berman et al. 2016b, Lemma 2.7; Darvas and Rubinstein 2017, Claim 7.11].

Proof. By [Darvas and Rubinstein 2017, Propositions 6.2 and 6.9] there exists $k \in \operatorname{Isom}_0(X,\omega)$ and a Hamiltonian vector field $X \in \operatorname{isom}(X,\omega)$ such that $g = k \exp_{\mathrm{Id}} \mathrm{J}X$, where $\exp_{\mathrm{Id}}$ is the exponential map of the Lie group G (recall that J is the complex structure of X). It is clear from the definition of the action of G on the level of potentials that $k^{-1}.0 = 0$. Thus we can write

$$C \geq d_1(g.0,0) = d_1(k\exp_{\mathrm{Id}}(\mathrm{J}X).0,0) = d_1(\exp_{\mathrm{Id}}(\mathrm{J}X).0,k^{-1}.0) = d_1(\exp_{\mathrm{Id}}(\mathrm{J}X).0,0).$$

As shown in [Darvas and Rubinstein 2017, Section 7.1], the curve $[0,\infty) \ni t \to \exp_{\mathrm{Id}}(t\mathrm{J}X).0 \in \mathcal{H}_0$ is a d_1-geodesic ray, and hence $\|X\|$ is bounded. Since $\operatorname{Isom}_0(X,\omega)$ is compact, we obtain that $g = k\exp_{\mathrm{Id}} \mathrm{J}X$ is contained in a geodesic ball $B \subset G$ centered at Id with radius $R := R(C) > 0$. □

For the sake of completeness we recall a version of the Chern–Lu inequality, going back to [Lu 1968], that gives the Laplacian estimate based on a C^0 estimate, elaborated in [Rubinstein 2008c, pp. 1539–1540]; see also [Jeffres et al. 2016, Lemma 7.2]. Since it is stated there in the context of incomplete edge metrics, we state here the simpler smooth version, which follows by setting $D = \varnothing$ in [Jeffres et al. 2016, Lemma 7.2] or [Rubinstein 2014, Corollary 7.8(i)]. Recall that $\operatorname{osc} f := \sup f - \inf f$.

Lemma 5.3. *Let $\varphi \in C^4(M) \cap \mathcal{H}_\omega$. Suppose that $\operatorname{Ric}\omega_\varphi \geq -C_1\omega - C_2\omega_\varphi$. Then for some $C = C(M,\omega,C_1,C_2,\operatorname{osc}\varphi) > 0$,*

$$\operatorname{tr}_{\omega_\varphi}\omega \leq C. \tag{28}$$

Proof. Let $f:(M,\omega_\varphi)\to(M,\omega)$ be the identity map. Then consider the Chern–Lu inequality, see, e.g., [Rubinstein 2014, Proposition 7.1],

$$|\partial f|^2\Delta_{\omega_\varphi}\log|\partial f|^2\geq(\operatorname{Ric}\omega_\varphi)^{\#}\otimes\omega(\partial f,\bar\partial f)-{\omega_\varphi}^{\#}\otimes{\omega_\varphi}^{\#}\otimes R_\omega(\partial f,\bar\partial f,\partial f,\bar\partial f),\tag{29}$$

whose meaning (and proof) in local coordinates we now explain. Write

$$\omega_\varphi=\sqrt{-1}g_{i\bar j}(z)dz^i\wedge\overline{dz^j},\quad \omega=\sqrt{-1}h_{i\bar j}(w)dw^i\wedge\overline{dw^j},$$

where we choose *two* holomorphic coordinate charts $(z_1,\dots,z_n)$ and $(w_1,\dots,w_n)$, respectively, centered at the same point $z_0=f(z_0)\in M$ such that the first is normal for ω_φ, while the second is normal for ω. Write $f:z=(z^1,\dots,z^n)\mapsto f(z)=(f^1(z),\dots,f^n(z))$. Then,

$$\partial f=f_i^j\,dz^i|_z\otimes\frac{\partial}{\partial w^j}\bigg|_{f(z)},$$

and the norm of ∂f induced from considering f as the map $f:(M,\omega_\varphi)\to(M,\omega)$ is then $|\partial f|^2=g^{i\bar l}(z)h_{j\bar k}(f(z))f_i^j(z)\overline{f_l^k(z)}$. Thus, at z_0,

$$\begin{aligned}\Delta_\omega|\partial f|^2&=\sum_{p,q}g^{p\bar q}\frac{\partial^2(g^{i\bar l}h_{j\bar k}f_i^j\,\overline{f_l^k})}{\partial z^p\,\overline{\partial z^p}}\\&=\sum_p g^{p\bar q}[g^{i\bar l}h_{j\bar k,d\bar e}f_i^j\,\overline{f_l^k}\,f_p^d\,\overline{f_q^e}-h_{j\bar k}g^{i\bar t}g^{s\bar l}g_{s\bar t,p\bar q}f_i^j\,\overline{f_l^k}+g^{i\bar l}h_{j\bar k}f_{ip}^j\overline{f_{lq}^k}]\\&=-\omega_\varphi^{\#}\otimes\omega_\varphi^{\#}\otimes R_\omega(\partial f,\bar\partial f,\partial f,\bar\partial f)+(\operatorname{Ric}\omega_\varphi)^{\#}\otimes\omega(\partial f,\bar\partial f)+g^{p\bar q}g^{i\bar l}h_{j\bar k}f_{ip}^j\overline{f_{lq}^k}.\end{aligned}\tag{30}$$

Here R_ω denotes the curvature tensor of η (of type $(0,4)$), while $\omega^{\#}$ denotes the metric g^{-1} on $T^{1,0\star}M$ (i.e., of type $(2,0)$), and similarly $(\operatorname{Ric}\omega)^{\#}$ denotes the $(2,0)$-type tensor obtained from $\operatorname{Ric}\omega_\varphi$ by raising indices using g. The proof of (29) now follows from (30), the identity $u\Delta_\omega\log u=\Delta_\omega u-u|\partial\log u|^2$, and the Cauchy–Schwarz inequality; see [Rubinstein 2014, p. 102].

We claim that (29) implies

$$\Delta_{\omega_\varphi}(\log\operatorname{tr}_{\omega_\varphi}\omega-(C_2+2C_3+1)\varphi)\geq-C_1-(C_2+2C_3+1)n+\operatorname{tr}_{\omega_\varphi}\omega,\tag{31}$$

where C_3 depends on the curvature of ω. Indeed, the assumption on $\operatorname{Ric}\omega_\varphi$ implies

$$\begin{aligned}(\operatorname{Ric}\omega_\varphi)^{\#}\otimes\omega(\partial f,\bar\partial f)&=g^{i\bar l}g^{k\bar j}R_{i\bar j}h_{k\bar l}\\&\geq-C_1g^{i\bar l}g^{k\bar j}g_{i\bar j}h_{k\bar l}-C_2g^{i\bar l}g^{k\bar j}h_{i\bar j}h_{k\bar l}\\&\geq-C_1\operatorname{tr}_{\omega_\varphi}\omega-C_2(\operatorname{tr}_{\omega_\varphi}\omega)^2.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}-{\omega_\varphi}^{\#}\otimes{\omega_\varphi}^{\#}\otimes R^\omega(\partial f,\bar\partial f,\partial f,\bar\partial f)&=-g^{i\bar j}g^{k\bar l}R^\omega_{i\bar jk\bar l}\\&\geq-C_3g^{i\bar j}g^{k\bar l}(h_{i\bar j}h_{k\bar l}+h_{i\bar l}h_{k\bar j})\geq-2C_3(\operatorname{tr}_{\omega_\varphi}\omega)^2,\end{aligned}$$

where C_3 is an upper bound for the bisectional curvature of ω. Finally, the claim follows since $\mathrm{tr}_{\omega_\varphi}\,\omega = \mathrm{tr}_{\omega_\varphi}(\omega_\varphi - \sqrt{-1}\,\partial\bar\partial\varphi) = n - \Delta_{\omega_\varphi}\varphi$.

Using the inequality now in (31) (at the point where the maximum of $\log \mathrm{tr}_{\omega_\varphi}\,\omega - (C_2 + 2C_3 + 1)\varphi$ is attained), the maximum principle thus gives an estimate on $\mathrm{tr}_{\omega_\varphi}\,\omega$, depending of course also on $\operatorname{osc}\varphi$. □

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