ANALYSIS & PDE

Volume 12 No. 3 2019

ANNALISA CESARONI AND MARCO CIRANT

CONCENTRATION OF GROUND STATES IN STATIONARY MEAN-FIELD GAMES SYSTEMS

msp

ANALYSIS AND PDE
Vol. 12, No. 3, 2019
dx.doi.org/10.2140/apde.2019.12.737

msp

CONCENTRATION OF GROUND STATES IN STATIONARY MEAN-FIELD GAMES SYSTEMS

ANNALISA CESARONI AND MARCO CIRANT

We provide the existence of classical solutions to stationary mean-field game systems in the whole space $\mathbb{R}^N$, with coercive potential and aggregating local coupling, under general conditions on the Hamiltonian. The only structural assumption we make is on the growth at infinity of the coupling term in terms of the growth of the Hamiltonian. This result is obtained using a variational approach based on the analysis of the nonconvex energy associated to the system. Finally, we show that in the vanishing viscosity limit, mass concentrates around the flattest minima of the potential. We also describe the asymptotic shape of the rescaled solutions in the vanishing viscosity limit, in particular proving the existence of ground states, i.e., classical solutions to mean-field game systems in the whole space without potential, and with aggregating coupling.

1. Introduction

We consider a class of ergodic mean-field games systems set on the whole space $\mathbb{R}^N$ with unbounded decreasing coupling: our problem is, given $\varepsilon > 0$ and $M > 0$, to find a constant $\lambda \in \mathbb{R}$ for which there exists a pair $(u, m) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$, for any $p > 1$, solving

$$\begin{cases} -\varepsilon\Delta u + H(\nabla u) + \lambda = f(m) + V(x), \\ -\varepsilon\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0 \quad \text{on } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m = M. \end{cases} \tag{1-1}$$

The aim of this work is two-fold. Firstly, for any fixed $\varepsilon > 0$, we prove the existence of classical ground states of (1-1). Secondly, we study their behavior in the vanishing viscosity limit $\varepsilon \to 0$.

MSC2010: primary 35J50; secondary 49N70, 35J47, 91A13, 35B25.
Keywords: ergodic mean-field games, semiclassical limit, concentration-compactness method, mass concentration, elliptic systems, variational methods.

The Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$ is strictly convex, $H \in C^2(\mathbb{R}^N \setminus \{0\})$ and it has superlinear growth: we assume that there exist $C_H > 0$, $K > 0$ and $\gamma > 1$ such that, for all $p \in \mathbb{R}^N$,

$$\begin{gathered} C_H|p|^\gamma - K \le H(p) \le C_H|p|^\gamma, \\ \nabla H(p)\cdot p - H(p) \ge K^{-1}|p|^\gamma - K \quad \text{and} \quad |\nabla H(p)| \le K|p|^{\gamma-1}. \end{gathered} \tag{1-2}$$

The coupling term $f : [0,+\infty) \to \mathbb{R}$ is a locally Lipschitz continuous function such that there exist $C_f > 0$ and $K > 0$ for which

$$-C_f m^\alpha - K \le f(m) \le -C_f m^\alpha + K, \tag{1-3}$$

with

$$0 < \alpha < \frac{\gamma}{N(\gamma-1)} = \frac{\gamma'}{N}, \tag{1-4}$$

where $\gamma' = \frac{\gamma}{\gamma-1}$ is the conjugate exponent of γ.

Finally, we assume that the potential V is a locally Hölder continuous function, and that there exist $b > 0$ and a constant $C_V > 0$ such that

$$C_V^{-1}(\max\{|x| - C_V, 0\})^b \le V(x) \le C_V(1+|x|)^b. \tag{1-5}$$

Note that the requirement of V to be nonnegative is not crucial; we just need it to be bounded from below.

Mean-field games (MFG) is a recent theory that models the behavior of a very large number of indistinguishable rational agents, aiming at minimizing a common cost. The theory was introduced in the seminal works by Lasry and Lions [2006a; 2006b; 2007] and by Huang, Malhamé and Caines [Huang et al. 2006], and has been rapidly growing during the last decade due to its mathematical challenges and several potential applications (from economics and finance, to engineering and models of social systems). In the ergodic MFG setting, the dynamics of a typical agent is given by the controlled stochastic differential equation

$$dX_s = -v_s\, ds + \sqrt{2\varepsilon}\, dB_s, \quad s > 0,$$

where v_s is the control and B_s is a Brownian motion, with initial state given by a random variable X_0. The cost (of long-time average form) is given by

$$\lim_{T\to\infty} \frac{1}{T}\, \mathbb{E}\int_0^T [L(v_s) + V(X_s) + f(m(X_s))]\, ds,$$

where the Lagrangian L is the Legendre transform of H, see (2-1), and $m(x)$ denotes the density of population of small agents at a position $x \in \mathbb{R}^N$. A typical agent minimizes his own cost, and the density of its corresponding distribution law $\mathcal{L}(X_s)$ converges, as $s \to \infty$, to a stationary density μ, which is independent of the initial distribution $\mathcal{L}(X_0)$. In an equilibrium regime, μ coincides with the population density m. This equilibrium is encoded from the PDE viewpoint in (1-1): a solution u of the Hamilton–Jacobi–Bellman (HJB) equation gives an optimal control for the typical agent in feedback form $\nabla H(\nabla u(\cdot))$, and the Kolmogorov equation provides the density m of the agents playing in an optimal way.

The two key points of our setting are the following: Firstly, the cost is monotonically *decreasing* with respect to the population distribution m; namely, agents are attracted toward congested areas. A large part of the MFG literature focuses on the study of systems with competition, namely when the coupling in the cost is monotonically increasing. This assumption is essential if one seeks for uniqueness of equilibria, and it is in general crucial in many existence and regularity arguments; see, e.g., [Gomes et al. 2016]. On the other hand, models with aggregation like (1-1) have been considered in few cases, see [Cesaroni and Cirant 2017; Cirant 2016; 2017; Cirant and Tonon 2018; Gomes et al. 2018].

Secondly, the state of a typical agent here is the *whole euclidean space* $\mathbb{R}^N$. Usually, the analysis of (1-1) is carried out in the periodic setting, in order to avoid boundary issues and the noncompactness of $\mathbb{R}^N$. Few investigations are available in the truly nonperiodic setting: see [Porretta 2017] for time-dependent problems, [Arapostathis et al. 2017] for the case of bounded controls, [Gomes and Pimentel 2016] for some regularity results and [Bardi and Priuli 2014] for the linear-quadratic framework. We observe that the noncompact setting is even more delicate for stationary (ergodic) problems like (1-1): a stable long-time regime of a typical player is ensured if the Brownian motion is compensated by the optimal velocity v_s. In other words, if a force that drives players to bounded states is missing, dissipation eventually leads their distribution to vanish on the whole $\mathbb{R}^N$. This phenomenon is impossible if the state space is compact. The main issue here is that the behavior of the optimal velocity $v_s(\cdot) = \nabla H(\nabla u(\cdot))$ is a priori unknown, and depends in an implicit way on V and the distribution m itself. Note that $V(\cdot)$ represents the spatial preference of a single agent; if it grows as $|x| \to \infty$, it discourages agents from being far away from the origin. At the PDE level, this will compensate the lack of compactness of $\mathbb{R}^N$. Let us mention that even without the coupling term $f(m^\alpha)$, the ergodic control problem in unbounded domains has received considerable attention; see, e.g., [Barles and Meireles 2016; Ichihara 2011; 2015].

In our analysis, we exploit the variational nature of the system (1-1), which has been pointed out already in the first papers on MFG, see [Lasry and Lions 2007], and the more recent work [Mészáros and Silva 2018]. Indeed, solutions to (1-1) can be put in correspondence with critical points of the energy

$$\mathcal{E}(m,w) := \begin{cases} \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V(x)m + F(m)\,dx & \text{if } (m,w) \in \mathcal{K}_{\varepsilon,M}, \\ +\infty & \text{otherwise,} \end{cases} \tag{1-6}$$

where $F(m) = \int_0^m f(n)\,dn$ for $m \geq 0$ and $F(m) = 0$ for $m \leq 0$ and

$$L\left(-\frac{w}{m}\right) := \begin{cases} \sup_{p\in\mathbb{R}^N}\left(-\frac{p\cdot w}{m} - H(p)\right) & \text{if } m > 0, \\ 0 & \text{if } m = 0,\ w = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{1-7}$$

Note that $mL\left(-\frac{\cdot}{m}\right)$ reads as the Legendre transform of $mH(\cdot)$. The constraint set is defined as

$$\mathcal{K}_{\varepsilon,M} := \Big\{(m,w) \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \times L^1(\mathbb{R}^N) : \\ \varepsilon \int_{\mathbb{R}^N} m(-\Delta\varphi)\,dx = \int_{\mathbb{R}^N} w\cdot\nabla\varphi\,dx \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N),\ \int_{\mathbb{R}^N} m\,dx = M,\ m \geq 0 \text{ a.e.}\Big\}, \tag{1-8}$$

with

$$q = \begin{cases} \frac{N}{N-\gamma'+1}, & \gamma' \leq N, \\ \gamma', & \gamma' > N. \end{cases}$$

Under assumption (1-3) on the coupling term, the energy $\mathcal{E}$ is not convex. Condition (1-4) is necessary for the problem $e_\varepsilon(M) := \min_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m,w)$ to be well-posed. Indeed, consider any

$(m_0, w_0) \in \mathcal{K}_{\varepsilon,M}$ such that m_0 has compact support. An easy computation shows that if $\alpha > \frac{\gamma'}{N}$, then

$$\mathcal{E}(\sigma^{-N} m_0(\sigma^{-1}\,\cdot), \sigma^{-(N+1)} w_0(\sigma^{-1}\,\cdot)) \to -\infty$$

as $\sigma \to 0$, so $\mathcal{E}$ is not bounded from below on $\mathcal{K}_{\varepsilon,M}$. We show that (1-4) is indeed sufficient for $e_\varepsilon(M)$ to be finite, and allows us to look for *ground states* of (1-1). This will be accomplished by a study of the Sobolev regularity of the Kolmogorov equation; see in particular Section 2B. Note that the critical case $\alpha = \frac{\gamma'}{N}$ is more delicate, and requires additional analysis. We also mention that another critical exponent is intrinsic in (1-1): if $\alpha > \frac{\gamma'}{N-\gamma'}$, one has to expect nonexistence of solutions; see [Cirant 2016]. We refer to our case as the *subcritical case*, in analogy with the L^2-subcritical regime in nonlinear Schrödinger equations with prescribed mass; see [Cirant 2016, Remark 2.9] for additional comments. The analogy can be made precise in the purely quadratic framework, that is when $H(p) = \frac{1}{2}|p|^2$. Indeed, as observed in [Lasry and Lions 2006a; 2006b], the so-called Hopf–Cole transformation permits us to reduce the number of unknowns in the system. Setting $v^2(x) := m(x) = ce^{-\frac{u(x)}{\varepsilon}}$, with c a normalizing constant, v is a solution to

$$-2\varepsilon^2 \Delta v + (V(x) - \lambda)v = -f(v^2)v,$$

with $\int_{\mathbb{R}^N} v^2(x)\,dx = M$. Then the energy reads $\mathcal{E}(v) = \int_{\mathbb{R}^N} \varepsilon^2 |\nabla v|^2 + \frac{1}{2}V(x)v^2 + \frac{1}{2}F(v^2)\,dx$.

In our approach, to construct solutions to (1-1), we look for minimizers $(m, w) \in \mathcal{K}_{\varepsilon,M}$ of the energy (1-6). These minimizers can be obtained by classical direct methods, by using in particular estimates and compactness in some L^p space for elements (m, w) in $\mathcal{K}_{\varepsilon,M}$ with bounded action, i.e., which satisfy $\int_{\mathbb{R}^N} mL\big(-\frac{w}{m}\big)\,dx \le C$, obtained in Section 2B. Then, the existence of a solution $(u_\varepsilon, \lambda_\varepsilon)$ of the HJB equation in (1-1) is obtained by considering another functional with linearized coupling (around the minimizer) and the associated dual functional in the sense of Fenchel and Rockafellar, as in [Briani and Cardaliaguet 2018]. One has to take care of the interplay between u and m as $|x| \to \infty$. To handle the lack of a priori regularity on the function m, we first regularize the problem, by applying standard regularizing convolution kernels on the coupling (see Section 3). We construct minimizers (m_k, w_k) of the regularized energy and associated solutions (u_k, m_k) of the regularized version of (1-1). Then, in order to come back to the initial problem, we provide some new a priori uniform L^∞ bounds on m_k, which in turn imply a priori uniform bounds on $|\nabla u_k|$ and (local) Hölder regularity of m_k that is uniform in k. This key a priori bound is provided by Theorem 4.1.

Note that we will consider classical solutions to this system (with a slight abuse of terminology), that is, $(u, m) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$ for all $p > 1$. The existence result, proved in Section 4, is the following.

Theorem 1.1. *Under the assumptions* (1-2), (1-3), (1-4) *and* (1-5), *for every* $\varepsilon > 0$ *there exists a classical solution* $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$, *for all* $p > 1$, *to* (1-1). *Moreover*, $(m_\varepsilon, -m_\varepsilon \nabla H(\nabla u_\varepsilon))$ *is a minimizer in the set* $\mathcal{K}_{\varepsilon,M}$ *of the energy* (1-6).

We observe (see Remarks 3.5 and 4.2) that Theorem 1.1 holds under more general conditions on H and f, that is, if there exist $C_H, C_f > 0$ and $K > 0$ such that

$$C_H^{-1}|p|^\gamma - K \le H(p) \le C_H(|p|^\gamma + 1), \quad -C_f m^\alpha - K \le f(m) \le C_f^{-1} m^\alpha + K, \tag{1-9}$$

where α satisfies (1-4).

In the second part of the work, in Section 5, we analyze the behavior of the triple $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ coming from a minimizer of $\mathcal{E}$ as $\varepsilon \to 0$, under the assumptions (1-2), (1-3). From the viewpoint of the model, this amounts to removing the Brownian noise from the agents' dynamics. Heuristically, if the diffusion becomes negligible, one should observe aggregation of players (induced by the decreasing monotonicity of coupling in the cost) towards minima of the potential V, which are the preferred sites. Moreover, in the case V has a finite number of minima and polynomial behavior (that is, when (1-13) holds) we specialize the result showing that the limit procedure selects the more stable minima of V, implying, e.g., full convergence in the case that there exists a unique flattest minimum.

In order to bring as much information as possible to the limit, we consider an appropriate rescaling of m, u, namely

$$\bar{m}_\varepsilon(\cdot) = \varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}} m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon), \quad \bar{u}_\varepsilon(\cdot) = \varepsilon^{\frac{N\alpha(\gamma'-1)-\gamma'}{\gamma'-\alpha N}} (u(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon) - u(x_\varepsilon)) \tag{1-10}$$

for all $\varepsilon > 0$. The rescaling is designed so that $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ solves an MFG system where the nonlinearities have the same behavior of the original ones; i.e., $H_\varepsilon \sim |p|^\gamma$ as $p \to \infty$, but the coefficient in front of the Laplacian is equal to 1 for all ε; see (5-19). Moreover, the pair $\bar{u}_\varepsilon, \bar{m}_\varepsilon$ is associated to a minimizer of a rescaled energy $\mathcal{E}_\varepsilon$; see (5-23). It turns out that in this rescaling process, the potential V becomes

$$V_\varepsilon(\cdot) = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot),$$

and vanishes (locally) as $\varepsilon \to 0$. Therefore, as one passes to the limit, the potential cannot compensate anymore for the lack of compactness of $\mathbb{R}^N$, and the convergence of $\bar{m}_\varepsilon$ in $L^1(\mathbb{R}^N)$ has to be proven by other methods. Heuristically, the aggregating force should be strong enough to overcome the dissipation effect, but the clustering point can be hard to predict by lack of spatial preference. This is why we also have to translate in (1-10) by x_ε. We will select x_ε to be the minimum of u_ε: heuristically, since u_ε is the value function, this is the point where most of the players should be located. In order to recover compactness for the sequence $\bar{m}_\varepsilon$, we implement some ideas of the celebrated *concentration-compactness* method [Lions 1984]. This principle states intuitively that if loss of compactness occurs, $\bar{m}_\varepsilon$ splits in (at least) two parts which are going infinitely far away from each other; that is,

$$\bar{m}_\varepsilon \sim \chi_{B_R(0)} \bar{m}_\varepsilon + \chi_{\mathbb{R}^N \setminus B_{2R}(0)} \bar{m}_\varepsilon, \tag{1-11}$$

with $R \to \infty$, $\int \chi_{B_R(0)} \bar{m}_\varepsilon \sim a$ and $\int \chi_{\mathbb{R}^N \setminus B_{2R}(0)} \bar{m}_\varepsilon \sim M - a$ for some $a \in (0, M)$ (a third possibility might happen, but it is easily ruled out here by local estimates). This induces a splitting in the energy $\mathcal{E}$; that is,

$$\inf_{\int m = M} \mathcal{E}_\varepsilon \gtrsim \inf_{\int m = a} \mathcal{E}_\varepsilon + \inf_{\int m = M-a} \mathcal{E}_\varepsilon. \tag{1-12}$$

One then exploits a special feature of $\mathcal{E}_\varepsilon$, which is called subadditivity:

$$\inf_{\int m = M} \mathcal{E}_\varepsilon < \inf_{\int m = a} \mathcal{E}_\varepsilon + \inf_{\int m = M-a} \mathcal{E}_\varepsilon,$$

which makes (1-12) impossible. While subadditivity is easy to prove for $\mathcal{E}_\varepsilon$ (see Lemma 5.5), the splitting (1-12) requires technical work, in particular due to the presence of the term $mL\left(-\frac{w}{m}\right)$ in $\mathcal{E}_\varepsilon$, which becomes increasingly singular as m approaches zero (a simple cut-off as in (1-11) is not useful). The property (1-12) is proven in Theorem 5.6. It relies on the Brezis–Lieb lemma and a perturbation argument. The L^1 convergence of $\bar{m}_\varepsilon$ enables us to obtain the full convergence of $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ to a limit MFG system. By a uniform control of the decay of $\bar{m}_\varepsilon$ as $|x| \to \infty$, which comes from a Lyapunov function built upon $\bar{u}_\varepsilon$, energy arguments and the crucial L^∞ estimate of Theorem 4.1, we are also able to keep track of x_ε. In terms of the nonrescaled density m_ε, x_ε is the point around which most of the mass is located.

The second main result of this work is stated in the following two theorems. The first one is about the concentration of m_ε.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exist sequences $\varepsilon \to 0$ and x_ε such that for all $\eta > 0$ there exist R and ε_0 for which, for all $\varepsilon < \varepsilon_0$,*

$$\int_{|x-x_\varepsilon|\le R\varepsilon^{\gamma'/(\gamma'-\alpha N)}} m_\varepsilon \, dx \ge M - \eta.$$

Moreover, $x_\varepsilon \to \bar{x}$, where $V(\bar{x}) = 0$, i.e., $\bar{x}$ is a minimum of V.

If, in addition, V has the form

$$V(x) = h(x) \prod_{j=1}^{n} |x - x_j|^{b_j}, \qquad C_V^{-1} \le h(x) \le C_V \quad \text{on } \mathbb{R}^N, \tag{1-13}$$

for some $x_j \in \mathbb{R}^N$, and $b_j > 0$ $\big(\text{with } \sum_{j=1}^{n} b_j = b\big)$, then $x_\varepsilon \to x_i$, with $i \in \{j = 1, \dots, n : b_j = \max_k b_k\}$.

Secondly, we describe the asymptotic profile of $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ as $\varepsilon \to 0$. Note that as a byproduct we obtain the existence of solutions to MFG systems without potential.

Theorem 1.3. *Up to subsequences, $(\bar{u}_\varepsilon, \bar{m}_\varepsilon)$ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^N) \times C_{\mathrm{loc}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, for all $p \ge 1$, to a solution $(\bar{u}, \bar{m})$ of*

$$\begin{cases} -\Delta u + C_H |\nabla u|^\gamma + \lambda = -C_f m^\alpha, \\ -\Delta m - C_H \gamma \operatorname{div}(m |\nabla u|^{\gamma-2} \nabla u) = 0, \\ \int_{\mathbb{R}^N} m = M. \end{cases} \tag{1-14}$$

The function $\bar{u}$ is globally Lipschitz continuous on $\mathbb{R}^N$, and there exist $c_1, c_2 > 0$ such that $0 < \bar{m}(x) \le c_1 e^{-c_2|x|}$.

Finally, if $\bar{w} = -C_H \gamma \bar{m} |\nabla \bar{u}|^{\gamma-2} \nabla \bar{u}$, then

$$\mathcal{E}_0(\bar{m}, \bar{w}) = \min\{\mathcal{E}_0(m, w) : (m, w) \in \mathcal{K}_{1,M},\ m(1 + |y|^b) \in L^1(\mathbb{R}^N)\}, \tag{1-15}$$

where

$$\mathcal{E}_0(m, w) = \int_{\mathbb{R}^N} C_L \frac{|w|^{\gamma'}}{m^{\gamma'-1}} - \frac{1}{\alpha+1} m^{\alpha+1} \, dy. \tag{1-16}$$

We finally observe that by analogous methods, one can prove existence of solutions to more general potential-free MFG systems; see Remark 5.9.

Notation. We will denote a classical solution to the system (1-1) by a triple

$$(u, m, \lambda) \in C^2(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times \mathbb{R} \quad \text{for all } p > 1.$$

For any given $p > 1$, we will denote by $p' = \frac{p}{p-1}$ the conjugate exponent of p, and set

$$p^* = \frac{Np}{N-p} \quad \text{if } p < N \qquad \text{and} \qquad p^* = +\infty \quad \text{if } p \geq N.$$

For all $R > 0$, $x \in \mathbb{R}^N$, we define $B_R(x) := \{y \in \mathbb{R}^N : |x - y| < R\}$. We will set $\omega_N := |B_1(0)|$. Finally, $C, C_1, K, K_1, \ldots$ denote (positive) constants we need not specify.

2. Some preliminary regularity results

Let L be the Legendre transform of H, i.e.,

$$L(q) = H^*(q) = \sup_{p \in \mathbb{R}^N} [p \cdot q - H(p)], \quad q \in \mathbb{R}^N. \tag{2-1}$$

The assumptions on H guarantee the following; see, e.g., [Cirant 2014, Proposition 2.1].

Proposition 2.1. *There exist $C_L, C_1, C_2 > 0$ depending on C_H and on γ such that for all $p, q \in \mathbb{R}^N$,*

(i) $L \in C^2(\mathbb{R}^N \setminus \{0\})$ *and it is strictly convex*,

(ii) $0 \leq C_L |q|^{\gamma'} \leq L(q) \leq C_L(|q|^{\gamma'} + 1)$,

(iii) $\nabla L(q) \cdot q - L(q) \geq C_1 |q|^{\gamma'} - C_1^{-1}$,

(iv) $C_1 q|^{\gamma'-1} - C_1^{-1} \leq |\nabla L(q)| \leq C_1^{-1}(|q|^{\gamma'-1} + 1)$,

(v) $C_2 |p|^{\gamma-1} - C_2^{-1} \leq |\nabla H(p)| \leq C_2^{-1}(|p|^{\gamma-1} + 1)$.

We will use the following (standard) result on Hölder functions vanishing at infinity.

Lemma 2.2. *Suppose that $m \geq 0$, $\|m\|_{C^{0,\theta}(\mathbb{R}^N)} \leq c_h$ for some $\theta, c_h > 0$, and $\int_{\mathbb{R}^N} m\,dx < \infty$. Then, $m(x) \to 0$ as $|x| \to \infty$. Moreover, if*

$$\int_{|x| \geq R} m\,dx < \eta$$

for some $\eta, R > 0$, then

$$\max_{|x| \geq R} m(x) \leq C \eta^{\frac{\theta}{\theta + N}}, \tag{2-2}$$

where $C > 0$ depends only on c_h, N.

Proof. By contradiction, suppose that there exists $\delta > 0$ and a sequence $|x_n| \to \infty$ such that $m(x_n) > \delta$ for all n. We may also assume that $|x_{n+1}| \geq |x_n| + 1$ for all n. By the Hölder regularity assumption,

$$m(x) \geq m(x_n) - c_h |x - x_n|^\theta \geq \tfrac{1}{2}\delta,$$

provided that $x \in B_r(x_n)$, and $r^\theta \le \frac{\delta}{2c_h}$. Choose $r = \min\{1, (\frac{\delta}{2c_h})^{\frac{1}{\theta}}\}$, so that $B_r(x_n) \cap B_r(x_m) = \varnothing$ for all $n \neq m$. Then,

$$\int_{\mathbb{R}^N} m\,dx \ge \sum_{n\in\mathbb{N}} \int_{B_r(x_n)} m\,dx \ge \sum_{n\in\mathbb{N}} \tfrac{1}{2}\delta |B_r(0)| = +\infty,$$

which is impossible.

As for the second part, let $M := \max_{|x|\ge R} m(x) = m(\bar{x})$, $|\bar{x}| \ge R$ (note that such a maximum is achieved as a consequence of the first part of the lemma). As before,

$$m(x) \ge m(\bar{x}) - c_h|x-\bar{x}|^\theta \ge \tfrac{1}{2}M$$

for all $x \in B_r(\bar{x})$, where $r = (\frac{M}{2c_h})^{\frac{1}{\theta}}$. Therefore,

$$\eta > \int_{|x|\ge R} m\,dx \ge \tfrac{1}{4}M|B_r(\bar{x})| = \tfrac{1}{4}M|B_1(0)|\left(\frac{M}{2c_h}\right)^{\frac{N}{\theta}},$$

and (2-2) follows. □

We recall the following well-known result, proved in [Brézis and Lieb 1983, Theorem 1].

Theorem 2.3. *Let $f_n \to f$ a.e. in $\mathbb{R}^N$ and assume $\|f_n\|_{L^p(\mathbb{R}^N)} \le C$ for all n and for some $p \in [1, +\infty)$. Then*

$$\lim_n [\|f_n\|^p_{L^p(\mathbb{R}^N)} - \|f_n - f\|^p_{L^p(\mathbb{R}^N)}] = \|f\|^p_{L^p(\mathbb{R}^N)}.$$

From classical elliptic regularity, we have the following result.

Proposition 2.4. *Let $p > 1$ and $m \in L^p(\mathbb{R}^N)$ be such that*

$$\left|\int_{\mathbb{R}^N} m\,\Delta\varphi\,dx\right| \le K\|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)} \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

for some $K > 0$. Then, $m \in W^{1,p}(\mathbb{R}^N)$ and there exists $C > 0$ depending only on p, such that

$$\|\nabla m\|_{L^p(\mathbb{R}^N)} \le CK.$$

Proof. Fix any $R > 1$. Let $\psi \in C_0^\infty(B_2(0))$, $\varphi(Rx) := \psi(x)$ (so, $\varphi \in C_0^\infty(B_{2R}(0))$) and $v(x) := m(Rx)$ on $\mathbb{R}^N$. Then,

$$\begin{aligned}\left|\int_{B_2(0)} v\,\Delta\psi\,dx\right| &= R^{2-N}\left|\int_{B_{2R}(0)} m\,\Delta\varphi\,dy\right| \le KR^{2-N}\left(\int_{B_{2R}(0)} |\nabla\varphi|^{p'}\,dy\right)^{\frac{1}{p'}} \\ &= KR^{1-N+\frac{N}{p}'}\left(\int_{B_2(0)} |\nabla\psi|^{p'}\,dx\right)^{\frac{1}{p'}} \le KR^{1-\frac{N}{p}}\|\psi\|_{W^{1,p'}(B_2(0))}.\end{aligned}$$

Hence, by [Agmon 1959, Theorem 6.1], $v \in W^{1,p}(B_1(0))$ and there exists a constant C, depending on p (but not on R), such that

$$\|\nabla v\|_{L^p(B_1(0))} \le \|v\|_{W^{1,p}(B_1(0))} \le C(KR^{1-\frac{N}{p}} + \|v\|_{L^p(B_2(0))}).$$

Therefore,

$$\left(\int_{B_R(0)} |\nabla m|^p \, dy\right)^{\frac{1}{p}} = R^{\frac{N}{p}-1}\left(\int_{B_1(0)} |\nabla v|^p \, dx\right)^{\frac{1}{p}} \le C\left[K + R^{\frac{N}{p}-1}\left(\int_{B_2(0)} |v|^p \, dx\right)^{\frac{1}{p}}\right]$$
$$= C(K + R^{-1}\|m\|_{L^p(B_{2R}(0))}).$$

Letting $R \to \infty$, we get that $|\nabla m| \in L^p(\mathbb{R}^n)$ and the desired estimate. □

2A. ***The Hamilton–Jacobi–Bellman equation on the whole space.*** In this section we provide some a priori regularity estimates and existence results for Hamilton–Jacobi–Bellman equations in whole spaces of ergodic type. In particular we will consider families of Hamilton–Jacobi–Bellman equations

$$-\Delta u_n + H_n(\nabla u_n) + \lambda_n = F_n(x) - f_n(x) \quad \text{on } \mathbb{R}^N, \tag{2-3}$$

where $F_n - f_n$ is locally Hölder continuous, $\lambda_n \in \mathbb{R}$ are equibounded in n, that is, $|\lambda_n| \le \lambda$, and $f_n \in L^\infty(\mathbb{R}^N)$, with $\|f_n\|_\infty \le c_f$ for some $c_f > 0$ independent of n. Moreover H_n is for every n a Hamiltonian which satisfies (1-2), with constants γ and C_H independent of n; finally, there exists $C_F \ge 0$ and $b \ge 0$ independent of n such that

$$C_F^{-1}(\max\{|x| - C_F, 0\})^b \le F_n(x) \le C_F(1 + |x|)^b \quad \text{for all } n \text{ and all } x \in \mathbb{R}^N. \tag{2-4}$$

Note that, differently from assumption (1-5) for the potential V, the function F_n can also be bounded, if $b = 0$.

Theorem 2.5. *Let $u_n \in C^2(\mathbb{R}^N)$ be a sequence of classical solutions of the HJB equations* (2-3). *Then there exists a constant $K > 0$ depending on $C_H, C_F, c_f, \gamma, N, \lambda$ such that*

$$|\nabla u_n(x)| \le K(1 + |x|)^{\frac{b}{\gamma}}, \tag{2-5}$$

where $b \ge 0$ is the growth of F_n appearing in (2-4) *and γ is the growth of H_n appearing in* (1-2).

Proof. Without loss of generality we may consider $H_n(p) = C_H|p|^\gamma$ for all n and p. Indeed, every v_n solves

$$-\Delta u_n + C_H|\nabla u_n|^\gamma + \lambda_n = F_n(x) - f_n(x) + C_H|\nabla u_n|^\gamma - H_n(\nabla u_n) \quad \text{on } \mathbb{R}^N,$$

and since $|C_H|\nabla u_n|^\gamma - H_n(\nabla u_n)| \le C_H$ by (1-2), we can redefine f_n to include $C_H|\nabla u_n|^\gamma - H_n(\nabla u_n)$, which then satisfies the bound $\|f_n\|_\infty \le c_f + C_H$.

We first claim that if $v \in C^2(B_2(0))$ satisfies

$$\left|-\Delta v + C_H|\nabla v|^\gamma\right| \le k \quad \text{on } B_2(0)$$

for some $k > 0$, then we have for any $r \in [1, \infty]$,

$$\|\nabla v\|_{L^r(B_1(0))} \le \widetilde{C}, \tag{2-6}$$

where $\widetilde{C}$ depends only on k, C_H, γ, N, r. If $r \in [1, \infty)$, this is proven in [Lasry and Lions 1989, Theorem A.1]; see also [Cirant 2015, Theorem 19]. The case $r = \infty$ follows by classical elliptic regularity, since if r in (2-6) is large enough, then $-\Delta v$ is bounded in $L^q(B_{\frac{3}{2}}(0))$ for some $q > N$, and the statement follows by Sobolev embeddings.

In view of these considerations, the gradient bound (2-5) easily follows if $b = 0$. For the case $b > 0$, fix $x_0 \in \mathbb{R}^N$, and let $\delta = (1+|x_0|)^{-\frac{b}{\gamma'}}$. Let

$$v_n(y) := \delta^{\frac{2-\gamma}{\gamma-1}} u_n(x_0 + \delta y) \quad \text{on } \mathbb{R}^N.$$

Then, v_n solves

$$-\Delta v_n + C_H |\nabla v_n|^\gamma = \delta^{\gamma'}(F_n(x_0+\delta y) - f_n(x_0+\delta y) - \lambda_n).$$

Since $\delta \leq 1$,

$$\delta^{\gamma'} |F_n(x_0+\delta y) - f_n(x_0+\delta y) - \lambda_n| \leq \frac{C_F(3+|x_0|)^b + c_f + \lambda}{(1+|x_0|)^b} \leq C_1$$

for all $y \in B_2(0)$ by (2-4) and the bound on f_n.

Therefore, by the first claim,

$$\|\nabla v_n\|_{L^\infty(B_1(0))} \leq \widetilde{C}$$

for all n. In particular, choosing $y = 0$,

$$|\nabla u_n(x_0)| = \delta^{-\frac{1}{\gamma-1}} |\nabla v_n(0)| \leq \widetilde{C}(1+|x_0|)^{\frac{b}{\gamma}},$$

and the desired estimate follows. □

Moreover, we prove the following a priori estimates on bounded-from-below solutions to (2-3).

Theorem 2.6. *Let $u_n \in C^2(\mathbb{R}^N)$ be a family of uniformly bounded-from-below classical solutions to* (2-3), *that is, for which there exists $C > 0$ such that $u_n \geq -C$ for every n.*

If $b = 0$ in (2-4), *we moreover assume that there exists $\delta > 0$ and $R > 0$ independent of n such that*

$$F_n(x) - f_n(x) - \lambda_n > \delta > 0 \quad \text{for all } |x| > R. \tag{2-7}$$

Then there exists $C > 0$ such that

$$u_n(x) \geq C|x|^{1+\frac{b}{\gamma}} - C^{-1} \quad \text{for all } n \in \mathbb{N}, x \in \mathbb{R}^N, \tag{2-8}$$

where $b \geq 0$ is the growth power appearing in (2-4) *and γ is the growth power appearing in* (1-2).

Proof. The proof is based on the same argument as in [Barles and Meireles 2016, Proposition 3.4], we sketch it briefly for completeness. Since u_n is bounded from below we can assume $u_n \geq 0$, up to addition of constant C (without changing the equation).

We assume by contradiction that (2-8) does not hold. Then there exist sequences x_l and u_{n_l} such that $|x_l| > 2R$, $|x_l| \to +\infty$, and

$$\frac{u_{n_l}(x_l)}{|x_l|^{1+\frac{b}{\gamma}}} \to 0.$$

Let $a_l = \frac{1}{2}|x_l|$ and we define the function

$$v^l(x) = \frac{1}{a_l^{1+\frac{b}{\gamma}}} u_{n_l}(x_l + a_l x).$$

By Theorem 2.5, we get $|\nabla u_{n_l}(x)| \le K(1+|x|)^{\frac{b}{\gamma}}$. Therefore, v^l, $|\nabla v^l|$ are uniformly bounded.

Moreover, v^l is a solution to

$$-a_l^{\frac{b}{\gamma}-1} \Delta v^l + H_{n_l}(a_l^{\frac{b}{\gamma}} \nabla v^l) + \lambda_{n_l} = F_{n_l}(x_l + a_l x) - f_{n_l}(x_l + a_l x).$$

In particular, recalling (1-2), we get that v^l is a supersolution to

$$-a_l^{\frac{b}{\gamma}-1-b} \Delta v^l + C_H |\nabla v^l|^\gamma \ge a_l^{-b}(-\lambda_{n_l} + F_{n_l}(x_l + a_l x) - f_{n_l}(x_l + a_l x)).$$

Note that, for every l sufficiently large, by (2-4) and by (2-7) (in the case $b = 0$) the right-hand side above satisfies

$$a_l^{-b}(-\lambda_{n_l} + F_{n_l}(x_l + a_l x) - f_{n_l}(x_l + a_l x)) > 0$$

for x such that $|x| \le 1$.

Moreover, passing eventually to a subsequence, we get $v^l \to v$ locally uniformly in n and $a_l^{\frac{b}{\gamma}-1-b} \to 0$. So v is a supersolution to $C_H |\nabla v|^\gamma \ge \delta > 0$ in $B(0, 1)$ with homogeneous boundary conditions (since $v \ge 0$). By comparison, recalling the explicit formula of the solution to the eikonal equation $|\nabla f|^\gamma = C$ in $B(0, 1)$ with homogeneous boundary conditions, we conclude that $v(x) \ge C^{\frac{1}{\gamma}}(1 - |x|)$ for all x such that $|x| \le 1$. Moreover, by uniform convergence, we get that, eventually enlarging C and taking l sufficiently large, $v^l(x) \ge C^{\frac{1}{\gamma}}(1 - |x|)$ for all x with $|x| \le 1$; in particular $v^l(0) \ge C^{\frac{1}{\gamma}}$. Recalling the definition of v^l, we get that $v^l(0) \to 0$, which yields a contradiction. □

Define

$$\bar{\lambda}_n := \sup\{\lambda \in \mathbb{R} : \text{(2-3) has a solution } u_n \in C^2(\mathbb{R}^N)\}.$$

Theorem 2.7. *Assume that for every n the function $F_n - f_n$ is bounded from below uniformly in n:*

(i) $\bar{\lambda}_n < \infty$ *for every n, and there exists, for every n, a solution $u_n \in C^2(\mathbb{R}^N)$ to* (2-3) *with $\lambda_n = \bar{\lambda}_n$. Moreover*

$$\bar{\lambda}_n := \sup\{\lambda \in \mathbb{R} : \text{(2-3) } \textit{has a subsolution } u_n \in C^2(\mathbb{R}^N)\}.$$

(ii) *If F_n satisfies* (2-4), *with $b > 0$, then, for every n, the solution u_n to* (2-3) *with $\lambda_n = \bar{\lambda}_n$ is unique up to addition of constants and satisfies* (2-8).

(iii) *If $F_n \equiv 0$, and there exists $\delta > 0$ independent of n such that*

$$\limsup_{|x| \to +\infty} f_n(x) + \bar{\lambda}_n < -\delta < 0, \tag{2-9}$$

then for every n there exists a solution to (2-3) *with $\lambda_n = \bar{\lambda}_n$ which satisfies* (2-8) *with $b = 0$.*

Proof. (i) The proof of this result can be obtained by a straightforward adaptation of the proof of Theorem 2.1 in [Barles and Meireles 2016], using the a priori estimates on the gradient given in Theorem 2.5. Observe that actually in that paper a stronger assumption on the regularity of $F_n - f_n$ is required, in particular local Lipschitz continuity. This assumption is used to derive a priori estimates on the gradient of solutions by using the so-called Bernstein method, see Appendix A in [Barles and Meireles 2016], which depends also on the L^∞ norm of $\nabla(F_n - f_n)$. In our case we can weaken this assumption to just Hölder continuity (so still ensuring classical elliptic regularity) since we are using a priori estimates on the gradient given in Theorem 2.5, which depends only on the L^∞ norm of $F_n - f_n$, and are obtained in [Lasry and Lions 1989] by the so-called integral Bernstein method.

(ii) For the proof we refer to [Ichihara 2011]; see also [Barles and Meireles 2016; Cirant 2014]. In particular in [Ichihara 2011], it is proved that u_n is bounded from below. By looking at the proof, it is easy to check that, due to the uniformity in n of the norms of coefficients, the bound can be taken independent of n, and by Theorem 2.6 we get the estimate on the growth.

(iii) By adapting the argument in [Barles and Meireles 2016, Theorem 2.6], we get that there exists a bounded-from-below solution to (2-3) with $\lambda_n = \bar{\lambda}_n$, with bound uniform in n. Then using Theorem 2.6, we get the estimate on the growth. We give a brief sketch of the proof of the existence of a bounded-from-below solution. For every $R > 0$, we consider the ergodic problem

$$\begin{cases} -\Delta u_n^R + H_n(\nabla u_n^R) + \lambda_n^R = -f, & |x| < R, \\ u_n^R(x) \to +\infty, & |x| \to R. \end{cases} \tag{2-10}$$

Using the result in [Barles et al. 2010], we get that for every $R > 0$ there exists a unique λ_n^R and a unique up to addition of constant solution $u_n^R \in C^2(B_R)$.

First of all we claim that $\lim_R \lambda_n^R = \bar{\lambda}_n$. It is easy to check that if $R' > R$, then $\lambda_n^{R'} \le \lambda_n^R$, and moreover that $\lambda_n^R \ge \bar{\lambda}_n$. So, the sequence λ_n^R is converging as $R \to +\infty$ to some $\lambda_n^\star \ge \bar{\lambda}_n$. Additionally, by the same argument as in Theorem 2.5, we get that for every compact $K \subset \mathbb{R}^N$, there exists a constant $C > 0$ such that $|\nabla u_n^R| \le C$ in K for every R sufficiently large and for all n. Without loss of generality we can assume that $u_n^R(0) = 0$ for every R. So, using the gradient bound, and elliptic regularity, we conclude that u_n^R is bounded in $C^2(K)$ by some constant independent of R. Hence, by the Ascoli–Arzelà theorem, and via a diagonalization procedure, we get that u_n^R converges locally in $\mathbb{R}^N$, with $u_n \in C^2(\mathbb{R}^N)$. Moreover, u_n is a solution to (2-3), with $\lambda = \lambda_n^\star$. Recalling the characterization of $\bar{\lambda}_n$ and the fact that $\lambda_n^\star \ge \bar{\lambda}_n$, we conclude that $\lambda_n^\star = \bar{\lambda}_n$.

Then, we consider $x_n^R \in B_R$ such that

$$u_n^R(x_n^R) = \min_{|x| \le R} u_n^R.$$

Recalling that u_n^R is a solution to (2-10), we get by computing the equation at x_n^R and by recalling that $H_n(0) \le 0$, that

$$\lambda_n^R + f(x_n^R) \ge H_n(0) + \lambda_n^R + f(x_n^R) \ge 0.$$

Using condition (2-9), and recalling that $\lambda_n^R \to \bar{\lambda}_n$, we get that there exists a compact set K (independent of R and of n) and $R_0 > 0$ such that for all $R > R_0$ we have $x_n^R \in K$.

Recalling that $u_n^R(0) = 0$ and $|\nabla u_n^R| \le C$ in K with C independent of n, R, we conclude that $u_n^R(x_R) \ge -C$ for some constant C independent of n, R. But, this implies, since $u_n^R(x) \ge u_n^R(x_n^R)$ for every R, that passing to the limit $u_n(x) \ge -C$, with C independent of n. □

2B. ***A priori estimates for the Kolmogorov equation.*** In this section we provide general a priori estimates for pairs $(m, w) \in (L^1(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)) \times L^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} m(x) = M$ and $-\varepsilon\Delta m + \operatorname{div} w = 0$, where

$$q = \begin{cases} \gamma', & \gamma' \ge N, \\ \frac{N}{N-\gamma'+1}, & \gamma' < N. \end{cases} \tag{2-11}$$

Lemma 2.8. *Let $\beta \le \frac{Nq}{N-q}$ for $q < N$, and $\beta < +\infty$ for $q \ge N$. We define $1 \le r \le \beta$ as follows*:

$$\frac{1}{r} = \frac{1}{\gamma'} + \left(1 - \frac{1}{\gamma'}\right)\frac{1}{\beta}. \tag{2-12}$$

Then, there exists a constant C, depending only on N and β, such that

$$\begin{aligned} \|m\|_{W^{1,r}(\mathbb{R}^N)} &\le C\left(\frac{1}{\varepsilon^{\gamma'}}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + M\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} \\ &\le C\left(\frac{C_L}{\varepsilon^{\gamma'}}\int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx + M\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}}, \end{aligned} \tag{2-13}$$

where $C_L = C_L(C_H, \gamma)$ is the constant appearing in Proposition 2.1.

We now assume that

$$1 < \beta < 1 + \frac{\gamma'}{N}. \tag{2-14}$$

Then, there exists $\delta > 0$ such that

$$\|m\|_{L^\beta(\mathbb{R}^N)}^{(1+\delta)\beta} \le C\frac{1}{\varepsilon^{\gamma'}}M^{(1+\delta)\beta-1}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right) \le C C_L\frac{1}{\varepsilon^{\gamma'}}M^{(1+\delta)\beta-1}\int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx, \tag{2-15}$$

where the constant C depends only on γ, N, and β.

Proof. Since $m \in W^{1,q}(\mathbb{R}^N)$, by Sobolev embedding and interpolation, we get that $m \in L^\beta(\mathbb{R}^N)$. Using $-\varepsilon\Delta m + \operatorname{div} w = 0$, we get for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\varepsilon\int_{\mathbb{R}^N} \nabla m \cdot \nabla\varphi\, dx = \int_{\mathbb{R}^N} w \cdot \nabla\varphi\, dx.$$

Using the Hölder inequality, recalling (2-12), we obtain

$$\begin{aligned} \left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N} w \cdot \nabla\varphi\, dx\right| &\le \int_{\mathbb{R}^N} \frac{1}{\varepsilon}\left|\frac{w}{m}\right| m^{\frac{1}{\gamma'}} m^{1-\frac{1}{\gamma'}} |\nabla\varphi|\, dx \\ &\le \left(\frac{1}{\varepsilon^{\gamma'}}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla\varphi\|_{L^{r'}(\mathbb{R}^N)}. \end{aligned}$$

Therefore, we get that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} \nabla m \cdot \nabla\varphi \, dx\right| \le \left(\frac{1}{\varepsilon\gamma'}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} \|\nabla\varphi\|_{r'}.$$

We apply then Proposition 2.4 and we obtain that $m \in W^{1,r}(\mathbb{R}^N)$ and that there exists a constant C, depending only on r, such that

$$\|\nabla m\|_{L^r(\mathbb{R}^N)} \le C\left(\frac{1}{\varepsilon\gamma'}\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}}. \tag{2-16}$$

From this inequality, using Proposition 2.1 and recalling that by interpolation, since $\|m\|_{L^1(\mathbb{R}^N)} = M$,

$$\|m\|_{L^r(\mathbb{R}^N)} \le \|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{1}{\gamma}} M^{\frac{1}{\gamma'}},$$

we conclude the desired inequality (2-13).

Now we fix η such that

$$\frac{1}{\eta} = \left(\frac{1}{r} - \frac{1}{N}\right)\frac{N}{N+1} + 1 - \frac{N}{N+1} = \frac{N}{N+1}\frac{1}{r}.$$

Note that, by a simple computation using (2-12), we get

$$\frac{1}{\eta} - \frac{1}{\beta} = \frac{N}{N+1}\frac{1}{\beta\gamma'}\left(\beta - 1 - \frac{\gamma'}{N}\right);$$

therefore, by (2-14), we conclude that $\eta > \beta$. By the Gagliardo–Nirenberg inequality, and recalling that $\|m\|_1 = M$, we get

$$\|m\|_{L^\eta(\mathbb{R}^N)} \le C\|\nabla m\|_{L^r(\mathbb{R}^N)}^{\frac{N}{N+1}} M^{\frac{1}{N+1}}. \tag{2-17}$$

Since $\eta > \beta$, by interpolation we get that there exists $\theta > 1$ such that $\|m\|_{L^\beta(\mathbb{R}^N)}^{\theta} \le \|m\|_{L^\eta(\mathbb{R}^N)} M^{\theta-1}$. Actually

$$\frac{1}{\theta} = \left(1 - \frac{1}{\beta}\right)(N+1)\frac{1}{1 + N\left(1-\frac{1}{\beta}\right)\left(1-\frac{1}{\gamma'}\right)}.$$

So, we substitute in (2-17) and (2-16) and we get, elevating both terms to $\gamma'\frac{N+1}{N}$,

$$\|m\|_{L^\beta(\mathbb{R}^N)}^{\theta\gamma'\frac{N+1}{N}} \le C\frac{1}{\varepsilon\gamma'} M^{\gamma'(\theta\frac{N+1}{N}-1)}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx\right)\|m\|_{L^\beta(\mathbb{R}^N)}^{\frac{\gamma'}{\gamma}}. \tag{2-18}$$

Now, since $\theta > 1$, by (2-14), we get

$$\theta\gamma'\frac{N+1}{N} - \frac{\gamma'}{\gamma} = \frac{\beta\gamma'}{N(\beta-1)} = \beta + \frac{\beta}{\beta-1}\left[\frac{\gamma'}{N} + 1 - \beta\right] > 0.$$

Therefore we deduce (2-15) from (2-18) with

$$\delta = \frac{1}{\beta-1}\left[\frac{\gamma'}{N} + 1 - \beta\right]. \tag{2-19}$$

This concludes the proof. □

Corollary 2.9. *For every $r < q$, there exists $C > 0$ depending on N, γ' and r such that*

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(C_L \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx + \varepsilon^{\gamma'} M\right). \tag{2-20}$$

Moreover, if $\gamma' > N$ (so $q > N$), then $m \in C^{0,\theta}(\mathbb{R}^N)$ and

$$\|m\|_{C^{0,\theta}(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(C_L \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx + \varepsilon^{\gamma'} M\right). \tag{2-21}$$

Proof. For $q \ge N$ (equivalently $\gamma' \ge N$), we fix $r < q$ and we choose β which satisfies (2-12) for such r. By the Sobolev embedding theorem, $W^{1,r}(\mathbb{R}^N)$ is continuously embedded in $L^\beta(\mathbb{R}^N)$. So, there exists C depending on N and r such that $\|m\|_{L^\beta(\mathbb{R}^N)} \le C\|m\|_{W^{1,r}(\mathbb{R}^N)}$. Using inequality (2-13), we get

$$\|m\|_{L^\beta(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \varepsilon^{\gamma'} M\right).$$

If we substitute again in (2-13) we get

$$\|m\|_{W^{1,r}(\mathbb{R}^N)} \le \frac{C}{\varepsilon^{\gamma'}}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \varepsilon^{\gamma'} M\right).$$

In particular for $q > N$, we can choose $r > N$ and by the Sobolev embedding theorem we get that there exists $\theta = 1 - \frac{N}{r}$ and a constant $C > 0$ depending on N and r such that

$$\begin{aligned}\|m\|_{C^{0,\theta}(\mathbb{R}^N)} &\le \frac{C}{\varepsilon^{\gamma'}}\left(\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx + \varepsilon^{\gamma'} M\right)\\ &\le \frac{C}{\varepsilon^{\gamma'}}\left(C_L \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx + \varepsilon^{\gamma'} M\right).\end{aligned}$$

For $q < N$, we fix $r < q$, and choose the corresponding β in (2-12), which satisfies $\beta < \frac{N}{N-\gamma'}$. Hence we conclude again using inequality (2-13). □

3. Regularization procedure and existence of approximate solutions for $\varepsilon > 0$

3A. ***The regularized problem.*** We consider the approximation of the system (1-1)

$$\begin{cases} -\varepsilon\Delta u + H(\nabla u) + \lambda = f_k[m](x) + V(x),\\ -\varepsilon\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0,\\ \int_{\mathbb{R}^N} m\,dx = M,\end{cases} \tag{3-1}$$

where

$$f_k[m](x) = f(m \star \chi_k) \star \chi_k(x) = \int_{\mathbb{R}^N} \chi_k(x-y) f\left(\int_{\mathbb{R}^N} m(z)\chi_k(y-z)\,dz\right) dy \tag{3-2}$$

and χ_k, for $k > 0$, is a sequence of standard symmetric mollifiers approximating the unit as $k \to \infty$.

We observe that $f_k[m](x)$ is the L^2-gradient of a C^1 potential $F_k : L^1(\mathbb{R}^N) \to \mathbb{R}$, defined as

$$F_k[m] := \int_{\mathbb{R}^N} F(m \star \chi_k(x))\,dx, \tag{3-3}$$

where $F(m) = \int_0^m f(n)\,dn$ for $m \geq 0$ and $F(m) = 0$ for $m \leq 0$. Note that using Jensen's inequality and (1-3), we get that for all $m \in L^1(\mathbb{R}^N)$ such that $m \geq 0$, and $\int_{\mathbb{R}^N} m(x)\,dx = M$,

$$-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}(x)\,dx - KM \leq F_k[m] \leq -\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} (m \star \chi_k(x))^{\alpha+1}\,dx + KM. \tag{3-4}$$

In order to construct solutions to the system, we follow a variational approach and we associate to (3-1) an energy, as already described in the Introduction. We define the energy

$$\mathcal{E}_k(m,w) := \begin{cases} \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V(x)m\,dx + F_k[m] & \text{if } (m,w) \in \mathcal{K}_{\varepsilon,M}, \\ +\infty & \text{otherwise,} \end{cases} \tag{3-5}$$

where $\mathcal{K}_{\varepsilon,M}$ is defined in (1-8) and L is defined in (1-7). We recall that the exponent q appearing in the definition of $\mathcal{K}_{\varepsilon,M}$ is

$$q = \begin{cases} \frac{N}{N-\gamma'+1}, & \gamma' \leq N, \\ \gamma', & \gamma' > N. \end{cases}$$

Therefore, $q \leq \gamma'$. Observe that, if $q < N$,

$$q^* = \frac{qN}{N-q} = \frac{N}{N-\gamma'},$$

and that $q^* > 1 + \frac{\gamma'}{N} > 1+\alpha$ by (1-4). If $q = \gamma' \geq N$, then we let $q^* = +\infty$.

3B. ***A priori estimates and energy bounds.*** In this section, we provide bounds from below for the energy $\mathcal{E}_k$, ensuring in particular that the minimum problem is well-defined.

Lemma 3.1. *Let $(m,w) \in \mathcal{K}_{\varepsilon,M}$. Then*

$$\mathcal{E}_k(m,w) \geq -K - C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}, \tag{3-6}$$

where $C, K > 0$ are constants depending only on $N, M, C_L, \gamma, \alpha, M$.

In particular there exists finite

$$e_{k,\varepsilon}(M) = \inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}_k(m,w).$$

Proof. Recalling that $V \geq 0$, using estimate (3-4) and applying (2-15) with $\alpha = \beta - 1$, we get

$$\begin{aligned} \mathcal{E}_k(m,w) &\geq \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right)dx - \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx - KM \\ &\geq C\varepsilon^{\gamma'} M^{1-(1+\delta)(1+\alpha)} \|m\|_{L^{\alpha+1}}^{(1+\alpha)(1+\delta)} - \frac{1}{\alpha+1}\|m\|_{L^{\alpha+1}}^{(1+\alpha)} - KM \\ &\geq -C\delta\varepsilon^{-\frac{\gamma'}{\delta}}\left(\frac{1}{(\delta+1)(\alpha+1)}\right)^{1+\frac{1}{\delta}} - KM, \end{aligned}$$

where C is a constant depending only on $N, M, C_L, \gamma, \alpha$ and

$$\delta = \frac{1}{\alpha}\left[\frac{\gamma'}{N} - \alpha\right]. \tag{3-7}$$

Therefore, substituting in the energy, we get

$$\mathcal{E}_k(m,w) \geq -C\,\frac{(\gamma'-\alpha N)}{\alpha N}\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}\left(\frac{\alpha N}{\gamma'(\alpha+1)}\right)^{\frac{\gamma'}{\gamma'-\alpha N}} - KM,$$

which gives the desired inequality. □

We get also a priori bounds on minimizers and minimizing sequences.

Proposition 3.2. *Let $(m,w)\in\mathcal{K}_{\varepsilon,M}$ be such that $e_{k,\varepsilon}(M)\geq\mathcal{E}_k(m,w)-\eta$ for some positive η. Then*

$$\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}dx \leq C\varepsilon^{-\frac{\gamma' N\alpha}{\gamma'-N\alpha}}+K, \tag{3-8}$$

$$\|m\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^N)} \leq C\varepsilon^{-\frac{\gamma' N\alpha}{\gamma'-N\alpha}}+K \tag{3-9}$$

for some C, K positive constants which depend only on α, N, V, C_L, η.

Proof. First of all we observe that there exists $C\geq 0$ depending on M, C_L, C_V such that

$$e_{k,\varepsilon}(M)\leq C. \tag{3-10}$$

Let $m=ce^{-|x|}$, where c is chosen to have $\int_{\mathbb{R}^n} m\,dx = M$, and $w=\varepsilon\nabla m$, so that $(m,w)\in\mathcal{K}_{\varepsilon,M}$. By assumption (1-5), we get $\int_{\mathbb{R}^n} mV(x)\,dx\leq C$ for some constant $C>0$, by (3-4) we get $F_k[m]\leq KM$ and by the properties of L in Proposition 2.1 we have

$$\int_{\mathbb{R}^n} mL\left(-\frac{w}{m}\right)dx \leq \left(\frac{\varepsilon^{\gamma'}}{c^{\gamma'}}+C_L\right)M.$$

So, in conclusion $e_{k,\varepsilon}(M)\leq\mathcal{E}_k(m,w)\leq C$ as required.

Note that if $(m,w)\in\mathcal{K}_{\varepsilon,M}$, and $e_\varepsilon(M)\geq\mathcal{E}(m,w)-\eta$ for some positive η, then, by (3-4), by the fact that $V\geq 0$, and by the properties of L in Proposition 2.1, we get

$$C+\eta\geq e_\varepsilon(M)+\eta\geq\mathcal{E}_k(m,w)\geq\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}-\frac{C_f}{\alpha+1}m^{\alpha+1}\,dx\ -KM. \tag{3-11}$$

We apply (2-15) with $\alpha=\beta-1$, and we obtain

$$\begin{aligned} C+\eta+KM &\geq \int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}-\frac{C_f}{\alpha+1}m^{\alpha+1}\,dx \\ &\geq C\varepsilon^{\gamma'}M^{1-(1+\delta)(1+\alpha)}\|m\|^{(1+\alpha)(1+\delta)}_{L^{\alpha+1}}-\frac{C_f}{\alpha+1}\|m\|^{(1+\alpha)}_{L^{\alpha+1}}. \end{aligned}$$

Recall that $\delta+1=\frac{\gamma'}{\alpha N}$, which can be computed using (2-19), so

$$\frac{\gamma'}{\delta}=\frac{\gamma' N\alpha}{\gamma'-N\alpha}.$$

Note that if we choose A sufficiently large (depending on δ, M, C_f, C_L), we get

$$C\varepsilon^{\gamma'}M^{1-(1+\delta)(1+\alpha)}(\varepsilon^{-\frac{\gamma'}{\delta}}A)^{1+\delta}-\frac{C_f}{\alpha+1}(\varepsilon^{-\frac{\gamma'}{\delta}}A)\geq C+\eta+KM,$$

from which we conclude that $\|m\|_{L^{\alpha+1}}^{(1+\alpha)} \le \varepsilon^{-\frac{\gamma'}{\delta}} A$, and so estimate (3-9) holds. Estimate (3-8) comes from (3-9) and (3-11). □

3C. *Existence of a solution.* We are now in the position to show existence of minimizers of the energy $\mathcal{E}_k$ in the class $\mathcal{K}_{\varepsilon,M}$ for every $\varepsilon, M > 0$.

Proposition 3.3. *For every $\varepsilon > 0$ and $M > 0$, there exists a minimizer $(m_k, w_k) \in \mathcal{K}_{\varepsilon,M}$ of $\mathcal{E}_k$, that is,*

$$\mathcal{E}_k(m_k, w_k) = \inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}_k(m, w).$$

Moreover, for every minimizer $(m_k, w_k) \in \mathcal{K}_{\varepsilon,M}$ of $\mathcal{E}_k$, there holds

$$m_k(1+|x|)^b \in L^1(\mathbb{R}^N), \quad w_k(1+|x|)^{\frac{b}{\gamma}} \in L^1(\mathbb{R}^N), \tag{3-12}$$

and there exist constants $C > 0$ and K, independent of ε and k, such that

$$\int_{\mathbb{R}^N} m_k \left|\frac{w_k}{m_k}\right|^{\gamma'} dx + \int_{R^N} m_k V(x)\,dx + \|m_k\|_{L^{\alpha+1}(\mathbb{R}^N)}^{\alpha+1} \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-N\alpha}} + K. \tag{3-13}$$

Proof. Let $(m_n, w_n) \in \mathcal{K}_{\varepsilon,M}$ be a minimizing sequence, that is, $\mathcal{E}_k(m_n, w_n) \to e_{k,\varepsilon}(M)$. This implies that, choosing n sufficiently large, $\mathcal{E}_k(m_n, w_n) \le e_\varepsilon(M) + 1$. From this and (3-4) we get

$$\begin{aligned}\int_{\mathbb{R}^N} m_n L\left(-\frac{w_n}{m_n}\right) dx + \int_{\mathbb{R}^N} V(x) m_n\,dx &\le \mathcal{E}_k(m_n, w_n) + \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m_n^{\alpha+1}\,dx + KM \\ &\le e_{k,\varepsilon}(M) + 1 + \frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m_n^{\alpha+1} + KM. \end{aligned}\tag{3-14}$$

By Proposition 3.2, we get

$$\|m_n\|_{L^{\alpha+1}} + \int_{\mathbb{R}^N} m_n^{1-\gamma'} |w_n|^{\gamma'}\,dx \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K.$$

We conclude also that

$$\int_{\mathbb{R}^N} V(x) m_n(x)\,dx \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K$$

for some $C, K > 0$. These estimates will imply (3-13), after passing to the limit, using Fatou's lemma.

Moreover, by Corollary 2.9, we have that there exists $C_\varepsilon > 0$ depending on ε such that for all $r < q$,

$$\|m_n\|_{W^{1,r}(\mathbb{R}^N)} \le C_\varepsilon.$$

Moreover, due to Sobolev embeddings, we get $\|m_n\|_{L^s(\mathbb{R}^N)} \le C_\varepsilon$ for all $s < q^*$. In addition, by applying the Hölder inequality, we get that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |w_n|^{\frac{\gamma'\alpha+\gamma'}{\gamma'+\alpha}}\,dx \le C\left(\int_{\mathbb{R}^N} m_n^{1-\gamma'}|w_n|^{\gamma'}\,dx\right)^{\frac{\alpha+1}{\gamma'+\alpha}} \|m_n\|_{L^{\alpha+1}(\mathbb{R}^N)}^{\frac{\gamma'-1}{(\alpha+1)(\gamma'+\alpha)}}.$$

By these estimates and Sobolev compact embeddings, we get that eventually extracting a subsequence via a diagonalization procedure, $m_n \to m_k$ weakly in $W^{1,r}(\mathbb{R}^N)$ for all $r < q$ and strongly in $L^s(K)$ for all $1 \le s < q^*$ and for every compact $K \subset \mathbb{R}^N$, and $w_n \to w_k$ weakly in $L^{\frac{\gamma'\alpha+\gamma'}{\gamma'+\alpha}}(\mathbb{R}^N)$. By using the fact that $\int_{\mathbb{R}^N} V(x)m_n(x)\,dx \le C_\varepsilon$ and (1-5), we get that for all $R > 1$,

$$C_\varepsilon \ge \int_{\mathbb{R}^N} m_n(x)V(x)\,dx \ge \int_{|x|>R} m_n(x)V(x)\,dx \ge CR^b \int_{|x|>R} m_n(x)\,dx.$$

So for every $\varepsilon > 0$ fixed and all $\eta > 0$, there exists $R > 0$ for which $\int_{|x|>R} m_n(x)\,dx \le \eta$: up to extracting a subsequence we get that $m_n \to m_k$ in $L^1(\mathbb{R}^N)$, and so $\int_{\mathbb{R}^N} m_k(x)\,dx = M$. By the boundedness of m_n in $L^s(\mathbb{R}^N)$ for all $1 \le s < q^*$, we then have $m_n \to m_k$ strongly in $L^{\alpha+1}(\mathbb{R}^N)$. Finally, observe that from (3-13), using (1-5), we conclude that $m_k(1+|x|^b) \in L^1(\mathbb{R}^N)$. Moreover, we get

$$\int_{\mathbb{R}^N} |w_k|\,dx \le \int_{\mathbb{R}^N} |w_k|(1+|x|)^{\frac{b}{\gamma}}\,dx \le \left(\int_{\mathbb{R}^N} \frac{|w_k|^{\gamma'}}{m_k^{\gamma'-1}}\,dx\right)^{\frac{1}{\gamma'}} \left(\int_{\mathbb{R}^N} m_k(1+|x|)^b\,dx\right)^{\frac{1}{\gamma}},$$

and so $w_k(1+|x|)^{\frac{b}{\gamma}} \in L^1(\mathbb{R}^N)$.

Therefore the convergence is sufficiently strong to ensure that $(m_k, w_k) \in \mathcal{K}_{\varepsilon,M}$. We conclude that (m_k, w_k) is a minimum of the energy, by the lower semicontinuity with respect to weak convergence of the functional $\int_{\mathbb{R}^N} mL\big(-\frac{w}{m}\big) + V(x)m\,dx$ and by using the fact that $F_k[m_n] \to F_k[m_k]$, since $m_n \to m_k$ strongly in $L^{\alpha+1}(\mathbb{R}^N)$. □

Using the minimizers we constructed in Proposition 3.3, we prove existence of a classical solution to (3-1).

Proposition 3.4. *There exists a classical solution (u_k, m_k, λ_k) to* (3-1) *that satisfies for some constant $C_{k,\varepsilon} > 0$ the inequalities*

$$|\nabla u_k(x)| \le C_{k,\varepsilon}(1+|x|^{\frac{b}{\gamma}}), \quad u_k(x) \ge C_{k,\varepsilon}^{-1}(1+|x|^{1+\frac{b}{\gamma}}) - C_{k,\varepsilon}. \tag{3-15}$$

Additionally there exist $C, K > 0$ not depending on ε, k such that

$$-K - C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} \le \lambda_k \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K. \tag{3-16}$$

Proof. Let (m_k, w_k) be a minimizer of $\mathcal{E}_k$. Define the space of test functions

$$\mathcal{A} = \mathcal{A}_{b,\gamma} := \left\{ \psi \in C^2(\mathbb{R}^N) : \limsup_{|x|\to\infty} \frac{|\nabla\psi(x)|}{|x|^{\frac{b}{\gamma}}} < \infty,\ \limsup_{|x|\to\infty} \frac{|\Delta\psi(x)|}{|x|^b} < \infty \right\}. \tag{3-17}$$

Note that we also have, for all $\psi \in \mathcal{A}$,

$$\limsup_{|x|\to\infty} \frac{|\psi(x)|}{|x|^{\frac{b}{\gamma}+1}} < \infty.$$

We claim that

$$-\varepsilon \int_{\mathbb{R}^N} m_k \Delta\psi\,dx = \int_{\mathbb{R}^N} w_k \nabla\psi\,dx \quad \text{for all } \psi \in \mathcal{A}. \tag{3-18}$$

Indeed, consider a radial smooth cutoff function $\chi(x)$ which is identically equal to 1 in $B_1(0)$ and identically zero in $\mathbb{R}^N \setminus B_2(0)$. Set $\chi_R(x) := \chi\big(\frac{x}{R}\big)$; we have $|\nabla \chi_R| \le CR^{-1}$ and $|\Delta \chi_R| \le CR^{-2}$ on $\mathbb{R}^N$ for some positive constant C.

Since the equality $\varepsilon \Delta m_k = \operatorname{div} w_k$ holds in the weak sense on $\mathbb{R}^N$, we may multiply it by $\chi_R \psi$ with $\psi \in \mathcal{A}$ and integrate by parts to obtain

$$-\varepsilon \int_{B_{2R}} m_k(\chi_R \Delta\psi + 2\nabla\psi \cdot \nabla\chi_R + \psi\Delta\chi_R)\,dx = \int_{B_{2R}} w_k \cdot (\chi_R \nabla\psi + \psi\nabla\chi_R)\,dx. \tag{3-19}$$

Note that for some positive C,

$$\int_{\mathbb{R}^N} |w_k \nabla\psi|\,dx \le C \int_{\mathbb{R}^N} |w_k|(1+|x|)^{\frac{b}{\gamma}}\,dx < \infty, \quad \int_{\mathbb{R}^N} m_k |\Delta\psi|\,dx \le C \int_{\mathbb{R}^N} m_k (1+|x|)^b\,dx < \infty$$

by the integrability properties (3-12). Moreover,

$$\begin{aligned}\int_{R\le|x|\le 2R} m_k |\psi| |\Delta\chi_R|\,dx &\le C \int_{R\le|x|\le 2R} m_k \frac{(1+|x|)^{\frac{b}{\gamma}+1}}{R^2}\,dx \\ &\le C_1 \int_{R\le|x|\le 2R} m_k (1+|x|)^{\frac{b}{\gamma}-1}\,dx \to 0 \quad \text{as } R\to\infty,\end{aligned}$$

because $\frac{b}{\gamma} - 1 \le b$. Reasoning in a similar way, we also have that $\int_{R\le|x|\le 2R} m_k \nabla\psi \cdot \nabla\chi_R$ and $\int_{R\le|x|\le 2R} w_k \cdot \psi \nabla\chi_R$ converge to zero as $R \to \infty$. Equality (3-18) then follows by passing to the limit in (3-19).

Therefore, recalling the integrability properties of m_k, w_k obtained in Proposition 3.3, the problem of minimizing $\mathcal{E}_k$ on $\mathcal{K}_{\varepsilon,M}$ is equivalent to minimizing $\mathcal{E}_k$ on $\mathcal{K}$, where

$$\mathcal{K} := \Big\{(w,m) \in (L^1 \cap W^{1,r})(\mathbb{R}^N) \times L^{\frac{\gamma'(\alpha+1)}{\gamma'+\alpha}}(\mathbb{R}^N) : (w,m) \text{ satisfies (3-12), (3-18), } m \ge 0,\ \textstyle\int_{\mathbb{R}^N} m = M\Big\}$$

for some $r < q$. As in [Briani and Cardaliaguet 2018, Proposition 3.1], the convexity of L implies that (m_k, w_k) is also a minimizer of the following convex functional on $\mathcal{K}$:

$$\widetilde{J}(m,w) = \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big) + (V(x) + f_k[m_k])m\,dx.$$

We now aim to prove that

$$\sup\big\{\lambda M : -\varepsilon\Delta\psi + H(\nabla\psi) + \lambda \le V(x) + f_k[m_k] \text{ on } \mathbb{R}^N \text{ for some } \psi \in \mathcal{A}\big\} = \min_{(w,m)\in\mathcal{K}} \widetilde{J}(m,w). \tag{3-20}$$

We proceed as in [Cardaliaguet and Graber 2015, Theorem 3.5]: Setting

$$\mathcal{L}(m,w,\lambda,\psi) := \widetilde{J}(m,w) + \int_{\mathbb{R}^N} \varepsilon m\Delta\psi + w\nabla\psi - \lambda m\,dx + \lambda M,$$

we have

$$\min_{(m,w)\in\mathcal{K}} \widetilde{J}(m,w) = \min_{(m,w)} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}} \mathcal{L}(m,w,\lambda,\psi),$$

where the minimum in the right-hand side has to be taken over pairs

$$(m,w)\in(L^1\cap W^{1,r})(\mathbb{R}^N)\times L^{\frac{\gamma'(\alpha+1)}{\gamma'+\alpha}}(\mathbb{R}^N)$$

for some $r<q$, satisfying (3-12). Note that $\mathcal{L}(\cdot,\cdot,\lambda,\psi)$ is convex, and $\mathcal{L}(m,w,\cdot,\cdot)$ is linear. Moreover, since $\mathcal{L}(\cdot,\cdot,\lambda,\psi)$ is weak-* lower semicontinuous, we can use the min-max theorem, see [Borwein and Vanderwerff 2010, Theorem 2.3.7], to get

$$\begin{aligned}
&\min_{(m,w)}\sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}}\mathcal{L}(m,w,\lambda,\psi)\\
&\quad=\sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}}\min_{(m,w)}\mathcal{L}(m,w,\lambda,\psi)\\
&\quad=\sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}}\min_{(m,w)}\int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)+(V(x)+f_k[m_k])m+\varepsilon m\Delta\psi+w\nabla\psi-\lambda m\,dx+\lambda M\\
&\quad=\sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}}\int_{\mathbb{R}^N}\min_{(m,w)\in\mathbb{R}\times\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)+(V(x)+f_k[m_k])m+\varepsilon m\Delta\psi+w\nabla\psi-\lambda m\,dx+\lambda M,
\end{aligned}$$

where the interchange of the min and the integration is possible by standard results in convex optimization. By computation, $\min_{(m,w)\in\mathbb{R}\times\mathbb{R}^N} mL\big(-\frac{w}{m}\big)+(V(x)+f_k[m_k])m+\varepsilon m\Delta\psi+w\nabla\psi-\lambda m$ is zero whenever $\varepsilon\Delta\psi-H(\nabla\psi)-\lambda+(V(x)+f_k[m_k])$ is positive, and it is $-\infty$ otherwise. Therefore, we have proven (3-20).

By Theorem 2.7(i)–(ii), there exists $u_k\in C^2(\mathbb{R}^N)$ such that

$$-\varepsilon\Delta u_k+H(\nabla u_k)+\lambda_k=V(x)+f_k[m_k]\quad\text{on }\mathbb{R}^N,\tag{3-21}$$

which satisfies

$$|\nabla u_k(x)|\le C_{k,\varepsilon}(1+|x|)^{\frac{b}{\gamma}},\quad u_k(x)\ge C_{k,\varepsilon}|x|^{\frac{b}{\gamma}+1}-C_{k,\varepsilon}^{-1}$$

for some $C_{k,\varepsilon}>0$.

Moreover,

$$\varepsilon|\Delta u_k(x)|\le|H(\nabla u_k(x))|+|\lambda_k|+V(x)-f_k[m_k]\le C_{k,\varepsilon}(1+|x|)^b\quad\text{on }\mathbb{R}^N,$$

so $u_k\in\mathcal{A}$. Thus, the supremum in the left-hand side of (3-20) is achieved by λ_k, and it holds true that

$$\lambda_k M=\widetilde{J}(m_k,w_k)=\mathcal{E}_k(m_k,w_k)+\int_{\mathbb{R}^N}f_k[m_k]m_k\,dx-F[m_k].\tag{3-22}$$

This gives in particular (3-16), using Lemma 3.1, estimates (3-10) and recalling Proposition 3.2 and assumptions (1-3), (3-2) and (3-4).

We now use (3-22), (3-21) and (3-18) with $\psi=u_k$ to get

$$\begin{aligned}
0&=\int_{\mathbb{R}^N}\Big(L\Big(-\frac{w_k}{m_k}\Big)+V(x)-m_k^\alpha-\lambda_k\Big)m_k\,dx\\
&=\int_{\mathbb{R}^N}\Big(L\Big(-\frac{w_k}{m_k}\Big)-\varepsilon\Delta u_k+H(\nabla u_k)\Big)m_k\,dx\\
&=\int_{\mathbb{R}^N}\Big(L\Big(-\frac{w_k}{m_k}\Big)+H(\nabla u_k)+\nabla u_k\cdot\frac{w_k}{m_k}\Big)m_k\,dx,
\end{aligned}$$

which implies

$$\frac{w_k}{m_k} = -\nabla H(\nabla u_k) \quad \text{on the set } \{m_k > 0\}.$$

Hence, the Kolmogorov equation $\varepsilon \Delta m_k + \operatorname{div}(m_k \nabla H(\nabla u_k)) = 0$ holds in the weak sense, and by elliptic regularity we conclude that (u_k, m_k, λ_k) is a classical solution to (1-1). □

Remark 3.5. Note that if we assume that the local term f satisfies (1-9) instead of (1-3), then the same argument as above applies. In particular there exists a classical solution (u_k, m_k, λ_k) to (3-1) such that

$$|\nabla u_k(x)| \le C_{k,\varepsilon}(1+|x|^{\frac{b}{\gamma}}), \quad u_k(x) \ge C_{k,\varepsilon}^{-1}(1+|x|^{1+\frac{b}{\gamma}}) - C_{k,\varepsilon},$$
$$\int_{\mathbb{R}^N} m_k^{\alpha+1}\,dx, \int_{\mathbb{R}^N} m_k(x)V(x)\,dx \le C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K.$$

We finally prove that every m_k is bounded from above in $\mathbb{R}^N$ (this is not obvious from Proposition 3.4 unless $\gamma' > N$). Note that the following result does not provide uniform bounds with respect to k. These will be produced in Theorem 4.1 using a much more involved argument.

Proposition 3.6. *Let (u_k, m_k, λ_k) be as in Proposition 3.4. Then, m_k is bounded in $L^\infty(\mathbb{R}^N)$.*

Proof. Let $\phi(x) = u_k(x)^p$, for $p > 1$ to be chosen later. Using the fact that u_k is a classical solution to the HJB equation, we get

$$\begin{aligned}
&-\varepsilon\Delta\phi + \nabla H(\nabla u_k)\cdot\nabla\phi \\
&\qquad = p u_k^{p-1}\left(-\Delta u_k - (p-1)\frac{|\nabla u_k|^2}{u_k} + \nabla H(\nabla u_k)\cdot\nabla u_k\right) \\
&\qquad = p u_k^{p-1}\left(-\Delta u_k + H(\nabla u_k) - (p-1)\frac{|\nabla u_k|^2}{u_k} - H(\nabla u_k) + \nabla H(\nabla u_k)\cdot\nabla u_k\right) \\
&\qquad = p u_k^{p-1}\left(-(p-1)\frac{|\nabla u_k|^2}{u_k} - H(\nabla u_k) + \nabla H(\nabla u_k)\cdot\nabla u_k - \lambda + f_k[m_k] + V\right). \qquad (3\text{-}23)
\end{aligned}$$

Observe that by (1-2), (1-5), (3-15) and the fact that $f_k[m_k]$ is bounded on $\mathbb{R}^N$, there exist large R and C such that

$$\begin{aligned}
G(x) &= -(p-1)\frac{|\nabla u_k|^2}{u_k} - H(\nabla u_k) + \nabla H(\nabla u_k)\cdot\nabla u_k - \lambda + f_k[m_k] + V(x) \\
&\ge K^{-1}|\nabla u_k|^\gamma - (p-1)\frac{|\nabla u_k|^2}{u_k} - K - \lambda + f_k[m_k] + V(x) \\
&\ge (p-1)|\nabla u_k|^\gamma\left(\frac{1}{K(p-1)} - \frac{|\nabla u_k|^{2-\gamma}}{u_k}\right) - C + C_V^{-1}|x|^b \ge 1 \quad \text{for all } |x| > R.
\end{aligned}$$

Hence, again by (3-15), for all $|x| > R$

$$-\varepsilon\Delta\phi + \nabla H(\nabla u_k)\cdot\nabla\phi \ge c|x|^{(1+\frac{b}{\gamma})(p-1)}.$$

In view of [Metafune et al. 2005, Proposition 2.6], we have $|x|^{(1+\frac{b}{\gamma})(p-1)} m_k \in L^1(\mathbb{R}^N)$. Recall now that $|\nabla H(\nabla u_k)| \le C(1+|x|)^{\frac{b}{\gamma'}}$ by (3-15). Therefore, by choosing p large enough, $|\nabla H(\nabla u_k)|^s m_k \in L^1(\mathbb{R}^N)$ for some $s > N$. We conclude the boundedness of m_k in L^∞ by [Metafune et al. 2005, Theorem 3.5]. □

4. Existence of a solution to the MFG system for $\varepsilon > 0$

Our aim is to pass to the limit $k \to \infty$ for solutions to (3-1).

4A. ***A priori L^∞ bounds.*** We need first a priori L^∞ bounds on m_k that are independent with respect to k. These will be achieved by a blow-up argument, as proposed in [Cirant 2016] for systems set on the flat torus $\mathbb{T}^N$. Here, the unbounded space $\mathbb{R}^N$ and the presence of the unbounded term V make the argument much more involved than the one in that paper. To control the points $x_k \in \mathbb{R}^N$ where $m_k(x_k)$ possibly explodes, some delicate estimates on the decay (in L^1) of its renormalization will be produced.

We provide a more general result, that will be used also in the rescaled framework (Section 5). Let r_k, s_k, t_k be bounded sequences of positive real numbers.

Theorem 4.1. *Let (u_k, λ_k, m_k) be a classical solution to the mean-field game system*

$$\begin{cases} -\Delta u + r_k^\gamma H(r_k^{-1}\nabla u) + \lambda_k = g_k[m] + s_k V(t_k x), \\ -\Delta m - \operatorname{div}(m\, r_k^{\gamma-1} \nabla H(r_k^{-1}\nabla u)) = 0, \\ \int_{\mathbb{R}^N} m\,dx = M, \end{cases}$$

where $g_k : L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ are such that for all $m \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and for all k,

$$\|g_k[m]\|_{L^\infty(\mathbb{R}^N)} \le K(\|m\|^\alpha_{L^\infty(\mathbb{R}^N)} + 1) \tag{4-1}$$

for some $K > 0$. Suppose also that for all k, u_k is bounded from below and m_k is bounded from above on $\mathbb{R}^N$. Then, there exists a constant C independent of k such that

$$\|m_k\|_{L^\infty} \le C.$$

Proof. We argue by contradiction, so we assume that

$$\sup_{\mathbb{R}^N} m_k = L_k \to +\infty.$$

We divide the proof into several steps.

Step 1: rescaling of the solutions. Let

$$\mu_k := L_k^{-\beta}, \quad \beta = \alpha\frac{\gamma-1}{\gamma} > 0.$$

So, observe that $\mu_k \to 0$ as $k \to 0$. Since u_k is bounded from below, up to adding a suitable constant we can assume that $\min_{\mathbb{R}^N} u_k = 0$. We define the rescaling

$$\begin{cases} v_k(x) = \mu_k^{\frac{2-\gamma}{\gamma-1}} u_k(\mu_k x) + 1, \\ n_k(x) = L_k^{-1} m_k(\mu_k x). \end{cases}$$

Note that $0 \le n_k(x) \le 1$. Moreover, due to (1-4),

$$\int_{\mathbb{R}^N} n_k(x)\,dx = M L_k^{\frac{\alpha N(\gamma-1)}{\gamma}-1} \to 0, \tag{4-2}$$

and $\min v_k = 1$. We define

$$H_k(q) = \mu_k^{\frac{\gamma}{\gamma-1}} r_k^{\gamma} H(r_k^{-1}\mu_k^{\frac{1}{1-\gamma}} q), \quad \text{so} \quad \nabla H_k(q) = \mu_k r_k^{\gamma-1} \nabla H(r_k^{-1}\mu_k^{\frac{1}{1-\gamma}} q).$$

Recalling (1-2) we have that for all $q \in \mathbb{R}^N$,

$$\begin{gathered} C_H|q|^\gamma - K \le H_k(q) \le C_H(|q|^\gamma + 1), \\ |\nabla H_k(q))| \le C_H|q|^{\gamma-1}, \\ \nabla H_k(q)\cdot q - H_k(q) \ge K^{-1}|q|^\gamma - K. \end{gathered} \tag{4-3}$$

Moreover, we define

$$\tilde{g}_k(x) = \mu_k^{\frac{\gamma}{\gamma-1}} g_k[m_k](\mu_k x).$$

Recalling that $0 \le m_k \le L_k$, by (4-1) we get that for all x and for all k,

$$|\tilde{g}_k(x)| \le \mu_k^{\frac{\gamma}{\gamma-1}} K(L_k^\alpha + 1) \le 2K, \tag{4-4}$$

where we used the fact that $\mu_k = L_k^{-\beta}$ with $\beta = \alpha\frac{\gamma-1}{\gamma}$. Finally, we let

$$\tilde{\lambda}_k = \mu_k^{\frac{\gamma}{\gamma-1}} \lambda_k = \frac{1}{L_k^\alpha}\lambda_k$$

and we observe that

$$|\tilde{\lambda}_k| \le C. \tag{4-5}$$

Finally, let

$$V_k(x) = \mu_k^{\frac{\gamma}{\gamma-1}} s_k V(\mu_k t_k x).$$

By assumption (1-5), we get

$$s_k \mu_k^{\frac{\gamma}{\gamma-1}} C_V^{-1}(\max\{|t_k\mu_k x| - C_V, 0\})^b \le V_k(x) \le C_V(1 + \sigma_k|x|^b), \tag{4-6}$$

where

$$\sigma_k := \mu_k^{\frac{\gamma}{\gamma-1}+b} s_k t_k^b \to 0 \quad \text{as } k \to \infty.$$

In particular we also have the following bound from below for V_k:

$$V_k(x) \ge \frac{C_V^{-1}}{2^b}\sigma_k|x|^b \quad \text{for all } |x| \ge 2C_V(t_k\mu_k)^{-1}. \tag{4-7}$$

An easy computation shows that by rescaling we have that $(v_k, n_k, \tilde{\lambda}_k)$ is a solution to

$$\begin{cases} -\Delta v_k + H_k(\nabla v_k) + \tilde{\lambda}_k = \tilde{g}_k(x) + V_k(x), \\ -\Delta n_k - \operatorname{div}(n_k \nabla H_k(\nabla v_k)) = 0. \end{cases} \tag{4-8}$$

Step 2: a priori bounds on the rescaled solution to the Hamilton–Jacobi equation. We observe that by Theorem 2.5 and (4-6), there exists $C > 0$, independent of k, such that

$$|\nabla v_k(x)| \le C(1 + \sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}}) \quad \text{on } \mathbb{R}^N. \tag{4-9}$$

We recall that we assumed $v_k(\hat{x}_k) = \min v_k = 1$. Since v_k is a classical solution to (4-8), at a minimum point $\hat{x}_k$ we have, by (4-3), (4-4), (4-5) and (4-7),

$$\sigma_k |\hat{x}_k|^b \le C.$$

Therefore, by using this estimate and (4-9), since $|v_k(0)| \le |v_k(\hat{x}_k)| + |\hat{x}_k| \sup_{|y| \le |\hat{x}_k|} |\nabla u_k(y)|$ we obtain

$$|v_k(0)| \le 1 + C(1 + \sigma_k^{\frac{1}{\gamma}} |\hat{x}_k|^{1+\frac{b}{\gamma}}) \le C_1(1 + \sigma_k^{-\frac{1}{b}})$$

and then again by (4-9),

$$|v_k(x)| \le C(1 + \sigma_k^{-\frac{1}{b}} + \sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}+1}) \quad \text{on } \mathbb{R}^N. \tag{4-10}$$

Let χ be a smooth function $\chi : [0, +\infty) \to [0, +\infty)$ such that $\chi \equiv 0$ in $\left(0, \frac{1}{2}\right) \cup \left(\frac{3}{2}, +\infty\right)$, $\chi(1) > 0$ and $|\chi'|, |\chi''| \le 1$. We fix $\tilde{x} \in \mathbb{R}^N$ such that $|\tilde{x}| > 4C_V(t_k\mu_k)^{-1}$, and we set

$$w(x) = \kappa \sigma_k^{\frac{1}{\gamma}} |\tilde{x}|^{1+\frac{b}{\gamma}} \chi\left(\frac{|x|}{|\tilde{x}|}\right),$$

where $\kappa \ge 0$ has to be chosen. We have that $w(x) \le v_k(x)$ for all x such that $|x| \ge \frac{3}{2}|\tilde{x}|$ or $|x| \le \frac{1}{2}|\tilde{x}|$. Moreover, for x such that $\frac{1}{2}|\tilde{x}| \le |x| \le \frac{3}{2}|\tilde{x}|$ we have $|x| > 2C_V(\mu_k t_k)^{-1}$, so using the estimates (4-3), (4-4), (4-5) and (4-7),

$$-\Delta w + H_k(\nabla w) + \tilde{\lambda}_k - \tilde{g}_k(x) - V_k(x) \le \kappa N \sigma_k^{\frac{1}{\gamma}} |\tilde{x}|^{\frac{b}{\gamma}-1} + C_H \kappa^\gamma \sigma_k |\tilde{x}|^b + C - \frac{C_V^{-1}}{2^b} \sigma_k |\tilde{x}|^b.$$

Note that there exist $\kappa > 0$ small and $C_2 > 0$ large, depending only C_V and C_H and not on $|\tilde{x}|$, k, such that the right-hand side of the last expression is negative if

$$\sigma_k |\tilde{x}|^b \ge C_2$$

(this also implies that $t_k \mu_k |\tilde{x}| > 4C_V$, as required). The test function w is then a subsolution of the HJB equation in (4-8); therefore by comparison we get

$$v_k(\tilde{x}) \ge \kappa \chi(1) \sigma_k^{\frac{1}{\gamma}} |\tilde{x}|^{1+\frac{b}{\gamma}}.$$

By the arbitrariness of $\tilde{x}$ we conclude that, for some $C > 0$,

$$v_k(x) \ge C \sigma_k^{\frac{1}{\gamma}} |x|^{\frac{b}{\gamma}+1} \quad \text{for all } \sigma_k |x|^b \ge C_2. \tag{4-11}$$

Step 3: estimates on the (approximate) maxima of n_k. We now fix $0 < \delta \ll 1$ and x_k such that $n_k(x_k) = 1 - \delta$. Two possibilities may arise: either $\lim_k \sigma_k |x_k|^b = +\infty$ up to some subsequence, or there exists $C > 0$ such that $\sigma_k |x_k|^b \le C$. We rule out the second possibility by contradiction. Suppose indeed that there exists $C > 0$ such that $\sigma_k |x_k|^b \le C$. By (4-9), $|\nabla v_k| \le C$ on $B_2(x_k)$ for some $C > 0$. Therefore, using the fact that n_k solves the second equation in (4-8), the elliptic estimates in Proposition 2.4, (4-3), the interpolation inequality $\|n\|_q \le \|n\|_1^{\frac{1}{q}} \|n\|_\infty^{1-\frac{1}{q}}$ and the fact that $0 \le n_k \le 1$, we get for all $q > 1$,

$$\|n_k\|_{W^{1,q}(B_1(x_k))} \le C(1 + \|\nabla H_k(\nabla v_k)\|_{L^\infty(B_2(x_k))})\|n_k\|_{L^1(B_2(x_k))}^{\frac{1}{q}} \le C_q \tag{4-12}$$

for some $C_q > 0$ depending on q. This implies, choosing $q > N$, that for all $\theta \in (0, 1)$ there exists C_θ depending on θ (but not on k) such that $\|n_k\|_{C^{0,\theta}(B_1(x_k))} \le C_\theta$. Recalling that $n_k(x_k) = 1 - \delta$, we can fix $r < 1$ such that $n_k(x) \ge \frac{1}{2}$ for all $x \in B_r(x_k)$. It is sufficient to choose $r = C_\theta^{-\frac{1}{\theta}} \left(\frac{1}{2} - \delta\right)^{\frac{1}{\theta}}$. Therefore we have, by (4-2),

$$0 < \tfrac{1}{2}\omega_N r^N \le \int_{B_r(x_k)} n_k(x)\,dx \le \int_{\mathbb{R}^N} n_k(x)\,dx = M L_k^{\frac{\alpha N(\gamma-1)}{\gamma}-1} \to 0.$$

This gives a contradiction. Then we deduce that, up to a subsequence,

$$\lim_k \sigma_k |x_k|^b = +\infty. \tag{4-13}$$

<u>Step 4</u>: construction of a Lyapunov function. Let $\phi(x) = v_k(x)^p$, for $p > 1$ to be chosen later. Using the fact that v_k is a classical solution to (4-8), arguing as in (3-23), we get

$$\begin{aligned} -\Delta\phi + \nabla H_k(\nabla v_k)\cdot\nabla\phi &= p v_k^{p-1}\left(-\Delta v_k - (p-1)\frac{|\nabla v_k|^2}{v_k} + \nabla H_k(\nabla v_k)\cdot\nabla v_k\right) \\ &= p v_k^{p-1}\left(-(p-1)\frac{|\nabla v_k|^2}{v_k} - H_k(\nabla v_k) + \nabla H_k(\nabla v_k)\cdot\nabla v_k - \tilde{\lambda}_k + \tilde{g}_k(x) + V_k(x)\right). \end{aligned}$$

We set

$$G_k(x) = -(p-1)\frac{|\nabla v_k|^2}{v_k} - H_k(\nabla v_k) + \nabla H_k(\nabla v_k)\cdot\nabla v_k - \tilde{\lambda}_k + \tilde{g}_k(x) + V_k(x). \tag{4-14}$$

Using the previous computation and the fact that n_k is a solution to (4-8), we get, by integrating by parts, that

$$0 = \int_{\mathbb{R}^N} n_k(x)\big(-\Delta\phi(x) + \nabla H_k(\nabla v_k(x))\cdot\nabla\phi(x)\big)\,dx = p\int_{\mathbb{R}^N} n_k(x) G_k(x) \phi^{\frac{p-1}{p}}(x)\,dx.$$

Therefore from this, for every $\Lambda > 0$ we get

$$\int_{\{\phi(x)\ge\Lambda^p\}} n_k(x) G_k(x)\phi^{\frac{p-1}{p}}(x)\,dx = -\int_{\{\phi(x)\le\Lambda^p\}} n_k(x) G_k(x)\phi^{\frac{p-1}{p}}(x)\,dx. \tag{4-15}$$

Observe that by (4-3), (4-4), (4-5) and (4-7) we get that for all $t_k\mu_k|x| \ge 2C_V$,

$$\begin{aligned} G_k(x) &\ge K^{-1}|\nabla v_k|^\gamma - (p-1)\frac{|\nabla v_k|^2}{v_k} - K - \tilde{\lambda}_k + \tilde{g}_k(x) + V_k(x) \\ &\ge (p-1)|\nabla v_k|^\gamma\left(\frac{1}{K(p-1)} - \frac{|\nabla v_k|^{2-\gamma}}{v_k}\right) - C + C_V\sigma_k|x|^b. \end{aligned} \tag{4-16}$$

We first claim that by (4-9) and (4-11),

$$\frac{1}{K(p-1)} - \frac{|\nabla v_k|^{2-\gamma}}{v_k} > 0 \quad \text{if } \sigma_k |x|^b \geq C_2,$$

eventually enlarging C_2 in (4-11). Indeed,

$$\frac{|\nabla v_k(x)|^{2-\gamma}}{v_k(x)} \leq C \frac{[1+\sigma_k^{\frac{1}{\gamma}}|x|^{\frac{b}{\gamma}}]^{2-\gamma}}{[\sigma_k^{\frac{1}{\gamma}}|x|^{\frac{b}{\gamma}}]|x|} \leq \frac{C_H}{p-1} \tag{4-17}$$

whenever $\sigma_k|x|^b$ is large enough. This implies that for all $\sigma_k|x|^b \geq C_2$, by (4-16) we have $G_k(x) \geq -C$. On the other hand, again by the gradient bounds in (4-9) we have that $|\nabla v_k(x)| \leq C(1+C_2)$ on the set $\sigma_k|x|^b \leq C_2$, so (4-16) and $\min v_k = 1$ again guarantee that $G_k(x) \geq -C_3$. In conclusion, there exists $C > 0$ such that

$$G_k(x) \geq -C \quad \text{for all } x \in \mathbb{R}^N.$$

Therefore, going back to (4-15), recalling (4-2), we obtain that

$$\begin{aligned}\int_{\{\phi(x)\geq\Lambda^p\}} n_k(x) G_k(x) \left(\frac{\phi(x)}{\Lambda^p}\right)^{\frac{p-1}{p}} dx &\leq C \int_{\{\phi(x)\leq\Lambda^p\}} n_k(x)\,dx \leq C \int_{\mathbb{R}^N} n_k(x)\,dx \\ &= CM\mu_k^{-N+\frac{\gamma}{\alpha(\gamma-1)}} \to 0\end{aligned} \tag{4-18}$$

as $k \to \infty$.

Note that by (4-16) and (4-17), if x is such that $G_k(x) \leq 0$, then necessarily $\sigma_k|x|^b \leq C$ for some $C > 0$. Hence, by (4-10), we get that $v_k(x) \leq C_3(1+\sigma_k^{-\frac{1}{b}})$. Therefore if we choose $\Lambda = \Lambda_k = K\sigma_k^{-\frac{1}{b}}$ for a sufficiently large $K > 0$, we get that $G_k(x) > 0$ in the set $\{x : \phi(x) \geq \Lambda^p\}$.

<u>Step 5</u>: integral estimates on n_k. Arguing as in the end of Step 4, we may choose K big enough so that $G_k(x) \geq 1$ in the set $\{x : \phi(x) \geq \Lambda_k^p\}$, where $\Lambda_k = K\sigma_k^{-\frac{1}{b}}$. If k is sufficiently large, by (4-11) and (4-13) it follows that for some $C > 0$,

$$v_k(x) \geq C\sigma_k^{\frac{1}{\gamma}}|x_k|^{1+\frac{b}{\gamma}} \quad \text{in } B_1(x_k), \quad \text{and} \quad B_1(x_k) \subseteq \{x : \phi(x) \geq \Lambda_k^p\}.$$

Therefore, we may conclude that

$$\begin{aligned}\int_{\{\phi(x)\geq\Lambda_k^p\}} n_k(x) G_k(x) \left(\frac{\phi(x)}{\Lambda_k^p}\right)^{\frac{p-1}{p}} dx &\geq C\left(\frac{\sigma_k^{\frac{1}{\gamma}}|x_k|^{1+\frac{b}{\gamma}}}{\sigma_k^{-\frac{1}{b}}}\right)^{p-1} \int_{B_1(x_k)} n_k(x)\,dx \\ &\geq C(\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}})^{p-1} \int_{B_1(x_k)} n_k(x)\,dx,\end{aligned} \tag{4-19}$$

which together with (4-18) gives

$$\int_{B_1(x_k)} n_k(x)\,dx \leq (\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}})^{1-p} \tag{4-20}$$

for all k large.

Reasoning as in Step 3, see in particular (4-12), by Proposition 2.4, (4-3), (4-9) and (4-20), we get that for all $q > 1$,

$$\|n_k\|_{W^{1,q}(B_{1/2}(x_k))} \le C(1 + \|\nabla H_k(\nabla v_k)\|_{L^\infty(B_1(x_k))})\|n_k\|_{L^1(B_1(x_k))}^{\frac{1}{q}} \le C_4[1 + (\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}})^{\gamma-1}](\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}})^{\frac{1-p}{q}} \le 1,$$

whenever p is such that $\gamma - 1 + \frac{1-p}{q} < 0$ and k is large (recall that we are supposing $\sigma_k^{\frac{1}{\gamma}}|x_k|^{\frac{b}{\gamma}} \to +\infty$).

Therefore, we may conclude as in Step 3: choosing $q > N$, for some $\theta \in (0, 1)$ there exists C_θ such that $\|n_k\|_{C^{0,\theta}(B_{1/2}(x_k))} \le C_\theta$. Since $n_k(x_k) = 1-\delta$, we can fix $r < 1$ such that $n_k(x) \ge \frac{1}{2}$ for all $x \in B_r(x_k)$. Finally, by (4-2)

$$0 < \tfrac{1}{2}\omega_N r^N \le \int_{B_r(x_k)} n_k(x)\,dx \le \int_{\mathbb{R}^N} n_k(x)\,dx = M L_k^{\frac{\alpha N(\gamma-1)}{\gamma-1}} \to 0.$$

That gives a contradiction and rules out the possibility that $\sigma_k|x_k|^b \to +\infty$. Therefore, $L_k \to +\infty$ is impossible. ☐

4B. ***Existence of a solution to the MFG system.*** Using the a priori bounds we obtained, we can pass to the limit in k in the MFG system (3-1) to get a solution to (1-1) for every $\varepsilon > 0$.

Proof of Theorem 1.1. First, by Proposition 3.4, the existence for all k of a classical solution (u_k, m_k, λ_k) to (3-1) follows. By (3-16), up to passing to a subsequence we have that $\lambda_k \to \lambda_\varepsilon$.

Note that by Propositions 3.4 and 3.6, u_k and m_k are bounded by below and above respectively, so due to Theorem 4.1 (with $g[m] = f_k[m]$ and $r_k = s_k = t_k = 1$ for all k), we get that there exists $C_\varepsilon > 0$ independent of k (but eventually dependent on $\varepsilon > 0$) such that $\|m_k\|_{L^\infty(\mathbb{R}^N)} \le C_\varepsilon$. Using Theorem 2.5, this implies $|\nabla u_k(x)| \le C_\varepsilon(1 + |x|^{\frac{b}{\gamma}})$ for some C_ε independent of k. We can normalize $u_k(0) = 0$ and using the Ascoli–Arzelà theorem we can extract by a diagonalization procedure a sequence u_k such that $u_k \to u_\varepsilon$ locally uniformly in $\mathbb{R}^N$. Moreover, by using the estimates and the equation we have that actually $u_k \to u_\varepsilon$ locally uniformly in C^1. Note that, denoting by x_k a minimum point of u_k on $\mathbb{R}^N$, we have by the HJB equation that

$$H(0) + \lambda_k - f_k[m_k](x_k) \ge V(x_k).$$

Coercivity (1-5) of V and uniform boundedness of λ_k and $f_k[m_k]$ guarantee that x_k remains bounded, in particular that $u_k \ge -C$ on $\mathbb{R}^N$ by gradient bounds. Theorem 2.6 then applies, and in particular $u_k(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1}$ for all k. This implies, passing to the limit, that

$$u_\varepsilon(x) \ge C|x|^{1+\frac{b}{\gamma}} - C^{-1} \quad \text{on } \mathbb{R}^N. \tag{4-21}$$

By the elliptic estimates in Proposition 2.4, we get that $m_k \to m_\varepsilon$ locally uniformly in $C^{0,\alpha}$ for all $\alpha \in (0, 1)$ and weakly in $W^{1,p}(B_R)$ for every $p > 1$ and $R > 0$. Therefore we get that u_ε is a solution in the viscosity sense of the Hamilton–Jacobi equation, by stability with respect to uniform convergence, and m_ε is a weak solution to the Fokker–Planck equation, by strong convergence of $\nabla u_k \to \nabla u_\varepsilon$. Finally this implies, again by using the regularity of the HJB equation, that $u_k \to u_\varepsilon$ locally uniformly in C^2. Therefore, $u_\varepsilon, m_\varepsilon$ solve in classical sense the system (1-1).

Now we show that $\int_{\mathbb{R}^N} m_\varepsilon(x)\,dx = M$. We have that $m_k \to m_\varepsilon$ locally uniformly in $C^{0,\alpha}$ for every $\alpha \in (0,1)$. Moreover, due to (3-13) and to (1-5), we get that for all $R > 1$,

$$C_\varepsilon \geq \int_{\mathbb{R}^N} m_k(x)V(x)\,dx \geq \int_{|x|>R} m_k(x)V(x)\,dx \geq CR^b \int_{|x|>R} m_k(x)\,dx.$$

This implies $\int_{|x|\leq R} m_k(x)\,dx \geq M - C_\varepsilon R^{-b}$ and then by uniform convergence we get that for every $\varepsilon > 0$, and $\eta > 0$, there exists $R > 0$ such that

$$\int_{|x|\leq R} m_\varepsilon(x)\,dx \geq M - \eta.$$

From this we can conclude that $m_k \to m_\varepsilon$ in $L^1(\mathbb{R}^N)$, that is, $\int_{\mathbb{R}^N} m_\varepsilon(x)\,dx = M$. By the boundedness of m_k in L^∞, it also follows that $m_k \to m_\varepsilon$ in $L^{\alpha+1}(\mathbb{R}^N)$.

Finally, we get that if $w_\varepsilon = -m_\varepsilon \nabla H(\nabla u_\varepsilon)$, then $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_{\varepsilon,M}$, due to the second equation in (1-1). Moreover, we have that if $m_k \to m$ strongly in $L^{\alpha+1}(\mathbb{R}^N)$, then, due to the Lebesgue dominated convergence theorem and (3-4), $F(m_k \star \chi_k) \to F(m)$ strongly in $L^1(\mathbb{R}^N)$. This implies that the energy $\mathcal{E}_k$ Γ-converges to the energy $\mathcal{E}$, from which we conclude that $(m_\varepsilon, w_\varepsilon)$ is a minimizer of $\mathcal{E}$ in the set $\mathcal{K}_{\varepsilon,M}$. □

Remark 4.2. Note that by the very same arguments, recalling Remark 3.5, we have the existence of solutions also in the more general case that condition (1-9) is satisfied.

We conclude proving some estimates on the solution $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ given in Theorem 1.1 that will be useful in the following.

Corollary 4.3. *Let $(u_\varepsilon, m_\varepsilon, \lambda_\varepsilon)$ be as in Theorem 1.1. There exist constants $C, C_1, C_2, K, K_1, K_2 > 0$ independent of ε such that*

$$\int_{\mathbb{R}^N} m_\varepsilon |\nabla u_\varepsilon|^\gamma\,dx + \int_{\mathbb{R}^N} m_\varepsilon^{\alpha+1}\,dx + \int_{\mathbb{R}^N} m_\varepsilon(x)V(x)\,dx \leq C\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + K, \tag{4-22}$$

$$-K_1 - C_1\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} \leq \lambda_\varepsilon \leq K_2 - C_2\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}. \tag{4-23}$$

Proof. We observe that, by the arguments in the proof of Theorem 1.1, $m_k \to m_\varepsilon$ and $|\nabla u_k| \to |\nabla u_\varepsilon|$ almost everywhere, and using the fact that $V(x) \geq 0$, we have that by Fatou's lemma

$$\int_{\mathbb{R}^N} m_\varepsilon(x)|\nabla u_\varepsilon|^\gamma\,dx \leq \liminf_k \int_{\mathbb{R}^N} m_k(x)|\nabla u_k|^\gamma\,dx,$$
$$\int_{\mathbb{R}^N} m_\varepsilon(x)V(x)\,dx \leq \liminf_k \int_{\mathbb{R}^N} m_k(x)V(x)\,dx,$$
$$\int_{\mathbb{R}^N} m_\varepsilon^{\alpha+1}\,dx \leq \liminf_k \int_{\mathbb{R}^N} m_k^{\alpha+1}\,dx.$$

So inequality (3-13) gives immediately (4-22).

Now we prove (4-23). Note that the estimate from below is a direct consequence of (3-16). So, it remains to show that

$$\lambda_\varepsilon \le C_2 - C_2 \varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}}.$$

Recalling that formula (3-22) holds and $\int f(m)m - F(m) \le 2KM$ by (1-3), it is sufficient to show that

$$\inf_{(m,w)\in\mathcal{K}_{\varepsilon,M}} \mathcal{E}(m,w) \le -C_2\varepsilon^{-\frac{\gamma'\alpha N}{\gamma'-\alpha N}} + C_2, \tag{4-24}$$

where C_2 is a constant depending only on $N, M, C_L, \gamma, \alpha, V$. We construct a pair $(m,w) \in \mathcal{K}_{\varepsilon,M}$ as follows. First of all we consider a smooth function $\phi : [0,+\infty) \to \mathbb{R}$ which solves the ordinary differential equation

$$\begin{cases} \phi'(r) = -\phi(r)(1+\phi(r)^\alpha)^{\frac{1}{\gamma'}}, \\ \phi(0) = \frac{1}{2}. \end{cases} \tag{$*$}$$

Then, it is easy to check that $0 < \phi(r) \le \frac{1}{2}e^{-r}$. We define $m(x) = A\phi(\tau|x|)$, where A, τ are constants to be fixed, and $w(x) = \varepsilon\nabla m(x)$.

First of all we impose

$$M = \int_{\mathbb{R}^N} m(x)\,dx = \frac{A}{\tau^N}\int_{\mathbb{R}^N} \phi(|y|)\,dy = \frac{A}{\tau^N}C^{-1},$$

recalling that ϕ is exponentially decreasing. So $A = M\tau^N C$, where $C^{-1} = \int_{\mathbb{R}^N} \phi(|y|)\,dy$.

Observe also that

$$\int_{\mathbb{R}^N} m^{\alpha+1}(x)\,dx = M^{\alpha+1}\tau^{\alpha N}C^{\alpha+1}\int_{\mathbb{R}^N} \phi^{\alpha+1}(|y|)\,dy = M^{\alpha+1}\tau^{\alpha N}C^{\alpha+1}C_\alpha, \tag{4-25}$$

where $C_\alpha = \int_{\mathbb{R}^N} \phi^{\alpha+1}(|y|)\,dy$.

We check, recalling the growth condition (1-5), that the following holds:

$$\int_{\mathbb{R}^N} m(x)V(x)\,dx = MC\int_{\mathbb{R}^N} V\left(\frac{y}{\tau}\right)\phi(|y|)\,dy = C_1\frac{1}{\tau^b}, \tag{4-26}$$

where K is a constant depending on N, ϕ, C_0.

Moreover, we compute, recalling that ϕ solves the ODE ($*$),

$$|w|^{\gamma'} = \left|\varepsilon\tau m\left(1+\frac{1}{M^\alpha C^\alpha \tau^{N\alpha}}m^\alpha\right)^{\frac{1}{\gamma'}}\right|^{\gamma'} = \varepsilon^{\gamma'}\tau^{\gamma'}m^{\gamma'}\left(1+\frac{1}{M^\alpha C^\alpha\tau^{N\alpha}}m^\alpha\right). \tag{4-27}$$

We consider the energy at (m,w)

$$\mathcal{E}(m,w) = \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + F(m) + mV(x)\,dx.$$

Observe that by (1-3),

$$F(m) \le -\frac{C_f}{\alpha+1}m^{\alpha+1} + Km.$$

Using Proposition 2.1, computation (4-27) and (4-25), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)+F(m)\,dx &\le \int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)\,dx-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx+KM \\
&\le C_L\int_{\mathbb{R}^N} m\frac{|w|^{\gamma'}}{m^{\gamma'}}\,dx+(C_L+K)M-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx \\
&= C_L\varepsilon^{\gamma'}\tau^{\gamma'}\Big(M+\int_{\mathbb{R}^N}\frac{1}{M^\alpha C^\alpha\tau^{N\alpha}}m^{\alpha+1}\,dx\Big)+(C_L+K)M-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N} m^{\alpha+1} \\
&= C_L\varepsilon^{\gamma'}\tau^{\gamma'}M+(C_L+K)M-\Big(\frac{C_f}{\alpha+1}-\frac{\varepsilon^{\gamma'}\tau^{\gamma'-N\alpha}}{M^\alpha C^\alpha}\Big)\int_{\mathbb{R}^N} m^{\alpha+1}\,dx \\
&= (MC_L+MCC_\alpha)\varepsilon^{\gamma'}\tau^{\gamma'}-\frac{C_f}{\alpha+1}M^{\alpha+1}C^{\alpha+1}C_\alpha\tau^{\alpha N}+(C_L+K)M.
\end{aligned}$$

We choose now τ such that $\tau=\frac{1}{a}\varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}}$, where a is sufficiently large, in such a way that

$$\int_{\mathbb{R}^N} mL\Big(-\frac{w}{m}\Big)\,dx+F(m)\,dx\le -C\varepsilon^{-\frac{\gamma' N\alpha}{\gamma'-N\alpha}}+C,$$

where C is a constant depending on α, C_L, M. Substituting this in the energy and recalling (4-26), we get the desired inequality. $\square$

5. Concentration phenomena

In the second part of this work, we are interested in the asymptotic analysis of solutions to (1-1) when $\varepsilon\to 0$.

5A. ***The rescaled problem.*** We consider the rescaling

$$\begin{cases}\tilde m(y):=\varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}}m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y),\\ \tilde u(y):=\varepsilon^{\frac{N\alpha(\gamma'-1)-\gamma'}{\gamma'-\alpha N}}u(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y),\\ \tilde\lambda:=\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}\lambda.\end{cases}\tag{5-1}$$

We introduce the rescaled potential

$$V_\varepsilon(y)=\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y).\tag{5-2}$$

Note that by (1-5), we get

$$C_V^{-1}\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(\max\{|\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y|-C_V,0\})^b\le V_\varepsilon(y)\le C_V\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(1+\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y|)^b.\tag{5-3}$$

The rescaled coupling term is given by

$$f_\varepsilon(\tilde m(y))=\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}f(\varepsilon^{-\frac{N\gamma'}{\gamma'-\alpha N}}m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y)).\tag{5-4}$$

Note that, using (1-3), we obtain

$$-C_f m^\alpha-K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}\le f_\varepsilon(m)\le -C_f m^\alpha+K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}.\tag{5-5}$$

Then we get that

$$\lim_{\varepsilon\to 0} f_\varepsilon(m) = -C_f m^\alpha \quad \text{uniformly in } [0,+\infty). \tag{5-6}$$

Moreover, we define $F_\varepsilon(m) = \int_0^m f_\varepsilon(n)\,dn$ if $m \geq 0$ and $F_\varepsilon(m) = 0$ otherwise, and we get

$$-\frac{C_f}{\alpha+1} m^{\alpha+1} - K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} m \leq F_\varepsilon(m) \leq -\frac{C_f}{\alpha+1} m^{\alpha+1} + K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} m. \tag{5-7}$$

We define also the rescaled Hamiltonian

$$H_\varepsilon(p) = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} H(\varepsilon^{-\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}} p). \tag{5-8}$$

By (1-2),

$$\begin{gathered} C_H|p|^\gamma - \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} K \leq H_\varepsilon(p) \leq C_H|p|^\gamma, \\ |\nabla H_\varepsilon(p)| \leq K|p|^{\gamma-1}. \end{gathered} \tag{5-9}$$

So, we get

$$\lim_{\varepsilon\to 0} H_\varepsilon(p) = H_0(p) := C_H|p|^\gamma \quad \text{uniformly in } \mathbb{R}^N. \tag{5-10}$$

Moreover, if we assume that ∇H_ε is locally bounded in $C^{0,\gamma-1}(\mathbb{R}^N)$, then

$$\nabla H_\varepsilon(p) \to \nabla H_0(p) = \frac{C_H}{\gamma}|p|^{\gamma-2}p \quad \text{locally uniformly.}$$

We can define L_ε as in (1-7), with H_ε in place of H and we obtain that condition (5-9) gives that there exists $C_L > 0$ such that

$$C_L|q|^{\gamma'} \leq L_\varepsilon(q) \leq C_L|q|^{\gamma'} + \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} C_L, \tag{5-11}$$

which in turns gives that

$$L_\varepsilon(q) \to L_0(q) = C_L|q|^{\gamma'} \quad \text{uniformly in } \mathbb{R}^N. \tag{5-12}$$

The rescalings (5-13) lead to the rescaled system

$$\begin{cases} -\Delta\tilde{u}_\varepsilon + H_\varepsilon(\nabla\tilde{u}_\varepsilon) + \tilde{\lambda}_\varepsilon = f_\varepsilon(\tilde{m}_\varepsilon) + V_\varepsilon(y), \\ -\Delta\tilde{m}_\varepsilon - \operatorname{div}(\tilde{m}_\varepsilon \nabla H_\varepsilon(\nabla\tilde{u}_\varepsilon)) = 0, \\ \int_{\mathbb{R}^N} \tilde{m}_\varepsilon = M. \end{cases} \tag{5-13}$$

Existence of a triple $(\tilde{u}_\varepsilon, \tilde{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ solving the previous system is an immediate consequence of Theorem 1.1. We first start by stating some a priori estimates.

Lemma 5.1. *There exist $C, C_1, C_2 > 0$ independent of ε such that the following hold*:

$$-C_1 \leq \tilde{\lambda}_\varepsilon \leq -C_2, \tag{5-14}$$

$$\int_{\mathbb{R}^N} \tilde{m}_\varepsilon |\nabla\tilde{u}_\varepsilon|^\gamma\,dy + \int_{\mathbb{R}^N} \tilde{m}_\varepsilon(y)V_\varepsilon(y)\,dy + \|\tilde{m}_\varepsilon\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^N)} \leq C, \tag{5-15}$$

$$\|\tilde{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq C. \tag{5-16}$$

Proof. Estimates (4-23), (4-22) give (5-14), (5-15) by rescaling.

We apply Theorem 4.1 with $g[m](x) = f_\varepsilon(m(x))$, $r_k = \varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}$, $s_k = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}$ and $t_k = \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}$, which are all bounded sequences, and we obtain (5-16). □

Using the a priori bounds on the solutions to (5-13), we want to pass to the limit $\varepsilon \to 0$. The problem is that these estimates are not sufficient to ensure that there is no loss of mass, namely that the limit of $\tilde{m}_\varepsilon$ still has L^1 norm equal to M. Therefore, we need to translate the reference system at a point around which the mass of $\tilde{m}_\varepsilon$ remains positive. This will be done as follows.

Let $y_\varepsilon \in \mathbb{R}^N$ be such that

$$\tilde{u}_\varepsilon(y_\varepsilon) = \min_{\mathbb{R}^N} \tilde{u}_\varepsilon(y), \tag{5-17}$$

note that this point exists due to (4-21).

We will define

$$\begin{aligned} \bar{u}_\varepsilon(y) &= \tilde{u}_\varepsilon(y + y_\varepsilon) - \tilde{u}_\varepsilon(y_\varepsilon), \\ \bar{m}_\varepsilon(y) &= \tilde{m}_\varepsilon(y + y_\varepsilon). \end{aligned} \tag{5-18}$$

Note that $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ is a classical solution to

$$\begin{cases} -\Delta \bar{u}_\varepsilon + H_\varepsilon(\nabla \bar{u}_\varepsilon) + \tilde{\lambda}_\varepsilon = f_\varepsilon(\bar{m}_\varepsilon) + V_\varepsilon(y + y_\varepsilon), \\ -\Delta \bar{m}_\varepsilon - \operatorname{div}(\bar{m}_\varepsilon \nabla H_\varepsilon(\nabla \bar{u}_\varepsilon)) = 0, \\ \int_{\mathbb{R}^N} \bar{m}_\varepsilon = M, \end{cases} \tag{5-19}$$

and in addition $\bar{u}_\varepsilon(0) = 0 = \min_{\mathbb{R}^N} \bar{u}_\varepsilon$.

5B. ***A preliminary convergence result.*** In this section, we provide some preliminary convergence results, where we are not preventing possible loss of mass in the limit. First of all we need some a priori estimates on the solutions to (5-19).

Proposition 5.2. *Let $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ be as in* (5-18). *Then there exists a constant $C > 0$ independent of ε such that the following hold*:

$$\varepsilon^{\frac{(N\alpha+b)\gamma'}{\gamma'-N\alpha}} |y_\varepsilon|^b \le C \quad \text{and} \quad 0 \le V_\varepsilon(y + y_\varepsilon) \le C(\varepsilon^{\frac{(N\alpha+b)\gamma'}{\gamma'-N\alpha}} |y|^b + 1), \tag{5-20}$$

$$|\nabla \bar{u}_\varepsilon(y)| \le C(1 + |y|)^{\frac{b}{\gamma}} \quad \text{and} \quad \bar{u}_\varepsilon(y) \ge C|y|^{1+\frac{b}{\gamma}} - C^{-1}, \tag{5-21}$$

$$\int_{B_R(0)} \bar{m}_\varepsilon(y)\, dy \ge C \quad \text{for all } R \ge 1. \tag{5-22}$$

Finally, if $\bar{w}_\varepsilon = -\bar{m}_\varepsilon \nabla H_\varepsilon(\nabla \bar{u}_\varepsilon)$, then $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ is a minimizer in the set $\mathcal{K}_{1,M}$ of the energy

$$\mathcal{E}_\varepsilon(m, w) = \int_{\mathbb{R}^N} m L_\varepsilon\left(-\frac{w}{m}\right) + V_\varepsilon(y + y_\varepsilon) m + F_\varepsilon(m)\, dy, \tag{5-23}$$

where L_ε and F_ε are defined in Section 5A.

Proof. Since $\bar{u}_\varepsilon$ is a classical solution, we can compute the equation in $y = 0$, obtaining

$$H_\varepsilon(0) + \tilde{\lambda}_\varepsilon \ge f_\varepsilon(\bar{m}_\varepsilon(0)) + V(y_\varepsilon).$$

Using the a priori estimates (5-14), (5-16), (5-9) and the assumptions (5-5), (5-3), this implies

$$\varepsilon^{\frac{(N\alpha+b)\gamma'}{\gamma'-N\alpha}}|y_\varepsilon|^b \le C,$$

and then, again by assumption (5-3), that (5-20) holds.

Using estimates (5-14), (5-16), and (5-20), we conclude by Theorem 2.5 that estimate (5-21) holds.

Again by the equation computed at $y=0$, recalling that $H_\varepsilon(0)\to 0$ and $V_\varepsilon \ge 0$ and estimate (5-14), we deduce that $-f_\varepsilon(\bar{m}_\varepsilon(0)) \ge -C_2 > 0$. So, by assumption (5-5), we get that there exists $C>0$ independent of ε, such that $\bar{m}_\varepsilon(0) > C > 0$. Using the estimates (5-21) and (5-16), by Proposition 2.4, we get that there exists a positive constant depending on p such that $\|\bar{m}_\varepsilon\|_{W^{1,p}(B_2(0))} \le C_p$ for all $p>1$. This, by Sobolev embeddings, gives that $\|\bar{m}_\varepsilon\|_{C^{0,\alpha}(B_2(0))} \le C_\alpha$ for every $\alpha \in (0,1)$ and for some positive constant depending on α. We choose now $R_0 \in (0,1]$ such that $\bar{m}_\varepsilon \ge \frac{1}{2}C$ in $B_{R_0}(0)$, using the C^α estimate and the fact that $\bar{m}_\varepsilon(0) > C > 0$. This implies immediately that $\int_{B_{R_0}(0)} \bar{m}_\varepsilon(y)\,dy \ge \frac{1}{2}C|B_{R_0}| > 0$. This gives the estimate (5-22), for all radii bigger than R_0.

Finally that $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ is a minimizer of (5-23) in $\mathcal{K}_{1,M}$ follows from Theorem 1.1, by rescaling. □

We get the first convergence result.

Proposition 5.3. *Let $(\bar{u}_\varepsilon, \bar{m}_\varepsilon, \tilde{\lambda}_\varepsilon)$ be the classical solution to* (5-19) *constructed above. Up to subsequences, we get that $\tilde{\lambda}_\varepsilon \to \bar{\lambda}$, and*

$$\bar{u}_\varepsilon \to \bar{u}, \quad \bar{m}_\varepsilon \to \bar{m}, \quad \nabla\bar{u}_\varepsilon \to \nabla\bar{u}, \quad \nabla H_\varepsilon(\nabla\bar{u}_\varepsilon) \to \nabla H_0(\nabla\bar{u}) \tag{5-24}$$

locally uniformly, where $\bar{u} \ge 0 = \bar{u}(0)$, and $(\bar{u}, \bar{m}, \bar{\lambda})$ is a classical solution to

$$\begin{cases} -\Delta\bar{u} + H_0(\nabla\bar{u}) + \bar{\lambda} = -C_f\bar{m}^\alpha + g(x), \\ -\Delta\bar{m} - \operatorname{div}(\bar{m}\nabla H_0(\nabla\bar{u})) = 0 \end{cases} \tag{5-25}$$

for a continuous function g such that $0 \le g(x) \le C$ on $\mathbb{R}^N$ for some $C>0$.

Moreover, there exist $a \in (0,M]$, $C, K, \kappa > 0$ such that $\int_{\mathbb{R}^N} \bar{m}\,dx = a$, and

$$\bar{u}(x) \ge C|x| - C, \qquad |\nabla\bar{u}| \le K \quad \text{on } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} e^{\kappa|x|}\bar{m}(x)\,dx < +\infty. \tag{5-26}$$

Proof. First of all observe that, since V is a locally Hölder continuous function, (5-20) implies that, up to a subsequence, $V_\varepsilon(x+y_\varepsilon) \to g(x)$ locally uniformly as $\varepsilon \to 0$, where g is a continuous function such that $0 \le g(x) \le C$ for some $C>0$.

Using the a priori estimate (5-21), and recalling that $\bar{u}_\varepsilon$ is a classical solution to (5-19), by classical elliptic regularity theory we obtain that $\bar{u}_\varepsilon$ is locally bounded in $C^{1,\alpha}$ in every compact set, uniformly with respect to ε. So, up to extracting a subsequence via a diagonalization procedure, we get that

$$\bar{u}_\varepsilon \to \bar{u}, \quad \nabla\bar{u}_\varepsilon \to \nabla\bar{u}, \quad \nabla H_\varepsilon(\nabla\bar{u}_\varepsilon) \to \nabla H_0(\nabla\bar{u})$$

locally uniformly, and $\tilde{\lambda}_\varepsilon \to \bar{\lambda}$. Using the estimates (5-21) and (5-16), by Proposition 2.4, and by Sobolev embeddings, for every compact set $K \subset \mathbb{R}^N$, we have that $\|\bar{m}_\varepsilon\|_{C^{0,\alpha}(K)} \le C_{K,\alpha}$ for every $\alpha \in (0,1)$ and for some positive constant depending on α and K. So, up to extracting a subsequence via a diagonalization procedure, we get that $\bar{m}_\varepsilon \to \bar{m}$ locally uniformly.

So, we can pass to the limit in (5-19) and obtain that $(\bar{u}, \bar{m}, \bar{\lambda})$ is a solution to (5-25), which is classical by elliptic regularity theory.

Using (5-22) and locally uniform convergence, we get that there exists $a \in (0, M]$ such that $\int_{\mathbb{R}^N} \bar{m}\, dy = a$.

Observe that $\bar{u}$ is a solution to

$$-\Delta \bar{u} + H_0(\nabla \bar{u}) + \bar{\lambda} = -C_f \bar{m}^\alpha + g(x).$$

By Theorem 2.5, we get that there exists a constant K depending on $\sup g$ and $-\bar{\lambda}$ such that $|\nabla \bar{u}| \le K$. Moreover, by construction $\bar{u} \ge 0$.

Since $\bar{m}$ is Hölder continuous and such that $\int_{\mathbb{R}^N} \bar{m}\, dx = a \in (0, M]$, by Lemma 2.2, we get that $\bar{m} \to 0$ as $|x| \to +\infty$. Therefore, we get that $\liminf_{|x|\to+\infty}(-\bar{m}^\alpha(x) + g(x) - \bar{\lambda} - H_0(0)) \ge -\lambda > 0$. So, by Theorem 2.6, recalling that by construction $\bar{u}(0) = 0 \le \bar{u}(y)$, we get that $\bar{u}$ satisfies

$$\bar{u}(x) \ge C|x| - C \tag{5-27}$$

for some $C > 0$.

To conclude, consider the function $\Phi(x) = e^{\kappa \bar{u}(x)}$. We claim that we can choose $\kappa > 0$ such that there exist $R > 0$ and $\delta > 0$ with

$$-\Delta\Phi + \nabla H_0(\nabla \bar{u}) \cdot \nabla\Phi > \delta\Phi, \quad |x| > R. \tag{5-28}$$

Indeed, since $\bar{u}$ solves the first equation in (5-25), we get

$$-\Delta\Phi + \nabla H_0(\nabla \bar{u}) \cdot \nabla\Phi \ge \kappa(-\bar{\lambda} - \kappa|\nabla \bar{u}|^2 - \bar{m}^\alpha)\Phi.$$

Using (5-27) and $\bar{m} \to 0$ as $|x| \to +\infty$, we obtain the claim. Reasoning as in [Ichihara 2015, Proposition 4.3], or [Metafune et al. 2005, Proposition 2.6], we get that $\int_{\mathbb{R}^N} e^{\kappa \bar{u}} \bar{m}\, dx < +\infty$, which concludes the estimate (5-26). □

Remark 5.4. With estimates (5-26) in force, the pointwise bounds stated in [Metafune et al. 2005, Theorem 6.1] hold; namely there exist positive constants c_1, c_2, such that

$$\bar{m}(x) \le c_1 e^{-c_2|x|} \quad \text{on } \mathbb{R}^N.$$

5C. ***Concentration-compactness.*** In this section we show that actually there is no loss of mass when passing to the limit as in Proposition 5.3. In order to do so, we apply a kind of concentration-compactness argument.

First of all we show that the functional $\mathcal{E}_\varepsilon(m, w)$ enjoys the following subadditivity property. Let us set

$$\tilde{e}_\varepsilon(M) = \min_{(m,w)\in\mathcal{K}_M} \mathcal{E}_\varepsilon(m, w).$$

Recalling (3-6), (4-24), and the rescaling (5-1), for every $M > 0$ there exist $C_1(M), C_2(M), K_1, K_2 > 0$ depending on M (and on the other constants of the problem) but not on ε such that there holds

$$-C_1(M) - K_1 \varepsilon^{\frac{N\alpha\gamma'}{\gamma' - N\alpha}} \le \tilde{e}_\varepsilon(M) \le -C_2(M) - K_2 \varepsilon^{\frac{N\alpha\gamma'}{\gamma' - N\alpha}}. \tag{5-29}$$

Lemma 5.5. *For all $a \in (0, M)$, there exist $\varepsilon_0 = \varepsilon_0(a)$ and a constant $C = C(a, M) \geq 0$, depending only on a, M and the data (not on ε), such that $C(M, M) = 0 = C(0, M)$, $C(a, M) > 0$ for $0 < a < M$ and*

$$\tilde{e}_\varepsilon(M) \leq \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - C(a, M) \quad \text{for all } \varepsilon \leq \varepsilon_0. \tag{5-30}$$

Proof. We assume that $a \geq \frac{1}{2}M$ (otherwise it suffices to replace a with $M - a$).

Let $c > 1$ and $B > 0$. For all admissible pairs $(m, w) \in \mathcal{K}_B$ we have, recalling (5-7),

$$\begin{aligned}
\tilde{e}_\varepsilon(cB) \leq \mathcal{E}_\varepsilon(cm, cw) &= \int_{\mathbb{R}^N} cmL_\varepsilon\left(-\frac{w}{m}\right) + F_\varepsilon(cm) + cV_\varepsilon(x + y_\varepsilon)m\,dx \\
&= c\mathcal{E}_\varepsilon(m, w) + \int_{\mathbb{R}^N} F_\varepsilon(cm) - cF_\varepsilon(m)\,dx \\
&\leq c\mathcal{E}_\varepsilon(m, w) - \frac{c(c^\alpha - 1)C_f}{\alpha + 1}\int_{\mathbb{R}^N} m^{\alpha+1}\,dx + 2KcB\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.
\end{aligned} \tag{5-31}$$

Let now (m_n, w_n) be a minimizing sequence of $\mathcal{E}_\varepsilon$ in $\mathcal{K}_B$ such that $\mathcal{E}_\varepsilon(m_n, w_n) \leq \tilde{e}_\varepsilon(B) + \frac{1}{4}C_2(B)$, where $C_2(B)$ is the constant appearing in (5-29), which depends on B and on the data of the problem. Recalling that $V_\varepsilon \geq 0$ and $L_\varepsilon \geq 0$, and using estimate (5-7), we get

$$\tilde{e}_\varepsilon(M) + \tfrac{1}{4}C_2(B) \geq \mathcal{E}_\varepsilon(m_n, w_n) \geq \int_{\mathbb{R}^N} F_\varepsilon(m_n)\,dx \geq -\frac{C_f}{\alpha + 1}\int_{\mathbb{R}^n} m^{\alpha+1}\,dx - KB\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

Using (5-29), we get, for all ε sufficiently small,

$$\frac{C_f}{\alpha + 1}\int_{\mathbb{R}^N} m_n^{\alpha+1}\,dx \geq \tfrac{3}{4}C_2(B) - K\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} > \tfrac{1}{2}C_2(B) > 0.$$

So, this estimate in particular holds for a minimizer of $\mathcal{E}_\varepsilon$. Therefore in (5-31) we get, taking (m, w) to be a minimizer of $\mathcal{E}_\varepsilon$ (which exists by Proposition 5.2),

$$\tilde{e}_\varepsilon(cB) < c\tilde{e}_\varepsilon(B) - c(c^\alpha - 1)\tfrac{1}{2}C_2(B) + 2KcB\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}. \tag{5-32}$$

Using (5-32) with $B = a$ and $c = \frac{M}{a}$ we get

$$\tilde{e}_\varepsilon(M) < \frac{M}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

If $a = \frac{1}{2}M$, this permits us to conclude, choosing ε sufficiently small (depending on a). If $a > \frac{1}{2}M$, we use (5-32) with $B = M - a$ and $c = \frac{a}{M-a}$ to get (multiplying everything by $\frac{M-a}{a}$)

$$\begin{aligned}
\frac{M-a}{a}\tilde{e}_\varepsilon(a) &< \tilde{e}_\varepsilon(M-a) - \left[\left(\frac{a}{M-a}\right)^\alpha - 1\right]\frac{C_2(M-a)}{2} + 2K(M-a)\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\
&< \tilde{e}_\varepsilon(M-a) - \left[\left(\frac{a}{M-a}\right)^\alpha - 1\right]\frac{C_2(M-a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \leq \tilde{e}_\varepsilon(M-a) + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.
\end{aligned}$$

So putting together the last two inequalities we get

$$\begin{aligned}\tilde{e}_\varepsilon(M) &< \frac{M}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &= \tilde{e}_\varepsilon(a) + \frac{M-a}{a}\tilde{e}_\varepsilon(a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 2KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &< \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{2} + 4KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \\ &\le \tilde{e}_\varepsilon(a) + \tilde{e}_\varepsilon(M-a) - \frac{M}{a}\left[\left(\frac{M}{a}\right)^\alpha - 1\right]\frac{C_2(a)}{4}\end{aligned}$$

for ε sufficiently small (depending on a). □

Theorem 5.6. *Let $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ be the minimizer of $\mathcal{E}_\varepsilon$ as in Proposition 5.2. Let $\bar{u}, \bar{m}$ as in Proposition 5.3, so that $\bar{m}_\varepsilon \to \bar{m}$, $\bar{w}_\varepsilon \to \bar{w} = -\bar{m}\nabla H_0(\nabla\bar{u})$ locally uniformly, and $\bar{m}$ satisfies the exponential decay* (5-26). *Then,*

$$\int_{\mathbb{R}^N} \bar{m}\,dx = M. \tag{5-33}$$

Proof. Assume by contradiction that $\int_{\mathbb{R}^N} \bar{m}\,dx = a$, with $0 < a < M$. We fix $\varepsilon_0(a)$ as in Lemma 5.5, and we consider from now on $\varepsilon \le \varepsilon_0(a)$. Let $\bar{c} > 0$ be such that $\bar{m} \le \bar{c}e^{-|x|}$ (such $\bar{c}$ exists by Remark 5.4).

For R sufficiently large (to be chosen later), we define

$$\nu_R(x) = \begin{cases}\bar{c}e^{-R}, & |x| \le R,\\ \bar{c}e^{-|x|}, & |x| > R.\end{cases} \tag{5-34}$$

So in particular $\bar{m}(x) \le \nu_R(x)$ for $|x| > R$.

We observe that as $R \to +\infty$

$$\int_{\mathbb{R}^n} \nu_R(x)\,dx = \bar{c}\omega_N e^{-R}R^N + \int_{\mathbb{R}^N\setminus B_R} \bar{c}e^{-|x|}\,dx \le Ce^{-R}R^N \to 0. \tag{5-35}$$

Since $\bar{m}_\varepsilon \to \bar{m}$ and $\nabla H_\varepsilon(\nabla\bar{u}_\varepsilon) \to \nabla H_0(\nabla\bar{u})$ locally uniformly, there exists $\varepsilon_0 = \varepsilon_0(R)$ such that for all $\varepsilon \le \varepsilon_0$,

$$|\bar{m}_\varepsilon - \bar{m}| + |\nabla H_\varepsilon(\nabla\bar{u}_\varepsilon) - \nabla H_0(\nabla\bar{u})| \le \bar{c}e^{-R}, \quad |x| \le R. \tag{5-36}$$

We observe that for all $\varepsilon \le \varepsilon_0$,

$$\bar{m}_\varepsilon - \bar{m} + 2\nu_R \ge \nu_R(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{5-37}$$

Indeed, if $|x| > R$, then $\bar{m}_\varepsilon - \bar{m} + 2\nu_R \ge \bar{m}_\varepsilon + \nu_R \ge \nu_R$, since $\bar{m} \le \nu_R$. On the other hand, if $|x| \le R$, then by (5-36) $\bar{m}_\varepsilon - \bar{m} + 2\nu_R \ge -\bar{c}e^{-R} + 2\bar{c}e^{-R} = \bar{c}e^{-R} = \nu_R$. From (5-37) we deduce that

$$|\bar{m}_\varepsilon - \bar{m}| \le \bar{m}_\varepsilon - \bar{m} + 2\nu_R. \tag{5-38}$$

Moreover, since $\bar{m}_\varepsilon \to \bar{m}$ a.e. by Theorem 2.3, recalling that $\int_{\mathbb{R}^N} \bar{m}_\varepsilon\,dx = M$, $\int_{\mathbb{R}^n} \bar{m} = a$ and using (5-35) and (5-38), we have

$$\int_{\mathbb{R}^N} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx = M - a + 2\int_{\mathbb{R}^N} \nu_R\,dx \to M - a \quad \text{as } R \to +\infty, \tag{5-39}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\bar{m}_\varepsilon^{\alpha+1}\,dx=\int_{\mathbb{R}^N}\bar{m}^{\alpha+1}\,dx+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|\bar{m}_\varepsilon-\bar{m}|^{\alpha+1}\,dx$$
$$\leq\int_{\mathbb{R}^N}\bar{m}^{\alpha+1}\,dx+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(\bar{m}_\varepsilon-\bar{m}+2\nu_R)^{\alpha+1}\,dx. \tag{5-40}$$

We claim that

$$\mathcal{E}_\varepsilon(\bar{m}_\varepsilon,\bar{w}_\varepsilon)\geq\mathcal{E}_\varepsilon(\bar{m},\bar{w})+\mathcal{E}_\varepsilon(\bar{m}_\varepsilon-\bar{m}+2\nu_R,\bar{w}_\varepsilon-\bar{w}+2\nabla\nu_R)+o_\varepsilon(1)+o_R(1), \tag{5-41}$$

where $o_\varepsilon(1)$ is an error such that $\lim_{\varepsilon\to 0}o_\varepsilon(1)=0$.

We consider the function $(m,w)\mapsto mL_\varepsilon\big(-\frac{w}{m}\big)$. This is a convex function in (m,w). We compute $\nabla_w\big(mL_\varepsilon\big(-\frac{w}{m}\big)\big)=-\nabla L_\varepsilon\big(-\frac{w}{m}\big)$, so in particular by (5-11) we get

$$C_L\left|\frac{w}{m}\right|^{\gamma'-1}-C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}\leq\left|\nabla_w\left(mL_\varepsilon\left(-\frac{w}{m}\right)\right)\right|\leq C_L^{-1}\left|\frac{w}{m}\right|^{\gamma'-1}+C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}. \tag{5-42}$$

Moreover, $\partial_m\big(mL_\varepsilon\big(-\frac{w}{m}\big)\big)=L_\varepsilon\big(-\frac{w}{m}\big)+\frac{w}{m}\cdot\nabla L_\varepsilon\big(-\frac{w}{m}\big)$, therefore, again by (5-11) we get

$$C_L\left|\frac{w}{m}\right|^{\gamma'}-C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}\leq\left|\partial_m\left(mL_\varepsilon\left(-\frac{w}{m}\right)\right)\right|\leq C_L^{-1}\left|\frac{w}{m}\right|^{\gamma'}+C_L^{-1}\varepsilon^{\frac{N\alpha(\gamma'-1)}{\gamma'-\alpha N}}. \tag{5-43}$$

Note that

$$\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)\bar{m}_\varepsilon\,dx$$
$$=\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)\bar{m}\,dx+\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)(\bar{m}_\varepsilon-\bar{m}+2\nu_R)\,dx-2\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)\nu_R\,dx.$$

Recalling the estimate (5-20) and the definition of ν_R, we have

$$2\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)\nu_R\,dx\leq CR^{b+N}e^{-R}.$$

Hence we obtain

$$\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)\bar{m}_\varepsilon\,dx$$
$$\geq\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)\bar{m}\,dx+\int_{\mathbb{R}^N}V_\varepsilon(y+y_\varepsilon)(\bar{m}_\varepsilon-\bar{m}+2\nu_R)\,dx-CR^{b+N}e^{-R}. \tag{5-44}$$

By (5-40) and (5-7) we get

$$\int_{\mathbb{R}^N}F_\varepsilon(\bar{m}_\varepsilon)\,dx\geq-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N}\bar{m}_\varepsilon^{\alpha+1}\,dx-KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}$$
$$\geq-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N}\bar{m}^{\alpha+1}\,dx-\frac{C_f}{\alpha+1}\int_{\mathbb{R}^N}(\bar{m}_\varepsilon-\bar{m}+2\nu_R)^{\alpha+1}\,dx+o_\varepsilon(1)$$
$$\geq\int_{\mathbb{R}^N}F_\varepsilon(\bar{m})\,dx+\int_{\mathbb{R}^N}F_\varepsilon(\bar{m}_\varepsilon-\bar{m}+2\nu_R)\,dx+o_\varepsilon(1). \tag{5-45}$$

Finally, we estimate the kinetic terms in the energy. Splitting

$$\int_{\mathbb{R}^N} \bar m_\varepsilon L_\varepsilon\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx = \int_{B_R} \bar m_\varepsilon L_\varepsilon\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx + \int_{\mathbb{R}^N\setminus B_R} \bar m_\varepsilon L\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx,$$

we proceed by estimating separately the two terms.

Estimates in $\mathbb{R}^N\setminus B_R$. First of all, note that by (5-26), (5-9) and (5-11), we get that $L_\varepsilon\big(-\frac{\bar w}{\bar m}\big) = L_\varepsilon(\nabla H_0(\nabla \bar u)) \le C$ for come constant $C>0$, just depending on the data. Moreover, recalling that $\bar m \le \bar c e^{-|x|}$, we get that, eventually enlarging C,

$$\int_{\mathbb{R}^N\setminus B_R} \bar m L_\varepsilon\Big(-\frac{\bar w}{\bar m}\Big)\,dx \le C\int_{|x|>R} e^{-|x|}\,dx \le CR^N e^{-R}. \tag{5-46}$$

By the convexity of the function $(m,w)\mapsto mL\big(-\frac{w}{m}\big)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R} \bar m_\varepsilon L\Big(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\Big)\,dx\\
&\quad\ge \int_{\mathbb{R}^N\setminus B_R} (\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\Big(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\Big)\,dx\\
&\qquad+\int_{\mathbb{R}^N\setminus B_R} \partial_m\Big((\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\Big(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\Big)\Big)(\bar m-2\nu_R)\,dx \qquad (5\text{-}47)\\
&\qquad+\int_{\mathbb{R}^N\setminus B_R} \nabla_w\Big[(\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\Big(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\Big)\Big]\cdot(\bar w-2\nabla\nu_R)\,dx. \qquad (5\text{-}48)
\end{aligned}$$

We recall that $|\bar w| = \bar m|\nabla H_0(\nabla\bar u)| \le C\bar m$ by (5-26) and $|\nabla\nu_R|\le C\nu_R$ by definition. Moreover, by (5-21) and (5-9),

$$|\bar w_\varepsilon| = \bar m_\varepsilon|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)| \le C\bar m_\varepsilon[(1+|x|)^{\frac{b}{\gamma}}]^{\gamma-1} \le C_1\bar m_\varepsilon(1+|x|)^{\frac{b}{\gamma'}}.$$

Using the triangle inequality we get the following, where the constant C can change from line to line:

$$\begin{aligned}
\Big|\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\Big| &\le \frac{\bar m_\varepsilon|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)|}{\bar m_\varepsilon-\bar m+2\nu_R}+\frac{\bar m|\nabla H_0(\nabla\bar u)|}{\bar m_\varepsilon-\bar m+2\nu_R}+\frac{C\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\\
&\le \frac{C\bar m_\varepsilon(1+|x|)^{\frac{b}{\gamma'}}}{\bar m_\varepsilon-\bar m+2\nu_R}+\frac{C\bar m}{\bar m_\varepsilon-\bar m+2\nu_R}+\frac{C\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R} \le C(1+|x|)^{\frac{b}{\gamma'}} \qquad (5\text{-}49)
\end{aligned}$$

on $\mathbb{R}^N\setminus B_R(0)$, where we used respectively the fact that $\bar m_\varepsilon-\bar m+2\nu_R\ge\bar m_\varepsilon$, $\bar m\le\nu_R$, and that $\bar m_\varepsilon-\bar m+2\nu_R\ge\nu_R$.

Now, using (5-43) and (5-49), we can estimate (5-47), and by (5-42) and (5-49) we can estimate (5-48). Indeed, we get

$$\int_{\mathbb{R}^N\setminus B_R}\Big|\partial_m\Big((\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\Big(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\Big)\Big)\Big|\,|\bar m-2\nu_R|\,dx \le C\int_{\mathbb{R}^N\setminus B_R}(1+|x|)^b\nu_R(x)\,dx$$

and

$$\int_{\mathbb{R}^N\setminus B_R}\Big|\nabla_w\Big[(\bar m_\varepsilon-\bar m+2\nu_R)L_\varepsilon\Big(-\frac{\bar w_\varepsilon-\bar w+2\nabla\nu_R}{\bar m_\varepsilon-\bar m+2\nu_R}\Big)\Big]\Big|(|\bar w|+2|\nabla\nu_R|)\,dx \le C\int_{\mathbb{R}^N\setminus B_R}(1+|x|)^{\frac{b}{\gamma}}\nu_R(x)\,dx,$$

because $\bar{w} \le C\bar{m}$ on $\mathbb{R}^N$. Therefore, we may conclude, possibly enlarging C, that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} \bar{m}_\varepsilon L\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx &\ge \int_{\mathbb{R}^N\setminus B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx - C\int_{\mathbb{R}^N\setminus B_R}(1+|x|)^b \nu_R(x)\,dx \\ &\ge \int_{\mathbb{R}^N\setminus B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx - CR^{N+b}e^{-R}. \qquad (5\text{-}50)\end{aligned}$$

Finally, putting together (5-46) and (5-50), we have, choosing C sufficiently large,

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} \bar{m}_\varepsilon L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx \ge \int_{\mathbb{R}^N\setminus B_R} \bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\,dx \\ + \int_{\mathbb{R}^N\setminus B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx - CR^{N+b}e^{-R}. \qquad (5\text{-}51)\end{aligned}$$

<u>Estimates in B_R</u>. Again by the convexity of the function $(m,w)\mapsto mL\big(-\frac{w}{m}\big)$, we get

$$\begin{aligned}\int_{B_R} \bar{m}_\varepsilon L\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx \ge \int_{B_R} \bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\,dx + \int_{B_R} \partial_m\Big(\bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\Big)(\bar{m}_\varepsilon - \bar{m})\,dx \\ + \int_{B_R} \nabla_w\Big[\bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\Big]\cdot(\bar{w}_\varepsilon - \bar{w})\,dx. \qquad (5\text{-}52)\end{aligned}$$

We now estimate (5-52). We recall that

$$\left|\frac{\bar{w}}{\bar{m}}\right| \le |\nabla H_0(\nabla\bar{u})| \le K$$

and also $|\nabla H_\varepsilon(\nabla\bar{u}_\varepsilon)| \le K$ for all $\varepsilon \le \varepsilon_0(R)$. Then, using these facts and (5-42) and (5-43) and recalling (5-36), we get

$$\int_{B_R} \left|\partial_m\Big(\bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\Big)\right| |\bar{m}_\varepsilon - \bar{m}|\,dx = \int_{B_R} |\partial_m(\bar{m} L_\varepsilon(\nabla H_0(\nabla\bar{u})))|\,|\bar{m}_\varepsilon - \bar{m}|\,dx \le Ce^{-R}R^N$$

and

$$\int_{B_R} |\nabla_w[\bar{m} L_\varepsilon(\nabla H_0(\nabla\bar{u}))]|\big(|\nabla H_\varepsilon(\nabla u_\varepsilon)|\,|\bar{m}_\varepsilon - \bar{m}| + |\nabla H_\varepsilon(\nabla\bar{u}_\varepsilon) - \nabla H_0(\nabla\bar{u})|\bar{m}\big)\,dx \le Ce^{-R}R^N.$$

This implies that for all $\varepsilon \le \varepsilon_0(R)$

$$\int_{B_R} \bar{m}_\varepsilon L\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big)\,dx \ge \int_{B_R} \bar{m} L_\varepsilon\Big(-\frac{\bar{w}}{\bar{m}}\Big)\,dx - Ce^{-R}R^N. \qquad (5\text{-}53)$$

Now we observe that by (5-11),

$$\int_{B_R} (\bar{m}_\varepsilon - \bar{m} + 2\nu_R) L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\Big)\,dx \le C\int_{B_R}\left[\left|\frac{\bar{w}_\varepsilon - \bar{w} + 2\nabla\nu_R}{\bar{m}_\varepsilon - \bar{m} + 2\nu_R}\right|^{\gamma'} + 1\right](\bar{m}_\varepsilon - \bar{m} + 2\nu_R)\,dx.$$

By (5-38) we get that $\bar m_\varepsilon - \bar m + 2\nu_R \le |\bar m_\varepsilon - \bar m| + 2\nu_R \le Ce^{-R}$, eventually enlarging C. Moreover, reasoning as in (5-49), we get

$$\left|\frac{\bar w_\varepsilon - \bar w + 2\nabla\nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\right| \le |\nabla H_\varepsilon(\nabla\bar u_\varepsilon)|\frac{|\bar m_\varepsilon - \bar m|}{\bar m_\varepsilon - \bar m + 2\nu_R} + \frac{|\nabla H_\varepsilon(\nabla\bar u_\varepsilon) - \nabla H_0(\nabla\bar u)|}{\bar m_\varepsilon - \bar m + 2\nu_R}\bar m \le C,$$

where we used that $\nabla\nu_R = 0$ for $|x| < R$, that $|\nabla H_\varepsilon(\nabla\bar u_\varepsilon)| \le K$, that by (5-38)

$$\frac{|\bar m_\varepsilon - \bar m|}{\bar m_\varepsilon - \bar m + 2\nu_R} \le 1,$$

and that by (5-37) and (5-36)

$$\frac{|\nabla H_\varepsilon(\nabla\bar u_\varepsilon) - \nabla H_0(\nabla\bar u)|}{\bar m_\varepsilon - \bar m + 2\nu_R} \le C.$$

So, we conclude that

$$\int_{B_R} (\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\left(-\frac{\bar w_\varepsilon - \bar w + 2\nabla\nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\right) dx \le Ce^{-R}R^N. \tag{5-54}$$

Putting together (5-53) and (5-54) we get, choosing C sufficiently large, for all $\varepsilon \le \varepsilon_0(R)$,

$$\int_{B_R} \bar m_\varepsilon L_\varepsilon\left(-\frac{\bar w_\varepsilon}{\bar m_\varepsilon}\right) dx \ge \int_{B_R} \bar m L_\varepsilon\left(-\frac{\bar w}{\bar m}\right) dx + \int_{B_R} (\bar m_\varepsilon - \bar m + 2\nu_R) L_\varepsilon\left(-\frac{\bar w_\varepsilon - \bar w + 2\nabla\nu_R}{\bar m_\varepsilon - \bar m + 2\nu_R}\right) dx - CR^N e^{-R}. \tag{5-55}$$

Therefore, summing up (5-55), (5-51), (5-44) and (5-45), we conclude for all $\varepsilon \le \varepsilon_0(R)$,

$$\mathcal{E}_\varepsilon(\bar m_\varepsilon, \bar w_\varepsilon) \ge \mathcal{E}_\varepsilon(\bar m, \bar w) + \mathcal{E}_\varepsilon(\bar m_\varepsilon - \bar m + 2\nu_R, \bar w_\varepsilon - \bar w + 2\nabla\nu_R) + o_\varepsilon(1) - CR^{b+N}e^{-R}. \tag{5-56}$$

Let now

$$c_R = \frac{M-a}{M-a+2\int_{\mathbb{R}^N} \nu_R\,dx}.$$

We have $c_R \to 1$ as $R \to +\infty$ and $c_R < 1$. In particular, $(c_R(\bar m_\varepsilon - \bar m + 2\nu_R), c_R(\bar w_\varepsilon - \bar w + 2\nabla\nu_R)) \in \mathcal{K}_{M-a}$. By the same computation as in (5-31), we get

$$\begin{aligned} &c_R\mathcal{E}_\varepsilon(\bar m_\varepsilon - \bar m + 2\nu_R, \bar w_\varepsilon - \bar w + 2\nabla\nu_R) \\ &= \mathcal{E}_\varepsilon(c_R(\bar m_\varepsilon - \bar m + 2\nu_R), c_R(\bar w_\varepsilon - \bar w + 2\nabla\nu_R)) + \int_{\mathbb{R}^N} c_R F_\varepsilon(\bar m_\varepsilon - \bar m + 2\nu_R) - F_\varepsilon(c_R(\bar m_\varepsilon - \bar m + 2\nu_R))\,dx \\ &\ge \mathcal{E}_\varepsilon(c_R(\bar m_\varepsilon - \bar m + 2\nu_R), c_R(\bar w_\varepsilon - \bar w + 2\nabla\nu_R)) \\ &\qquad + c_R\frac{c_R^\alpha - 1}{\alpha+1}C_f \int_{\mathbb{R}^N} (\bar m_\varepsilon - \bar m + 2\nu_R)^{\alpha+1}\,dx - 2K\left(M - a + 2\int_{\mathbb{R}^N} \nu_R\,dx\right)\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}. \end{aligned} \tag{5-57}$$

Observe that by (5-15) there exists C independent of ε such that

$$0 \le \int_{\mathbb{R}^N} (\bar m_\varepsilon - \bar m + 2\nu_R)^{\alpha+1}\,dx \le (\|\bar m_\varepsilon\|_{\alpha+1} + \|\bar m\|_{\alpha+1} + \|2\nu_R\|_{\alpha+1})^{\alpha+1} \le C.$$

Therefore, (5-57) reads (recalling that $c_R < 1$ and enlarging the constants C, K)

$$c_R\mathcal{E}_\varepsilon(\bar m_\varepsilon - \bar m + 2\nu_R, \bar w_\varepsilon - \bar w + 2\nabla\nu_R) \geq \mathcal{E}_\varepsilon(c_R(\bar m_\varepsilon - \bar m + 2\nu_R), c_R(\bar w_\varepsilon - \bar w + 2\nabla\nu_R)) + c_R\frac{c_R^\alpha - 1}{\alpha+1}C - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \geq \tilde e_\varepsilon(M-a) + c_R\frac{c_R^\alpha - 1}{\alpha+1}C - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

Using this inequality, and using the fact that $\mathcal{E}_\varepsilon(\bar m_\varepsilon, \bar w_\varepsilon) = \tilde e_\varepsilon(M)$ and that $\mathcal{E}_\varepsilon(\bar m, \bar w) \geq \tilde e_\varepsilon(a)$, we obtain from (5-56)

$$\tilde e_\varepsilon(M) \geq \tilde e_\varepsilon(a) + \tilde e_\varepsilon(M-a) + (1-c_R)\mathcal{E}_\varepsilon(\bar m_\varepsilon - \bar m + 2\nu_R, \bar w_\varepsilon - \bar w + 2\nabla\nu_R) + Cc_R\frac{c_R^\alpha - 1}{\alpha+1} - KM\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} + o_\varepsilon(1) - CR^{b+N}e^{-R}.$$

Moreover by (5-29) we get that there exists $K = K(M-a) > 0$ such that

$$\mathcal{E}_\varepsilon(\bar m_\varepsilon - \bar m + 2\nu_R, \bar w_\varepsilon - \bar w + 2\nabla\nu_R) \geq -K;$$

therefore the previous inequality gives

$$\tilde e_\varepsilon(M) \geq \tilde e_\varepsilon(a) + \tilde e_\varepsilon(M-a) - (1-c_R)K + Cc_R\frac{c_R^\alpha - 1}{\alpha+1} + o_\varepsilon(1) - CR^{b+N}e^{-R}. \tag{5-58}$$

By Lemma 5.5, we get

$$\tilde e_\varepsilon(M) \leq \tilde e_\varepsilon(a) + \tilde e_\varepsilon(M-a) - C(a, M),$$

where $C(a, M) > 0$ for $a < M$ and $C(M, M) = 0$. This implies in particular that

$$0 > -C(a, M) \geq -(1-c_R)K + Cc_R\frac{c_R^\alpha - 1}{\alpha+1} + o_\varepsilon(1) - CR^{b+N}e^{-R}.$$

Recalling that $c_R \to 1$ as $R \to +\infty$, this gives a contradiction, choosing R sufficiently large and $\varepsilon < \varepsilon_0(R)$. □

An immediate corollary of the previous theorem is the following convergence result.

Corollary 5.7. *Let* $(\bar u_\varepsilon, \bar m_\varepsilon, \tilde\lambda_\varepsilon)$ *and* $(\bar u, \bar m, \bar\lambda)$ *be as in Proposition 5.3. Then,*

$$\bar m_\varepsilon \to \bar m \quad \text{in } L^1(\mathbb{R}^N) \text{ and } L^{\alpha+1}(\mathbb{R}^N). \tag{5-59}$$

Finally for all $\eta > 0$, *there exist* $R > 0$ *and* ε_0 *such that for all* $\varepsilon \leq \varepsilon_0$,

$$\int_{B(0,R)} \bar m_\varepsilon \, dx \geq M - \eta. \tag{5-60}$$

Proof. By Proposition 5.3 we get that $\bar m_\varepsilon \to \bar m$ almost everywhere, and by Theorem 5.6, $\int_{\mathbb{R}^N} \bar m_\varepsilon = M = \int_{\mathbb{R}^N} \bar m$. This implies the convergence in $L^1(\mathbb{R}^N)$. Indeed, by Fatou's lemma

$$2M \leq \liminf_\varepsilon \int_{\mathbb{R}^N} \bar m_\varepsilon + \bar m - |\bar m_\varepsilon - \bar m| \, dx \leq 2M - \limsup_\varepsilon \int_{\mathbb{R}^N} |\bar m_\varepsilon - \bar m| \, dx.$$

Moreover, recalling (5-16), we get

$$\|\bar m_\varepsilon-\bar m\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{R}^N)}\le \|\bar m_\varepsilon-\bar m\|_{L^1(\mathbb{R}^N)}(\|\bar m\|_{L^\infty(\mathbb{R}^N)}+\|\bar m_\varepsilon\|_{L^\infty(\mathbb{R}^N)})\to 0.$$

Finally observe that for all R, by Remark 5.4,

$$\int_{B_R(0)}\bar m_\varepsilon\,dy\ge\int_{B_R(0)}\bar m\,dy-\int_{B_R(0)}|\bar m_\varepsilon-\bar m|\,dy\ge M-CR^{N-1}e^{-R}-\int_{\mathbb{R}^N}|\bar m_\varepsilon-\bar m|\,dy.$$

So, using the L^1 convergence we conclude the desired estimate. □

5D. ***Existence of ground states.*** In this subsection we aim at proving that as ε goes to zero, $(\bar u_\varepsilon,\bar m_\varepsilon,\tilde\lambda_\varepsilon)$ converges to a solution of the limiting MFG system (1-14), without potential terms. In particular, we will prove Theorem 1.3.

We first need a Γ-convergence-type result, proved in the following lemma.

Lemma 5.8. *Let $(m_\varepsilon,w_\varepsilon),(m,w)\in\mathcal{K}_{1,M}$ be such that $m_\varepsilon\to m$ in $L^1\cap L^{\alpha+1}(\mathbb{R}^N)$ and $w_\varepsilon\rightharpoonup w$ weakly in $L^q(\mathbb{R}^N)$ for some $q>1$. Then*

$$\liminf_\varepsilon \mathcal{E}_\varepsilon(m_\varepsilon,w_\varepsilon)\ge\mathcal{E}_0(m,w),\tag{5-61}$$

where $\mathcal{E}_0$ is defined in (1-16).

Let $(m,w)\in\mathcal{K}_{1,M}$ be such that $m(1+|y|^b)\in L^1(\mathbb{R}^N)$. Then

$$\lim_\varepsilon \mathcal{E}_\varepsilon(m(\cdot-y_\varepsilon),w(\cdot-y_\varepsilon))\le\mathcal{E}_0(m,w).\tag{5-62}$$

Proof. We recall that $L_\varepsilon(q)\to C_L|q|^{\gamma'}$ uniformly in $\mathbb{R}^N$ by (5-11) and $F_\varepsilon(m)\to-\frac{1}{\alpha+1}m^{\alpha+1}$ uniformly in $[0,+\infty)$ by (5-7). Moreover we observe that the energy $\mathcal{E}_0$ is lower semicontinuous with respect to weak L^q convergence of w and strong $L^{\alpha+1}\cap L^1$ convergence of m. Since $V\ge 0$, we get

$$\begin{aligned}\liminf_\varepsilon \mathcal{E}_\varepsilon(m_\varepsilon,w_\varepsilon)&\ge\liminf_\varepsilon\int_{\mathbb{R}^N}m_\varepsilon L_\varepsilon\Big(-\frac{w_\varepsilon}{m_\varepsilon}\Big)+F_\varepsilon(m_\varepsilon)\,dx\\&\ge\liminf_\varepsilon\int_{\mathbb{R}^N}C_Lm_\varepsilon^{1-\gamma'}|w_\varepsilon|^{\gamma'}-\frac{C_f}{\alpha+1}m_\varepsilon^{\alpha+1}\,dx\\&\ge\int_{\mathbb{R}^N}C_Lm^{1-\gamma'}|w|^{\gamma'}-\frac{C_f}{\alpha+1}m^{\alpha+1}\,dx=\mathcal{E}_0(m,w).\end{aligned}$$

Now we observe that for all m such that $m(1+|y|^b)\in L^1(\mathbb{R}^N)$, using (5-3), we get

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^N}m(y+y_\varepsilon)V_\varepsilon(y+y_\varepsilon)\,dy\le\lim_\varepsilon C_V\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}\int_{\mathbb{R}^N}(1+|y|)^bm(y)\,dy=0.\tag{5-63}$$

Therefore, recalling again the uniform convergence of $L_\varepsilon(q)\to C_L|q|^{\gamma'}$ and $F_\varepsilon(m)\to-\frac{1}{\alpha+1}m^{\alpha+1}$, we conclude (noting that if we translate m,w of y_ε the energy $\mathcal{E}_0$ remains the same)

$$\lim_\varepsilon\mathcal{E}_\varepsilon(m(\cdot-y_\varepsilon),w(\cdot-y_\varepsilon))=\mathcal{E}_0(m,w)+\lim_{\varepsilon\to0}\int_{\mathbb{R}^N}m(y+y_\varepsilon)V_\varepsilon(y+y_\varepsilon)\,dy\le\mathcal{E}_0(m,w).$$ □

Proof of Theorem 1.3. We first show that $(\bar{u}, \bar{m})$ obtained in Proposition 5.3 are associated to minimizers of an appropriate energy, without potential term, so that (1-15) holds.

Note that $(\bar{m}, \bar{w}) \in \mathcal{K}_{1,M}$, where $\bar{w} = -\bar{m}\nabla H_0(\nabla \bar{u})$, due to Proposition 5.3 and Theorem 5.6 and $\bar{m}(1+|y|^b) \in L^1(\mathbb{R}^N)$ by the exponential decay (5-26). Moreover $\bar{m}_\varepsilon \to \bar{m}$ in $L^1 \cap L^{\alpha+1}$ by Corollary 5.7 and

$$\bar{w}_\varepsilon = -\bar{m}_\varepsilon \nabla H_\varepsilon(\nabla \bar{u}_\varepsilon) \to \bar{w} = -\bar{m}\nabla H_0(\nabla \bar{u})$$

locally uniformly (by Proposition 5.3) and weakly in $L^{\frac{\gamma'(\alpha+1)}{\gamma'+\alpha}}$ by the same argument as in the proof of Proposition 3.3.

Let now $(m, w) \in \mathcal{K}_{1,M}$ be such that $m(1+|y|^b) \in L^1(\mathbb{R}^N)$. Using the minimality of $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$, (5-61) and (5-62), we conclude that

$$\mathcal{E}_0(m, w) \geq \lim_\varepsilon \mathcal{E}_\varepsilon(m(\cdot - y_\varepsilon), w(\cdot - y_\varepsilon)) \geq \lim_\varepsilon \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) \geq \mathcal{E}_0(\bar{m}, \bar{w}).$$

This implies (1-15).

To obtain the first part of the theorem, that is, the existence of a solution to (1-14), we need to prove that the function g appearing in Proposition 5.3 is actually zero on $\mathbb{R}^N$. To do that, we derive a better estimate on the term $V_\varepsilon(y + y_\varepsilon)$; in particular we show that $V_\varepsilon(y + y_\varepsilon) \to 0$ locally uniformly in $\mathbb{R}^N$.

By the minimality of $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ and $(\bar{m}, \bar{w})$, (5-11), (5-7) and (5-63) we get

$$\begin{aligned}
\mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) &\leq \mathcal{E}_\varepsilon(\bar{m}(\cdot + y_\varepsilon), \bar{w}(\cdot + y_\varepsilon)) \\
&\leq \mathcal{E}_0(\bar{m}, \bar{w}) + \int_{\mathbb{R}^N} \bar{m}(y + y_\varepsilon) V_\varepsilon(y + y_\varepsilon)\, dy + C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \leq \mathcal{E}_0(\bar{m}_\varepsilon, \bar{w}_\varepsilon) + C_1 \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.
\end{aligned}$$

Again using (5-7) and (5-11) we get

$$\mathcal{E}_0(\bar{m}_\varepsilon, \bar{w}_\varepsilon) + C_1\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}} \leq \int_{\mathbb{R}^N} \bar{m}_\varepsilon L_\varepsilon\biggl(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\biggr) + F_\varepsilon(\bar{m}_\varepsilon)\, dy + C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} M + C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}.$$

So, putting together the last two inequalities, we conclude that

$$\int_{\mathbb{R}^N} \bar{m}_\varepsilon V_\varepsilon(y + y_\varepsilon)\, dy \leq C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-N\alpha}}. \tag{5-64}$$

Recalling (5-2), this implies that for all $R > 0$, we get

$$C_V^{-1}(\max\{\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y_\varepsilon| - \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} R - C_V, 0\})^b \int_{B(0,R)} \bar{m}_\varepsilon\, dy \leq C.$$

Using (5-60), we conclude that there exists $C > 0$ such that

$$\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} |y_\varepsilon| \leq C. \tag{5-65}$$

In turn this gives, recalling again (5-2), that

$$0 \leq V_\varepsilon(y + y_\varepsilon) \leq C_V \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} (1 + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y| + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}|y_\varepsilon|)^b \leq C\varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}}(1 + |y|)^b,$$

which implies that $V_\varepsilon(y + y_\varepsilon) \to 0$ locally uniformly. □

Remark 5.9. If H and f satisfy the growth conditions (1-2) and (1-3), arguing as before one has that there exists a classical solution to the potential-free version of (1-1),

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(m), \\ -\Delta m - \operatorname{div}(\nabla H(\nabla u) m) = 0, \\ \int_{\mathbb{R}^N} m = M. \end{cases} \tag{5-66}$$

In addition, $(m, -\nabla H(\nabla u)m)$ is a minimizer of

$$(m, w) \mapsto \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + F(m)\, dx$$

among $(m, w) \in \mathcal{K}_{1,M}$, $m(1 + |y|^b) \in L^1(\mathbb{R}^N)$. This can be done as follows: Start with a sequence $(u_\delta, m_\delta, \lambda_\delta)$ solving

$$\begin{cases} -\Delta u_\delta + H(\nabla u_\delta) + \lambda_\delta = f(m_\delta) + \delta |x|^b, \\ -\Delta m_\delta - \operatorname{div}(\nabla H(\nabla u_\delta) m_\delta) = 0, \\ \int_{\mathbb{R}^N} m_\delta = M, \end{cases} \tag{5-67}$$

with $\delta = \delta_n \to 0$. Such a sequence exists by Theorem 1.1. The problem of passing to the limit in (5-67) to obtain (5-66) is the same as passing to the limit in (5-13), and it is even simpler: in (5-13), one has to be careful as the Hamiltonian H_ε and the coupling f_ε vary as $\varepsilon \to 0$ (still, they converge uniformly), while in (5-67) they are fixed, and only the potential is vanishing. We observe that $b > 0$ could be chosen arbitrarily; the perturbation $\delta|x|^b$ always disappears in the limit. Still, the limit m, u somehow retains a memory of b in terms of energy properties: m minimizes an energy among competitors satisfying $m(1 + |y|^b) \in L^1(\mathbb{R}^N)$.

Remark 5.10. We stress that uniqueness of solutions for (1-14) does not hold in general; for example, a triple (u, m, λ) solving the system may be translated in space to obtain a full family of solutions. On the other hand, a more subtle issue is the uniqueness of m in the second equation (with ∇u fixed); that is, if (u, m_1, λ) and (u, m_2, λ) are solutions, then $m_1 \equiv m_2$. This property is intimately related to the ergodic behavior of the optimal trajectory $dX_s = -\nabla H_0(\nabla u(X_s))\, ds + \sqrt{2\varepsilon}\, dB_s$; see, for example, [Cirant 2014]. It is well known that uniqueness for the Kolmogorov equation is guaranteed by the existence of a so-called Lyapunov function; in our cases, it can be checked that u itself (or increasing functions of u, as in (5-28)) acts as a Lyapunov function, so uniqueness of m and ergodicity hold for (1-14) and (1-1).

5E. ***Concentration of mass.*** The last problem we address is the localization of the point y_ε, to conclude the proof of Theorem 1.2. Rewriting (5-60) in view of (5-1) and (5-18), we get that for all $\eta > 0$ there exist R, ε_0 such that for all $\varepsilon \le \varepsilon_0$,

$$\int_{B(\varepsilon^{\frac{\gamma'}{\gamma' - \alpha N}} y_\varepsilon, \varepsilon^{\frac{\gamma'}{\gamma' - \alpha N}} R)} m(x)\, dx \ge M - \eta, \tag{5-68}$$

where m is the classical solution to (1-1) given in Theorem 1.1, and

$$\bar{m}_\varepsilon(y) = \varepsilon^{\frac{N\gamma'}{\gamma' - \alpha N}} m(\varepsilon^{\frac{\gamma'}{\gamma' - \alpha N}} y + \varepsilon^{\frac{\gamma'}{\gamma' - \alpha N}} y_\varepsilon).$$

By (5-65), we know that, up to subsequences, $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon \to \bar{x}$. Our aim is to locate this point, which is the point where mass concentrates. We need a preliminary lemma stating the existence of suitable competitors that will be used in the sequel.

Lemma 5.11. *For all $\varepsilon \le \varepsilon_0$, there exists $(\hat{m}_\varepsilon, \hat{w}_\varepsilon) \in \mathcal{K}_{1,M}$ that minimizes*

$$(m, w) \mapsto \int_{\mathbb{R}^N} m L_\varepsilon\left(-\frac{w}{m}\right) + F_\varepsilon(m)\,dy \tag{5-69}$$

among $(m, w) \in \mathcal{K}_{1,M}$, $m(1+|y|^b) \in L^1(\mathbb{R}^N)$. Moreover, for some positive constants c_1, c_2 independent of ε,

$$\hat{m}_\varepsilon(y) \le c_1 e^{-c_2|y|} \quad \textit{on } \mathbb{R}^N. \tag{5-70}$$

Proof. The existence of $(\hat{m}_\varepsilon, \hat{w}_\varepsilon)$ is stated in Remark 5.9, together with a solution $(\hat{u}_\varepsilon, \hat{m}_\varepsilon, \hat{\lambda}_\varepsilon)$ to the associated MFG system, as the optimality conditions; see (5-71) below. To obtain the uniform exponential decay, we can argue by Lyapunov functions as in Proposition 5.3; here, we have to be careful, since the argument in Proposition 5.3 mainly requires

$$f_\varepsilon(\hat{m}_\varepsilon) - \hat{\lambda}_\varepsilon - H_\varepsilon(0) \ge -\tfrac{1}{2}\hat{\lambda}_\varepsilon > 0$$

outside some fixed ball $B_r(0)$. This claim can be proved as follows: First, $-\hat{\lambda}_\varepsilon$ is bounded away from zero for ε small. Indeed,

$$\hat{\lambda}_\varepsilon M = \int_{\mathbb{R}^N} \hat{m}_\varepsilon L_\varepsilon\left(-\frac{\hat{w}_\varepsilon}{\hat{m}_\varepsilon}\right) + f_\varepsilon(\hat{m}_\varepsilon)\hat{m}_\varepsilon\,dy \le \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) + o_\varepsilon(1) \le -C.$$

The inequality follows by the minimality of $(\hat{m}_\varepsilon, \hat{w}_\varepsilon)$ and $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$, and (rescaled) (4-24).

We now prove that $\hat{m}_\varepsilon$ decays as $|x| \to \infty$ uniformly in ε. Note that $\hat{w}_\varepsilon = -\nabla H_\varepsilon(\nabla \hat{u}_\varepsilon)\hat{m}_\varepsilon$, where $(\hat{u}_\varepsilon, \hat{m}_\varepsilon, \hat{\lambda}_\varepsilon)$ solves

$$\begin{cases} -\Delta \hat{u}_\varepsilon + H_\varepsilon(\nabla \hat{u}_\varepsilon) + \lambda = f_\varepsilon(\hat{m}_\varepsilon), \\ -\Delta \hat{m}_\varepsilon - \operatorname{div}(\nabla H_\varepsilon(\nabla \hat{u}_\varepsilon)\hat{m}_\varepsilon) = 0, \\ \int_{\mathbb{R}^N} \hat{m}_\varepsilon = M. \end{cases} \tag{5-71}$$

We derive local estimates for $\hat{u}_\varepsilon$ and $\hat{m}_\varepsilon$. We shift the x-variable so that $\hat{u}_\varepsilon(0) = 0 = \min_{\mathbb{R}^N} \hat{u}_\varepsilon$ for all ε. Choose $p > N$ such that

$$\alpha < \frac{\gamma'}{p} < \frac{\gamma'}{N}.$$

If one considers the HJB equation solved by $\hat{u}_\varepsilon$, recalling (5-5) and (5-9), Theorem 2.5 gives the existence of $C > 0$ such that

$$\|\nabla \hat{u}_\varepsilon\|_{L^\infty(B_{2R}(x_0))} \le K(\|\hat{m}_\varepsilon\|^\alpha_{L^\infty(B_{4R}(x_0))} + 1)^{\frac{1}{\gamma}}.$$

Note that $C > 0$ does not depend on ε and x_0. Turning to the Kolmogorov equation, again by (5-9) and Proposition 2.4,

$$\|\hat{m}_\varepsilon\|_{W^{1,p}(B_R(x_0))} \le C(\|\nabla \hat{u}_\varepsilon\|^{\gamma-1}_{L^\infty(B_{2R}(x_0))} + 1)\|m_\varepsilon\|_{L^p(B_{2R}(x_0))}.$$

By the previous L^∞ estimate on ∇u_ε and interpolation of the L^p norm of m between L^1 and L^∞ we get

$$\|\hat{m}_\varepsilon\|_{W^{1,p}(B_R(x_0))} \le C(\|\hat{m}_\varepsilon\|_{L^\infty(B_{4R}(x_0))}^{\frac{\alpha}{\gamma'}} + 1)\|\hat{m}_\varepsilon\|_{L^1(B_{4R}(x_0))}^{\frac{1}{p}} \|\hat{m}_\varepsilon\|_{L^\infty(B_{4R}(x_0))}^{1-\frac{1}{p}}.$$

Recall that $\|\hat{m}_\varepsilon\|_{L^1(B_{4R}(x_0))} \le M$; then, since $p > N$, by Sobolev embeddings we obtain that for some $\beta > 0$,

$$\|\hat{m}_\varepsilon\|_{C^{0,\beta}(B_R(x_0))} \le C(\|\hat{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}^{\frac{\alpha}{\gamma'}} + 1)\|\hat{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{1}{p}}. \tag{5-72}$$

First, since C does not depend on x_0, this yields $\|\hat{m}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C$, by the choice of $p < \frac{\gamma'}{\alpha}$. Secondly, plugging this estimate back into (5-72), we conclude

$$\|\hat{m}_\varepsilon\|_{C^{0,\beta}(\mathbb{R}^N)} \le C.$$

Then, using these estimates, we get that up to subsequences, $\hat{\lambda}_\varepsilon \to \hat{\lambda}$, $\hat{u}_\varepsilon \to \hat{u}$ locally uniformly in C^1, and $\hat{m}_\varepsilon \to \hat{m}$ locally uniformly, where $(\hat{u}, \hat{m}, \hat{\lambda})$ is a solution to (5-25) with $g \equiv 0$. Arguing exactly as in Proposition 5.3, we get that $\tilde{u}$, $\tilde{m}$ satisfy the estimates (5-26) (eventually modifying the constants). Moreover,

$$\int_{\mathbb{R}^N} \hat{m}\, dx = a \in (0, M].$$

Observe now that Lemma 5.5 and Theorem 5.6 hold also for the energy (5-69), since it coincides with the energy $\mathcal{E}_\varepsilon$ without the potential term $\int_{\mathbb{R}^N} V_\varepsilon m\, dx$. Therefore we can apply Theorem 5.6 to $\hat{m}$ to conclude that actually $\int_{\mathbb{R}^N} \hat{m}\, dx = M$. So, by Corollary 5.7, we obtain that for all $\eta > 0$, there exist $R > 0$ and ε_0 such that for all $\varepsilon \le \varepsilon_0$,

$$\int_{B(0,R)} \hat{m}_\varepsilon\, dx \ge M - \eta. \tag{5-73}$$

By (5-72) and (5-73), using Lemma 2.2, we get

$$f_\varepsilon(\hat{m}_\varepsilon) \ge \tfrac{1}{4}\hat{\lambda}_\varepsilon$$

outside a ball $B_r(0)$. Since $H_\varepsilon(0) \to 0$, the claim

$$f_\varepsilon(\hat{m}_\varepsilon) - \hat{\lambda}_\varepsilon - H_\varepsilon(0) \ge -\tfrac{1}{2}\hat{\lambda}_\varepsilon > 0 \tag{5-74}$$

outside a ball $B_r(0)$ follows. As previously mentioned, we may now proceed and conclude as in Proposition 5.3; basically, (5-74) implies that $x \mapsto e^{k\hat{u}_\varepsilon(x)}$ acts as a Lyapunov function for $\hat{m}_\varepsilon$ for some small $k > 0$, giving

$$c \int_{\mathbb{R}^N} e^{k|x|-k_1}\hat{m}_\varepsilon \le \int_{\mathbb{R}^N} e^{k\hat{u}_\varepsilon}\hat{m}_\varepsilon \le C$$

for all ε small, which easily implies the pointwise exponential decay (5-70) of $\hat{m}_\varepsilon$ by the Hölder regularity of $\hat{m}_\varepsilon$ itself. □

For general potentials, the point where mass concentrates is a minimum for V.

Proposition 5.12. *Up to subsequences,* $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon \to \bar{x}$, *where* $V(\bar{x}) = 0$, *i.e.,* $\bar{x}$ *is a minimum of* V.

Proof. Fix a generic $z \in \mathbb{R}^N$ and observe that $(\hat{m}_\varepsilon(\cdot+z), \hat{w}_\varepsilon(\cdot+z))$ is still a minimizer of $\int mL_\varepsilon\big(-\frac{w}{m}\big) + F_\varepsilon(m)$. By the minimality of $(\bar{m}_\varepsilon, \bar{w}_\varepsilon)$ and of $(\hat{m}_\varepsilon(\cdot+z), \hat{w}_\varepsilon(\cdot+z))$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{m}_\varepsilon L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big) + F_\varepsilon(\bar{m}_\varepsilon)\,dy + \int_{\mathbb{R}^N} \bar{m}_\varepsilon(y)V_\varepsilon(y+y_\varepsilon)\,dy& \\ = \mathcal{E}_\varepsilon(\bar{m}_\varepsilon, \bar{w}_\varepsilon) &\le \mathcal{E}_\varepsilon(\hat{m}_\varepsilon(\cdot+z), \hat{w}_\varepsilon(\cdot+z)) \\ &\le \int_{\mathbb{R}^N} \bar{m}_\varepsilon L_\varepsilon\Big(-\frac{\bar{w}_\varepsilon}{\bar{m}_\varepsilon}\Big) + F_\varepsilon(\bar{m}_\varepsilon) + \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y+z)V_\varepsilon(y+y_\varepsilon)\,dy.\end{aligned}$$

In particular this gives

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{m}_\varepsilon(y)V_\varepsilon(y+y_\varepsilon)\,dy &\le \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y+z)V_\varepsilon(y+y_\varepsilon)\,dy \\ &= \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y)V_\varepsilon(y+y_\varepsilon-z)\,dy \quad \text{for all } z \in \mathbb{R}^N. \end{aligned} \tag{5-75}$$

Recalling the rescaling of V_ε and of $\bar{m}_\varepsilon$ in (5-1), this is equivalent to

$$\int_{\mathbb{R}^N} m(x)V(x)\,dx \le \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y)V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon - \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}z)\,dy \quad \text{for all } z \in \mathbb{R}^N, \tag{5-76}$$

where m is the classical solution to (1-1) given in Theorem 1.1 such that

$$\bar{m}_\varepsilon(y) = \varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}} m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon).$$

By (5-65), we get that up to passing to a subsequence, $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^N$. Then by (5-68), we get

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} m(x)V(x)\,dx \ge \liminf_{\varepsilon\to 0} \int_{B(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon, \varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}R)} m(x)V(x)\,dx \ge (M-\eta)V(\bar{x}). \tag{5-77}$$

We fix $\bar{z}$ such that $V(\bar{z}) = 0$ and we choose in (5-76) $z = y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}}\bar{z}$. We have, by the Lebesgue convergence theorem and (5-70),

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y)V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \bar{z})\,dy \le \limsup_{\varepsilon\to 0}\, c_1 \int_{\mathbb{R}^N} e^{-c_2|y|}V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y + \bar{z})\,dy = 0. \tag{5-78}$$

By (5-77), (5-78) and (5-76), we conclude $V(\bar{x}) = 0$. □

If we assume that the potential V has a finite number of minima and polynomial behavior, that is, it satisfies assumption (1-13), then we get that at the limit $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon$ selects at the limit the more stable minima of V, as we will show in the next proposition.

Proposition 5.13. *Assume that V satisfies assumption* (1-13). *Then, up to subsequences, there holds*

$$\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon \to x_i \quad \text{as } \varepsilon \to 0,$$

where $i \in \{j = 1, \dots, n : b_j = \max_k b_k\}$.

Proof. By Proposition 5.12, we know that up to subsequences, $\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} y_\varepsilon \to x_\iota$ for some $\iota = 1, \dots n$. It remains to prove that $b_\iota = \max_i b_i$. Assume by contradiction that it is not true, and then $b_\iota < \max_i b_i$.

We compute for $j \in 1, \dots n$, recalling the uniform exponential decay of $\hat{m}_\varepsilon$ given in (5-70),

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{m}_\varepsilon(y + y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j) V_\varepsilon(y + y_\varepsilon)\, dy \\ &= \int_{\mathbb{R}^n} \hat{m}_\varepsilon(y) V_\varepsilon(y + \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j)\, dy \\ &\le C_V \varepsilon^{\frac{\gamma' N\alpha}{\gamma'-N\alpha}} \int_{\mathbb{R}^n} \hat{m}_\varepsilon(y) \varepsilon^{\frac{b_j\gamma'}{\gamma'-N\alpha}} |y|^{b_j} \prod_{i\neq j} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y - x_i + x_j|^{b_i}\, dy \\ &\le C \varepsilon^{\frac{\gamma'(N\alpha+b_j)}{\gamma'-N\alpha}} \int_{\mathbb{R}^n} \hat{m}_\varepsilon(y) |y|^{b_j} \prod_{i\neq j} |y - x_i + x_j|^{b_i}\, dy \le C \varepsilon^{\frac{\gamma'(N\alpha+b_j)}{\gamma'-N\alpha}}. \end{aligned} \tag{5-79}$$

Note in particular that we can choose in the previous inequality $b_j = \max_i b_i$.

We get from (5-75) applied to $z = y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j$, where j is such that $b_j = \max_i b_i$, and from (5-79) the following improvement of (5-64):

$$\int_{B(0,R)} \bar{m}_\varepsilon V_\varepsilon(y + y_\varepsilon)\, dy \le \int_{\mathbb{R}^N} \hat{m}_\varepsilon(y + y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-\alpha N}} x_j) V_\varepsilon(y + y_\varepsilon)\, dy \le C \varepsilon^{\frac{(N\alpha+\max b_j)\gamma'}{\gamma'-N\alpha}} \tag{5-80}$$

for all $R \ge 0$. We choose $R > 0$ sufficiently large such that $\int_{B(0,R)} \bar{m}_\varepsilon\, dy \ge \frac{1}{2} M$. Recalling the rescaling of V, (5-80) implies

$$C \varepsilon^{\frac{\max b_j \gamma'}{\gamma'-N\alpha}} \ge \tfrac{1}{2} M C_V^{-1} \min_{y\in B(0,R)} \prod_{j=1}^{n} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_j|^{b_j}. \tag{5-81}$$

Note that for ε sufficiently small $|\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_j| \ge \delta > 0$ for all $i \neq \iota$ and all $y \in B(0, R)$. So, by (5-81) we get that there exists $C > 0$ for which

$$\min_{y\in B(0,R)} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_\iota|^{b_\iota} \le C \varepsilon^{\frac{\max b_j \gamma'}{\gamma'-N\alpha}}$$

and then

$$|\hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota} = \min_{y\in B(0,R)} |y + y_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota} \le C \varepsilon^{\frac{(\max b_j - b_\iota)\gamma'}{\gamma'-N\alpha}} \to 0 \tag{5-82}$$

for some $\hat{y}_\varepsilon \in B(y_\varepsilon, R)$. Let $z_\varepsilon = \hat{y}_\varepsilon - y_\varepsilon \in B(0, R)$. Up to subsequences we can assume that $z_\varepsilon \to \bar{z} \in B(0, R)$.

We use now (5-80), recalling assumption (1-13), and we get

$$\begin{aligned} C \varepsilon^{\frac{\max b_j \gamma'}{\gamma'-N\alpha}} &\ge C_V^{-1} \int_{B(0,R)} \bar{m}_\varepsilon(y) \prod_{j=1}^{n} |\varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y + \varepsilon^{\frac{\gamma'}{\gamma'-N\alpha}} y_\varepsilon - x_j|^{b_j}\, dy \\ &\ge c_1 \varepsilon^{\frac{b_\iota \gamma'}{\gamma'-N\alpha}} \int_{B(0,R)} \bar{m}_\varepsilon(y) |y - z_\varepsilon + \hat{y}_\varepsilon - \varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}} x_\iota|^{b_\iota}\, dy. \end{aligned}$$

In particular this implies

$$\lim_{\varepsilon\to 0}\int_{B(0,R)}\bar{m}_\varepsilon(y)|y-z_\varepsilon+\hat{y}_\varepsilon-\varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}}x_\iota|^{b_\iota}\,dy=0. \tag{5-83}$$

Recalling that $\bar{m}_\varepsilon\to\bar{m}$ locally uniformly, see (5-24), that $\hat{y}_\varepsilon-\varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}}x_\iota\to 0$ by (5-82), and that $z_\varepsilon\to\bar{z}$, we get

$$\lim_{\varepsilon\to 0}\int_{B(0,R)}\bar{m}_\varepsilon(y)|y-z_\varepsilon+\hat{y}_\varepsilon-\varepsilon^{-\frac{\gamma'}{\gamma'-N\alpha}}x_\iota|^{b_\iota}\,dy=\int_{B(0,R)}\bar{m}(y)|y-\bar{z}|^{b_\iota}\,dy>0.$$

This gives a contradiction with (5-83). □

As a consequence of the previous results, we can conclude with the following.

Proof of Theorem 1.2. Setting $x_\varepsilon=\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}y_\varepsilon$, it suffices to recall (5-68) and Propositions 5.12, 5.13. □

Acknowledgements

The authors are partially supported by the Fondazione CaRiPaRo Project "Nonlinear Partial Differential Equations: Asymptotic Problems and Mean-Field Games" and PRAT CPDA157835 of the University of Padova "Mean-Field Games and Nonlinear PDEs".

References

[Agmon 1959] S. Agmon, "The L_p approach to the Dirichlet problem, I: Regularity theorems", *Ann. Scuola Norm. Sup. Pisa* (3) **13**:4 (1959), 405–448. MR Zbl

[Arapostathis et al. 2017] A. Arapostathis, A. Biswas, and J. Carroll, "On solutions of mean field games with ergodic cost", *J. Math. Pures Appl.* (9) **107**:2 (2017), 205–251. MR Zbl

[Bardi and Priuli 2014] M. Bardi and F. S. Priuli, "Linear-quadratic N-person and mean-field games with ergodic cost", *SIAM J. Control Optim.* **52**:5 (2014), 3022–3052. MR Zbl

[Barles and Meireles 2016] G. Barles and J. Meireles, "On unbounded solutions of ergodic problems in $\mathbb{R}^m$ for viscous Hamilton–Jacobi equations", *Comm. Partial Differential Equations* **41**:12 (2016), 1985–2003. MR Zbl

[Barles et al. 2010] G. Barles, A. Porretta, and T. T. Tchamba, "On the large time behavior of solutions of the Dirichlet problem for subquadratic viscous Hamilton–Jacobi equations", *J. Math. Pures Appl.* (9) **94**:5 (2010), 497–519. MR Zbl

[Borwein and Vanderwerff 2010] J. M. Borwein and J. D. Vanderwerff, *Convex functions: constructions, characterizations and counterexamples*, Encyclopedia of Math. and Its Appl. **109**, Cambridge Univ. Press, 2010. MR Zbl

[Brézis and Lieb 1983] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals", *Proc. Amer. Math. Soc.* **88**:3 (1983), 486–490. MR Zbl

[Briani and Cardaliaguet 2018] A. Briani and P. Cardaliaguet, "Stable solutions in potential mean field game systems", *Nonlinear Differential Equations Appl.* **25**:1 (2018), art. id. 1. MR Zbl

[Cardaliaguet and Graber 2015] P. Cardaliaguet and P. J. Graber, "Mean field games systems of first order", *ESAIM Control Optim. Calc. Var.* **21**:3 (2015), 690–722. MR Zbl

[Cesaroni and Cirant 2017] A. Cesaroni and M. Cirant, "Introduction to variational methods for viscous ergodic mean-field games with local coupling", lecture notes, Istituto Nazionale di Alta Matematica, 2017, available at https://tinyurl.com/cesaindam.

[Cirant 2014] M. Cirant, "On the solvability of some ergodic control problems in $\mathbb{R}^d$", *SIAM J. Control Optim.* **52**:6 (2014), 4001–4026. MR Zbl

[Cirant 2015] M. Cirant, "Multi-population mean field games systems with Neumann boundary conditions", *J. Math. Pures Appl.* (9) **103**:5 (2015), 1294–1315. MR Zbl

[Cirant 2016] M. Cirant, "Stationary focusing mean-field games", *Comm. Partial Differential Equations* **41**:8 (2016), 1324–1346. MR Zbl

[Cirant 2017] M. Cirant, "On the existence of oscillating solutions in non-monotone mean-field games", preprint, 2017. arXiv

[Cirant and Tonon 2018] M. Cirant and D. Tonon, "Time-dependent focusing mean-field games: the sub-critical case", *J. Dynam. Differential Equations* (online publication April 2018).

[Gomes and Pimentel 2016] D. A. Gomes and E. Pimentel, "Local regularity for mean-field games in the whole space", *Minimax Theory Appl.* **1**:1 (2016), 65–82. MR Zbl

[Gomes et al. 2016] D. A. Gomes, E. A. Pimentel, and V. Voskanyan, *Regularity theory for mean-field game systems*, Springer, 2016. MR Zbl

[Gomes et al. 2018] D. A. Gomes, L. Nurbekyan, and M. Prazeres, "One-dimensional stationary mean-field games with local coupling", *Dyn. Games Appl.* **8**:2 (2018), 315–351. MR Zbl

[Huang et al. 2006] M. Huang, R. P. Malhamé, and P. E. Caines, "Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle", *Commun. Inf. Syst.* **6**:3 (2006), 221–251. MR Zbl

[Ichihara 2011] N. Ichihara, "Recurrence and transience of optimal feedback processes associated with Bellman equations of ergodic type", *SIAM J. Control Optim.* **49**:5 (2011), 1938–1960. MR Zbl

[Ichihara 2015] N. Ichihara, "The generalized principal eigenvalue for Hamilton–Jacobi–Bellman equations of ergodic type", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32**:3 (2015), 623–650. MR Zbl

[Lasry and Lions 1989] J.-M. Lasry and P.-L. Lions, "Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints, I: The model problem", *Math. Ann.* **283**:4 (1989), 583–630. MR Zbl

[Lasry and Lions 2006a] J.-M. Lasry and P.-L. Lions, "Jeux à champ moyen, I: Le cas stationnaire", *C. R. Math. Acad. Sci. Paris* **343**:9 (2006), 619–625. MR Zbl

[Lasry and Lions 2006b] J.-M. Lasry and P.-L. Lions, "Jeux à champ moyen, II: Horizon fini et contrôle optimal", *C. R. Math. Acad. Sci. Paris* **343**:10 (2006), 679–684. MR Zbl

[Lasry and Lions 2007] J.-M. Lasry and P.-L. Lions, "Mean field games", *Jpn. J. Math.* **2**:1 (2007), 229–260. MR Zbl

[Lions 1984] P.-L. Lions, "The concentration-compactness principle in the calculus of variations: the locally compact case, I", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1**:2 (1984), 109–145. MR Zbl

[Mészáros and Silva 2018] A. R. Mészáros and F. J. Silva, "On the variational formulation of some stationary second-order mean field games systems", *SIAM J. Math. Anal.* **50**:1 (2018), 1255–1277. MR Zbl

[Metafune et al. 2005] G. Metafune, D. Pallara, and A. Rhandi, "Global properties of invariant measures", *J. Funct. Anal.* **223**:2 (2005), 396–424. MR Zbl

[Porretta 2017] A. Porretta, "On the weak theory for mean field games systems", *Boll. Unione Mat. Ital.* **10**:3 (2017), 411–439. MR Zbl

Received 16 Aug 2017. Revised 25 May 2018. Accepted 29 Jun 2018.

ANNALISA CESARONI: annalisa.cesaroni@unipd.it
Dipartimento di Scienze Statistiche, Università di Padova, Padova, Italy

MARCO CIRANT: cirant@math.unipd.it
Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Padova, Italy

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2019 is US $310/year for the electronic version, and $520/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 12 No. 3 2019

2157-5045(2019)12:3;1-N