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A RIGOROUS DERIVATION FROM THE KINETIC CUCKER–SMALE MODEL TO THE PRESSURELESS EULER SYSTEM WITH NONLOCAL ALIGNMENT

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We consider the kinetic Cucker–Smale model with local alignment as a mesoscopic description for the flocking dynamics. The local alignment was first proposed by Karper, Mellet and Trivisa (2014), as a singular limit of a normalized nonsymmetric alignment introduced by Motsch and Tadmor (2011). The existence of weak solutions to this model was obtained by Karper, Mellet and Trivisa (2014), and in the same paper they showed the time-asymptotic flocking behavior. Our main contribution is to provide a rigorous derivation from a mesoscopic to a macroscopic description for the Cucker–Smale flocking models. More precisely, we prove the hydrodynamic limit of the kinetic Cucker–Smale model with local alignment towards the pressureless Euler system with nonlocal alignment, under a regime of strong local alignment. Based on the relative entropy method, a main difficulty in our analysis comes from the fact that the entropy of the limit system has no strict convexity in terms of density variable. To overcome this, we combine relative entropy quantities with the 2-Wasserstein distance.

1. Introduction

This article is mainly devoted to providing a rigorous justification of the hydrodynamic limit of the kinetic Cucker–Smale model to the pressureless Euler system with nonlocal alignment force. Cucker and Smale [2007] introduced an agent-based model capturing a flocking phenomenon observed within complex systems, such as flocks of birds, schools of fish and swarms of insects. The Cucker–Smale (CS) model has received extensive attention in the mathematical community, as well as physics, biology, engineering and social science, etc.; see for instance [Carlen et al. 2015; Cañizo et al. 2011; Carrillo et al. 2010; Duan et al. 2010; Fornasier et al. 2011; Ha et al. 2014c; 2017; Ha and Tadmor 2008; Poyato and Soler 2017; Zavlanos et al. 2011]. Motsch and Tadmor [2011] proposed a modified Cucker–Smale model by replacing the original CS alignment by a normalized nonsymmetric alignment. Karper, Mellet, and Trivisa [Karper et al. 2014] proposed a new kinetic flocking model as a combination of the CS alignment and a local alignment interaction, where the latter was obtained as a singular limit of the nonsymmetric alignment introduced by Motsch and Tadmor.

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We consider the kinetic flocking model without Brownian noise, proposed by Karper, Mellet and Trivisa [Karper et al. 2013] on $\mathbb{T}^d \times \mathbb{R}^d$:

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(L[f]f) + \nabla_v\cdot((u-v)f) = 0,\\ L[f](t,x,v) = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \psi(x-y) f(t,y,w)(w-v)\,dw\,dy,\\ u(t,x) = \frac{\int_{\mathbb{R}^d} vf\,dv}{\int_{\mathbb{R}^d} f\,dv}, \quad \|f(0)\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = 1. \end{aligned} \tag{1-1}$$

Here $\psi : \mathbb{T}^d \to \mathbb{R}^d$ is a Lipschitz communication weight that is positive and symmetric, i.e., $\psi(x-y) = \psi(y-x)$. The term $\nabla_v\cdot(L[f]f)$ describes a nonlocal alignment due to the original Cucker–Smale flocking mechanism, while the last term $\nabla_v\cdot((u-v)f)$ describes a local alignment interaction, because of the averaged local velocity u. The global existence of weak solutions to (1-1) was proved in [Karper et al. 2013]. The flocking behaviors of (1-1), however, have not been studied so far. We here provide its time-asymptotic behavior.

As a mesoscopic description, the kinetic model (1-1) is posed in $(t,x,v) \in \mathbb{R}\times\mathbb{T}^d\times\mathbb{R}^d$, i.e., in $2d+1$ dimensions. This feature provides an accurate description for a significant number of particles. However, its numerical test is very costly with respect to an associated macroscopic description. Hence, it is very important to find a suitable parameter regime on which the complexity of (1-1) is reduced.

The main goal of this article is to show a singular limit of (1-1) in a regime of strong local alignment:

$$\begin{aligned} \partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \nabla_v\cdot(L[f^\varepsilon]f^\varepsilon) + \frac{1}{\varepsilon}\nabla_v\cdot((u^\varepsilon-v)f^\varepsilon) = 0,\\ L[f^\varepsilon](t,x,v) = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \psi(x-y) f^\varepsilon(t,y,w)(w-v)\,dw\,dy,\\ u^\varepsilon = \frac{\int_{\mathbb{R}^d} vf^\varepsilon\,dv}{\int_{\mathbb{R}^d} f^\varepsilon\,dv},\\ f^\varepsilon|_{t=0} = f_0^\varepsilon, \quad \|f_0^\varepsilon\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = 1. \end{aligned} \tag{1-2}$$

As $\varepsilon \to 0$, it is expected that the solution f^ε of (1-2) converges, in some weak sense, to a monokinetic distribution

$$\delta_{v=u(t,x)} \otimes \rho(t,x); \tag{1-3}$$

see Remark 1.1. Here, $\delta_{v=u(t,x)}$ denotes a Dirac mass in v centered on $u(t,x)$. Also, as we shall explain later, at least formally ρ and u should solve the associated limit system given by the pressureless Euler system with nonlocal flocking dissipation:

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho u) = 0,\\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) = \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\,dy,\\ \rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad \|\rho_0\|_{L^1(\mathbb{T}^d)} = 1. \end{aligned} \tag{1-4}$$

The main difficulty in the justification of this limit comes from the singularity of the monokinetic distribution. To the best of our knowledge, there is no general method to handle the hydrodynamic limit from some kinetic equations to the pressureless Euler systems, no matter what regime is considered. Indeed, there are few results on this kinds of limit; see [Jabin and Rey 2017; Kang 2018; Kang and Vasseur 2015] (see also [Jabin 2000] for a general treatment of similar regimes that lead to the Dirac formation and pressureless gases equations).

Remark 1.1. In this paper we will use the symbol $\otimes$ in two different contexts: if μ is a measure on a complete metric space X, and $\{\nu_x\}_{x\in X}$ is a family of measures on a complete metric space Y, then $\nu_x\otimes\mu$ denotes the measure on $X\times Y$ defined as

$$\int_{X\times Y}\varphi\, d[\nu_x\otimes\mu]=\int_X\left(\int_Y\varphi(x,y)\,d\nu_x(y)\right)d\mu(x)\quad \text{for all } \varphi\in C_c(X\times Y).$$

When ν_x is independent of x (that is, $\nu_x=\nu$ for all x), we use the more standard notation $\mu\otimes\nu$ (instead of $\nu\otimes\mu$, as done before) to denote the product measure:

$$\int_{X\times Y}\varphi\, d[\mu\otimes\nu]=\int_X\left(\int_Y\varphi(x,y)\,d\nu(y)\right)d\mu(x)\quad \text{for all } \varphi\in C_c(X\times Y).$$

Finally, if $a,b\in\mathbb{R}^d$ are vectors, then $a\otimes b$ denotes the $(d\times d)$-matrix with entries

$$(a\otimes b)_{ij}=a_ib_j\quad \text{for all } i,j=1,\dots,d.$$

The meaning will always be clear from the context.

It is worth mentioning that the pressureless Euler system without the nonlocal alignment has been used for the formation of large-scale structures in astrophysics and the aggregation of sticky particles [Silk et al. 1983; Zeldovich 1970]. For more theoretical studies on the pressureless gases, we for example refer to [Bouchut 1994; Bouchut and James 1999; Boudin 2000; Brenier and Grenier 1998; Huang and Wang 2001; Poupaud and Rascle 1997; Weinan et al. 1996].

The macroscopic flocking model (1-4) or its variants have been formally derived under a monokinetic ansatz (1-3), and studied in various topics; see for example [Do et al. 2018; Ha et al. 2014a; 2014b; 2015; Tadmor and Tan 2014]. In [Ha et al. 2014b], the authors showed the global well-posedness of (1-4) with suitably smooth and small initial data, and the time-asymptotic flocking behavior. In [Ha et al. 2015], the authors dealt with a moving boundary problem of (1-4) with compactly supported initial density. We also refer to [Ha et al. 2014a] for a reformulation of (1-4) into hyperbolic conservation laws with damping in one dimension.

In [Karper et al. 2015], the authors showed the hydrodynamic limit of the kinetic flocking model (1-1) with Brownian motion, that is, a Vlasov–Fokker–Planck-type equation, under the regime of strong local alignment and strong Brownian motion:

$$\partial_t f^\varepsilon+v\cdot\nabla_x f^\varepsilon+\nabla_v\cdot(L[f^\varepsilon]f^\varepsilon)+\frac{1}{\varepsilon}\nabla_v\cdot((u^\varepsilon-v)f^\varepsilon)-\frac{1}{\varepsilon}\Delta_v f^\varepsilon=0. \tag{1-5}$$

In this case, as $\varepsilon\to 0$, f^ε converges to a smooth local equilibrium given by a local Maxwellian, contrary to (1-3). There, the authors used the relative entropy method, heavily relying on a strict convexity of the entropy of the isothermal Euler system (as a limit system of (1-5)):

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho = \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\,dy.$$

The relative entropy method based on a strict convex entropy has been successfully used to prove the hydrodynamic limit of Vlasov–Fokker–Planck-type equations; we refer to [Berthelin and Vasseur 2005; Carrillo et al. 2016; Goudon et al. 2004; Mellet and Vasseur 2008; Vasseur 2008].

On the other hand, the pressureless Euler system (1-4) has a convex entropy given by

$$\eta(\rho, \rho u) = \rho \tfrac{1}{2}(|u|^2), \tag{1-6}$$

which is not strictly convex with respect to ρ. For this reason, the associated relative entropy (1-6) is not enough to control the convergence of the nonlocal alignment term (compare with [Kang and Vasseur 2015], where the nonlocal alignment is not present). To overcome this difficulty, we first estimate an L^2-distance of characteristics generated by vector fields u^ε and u that controls the 2-Wasserstein distance of densities, and then combine the estimates of the relative entropy and the L^2-distance of characteristics.

As a related work on (1-5), we refer to [Carrillo et al. 2016], where the authors studied the flocking behavior and hydrodynamic limit of a coupled system of (1-5) and fluid equations via drag force.

The rest of this paper is organized as follows. In Section 2, we mention different scales of Cucker–Smale models from a microscopic level to a macroscopic level, and then specify some known existence results on the two descriptions (1-1) and (1-4). In Section 3, we present our main theorem on the hydrodynamic limit, and collect some useful results on the relative entropy method and the optimal transportation theory that are used in the proof of the main theorem. In Section 4, we present some structural hypotheses to guarantee the hydrodynamic limit in a general setting. Then we apply the general result to our systems by verifying the hypotheses in Section 5. In the Appendix, we provide the proof of the long time-asymptotic flocking dynamics and the existence of monokinetic solutions for the kinetic model (1-1).

2. Various scales of Cucker–Smale models

We first present various scales of Cucker–Smale models, from a microscopic level to a macroscopic level. Then we state some known results on global existence of weak solutions to the kinetic description (1-1), and local existence of smooth solutions to the limit system (1-4). Those results are crucially used in the proof of the main theorem. Finally, in Theorem 2.2, we present the time-asymptotic flocking behavior of the kinetic model (1-1).

Variants of Cucker–Smale models. We briefly present the kinetic CS model and its variants. Cucker and Smale [2007] proposed a mathematical model to explain the flocking phenomenon:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad i = 1, \dots, N, \\ \frac{dv_i}{dt} &= \frac{1}{N}\sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i), \end{aligned} \tag{2-1}$$

where $x_i, v_i \in \mathbb{R}^d$ denote the spatial position and velocity of the i-th particle for an ensemble of N self-propelled particles. The kernel $\psi(|x_j - x_i|)$ is a communication weight given by

$$\psi(x_j - x_i) = \frac{\lambda}{(1+|x_j - x_i|^2)^\beta}, \quad \beta \geq 0, \ \lambda > 0. \tag{2-2}$$

The system (2-1) with (2-2) was used as an analytical description of the Vicsek model [Vicsek et al. 1995] without resorting to the first principle of physics.

When the number of particles is sufficiently large, the ensemble of particles can be described by the one-particle density function $f = f(t, x, v)$ at the spatial-velocity position $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ at time t. Then, the evolution of f is governed by the following Vlasov-type equation:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = 0, \\ &L[f](t, x, v) = \int_{\mathbb{R}^{2d}} \psi(x-y) f(t, y, w)(w - v)\, dw\, dy. \end{aligned} \tag{2-3}$$

This was first introduced by Ha and Tadmor [2008] using the BBGKY hierarchy from the particle CS model (2-1). A rigorous mean-field limit was given in [Ha and Liu 2009].

Motsch and Tadmor [2011] recognized a drawback of the CS model (2-1), which is due to the normalization factor $1/N$. More precisely, when a small group of agents are located far away from a much larger group of agents, the internal dynamics of the small group is almost halted since the total number of agents is relatively very large. To solve this issue, they replaced the nonlocal alignment $L[f]$ by a normalized nonsymmetric alignment operator:

$$\bar{L}[f](t, x, v) := \frac{\int_{\mathbb{R}^{2d}} K^r(x-y) f(t, y, w)(w-v)\, dw\, dy}{\int_{\mathbb{R}^{2d}} K^r(x-y) f(t, y, w)\, dw\, dy},$$

where the kernel K^r is a communication weight and r denotes the radius of influence of K^r.

In [Karper et al. 2014], the authors considered the case when the communication weight is extremely concentrated near each agent, so that the alignment term $\bar{L}[f]$ corresponds to a short-range interaction. More precisely, they rigorously justified the singular limit $r \to 0$, i.e., as K^r converges to the Dirac distribution δ_0, in which case $\bar{L}[f]$ converges to a local alignment term:

$$\bar{L}[f](t, x, v) \to \frac{\int_{\mathbb{R}^d} f(t, x, w)(w-v)\, dw}{\int_{\mathbb{R}^d} f(t, x, w)\, dw} = u(t, x) - v,$$

where $u(t, x)$ denotes the averaged local velocity defined as

$$u(t, x) = \frac{\int_{\mathbb{R}^d} v f(t, x, v)\, dv}{\int_{\mathbb{R}^d} f(t, x, v)\, dv}.$$

Hence, their new model became (1-1), which consists of two kinds of alignment force: a nonlocal alignment due to the original CS model, plus a local alignment.

***Existence of weak solutions to* (1-2).** In [Karper et al. 2013], the authors showed the existence of weak solutions to the kinetic Cucker–Smale model with local alignment, noise, self-propulsion, and friction:

$$\begin{aligned}\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(L[f]f) + \nabla_v\cdot((u-v)f) &= \sigma\Delta_v f - \nabla_v\cdot((a-b|v|^2)vf),\\ L[f] &= \int_{\mathbb{R}^{2d}} \psi(x-y) f(t,y,w)(w-v)\,dw\,dy,\end{aligned} \tag{2-4}$$

where the kernel ψ is the same as (1-2) and a, b, and σ are nonnegative constants. By their result applied with $a=b=\sigma=0$ inside the periodic domain $\mathbb{T}^d$, we obtain existence of solutions for (1-2). To precisely state such an existence result, we need to define a (mathematical) entropy $\mathcal{F}(f^\varepsilon)$ and kinetic dissipations $\mathcal{D}_1(f^\varepsilon)$, $\mathcal{D}_2(f^\varepsilon)$ for (1-2):

$$\begin{aligned}\mathcal{F}(f^\varepsilon) &:= \int_{\mathbb{R}^d} \frac{|v|^2}{2} f^\varepsilon\,dv,\\ \mathcal{D}_1(f^\varepsilon) &:= \int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon |u^\varepsilon - v|^2\,dv\,dx,\\ \mathcal{D}_2(f^\varepsilon) &:= \frac{1}{2}\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(x,v) f^\varepsilon(y,w)|v-w|^2\,dx\,dy\,dv\,dw.\end{aligned} \tag{2-5}$$

Proposition 2.1. *For any $\varepsilon>0$, assume that f_0^ε satisfies*

$$f_0^\varepsilon \ge 0, \qquad f_0^\varepsilon \in L^1\cap L^\infty(\mathbb{R}^{2d}), \qquad |v|^2 f_0^\varepsilon \in L^1(\mathbb{R}^{2d}). \tag{2-6}$$

Then there exists a weak solution $f^\varepsilon \ge 0$ of (1-2) such that

$$\begin{aligned}f^\varepsilon &\in C(0,T;L^1(\mathbb{R}^{2d}))\cap L^\infty((0,T)\times\mathbb{R}^{2d}),\\ |v|^2 f^\varepsilon &\in L^\infty(0,T;L^1(\mathbb{R}^{2d})),\end{aligned} \tag{2-7}$$

and (1-2) holds in the sense of distributions, that is, for any $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^{2d})$, the weak formulation holds:

$$\int_0^t\int_{\mathbb{R}^{2d}} f^\varepsilon\Big(\partial_t\varphi + v\cdot\nabla_x\varphi + L[f^\varepsilon]\cdot\nabla_v\varphi + \frac{1}{\varepsilon}(u^\varepsilon - v)\cdot\nabla_v\varphi\Big)\,dv\,dx\,ds + \int_{\mathbb{R}^{2d}} f_0^\varepsilon\varphi(0,\cdot)\,dv\,dx = 0. \tag{2-8}$$

Moreover, f^ε preserves the total mass and satisfies the entropy inequality

$$\int_{\mathbb{T}^d}\mathcal{F}(f^\varepsilon)(t)\,dx + \frac{1}{\varepsilon}\int_0^t \mathcal{D}_1(f^\varepsilon)(s)\,ds + \int_0^t\mathcal{D}_2(f^\varepsilon)(s)\,ds \le \int_{\mathbb{T}^d}\mathcal{F}(f_0^\varepsilon)\,dx. \tag{2-9}$$

The entropy inequality (2-9) is crucially used in the proof of Theorem 3.1.

***Flocking behavior of the kinetic model* (1-1).** We now present the time-asymptotic flocking behavior of solutions to the kinetic model (1-1). For that, we define the following two Lyapunov functionals:

$$\begin{aligned}\mathcal{E}_1(t) &:= \int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)|u(t,x)-v|^2\,dv\,dx,\\ \mathcal{E}_2(t) &:= \int_{\mathbb{T}^{2d}} \rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy,\end{aligned}$$

where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$. We remark that $\mathcal{E}_1$ measures a local alignment, and $\mathcal{E}_2$ measures alignment of the averaged local velocities. Then, for the flocking estimate, we combine the two functionals as follows:

$$\mathcal{E}(t) := \mathcal{E}_1(t) + \tfrac{1}{2}\mathcal{E}_2(t). \tag{2-10}$$

Theorem 2.2. *Let f be a solution to (1-1). Then, we have the time-asymptotic flocking estimate*

$$\mathcal{E}(t) \le \mathcal{E}(0)\exp(-2\min\{1,\psi_m\}t), \qquad t > 0, \tag{2-11}$$

where ψ_m is the minimum communication weight:

$$\psi_m := \min_{x,y\in\mathbb{T}^d} \psi(x-y) > 0.$$

In addition, if u is uniformly Lipschitz continuous on a time interval $[0,T]$, namely

$$\ell_T := \sup_{t\in[0,T]} \|\nabla_x u\|_{L^\infty(\mathbb{T}^d)} < \infty,$$

then

$$\mathcal{E}_1(t) \le \mathcal{E}_1(0)e^{2(\ell_T-1)} \quad \textit{for all } t\in[0,T]. \tag{2-12}$$

Proof. We postpone the proof to the Appendix. □

Remark 2.3. As an interesting consequence of (2-12) one obtains that, for smooth solutions, $\mathcal{E}_1(0) = 0$ implies that $\mathcal{E}_1(t) = 0$ for all $t\in[0,T]$. In other words, monokinetic initial conditions remain monokinetic as long as the velocity field is Lipschitz. One can note that monokinetic solutions to (1-1) simply correspond to solutions of the pressureless Euler system (1-4); hence the short time existence of Lipschitz solutions is guaranteed by Proposition 2.4 and Remark 2.5 below.

***Formal derivation of the hydrodynamic Cucker–Smale system* (1-4).** We consider the hydrodynamic variables $\rho^\varepsilon := \int_{\mathbb{R}^d} f^\varepsilon\,dv$ and $\rho^\varepsilon u^\varepsilon := \int_{\mathbb{R}^d} v f^\varepsilon\,dv$.

First of all, integrating (1-2) with respect to v, we get the continuity equation

$$\partial_t\rho^\varepsilon + \nabla_x\cdot(\rho^\varepsilon u^\varepsilon) = 0.$$

Multiplying (1-2) by v, and then integrating it with respect to v, we have

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \nabla_x\cdot\left(\int_{\mathbb{R}^d} v\otimes v f^\varepsilon\,dv\right) = \int_{\mathbb{T}^d}\psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)(u^\varepsilon(t,y)-u^\varepsilon(t,x))\,dy,$$

where we used

$$u^\varepsilon = \frac{\int_{\mathbb{R}^d} v f^\varepsilon\,dv}{\int_{\mathbb{R}^d} f^\varepsilon\,dv}.$$

Then, we rewrite the system for ρ^ε and u^ε as

$$\begin{aligned}\partial_t\rho^\varepsilon + \nabla_x\cdot(\rho^\varepsilon u^\varepsilon) &= 0,\\ \partial_t(\rho^\varepsilon u^\varepsilon) + \nabla_x\cdot(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon + P^\varepsilon) &= \int_{\mathbb{T}^d}\psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)(u^\varepsilon(t,y)-u^\varepsilon(t,x))\,dy,\end{aligned} \tag{2-13}$$

where P^ε is the stress tensor given by

$$P^\varepsilon := \int_{\mathbb{R}^d} (v-u^\varepsilon)\otimes(v-u^\varepsilon) f^\varepsilon \, dv.$$

If we take $\varepsilon \to 0$ in (1-2), the local alignment term $\nabla_v \cdot ((u^\varepsilon - v) f^\varepsilon)$ converges to 0. Hence, if $\rho^\varepsilon \to \rho$ and $\rho^\varepsilon u^\varepsilon \to \rho u$ for some limiting functions ρ and u, we have that $f^\varepsilon \to \delta_{v=u} \otimes \rho$ (in some suitable sense). Hence, the stress tensor P^ε should vanish in the limit, since

$$\int_{\mathbb{R}^d} (v-u)\otimes(v-u)\delta_{v=u}\rho \, dv = 0.$$

Therefore, at least formally, the limit quantities ρ and u satisfy the pressureless Euler system with nonlocal alignment:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) &= \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\, dy. \end{aligned}$$

Existence of classical solutions to (1-4). We present here the local existence of classical solutions to the pressureless Euler system (1-4).

Proposition 2.4. *Assume that*

$$\rho_0 > 0 \quad \text{in } \mathbb{T}^d \qquad \text{and} \qquad (\rho_0, u_0) \in H^s(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d) \quad \text{for } s > \tfrac{1}{2}d + 1. \tag{2-14}$$

Then, there exists $T_* > 0$ such that (1-4) has a unique classical solution (ρ, u) satisfying

$$\begin{aligned} &\rho \in C^0([0,T_*]; H^s(\mathbb{T}^d)) \cap C^1([0,T_*]; H^{s-1}(\mathbb{T}^d)), \\ &u \in C^0([0,T_*]; H^{s+1}(\mathbb{T}^d)) \cap C^1([0,T_*]; H^s(\mathbb{T}^d)). \end{aligned} \tag{2-15}$$

Remark 2.5. Since $s > \frac{1}{2}d + 1$, by the Sobolev inequality it follows that $(\rho, u) \in C^1([0,T_*] \times \mathbb{T}^d)$.

Proposition 2.4 has been proven in [Ha et al. 2014b]. There, the authors obtained also a global well-posedness of classical solutions, provided an initial datum is suitably smooth and small.

3. Main result and preliminaries

We first present our main result on the hydrodynamic limit of (1-2). We next present useful results on the relative entropy method and the optimal transportation theory, which are used as main tools in the next section.

Main result. For the hydrodynamic limit, we consider a well-prepared initial data f_0^ε satisfying (2-6) and

($\mathcal{A}1$) $\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} (f_0^\varepsilon \frac{1}{2}|v|^2 - \rho_0 \frac{1}{2}|u_0|^2)\, dv\, dx = \mathcal{O}(\varepsilon)$,

($\mathcal{A}2$) $\|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{T}^d)} = \mathcal{O}(\varepsilon)$,

($\mathcal{A}3$) $\|u_0^\varepsilon - u_0\|_{L^\infty(\mathbb{T}^d)} = \mathcal{O}(\varepsilon)$.

We now specify our main result on the hydrodynamic limit.

Theorem 3.1. *Assume that the initial data f_0^ε and (ρ_0, u_0) satisfy (2-6), (2-14), and $(\mathcal{A}1)$–$(\mathcal{A}3)$. Let f^ε be a weak solution to (1-2) satisfying (2-9), and (ρ, u) be a local-in-time smooth solution to (1-4) satisfying (2-15) up to the time T_*. Then, there exists a positive constant C_* (depending on T_*) such that, for all $t \le T_*$,*

$$\int_{\mathbb{T}^d} \rho^\varepsilon(t)|(u^\varepsilon - u)|^2(t)\,dx + W_2^2(\rho^\varepsilon(t), \rho(t)) \le C_*\varepsilon, \tag{3-1}$$

where $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon\,dv$, $\rho^\varepsilon u^\varepsilon = \int_{\mathbb{R}^d} v f^\varepsilon\,dv$, and W_2 denotes the 2-Wasserstein distance.

Therefore, we have

$$f^\varepsilon \rightharpoonup \delta_{v=u(t,x)} \otimes \rho(t,x) \quad \text{in } \mathcal{M}((0,T_*)\times\mathbb{T}^d\times\mathbb{R}^d), \tag{3-2}$$

where $\mathcal{M}((0,T_*)\times\mathbb{T}^d\times\mathbb{R}^d)$ is the space of nonnegative Radon measures on $(0,T_*)\times\mathbb{T}^d\times\mathbb{R}^d$.

The proof of this result is postponed to Section 5. In the next subsections we collect some preliminary facts that will be used later in the proof.

Relative entropy method. First of all, we rewrite the limit system (1-4) in an abstract form using the notation

$$P = \rho u, \quad U = \begin{pmatrix} \rho \\ P \end{pmatrix}, \quad A(U) = \begin{pmatrix} P^T \\ (P\otimes P)/\rho \end{pmatrix},$$

$$F(U) = \begin{pmatrix} 0 \\ \int_{\mathbb{T}^d} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\,dy \end{pmatrix}.$$

Then we can rewrite (1-4) as the balance law

$$\partial_t U + \operatorname{div}_x A(U) = F(U). \tag{3-3}$$

We consider the relative entropy and relative flux

$$\begin{aligned} \eta(V \mid U) &= \eta(V) - \eta(U) - D\eta(U)\cdot(V-U), \\ A(V \mid U) &= A(V) - A(U) - DA(U)\cdot(V-U), \end{aligned} \tag{3-4}$$

where $DA(U)\cdot(V-U)$ is a matrix defined as

$$(DA(U)\cdot(V-U))_{ij} = \sum_{k=1}^{d+1} \partial_{U_k} A_{ij}(U)(V_k - U_k), \quad 1 \le i \le d+1,\ 1 \le j \le d.$$

By the theory of conservation laws, the system (3-3) has a convex entropy $\eta(U) = \rho\,\frac{1}{2}|u|^2$ with entropy flux G given by the identity

$$\partial_{U_i} G_j(U) = \sum_{k=1}^{d+1} \partial_{U_k}\eta(U)\,\partial_{U_i} A_{kj}(U), \quad 1 \le i \le d+1,\ 1 \le j \le d.$$

Since $\eta(U) = |P|^2/(2\rho)$, and

$$D\eta(U) = \begin{pmatrix} D_\rho \eta \\ D_P \eta \end{pmatrix} = \begin{pmatrix} -|P|^2/(2\rho^2) \\ P/\rho \end{pmatrix} = \begin{pmatrix} -|u|^2/2 \\ u \end{pmatrix}, \tag{3-5}$$

for given $V = \binom{q}{qw}$, $U = \binom{\rho}{\rho u}$, we have

$$\begin{aligned}\eta(V \mid U) &= \tfrac{1}{2}q|w|^2 - \tfrac{1}{2}\rho|u|^2 + \tfrac{1}{2}|u|^2(q-\rho) - u(qw - \rho u) \\ &= \tfrac{1}{2}q|u-w|^2.\end{aligned} \tag{3-6}$$

The next proposition provides a cornerstone to verify the hydrodynamic limit through the relative entropy method. For its proof, we refer to the proof of Proposition 4.2 in [Karper et al. 2015]; see also [Vasseur 2008].

Proposition 3.2. *Let U be a strong solution to balance law (3-3) and V be any smooth function. Then, the following holds*:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} \eta(V \mid U)\,dx = \frac{d}{dt}\int_{\mathbb{T}^d} \eta(V)\,dx &- \int_{\mathbb{T}^d} \nabla_x\big(D\eta(U)\big) : A(V \mid U)\,dx \\ &- \int_{\mathbb{T}^d} D\eta(U)\cdot[\partial_t V + \mathrm{div}_x\, A(V) - F(V)]\,dx \\ &- \int_{\mathbb{T}^d} [D^2\eta(U)F(U)(V-U) + D\eta(U)F(V)]\,dx.\end{aligned}$$

Wasserstein distance and representation formulae for solutions of the continuity equation. For $p \geq 1$, the p-Wasserstein distance between two probability measures μ_1 and μ_2 on $\mathbb{R}^d$ is defined by

$$W_p^p(\mu_1, \mu_2) := \inf_{\nu\in\Lambda(\mu_1,\mu_2)} \int_{\mathbb{R}^{2d}} |x-y|^2\,d\nu(x,y),$$

where $\Lambda(\mu_1, \mu_2)$ denotes the set of all probability measures ν on $\mathbb{R}^{2d}$ with marginals μ_1 and μ_2, i.e,

$$\pi_{1\#}\nu = \mu_1, \qquad \pi_{2\#}\nu = \mu_2,$$

where $\pi_1 : (x, y) \mapsto x$ and $\pi_2 : (x, y) \mapsto y$ are the natural projections from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$, and $\pi_{\#}\nu$ denotes the push forward of ν through a map π, i.e., $\pi_{\#}\nu(B) := \nu(\pi^{-1}(B))$ for any Borel set B. This same definition can be extended to measures on the torus $\mathbb{T}^d$ with the understanding that $|x-y|$ denotes the distance on the torus.

To make a connection between the L^2-distance of velocities and the 2-Wasserstein distance of densities (see Lemma 5.2), we will use two different representation formulas for solutions to the continuity equation

$$\partial_t\mu_t + \mathrm{div}_x(u_t\mu_t) = 0. \tag{3-7}$$

Let us recall that, if the velocity field $u_t : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz with respect to x, uniformly in t, then for any x there exists a global-in-time unique characteristic X generated by u_t starting from x,

$$\dot{X}(t,x) = u_t(X(t,x)), \qquad X(0,x) = x,$$

and the solution μ_t of (3-7) is the push forward of the initial data μ_0 through $X(t)$, i.e.,

$$\mu_t = X(t)_{\#}\mu_0; \tag{3-8}$$

e.g., see [Ambrosio et al. 2005, Proposition 8.1.8]. On the other hand, if the velocity field u_t is not Lipschitz with respect to x, the uniqueness of the characteristics is not guaranteed anymore. Still, a probabilistic representation formula for solutions to (3-7) holds (recall that a curve of probability measures in $\mathbb{R}^d$ is called narrowly continuous if it is continuous in the duality with continuous bounded functions):

Proposition 3.3. *For a given $T > 0$, let $\mu_t : [0, T] \to \mathcal{P}(\mathbb{R}^d)$ be a narrowly continuous solution of (3-7) for a Borel vector field u_t satisfying*

$$\int_0^T \int_{\mathbb{R}^d} |u_t(x)|^p \, d\mu_t(x) \, dt < \infty \quad \text{for some } p > 1.$$

Let Γ_T denote the space of continuous curves from $[0, T]$ into $\mathbb{R}^d$. Then, there exists a probability measure η on $\Gamma_T \times \mathbb{R}^d$ satisfying the following properties:

(i) *η is concentrated on the set of pairs (γ, x) such that γ is an absolutely continuous curve solving the ODE*

$$\dot{\gamma}(t) = u_t(\gamma(t)) \quad \text{for a.e. } t \in (0, T), \text{ with } \gamma(0) = x.$$

(ii) *μ_t satisfies*

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu_t(x) = \int_{\Gamma_T \times \mathbb{R}^d} \varphi(\gamma(t)) \, d\eta(\gamma, x) \quad \text{for all } \varphi \in C_b^0(\mathbb{R}^d), \ t \in [0, T].$$

Again, this result readily extends on the torus.

Note that, in the case when u_t is Lipschitz, there exists a unique curve γ solving the ODE and starting from x (i.e., $\gamma = X(\,\cdot\,, x)$), so the measure η is given by the formula

$$d\eta(\gamma, x) = \delta_{\gamma = X(\,\cdot\,, x)} \otimes d\mu_0(x).$$

We refer to [Ambrosio et al. 2005, Theorem 8.2.1] for more details and a proof.

Useful inequality. We here present a standard inequality that is used in the proof of Lemma 5.2, for the convenience of the reader:

Lemma 3.4. *Let $\rho_1, \rho_2 : \mathbb{T}^d \to \mathbb{R}$ be two probability densities. Then*

$$W_2^2(\rho_1, \rho_2) \le \tfrac{1}{8} d \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)}.$$

Proof. The idea is simple: to estimate the transportation cost from ρ_1 to ρ_2 it suffices to consider a transport plan that keeps at rest all the mass in common between ρ_1 and ρ_2 (namely $\min\{\rho_1, \rho_2\}$) and sends $\rho_1 - \min\{\rho_1, \rho_2\}$ onto $\rho_2 - \min\{\rho_1, \rho_2\}$ in an arbitrary way. For instance, assuming without loss of generality that $\rho_1 \ne \rho_2$ (otherwise the result is trivial), we set

$$m := \|\rho_1 - \min\{\rho_1, \rho_2\}\|_{L^1(\mathbb{T}^d)} = \|\rho_2 - \min\{\rho_1, \rho_2\}\|_{L^1(\mathbb{T}^d)} = \tfrac{1}{2} \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} > 0.$$

Then, a possible choice of transport plan between ρ_1 and ρ_2 is given by

$$\pi(dx, dy) := \delta_{x=y}(dy) \otimes \min\{\rho_1(x), \rho_2(x)\} dx \\ + \frac{1}{m} [\rho_1(x) - \min\{\rho_1(x), \rho_2(x)\}] dx \otimes [\rho_2(y) - \min\{\rho_1(y), \rho_2(y)\}] dy.$$

Since the diameter of $\mathbb{T}^d$ is bounded by $\frac{1}{2}\sqrt{d}$, we deduce that the W_2^2-cost to transport $\rho_1 - \min\{\rho_1, \rho_2\}$ onto $\rho_2 - \min\{\rho_1, \rho_2\}$ is at most

$$\int_{\mathbb{T}^{2d}} |x-y|^2 \, d\pi(x,y) = \frac{1}{m} \int_{\mathbb{T}^{2d}} |x-y|^2 (\rho_1(x) - \min\{\rho_1(x), \rho_2(x)\})(\rho_2(y) - \min\{\rho_1(y), \rho_2(y)\}) \, dx \, dy$$
$$\leq \tfrac{1}{4} d \|\rho_1 - \min\{\rho_1, \rho_2\}\|_{L^1(\mathbb{T}^d)} = \tfrac{1}{8} d \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)},$$

as desired. □

4. Structural lemma

In a general system, we first present some structural hypotheses to provide a Gronwall-type inequality on the relative entropy that is also controlled by 2-Wasserstein distance.

Hypotheses. Let f^ε be a solution to a given kinetic equation KE_ε scaled with $\varepsilon > 0$ corresponding to initial data f_0^ε. Let U^ε and U_0^ε consist of hydrodynamic variables of f^ε and f_0^ε respectively.

Let U be a solution to a balance law (as a limit system of KE_ε):

$$\partial_t U + \mathrm{div}_x \, A(U) = F(U), \quad U|_{t=0} = U_0.$$

($\mathcal{H}1$) The kinetic equation KE_ε has a kinetic entropy $\mathcal{F}$ such that $\int \mathcal{F}(f^\varepsilon)(t) \, dx \geq 0$ and

$$\int \mathcal{F}(f^\varepsilon)(t) \, dx + \frac{1}{\varepsilon} \int_0^t D_1(f^\varepsilon)(s) \, ds + \int_0^t D_2(f^\varepsilon)(s) \, ds \leq \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon) \, dx,$$

where $D_1, D_2 \geq 0$ are some dissipations.

($\mathcal{H}2$) There exists a constant $C > 0$ (independent of ε) such that

$$\int \eta(U_0^\varepsilon \mid U_0) \, dx \leq C\varepsilon, \quad \int (\mathcal{F}(f_0^\varepsilon) - \eta(U_0^\varepsilon)) \, dx \leq C\varepsilon, \quad \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon) \, dx \leq C.$$

($\mathcal{H}3$) The balance law has a convex entropy η, and the following minimization property holds:

$$\eta(U^\varepsilon) \leq \mathcal{F}(f^\varepsilon).$$

($\mathcal{H}4$) There exists a constant $C > 0$ (independent of ε) such that

$$\left| \int \nabla_x (D\eta(U)) : A(U^\varepsilon \mid U) \, dx \right| \leq C \int \eta(U^\varepsilon \mid U) \, dx.$$

($\mathcal{H}5$) There exists a constant $C > 0$ (independent of ε) such that

$$\left| \int D\eta(U) \cdot [\partial_t U^\varepsilon + \mathrm{div}_x \, A(U^\varepsilon) - F(U^\varepsilon)] \, dx \right| \leq C D_1(f^\varepsilon).$$

($\mathcal{H}6$) Let ρ^ε be the hydrodynamic variable of f^ε as the local mass, and ρ be the corresponding variable for the balance law. Then,

$$-\int [D^2\eta(U) F(U)(U^\varepsilon - U) + D\eta(U) F(U^\varepsilon)] \, dx \leq D_2(f^\varepsilon) + C W_2^2(\rho^\varepsilon, \rho) + C \int \eta(U^\varepsilon \mid U) \, dx.$$

($\mathcal{H}7$) There exists a constant $C > 0$ (independent of ε) such that

$$W_2^2(\rho^\varepsilon, \rho)(t) \le C \int_0^t \int \eta(U^\varepsilon \,|\, U)\, dx\, ds + C\varepsilon.$$

Remark 4.1. (1) The hypotheses ($\mathcal{H}1$)–($\mathcal{H}5$) provide a basic structure in applying the relative entropy method to hydrodynamic limits as in previous results, for example, [Kang and Vasseur 2015; Karper et al. 2015; Mellet and Vasseur 2008]. On the other hand, the hypotheses ($\mathcal{H}6$)–($\mathcal{H}7$) provide a crucial connection between the relative entropy and Wasserstein distance.

(2) The (kinetic) entropy inequality ($\mathcal{H}1$) plays an important role in controlling the dissipations D_1, D_2 in ($\mathcal{H}5$) and ($\mathcal{H}6$).

(3) ($\mathcal{H}2$) is related to a kind of well-prepared initial data.

Lemma 4.2. *Assume the hypotheses* ($\mathcal{H}1$)–($\mathcal{H}7$). *Then, for a given* $T > 0$, *there exists a constant* $C > 0$ *such that*

$$\int \eta(U^\varepsilon \,|\, U)(t)\, dx + W_2^2(\rho^\varepsilon, \rho)(t) \le C\varepsilon, \quad t \le T.$$

Proof. First of all, using Proposition 3.2, we have

$$\int_{\mathbb{T}^d} \eta(U^\varepsilon \,|\, U)(t)\, dx \le I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned}
I_1 &:= \int_{\mathbb{T}^d} \eta(U_0^\varepsilon \,|\, U_0)\, dx,\\
I_2 &:= \int_{\mathbb{T}^d} (\eta(U^\varepsilon)(t) - \eta(U_0^\varepsilon))\, dx,\\
I_3 &:= -\int_0^t \int_{\mathbb{T}^d} \nabla_x (D\eta(U)) : A(U^\varepsilon \,|\, U)\, dx\, ds,\\
I_4 &:= -\int_0^t \int_{\mathbb{T}^d} D\eta(U) \cdot [\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon)]\, dx\, ds,\\
I_5 &:= -\int_0^t \int_{\mathbb{T}^d} [D^2\eta(U) F(U)(U^\varepsilon - U) + D\eta(U) F(U^\varepsilon)]\, dx\, ds.
\end{aligned}$$

It follows from ($\mathcal{H}2$) that $I_1 \le C\varepsilon$.

We decompose I_2 as

$$I_2 = \underbrace{\int_{\mathbb{T}^d} (\eta(U^\varepsilon)(t) - \mathcal{F}(f^\varepsilon)(t))\, dx}_{=: I_2^1} + \underbrace{\int_{\mathbb{T}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon))\, dx}_{=: I_2^2} + \underbrace{\int_{\mathbb{T}^d} (\mathcal{F}(f_0^\varepsilon) - \eta(U_0^\varepsilon))\, dx}_{=: I_2^3}. \tag{4-1}$$

First, $I_2^1 \le 0$ by ($\mathcal{H}3$).

Since ($\mathcal{H}1$) yields

$$I_2^2 \le -\int_0^t D_2(f^\varepsilon)\, ds,$$

it follows from ($\mathcal{H}6$) that

$$I_2^2 + I_5 \le C\int_0^t W_2^2(\rho^\varepsilon, \rho)\,ds + C\int_0^t\int_{\mathbb{T}^d} \eta(U^\varepsilon \,|\, U)\,dx\,ds.$$

By ($\mathcal{H}2$), $I_2^3 \le C\varepsilon$.

It follows from ($\mathcal{H}4$) that

$$I_3 \le C\int_0^t\int_{\mathbb{T}^d} \eta(U^\varepsilon \,|\, U)\,dx\,ds.$$

Since ($\mathcal{H}1$) and ($\mathcal{H}2$) imply

$$\int_0^t D_1(f^\varepsilon)(s)\,ds \le C\varepsilon,$$

we have $I_4 \le C\varepsilon$.

Therefore, we have

$$\int \eta(U^\varepsilon \,|\, U)(t)\,dx \le C\varepsilon + C\int_0^t \left[\int \eta(U^\varepsilon \,|\, U)(s)\,dx\,ds + W_2^2(\rho^\varepsilon, \rho)\right] ds.$$

Hence, combining it with ($\mathcal{H}7$), and using Gronwall's inequality, we have the desired result. □

5. Proof of Theorem 3.1

The main part of the proof consists in proving the estimate (3-1).

***Proof of* (3-1).** This will be done by verifying the hypotheses ($\mathcal{H}1$)–($\mathcal{H}7$), and then completed by Lemma 4.2.

Verification of ($\mathcal{H}1$). ($\mathcal{H}1$) is satisfied thanks to Lemma 5.1 below. There we show that one can replace the nonlocal dissipation $\mathcal{D}_2$ in the kinetic entropy inequality (2-9) by another dissipation $\widetilde{\mathcal{D}}_2$ defined in terms of the hydrodynamic variables ρ^ε and u^ε.

Lemma 5.1. *For any $\varepsilon > 0$, assume that f_0^ε satisfies*

$$f_0^\varepsilon \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d), \quad |v|^2 f_0^\varepsilon \in L^1(\mathbb{T}^d \times \mathbb{R}^d).$$

Then the weak solution f^ε in Proposition 2.1 also satisfies

$$\int_{\mathbb{T}^d} \mathcal{F}(f^\varepsilon)(t)\,dx + \frac{1}{\varepsilon}\int_0^t \mathcal{D}_1(f^\varepsilon)(s)\,ds + \int_0^t \widetilde{\mathcal{D}}_2(f^\varepsilon)(s)\,ds \le \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx, \tag{5-1}$$

where $\mathcal{F}$ and $\mathcal{D}_1$ are as in (2-5), *and*

$$\widetilde{\mathcal{D}}_2(f^\varepsilon) := \frac{1}{2}\int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)|u^\varepsilon(t,x) - u^\varepsilon(t,y)|^2\,dx\,dy.$$

Proof. Recalling (2-9), it is enough to show $\widetilde{\mathcal{D}}_2(f^\varepsilon) \le \mathcal{D}_2(f^\varepsilon)$. We first rewrite $\widetilde{\mathcal{D}}_2(f^\varepsilon)$ in terms of the mesoscopic variables as follows: using $\psi(x-y) = \psi(y-x)$, we have

$$\begin{aligned}\widetilde{\mathcal{D}}_2(f^\varepsilon) &= \frac{1}{2}\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot(u^\varepsilon(t,x)-u^\varepsilon(t,y))\,dv\,dw\,dx\,dy \\ &= \int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot u^\varepsilon(t,x)\,dv\,dw\,dx\,dy \\ &= \underbrace{\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot v\,dv\,dw\,dx\,dy}_{=:\mathcal{I}_1} \\ &\qquad + \underbrace{\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)(v-w)\cdot(u^\varepsilon(t,x)-v)\,dv\,dw\,dx\,dy}_{=:\mathcal{I}_2}.\end{aligned}$$

First, we have

$$\mathcal{I}_1 = \frac{1}{2}\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)|v-w|^2\,dx\,dy\,dv\,dw = \mathcal{D}_2(f^\varepsilon).$$

We next claim $\mathcal{I}_2 \le 0$.

Indeed, since

$$\rho^\varepsilon|u^\varepsilon|^2 = \frac{\left(\int_{\mathbb{R}^d} v f^\varepsilon\,dv\right)^2}{\int_{\mathbb{R}^d} f^\varepsilon\,dv} \le \int_{\mathbb{R}^d} |v|^2 f^\varepsilon\,dv, \tag{5-2}$$

we have

$$\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w)|v|^2\,dv\,dw\,dx\,dy \ge \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,y)\rho^\varepsilon(t,x)|u^\varepsilon(t,x)|^2\,dx\,dy.$$

Then, since

$$\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w) u^\varepsilon(t,x)\cdot w\,dv\,dw\,dx\,dy = \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y) u^\varepsilon(t,x)\cdot u^\varepsilon(t,y)\,dx\,dy,$$

$$\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w) u^\varepsilon(t,x)\cdot v\,dv\,dw\,dx\,dy = \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)|u^\varepsilon(t,x)|^2\,dx\,dy,$$

$$\int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y) f^\varepsilon(t,x,v) f^\varepsilon(t,y,w) v\cdot w\,dv\,dw\,dx\,dy = \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y) u^\varepsilon(t,x)\cdot u^\varepsilon(t,y)\,dx\,dy,$$

we conclude that $\mathcal{I}_2 \le 0$, as desired. □

Verification of $(\mathcal{H}2)$. We show that the assumptions $(\mathcal{A}1)$–$(\mathcal{A}3)$ for initial data imply $(\mathcal{H}2)$. Using (3-6) and assumption $(\mathcal{A}3)$, we have

$$\int_{\mathbb{T}^d} \eta(U_0^\varepsilon \mid U_0)\,dx = \frac{1}{2}\int_{\mathbb{T}^d} \rho_0^\varepsilon |u_0^\varepsilon - u_0|^2\,dx \le C\varepsilon^2 \int_{\mathbb{T}^d} \rho_0^\varepsilon\,dx \le C\varepsilon^2.$$

Since it follows from $(\mathcal{A}1)$–$(\mathcal{A}3)$ that

$$\int_{\mathbb{T}^d} (\mathcal{F}(f_0^\varepsilon) - \eta(U_0))\,dx = \mathcal{O}(\varepsilon),$$

and

$$\begin{aligned}\int_{\mathbb{T}^d} (\eta(U_0) - \eta(U_0^\varepsilon))\,dx &= \frac{1}{2}\int_{\mathbb{R}^d} (\rho_0 |u_0|^2 - \rho_0^\varepsilon |u_0^\varepsilon|^2) \\ &\le \frac{1}{2}\int_{\mathbb{T}^d} |\rho_0 - \rho_0^\varepsilon|\,|u_0|^2 + \frac{1}{2}\int_{\mathbb{T}^d} \rho_0^\varepsilon \big| |u_0^\varepsilon|^2 - |u_0|^2 \big| = \mathcal{O}(\varepsilon),\end{aligned}$$

we have

$$\int_{\mathbb{T}^d} (\mathcal{F}(f_0^\varepsilon) - \eta(U_0^\varepsilon))\,dx = \mathcal{O}(\varepsilon).$$

It is obvious that $(\mathcal{A}1)$ implies

$$\int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx \le C.$$

Verification of $(\mathcal{H}3)$. It follows from (5-2) that

$$\eta(U^\varepsilon) = \rho^\varepsilon \tfrac{1}{2}|u^\varepsilon|^2 \le \int_{\mathbb{R}^d} \tfrac{1}{2}|v|^2 f^\varepsilon\,dv = \mathcal{F}(f^\varepsilon). \tag{5-3}$$

Verification of $(\mathcal{H}4)$. Since

$$A(U) = \begin{pmatrix} P^T \\ (P\otimes P)/\rho \end{pmatrix},$$

we have

$$\begin{aligned} DA(U)\cdot(U^\varepsilon - U) &= D_\rho A(U)(\rho^\varepsilon - \rho) + D_{P_i} A(U)(P_i^\varepsilon - P_i) \\ &= \begin{pmatrix} (P^\varepsilon - P)^T \\ -((\rho^\varepsilon - \rho)/\rho^2) P\otimes P + (1/\rho) P \otimes (P^\varepsilon - P) + (1/\rho)(P^\varepsilon - P)\otimes P \end{pmatrix}, \end{aligned}$$

which yields

$$\begin{aligned} &A(U^\varepsilon \mid U) \\ &= \begin{pmatrix} 0 \\ (1/\rho^\varepsilon) P^\varepsilon \otimes P^\varepsilon - (1/\rho) P\otimes P + ((\rho^\varepsilon - \rho)/\rho^2) P\otimes P - (1/\rho) P\otimes (P^\varepsilon - P) - (1/\rho)(P^\varepsilon - P)\otimes P \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \rho^\varepsilon (u^\varepsilon - u)\otimes (u^\varepsilon - u) \end{pmatrix}. \end{aligned}$$

Therefore, using (3-5) and (3-6), we have

$$\begin{aligned} \left| \int \nabla_x (D\eta(U)) : A(U^\varepsilon \mid U)\,dx \right| &= \left| \int_0^t \int_{\mathbb{T}^d} \rho^\varepsilon (u^\varepsilon - u)\otimes (u^\varepsilon - u) : \nabla_x u\,dx\,ds \right| \\ &\le C \|\nabla_x u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)} \int_0^t \int_{\mathbb{T}^d} \eta(U^\varepsilon \mid U)\,dx\,ds. \end{aligned}$$

Verification of ($\mathcal{H}5$). For a weak solution f^ε to (1-2), it follows from (2-13) that $U^\varepsilon = \binom{\rho^\varepsilon}{P^\varepsilon}$ solves the system

$$\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon) = \operatorname{div}_x \begin{pmatrix} 0 \\ -\int_{\mathbb{R}^d} (v-u^\varepsilon)\otimes(v-u^\varepsilon) f^\varepsilon\, dv \end{pmatrix}, \tag{5-4}$$

where the equality holds in the sense of distributions; see (2-8). Therefore, we have

$$\begin{aligned}\left|\int D\eta(U)\cdot[\partial_t U^\varepsilon + \operatorname{div}_x A(U^\varepsilon) - F(U^\varepsilon)]\,dx\right| \\ &= \left|\int_{\mathbb{T}^d} \nabla_x u : \left(\int_{\mathbb{R}^d} (v-u^\varepsilon)\otimes(v-u^\varepsilon) f^\varepsilon\,dv\right)dx\right| \\ &\le C\|\nabla_x u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d} |v-u^\varepsilon|^2 f^\varepsilon\,dv\,dx = C\|\nabla_x u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)}\mathcal{D}_1(f^\varepsilon).\end{aligned}$$

Verification of ($\mathcal{H}6$). From the proof of Proposition 4.2 in [Karper et al. 2015], we see

$$-\int_{\mathbb{T}^d} [D^2\eta(U)F(U)(U^\varepsilon - U) + D\eta(U)F(U^\varepsilon)]\,dx = K_1 + K_2 + K_3,$$

where

$$\begin{aligned} K_1 &:= -\frac{1}{2}\int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(x)\rho^\varepsilon(t,y)\big|(u^\varepsilon(x)-u(x)) - (u^\varepsilon(y)-u(y))\big|^2\,dx\,dy,\\ K_2 &:= \frac{1}{2}\int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(x)\rho^\varepsilon(y)\,|u^\varepsilon(x)-u^\varepsilon(y)|^2\,dx\,dy,\\ K_3 &:= \int_{\mathbb{T}^{2d}} \psi(x-y)\rho^\varepsilon(x)(\rho^\varepsilon(y)-\rho(y))(u(y)-u(x))(u^\varepsilon(x)-u(x))\,dx\,dy.\end{aligned}$$

Notice that $K_1 \le 0$, and $K_2 = \widetilde{\mathcal{D}}_2(f^\varepsilon)$ where $\widetilde{\mathcal{D}}_2(f^\varepsilon)$ is in Lemma 5.1.

To estimate K_3, we separate it into two parts:

$$\begin{aligned} K_3 = \int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d} \psi(x-y)u(y)(\rho^\varepsilon(y)-\rho(y))\,dy\right)\rho^\varepsilon(x)(u^\varepsilon(x)-u(x))\,dx \\ - \int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d} \psi(x-y)(\rho^\varepsilon(y)-\rho(y))\,dy\right)u(x)\rho^\varepsilon(x)(u^\varepsilon(x)-u(x))\,dx.\end{aligned}$$

Since ψ and u are Lipschitz, we use the Kantorovich–Rubinstein theorem, see [Villani 2009, Theorem 5.10 and Particular Case 5.16], to estimate

$$\begin{aligned} K_3 \le W_1(\rho^\varepsilon,\rho)\Big(\sup_{x\in\mathbb{T}^d}\|\psi(x-\cdot)u\|_{L^\infty(0,T_*;W^{1,\infty}(\mathbb{T}^d))} + \|\psi\|_{L^\infty(0,T_*;W^{1,\infty}(\mathbb{T}^d))}\|u\|_{L^\infty((0,T_*)\times\mathbb{T}^d)}\Big) \\ \times\int_{\mathbb{T}^d} \rho^\varepsilon(x)|u^\varepsilon(x)-u(x)|\,dx.\end{aligned}$$

Therefore, since $W_1(\rho^\varepsilon,\rho) \le W_2(\rho^\varepsilon,\rho)$, we obtain

$$K_3 \le C\left(W_2^2(\rho^\varepsilon,\rho) + \int_{\mathbb{T}^d} \rho^\varepsilon(x)|u^\varepsilon(x)-u(x)|^2\,dx\right).$$

Hence we have verified ($\mathcal{H}6$).

Verification of ($\mathcal{H}7$). This will be shown by Lemma 5.2 below. We first derive some estimates for the characteristics generated by the velocity fields u^ε and u.

For the velocity u in the limit system (1-4), let X be a characteristic generated by it, that is,

$$\dot{X}(t,x) = u(t, X(t,x)), \qquad X(0,x) = x. \tag{5-5}$$

Then, thanks to the smoothness of u, it follows from (3-8) that

$$X(t)_\# \rho_0(x)\,dx = \rho(t,x)\,dx.$$

On the other hand, since u^ε is not Lipschitz with respect to x, we use a probabilistic representation for ρ^ε as a solution of the continuity equation in (3-3). More precisely, (5-3) and (2-9) imply

$$\int_{\mathbb{T}^d} |u^\varepsilon(t)|^2 \rho^\varepsilon(t)\,dx \le \int_{\mathbb{T}^d} \mathcal{F}(f^\varepsilon)(t)\,dx \le \int_{\mathbb{T}^d} \mathcal{F}(f_0^\varepsilon)\,dx < \infty,$$

so it follows from Proposition 3.3 that there exists a probability measure η^ε in $\Gamma_{T_*} \times \mathbb{T}^d$ that is concentrated on the set of pairs (γ, x) such that γ is a solution of the ODE

$$\dot{\gamma}(t) = u^\varepsilon(\gamma(t)), \qquad \gamma(0) = x, \tag{5-6}$$

and

$$\int_{\mathbb{T}^d} \varphi(x)\rho^\varepsilon(t,x)\,dx = \int_{\Gamma_{T_*}\times\mathbb{T}^d} \varphi(\gamma(t))\,d\eta^\varepsilon(\gamma,x) \quad \text{for all } \varphi \in C^0(\mathbb{T}^d),\ t \in [0, T_*]. \tag{5-7}$$

In particular, this says that the time marginal of the measure η^ε at time 0 is given by $\rho^\varepsilon(0) = \rho_0^\varepsilon$. Hence, by the disintegration theorem of measures, see for instance [Ambrosio et al. 2005, Theorem 5.3.1] and the comments at the end of Section 8.2 in [Ambrosio et al. 2005], we can write

$$d\eta^\varepsilon(\gamma, x) = \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx,$$

where $\{\eta_x^\varepsilon\}_{x\in\mathbb{T}^d}$ is a family of probability measures on Γ_{T^*} concentrated on solutions of (5-6).

For the flow X in (5-5), we also consider the densities $\tilde{\rho}^\varepsilon(t)$ defined as

$$\tilde{\rho}^\varepsilon(t,x)\,dx = X(t)_\# \rho_0^\varepsilon(x)\,dx. \tag{5-8}$$

Note that, since

$$\begin{aligned} \|\rho(t) - \tilde{\rho}^\varepsilon(t)\|_{L^1(\mathbb{T}^d)} &= \sup_{\|\varphi\|_\infty \le 1} \int_{\mathbb{T}^d} \varphi(x)[\rho(t,x) - \tilde{\rho}^\varepsilon(t,x)]\,dx \\ &= \sup_{\|\varphi\|_\infty \le 1} \int_{\mathbb{T}^d} \varphi(X(t,x))[\rho_0(x) - \rho_0^\varepsilon(x)]\,dx \le \|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{T}^d)}, \end{aligned}$$

we have

$$\|\rho(t) - \tilde{\rho}^\varepsilon(t)\|_{L^1(\mathbb{T}^d)} \le \|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{T}^d)}. \tag{5-9}$$

We now consider the measure ν^ε on $\Gamma_{T_*} \times \Gamma_{T_*} \times \mathbb{T}^d$ defined as

$$d\nu^\varepsilon(\gamma, \sigma, x) = \eta_x^\varepsilon(d\gamma) \otimes \delta_{X(\,\cdot\,,x)}(d\sigma) \otimes \rho_0^\varepsilon(x)\,dx.$$

If we consider the evaluation map

$$E_t : \Gamma_{T_*} \times \Gamma_{T_*} \times \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{T}^d, \qquad E_t(\gamma, \sigma, x) = (\gamma(t), \sigma(t)),$$

it follows that the measure $\pi_t^\varepsilon := (E_t)_\# \nu^\varepsilon$ on $\mathbb{T}^d \times \mathbb{T}^d$ has marginals $\rho^\varepsilon(t,x)\,dx$ and $\tilde{\rho}^\varepsilon(t,y)\,dy$ for all $t \geq 0$. Therefore, we have

$$\begin{aligned}\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - X(t,x)|^2 \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx &= \int_{\Gamma_{T_*}\times\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - \sigma(t)|^2\,d\nu^\varepsilon(\gamma,\sigma,x)\\ &= \int_{\mathbb{T}^{2d}} |x-y|^2\,d\pi_t^\varepsilon(x,y)\\ &\geq W_2^2(\rho^\varepsilon(t), \tilde{\rho}^\varepsilon(t)). \end{aligned} \tag{5-10}$$

We now use the above results to prove the following lemma.

Lemma 5.2. *Under the same assumptions as in Theorem 3.1, we have*

$$W_2^2(\rho^\varepsilon(t), \rho(t)) \leq Ce^{T_*}\int_0^t \int_{\mathbb{T}^d} |u^\varepsilon(s,x) - u(s,x)|^2 \rho^\varepsilon(s,x)\,dx\,ds + \mathcal{O}(\varepsilon), \qquad t \leq T_*. \tag{5-11}$$

Proof. Let $\tilde{\rho}^\varepsilon$ be defined as in (5-8). We begin by observing that, thanks to Lemma 3.4, (5-9), and assumption $(\mathcal{A}2)$, it follows that

$$W_2^2(\tilde{\rho}^\varepsilon(t), \rho(t)) \leq \mathcal{O}(\varepsilon).$$

Hence, to prove (5-11), it is enough to bound $W_2^2(\rho^\varepsilon(t), \tilde{\rho}^\varepsilon(t))$.

To this aim, we try to get a Gronwall-type inequality on

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - X(t,x)|^2 \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx.$$

Since

$$\dot{\gamma}(t) - \dot{X}(t,x) = (u^\varepsilon(\gamma(t)) - u(\gamma(t))) + (u(\gamma(t)) - u(X(t,x)))$$

by (5-5) and (5-6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx &\leq \int_{\Gamma_{T_*}\times\mathbb{T}^d} |u^\varepsilon(\gamma(t)) - u(\gamma(t))|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx\\ &\quad + \int_{\Gamma_{T_*}\times\mathbb{T}^d} |u(\gamma(t)) - u(X(t,x))|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx\\ &\quad + 2\int_{\Gamma_{T_*}\times\mathbb{T}^d} |\gamma(t) - X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx.\end{aligned}$$

Notice that, thanks to (5-7),

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |u^\varepsilon(\gamma(t)) - u(\gamma(t))|^2\,d\eta_x^\varepsilon(\gamma) \otimes \rho_0^\varepsilon(x)\,dx = \int_{\mathbb{T}^d} |u^\varepsilon(t,x) - u(t,x)|^2 \rho^\varepsilon(t,x)\,dx.$$

Moreover, since

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d} |u(\gamma(t))-u(X(t,x))|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \\ \le \|u\|_{L^\infty(0,T_*;W^{1,\infty}(\mathbb{T}^d))}\int_{\Gamma_{T_*}\times\mathbb{T}^d}|\gamma(t)-X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx,$$

we have

$$\frac{d}{dt}\int_{\Gamma_{T_*}\times\mathbb{T}^d}|\gamma(t)-X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx \\ \le C\int_{\Gamma_{T_*}\times\mathbb{T}^d}|\gamma(t)-X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx+\int_{\mathbb{T}^d}|u^\varepsilon(t,x)-u(t,x)|^2\rho^\varepsilon(t,x)\,dx.$$

Therefore, using Gronwall's inequality together with $\gamma(0)=X(0,x)=x$ for η_x^ε-a.e. γ, we obtain

$$\int_{\Gamma_{T_*}\times\mathbb{T}^d}|\gamma(t)-X(t,x)|^2\,d\eta_x^\varepsilon(\gamma)\otimes\rho_0^\varepsilon(x)\,dx\le Ce^{T_*}\int_0^t\int_{\mathbb{T}^d}|u^\varepsilon(s,x)-u(s,x)|^2\rho^\varepsilon(s,x)\,dx\,ds,\qquad t\le T_*.$$

Hence, using (5-10) we get the desired control on $W_2^2(\rho^\varepsilon(t),\tilde\rho^\varepsilon(t))$, which concludes the proof. □

***Proof of* (3-2).** Here we use the estimate (3-1) to show the convergence (3-2).

First, since (5-1) and ($\mathcal{A}$1) imply

$$\int_0^t \mathcal{D}_1(f^\varepsilon)(s)\,ds\le C\varepsilon,$$

using (3-1), we have

$$\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon|v-u|^2\,dx\,dv\,ds\le 2\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^\varepsilon(|v-u^\varepsilon|^2+|u^\varepsilon-u|^2)\,dx\,dv\,ds\le C(1+T_*)\varepsilon. \tag{5-12}$$

Then, for any $\varphi\in C_c^1((0,T_*)\times\mathbb{T}^d\times\mathbb{R}^d)$,

$$\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(s,x,v)f^\varepsilon\,dx\,dv\,ds-\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(s,x,v)\rho\,\delta_u(dv)\,dx\,ds \\ =\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}\varphi(s,x,v)f^\varepsilon\,dx\,dv\,ds-\int_0^{T_*}\int_{\mathbb{T}^d}\varphi(s,x,u)\rho\,dx\,ds \\ =\underbrace{\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}f^\varepsilon(\varphi(s,x,v)-\varphi(s,x,u))\,dx\,dv\,ds}_{=:I_1^\varepsilon}+\underbrace{\int_0^{T_*}\int_{\mathbb{T}^d}\varphi(s,x,u)(\rho^\varepsilon-\rho)\,dx\,ds}_{=:I_2^\varepsilon}.$$

Using (5-12), we have

$$I_1^\varepsilon\le\|\nabla_v\varphi\|_\infty\int_0^{T_*}\int_{\mathbb{T}^d\times\mathbb{R}^d}f^\varepsilon|v-u|\,dx\,dv\,ds \\ =\|\nabla_v\varphi\|_\infty\left(\int_0^{T_*}\int_{|v-u|\le\sqrt{\varepsilon}}f^\varepsilon|v-u|\,dx\,dv\,ds+\int_0^{T_*}\int_{|v-u|>\sqrt{\varepsilon}}f^\varepsilon|v-u|\,dx\,dv\,ds\right)$$

$$\leq \|\nabla_v \varphi\|_\infty \left(\sqrt{\varepsilon} T_* + \frac{1}{\sqrt{\varepsilon}} \int_0^{T_*} \int_{|v-u|>\sqrt{\varepsilon}} f^\varepsilon |v-u|^2 \, dv \, dx \, ds \right)$$
$$\leq C(1+T_*)\sqrt{\varepsilon}.$$

Since $W_1(\rho^\varepsilon, \rho) \leq W_2(\rho^\varepsilon, \rho) \to 0$ by (3-1), we also have $I_2^\varepsilon \to 0$ as $\varepsilon \to 0$.

Hence, this completes the proof of (3-2).

Appendix: Proof of Theorem 2.2

We first estimate $(d/dt)\mathcal{E}_1$ as follows:

$$\frac{d}{dt}\mathcal{E}_1 = 2\int_{\mathbb{T}^d \times \mathbb{R}^d} f(u-v)\, \partial_t u \, dv \, dx + \int_{\mathbb{T}^d \times \mathbb{R}^d} \partial_t f |u-v|^2 \, dv \, dx := I_1 + I_2.$$

First of all, by the definition of u, we have $\int f(u-v)\, dv = 0$; hence $I_1 = 0$.

Concerning I_2, it follows from (1-1) that

$$\begin{aligned} I_2 &= \int_{\mathbb{T}^d \times \mathbb{R}^d} |u-v|^2 \big(-\nabla_x \cdot (vf) - \nabla_v \cdot (L[f]f) - \nabla_v \cdot ((u-v)f) \big) \, dv \, dx \\ &= \underbrace{2\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x u (u-v) \cdot vf \, dv \, dx}_{=:I_{21}} \underbrace{-2\int_{\mathbb{T}^d \times \mathbb{R}^d} (u-v) \cdot L[f]f \, dv \, dx}_{=:I_{22}} \underbrace{-2\int_{\mathbb{T}^d \times \mathbb{R}^d} |u-v|^2 f \, dv \, dx}_{=-2\mathcal{E}_1}. \end{aligned}$$

Then, we use the stress tensor $P = \int_{\mathbb{R}^d} (v-u) \otimes (v-u) f \, dv$ to rewrite I_{21} as

$$I_{21} = 2\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x u (u-v) \cdot (v-u) f \, dv \, dx = 2\int_{\mathbb{T}^d} (\nabla_x \cdot P) \cdot u \, dx.$$

Thanks to the estimate on $\mathcal{I}_2$ in the proof of Lemma 5.1, we see that

$$I_{22} = -2\int_{\mathbb{T}^{2d} \times \mathbb{R}^{2d}} \psi(x-y) f(t,x,v) f(t,y,w)(u(t,x)-v) \cdot (w-v) \, dv \, dw \, dx \, dy \leq 0.$$

Therefore, we have

$$\frac{d}{dt}\mathcal{E}_1 \leq 2\int_{\mathbb{T}^d} (\nabla_x \cdot P) \cdot u \, dx - 2\mathcal{E}_1. \tag{A-1}$$

We next estimate $(d/dt)\mathcal{E}_2$ as follows:

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2 &= 2\int_{\mathbb{T}^{2d}} \partial_t \rho(t,x) \rho(t,y) |u(t,x) - u(t,y)|^2 \, dx \, dy \\ &\qquad + 2\int_{\mathbb{T}^{2d}} \rho(t,x)\rho(t,y)(u(t,x)-u(t,y)) \partial_t (u(t,x)-u(t,y)) \, dx \, dy \\ &:= J_1 + J_2. \end{aligned}$$

Since it follows from (2-13) with $\varepsilon = 1$ that

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\rho\, \partial_t u + \rho u \cdot \nabla_x u + \nabla_x \cdot P = \int_{\mathbb{R}^d} L[f] f \, dv,$$

we obtain (recall that $\|\rho\|_{L^1(\mathbb{T}^d)} = 1$)

$$J_1 = -2\int_{\mathbb{T}^{2d}} \nabla_x \cdot (\rho u)(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy$$
$$= 4\int_{\mathbb{T}^d} \rho u\cdot\nabla_x u\cdot u\,dx - 4\int_{\mathbb{T}^d}\rho u\cdot\nabla_x u\,dx\cdot\int_{\mathbb{T}^d}\rho u\,dx,$$

and

$$J_2 = 4\int_{\mathbb{T}^{2d}}\rho(t,y)u(t,x)\rho(t,x)\partial_t u(t,x)\,dx\,dy - 4\int_{\mathbb{T}^{2d}}\rho(t,y)u(t,y)\rho(t,x)\partial_t u(t,x)\,dx\,dy$$
$$= -4\int_{\mathbb{T}^d}\rho u\cdot\nabla_x u\cdot u\,dx - 4\int_{\mathbb{T}^d}\nabla_x\cdot P\cdot u\,dx + 4\underbrace{\int_{\mathbb{T}^d\times\mathbb{R}^d} u\cdot L[f]f\,dx\,dv}_{:=J_{21}} + 4\int_{\mathbb{T}^d}\rho u\cdot\nabla_x u\,dx\cdot\int_{\mathbb{T}^d}\rho u\,dx$$
$$+ 4\underbrace{\int_{\mathbb{T}^d}\nabla_x\cdot P\,dx}_{=0}\cdot\int_{\mathbb{T}^d}\rho u\,dx - 4\underbrace{\int_{\mathbb{T}^d\times\mathbb{R}^d} L[f]f\,dx\,dv}_{:=J_{22}}\cdot\int_{\mathbb{T}^d}\rho u\,dx.$$

Now, we compute the above terms J_{21} and J_{22} as follows:

$$J_{21} = \int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y)f(t,x,v)f(t,y,w)(w-v)\cdot u(t,x)\,dv\,dw\,dx\,dy$$
$$= \int_{\mathbb{T}^{2d}} \psi(x-y)\rho(t,x)\rho(t,y)(u(t,y)-u(t,x))\cdot u(t,x)\,dx\,dy$$
$$= -\frac{1}{2}\int_{\mathbb{T}^{2d}} \psi(x-y)\rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy,$$

$$J_{22} = \int_{\mathbb{T}^{2d}\times\mathbb{R}^{2d}} \psi(x-y)f(t,x,v)f(t,y,w)(w-v)\,dv\,dw\,dx\,dy = 0.$$

Therefore, we have

$$\frac{d}{dt}\mathcal{E}_2 = -4\int_{\mathbb{T}^d}\nabla_x\cdot P\cdot u\,dx - 2\int_{\mathbb{T}^{2d}}\psi(x-y)\rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy.$$

Recalling (A-1), proves that

$$\frac{d}{dt}\mathcal{E} \le -2\mathcal{E}_1 - \int_{\mathbb{T}^{2d}}\psi(x-y)\rho(t,x)\rho(t,y)|u(t,x)-u(t,y)|^2\,dx\,dy$$
$$\le -2\mathcal{E}_1 - \psi_m\mathcal{E}_2 \le -2\min\{1,\psi_m\}\mathcal{E},$$

which completes the proof of (2-11).

To show the second bound (2-12), note that if $\ell_T := \sup_{t\in[0,T]}\|\nabla_x u\|_{L^\infty(\mathbb{T}^d)} < \infty$ then (A-1) yields

$$\frac{d}{dt}\mathcal{E}_1(t) \le -2\int_{\mathbb{T}^d}\nabla_x u : P\,dx - 2\mathcal{E}_1 \le 2\ell_T\int_{\mathbb{T}^d\times\mathbb{R}^d}|u-v|^2 f\,dv\,dx - 2\mathcal{E}_1(t) = 2(\ell_T-1)\mathcal{E}_1(t),$$

which proves (2-12). □

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