ANALYSIS & PDE

Volume 12 No. 4 2019

RUI HAN AND SVETLANA JITOMIRSKAYA

QUANTUM DYNAMICAL BOUNDS FOR ERGODIC POTENTIALS
WITH
UNDERLYING DYNAMICS OF ZERO TOPOLOGICAL ENTROPY

msp

ANALYSIS AND PDE
Vol. 12, No. 4, 2019
dx.doi.org/10.2140/apde.2019.12.867

QUANTUM DYNAMICAL BOUNDS FOR ERGODIC POTENTIALS WITH UNDERLYING DYNAMICS OF ZERO TOPOLOGICAL ENTROPY

RUI HAN AND SVETLANA JITOMIRSKAYA

We show that positive Lyapunov exponents imply upper quantum dynamical bounds for Schrödinger operators $H_{f,\theta}u(n) = u(n+1) + u(n-1) + \phi(f^n\theta)u(n)$, where $\phi : \mathcal{M} \to \mathbb{R}$ is a piecewise Hölder function on a compact Riemannian manifold $\mathcal{M}$, and $f : \mathcal{M} \to \mathcal{M}$ is a uniquely ergodic volume-preserving map with zero topological entropy. As corollaries we also obtain localization-type statements for shifts and skew-shifts on higher-dimensional tori with arithmetic conditions on the parameters. These are the first localization-type results with precise arithmetic conditions for multifrequency quasiperiodic and skew-shift potentials.

1. Introduction

Positive Lyapunov exponents are generally viewed as a signature of localization. While it is known that they can coexist even with almost ballistic transport [Last 1996; del Rio et al. 1996], vanishing of certain dynamical exponents has been identified as a reasonable expected consequence of hyperbolicity of the corresponding transfer-matrix cocycle. Results in this direction were obtained in [Damanik and Tcheremchantsev 2007; 2008] for one-frequency trigonometric polynomials, and recently in [Jitomirskaya and Mavi 2017] for one-frequency quasiperiodic potentials under very mild assumptions on regularity of the sampling function. In this paper we identify a general property responsible for positive Lyapunov exponents implying vanishing of the dynamical quantities in the rather general case of underlying dynamics defined by volume-preserving maps of Riemannian manifolds with zero topological entropy, and under very minimal regularity assumptions. This work presents the first localization-type results that hold in such generality. We expect that positive topological entropy should also lead to vanishing of the dynamical quantities for a.e. (but not every!) phase, but this should be approached by completely different methods and will be explored in a future work.

Our general results allow us, in particular, to obtain localization-type statements for potentials defined by shifts and skew-shifts of higher-dimensional tori. Pure point spectrum with exponentially decaying eigenfunctions has been obtained for almost all multifrequency shifts in the regime of positive Lyapunov exponents in [Bourgain and Goldstein 2000] and for the skew-shift on $\mathbb{T}^2$ with a perturbative condition in [Bourgain et al. 2001], both very delicate results. While bounds on transport exponents are certainly weaker than dynamical localization that often (albeit not always [Jitomirskaya et al. 2003]) accompanies pure point spectrum [Bourgain and Jitomirskaya 2000], we note that pure point spectrum can be destroyed

MSC2010: 47B36, 81Q10.
Keywords: transport exponent, multifrequency quasiperiodic, skew-shift.

by generic rank-1 perturbations [Del Rio et al. 1994], while vanishing of the transport exponents is robust in this respect [Damanik and Tcheremchantsev 2007]. Finally, our results are the first ones for both of these families that hold under purely arithmetic conditions and the first nonperturbative ones for the skew-shift.

Let $(\mathcal{M}, g)$ be a d-dimensional compact (smooth) Riemannian manifold with a metric g. Let Vol_g be its Riemannian volume density; see (2-1). Let f be a uniquely ergodic volume-preserving map on $\mathcal{M}$, which means Vol_g is its unique invariant probability measure. We will study the dynamical properties of the Schrödinger operator acting on $l^2(\mathbb{Z})$,

$$H_{f,\theta}u(n) = u(n+1) + u(n-1) + \phi(f^n\theta)u(n), \tag{1-1}$$

where $\theta \in \mathcal{M}$ is the phase.

The time-dependent Schrödinger equation

$$i\,\partial_t u = H_\theta u,$$

leads to a unitary dynamical evolution

$$u(t) = e^{-itH_\theta}u(0).$$

Under the time evolution, the wavepacket will in general spread out with time. For operators with absolutely continuous spectrum, scattering theory leads to a good understanding of the quantum dynamics. In this paper we will study the spreading of the wavepacket under the assumption of positive Lyapunov exponent, which automatically implies the absence of absolutely continuous spectrum.

Let $e^{-itH_\theta}\delta_0$ be the time evolution with the localized initial state δ_0. Let

$$a_\theta(n,t) = |\langle e^{-itH_\theta}\delta_0, \delta_n\rangle|^2;$$

$a_\theta(n,t)$ describes the probability of finding the wavepacket at site n at time t. We denote the p-th moment of $a_\theta(n,t)$ by

$$\langle |X|_\theta^p(t)\rangle = \sum_n (1+|n|)^p a_\theta(n,t).$$

Dynamical localization is defined as boundedness of $\langle |X|_\theta^p(t)\rangle$ in time t. This implies purely point spectrum; therefore for general operators with positive Lyapunov exponent such a strong control of the wavepacket is not possible. Thus we need to define proper transport exponents which describe the rate of the spreading of the wavepacket. For $p > 0$ define the upper and lower transport exponents

$$\beta_\theta^+(p) = \limsup_{t\to\infty} \frac{\ln\langle |X|_\theta^p(t)\rangle}{p\ln t}, \quad \beta_\theta^-(p) = \liminf_{t\to\infty} \frac{\ln\langle |X|_\theta^p(t)\rangle}{p\ln t}.$$

Obtaining upper bounds for the two transport exponents above implies a power-law control of the spreading rate of the entire wavepacket.

It is also interesting to consider a portion of the wavepacket. For a nonnegative function $A(t)$ of time, let

$$\langle A(t)\rangle_T = \frac{2}{T}\int_0^\infty e^{-2t/T} A(t)\,dt$$

be its time average. Set

$$P_{\theta,T}(L) = \sum_{|n|\le L} \langle a_\theta(n,t)\rangle_T.$$

Roughly speaking, $P_{\theta,T}(T^a) > \tau$ means that, in average, over time T, a portion of the wavepacket stays inside a box of size T^a. Let us consider two other scaling exponents

$$\bar{\xi}_\theta = \lim_{\tau\to 0} \limsup_{T\to\infty} \frac{\ln \inf\{L : P_{\theta,T}(L) + P_{f\theta,T}(L) > \tau\}}{\ln T},$$
$$\underline{\xi}_\theta = \lim_{\tau\to 0} \liminf_{T\to\infty} \frac{\ln \inf\{L : P_{\theta,T}(L) + P_{f\theta,T}(L) > \tau\}}{\ln T}$$

introduced, in the half-line case, in [Killip et al. 2003].

The vanishing of $\beta^\pm$ and $\bar{\xi}$, $\underline{\xi}$ can be viewed as localization-type statements. If $\mathcal{M} = \mathbb{T}$ is the 1-dimensional torus and $f : \theta \to \theta + \alpha$ is the irrational rotation, the Lebesgue measure m is the unique invariant probability measure of f. It was first proved in [Damanik and Tcheremchantsev 2007; 2008] that in this case, for ϕ a trigonometric polynomial, under the assumption of positive Lyapunov exponent, $\beta_\theta^+(p) = 0$ for all $p > 0$, all θ and Diophantine α, and $\beta_\theta^-(p) = 0$ for all $p > 0$, all θ and all α. It was recently proved in [Jitomirskaya and Mavi 2017] that under very mild restrictions on the regularity of the potential, under the assumption of positivity and continuity of the Lyapunov exponent, $\beta_\theta^+(p) = 0$ for all $p > 0$, all θ and Diophantine α, and $\beta_\theta^-(p) = 0$ for all $p > 0$, all θ and all α. It was also proved in that paper that for piecewise Hölder functions, under the assumption of positive Lyapunov exponent, $\bar{\xi}_\theta = 0$ for a.e. θ and Diophantine α, and $\underline{\xi}_\theta = 0$ for a.e. θ and all α.

Remark 1.1. The two Diophantine sets of α are different between [Damanik and Tcheremchantsev 2007; 2008] and [Jitomirskaya and Mavi 2017]. They are both full-measure sets, but [Jitomirskaya and Mavi 2017] covers a slightly thinner set of frequencies because of the need to handle potentials with weaker regularity.

In this paper we consider a d-dimensional compact Riemannian manifold $\mathcal{M}$ and a uniquely ergodic volume-preserving map f. We consider maps with the following volume-scaling property. For $1 \le l \le d$, for a smooth map $\sigma : Q^l \to \mathcal{M}$, where $Q^l = [0,1]^l$, let

$$\mathrm{Vol}_{g,l}(\sigma) := \int_{Q^l} \overline{\mathrm{Vol}}_{g,l}(d\sigma),$$

where $\overline{\mathrm{Vol}}_{g,l}(d\sigma)$ is the volume form on Q^l induced by σ from the given Riemannian metric g on $\mathcal{M}$. Let $\Sigma(l)$ be the set of all C^∞ mappings $\sigma : Q^l \to \mathcal{M}$. For $n = 1, 2, \ldots$ and $1 \le l \le d$, let

$$V_l(f) := \sup_{\sigma\in\Sigma(l)} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}_{g,l}(f^n\sigma) \quad \text{and} \quad V(f) := \max_l V_l(f). \tag{1-2}$$

A volume-preserving f always satisfies $V_d(f) = V_d(f^{-1}) = 0$. Here we need to make an extra assumption that $V(f) = V(f^{-1}) = 0$. It is known that for a smooth invertible map f, $V(f) = V(f^{-1})$

is equal to the *topological entropy* of f [Yomdin 1987]; thus our class of maps includes all smooth maps with zero topological entropy. In particular, it includes both the irrational rotation and the skew-shift.

For such maps we will assume that f has a bounded discrepancy.

Let $J_N(\theta) = J(\theta, f\theta, \dots, f^{N-1}\theta)$, see (2-16), be the isotropic discrepancy function of the sequence $\{f^n\theta\}_{n=0}^{N-1}$. For $\delta > 0$, we will say f has *strongly δ-bounded isotropic discrepancy* if $J_N(\theta) \le |N|^{-\delta}$ uniformly in θ for $|N| > N_0$; f has *weakly δ-bounded isotropic discrepancy* if there exists a sequence $\{N_j\}$ such that $J_{N_j}(\theta) \le |N_j|^{-\delta}$ uniformly in θ. It turns out many concrete dynamical systems feature these properties. We will show in Lemmas 3.7–3.9 that the following hold:

- A shift of higher-dimensional tori, $f : \theta \to \theta + \alpha$, has strongly bounded isotropic discrepancy for Diophantine α.
- A skew-shift, $f : (y_1, y_2, \dots, y_d) \to (y_1 + \alpha, y_2 + y_1, \dots, y_d + y_{d-1})$, has strongly bounded isotropic discrepancy for Diophantine α, and weakly bounded isotropic discrepancy for Liouvillean α.

Under the assumption of boundedness of discrepancy and a scaling property of f, we are ready to formulate the following two abstract results.

Let μ_θ be the spectral measure of H_θ corresponding to δ_0. Let $N = \int_{\mathcal{M}} \mu_\theta \, \mathrm{d}\,\mathrm{Vol}_g$ be the integrated density of states. Let $L(E)$ be the Lyapunov exponent; see (2-6).

Theorem 1. *Let ϕ be a piecewise Hölder function. Suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Suppose f is a uniquely ergodic volume-preserving map satisfying $V(f) = V(f^{-1}) = 0$. We have*:

- *If, for some $\delta > 0$, f has weakly δ-bounded isotropic discrepancy, then $\underline{\xi}_\theta = 0$, for* Vol_g*-a.e. $\theta \in \mathcal{M}$*;
- *If, for some $\delta > 0$, f has strongly δ-bounded isotropic discrepancy, then $\overline{\xi}_\theta = 0$ for* Vol_g*-a.e. $\theta \in \mathcal{M}$.*

Remark 1.2. The full-measure set of θ appearing in Theorem 1 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Theorem 2. *Under the assumptions of Theorem 1, assume also $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. We have*:

- *If, for some $\delta > 0$, f has weakly δ-bounded isotropic discrepancy, then $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$;*
- *If, for some $\delta > 0$, f has strongly δ-bounded isotropic discrepancy, then $\beta_\theta^+(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.*

Remark 1.3. Strongly δ-bounded isotropic discrepancy is essential for vanishing of $\overline{\xi}$ and $\beta_\theta^+(p)$, see Remarks 1.6 and 1.9. However, it is not yet clear whether weakly δ-bounded isotropic discrepancy (or any condition at all other than mere positivity of the Lyapunov exponent) is essential for vanishing of the $\underline{\xi}$ or of β_θ^-.

Theorems 1 and 2 extend the results of [Damanik and Tcheremchantsev 2007; 2008; Jitomirskaya and Mavi 2017] from irrational rotations of the circle to general uniquely ergodic maps of compact Riemannian manifolds with zero topological entropy and bounded discrepancy. One key to achieving such generality is a new argument that does not rely on harmonic analysis/approximation by trigonometric polynomials.

By [Damanik and Tcheremchantsev 2003], $\beta_\theta^-(p) \geq p \dim_H(\mu_\theta)$, where $\dim_H(\mu)$ is the Hausdorff dimension of μ. Thus as a consequence of $\beta_\theta^-(p) = 0$ we have the following.

Corollary 1.4. *Under the assumptions of Theorem 2*, $\dim_H(\mu_\theta) = 0$ *for all* $\theta \in \mathcal{M}$.

Remark 1.5. The point here is that we obtain zero Hausdorff dimension of the spectral measure for *all* rather than a.e. $\theta \in \mathcal{M}$ (the latter is known for general ergodic potentials [Simon 2007]). The statement for all θ has only been known for irrational rotations of $\mathbb{T}^1$ (proved for trigonometric polynomials in [Jitomirskaya and Last 2000], and follows easily for piecewise functions from the results of [Jitomirskaya and Mavi 2017]).

The following Theorems 3–6 are all corollaries of our abstract results. Theorems 7 and 8 depend on a somewhat different technique (bypassing the discrepancy considerations), which allows us to cover more frequencies in the case of the shift of $\mathbb{T}^2$. To our knowledge, Theorems 3–8 are the first arithmetic localization-type results.

Let us introduce the Diophantine condition (DC) and the weak Diophantine condition (WDC) on $\mathbb{T}^d$:

$$\mathrm{DC}(\tau) = \bigcup_{c>0} \mathrm{DC}(c,\tau) = \bigcup_{c>0} \left\{ (\alpha_1, \dots, \alpha_d) : \|\langle \vec{h}, \alpha \rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{c}{r(\vec{h})^\tau} \text{ for any } \vec{h} \neq \vec{0} \right\},$$

where $r(\vec{h}) = \prod_{i=1}^d \max(|h_i|, 1)$ (it is well known that when $\tau > 1$, $\mathrm{DC}(\tau)$ is a full-measure set), and

$$\mathrm{WDC}(\tau) = \bigcup_{c>0} \mathrm{WDC}(c,\tau) = \bigcup_{c>0} \left\{ (\alpha_1, \dots, \alpha_d) : \max\{\|h\alpha_i\|_{\mathbb{R}/\mathbb{Z}}\} \geq \frac{c}{|h|^\tau} \text{ for any } h \neq 0 \right\}, \quad h \in \mathbb{Z}$$

(it is well known that when $\tau > 1/d$, $\mathrm{WDC}(\tau)$ is a full-measure set).

Theorem 1 reduces vanishing of (upper or lower) ξ_θ to bounds on the isotropic discrepancy. As corollaries, we obtain:

Theorem 3. *Let f be an irrational shift on $\mathbb{T}^d$. For piecewise Hölder ϕ, suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then if $\alpha \in \mathrm{DC}(\tau) \subset \mathbb{T}^d$, $\tau > 1$, we have $\bar{\xi}_\theta = 0$ for a.e. $\theta \in \mathbb{T}^d$.*

Remark 1.6. The Diophantine condition is essential for the vanishing of $\bar{\xi}$ [Jitomirskaya and Zhang 2015].

Theorem 4. *Let f be a skew-shift. For piecewise Hölder ϕ, suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then*:

- *For all irrational α, we have $\underline{\xi}_{\vec{y}} = 0$ for a.e. $\vec{y} \in \mathbb{T}^d$.*
- *If $\alpha \in \mathrm{DC}(\tau)$ for some $\tau > 1$, then $\bar{\xi}_{\vec{y}} = 0$ for a.e. $\vec{y} \in \mathbb{T}^d$.*

Remark 1.7. The full-measure set appearing in Theorems 3 and 4 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Similarly, for systems with continuous Lyapunov exponent, Theorem 2 reduces vanishing of $\beta_\theta^\pm(p)$ to the same discrepancy bounds, and we obtain:

Theorem 5. *Under the assumptions of Theorem 3, assume in addition that $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. Then if $\alpha \in \mathrm{DC}(\tau) \subset \mathbb{T}^d$, we have $\beta_\theta^+(p) = 0$ for all $\theta \in \mathbb{T}^d$, $p > 0$.*

Corollary 1.8. *Under the assumptions of Theorem 5, if* $\alpha \in \mathrm{DC}(\tau)$, *then* $\dim_H(\mu_\theta) = 0$ *for all* $\theta \in \mathbb{T}^d$.

Remark 1.9. The Diophantine condition is essential for $\beta^+ = 0$ [Jitomirskaya and Zhang 2015].

Theorem 6. *Under the assumptions of Theorem 4, assume in addition that* $L(E)$ *is continuous in* E *and* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then*:

- *For all irrational* α, *we have* $\beta^-_{\vec{y}}(p) = 0$ *for all* $\vec{y} \in \mathbb{T}^d$, $p > 0$.
- *If* $\alpha \in \mathrm{DC}(\tau)$ *for some* $\tau > 1$, *then* $\beta^+_{\vec{y}}(p) = 0$ *for all* $\vec{y} \in \mathbb{T}^d$, $p > 0$.

Corollary 1.10. *Under the assumptions of Theorem 6, for all irrational* α, *we have* $\dim_H(\mu_{\vec{y}}) = 0$ *for all* $\vec{y} \in \mathbb{T}^d$.

Finally, for the case of an irrational shift on $\mathbb{T}^2$ we can make two more delicate statements, using a different technique to obtain arithmetic estimates.

Theorem 7. *Let* f *be an irrational shift on* $\mathbb{T}^2$. *For piecewise Hölder* ϕ, *suppose* $L(E)$ *is positive on a Borel subset* U *with* $N(U) > 0$. *Then if* $\alpha = (\alpha_1, \alpha_2) \in \bigcup_{\tau>1} \mathrm{WDC}(\tau)$, *we have* $\underline{\xi}_\theta = 0$ *for a.e.* $\theta \in \mathbb{T}^2$.

Remark 1.11. The full-measure set appearing in Theorem 7 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Theorem 8. *Under the assumptions of Theorem 7, assume in addition that* $L(E)$ *is continuous in* E *and* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then if* $\alpha = (\alpha_1, \alpha_2) \in \bigcup_{\tau>1} \mathrm{WDC}(\tau)$, *we have* $\beta^-_\theta(p) = 0$ *for all* $\theta \in \mathbb{T}^2$, $p > 0$.

Corollary 1.12. *Under the assumptions of Theorem 8, if* $\alpha \in \bigcup_{\tau>1} \mathrm{WDC}(\tau)$, *we have* $\dim_H(\mu_\theta) = 0$ *for all* $\theta \in \mathbb{T}^2$.

The most technically complex part of the paper consists in obtaining arithmetic estimates on the covering of the torus by the trajectory of a small ball in a polynomial (in the inverse radius) time, which we obtain by estimating the discrepancy in Theorems 3–6, and by the bounded remainder set technique in Theorems 7 and 8. The discrepancy estimates are standard for the Diophantine shifts and are ideologically similar to the known results on the equidistribution of $n^k\alpha$ for the case of higher-dimensional Diophantine skew-shifts. We still develop the proof for the Diophantine skew-shift case in full detail because we did not find it in the literature and also because it serves as a good preparation for the Liouville higher-dimensional skew-shift, for which, to the best of our knowledge, our estimates are new. We note that for the *Diophantine* skew-shift of $\mathbb{T}^2$ and shifts of $\mathbb{T}^d$ the results on the covering of the torus by the trajectory of a ball are shown in [Avila et al. 2014] by a completely different technique. The authors therein considered smoothed-out indicator functions of small disks, and converted the covering problem to solving cohomological equations. It is unclear to us if that technique is extendable to the Liouville or weakly Diophantine case.

We organize this paper as follows: In Section 2 we introduce some basic definitions. Some of them have been mentioned in the Introduction but not in detail. In Section 3 we will present some key lemmas and prove Theorems 1–8. In Sections 4–7 we prove the key lemmas that are listed in Section 3.

2. Preparation

Riemannian manifolds. Let $\mathcal{M}$ be a d-dimensional compact Riemannian manifold with a Riemannian metric g.

Let K be a compact set in some coordinate patch $(U, x^1, \dots, x^d)$. We define the volume of K to be

$$\mathrm{Vol}_g(K) := \int_{x(K)} \sqrt{|G \circ x^{-1}|}\, dx^1 \cdots dx^d,$$

where $G = \det g_{ij}$, $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ and $dx^1 \cdots dx^d$ is the Lebesgue measure on $\mathbb{R}^d$. This definition is free of the choice of coordinate. If K is not contained in a single coordinate patch, one could apply a partition of unity to define $\mathrm{Vol}_g(K)$. More precisely, we pick an atlas $(U_\alpha, x^1_\alpha, \dots, x^d_\alpha)$ of $\mathcal{M}$ and a partition of unity $\{\rho_\alpha\}$ subordinate to this atlas. Now we can set

$$\mathrm{Vol}_g(K) = \sum_\alpha \int_{x^\alpha(K \cap U_\alpha)} (\rho_\alpha \sqrt{|G^\alpha|}) \circ (x^\alpha)^{-1}\, dx^1_\alpha \cdots dx^d_\alpha.$$

The *Riemannian volume density*, see, e.g., [Nicolaescu 2007, Section 3.4], on $(\mathcal{M}, g)$ is

$$\mathrm{d}\,\mathrm{Vol}_g = \sum_\alpha (\rho_\alpha \sqrt{|G^\alpha|}) \circ (x^\alpha)^{-1}\, dx^1_\alpha \cdots dx^d_\alpha. \tag{2-1}$$

With a rescaling, we could always assume $\mathrm{d}\,\mathrm{Vol}_g$ is a probability measure on $\mathcal{M}$. In the above definition, we do not assume $\mathcal{M}$ to be oriented. If $\mathcal{M}$ is oriented, then the volume density is actually a positive n-form, called the volume form.

If $\varrho : [a, b] \to \mathcal{M}$ is a continuously differentiable curve in the Riemannian manifold $\mathcal{M}$, then we define its length $l(\varrho)$ by

$$l(\varrho) = \int_a^b \sqrt{g_{\varrho(t)}(\dot\varrho(t), \dot\varrho(t))}\, dt,$$

where $g_{\varrho(t)}$ is the inner product g at the point $\varrho(t)$. One could define the distance between any two points $x, y \in \mathcal{M}$ as

$\mathrm{dist}(x, y) = \inf\{l(\varrho) : \varrho$ is a continuous, piecewise continuously differentiable curve connecting x and $y\}$.

With the definition of distance, *geodesics* in a Riemannian manifold are then the locally distance-minimizing paths.

Let $v \in \mathrm{T}_x\mathcal{M}$ be a tangent vector to the manifold $\mathcal{M}$ at x. Then there is a unique geodesic ϱ_v satisfying $\varrho_v(0) = x$ with initial tangent vector $\dot\varrho_v(0) = v$. The corresponding *exponential map* is defined by $\exp_x(v) = \varrho_v(1)$.

Let $B_r(x) = \{y \in \mathcal{M} : \mathrm{dist}(x, y) < r\}$ be a *geodesic ball* centered at $x \in \mathcal{M}$ with radius r. It is known that $B_r(x) = \exp_x(B(0, r))$, where $B(0, r) = \{v \in \mathrm{T}_x\mathcal{M} : g_x(v, v) < r\}$.

Proposition 2.1. *There exists $r_g > 0$ such that, for all $r < r_g$, there exist positive constants C_g and c_g which are independent of $x \in \mathcal{M}$ so that*

$$c_g r^d \le \mathrm{Vol}_g(B_r(x)) \le C_g r^d \quad \textit{for any } x \in \mathcal{M}. \tag{2-2}$$

Proof. We will discuss the proof briefly. We can identify the tangent space $\mathrm{T}_x\mathcal{M}$ isometrically with $\mathbb{R}^d$. Now $\exp_x : \mathbb{R}^d \to \mathcal{M}$ is a diffeomorphism on some small ball $B_{\mathbb{R}^d}(0,r)$. On this ball, straight lines are mapped to length-minimizing geodesics [do Carmo 1992, Proposition 3.6], and thus Euclidean balls are mapped to geodesic balls of the same radius. Taking r smaller if necessary, we can assume the Jacobian of $\exp_x$ is bounded away from 0 and ∞ on $B_{\mathbb{R}^d}(0,r)$; thus for $r < r_x$ we have that $c_{g_x} r^d \le \mathrm{Vol}_g(B_r(x)) \le C_{g_x} r^d$. Since $\mathcal{M}$ is a compact manifold, we can take r_x, c_{g_x}, C_{g_x} independent of $x \in \mathcal{M}$. □

A subset C of $\mathcal{M}$ is said to be a *geodesically convex set* if, given any two points in C, there is a minimizing geodesic contained within C that joins those two points.

The *convexity radius at a point* $x \in \mathcal{M}$ is the supremum (which may be $+\infty$) of $r_x \in \mathbb{R}$ such that for all $r < r_x$ the geodesic ball $B_{r_x}(x)$ is geodesically convex. The *convexity radius of* $(\mathcal{M}, g)$ is the infimum over the points $x \in \mathcal{M}$ of the convexity radii at these points.

Proposition 2.2 [Berger 2003]. *For a compact manifold* $\mathcal{M}$, *the convexity radius* r'_g *of* $(\mathcal{M}, g)$ *is positive.*

This clearly implies that for any $x \in \mathcal{M}$ and any $r < r'_g$, $B_r(x)$ is geodesically convex.

Piecewise Hölder functions. Let $L_\gamma(\mathcal{M})$ be the space of γ-Lipschitz functions on $\mathcal{M}$. For $\phi \in L_\gamma(\mathcal{M})$ define

$$\|\phi\|_{L_\gamma} = \|\phi\|_\infty + \sup_{\theta_1,\theta_2 \in M} \frac{|\phi(\theta_1) - \phi(\theta_2)|}{\mathrm{dist}\,(\theta_1, \theta_2)^\gamma}. \tag{2-3}$$

We say ϕ is piecewise Hölder if there exists $\gamma > 0$, a positive integer K and $\{\phi_j\}_{j=1}^K \subset L_\gamma(\mathcal{M})$ such that

$$\phi(\theta) = \sum_{j=1}^K \chi_{S_j}(\theta)\phi_j(\theta),$$

where $\{S_j\}_{j=1}^M$ are sets with "good boundary", namely $\{\partial S_j\}_{j=1}^K$ are $(d-1)$-dimensional smooth submanifolds of $\mathcal{M}$. Clearly the discontinuity set J_ϕ of ϕ is $\bigcup_{j=1}^K \partial S_j$, and

$$\mathrm{Vol}_{g,d-1}(J_\phi) \le \sum_{j=1}^K \mathrm{Vol}_{g,d-1}(\partial S_j) < \infty. \tag{2-4}$$

Clearly for any two points θ_1, θ_2 such that $\mathrm{dist}(\theta_i, J_\phi) \ge r$, if $\mathrm{dist}(\theta_1, \theta_2) < r$ then we have

$$|\phi(\theta_1) - \phi(\theta_2)| \le \mathrm{dist}(\theta_1, \theta_2)^\gamma \sum_{j=1}^K \|\phi_j\|_{L_\gamma}. \tag{2-5}$$

Cocycles and Lyapunov exponent. We now introduce the Lyapunov exponent. For a given $z \in \mathbb{C}$, a formal solution u of $Hu = zu$ can be reconstructed using the transfer matrix

$$A(\theta, z) = \begin{pmatrix} z - \phi(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

via the equation

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(f^n\theta, z) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}.$$

Indeed, let $A_k(\theta, z)$ be the product of consecutive transfer matrices:

$$A_k(\theta, z) = \begin{cases} A(f^{k-1}\theta, z) \cdots A(f\theta, z) A(\theta, z) & \text{if } k > 0, \\ I & \text{if } k = 0, \\ (A_{-k}(f^k\theta, z))^{-1} & \text{if } k < 0. \end{cases}$$

Then for any $k \in \mathbb{Z}$ we have the relation

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A_k(\theta, z) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}.$$

We define the Lyapunov exponent

$$L(z) = \lim_k \frac{1}{k} \int_{\mathcal{M}} \ln \|A_k(\theta, z)\| \,\mathrm{d}\,\mathrm{Vol}_g(\theta) = \inf_k \frac{1}{k} \int_{\mathcal{M}} \ln \|A_k(\theta, z)\| \,\mathrm{d}\,\mathrm{Vol}_g(\theta). \tag{2-6}$$

Furthermore, $L(z) = \lim_k (1/k) \ln \|A_k(\theta, z)\|$ for Vol_g-a.e. $\theta \in \mathcal{M}$.

Spectral measure and integrated density of states. Let μ_θ be the spectral measure of H_θ corresponding to δ_0 defined by

$$\langle (H_\theta - z)^{-1}\delta_0, \delta_0 \rangle = \int_{\mathbb{R}} \frac{d\mu_\theta(x)}{x - z}.$$

Then clearly $\mu_{f\theta}$ is the spectral measure of H_θ corresponding to δ_1. Let $N = \int_{\mathcal{M}} \mu_\theta \,\mathrm{d}\,\mathrm{Vol}_g(\theta)$ be the integrated density of states. Then $N = \int_{\mathcal{M}} \frac{1}{2}(\mu_\theta + \mu_{f\theta}) \,\mathrm{d}\,\mathrm{Vol}_g(\theta)$, so $N(U) > 0$ for some set U implies $\frac{1}{2}(\mu_\theta + \mu_{f\theta})(U) > 0$ for Vol_g-a.e. $\theta \in \mathcal{M}$.

Rational approximation.

Single frequency. Let α be an irrational number and let $\{p_n/q_n\}$ be its continued fraction approximants. We have the following properties; see, e.g., [Khinchin 1964]:

$$\frac{1}{2q_{n+1}} \le \|q_n\alpha\|_{\mathbb{T}} \le \frac{1}{q_{n+1}}, \tag{2-7}$$

$$\|k\alpha\| > \|q_n\alpha\| \quad \text{for } q_n < k < q_{n+1}. \tag{2-8}$$

(1) If $\alpha \in \mathrm{DC}(c, \tau)$ for some $c > 0$, we have

$$\|k\alpha\|_{\mathbb{T}} \ge \frac{c}{|k|^\tau} \quad \text{for any } k \neq 0. \tag{2-9}$$

In particular, combining (2-7) with (2-9) we have

$$cq_{n+1} \le q_n^\tau. \tag{2-10}$$

(2) If $\alpha \notin \mathrm{DC}(\tau)$, there exists a subsequence of the continued fraction approximants $\{p_{n_k}/q_{n_k}\}$ such that

$$q_{n_k+1} > q_{n_k}^\tau. \tag{2-11}$$

Multiple frequencies. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be a set of irrational frequencies. Let $\{\vec{p}_n/q_n\}$ be its best simultaneous approximation with respect to the Euclidean norm on $\mathbb{T}^d$, namely,

$$\sum_{j=1}^{d} \|q_n\alpha_j\|_{\mathbb{T}}^2 < \sum_{j=1}^{d} \|k\alpha_j\|_{\mathbb{T}}^2 \quad \text{for any } |k| < q_n.$$

Clearly by the pigeonhole principle, we have

$$\sqrt{\sum_{j=1}^{d} \|q_n\alpha_j\|_{\mathbb{T}}{}^2} \le \frac{2\Gamma\left(\frac{1}{2}d+1\right)^{1/d}}{\sqrt{\pi}\, q_{n+1}^{1/d}}. \tag{2-12}$$

We say that

(1) $\alpha \in \mathrm{DC}(c, \tau)$ if
$$\|\langle \vec{k}, \alpha\rangle\|_{\mathbb{T}} \ge \frac{c}{r(\vec{k})^{\tau}} \quad \text{for any } \vec{k} \in \mathbb{Z}^d \backslash \{\vec{0}\}, \tag{2-13}$$

(2) $\alpha \in \mathrm{WDC}(c, \tau)$ if
$$\max_{1\le j\le d} \|k\alpha_j\|_{\mathbb{T}} \ge \frac{c}{|k|^{\tau}} \quad \text{for any } k \in \mathbb{Z}\backslash\{\vec{0}\}. \tag{2-14}$$

Discrepancy. Let $\vec{x}_1, \dots, \vec{x}_N \in \mathcal{M}$. For a subset C of $\mathcal{M}$, let $A(C; \{\vec{x}_n\})$ be the counting function

$$A(C; \{\vec{x}_n\}_{n=1}^N) = \sum_{n=1}^{N} \chi_C(\vec{x}_n). \tag{2-15}$$

The *isotropic discrepancy* $J_N(\{\vec{x}_n\}_{n=1}^N)$ is defined as

$$J_N(\{\vec{x}_n\}_{n=1}^N) = \sup_{C\in\mathscr{C}} \left| \frac{A(C; \{\vec{x}_n\}_{n=1}^N)}{N} - \mathrm{Vol}_g(C)\right|, \tag{2-16}$$

where $\mathscr{C}$ is the family of all geodesically convex subsets of $\mathcal{M}$.

For a point $\theta \in \mathcal{M}$, let $J_N(\theta) = J(\{f^n\theta\}_{n=0}^{N-1})$. We say a map $f : \mathcal{M} \to \mathcal{M}$ has *strongly δ-bounded isotropic discrepancy* if, for some $N > N_0$, we have $J_N(\theta) \le N^{-\delta}$ uniformly in $\theta \in \mathcal{M}$. We say f has *weakly δ-bounded isotropic discrepancy* if there is a subsequence $\{N_j\}$ such that $J_{N_j}(\theta) \le N_j^{-\delta}$ uniformly in $\theta \in \mathcal{M}$.

If $\mathcal{M} = \mathbb{T}^d$ is the d-dimensional torus, we define the *discrepancy* $D_N(\{\vec{x}_n\}_{n=1}^N)$ as

$$D(\{\vec{x}_n\}_{n=1}^N) = \sup_{C\in\mathscr{J}} \left| \frac{A(C; \{\vec{x}\}_{n=1}^N)}{N} - m(C)\right|, \tag{2-17}$$

where $\mathscr{J}$ is the family of boxes C of the form $C = \{(\theta_1, \dots, \theta_d) \in \mathbb{T}^d : \beta_i \le \theta_i < \kappa_i \text{ for } 1 \le i \le d\}$.

For a point $\theta \in \mathbb{T}^d$, let $D_N(\theta) = D(\{f^n\theta\}_{n=0}^{N-1})$. We say a map $f : \mathbb{T}^d \to \mathbb{T}^d$ has *strongly δ-bounded discrepancy* if for some N_0 and all $N > N_0$, we have $D_N(\theta) \le N^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$. We say f has *weakly δ-bounded discrepancy* if there is a subsequence $\{N_j\}$ such that $D_{N_j}(\theta) \le N_j^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$.

When $\mathcal{M} = \mathbb{T}^d$, the isotropic discrepancy and discrepancy can be tightly controlled by each other:

Lemma 2.3 [Kuipers and Niederreiter 1974, Theorem 1.6 in Chapter 2]. *For any sequence $\{\vec{x}_n\}_{n=1}^N$ in $\mathbb{T}^d$, we have*

$$D_N(\{\vec{x}_n\}_{n=1}^N) \le J_N(\{\vec{x}_n\}_{n=1}^N) \le (4d\sqrt{d}+1) D_N(\{\vec{x}_n\}_{n=1}^N)^{1/d}. \tag{2-18}$$

Therefore, by (2-18), when $\mathcal{M} = \mathbb{T}^d$:

Proposition 2.4. *A map f has strongly* (*weakly*) *δ-bounded isotropic discrepancy for some $\delta > 0$ if and only if f has strongly* (*weakly*) *$\tilde{\delta}$-bounded discrepancy for some $\tilde{\delta} > 0$.*

In Section 5 and the Appendix we are going to apply the following two inequalities to estimate the discrepancy from above. Recall that $r(\vec{h}) = \prod_{i=1}^d \max(|h_i|, 1)$.

Lemma 2.5 (Erdős–Turán–Koksma [Koksma 1950]). *For any positive integer H_0, we have*

$$D(\{\vec{x}_n\}_{n=1}^N) \le C_d \left(\frac{1}{H_0} + \sum_{0<|\vec{h}|\le H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i \langle \vec{h}, \vec{x}_n \rangle} \right| \right), \tag{2-19}$$

where $|\vec{h}| = \max_{j=1}^d |h_j|$.

Lemma 2.6 (Van der Corput's fundamental inequality; see, e.g., [Kuipers and Niederreiter 1974], Lemma 3.1 in Chapter 1). *For any integer $1 \le H \le N$, we have*

$$\left| \frac{1}{N} \sum_{n=1}^N u_n \right|^2 \le \frac{N+H-1}{N^2 H} \sum_{n=1}^N |u_n|^2 + \frac{2(N+H-1)}{N^2 H^2} \sum_{k=1}^{H-1} (H-k) \operatorname{Re} \sum_{n=1}^{N-k} u_n \bar{u}_{n+k}. \tag{2-20}$$

3. Key lemmas and proofs of Theorems 1–8

Covering $\mathcal{M}$ with the orbit of a geodesic ball and proofs of Theorems 1, 7, 2 and 8.

Lemma 3.1. *Let ϕ be a piecewise Hölder function with $1 \ge \gamma > 0$. Suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$:*

(1) *If there exists a sequence $r_k \to 0$ such that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then $\underline{\xi}_\theta = 0$ for Vol_g-a.e. $\theta \in \mathcal{M}$.*

(2) *If, for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then $\bar{\xi}_\theta = 0$ for Vol_g-a.e. $\theta \in \mathcal{M}$.*

Lemma 3.2. *Let ϕ be a piecewise Hölder function with $1 \ge \gamma > 0$. Suppose $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$:*

(1) *If there exists a sequence $r_k \to 0$ such that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.*

(2) *If, for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then $\beta_\theta^+(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.*

Lemmas 3.1 and 3.2 are key to our abstract argument. They are proved in Section 4. The connection to bounded discrepancy comes in the following:

Let r_g be as in Proposition 2.1 and r'_g as in Proposition 2.2.

Lemma 3.3. *If f has weakly δ-bounded isotropic discrepancy, then there exists $r_k \to 0$ as $k \to \infty$ such that any geodesic ball in $\mathcal{M}$ with radius r_k will cover the whole $\mathcal{M}$ in $r_k^{-2d/\delta}$ steps.*

Proof. There exists a sequence $\{N_k\}$ and $k_0 > 0$ such that for any $k > k_0$ we have $J_{N_k}(\{f^n\theta\}_{n=0}^{N-1}) \le N_k^{-\delta}$. This means for any geodesically convex set $C \subset \mathcal{M}$,

$$\frac{\sum_{n=0}^{N_k-1} \chi_C(f^n\theta)}{N_k} - \mathrm{Vol}_g(C) \ge -N_k^{-\delta}$$

holds for all $\theta \in \mathcal{M}$. Thus if we take $r_k = N_k^{-\delta/(2d)} < \min(r_g, r'_g)$, then by Proposition 2.2, we know $B_{r_k}(\theta)$ is geodesically convex. By Proposition 2.1, $\mathrm{Vol}_g(B_{r_k}(\theta)) \ge c_g r_k^d = c_g N_k^{-\delta/2} > N_k^{-\delta}$. Thus

$$\sum_{n=0}^{r_k^{-2d/\delta}-1} \chi_{B_{r_k}(\theta)}(f^n\theta) > 0$$

for any $\theta \in \mathcal{M}$. □

Lemma 3.4. *If f has strongly δ-bounded isotropic discrepancy, then for any $0 < r < \min(r_g, r'_g)$, any geodesic ball in $\mathcal{M}$ with radius r will cover the whole $\mathcal{M}$ in $r^{-2d/\delta}$ steps.*

Proof. There exists N_0 such that for any $N > N_0$ we have $J_N(\{f^n\theta\}_{n=0}^{N-1}) \le N^{-\delta}$ for all $\theta \in \mathcal{M}$. This means for any $0 < r < \min(r_g, r'_g)$, any geodesic ball $B_r(\theta)$ (it is geodesically convex by Proposition 2.2) and $N = r^{-2d/\delta}$ we have

$$\frac{\sum_{n=0}^{r^{-2d/\delta}-1} \chi_{B_r(\theta)}(f^n\theta)}{r^{-2d/\delta}} - \mathrm{Vol}_g(B_r(\theta)) \ge -r^{2d}.$$

Since by Proposition 2.1, $\mathrm{Vol}_g(B_r(\theta)) \ge c_g r^d > r^{2d}$, we have

$$\sum_{n=0}^{r^{-2d/\delta}-1} \chi_{B_r(\theta)}(f^n\theta) > 0$$

for any $\theta \in \mathcal{M}$. □

In the case of 2-dimensional irrational rotation, we also have:

Lemma 3.5. *For any $(\alpha_1, \alpha_2) \in \bigcup_{\tau>1} \mathrm{WDC}(\tau)$, there exists $r_k(\alpha_1, \alpha_2, \tau) \to 0$ as $k \to \infty$ such that any Euclidean ball with radius r_k covers the whole $\mathbb{T}^2$ in $r_k^{-800\tau^4}$ steps.*

Remark 3.6. This lemma will be proved in Section 7.

We are now ready to complete the proofs of the main theorems.

Proof of Theorem 1. Combine Lemmas 3.3 and 3.4 with Lemma 3.1. □

Proof of Theorem 7. Combine Lemma 3.5 with Lemma 3.1. □

Proof of Theorem 2. Combine Lemmas 3.3 and 3.4 with Lemma 3.2. □

Proof of Theorem 8. Combine Lemma 3.5 with Lemma 3.2. □

Estimation of discrepancy and proofs of Theorems 3, 5, 4 and 6. We have the following control of the discrepancies of irrational rotation and skew-shift.

Lemma 3.7. *If $\alpha \in \mathrm{DC}(\tau)$, then for some constant $\delta > 0$ we have $D_N(\{\theta + n\alpha\}_{n=0}^{N-1}) \leq N^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$.*

Let

$$\vec{Y}_n = \left(y_1 + \binom{n}{1}\alpha,\ y_2 + \binom{n}{1}y_1 + \binom{n}{2}\alpha,\ \ldots,\ y_d + \binom{n}{1}y_{d-1} + \cdots + \binom{n}{d}\alpha\right) = f^n(y_1, \cdots, y_d),$$

where f is the skew-shift.

Lemma 3.8. *If $\alpha \in \mathrm{DC}(\tau)$, then for some constant $\delta > 0$ we have $D_N(\{\vec{Y}_n\}_{n=1}^{N}) \leq N^{-\delta}$ uniformly in $(y_1, \ldots, y_d) \in \mathbb{T}^d$.*

Lemma 3.9. *If $\alpha \notin \mathrm{DC}(d)$, then for some constant $\delta > 0$ there exists a sequence $\{N_j\}$ such that $D_{N_j}(\{\vec{Y}_n\}_{n=1}^{N_j}) \leq N_j^{-\delta}$ uniformly in $(y_1, \ldots, y_d) \in \mathbb{T}^d$.*

Remark 3.10. Lemma 3.7 is standard. Its proof will be given in the Appendix. Proofs of Lemmas 3.8 and 3.9 will be given in Section 5.

Proof of Theorems 3, 5. These follow from Lemma 3.7 and Theorems 1 and 2. □

Proof of Theorems 4, 6. These follow from Lemmas 3.8 and 3.9 and Theorems 1 and 2. □

4. Proofs of Lemmas 3.1 and 3.2

Upper and lower bounds on transfer matrices. The following lemma on the uniform upper bound of the transfer matrix is essentially from [Jitomirskaya and Mavi 2017]. We have adapted it into the following form for convenience.

Lemma 4.1 [Jitomirskaya and Mavi 2017, Theorem 3.1]. *Let ϕ be a function whose discontinuity set has measure 0 and f be a uniquely ergodic map on $\mathcal{M}$. Then*:

4.1.1. *Let $L(E)$ be positive on a Borel set U and μ be a measure with $\mu(U) > 0$. Then for any $\zeta > 0$ there exists a number $D_\zeta > 0$, and for any $\epsilon > 0$ there exists a set $B_{\zeta,\epsilon}$ with $0 < \mu(B_{\zeta,\epsilon}) < \zeta$, and an integer $N_{\zeta,\epsilon}$ such that for any $E \in U \setminus B_{\zeta,\epsilon}$*

(1) $L(E) \geq D_\zeta$,

(2) *for $n > N_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$, we have $(1/n) \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.*

4.1.2. *Furthermore, if $L(E)$ is continuous in E and U is a compact set, there exists $D > 0$ and for any $\epsilon > 0$ there exists an integer N_ϵ such that for any $E \in U$*

(1) $L(E) \geq D$,

(2) *for $n > N_\epsilon$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$, we have $(1/n) \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.*

We are also able to formulate the following lower bound for the norm of transfer matrices.

Lemma 4.2. *Let ϕ be a piecewise Hölder function with $1 \geq \gamma > 0$ and f be a uniquely ergodic volume-preserving map on $\mathcal{M}$ with $V(f) = V(f^{-1}) = 0$. Then*:

4.2.1. *Let $L(E)$ be positive on a Borel set U and μ be a measure with $\mu(U) > 0$. Then for any $\zeta, \epsilon > 0$, let D_ζ, $B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in* Lemma 4.1.1:

(1) *If there exists a sequence $r_k \to 0$ such that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then there exists a sequence $\{n_k(\epsilon)\}$ such that for $k > k_{\zeta,\epsilon}$, any $E \in U \setminus B_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n_k}$ and $\theta \in \mathcal{M}$ we have*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \dots, e^{(5M\epsilon/\gamma)n_k}} \|A_{n_k}(f^j \theta, z)\| \geq e^{n_k(L(E) - 3\epsilon)}.$$

(2) *If, for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then for $n > N'_{\zeta,\epsilon}$, any $E \in U \setminus B_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$ we have*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \dots, e^{(5M\epsilon/\gamma)n}} \|A_n(f^j \theta, z)\| \geq e^{n(L(E) - 3\epsilon)}.$$

4.2.2. *Furthermore, if $L(E)$ is continuous in E and U is a compact set, let D be defined as in* Lemma 4.1.2 *and for any $\epsilon > 0$ let N_ϵ be defined as in* Lemma 4.1.2. *Then for any $E \in U$ we have $L(E) \geq D$ and for any $|z - E| < e^{-4\epsilon n}$ we have*:

(1) *If there exists a sequence $r_k \to 0$ such that any geodesic ball in $\mathcal{M}$ with radius r_k covers the whole $\mathcal{M}$ in r_k^{-M} steps, then there exists a sequence $\{n_k(\epsilon)\}$ such that for $k > k_\epsilon$ and any $\theta \in \mathcal{M}$,*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \dots, e^{(5M\epsilon/\gamma)n_k}} \|A_{n_k}(f^j \theta, z)\| \geq e^{n_k(L(E) - 3\epsilon)}.$$

(2) *If, for any small $r > 0$, any geodesic ball with radius r covers the whole $\mathcal{M}$ in r^{-M} steps, then for $n > N'_\epsilon$ and any $\theta \in \mathcal{M}$,*

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0, \dots, e^{(5M\epsilon/\gamma)n}} \|A_n(f^j \theta, z)\| \geq e^{n(L(E) - 3\epsilon)}.$$

Proof of Lemma 4.2. We will focus on the proof of part (1) of Lemma 4.2.1. The other three proofs will be discussed briefly at the end of this section.

For any $E \in U \setminus B_{\zeta,\epsilon}$ and $n > N_{\zeta,\epsilon}$, by Lemma 4.1.1 we have $(1/n)\|A_n(\theta, E)\| < L(E) + \epsilon$. Since $\int_{\mathcal{M}} (1/n) \ln \|A_n(\theta, E)\| \, \mathrm{d}\,\mathrm{Vol}_g(\theta) \geq L(E)$, we have

$$\mathrm{Vol}_g(M_{n,E,L(E),\epsilon}) := \mathrm{Vol}_g\left(\left\{\theta \in \mathcal{M} : \frac{1}{n} \ln \|A_n(\theta, E)\| > L(E) - \epsilon\right\}\right) > \frac{1}{2}. \tag{4-1}$$

Now we take any $\theta \in M_{n,E,L(E),\epsilon}$ and $|z - E| < e^{-4\epsilon n}$. When $n > 2N_{\zeta,\epsilon} + 3$, by standard telescoping we have

$$\begin{aligned} \|A_n(\theta, z)\| &\geq \|A_n(\theta, E)\| - \|A_n(\theta, z) - A_n(\theta, E)\| \\ &\geq e^{n(L(E) - \epsilon)} - (n + 2(N_{\zeta,\epsilon} + 1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E) - 3\epsilon)} \\ &> e^{n(L(E) - 2\epsilon)} \end{aligned}$$

for large enough $n > N'_{\zeta,\epsilon}$. This means

$$M_{n,E,L(E),\epsilon} \subset M_{n,z,L(E),2\epsilon}. \tag{4-2}$$

We know the discontinuity set of $(1/n)\ln\|A_n(\theta,z)\|$ is $J_n = \bigcup_{l=0}^{n-1} f^{-l}(J_\phi)$, where $J_\phi = \bigcup_{j=1}^{K} \partial S_j$ is defined in the subsection on piecewise Hölder functions on page 874. By our assumption (2-4) and the fact the $V_{d-1}(f^{-1}) = 0$ (by the definition (1-2) of $V(f^{-1})$), for n large enough, we have

$$\mathrm{Vol}_{g,d-1}(J_n) \le e^{n\epsilon}\,\mathrm{Vol}_{g,d-1}(J_\phi); \tag{4-3}$$

note that the largeness depends only on f. Define

$$\widetilde{M}_{n,z,L(E),2\epsilon} = M_{n,z,L(E),2\epsilon} \setminus \overline{F_{2e^{-5\epsilon n/\gamma}}(J_n)},$$

where a neighborhood is defined as

$$F_r(A) = \{\theta \in \mathcal{M} : \mathrm{dist}(\theta, A) < r\}.$$

Then by (4-3),

$$\begin{aligned}\mathrm{Vol}_g(\widetilde{M}_{n,z,L(E),2\epsilon}) &\ge \mathrm{Vol}_g(M_{n,z,L(E),2\epsilon}) - 4e^{-5\epsilon n/\gamma}\,\mathrm{Vol}_{g,d-1}(J_n)\\ &\ge \mathrm{Vol}_g(M_{n,z,L(E),2\epsilon}) - 4e^{-n(5\epsilon/\gamma-\epsilon)}\,\mathrm{Vol}_{g,d-1}(J_\phi) > \tfrac{2}{5}.\end{aligned}$$

In particular, it is a nonempty set. Now we take any $\tilde\theta \in \widetilde{M}_{n,z,L(E),2\epsilon}$ and $\theta \in B_{e^{-5\epsilon n/\gamma}}(\tilde\theta)$. We have, by telescoping, (2-5) and the fact that $V_1(f) = 0$ (by the definition (1-2) of $V(f)$),

$$\begin{aligned}&\|A_n(\theta,z)\|\\ &\quad\ge \|A_n(\tilde\theta,z)\| - \|A_n(\theta,z) - A_n(\tilde\theta,z)\|\\ &\quad\ge e^{n(L(E)-2\epsilon)} - \Bigl(\sum_{l=1}^{K}\|\phi_l\|_{L^\gamma}\Bigr)(n + 2(N_{\zeta,\epsilon}+1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)+\epsilon)} \max_{j=0,\dots,n-1}(\mathrm{dist}(f^j\theta, f^j\tilde\theta))^\gamma\\ &\quad\ge e^{n(L(E)-2\epsilon)} - \Bigl(\sum_{l=1}^{K}\|\phi_l\|_{L^\gamma}\Bigr)(\mathrm{dist}(\theta,\tilde\theta))^\gamma(n + 2(N_{\zeta,\epsilon}+1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)+\epsilon+\gamma\epsilon)}\\ &\quad> e^{n(L(E)-3\epsilon)}\end{aligned}$$

for $n > N''_{\zeta,\epsilon}$. This means

$$F_{e^{-5\epsilon n/\gamma}}(\widetilde{M}_{n,z,L(E),2\epsilon}) \subset M_{n,z,L(E),3\epsilon}.$$

Hence for $E \in U \setminus B_{\zeta,\epsilon}$, $n > N''_{\zeta,\epsilon}$ and $|z-E| < e^{-4\epsilon n}$, we know $M_{n,z,L(E),3\epsilon}$ contains a geodesic ball with radius $e^{-(5\epsilon/\gamma)n}$. Then there exists a sequence $\{n_k(\epsilon)\}$ such that a geodesic ball with radius $e^{-(5\epsilon/\gamma)n_k} \sim r_k$ covers the whole $\mathcal{M}$ in at most $e^{(5M\epsilon/\gamma)n_k}$ steps. Thus for $E \in U \setminus B_{\zeta,\epsilon}$, $k > k_{\zeta,\epsilon}$ such that $n_k(\epsilon) > N''_{\zeta,\epsilon}$, any $|z-E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathbb{T}^d$ we have

$$\min_{\iota\in\{-1,1\}}\ \max_{\iota j=0,\dots,e^{(5M\epsilon/\gamma)n_k}} \|A_{n_k}(f^j\theta, z)\| > e^{n_k(L(E)-3\epsilon)}.$$

Remark 4.3. Notice that part (2) of Lemma 4.2.1 follows without taking a subsequence $\{n_k(\epsilon)\}$. Also, Lemma 4.2.2 follows without excluding the set $B_{\zeta,\epsilon}$. □

Dynamical bounds on ξ_θ. The key to estimating ξ_θ is to apply the following lemma by Killip, Kiselev and Last.

Following [Jitomirskaya and Last 1999], for $f : \mathbb{Z} \to H$ where H is a Banach space, the truncated l^2 norms in the positive and negative directions are defined by

$$\|f\|_L^2 = \sum_{n=1}^{\lfloor L\rfloor} |f(n)|^2 + (L-\lfloor L\rfloor)|f(\lfloor L\rfloor+1)|^2 \qquad \text{for } L>0,$$

$$\|f\|_L^2 = \sum_{n=0}^{\lfloor L\rfloor+1} |f(n)|^2 + (\lfloor L\rfloor+1-L)|f(\lfloor L\rfloor)|^2 \quad \text{for } L<0.$$

The truncated l^2 norm in both directions is defined by

$$\|f\|_{L_1,L_2}^2 = \sum_{n=-\lfloor L_1\rfloor}^{\lfloor L_2\rfloor} |f(n)|^2 + (L_1-\lfloor L_1\rfloor)|f(-\lfloor L_1\rfloor-1)|^2 + (L_2-\lfloor L_2\rfloor)|f(\lfloor L_2\rfloor+1)|^2 \quad \text{for } L_1,L_2\ge 1.$$

With $A_\bullet(\theta,z)$ being a function on $\mathbb{Z}$, define $\widetilde{L}_\epsilon^+(\theta,z)\in\mathbb{R}^+$ and $\widetilde{L}_\epsilon^-(\theta,z)\in\mathbb{R}^-$ by requiring

$$\|A_\bullet(\theta,z)\|_{\widetilde{L}_\epsilon^\pm(\theta,z)} = 2\|A(\theta,z)\|\epsilon^{-1}.$$

Lemma 4.4 [Killip et al. 2003, Theorem 1.5]. *Let H_θ be a Schrödinger operator and μ_θ be the spectral measure of H_θ and δ_0. Let $T>0$ and $L_1, L_2 > 2$. Then*

$$\left\langle \tfrac{1}{2}(\|e^{-itH_\theta}\delta_0\|_{L_1,L_2}^2 + \|e^{-itH_\theta}\delta_1\|_{L_1,L_2}^2)\right\rangle_T > C\,\tfrac{1}{2}(\mu_\theta+\mu_{f\theta})(\{E : |\widetilde{L}_{T^{-1}}^-|\le L_1,\ \widetilde{L}_{T^{-1}}^+\le L_2\}), \tag{4-4}$$

where C is an universal constant.[1]

This lemma directly implies

$$P_{\theta,T}(L) + P_{f\theta,T}(L) > C\,\tfrac{1}{2}(\mu_\theta+\mu_{f\theta})(\{E : \|A_\bullet(\theta,z)\|_{\pm L} > 2\|A(\theta,z)\|T\}).$$

The plan is to show that for any $\eta>1$ and any θ_0 satisfying $(\mu_{\theta_0}+\mu_{f\theta_0})(U)>0$, we have

$$(\mu_{\theta_0}+\mu_{f\theta_0})(\{E : \|A_\bullet(\theta_0,z)\|_{\pm T} > T^\eta\}) \gtrsim (\mu_{\theta_0}+\mu_{f\theta_0})(U).$$

Proof of Lemma 3.1. We will prove part (1) in detail. Part (2) will be discussed briefly at the end of this proof.

Fix $\eta>1$ and θ_0 such that $(\mu_{\theta_0}+\mu_{f\theta_0})(U)>0$. Let $\zeta = \frac{1}{2}(\mu_{\theta_0}+\mu_{f\theta_0})(U)$, $D=D_\zeta$ from Lemma 4.1, and $\epsilon = \min(\gamma D/(40M\eta), D/6)$. Then by Lemma 4.1, there exists a set B, $0<|B|<\frac{1}{2}(\mu_{\theta_0}+\mu_{V\theta_0})(U)$, and a sequence $\{n_k\}$ such that $L(E)\ge D$ on $U\setminus B$ and for $E\in U\setminus B$, $k\ge k_0$, $|z-E|<e^{-4\epsilon n_k}$ and

[1] Here we formulate this lemma for operators with potential $V(n)=\phi(f^n\theta)$. This covers arbitrary bounded potentials by taking f to be a corresponding subshift.

any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0,\dots,e^{(5M\epsilon/\gamma)n_k}} \|A_{n_k}(f^j\theta, z)\| > e^{n_k(L(E)-3\epsilon)}.$$

Using that $A_{s+t}(\theta, z) = A_t(f^s(\theta), z)A_s(\theta, z)$, this implies, by the condition on ϵ,

$$\|A_\bullet(\theta, z)\|_{\pm e^{(10M\epsilon/\gamma)n_k}} > e^{\frac{1}{2}n_k(L(E)-3\epsilon)} \geq e^{(10M\epsilon/\gamma)n_k\eta}.$$

If we take $T_k = e^{(10M\epsilon/\gamma)n_k}$, then $U \setminus B \subset \{E : \|A_\bullet(\theta, E)\|_{\pm T_k} > T_k^\eta\}$ for any θ, in particular θ_0. Then by (4-4),

$$P_{\theta_0,T_k^\eta}(T_k) + P_{f\theta_0,T_k^\eta}(T_k) \geq C\tfrac{1}{2}(\mu_{\theta_0} + \mu_{f\theta_0})(\{E : \|A_\bullet(\theta_0, E)\|_{\pm T_k} > T_k^\eta\}) \geq \widetilde{C}\tfrac{1}{2}(\mu_{\theta_0} + \mu_{f\theta_0})(U).$$

This implies $\underline{\xi}_\theta = 0$ for all $\theta \in \mathcal{M}$ such that $(\mu_\theta + \mu_{f\theta})(U) > 0$.

Remark 4.5. Using Lemmas 4.1.1(2) and 4.2.1(2) instead of 4.1.1(1) and 4.2.1(1), part (2) can be proved without taking a subsequence n_k; therefore the conclusion holds for all T large enough rather than a sequence T_k. □

Bounds on $\boldsymbol{\beta}$. The key to the bounds on β is to apply the following lemma.

Lemma 4.6 [Damanik and Tcheremchantsev 2007, Theorem 1; 2008, Corollary 1]. *Let H be the Schrödinger operator, with f real-valued and bounded, and $K \geq 4$ such that $\sigma(H) \subset [-K+1, K-1]$. Suppose for all $\rho \in (0,1)$ we have*

$$\int_{-K}^{K} \left(\min_{\iota \in \{-1,1\}} \max_{1 \leq \iota n \leq T^\rho} \left\| A_n\left(E + \frac{i}{T}\right) \right\|^2 \right)^{-1} \mathrm{d}E = O(T^{-\eta}) \tag{4-5}$$

for any $\eta \geq 1$. Then $\beta^+(p) = 0$ for all $p > 0$. If (4-5) *is satisfied for a sequence $T_k \to \infty$, then $\beta^-(p) = 0$ for all $p > 0$.*

Proof of Lemma 3.2. We will prove part (1) in detail. A modification needed for part (2) is discussed briefly at the end of this proof.

It suffices to consider small $\rho \in (0,1)$. Fix any $\rho \in (0,1)$ small and $\eta \geq 1$. Assume $\sigma(H) \subset [-K+1, K-1]$. Since $L(E)$ is continuous in E on a compact set $[-K, K]$, we have $L(E) \geq D > 0$ on $[-K, K]$. Fix $\epsilon_\eta = \min(\rho\gamma D/(20M\eta), D/6)$. By Lemma 4.2.2 there exists a sequence $\{n_{\eta,k}\}$ such that for any $E \in [-K, K]$, $k > k_\eta$, any $|z - E| < e^{-4\epsilon_\eta n_{\eta,k}}$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1,1\}} \max_{\iota j = 0,\dots,e^{(5M\epsilon_\eta/\gamma)n_{\eta,k}}} \|A_{n_{\eta,k}}(f^j\theta, z)\| > e^{n_{\eta,k}(L(E)-3\epsilon_\eta)}.$$

Thus

$$\min_{\iota \in \{-1,1\}} \max_{j = 0,\dots,e^{(10M\epsilon_\eta/\gamma)n_{\eta,k}}} \|A_j(\theta, z)\|^2 \geq e^{n_{\eta,k}(L(E)-3\epsilon_\eta)} \geq e^{(10M\epsilon_\eta/(\gamma\rho))n_{\eta,k}\eta}$$

holds for any $\theta \in \mathcal{M}$, any $E \in [-K, K]$ and $|z - E| < e^{-4\epsilon_\eta n_{\eta,k}}$. Now we take $T_{\eta,k} = e^{(10M\epsilon_\eta/(\gamma\rho))n_{\eta,k}}$,

$$\left| E + \frac{i}{T_{\eta,k}} - E \right| = \frac{1}{T_{\eta,k}} < e^{-4\epsilon_\eta n_{\eta,k}}.$$

Thus

$$\min_{\iota\in\{-1,1\}} \max_{\iota j=0,\dots,T_{\eta,k}^{\rho}} \left\| A_j\left(\theta, E+\frac{i}{T_{\eta,k}}\right)\right\|^2 \geq T_{\eta,k}^{\eta}$$

holds for any $E \in [-K, K]$. Therefore

$$\int_{-K}^{K} \left(\min_{\iota\in\{-1,1\}} \max_{1\leq \iota n\leq T_{\eta,k}^{\rho}} \left\| A_n\left(\theta, E+\frac{i}{T_{\eta,k}}\right)\right\|^2\right)^{-1} dE \leq 2K T_{\eta,k}^{-\eta}.$$

Now take a sequence $\{k_i\}$ such that $T_{1,k_1} < T_{2,k_2} < \cdots$. Let $T_m = T_{m,k_m}$. Then

$$\int_{-K}^{K} \left(\min_{\iota\in\{-1,1\}} \max_{1\leq \iota n\leq T_{m}^{\rho}} \left\| A_n\left(\theta, E+\frac{i}{T_{m}}\right)\right\|^2\right)^{-1} dE \leq 2K T_{m}^{-m}.$$

By (4-5), we have $\beta_\theta^-(p) \leq \rho$ for all $\theta \in \mathcal{M}$, any $\rho \in (0,1)$ and any $p > 0$; thus $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and any $p > 0$.

Remark 4.7. Using Lemmas 4.1.2(2) and 4.2.2(2), part (2) follows without taking a subsequence $\{n_{\eta,k}\}$. Therefore the conclusion holds for all T large rather than a sequence T_k. □

5. Skew-shift: proofs of Lemmas 3.8 and 3.9

In this section, we obtain the discrepancy bounds for the skew-shift. While the Diophantine case is likely known, we didn't find this in the literature. We thus present a detailed proof, especially since we build our proof for the Liouvillean case on some of the same considerations.

Skew-shift. Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be defined as

$$f(y_1, y_2, \dots, y_d) = (y_1+\alpha, y_2+y_1, \dots, y_d+y_{d-1}).$$

Let $\vec{Y}_n = f^n(y_1, \dots, y_d)$; then

$$\vec{Y}_n = \left(y_1 + \binom{n}{1}\alpha,\ y_2 + \binom{n}{1}y_1 + \binom{n}{2}\alpha, \dots, y_d + \binom{n}{1}y_{d-1} + \cdots + \binom{n}{d}\alpha\right), \tag{5-1}$$

where $\binom{n}{m} = 0$ if $n < m$.

Preparation*: *combinatorial identities.

Lemma 5.1. *Let $r_t \in \mathbb{N}$ for $1 \leq t \leq s$. Then we have*

$$\sum_{1\leq t\leq s}^{l_t=0,1} (-1)^{s-\sum_{t=1}^{s} l_t} \binom{\sum_{t=1}^{s} l_t r_t}{s-1} = 0, \tag{5-2}$$

$$\sum_{1\leq t\leq s}^{l_t=0,1} (-1)^{s-\sum_{t=1}^{s} l_t} \binom{\sum_{t=1}^{s} l_t r_t}{s} = \prod_{t=1}^{s} r_t. \tag{5-3}$$

Proof. Let us consider the coefficient C_a of x^a in the product $(1+x)^{r_1}(1+x)^{r_2}\cdots(1+x)^{r_s} = (1+x)^{\sum_{i=1}^s r_i}$. Let us define

$$A^{(a)} = \left\{(\vec{j}_1, \vec{j}_2, \dots, \vec{j}_s) : \vec{j}_t = (j_{t,1}, j_{t,2}, \dots, j_{t,r_t}),\ j_{t,k} \in \{0,1\},\ \sum_{t=1}^{s}\sum_{k=1}^{r_t} j_{t,k} = a\right\}. \tag{5-4}$$

Each element in $A^{(a)}$ corresponds to one way of choosing 1 or x in each term of the product $(1+x)^{r_1}\cdot(1+x)^{r_2}\cdots(1+x)^{r_s}$ in order to get x^a, where $j_{t,k}=0$ means we choose 1 out of the k-th $(1+x)$ from $(1+x)^{r_t}$, and $j_{t,k}=1$ means we choose x instead of 1. Thus the capacity of $A^{(a)}$, denoted by $|A^{(a)}|$, is equal to $C_\alpha = \binom{\sum_{t=1} r_t}{a}$. Let us further define

$$A_t^{(a)} = A^{(a)} \cap \{\vec{j}_t = \vec{0}\}. \tag{5-5}$$

For $a = s-1$, since it is impossible to obtain x^{s-1} with $\vec{j}_t \neq \vec{0}$ for any $1 \le t \le s$, we have

$$A^{(s-1)} \setminus \left(\bigcup_{t=1}^{s} A_t^{(s-1)}\right) = \varnothing. \tag{5-6}$$

For $a = s$,

$$A^{(s)} \setminus \left(\bigcup_{t=1}^{s} A_t^{(s)}\right) = D, \tag{5-7}$$

where

$$D = \left\{(\vec{j}_1, \vec{j}_2, \dots, \vec{j}_t) : \sum_{k=1}^{r_t} j_{t,k} = 1 \text{ for } 1 \le t \le s\right\}. \tag{5-8}$$

Clearly,

$$\left|\bigcup_{t=1}^{s} A_t^{(a)}\right| = \sum_{i=1}^{s}(-1)^{i-1}\sum_{1\le t_1<t_2<\cdots<t_i\le s}\left|\bigcap_{l=1}^{i} A_{t_l}^{(a)}\right|, \tag{5-9}$$

in which

$$\sum_{1\le t_1<t_2<\cdots<t_i\le s}\left|\bigcap_{l=1}^{i} A_{t_l}^{(a)}\right| = \sum_{\sum_{t=1}^{s} l_t = s-i}^{l_t=0,1}\binom{\sum_{t=1}^{s} l_t r_t}{a}. \tag{5-10}$$

Thus

$$\begin{aligned}\left|A^{(a)} \setminus \left(\bigcup_{t=1}^{s} A_t^{(a)}\right)\right| &= \binom{\sum_{t=1}^{s} r_t}{a} + \sum_{i=1}^{s}(-1)^i \sum_{\sum_{t=1}^{s} l_t = s-i}^{l_t=0,1}\binom{\sum_{t=1}^{s} l_t r_t}{a} \\ &= \sum_{1\le t\le s}^{l_t=0,1}(-1)^{s-\sum_{t=1}^{s} l_t}\binom{\sum_{t=1}^{s} l_t r_t}{a}.\end{aligned} \tag{5-11}$$

For $a = s-1$, (5-2) follows directly from (5-6) and (5-11). For $a = s$, (5-3) follows from (5-7), (5-11) and the fact that $|D| = \prod_{t=1}^{s} r_t$. $\square$

Diophantine α.

Proof of Lemma 3.8. For $\alpha \in \mathrm{DC}(\tau)$, we take integers

$$H_j \sim N^{2^j/((2^d-1)(\tau+\epsilon))} \quad \text{for } 0 \le j \le d-1. \tag{5-12}$$

By Lemma 2.5,

$$\begin{aligned} D(\vec{Y}_1, \dots, \vec{Y}_N) &\le C_d\Bigg(\frac{1}{H_0} + \sum_{0<|\vec{h}|\le H_0} \frac{1}{r(\vec{h})}\bigg|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \langle \vec{h}, \vec{Y}_n\rangle}\bigg|\Bigg) \\ &= C_d\Bigg(\frac{1}{H_0} + \sum_{0<|\vec{h}|\le H_0} \frac{1}{r(\vec{h})}\bigg|\frac{1}{N}\sum_{n=1}^{N} u_n^{(0)}\bigg|\Bigg), \end{aligned} \tag{5-13}$$

where

$$u_n^{(0)} = \exp\Bigg\{2\pi i \sum_{j=1}^{d}\Bigg(h_j\alpha + \sum_{r=1}^{d-j} h_{j+r}y_r\Bigg)\binom{n}{j}\Bigg\}. \tag{5-14}$$

Let

$$u_{k_1,n}^{(1)} = \exp\Bigg\{2\pi i \sum_{j=2}^{d}\Bigg(h_j\alpha + \sum_{r=1}^{d-j} h_{j+r}y_r\Bigg)\sum_{l_1=0}^{1}(-1)^{1-l_1}\binom{n+l_1k_1}{j}\Bigg\}. \tag{5-15}$$

In general, if $d \ge 3$, we define the following for $1 \le s \le d-2$:

$$u_{k_1,\dots,k_s,n}^{(s)} = \exp\Bigg\{2\pi i \sum_{j=s+1}^{d}\Bigg(h_j\alpha + \sum_{r=1}^{d-j} h_{j+r}y_r\Bigg)\sum_{1\le t\le s}^{l_t=0,1}(-1)^{s-\sum_{t=1}^{s} l_t}\binom{n+\sum_{t=1}^{s} l_tk_t}{j}\Bigg\}. \tag{5-16}$$

Next, we illustrate the steps of the proof without details for two simple cases $d = 2$ and $d = 3$. After that, we give a detailed derivation for arbitrary d.

Applying Lemma 2.6 to the $\big|\sum_{n=1}^{N} u_n^{(0)}/N\big|$ term in (5-13), we obtain

$$\bigg|\frac{1}{N}\sum_{n=1}^{N} u_n^{(0)}\bigg|^2 \lesssim \frac{1}{H_1} + \frac{1}{NH_1^2}\sum_{k_1=1}^{H_1}(H_1-k_1)\bigg|\sum_{n=1}^{N-k_1} u_n^{(0)}\overline{u_{n+k_1}^{(0)}}\bigg|. \tag{5-17}$$

<u>The $d = 2$ case</u>: Estimating the $\big|\sum_{n=1}^{N-k_1} u_n^{(0)}\overline{u_{n+k_1}^{(0)}}\big|$ term on the right-hand-side of (5-17) (see (5-27) with $d = 2$) we have

$$\bigg|\sum_{n=1}^{N-k_1} u_n^{(0)}\overline{u_{n+k_1}^{(0)}}\bigg| \lesssim \frac{1}{\|h_2k_1\alpha\|_{\mathbb{T}}}. \tag{5-18}$$

The Diophantine condition on α implies that, see (5-28),

$$\sum_{k_1=1}^{H_1}\frac{1}{\|h_2k_1\alpha\|_{\mathbb{T}}} \lesssim \sum_{j=1}^{H_1}\frac{\prod_{l=0}^{1} H_l^{\tau}}{j} \le H_0^{\tau}H_1^{\tau+\epsilon}. \tag{5-19}$$

Thus combining (5-17), (5-18) with (5-19), we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_n^{(0)}\right|^2 \lesssim \frac{1}{H_1} = \frac{1}{H_0^2}.$$

Plugging this estimate into (5-13) yields the claimed result for $d = 2$.

The $d = 3$ case: The difference between the cases $d \geq 3$ and $d = 2$ is that for $d = 2$ we can directly estimate (5-17) via (5-18). However, for $d \geq 3$, we need to iteratively apply Lemma 2.6 to reduce the dimension. Now let us illustrate the proof for $d = 3$.

To estimate the right-hand-side of (5-17), we compute as in (5-23),

$$\left|\sum_{n=1}^{N-k_1} u_n^{(0)}\overline{u_{n+k_1}^{(0)}}\right| = \left|\sum_{n=1}^{N-k_1} u_{k_1,n}^{(1)}\right|. \tag{5-20}$$

Applying Lemma 2.6 to the right-hand-side of the equation above, we obtain

$$\left|\frac{1}{N-k_1}\sum_{n=1}^{N-k_1} u_{k_1,n}^{(1)}\right|^2 \lesssim \frac{1}{H_2} + \frac{1}{(N-k_1)H_2^2}\sum_{k_2=1}^{H_2}(H_2-k_2)\left|\sum_{n=1}^{N-\sum_{t=1}^{2}k_t} u_{k_1,n}^{(1)}\overline{u_{k_1,n+k_2}^{(1)}}\right|.$$

As in (5-27), we compute

$$\left|\sum_{n=1}^{N-\sum_{t=1}^{2}k_t} u_{k_1,n}^{(1)}\overline{u_{k_1,n+k_2}^{(1)}}\right| \lesssim \frac{1}{\|h_3k_1k_2\alpha\|_{\mathbb{T}}}.$$

Proceeding as in the $d = 2$ case via the Diophantine condition, we arrive at

$$\left|\frac{1}{N-k_1}\sum_{n=1}^{N-k_1} u_{k_1,n}^{(1)}\right|^2 \lesssim \frac{1}{H_2} = \frac{1}{H_1^2}.$$

Combining (5-17), (5-20) with the estimate above, we obtain

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_n^{(0)}\right|^2 \lesssim \frac{1}{H_1} = \frac{1}{H_0^2}.$$

This proves the claimed result for $d = 3$.

The general case: As we explained above, the general strategy is to use Lemma 2.6 to reduce

$$u^{(0)} \to u^{(1)} \to u^{(2)} \to \cdots \to u^{(d-2)}.$$

We stop when we reach $u^{(d-2)}$, as we can apply (5-27) to these terms.

With the $u^{(s)}$ terms, $0 \leq s \leq d-3$, defined in (5-16), Lemma 2.6 implies,

$$\left|\frac{1}{N-\sum_{t=1}^{s}k_t}\sum_{n=1}^{N-\sum_{t=1}^{s}k_t} u_{k_1,\dots,k_s,n}^{(s)}\right|^2$$

$$\lesssim \frac{1}{H_{s+1}} + \frac{1}{\left(N-\sum_{t=1}^{s}k_t\right)H_{s+1}^2}\sum_{k_{s+1}=1}^{H_{s+1}}(H_{s+1}-k_{s+1})\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} u_{k_1,\dots,k_s,n}^{(s)}\overline{u_{k_1,\dots,k_s,n+k_{s+1}}^{(s)}}\right|. \tag{5-21}$$

Here

$$\begin{aligned}
&\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t} u^{(s)}_{k_1,\dots,k_s,n}\overline{u^{(s)}_{k_1,\dots,k_s,n+k_{s+1}}}\right| \\
&\quad=\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}\exp\left\{2\pi i\sum_{j=s+1}^{d}\left(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r\right)\right.\right. \\
&\qquad\qquad\left.\left.\sum_{\substack{l_t=0,1\\1\le t\le s}}(-1)^{s-\sum_{t=1}^{s}l_t}\left(\binom{n+\sum_{t=1}^{s}l_tk_t}{j}-\binom{n+k_{s+1}+\sum_{t=1}^{s}l_tk_t}{j}\right)\right\}\right| \\
&\quad=\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}\exp\left\{2\pi i\sum_{j=s+1}^{d}\left(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r\right)\sum_{\substack{l_t=0,1\\1\le t\le s+1}}(-1)^{s+1-\sum_{t=1}^{s+1}l_t}\binom{n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}\right| \\
&\quad=\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}\exp\left\{2\pi i\sum_{j=s+1}^{d}\left(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r\right)\sum_{\substack{l_t=0,1\\0\le t\le s+1}}(-1)^{s+2-\sum_{t=0}^{s+1}l_t}\binom{l_0n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}\right| \\
&\quad=\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}\exp\left\{2\pi i\sum_{j=s+2}^{d}\left(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r\right)\sum_{\substack{l_t=0,1\\0\le t\le s+1}}(-1)^{s+2-\sum_{t=0}^{s+1}l_t}\binom{l_0n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}\right| \qquad (5\text{-}22) \\
&\quad=\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}\exp\left\{2\pi i\sum_{j=s+2}^{d}(h_j\alpha+\sum_{r=1}^{d-j}h_{j+r}y_r)\sum_{\substack{l_t=0,1\\1\le t\le s+1}}(-1)^{s+1-\sum_{t=1}^{s+1}l_t}\binom{n+\sum_{t=1}^{s+1}l_tk_t}{j}\right\}\right| \\
&\quad=\left|\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}u^{(s+1)}_{k_1,\dots,k_{s+1},n}\right|. \qquad (5\text{-}23)
\end{aligned}$$

Notice that in (5-22), we applied (5-3),

$$\exp\left\{\left(h_{s+1}\alpha+\sum_{r=1}^{d-s-1}h_{s+1+r}y_r\right)\sum_{\substack{l_t=0,1\\0\le t\le s+1}}(-1)^{s+2-\sum_{t=0}^{s+1}l_t}\binom{l_0n+\sum_{t=1}^{s+1}l_tk_t}{s+1}\right\}=1.$$

Combining (5-21) with (5-23), we get for any $0\le s\le d-3$,

$$\begin{aligned}
&\left|\frac{1}{N-\sum_{t=1}^{s}k_s}\sum_{n=1}^{N-\sum_{t=1}^{s}k_t}u^{(s)}_{k_1,\dots,k_s,n}\right|^2 \\
&\quad\le\frac{1}{H_{s+1}}+\frac{1}{\left(N-\sum_{t=1}^{s}k_t\right)H_{s+1}^2}\sum_{k_{s+1}=1}^{H_{s+1}}(H_{s+1}-k_{s+1})\left(N-\sum_{t=1}^{s+1}k_t\right) \\
&\qquad\qquad\times\left|\frac{1}{N-\sum_{t=1}^{s+1}k_t}\sum_{n=1}^{N-\sum_{t=1}^{s+1}k_t}u^{(s+1)}_{k_1,\dots,k_{s+1},n}\right|. \qquad (5\text{-}24)
\end{aligned}$$

By (5-21), for $s = d-2$,

$$\begin{aligned}
&\left|\frac{1}{N-\sum_{l=1}^{d-2} k_l} \sum_{n=1}^{N-\sum_{l=1}^{d-2} k_l} u^{(d-2)}_{k_1,\ldots,k_{d-2},n}\right|^2 \\
&\quad \lesssim \frac{1}{H_{d-1}} + \frac{1}{\left(N-\sum_{l=1}^{d-2} k_l\right) H_{d-1}^2} \sum_{k_{d-1}=1}^{H_{d-1}} (H_{d-1}-k_{d-1}) \left| \sum_{n=1}^{N-\sum_{l=1}^{d-1} k_l} u^{(d-2)}_{k_1,\ldots,k_{d-2},n} \overline{u^{(d-2)}_{k_1,\ldots,k_{d-2},n+k_{d-1}}}\right| \\
&\quad \lesssim \frac{1}{H_{d-1}} + \frac{1}{\left(N-\sum_{l=1}^{d-2} k_l\right) H_{d-1}} \sum_{k_{d-1}=1}^{H_{d-1}} \left| \sum_{n=1}^{N-\sum_{l=1}^{d-1} k_l} u^{(d-2)}_{k_1,\ldots,k_{d-2},n} \overline{u^{(d-2)}_{k_1,\ldots,k_{d-2},n+k_{d-1}}}\right|, \qquad (5\text{-}25)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \sum_{n=1}^{N-\sum_{l=1}^{d-1} k_l} u^{(d-2)}_{k_1,\ldots,k_{d-2},n} \overline{u^{(d-2)}_{k_1,\ldots,k_{d-2},n+k_{d-1}}}\right| \\
&\quad = \left| \sum_{n=1}^{N-\sum_{l=1}^{d-1} k_l} \exp\left\{2\pi i h_d \alpha \sum_{\substack{j_l=0,1 \\ 1\le l\le d-1}} (-1)^{d-1-\sum_{l=1}^{d-1} j_l} \binom{n+\sum_{j=1}^{d-1} j_l k_l}{d}\right\}\right| \\
&\quad = \left| \sum_{n=1}^{N-\sum_{l=1}^{d-1} k_l} \exp\left\{2\pi i h_d \alpha \sum_{\substack{j_l=0,1 \\ 0\le l\le d-1}} (-1)^{d-\sum_{l=0}^{d-1} j_l} \binom{l_0 n+\sum_{j=1}^{d-1} j_l k_l}{d}\right\}\right| \\
&\quad = \left| \sum_{n=1}^{N-\sum_{l=1}^{d-1} k_l} \exp\left\{2\pi i h_d n \alpha \prod_{l=1}^{d-1} k_l\right\}\right| \qquad (5\text{-}26) \\
&\quad \lesssim \frac{1}{\left\| h_d \alpha \prod_{l=1}^{d-1} k_l \right\|_{\mathbb{T}}}, \qquad (5\text{-}27)
\end{aligned}$$

where in (5-26) we used (5-3).

Since $\alpha \in \mathrm{DC}(\tau)$, by the property of the Diophantine condition (2-9) and since $|h_i| \le H_0,\ 1 \le k_i \le H_i$, we have

$$\sum_{k_{d-1}=1}^{H_{d-1}} \frac{1}{\left\| h_d \alpha \prod_{l=1}^{d-1} k_l \right\|_{\mathbb{T}}} \lesssim \sum_{j=1}^{H_{d-1}} \frac{\prod_{l=0}^{d-1} H_l^\tau}{j} \le H_{d-1}^{\tau+\epsilon} \prod_{l=0}^{d-2} H_l^\tau. \qquad (5\text{-}28)$$

Thus combining (5-25), (5-27) with (5-28), we have

$$\left|\frac{1}{N-\sum_{l=1}^{d-2} k_l} \sum_{n=1}^{N-\sum_{l=1}^{d-2} k_l} u^{(d-2)}_{k_1,\ldots,k_{d-2},n}\right|^2 \lesssim \frac{1}{H_{d-1}} + \frac{H_{d-1}^{\tau+\epsilon} \prod_{l=0}^{d-2} H_l^\tau}{H_{d-1}\left(N-\sum_{l=1}^{d-2} H_l\right)} \lesssim \frac{1}{H_{d-1}} = \frac{1}{H_{d-2}^2}.$$

Lemma 5.2. *For any $\alpha \in \mathbb{T}$, if, for any $1 \le k_s \le H_s$,*

$$\left|\frac{1}{N-\sum_{l=1}^{s} k_l} \sum_{n=1}^{N-\sum_{l=1}^{s} k_l} u^{(s)}_{k_1,\ldots,k_s,n}\right|^2 \lesssim \frac{1}{H_s^2},$$

then for any $0\le t\le s-1,\ 1\le k_t\le H_t$, *we have*

$$\left|\frac{1}{N-\sum_{l=1}^{t}k_l}\sum_{n=1}^{N-\sum_{l=1}^{t}k_l}u^{(t)}_{k_1,\dots,k_t,n}\right|^2\lesssim\frac{1}{H_t^2}.$$

Proof. For $t=s-1$, by (5-24),

$$\begin{aligned}\left|\frac{1}{N-\sum_{l=1}^{s-1}k_l}\sum_{n=1}^{N-\sum_{l=1}^{s-1}k_l}u^{(s-1)}_{k_1,\dots,k_{s-1},n}\right|^2&\\ &\lesssim\frac{1}{H_s}+\frac{1}{\left(N-\sum_{l=1}^{s-1}k_l\right)H_s^2}\sum_{k_s=1}^{H_s}(H_s-k_s)\left(N-\sum_{l=1}^{s}k_l\right)\left|\frac{\sum_{n=1}^{N-\sum_{l=1}^{s}k_l}u^{(s)}_{k_1,\dots,k_s,n}}{\left(N-\sum_{l=1}^{s}k_l\right)}\right|\\ &\lesssim\frac{1}{H_s}=\frac{1}{H_{s-1}^2}.\end{aligned}$$

Then we proceed by reverse induction. $\square$

At the final step we obtain

$$\left|\frac{1}{N}\sum_{n=1}^{N}u^{(0)}_n\right|^2\lesssim\frac{1}{H_0^2}.$$

Plugging it into (5-13), we have

$$D(\vec{Y}_1,\dots,\vec{Y}_N)\lesssim\frac{1}{H_0}+\sum_{0<|\vec{h}|\le H_0}\frac{1}{r(\vec{h})}\frac{1}{H_0}\lesssim\frac{1}{H_0^{1-\epsilon}}\sim N^{-(1-\epsilon)/((2^d-1)(\tau+\epsilon))}.\qquad\square$$

Liouvillean α.

Proof of Lemma 3.9. For $\alpha\notin\mathrm{DC}(d)$, by property (2-11), we can find a subsequence $\{p_n/q_n\}$ of the continued fraction approximants of α such that $q_{n+1}>q_n^d$. In the following we will use q instead of q_n and $\tilde{q}$ instead of q_{n+1} for simplicity. Here we would like to show $D_q(\vec{Y}_1,\dots,\vec{Y}_q)\le q^{-\delta}$ for some $\delta>0$. Take

$$H_j\sim q^{2^j/2^d}\quad\text{for } 0\le j\le d-2\qquad\text{and}\qquad H_{d-1}\sim q^{2^{d-1}(1+\epsilon)/2^d},\tag{5-29}$$

where $\epsilon>0$ is small enough such that

$$\prod_{l=0}^{d-1}H_l=q^{(2^d-1+2^{d-1}\epsilon)/2^d}<q.\tag{5-30}$$

Now by Lemma 2.5,

$$D(\vec{Y}_1,\dots,\vec{Y}_q)\le C_d\left(\frac{1}{H_0}+\sum_{0<|\vec{h}|\le H_0}\frac{1}{r(\vec{h})}\left|\frac{1}{q}\sum_{n=1}^{q}\exp\left\{2\pi i\sum_{j=1}^{d}(h_j\alpha+h_{j+1}y_1+\dots+h_dy_{d-j})\binom{n}{j}\right\}\right|\right).\tag{5-31}$$

Consider the difference

$$\begin{aligned}\frac{1}{q}\bigg|\sum_{n=1}^{q}\exp\bigg\{2\pi i\sum_{j=1}^{d}(h_j\alpha+h_{j+1}y_1+\cdots+h_d y_{d-j})\binom{n}{j}\bigg\}\\ -\sum_{n=1}^{q}\exp\bigg\{2\pi i\sum_{j=1}^{d}\Big(h_j\frac{p}{q}+h_{j+1}y_1+\cdots+h_d y_{d-j}\Big)\binom{n}{j}\bigg\}\bigg|\\ \le\frac{1}{q}\sum_{n=1}^{q}\bigg|\exp\bigg\{2\pi i\sum_{j=1}^{d}h_j\Big(\alpha-\frac{p}{q}\Big)\binom{n}{j}\bigg\}-1\bigg|\\ \lesssim\frac{1}{q}\sum_{n=1}^{q}\sum_{j=1}^{d}\binom{n}{j}H_0\Big|\alpha-\frac{p}{q}\Big|\\ \lesssim\frac{H_0}{q},\end{aligned}\tag{5-32}$$

where in the last step we use (2-7),

$$\Big|\alpha-\frac{p}{q}\Big|\le\frac{1}{q\tilde{q}}<\frac{1}{q^{d+1}}.$$

Then combining (5-31) with (5-32), we have

$$D(\vec{Y}_1,\dots,\vec{Y}_q)\lesssim C_d\bigg(\frac{1}{H_0}+\sum_{0<|\vec{h}|\le H_0}\frac{1}{r(\vec{h})}\bigg|\frac{1}{q}\sum_{n=1}^{q}u_n^{(0)}\bigg|\bigg)+\frac{H_0}{q},\tag{5-33}$$

where

$$\tilde{u}_n^{(0)}=\exp\bigg\{2\pi i\sum_{j=1}^{d}\Big(h_j\frac{p}{q}+h_{j+1}y_1+\cdots+h_d y_{d-j}\Big)\binom{n}{j}\bigg\},$$

that is, $u_n^{(0)}$ as in (5-14) with α replaced with p/q. Thus with $\tilde{u}^{(s)}_{k_1,\dots,k_s,n}$ defined as in (5-16) with α replaced with p/q, similar to (5-25) and (5-26), we have

$$\begin{aligned}\bigg|\frac{1}{N-\sum_{l=1}^{d-2}k_l}\sum_{n=1}^{N-\sum_{l=1}^{d-2}k_l}\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n}\bigg|^2\\ \lesssim\frac{1}{H_{d-1}}+\frac{1}{\big(N-\sum_{l=1}^{d-2}k_l\big)H_{d-1}}\sum_{k_{d-1}=1}^{H_{d-1}}\bigg|\sum_{n=1}^{N-\sum_{l=1}^{d-1}k_l}\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n}\overline{\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n+k_{d-1}}}\bigg|,\end{aligned}\tag{5-34}$$

and

$$\begin{aligned}\bigg|\sum_{n=1}^{q-\sum_{l=1}^{d-1}k_l}\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n}\overline{\tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n+k_{d-1}}}\bigg|=\bigg|\sum_{n=1}^{q-\sum_{l=1}^{d-1}k_l}\exp\bigg\{2\pi i h_d n\frac{p}{q}\prod_{l=1}^{d-1}k_l\bigg\}\bigg|\\ \lesssim\frac{1}{\big\|h_d(p/q)\prod_{l=1}^{d-1}k_l\big\|_{\mathbb{R}/\mathbb{Z}}}.\end{aligned}\tag{5-35}$$

Since $|h_d| \le H_0,\ 1 \le k_i \le H_i$, and by (5-30), for any $1 \le k \le H_{d-1}$ we have

$$\left\| k h_d \frac{p}{q} \prod_{l=1}^{d-2} k_l \right\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{1}{q}.$$

Thus

$$\sum_{k_{d-1}=1}^{H_{d-1}} \frac{1}{\|h_d(p/q)\prod_{l=1}^{d-1} k_l\|_{\mathbb{R}/\mathbb{Z}}} \lesssim \sum_{j=1}^{H_{d-1}} \frac{q}{j} \le q \ln H_{d-1}. \tag{5-36}$$

Then combining (5-34), (5-35) with (5-36), we get

$$\begin{aligned} \left| \frac{1}{q-\sum_{l=1}^{d-2} k_l} \sum_{n=1}^{q-\sum_{l=1}^{d-2} k_l} \tilde{u}^{(d-2)}_{k_1,\dots,k_{d-2},n} \right|^2 &\lesssim \frac{1}{H_{d-1}} + \frac{q \ln H_{d-1}}{\big(q-\sum_{l=1}^{d-2} H_l\big) H_{d-1}} \\ &\lesssim \frac{1}{H_{d-1}^{1/(1+\epsilon)}} = \frac{1}{H_{d-2}^2}. \end{aligned} \tag{5-37}$$

By Lemma 5.2,

$$\left| \frac{1}{q} \sum_{n=1}^{q} \tilde{u}^{(0)}_n \right|^2 \lesssim \frac{1}{H_0}.$$

Plugging it into (5-33), we get

$$D(\vec{Y}_1,\dots,\vec{Y}_q) \lesssim \frac{1}{H_0} + \frac{(\log H_0)^d}{H_0} + \frac{H_0}{q} \lesssim \frac{1}{q^{(1-\epsilon)/2^d}}. \qquad \square$$

6. Bounded remainder sets

Most of the material covered in this section comes from [Grepstad and Lev 2015]. We briefly discuss it here for completeness and readers' convenience. From now on we restrict our attention to irrational rotation on $\mathbb{T}^d$. For a measurable set $U \subset \mathbb{T}^d$, consider the function

$$A_N(U,\vec{x}) - N|U| := A(U, \{\vec{x}+n\alpha\}_{n=0}^{N-1}) - N|U| = \sum_{n=0}^{N-1} \chi_U(\vec{x}+n\alpha) - N|U|.$$

We will say U is a *bounded remainder set* (BRS) with respect to α if there exists a constant $C(U,\alpha) > 0$ such that

$$|A_N(U,\vec{x}) - N|U|| \le C(U,\alpha)$$

for any N and a.e. $\vec{x} \in \mathbb{T}^d$. We will call a measurable function g on $\mathbb{T}^d$ a transfer function for U if its characteristic function satisfies

$$\chi_U(\vec{x}) - |U| = g(\vec{x}) - g(\vec{x}-\alpha) \quad \text{a.e.}$$

Obviously if g is a transfer function for U, then its Fourier coefficients satisfy

$$\hat{g}(\vec{m}) = \frac{\hat{\chi}_U(\vec{m})}{1-e^{-2\pi i \langle \vec{m},\alpha\rangle}}, \quad \vec{m} \ne 0. \tag{6-1}$$

Proposition 6.1 [Grepstad and Lev 2015]. *For a measurable set $U \subset \mathbb{T}^d$, the following are equivalent*:

- *U is a bounded remainder set.*
- *U has a bounded transfer function g.*

Theorems 9, 10 and Corollary 6.2 are presented in [Grepstad and Lev 2015] without explicit bounds on the transfer functions. We present the proofs in order to extract the needed estimates.

Theorem 9. *Any interval $I \subset \mathbb{T}$ of length $0 < |q\alpha - p| < 1$ is a BRS with respect to α. Furthermore its transfer function g satisfies $\|g\|_\infty \le |q|$.*

Proof. Without loss of generality, we consider an interval $I = [0, \kappa]$, where $\kappa = q\alpha - p > 0$. Then

$$\begin{aligned}\chi_I(x) - |I| &= -\{x\} + \{x - \kappa\}\\ &= -\{x\} + \{x - q\alpha\}\\ &= (-\{x\} - \cdots - \{x - (q-1)\alpha\}) + (\{x - \alpha\} + \cdots + \{x - q\alpha\})\\ &= g(x) - g(x - \alpha),\end{aligned}$$

where $g(x) = -\sum_{j=0}^{q-1}\{x - j\alpha\}$, $\|g\|_\infty \le |q|$. ☐

Theorem 10. *Let $\vec{v} = (v_1, v_2, \ldots, v_d) = q\alpha - \vec{p} \in \mathbb{Z}\alpha + \mathbb{Z}^d$, $v \notin \mathbb{Z}^d$, and let $\Sigma \in \mathbb{T}^{d-1}$ be a BRS with respect to the vector $(v_1/v_d, v_2/v_d, \ldots, v_{d-1}/v_d)$ with transfer function h. Then the set*

$$U = U(\Sigma, \vec{v}) = \{(\vec{x}, 0) + t\vec{v} : \vec{x} \in \Sigma,\ 0 \le t < 1\}$$

is a BRS with respect to α, whose transfer function g satisfies $\|g\|_\infty \le |q|(\|h\|_\infty + 1)$.

Proof. Let $\vec{v}_0 = (v_1, \ldots, v_{d-1})$ be the vector in $\mathbb{T}^{d-1}$ which consists of the first $d-1$ entries of $\vec{v}$. First, we wish to find a bounded function $\tilde{g}$ on $\mathbb{T}^d$ satisfying the cohomological equation

$$\chi_U(\vec{x}, y) - |U| = \tilde{g}(\vec{x}, y) - \tilde{g}(\vec{x} - \vec{v}_0, y - v_d) \quad \text{for a.e. } (\vec{x}, y) \in \mathbb{T}^{d-1} \times \mathbb{T}.$$

This means the Fourier coefficients satisfy the equation

$$\begin{aligned}&\hat{\tilde{g}}(\vec{m}, n)(1 - e^{-2\pi i(\langle \vec{m}, \vec{v}_0\rangle + n v_d)})\\ &\qquad = \int_0^{v_d}\int_{\Sigma + (y/v_d)\vec{v}_0} e^{-2\pi i\langle \vec{m}, \vec{x} + (y/v_d)\vec{v}_0\rangle}\, \mathrm{d}\vec{x}\; e^{-2\pi i n y}\, \mathrm{d}y, \quad (\vec{m}, n) \ne (\vec{0}, 0), \end{aligned} \tag{6-2}$$

which implies

$$\hat{\tilde{g}}(\vec{m}, n) = \frac{\hat{\chi}_\Sigma(\vec{m})}{2\pi i(\langle \vec{m}, \vec{v}_0\rangle / v_d + n)}, \quad (\vec{m}, n) \ne (\vec{0}, 0). \tag{6-3}$$

We know Σ is a BRS with respect to $\vec{v}_0/v_d$; by (6-1) its transfer function $h : \mathbb{T}^{d-1} \to \mathbb{R}$ satisfies

$$\hat{h}(\vec{m}) = \frac{\hat{\chi}_\Sigma(\vec{m})}{1 - e^{-2\pi i\langle \vec{m}, \vec{v}_0\rangle / v_d}}, \quad \vec{m} \ne 0.$$

It is straightforward to check that the bounded function $\tilde{g}$ defined by

$$\tilde{g}(\vec{x}, y) = h\left(\vec{x} - \frac{\vec{v}_0}{v_d}\{y\}\right) - |\Sigma| \cdot \{y\}$$

satisfies the cohomological equation (6-3). Hence $\tilde{g}$ is a bounded transfer function for U with respect to $\vec{v}$.

Indeed, $\|\tilde{g}\|_\infty \le \|h\|_\infty + 1$. Since $\vec{v} = q\alpha - \vec{p}$, letting $g(\vec{x}) = \tilde{g}(\vec{x}) + \tilde{g}(\vec{x} - \alpha) + \cdots + \tilde{g}(\vec{x} - (q-1)\alpha)$ we have that U is a BRS with respect to α with bounded transfer function g satisfying $\|g\|_\infty \le |q| \|\tilde{g}\|_\infty \le |q|(\|h\|_\infty + 1)$. □

The following corollary will be used several times in Section 7.

Corollary 6.2. *Let $U \subset \mathbb{T}^2$ be the parallelogram spanned by two vectors*

$$m(\alpha_1, \alpha_2) - (l_1, l_2) \quad \textit{and} \quad \left(q\frac{m\alpha_1 - l_1}{m\alpha_2 - l_2} - p,\ 0\right).$$

Then U is a BRS with respect to (α_1, α_2) with transfer function g satisfying $\|g\|_\infty \le |m|(|q|+1) \le 2|mq|$.

Proof. In this case

$$v = (v_1, v_2) = m(\alpha_1, \alpha_2) - (l_1, l_2) \in \mathbb{Z}\alpha + \mathbb{Z}^2, \quad \Sigma = \left[0, q\frac{v_1}{v_2} - p\right] \times \{0\}.$$

We know the transfer function h of Σ with respect to v_1/v_2 satisfies $\|h\|_\infty \le |q|$. Thus $\|g\|_\infty \le |m|(|q|+1) \le 2|mq|$. □

7. 2-dimensional irrational rotation with weak Diophantine frequencies

In this section we deal with 2-dimensional weakly Diophantine frequencies. Our goal is to prove Lemma 3.5.

Proof of Lemma 3.5. Assume $(\alpha_1, \alpha_2) \in \mathrm{WDC}(c_0, \tau/4)$, for some $\tau > 4$ and $c_0 > 0$. We divide the discussion into two parts.

First, we introduce the coprime Diophantine condition:

$$\begin{aligned} \mathrm{PDC}(\tau) &= \bigcup_{c>0} \mathrm{PDC}(c, \tau) \\ &= \bigcup_{c>0} \left\{ (\alpha_1, \alpha_2) : \|\langle \vec{h}, \boldsymbol{\alpha} \rangle\|_{\mathbb{T}} \ge \frac{c}{|\vec{h}|^\tau} \text{ for any } \gcd(h_1, h_2) = 1 \text{ or } h_1 h_2 = 0 \text{ but } \vec{h} \ne \vec{0} \right\}. \end{aligned} \tag{7-1}$$

Obviously if $\alpha \in \mathrm{PDC}(c, \tau)$, both α_1 and α_2 belong to $\mathrm{DC}(c, \tau)$.

Next we will distinguish two different cases: PDC or non-PDC. Roughly speaking, in the PDC setting, we use bounded remainder sets technique presented in Section 6 and work directly with the 2-dimensional problem. In the non-PDC but WDC setting, we are able to reduce the 2-dimensional problem to the 1-dimensional problem, which is much easier to analyze.

Case A: $(\alpha_1, \alpha_2) \in \mathrm{PDC}(c_1, \tau)$ for some $c_1 > 0$. Let's take the best simultaneous approximation $\{(l_{1,n}/m_n, l_{2,n}/m_n)\}$ of (α_1, α_2). It has the following property.

Lemma 7.1 [Lagarias 1982, Theorem 3.5]. *If $\{1, \alpha_1, \alpha_2\}$ is linearly independent over $\mathbb{Q}$, then there are infinitely many n_k such that*

$$\begin{vmatrix} m_{n_k} & l_{1,n_k} & l_{2,n_k} \\ m_{n_k+1} & l_{1,n_k+1} & l_{2,n_k+1} \\ m_{n_k+2} & l_{1,n_k+2} & l_{2,n_k+2} \end{vmatrix} \neq 0.$$

Now we take $r_k > 0$ such that

$$m_{n_k} \leq \frac{4}{\pi} r_k^{-2} < m_{n_k+1}. \tag{7-2}$$

By (2-12), the choice of r_k guarantees that for $n \geq n_k$,

$$(m_n\alpha_1 - l_{1,n}, m_n\alpha_2 - l_{2,n}) \in B_{r_k}(0,0), \tag{7-3}$$

where

$$B_r(x_1, x_2) := \{y = (y_1, y_2) \in \mathbb{T}^2 : \|y_1 - x_1\|_{\mathbb{T}}^2 + \|y_2 - x_2\|_{\mathbb{T}}^2 < r_k^2\}.$$

Let $\{p_{n,s}/q_{n,s}\}_{s=1}^{\infty}$ be the continued fraction approximants of $(m_n\alpha_1 - l_{1,n})/(m_n\alpha_2 - l_{2,n})$. For each n choose s_n such that

$$q_{n,s_n} \leq r_k^{-1} < q_{n,s_n+1}. \tag{7-4}$$

By (2-7), the choice of s_n guarantees that

$$\left(q_{n,s_n} \frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}} - p_{n,s_n}, 0\right) \in B_{r_k}(0,0). \tag{7-5}$$

By (2-12) and (2-14) we have

$$\frac{c_0}{m_n^{\tau/4}} \leq \max\{|m_n\alpha_1 - l_{1,n}|, |m_n\alpha_2 - l_{2,n}|\} \leq \frac{2}{\sqrt{\pi}\sqrt{m_{n+1}}}, \tag{7-6}$$

and by (7-2) we have $m_{n_k} \leq (4/\pi) r_k^{-2}$. Thus

$$\max\,(m_{n_k}, m_{n_k+1}, m_{n_k+2}) \leq C_{c_0,\tau} r_k^{-\tau^2/2}. \tag{7-7}$$

We have:

Lemma 7.2. *For some $n \in \{n_k, n_k+1, n_k+2\}$, we have $q_{n,s_n+1} \leq r_k^{-2\tau^4}$.*

Let us postpone the proof of this lemma and finish the proof of Case A first.

Let U be the parallelogram spanned by the two vectors

$$m_n(\alpha_1, \alpha_2) - (l_{1,n}, l_{2,n}) \quad \text{and} \quad \left(q_{n,s_n} \frac{m_n\alpha_1 - l_{1,n}}{m_n\alpha_2 - l_{2,n}} - p_{n,s_n},\ 0\right).$$

By (7-3) and (7-5), $U\subset B_{2r_k}(0,0)$. Corollary 6.2 implies

$$\left|\sum_{j=0}^{M-1}\chi_U(x+j\alpha_1,y+j\alpha_2)-M|U|\right|\le 4|m_nq_{n,s_n}|$$

for a.e. (x,y). Thus as long as $M>4|m_nq_{n,s_n}|/|U|$, we have $\bigcup_{j=0}^{M-1}U-(j\alpha_1,j\alpha_2)$ covers the whole $\mathbb{T}^2$ up to a measure zero set. Then

$$\mathbb{T}^2\subseteq\bigcup_{j=0}^{M-1}B_{2r_k}(-j\alpha_1,-j\alpha_2)\quad\text{for } M>\frac{4|m_nq_{n,s_n}|}{|U|}.\tag{7-8}$$

Now we want to estimate $|U|$. Since $\alpha_2\in\mathrm{DC}(c_1,\tau)$, by (2-9) we have

$$|U|=|m_n\alpha_2-l_{2,n}|\cdot\left|q_{n,s_n}\frac{m_n\alpha_1-l_{1,n}}{m_n\alpha_2-l_{2,n}}-p_{n,s_n}\right|\ge\frac{c_1}{|m_n|^\tau}\frac{1}{2q_{n,s_n+1}}.$$

Thus by (7-4) and (7-7),

$$\frac{4|m_n|q_{n,s_n}}{|S|}\le\frac{8}{c_1}|m_n|^{1+\tau}q_{n,s_n}q_{n,s_n+1}\le C_{c_0,c_1,\tau}r_k^{-3\tau^4}.$$

This means it takes $B_{2r_k}(0,0)$ at most $C_{\alpha_1,\alpha_2,\tau}r_k^{-3\tau^4}$ steps to cover the whole $\mathbb{T}^2$.

Proof of Lemma 7.2. We will show it is impossible to have $q_{n,s_n+1}>r_k^{-2\tau^4}$ for all $n\in\{n_k,n_k+1,n_k+2\}$. In this case by (2-7), (2-12) and (7-2), we have

$$\begin{aligned}|q_{n,s_n}m_n\alpha_1-p_{n,s_n}m_n\alpha_2+M_n|&=|m_n\alpha_2-l_{2,n}|\cdot\left|q_{n,s_n}\frac{m_n\alpha_1-l_{1,n}}{m_n\alpha_2-l_{2,n}}-p_{n,s_n}\right|\\&<\frac{2}{\sqrt{\pi}\sqrt{|m_{n+1}|}q_{n,s_n}}<r_k^{2\tau^4+1},\end{aligned}\tag{7-9}$$

where $M_n=p_{n,s_n}l_{2,n}-q_{n,s_n}l_{1,n}$.

We have the following estimates on the upper bounds of p_{n,s_n} and M_n. Combining (2-9), (7-2), (7-4), (7-6) with (7-7),

$$|p_{n,s_n}|\le q_{n,s_n}\left|\frac{m_n\alpha_1-l_{1,n}}{m_n\alpha_2-l_{2,n}}\right|+\frac{1}{q_{n,s_n+1}}\le\frac{2q_{n,s_n}|m_n|^\tau}{c_1\sqrt{\pi}\sqrt{|m_{n+1}|}}+r_k^{2\tau^4}\le C_{c_0,c_1,\tau}r_k^{-\tau^3/2}.\tag{7-10}$$

By (7-9), (7-2), (7-7), (7-4) and (7-10),

$$|M_n|<|q_{n,s_n}m_n\alpha_1-p_{n,s_n}m_n\alpha_2|+r_k^{2\tau^4}\le C_{c_0,c_1,\tau}r_k^{-\tau^3}.\tag{7-11}$$

Case 1: If $p_{n,s_n}=0$ for some $n\in\{n_k,n_k+1,n_k+2\}$, then by (2-7), (2-12) and (7-1), (2-9),(7-2), (7-7), we have

$$r_k^{2\tau^4}>\frac{1}{q_{n,s_n+1}}\ge\left|q_{n,s_n}\frac{m_n\alpha_1-l_{1,n}}{m_n\alpha_2-l_{2,n}}\right|\ge\frac{c_1\sqrt{\pi}\sqrt{|m_{n+1}|}}{2m_n^\tau}\ge C_{c_0,c_1,\tau}r_k^{\tau^3/2+1},$$

which is a contradiction.

Case 2: If $M_n = 0$ for some $n \in \{n_k, n_k+1, n_k+2\}$, then by (7-9), (7-2), (7-10), and the fact that $(\alpha_1, \alpha_2) \in \mathrm{PDC}(c_1, \tau)$, we have

$$r_k^{2\tau^4} > |m_n|\,|q_{n,s_n}\alpha_1 - p_{n,s_n}\alpha_2| \geq \frac{c_1|m_n|}{\max(p_{n,s_n}, q_{n,s_n})^\tau} \geq C_{c_0,c_1,\tau} r_k^{\tau^4/2},$$

again a contradiction.

Case 3: If $p_{n,s_n} \neq 0$ and $M_n \neq 0$ for any $n \in \{n_k, n_k+1, n_k+2\}$, then for any $i, j \in \{n_k, n_k+1, n_k+2\}$, we have

$$\begin{aligned} |(q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i)\alpha_1 - (p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i)\alpha_2| \\ \leq |(q_{i,s_i} m_i \alpha_1 - p_{i,s_i} m_i \alpha_2 + M_i) M_j| + |(q_{j,s_j} m_j \alpha_1 - p_{j,s_j} m_j \alpha_2 + M_j) M_i| \\ < (|M_i| + |M_j|) r_k^{2\tau^4}. \end{aligned} \tag{7-12}$$

Case 3.1: $(q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i,\ p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i) \neq (0,0)$ for some $i, j \in \{n_k, n_k+1, n_k+2\}$. In this case let $h = \gcd(q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i,\ p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i)$ be the greatest common divisor of the two numbers if they are both nonzero, and $h = 1$ otherwise. Then by (7-12),

$$\left| \frac{q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i}{h}\alpha_1 - \frac{p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i}{h}\alpha_2 \right| < \frac{|M_i| + |M_j|}{h} r_k^{2\tau^4}.$$

However on one hand by (7-11),

$$\frac{|M_i| + |M_j|}{h} r_k^{2\tau^4} \leq (|M_i| + |M_j|) r_k^{2\tau^4} \leq C_{c_0,c_1,\tau} r_k^{2\tau^4 - \tau^3}.$$

On the other hand, by the fact that $(\alpha_1, \alpha_2) \in \mathrm{PDC}(c_1, \tau)$ and (7-2), (7-7), (7-10), (7-11),

$$\begin{aligned} &\left| \frac{q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i}{h}\alpha_1 - \frac{p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i}{h}\alpha_2 \right| \\ &\qquad \geq \frac{c_1 h^\tau}{|(q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i,\, p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i)|^\tau} \\ &\qquad \geq C_{c_0,c_1,\tau} r_k^{7\tau^4/4}, \end{aligned}$$

a contradiction.

Case 3.2: For any $i, j \in \{n_k, n_k+1, n_k+2\}$

$$q_{i,s_i} m_i M_j = q_{j,s_j} m_j M_i,$$
$$p_{i,s_i} m_i M_j = p_{j,s_j} m_j M_i.$$

Then for $n = n_k$,

$$\frac{p_{n,s_n}}{q_{n,s_n}} = \frac{p_{n+1,s_{n+1}}}{q_{n+1,s_{n+1}}} = \frac{p_{n+2,s_{n+2}}}{q_{n+2,s_{n+2}}}.$$

Hence we can let $p = p_{n,s_n} = p_{n+1,s_{n+1}} = p_{n+2,s_{n+2}}$ and $q = q_{n,s_n} = q_{n+1,s_{n+1}} = q_{n+2,s_{n+2}}$. Then we have (after plugging in $M_n = q l_{1,n} - p l_{2,n}$)

$$q(m_n l_{1,n+1} - m_{n+1} l_{1,n}) = p(m_n l_{2,n+1} - m_{n+1} l_{2,n}), \tag{7-13}$$

$$q(m_n l_{1,n+2} - m_{n+2} l_{1,n}) = p(m_n l_{2,n+2} - m_{n+2} l_{2,n}), \tag{7-14}$$

$$q(m_{n+1} l_{1,n+2} - m_{n+2} l_{1,n+1}) = p(m_{n+1} l_{2,n+2} - m_{n+2} l_{2,n+1}). \tag{7-15}$$

Then considering (7-13) $\cdot (-l_{1,n+2})$ + (7-14) $\cdot\, l_{1,n+1}$ + (7-15) $\cdot (-l_{1,n})$, we get

$$p \cdot \begin{vmatrix} m_{n_k} & l_{1,n_k} & l_{2,n_k} \\ m_{n_k+1} & l_{1,n_k+1} & l_{2,n_k+1} \\ m_{n_k+2} & l_{1,n_k+2} & l_{2,n_k+2} \end{vmatrix} = q \cdot 0 = 0,$$

a contradiction with the choice of n_k. □

<u>Case B</u>: $(\alpha_1, \alpha_2) \notin \mathrm{PDC}(\tau)$. By the definition of $\mathrm{PDC}(\tau)$, the sequence $\vec{h}_n = (h_{1,n}, h_{2,n})$ for which (7-1) fails has to satisfy either $\gcd(h_{1,n}, h_{2,n}) = 1$ (Case B.1) or $h_{1,n} h_{2,n} = 0$ (Case B.2).

<u>Case B.1</u>: We can find a sequence $\{n_j\}$ such that $|\vec{h}_{n_j}| = \max(|h_{1,n_j}|, |h_{2,n_j}|) \to \infty$ as $j \to \infty$, $\gcd(h_{1,n_j}, h_{2,n_j}) = 1$ and

$$\|h_{1,n_j}\alpha_1 + h_{2,n_j}\alpha_2\|_{\mathbb{T}} < \frac{1}{|\vec{h}_{n_j}|^\tau}.$$

Without loss of generality, we can assume $|h_{1,n_j}| = |\vec{h}_{n_j}|$. In this case we can take $r_{n_j} = 1/|h_{1,n_j}|$. For simplicity we will denote n_j by n.

Now that $\|h_{1,n}\alpha_1 + h_{2,n}\alpha_2\|_{\mathbb{T}} < 1/|h_{1,n}|^\tau$, we can find $l_{1,n}, l_{2,n} \in \mathbb{Z}$ such that $|h_{1,n}(\alpha_1 - l_{1,n}) + h_{2,n}(\alpha_2 - l_{2,n})| < 1/|h_{1,n}|^\tau$. Since replacing (α_1, α_2) with $(\alpha_1 + l_{1,n}, \alpha_2 + l_{2,n})$ does not change anything, we will assume $|h_{1,n}\alpha_1 + h_{2,n}\alpha_2| < 1/|h_{1,n}|^\tau$. Then

$$\left|\frac{\alpha_2}{\alpha_1} - \left(-\frac{h_{1,n}}{h_{2,n}}\right)\right| < \frac{1}{|h_{1,n}|^\tau \alpha_1}. \tag{7-16}$$

We consider the following two lines on $\mathbb{T}^2$:

$$l_1(t) = \left(\{t\}, \left\{\frac{\alpha_2}{\alpha_1} t\right\}\right) \quad \text{and} \quad l_2(t) = \left(\{t\}, \left\{-\frac{h_{1,n}}{h_{2,n}} t\right\}\right).$$

These two lines are close to each other in the sense that for $|t| \le |h_{1,n}|^{3\tau/4}$, by (7-16),

$$\left\| \left\{\frac{\alpha_2}{\alpha_1} t\right\} - \left\{-\frac{h_{1,n}}{h_{2,n}} t\right\} \right\|_{\mathbb{T}} \le \left|\frac{\alpha_2}{\alpha_1} t + \frac{h_{1,n}}{h_{2,n}} t\right| \le \frac{|t|}{|h_{1,n}|^\tau \alpha_1} \le \frac{1}{|h_{1,n}|^{\tau/4} \alpha_1}.$$

The graph of $l_2(t)$ is the hypotenuse of a right triangle with two legs of lengths $|h_{1,n}|$ and $|h_{2,n}|$ (mod $\mathbb{Z}^2$). We consider the orbit of $(\alpha_1, -(h_{1,n}/h_{2,n})\alpha_1)$ under the rotation $(\alpha_1, -(h_{1,n}/h_{2,n})\alpha_1)$. These points lie on $l_2(t)$. Under this rotation the point moves a distance $(\sqrt{h_{1,n}^2 + h_{2,n}^2}/|h_{2,n}|)\alpha_1$ at each step by a big interval with length $\sqrt{h_{1,n}^2 + h_{2,n}^2}$. Let $\{p_m/q_m\}_{m=1}^\infty$ be the continued fraction approximants of $\alpha_1/h_{2,n}$. Choose m such that

$$q_{m-1} \le |h_{1,n}| \sqrt{h_{1,n}^2 + h_{2,n}^2} < q_m. \tag{7-17}$$

Then it would take a point on $\mathbb{T}$ at most $q_m + q_{m-1}$ steps (under the $(\alpha_1/h_{2,n})$-rotation) to enter each interval of length $1/(|h_{1,n}|\sqrt{h_{1,n}^2 + h_{2,n}^2})$ on $\mathbb{T}$, see, e.g., [Jitomirskaya and Last 2000], which means it would take a point on $l_2(t)$ at most $q_m + q_{m-1} - 1$ steps (under the $(\sqrt{h_{1,n}^2 + h_{2,n}^2}\alpha_1/|h_{2,n}|)$-rotation) to enter each interval of length $1/|h_{1,n}| = r_n$ on the graph of $l_2(t)$. Moreover, it is easy to see that the distance from any $x \in \mathbb{T}^2$ to $l_2(t)$ is bounded by $1/\sqrt{h_{1,n}^2 + h_{2,n}^2} < r_n$. Thus

$$\mathbb{T}^2 \subseteq \bigcup_{k=0}^{q_m+q_{m-1}} B_{2r_n}\left(k\alpha_1, -\frac{h_{1,n}}{h_{2,n}}k\alpha_1\right). \tag{7-18}$$

By (2-7) and (7-16),

$$\left| p_{m-1} + q_{m-1}\frac{\alpha_2}{h_{1,n}} \right| = \left| p_{m-1} - q_{m-1}\frac{\alpha_1}{h_{2,n}} + q_{m-1}\left(\frac{\alpha_1}{h_{2,n}} + \frac{\alpha_2}{h_{1,n}}\right) \right| \le \frac{1}{q_m} + \frac{q_{m-1}}{|h_{1,n}|^{\tau-1}}.$$

This implies, by (2-7) and (7-17),

$$\begin{aligned} \|q_{m-1}\alpha_1\|_{\mathbb{T}} &\le |q_{m-1}\alpha_1 - h_{2,n}p_{m-1}| \le \frac{|h_{2,n}|}{q_m}, \\ \|q_{m-1}\alpha_2\|_{\mathbb{T}} &\le \frac{|h_{1,n}|}{q_m} + \frac{2}{|h_{1,n}|^{\tau-4}}. \end{aligned}$$

Then by the fact that $\alpha \in \mathrm{WDC}(c_0, \tau/4)$ and (7-17),

$$\max\left\{ \frac{|h_{2,n}|}{q_m}, \frac{|h_{1,n}|}{q_m} + \frac{2}{|h_{1,n}|^{\tau-4}} \right\} \ge \max\left(\|q_{m-1}\alpha_1\|_{\mathbb{T}}, \|q_{m-1}\alpha_2\|_{\mathbb{T}}\right) \ge \frac{c_0}{q_{m-1}^{\tau/4}} \ge \frac{c_0}{2^{\tau/4}|h_{1,n}|^{\tau/2}}.$$

This implies

$$q_m + q_{m-1} < 2q_m \le \frac{2^{\tau/4+2}}{c_0}|h_{1,n}|^{\tau/2+1}. \tag{7-19}$$

Since

$$0 \le k \le \frac{2^{\tau/4+2}}{c_0}|h_{1,n}|^{\tau/2+1} < r_n^{-3\tau/4},$$

by (7-16) the points $(k\alpha_1, k\alpha_2)$ and $(k\alpha_1, -(h_{1,n}/h_{2,n})k\alpha_1)$ differ at most by $r_n^{\tau/4}$, so we obtain using (7-18) and (7-19),

$$\mathbb{T}^2 \subseteq \bigcup_{k=0}^{r_n^{-3\tau/4}} B_{3r_n}(k\alpha_1, k\alpha_2).$$

Case B.2: We can find a sequence $\{n_j\}$ such that $h_{2,n_j} \equiv 0$ and $|h_{1,n_j}| \to \infty$ such that

$$\|h_{1,n_j}\alpha_1\|_{\mathbb{T}} < \frac{1}{|h_{1,n_j}|^{\tau}}. \tag{7-20}$$

For simplicity we will replace n_j with n. We can find M_n such that $|h_{1,n}\alpha_1 - M_n| < 1/|h_{1,n}|^{\tau}$. Let $d_n = \gcd(h_{1,n}, M_n)$ be the greatest common divisor. Let $\tilde{h}_{1,n} = h_{1,n}/d_n$ and $\widetilde{M}_n = M_n/d_n$. We have

$$\left| \alpha_1 - \frac{\widetilde{M}_n}{\tilde{h}_{1,n}} \right| < \frac{1}{|h_{1,n}|^{\tau+1}} \to 0. \tag{7-21}$$

If $\tilde{h}_{1,n}$ is bounded in n, then α_1 can be approximated arbitrarily closely by rationals with bounded denominators, which is impossible. Thus $|\tilde{h}_{1,n}| \to \infty$. Now take radius $r_n = 1/|\tilde{h}_{1,n}|$. For each $0 \le i \le \tilde{h}_{1,n} - 1$ consider $\{(i\alpha_1 + k\tilde{h}_{1,n}\alpha_1,\ i\alpha_2 + k\tilde{h}_{1,n}\alpha_2)\}_{k=0}^{\infty}$. Let $\{p_m/q_m\}_{m=1}^{\infty}$ be the continued fraction approximants of $\tilde{h}_{1,n}\alpha_2$. Choose m such that

$$q_{m-1} \le |\tilde{h}_{1,n}| = r_n^{-1} < q_m. \tag{7-22}$$

Then it takes any point on $\mathbb{T}$ at most $q_m + q_{m-1} - 1$ steps (under the $\tilde{h}_{1,n}\alpha_2$–rotation) to enter each interval of length r_n; see, e.g., [Jitomirskaya and Last 2000]. By (2-7),

$$|p_{m-1} - q_{m-1}\tilde{h}_{1,n}\alpha_2| \le \frac{1}{q_m}. \tag{7-23}$$

By (7-20), (7-22) and since $\tau > 4$, we have

$$\|q_{m-1}\tilde{h}_{1,n}\alpha_1\| \le \frac{q_{m-1}}{|\tilde{h}_{1,n}|^{\tau}} < \frac{c_0}{(q_{m-1}|\tilde{h}_{1,n}|)^{\tau/4}}.$$

By the fact that $\alpha \in \mathrm{WDC}(c_0, \tau/4)$,

$$\|q_{m-1}\tilde{h}_{1,n}\alpha_2\| \ge \frac{c_0}{(q_{m-1}|\tilde{h}_{1,n}|)^{\tau/4}}.$$

By (7-23) and (7-22), we have

$$q_m \le \frac{1}{c_0}|\tilde{h}_{1,n}|^{\tau/2}. \tag{7-24}$$

Now for $0 \le k \le q_m + q_{m-1} - 1$, by (7-21), (7-20) and (7-24),

$$\left\| i\alpha_1 + k\tilde{h}_{1,n}\alpha_1 - \frac{i\widetilde{M}_n}{\tilde{h}_{1,n}} \right\|_{\mathbb{T}} \le \frac{C}{|\tilde{h}_{1,n}|^{\tau/2}} = C r_n^{\tau/2}.$$

Since $\gcd(\tilde{h}_{1,n}, \widetilde{M}_n) = 1$, any interval of length $r_n = 1/|\tilde{h}_{1,n}|$ contains $i\widetilde{M}_n/\tilde{h}_{1,n}$ for some $0 \le i \le \tilde{h}_{1,n} - 1$. Thus

$$\mathbb{T}^2 \subseteq \bigcup_{k=0}^{(q_m+q_{m-1})|\tilde{h}_{1,n}|} B_{r_n}(k\alpha_1, k\alpha_2).$$

By (7-24), $(q_m + q_{m-1})|\tilde{h}_{1,n}| \le r_n^{-\tau}$, so we have

$$\mathbb{T}^2 \subseteq \bigcup_{k=0}^{r_n^{-\tau}} B_{r_n}(k\alpha_1, k\alpha_2), \tag{7-25}$$

completing the proof of Case B.2 and thus of Lemma 3.5. □

Appendix

Proof of Lemma 3.7. For sufficiently small $\epsilon > 0$, fix an integer $H_0 \sim N^{1/(d(\tau-1)+1+d\epsilon)}$, define $g(n) = 1/(n(n+1))$ for $1 \le n < H_0$ and $g(H_0) = 1/H_0$. For $(n_1, \dots, n_d) \in \mathbb{Z}^d$ with $1 \le n_i \le H_0$, define $f(n_1,\dots,n_d)=\prod_{i=1}^d g(n_i)$. By Lemma 2.5, we have

$$\begin{aligned}
D_N(\theta) &\le C_d\biggl(\frac{1}{H_0}+\sum_{0<|h|\le H_0}\frac{1}{r(\vec{h})}\biggl|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i\langle\vec{h},\alpha\rangle n}\biggr|\biggr)\\
&\le \widetilde{C}_d\biggl(\frac{1}{H_0}+\frac{1}{N}\sum_{0<|h|\le H_0}\frac{1}{r(\vec{h})}\frac{1}{\|\langle\vec{h},\alpha\rangle\|_{\mathbb{T}}}\biggr)\\
&= \widetilde{C}_d\biggl(\frac{1}{H_0}+\frac{1}{N}\sum_{n_1,\dots,n_d=1}^{H_0}f(n_1,\dots,n_d)\sum_{\vec{h}=(h_1,\dots,h_d)\neq\vec{0},\,|h_j|\le n_j}\frac{1}{\|\langle\vec{h},\alpha\rangle\|_{\mathbb{T}}}\biggr)\\
&\le \widetilde{C}_d\biggl(\frac{1}{H_0}+\frac{1}{N}\sum_{n_1,\dots,n_d=1}^{H_0}f(n_1,\dots,n_d)\sum_{j=1}^{3^d r(\vec{n})}\frac{r(\vec{n})^\tau}{j}\biggr)\\
&\le \widetilde{C}_d\biggl(\frac{1}{H_0}+\frac{1}{N}\sum_{n_1,\dots,n_d=1}^{H_0}f(n_1,\dots,n_d)r(\vec{n})^\tau\log r(\vec{n})\biggr)\\
&\le \widetilde{C}_d\biggl(\frac{1}{H_0}+\frac{{H_0}^{d(\tau-1+\epsilon)}}{N}\biggr)\\
&\lesssim N^{-1/(d(\tau-1)+1+d\epsilon)}.
\end{aligned}$$

□

Acknowledgement

Han would like to thank Anton Gorodetski for valuable discussions. This research was partially supported by the NSF grants DMS-1401204 and DMS-1800689. Jitomirskaya would like to thank the support of the Simons Foundation where she was a Fellow in 2014–15. We are also grateful to the Isaac Newton Institute for Mathematical Sciences, Cambridge, for its hospitality, supported by EPSRC grant number EP/K032208/1, during the programme Periodic and Ergodic Spectral Problems, where this work was started.

References

[Avila et al. 2014] A. Avila, D. Damanik, and Z. Zhang, "Singular density of states measure for subshift and quasi-periodic Schrödinger operators", *Comm. Math. Phys.* **330**:2 (2014), 469–498. MR Zbl

[Berger 2003] M. Berger, *A panoramic view of Riemannian geometry*, Springer, 2003. MR Zbl

[Bourgain and Goldstein 2000] J. Bourgain and M. Goldstein, "On nonperturbative localization with quasi-periodic potential", *Ann. of Math.* (2) **152**:3 (2000), 835–879. MR Zbl

[Bourgain and Jitomirskaya 2000] J. Bourgain and S. Jitomirskaya, "Anderson localization for the band model", pp. 67–79 in *Geometric aspects of functional analysis*, edited by V. D. Milman and G. Schechtman, Lecture Notes in Math. **1745**, Springer, 2000. MR Zbl

[Bourgain et al. 2001] J. Bourgain, M. Goldstein, and W. Schlag, "Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by the skew-shift", *Comm. Math. Phys.* **220**:3 (2001), 583–621. MR Zbl

[do Carmo 1992] M. P. do Carmo, *Riemannian geometry*, Birkhäuser, Boston, 1992. MR Zbl

[Damanik and Tcheremchantsev 2003] D. Damanik and S. Tcheremchantsev, "Power-law bounds on transfer matrices and quantum dynamics in one dimension", *Comm. Math. Phys.* **236**:3 (2003), 513–534. MR Zbl

[Damanik and Tcheremchantsev 2007] D. Damanik and S. Tcheremchantsev, "Upper bounds in quantum dynamics", *J. Amer. Math. Soc.* **20**:3 (2007), 799–827. MR Zbl

[Damanik and Tcheremchantsev 2008] D. Damanik and S. Tcheremchantsev, "Quantum dynamics via complex analysis methods: general upper bounds without time-averaging and tight lower bounds for the strongly coupled Fibonacci Hamiltonian", *J. Funct. Anal.* **255**:10 (2008), 2872–2887. MR Zbl

[Del Rio et al. 1994] R. Del Rio, N. Makarov, and B. Simon, "Operators with singular continuous spectrum, II: Rank one operators", *Comm. Math. Phys.* **165**:1 (1994), 59–67. MR Zbl

[Grepstad and Lev 2015] S. Grepstad and N. Lev, "Sets of bounded discrepancy for multi-dimensional irrational rotation", *Geom. Funct. Anal.* **25**:1 (2015), 87–133. MR Zbl

[Jitomirskaya and Last 1999] S. Jitomirskaya and Y. Last, "Power-law subordinacy and singular spectra, I: Half-line operators", *Acta Math.* **183**:2 (1999), 171–189. MR Zbl

[Jitomirskaya and Last 2000] S. Y. Jitomirskaya and Y. Last, "Power law subordinacy and singular spectra, II: Line operators", *Comm. Math. Phys.* **211**:3 (2000), 643–658. MR Zbl

[Jitomirskaya and Mavi 2017] S. Jitomirskaya and R. Mavi, "Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials", *Int. Math. Res. Not.* **2017**:1 (2017), 96–120. MR Zbl

[Jitomirskaya and Zhang 2015] S. Jitomirskaya and S. Zhang, "Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators", preprint, 2015. arXiv

[Jitomirskaya et al. 2003] S. Jitomirskaya, H. Schulz-Baldes, and G. Stolz, "Delocalization in random polymer models", *Comm. Math. Phys.* **233**:1 (2003), 27–48. MR Zbl

[Khinchin 1964] A. Y. Khinchin, *Continued fractions*, The University of Chicago Press, 1964. MR

[Killip et al. 2003] R. Killip, A. Kiselev, and Y. Last, "Dynamical upper bounds on wavepacket spreading", *Amer. J. Math.* **125**:5 (2003), 1165–1198. MR Zbl

[Koksma 1950] J. F. Koksma, *Some theorems on Diophantine inequalities*, Stichting Mathematisch Centrum **5**, Math. Centrum Amsterdam, 1950. MR Zbl

[Kuipers and Niederreiter 1974] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, New York, 1974. Pure and Applied Mathematics. MR Zbl

[Lagarias 1982] J. C. Lagarias, "Best simultaneous Diophantine approximations, II: Behavior of consecutive best approximations", *Pacific J. Math.* **102**:1 (1982), 61–88. MR Zbl

[Last 1996] Y. Last, "Quantum dynamics and decompositions of singular continuous spectra", *J. Funct. Anal.* **142**:2 (1996), 406–445. MR Zbl

[Nicolaescu 2007] L. I. Nicolaescu, *Lectures on the geometry of manifolds*, 2nd ed., World Scientific, Hackensack, NJ, 2007. MR Zbl

[del Rio et al. 1996] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, "Operators with singular continuous spectrum, IV: Hausdorff dimensions, rank one perturbations, and localization", *J. Anal. Math.* **69** (1996), 153–200. MR Zbl

[Simon 2007] B. Simon, "Equilibrium measures and capacities in spectral theory", *Inverse Probl. Imaging* **1**:4 (2007), 713–772. MR Zbl

[Yomdin 1987] Y. Yomdin, "Volume growth and entropy", *Israel J. Math.* **57**:3 (1987), 285–300. MR Zbl

Received 6 Dec 2016. Revised 30 May 2018. Accepted 5 Jul 2018.

RUI HAN: rui.han@math.gatech.edu
Department of Mathematics, Georgia Institute of Technology, Atlanta, GA, United States

SVETLANA JITOMIRSKAYA: szhitomi@math.uci.edu
Department of Mathematics, University of California, Irvine, CA, United States

Analysis & PDE
msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard
patrick.gerard@math.u-psud.fr
Université Paris Sud XI
Orsay, France

BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Clément Mouhot	Cambridge University, UK c.mouhot@dpmms.cam.ac.uk
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Alessio Figalli	ETH Zurich, Switzerland alessio.figalli@math.ethz.ch	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	András Vasy	Stanford University, USA andras@math.stanford.edu
Richard B. Melrose	Massachussets Inst. of Tech., USA rbm@math.mit.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2019 is US $310/year for the electronic version, and $520/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2019 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 12 No. 4 2019

2157-5045(2019)12:4;1-M