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TWO-DIMENSIONAL GRAVITY WATER WAVES WITH CONSTANT VORTICITY I: CUBIC LIFESPAN

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This article is concerned with the incompressible, infinite-depth water wave equation in two space dimensions, with gravity and constant vorticity but with no surface tension. We consider this problem expressed in position-velocity potential holomorphic coordinates, and prove local well-posedness for large data, as well as cubic lifespan bounds for small-data solutions.

1. Introduction

The motion of water in contact with air is well described by the incompressible Euler equations in the fluid domain, combined with two boundary conditions on the free surface, i.e., the interface with air. In special, but still physically relevant cases, the equations of motion can be viewed as evolution equations for the free surface. These equations are commonly referred to as the water wave equations. Most notably, this is the case for irrotational flows. However, in two space dimensions there is a natural extension of these equations to flows with constant vorticity.

In previous work [Hunter, Ifrim and Tataru 2016; Ifrim and Tataru 2016] we have considered the local and long-time behavior for the irrotational gravity wave equations with infinite depth in two space dimensions. In this article we take a first step toward understanding the more difficult question of the long-time behavior of gravity waves with infinite depth and constant vorticity, either in $\mathbb{R} \times \mathbb{R}$ or in the periodic case $\mathbb{R} \times \mathbb{T}$. We begin by establishing a local well-posedness result. Then we consider the lifespan of small-data solutions, where, like in the zero vorticity case [Hunter, Ifrim and Tataru 2016],

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we are able to prove cubic lifespan bounds. To the best of our knowledge, this is the first long-time well-posedness result for this problem.

We remark that it is of further interest to consider small localized data, and establish an almost global in time result, as it was done in the irrotational case in [Hunter, Ifrim and Tataru 2016], improving an earlier result of [Wu 2009]. However, in the constant vorticity case this problem presents some interesting new challenges. We hope to consider this in subsequent work.

The motivation to study the constant vorticity problem comes from multiple sources. On one hand, from a mathematical perspective, it provides us with the possibility to consider vorticity effects in a framework where the equations of motion can still be described in terms of the water/air interface, while allowing for a larger range of dynamic behavior, which is particularly interesting over large time scales. On the other hand, from a practical perspective, constant vorticity flows are good models for the water motion in the presence of countercurrents. An interesting example of this type is provided by tidal effects.

The constant vorticity problem is a subset of the full vorticity problem, and as such, local well-posedness for regular enough data can be viewed as a consequence of results for the general problem. For this we refer the reader to [Christodoulou and Lindblad 2000; Lannes 2005; Lindblad 2005; Coutand and Shkoller 2007; Shatah and Zeng 2008; Zhang and Zhang 2008]. There has also been a considerable body of work devoted to the study of solitary waves in constant vorticity flows. A good source of information in this direction is provided by several recent articles [Constantin and Varvaruca 2011; Kozlov, Kuznetsov and Lokharu 2014; Constantin, Kalimeris and Scherzer 2015], as well as the survey article [Strauss 2010]. Various ways of formulating the equations have been described in [Wahlén 2007; Ehrnström 2008; Ashton and Fokas 2011].

The conformal formulation for two-dimensional water waves, which we adopt here, following our previous work [Hunter, Ifrim and Tataru 2016], originates in early work on traveling waves; see, e.g., [Levi-Civita 1925]. For the dynamical problem, to the best of our knowledge it first appears in [Ovsjannikov 1974], but was better developed later in [Wu 1997] and also in [Dyachenko, Kuznetsov, Spector and Zakharov 1996]. It has been widely used since then in order to study a variety of water wave problems. However, as far as we know, this is the first article where this formulation is fully implemented in the constant vorticity case.

1A. ***Equations of motion.*** The water flow is governed by the incompressible Euler equations in the fluid domain Ω_t, with a dynamic and a kinematic boundary condition on the free surface of the fluid Γ_t. Denoting the fluid velocity by $\boldsymbol{u}(t,x,y) = (u(t,x,y), v(t,x,y))$, and the pressure by $p(t,x,y)$, the equations of motion are the equation of mass conservation

$$u_x + v_y = 0 \quad \text{in } \Omega(t), \tag{1-1}$$

and Euler's equations also in $\Omega(t)$

$$\begin{cases} u_t + uu_x + vu_y = -p_x, \\ v_t + uv_x + vv_y = -p_y - g, \end{cases} \tag{1-2}$$

where g is the gravitational constant. The boundary conditions for capillary-gravity waves are the dynamic boundary condition

$$p = p_0 \quad \text{on } \Gamma(t), \tag{1-3}$$

p_0 being the constant atmospheric pressure, and the kinematic boundary condition, which asserts that the free boundary $\Gamma(t)$ is transported along the flow. Since we are in the two-dimensional case, the vorticity will also be transported along the flow. This makes it possible to study flows with a nonzero constant vorticity field,

$$\omega = u_y - v_x = -c, \quad \text{where } c \text{ is a constant.}$$

Then the velocity field $\boldsymbol{u}$ can be represented as

$$\boldsymbol{u} = (cy + \varphi_x, \varphi_y),$$

where $\varphi(t, x, y)$ is called the (generalized) velocity potential. Here φ it is defined up to an arbitrary function of time and, by the incompressibility condition, it satisfies the Laplace equation in $\Omega(t)$

$$\triangle\varphi(t, x, y) = 0.$$

This brings us to our boundary condition on the bottom, which asserts that

$$\lim_{y\to-\infty} \varphi(x, y) = 0, \quad \text{uniformly in } x.$$

Then φ is uniquely determined by its values on the free surface $\Gamma(t)$.

Introducing its harmonic conjugate $\theta(t, x, y)$,

$$\varphi_x = \theta_y, \quad \varphi_y = -\theta_x,$$

we can rewrite the equations in (1-2) as a single scalar equation in the fluid domain:

$$\nabla\big(\varphi_t - c\theta + cy\varphi_x + \tfrac{1}{2}(\varphi_x^2 + \varphi_y^2) + gy\big) = 0.$$

Since φ is only defined up to an arbitrary function of time, we can absorb a function of time into φ. Using the fact that the pressure is constant on $\Gamma(t)$, we obtain the following analogue of Bernoulli's equation:

$$\varphi_t - c\theta + cy\varphi_x + \tfrac{1}{2}(\varphi_x^2 + \varphi_y^2) + gy = 0 \quad \text{on } \Gamma(t). \tag{1-4}$$

This is the equation that makes possible reduction of dimensionality of the problem, in the same manner as for purely potential flows. This is in combination with the kinematic boundary condition, which is already restricted to $\Gamma(t)$. We remark that expressing θ and the full gradient $\nabla\varphi$ on $\Gamma(t)$ in terms of the restriction of φ to $\Gamma(t)$ requires using the Dirichlet-to-Neumann map associated to the fluid domain, and in turn makes our equations nonlocal.

1B. ***The equations in holomorphic coordinates.*** The first issue we need to address is the choice of coordinates. Here we take our cue from [Hunter, Ifrim and Tataru 2016], henceforth abbreviated [HIT16], and work in holomorphic coordinates. There are also other ways of expressing the equations, for instance in Cartesian coordinates using the Dirichlet-to-Neumann map associated to the water domain; see, e.g., [Alazard, Burq and Zuily 2011; 2014]. However, we prefer the holomorphic coordinates due to the simpler form of the equations; in particular, in these coordinates the Dirichlet-to-Neumann map is diagonalized and given in terms of the standard Hilbert transform.

We obtain a system which models the time evolution of the free surface, described via a function W as the graph of the function $\alpha \to W(\alpha)+\alpha$, and that of the holomorphic velocity potential $Q=\varphi+i\theta$ restricted to the free surface. The derivation of the equations is relegated to Appendix B, also using some of the analysis from the Appendix in [HIT16]. Some minor changes are needed for the periodic case; these are also described in [HIT16, Appendix B]. The outcome of this computation is the following set of equations:

$$\begin{cases} W_t+(W_\alpha+1)\underline{F}+i\frac{c}{2}W=0,\\ Q_t-igW+\underline{F}Q_\alpha+icQ+\boldsymbol{P}\left[\frac{|Q_\alpha|^2}{J}\right]-i\frac{c}{2}T_1=0, \end{cases} \tag{1-5}$$

where $J:=|1+W_\alpha|^2$, $\boldsymbol{P}$ is the projection onto negative wave numbers

$$\boldsymbol{P}=\tfrac{1}{2}(I-iH),\quad \text{with } H \text{ the Hilbert transform,}$$

and

$$F:=\boldsymbol{P}\left[\frac{Q_\alpha-\bar{Q}_\alpha}{J}\right],\quad F_1:=\boldsymbol{P}\left[\frac{W}{1+\overline{W}_\alpha}+\frac{\overline{W}}{1+W_\alpha}\right],$$
$$\underline{F}:=F-i\frac{c}{2}F_1,\qquad T_1:=\boldsymbol{P}\left[\frac{W\bar{Q}_\alpha}{1+\overline{W}_\alpha}-\frac{\overline{W}Q_\alpha}{1+W_\alpha}\right].$$

These equations are considered either in $\mathbb{R}\times\mathbb{R}$ or in $\mathbb{R}\times\mathbb{S}^1$. They model an evolution in the space of functions which admit bounded holomorphic extensions to the lower half-space; equivalently, their Fourier transform is supported on the negative real line. By a slight abuse, we call such functions holomorphic.

This is a Hamiltonian system, where the Hamiltonian has the form

$$\mathcal{E}(W,Q)=\Re\int g|W|^2(1+W_\alpha)-iQ\bar{Q}_\alpha+cQ_\alpha(\Im W)^2-\frac{c^3}{2i}|W|^2W(1+W_\alpha)\,d\alpha. \tag{1-6}$$

A second conserved quantity is the horizontal momentum,

$$\mathcal{P}(W,Q)=\int\left\{\frac{1}{i}(\bar{Q}W_\alpha-Q\overline{W}_\alpha)-c|W|^2+\frac{c}{2}(W^2\overline{W}_\alpha+\overline{W}^2W_\alpha)\right\}d\alpha, \tag{1-7}$$

which is the Hamiltonian for the group of horizontal translations.

We remark that if $c=0$ then the equations (1-5) are the gravity water wave equations studied in [HIT16]. The sign of c is not important, as it can be switched via the flip $\alpha\to-\alpha$. For convenience we assume $c>0$. The space-time scaling

$$(W(t,\alpha),Q(t,\alpha))\to(\lambda^{-2}W(\lambda t,\lambda^2x),\lambda^{-3}Q(\lambda t,\lambda^2x))$$

is a symmetry for gravity waves (thus it leaves g unchanged). Here it changes $c \to \lambda c$. Finally, the purely spatial scaling

$$(W(t,\alpha), Q(t,\alpha)) \to (\lambda^{-2} W(t, \lambda^2 x), \lambda^{-3} Q(t, \lambda^2 x))$$

has the effect of leaving c unaffected, but it changes $g \to \lambda^{-1} g$. Thus one could use scaling considerations to set both $c = 1$ and $g = 1$. However, we choose not to do that, and instead use the coefficients c and g to keep better track of the expressions arising in our analysis. In this context, it is useful to observe that one can interpret c^2/g as an (inverse) semiclassical parameter, so that all energy expressions can be viewed as homogeneous.

We further remark that the terms involving c are lower-order, though they cannot be viewed as bounded. Thus, it is natural to expect that the local theory for this problem is not very different from the gravity waves; indeed, our results in this regard are similar to [HIT16]. However, we will see that the long-time behavior is quite different in the constant vorticity case.

To further motivate our expectations concerning this system, we note that the linearization of the system (1-5) around the zero solution is

$$\begin{cases} w_t + q_\alpha = 0, \\ q_t + icq - igw = 0, \end{cases} \tag{1-8}$$

restricted to holomorphic functions (in our terminology, these are functions with negative spectrum). It is not difficult to see that this is a dispersive equation. Expressed as a second-order equation this becomes

$$w_{tt} + icw_t + igw_\alpha = 0. \tag{1-9}$$

From here we can deduce the associated dispersion relation,

$$\tau^2 + c\tau + g\xi = 0, \quad \xi \le 0. \tag{1-10}$$

This is the intersection of a lateral parabola with the left half-space. It has two branches, intersecting the axis $\xi = 0$ at $\tau = 0$ and $\tau = -c$.

The conserved energy and momentum for (1-8) are, respectively,

$$\mathcal{E}_0(w,r) = \int |w|^2 - iq\bar{q}_\alpha \, d\alpha = \|w\|_{L^2}^2 + \|q\|_{\dot{H}^{1/2}}^2,$$

$$\mathcal{P}_0(w,r) = \int \frac{1}{i}(\bar{q} w_\alpha - q\bar{w}_\alpha) - c|w|^2 \, d\alpha.$$

The former motivates the functional setting where we will study the equations (1-5). The system (1-8) is a well-posed linear evolution in the space $\dot{\mathcal{H}}_0$ of holomorphic functions endowed with the $L^2 \times \dot{H}^{1/2}$-norm. To measure higher regularity we will use the spaces $\dot{\mathcal{H}}_n$ endowed with the norm

$$\|(w,r)\|_{\dot{\mathcal{H}}_n}^2 := \sum_{k=0}^{n} \|\partial_\alpha^k (w,r)\|_{L^2 \times \dot{H}^{1/2}}^2,$$

where $n \ge 1$.

1C. ***The differentiated equations and diagonalization.*** As the system (1-5) is fully nonlinear, a standard procedure is to convert it into a quasilinear system by differentiating it. In the case of gravity waves this yields a self-contained first-order quasilinear system for (W_α, Q_α). This is no longer true here, precisely due to the contributions from F_1 and T_1.

To write the differentiated system, as well as the linearized system later on, we introduce several notations. First, as in [HIT16], we set

$$\mathbf{W} = W_\alpha, \qquad R = \frac{Q_\alpha}{1+\mathbf{W}}, \qquad Y = \frac{\mathbf{W}}{1+\mathbf{W}}.$$

The expression R has an intrinsic meaning; namely it is the complex velocity on the water surface. Y, on the other hand, is introduced for computational reasons only, in order to avoid rational expressions in many places in the sequel.

We also need two key auxiliary real functions. The first is $\underline{b}$, which we call the *advection velocity*, and is given by

$$\underline{b} := b - i\frac{c}{2}b_1, \quad b := \mathbf{P}\left[\frac{Q_\alpha}{J}\right] + \bar{\mathbf{P}}\left[\frac{\bar{Q}_\alpha}{J}\right], \quad b_1 := \mathbf{P}\left[\frac{W}{1+\bar{\mathbf{W}}}\right] - \bar{\mathbf{P}}\left[\frac{\bar{W}}{1+\mathbf{W}}\right]. \tag{1-11}$$

The second is the *frequency-shift* a, given by

$$\underline{a} := a + \frac{c}{2}a_1, \quad a := i(\bar{\mathbf{P}}[\bar{R}R_\alpha] - \mathbf{P}[R\bar{R}_\alpha]), \quad a_1 = R + \bar{R} - N, \tag{1-12}$$

where

$$N := \mathbf{P}[W\bar{R}_\alpha - \bar{\mathbf{W}}R] + \bar{\mathbf{P}}[\bar{W}R_\alpha - \mathbf{W}\bar{R}]. \tag{1-13}$$

The functions $\underline{a}$ and $\underline{b}$ are the leading-order coefficients in the linearized equation, and thus fully describe the quasilinear nature of the problem. The linearized equation is more involved and is described in full later, but, as a baseline, the reader should keep in mind the linear system

$$\begin{cases} w_t + \underline{b}w_\alpha + q_\alpha = 0, \\ q_t + icq - i(g + \underline{a})w = 0. \end{cases} \tag{1-14}$$

The real function $g + \underline{a}$ has a physical interpretation as the normal derivative of the pressure on the interface; this is proved in Appendix A. For more regular irrotational waves (i.e., $c = 0$) this was proved by Wu [1997] to be positive; an alternate proof was provided in [HIT16] in the context of holomorphic coordinates, assuming only scale-invariant regularity $(\mathbf{W}, R) \in \dot{\mathcal{H}}_{1/2}$. This positivity, called the *Taylor sign condition* [1950], was crucial for the well-posedness of the water wave system, both with gravity [Wu 1997; Hunter, Ifrim and Tataru 2016] and with surface tension [Ifrim and Tataru 2017].

For the present problem we still need to know that $g + \underline{a}$ is nonnegative, which corresponds with the normal derivative of the pressure being bounded away from zero. But it is no longer the case that this comes for free. Thus, we will impose the positivity condition on the normal derivative of the pressure and solve the equations for as long as this condition holds uniformly. We remark that this is always the case if we assume that R is small in L^∞, which is the case for our small-data result.

Differentiating with respect to α yields a system for (W_α, Q_α), which turns out to be degenerate hyperbolic with a double speed $\underline{b}$. This is explained in detail in the context of irrotational gravity waves in [HIT16], and easily carries over here as the highest-order terms in the equations are the same. Then it is natural to diagonalize it. This is also done exactly as in [HIT16], using the operator

$$\boldsymbol{A}(w, q) := (w, q - Rw), \qquad R := \frac{Q_\alpha}{1 + W_\alpha}. \tag{1-15}$$

Noting that

$$\boldsymbol{A}(W_\alpha, Q_\alpha) = (\boldsymbol{W}, R), \qquad \boldsymbol{W} := W_\alpha,$$

it follows that the pair $(\boldsymbol{W}, R)$ diagonalizes the differentiated system. Thus, rather than repeating the more extensive computations in [HIT16], here we directly take advantage of this knowledge to arrive more efficiently at the differentiated system for the diagonal variables $(\boldsymbol{W}, R)$.

We first introduce $\underline{b}$ into the equations using the relations

$$F = b - \frac{\bar{R}}{1 + \boldsymbol{W}}, \qquad F_1 = b_1 + \frac{\bar{W}}{1 + \boldsymbol{W}}, \tag{1-16}$$

so that the system (1-5) is written in the form

$$\begin{cases} W_t + \underline{b}(1 + W_\alpha) + i\frac{c}{2} W = \bar{R} + i\frac{c}{2}\bar{W}, \\ Q_t + \underline{b} Q_\alpha - igW + icQ - i\frac{c}{2}\bar{R}W = \bar{\boldsymbol{P}}[|R|^2] - i\frac{c}{2}\bar{\boldsymbol{P}}[W\bar{R} - \bar{W}R]. \end{cases}$$

Here the terms on the right are antiholomorphic and disappear when the equations are projected onto the holomorphic space.

Next we differentiate the equations. For the first equation we need the expression for $\underline{b}_\alpha$, for which we introduce one last set of quadratic auxiliary functions M, M_1 and $\underline{M}$ as follows:

$$\begin{aligned} \underline{M} &:= M - i\frac{c}{2} M_1, \\ M &:= \frac{R_\alpha}{1 + \bar{\boldsymbol{W}}} + \frac{\bar{R}_\alpha}{1 + \boldsymbol{W}} - b_\alpha = \bar{\boldsymbol{P}}[\bar{R}Y_\alpha - R_\alpha \bar{Y}] + \boldsymbol{P}[R\bar{Y}_\alpha - \bar{R}_\alpha Y], \\ M_1 &:= \boldsymbol{W} - \bar{\boldsymbol{W}} - b_{1,\alpha} = \boldsymbol{P}[W\bar{Y}]_\alpha - \bar{\boldsymbol{P}}[\bar{W}Y]_\alpha. \end{aligned} \tag{1-17}$$

Thus, we can substitute $\underline{b}_\alpha$ with

$$\underline{b}_\alpha = \frac{R_\alpha}{1 + \bar{\boldsymbol{W}}} + \frac{\bar{R}_\alpha}{1 + \boldsymbol{W}} - i\frac{c}{2}(\boldsymbol{W} - \bar{\boldsymbol{W}}) - \underline{M}. \tag{1-18}$$

For the second equation we switch directly to R, and then $\underline{b}_\alpha$ is no longer needed in view of the identity

$$(\partial_t + \underline{b}\partial_\alpha) R = \partial_\alpha(\partial_t + \underline{b}\partial_\alpha) Q - R\partial_\alpha(W_t + \underline{b}(1 + W_\alpha)).$$

Taking the above discussion into account, after some straightforward computations, one arrives at the system

$$\begin{cases} \mathbf{W}_t + \underline{b}\mathbf{W}_\alpha + \dfrac{(1+\mathbf{W})R_\alpha}{1+\bar{\mathbf{W}}} = (1+\mathbf{W})\underline{M} + i\dfrac{c}{2}\mathbf{W}(\mathbf{W}-\bar{\mathbf{W}}), \\ R_t + \underline{b}R_\alpha + icR - i\dfrac{g\mathbf{W}-a}{1+\mathbf{W}} = i\dfrac{c}{2}\dfrac{R\mathbf{W}+\bar{R}\mathbf{W}+N}{1+\mathbf{W}}. \end{cases} \tag{1-19}$$

This governs an evolution in the space of holomorphic functions, and will be used both directly and in its projected version. Obviously the two forms are algebraically equivalent.

We remark that while the transport coefficient $\underline{b}$ appears explicitly in these equations, the frequency shift $\underline{a}$ does not. This is due to the fact that the right-hand side of the second equation is still fully nonlinear in $\mathbf{W}$. To understand better the role played by $\underline{a}$ one needs either to consider the linearized equations, which are discussed in Section 3, or to further differentiate (1-19), as in Section 4.

1D. ***The main results.*** To describe the lifespan of the solutions we begin with the control norms in [HIT16], namely

$$A := \|\mathbf{W}\|_{L^\infty} + \|Y\|_{L^\infty} + \||D|^{1/2}R\|_{L^\infty \cap B^{0,\infty}_2}, \tag{1-20}$$

$$B := \||D|^{1/2}\mathbf{W}\|_{\mathrm{BMO}} + \|R_\alpha\|_{\mathrm{BMO}}, \tag{1-21}$$

where $|D|$ represents the multiplier with symbol $|\xi|$. In order to estimate lower-order terms introduced in conjunction with c we also introduce

$$A_{-1/2} := \||D|^{1/2}W\|_{L^\infty} + \|R\|_{L^\infty}, \tag{1-22}$$

$$A_{-1} := \|W\|_{L^\infty}. \tag{1-23}$$

It is also useful to introduce the notations

$$\begin{aligned} \underline{B} &:= B + cA + c^2 A_{-1/2}, \\ \underline{A} &:= A + cA_{-1/2} + c^2 A_{-1}. \end{aligned} \tag{1-24}$$

Here A is a scale-invariant quantity, while B corresponds to the homogeneous $\dot{\mathcal{H}}_1$-norm of $(\mathbf{W}, R)$ and $A_{-1/2}$ corresponds to the homogeneous $\dot{\mathcal{H}}_0$-norm of $(\mathbf{W}, R)$. We note that B, and all but the Y-component of A are controlled by the energy and $\dot{\mathcal{H}}_1$-norm of $(\mathbf{W}, R)$.

Now we are ready to state our main results. We begin with the local well-posedness result:

Theorem 1. *Let $n \geq 1$. The system (1-5) is locally well-posed for initial data (W_0, Q_0) with the regularity*

$$(W_0, Q_0) \in \dot{\mathcal{H}}_0, \quad (\mathbf{W}_0, R_0) \in \dot{\mathcal{H}}_1,$$

and satisfying the pointwise constraints

$$|\mathbf{W}(\alpha) + 1| > \delta > 0 \quad (\textit{no interface singularities}), \tag{1-25}$$

$$g + \underline{a}(\alpha) > \delta > 0 \quad (\textit{Taylor sign condition}). \tag{1-26}$$

Further, the solution can be continued for as long as $\underline{A}$ and $\underline{B}$ remain bounded and the pointwise conditions above hold uniformly. The same result holds in the periodic setting.

The well-posedness above should be interpreted in the sense of Hadamard. To be more precise, it means that there exists some time $T > 0$, depending only on the initial data size and on the constant δ in the above pointwise constraints, so that the following properties hold:

(1) (regular data) For each data (W_0, Q_0), which is as above, but with additional regularity $(\mathbf{W}_0, R_0) \in \dot{\mathcal{H}}_n$, with $n \geq 2$, there exists a unique solution (W, Q) in $[0, T]$, with the property that

$$\|(\mathbf{W}, R)\|_{C[0,T;\dot{\mathcal{H}}_k]} \lesssim \|(\mathbf{W}_0, R_0)\|_{\dot{\mathcal{H}}_k}, \quad 0 \leq k \leq n.$$

(2) (rough data) For each data (W_0, Q_0) as above there exists a solution (W, Q) in $[0, T]$, with the property that

$$\|(\mathbf{W}, R)\|_{C[0,T;\dot{\mathcal{H}}_k]} \lesssim \|(\mathbf{W}_0, R_0)\|_{\dot{\mathcal{H}}_k}, \quad k = 0, 1.$$

Further this solution is the unique limit of regular solutions, and depends continuously on the initial data.

(3) (weak Lipschitz dependence) The solution (W, Q) has a Lipschitz dependence on the initial data in the $\dot{\mathcal{H}}_1$ topology.

The implicit constants in all the estimates above depend only on the initial data size and on the constant δ in the above pointwise constraints. Further, the last part of the Theorem 1 asserts that in effect these constants can alternatively be estimated purely in terms of our uniform control parameters A and B, rather than the full Sobolev norm of the data.

Our second result in this paper is a cubic lifespan bound for the small-data problem:

Theorem 2. *Let (W, Q) be a solution for the system (1-5) whose initial data satisfies*

$$\|(W_0, Q_0)\|_{\dot{\mathcal{H}}_0} + \|(\mathbf{W}_0, R_0)\|_{\dot{\mathcal{H}}_1} \leq \epsilon \ll 1. \tag{1-27}$$

Then the solution exists for a time $T_\epsilon \approx \epsilon^{-2}$, with bounds

$$\|(W, Q)(t)\|_{\dot{\mathcal{H}}_0} + \|(\mathbf{W}, R)(t)\|_{\dot{\mathcal{H}}_1} \lesssim \epsilon, \quad |t| < T_\epsilon. \tag{1-28}$$

Further, higher regularity is also preserved,

$$\|(\mathbf{W}, R)(t)\|_{\dot{\mathcal{H}}_n} \lesssim \|(\mathbf{W}, R)(0)\|_{\dot{\mathcal{H}}_n}, \quad |t| < T_\epsilon, \tag{1-29}$$

whenever the norm on the right is finite.

To the best of our knowledge this is the first nontrivial lifespan bound for solutions to this problem. A similar result for the zero-vorticity problem was proved in our earlier article [HIT16]. The problem here is considerably more difficult than the one in [HIT16], both technically and conceptually. At the technical level, the normal form for the vorticity problem is much more involved, and quite nontrivial to compute (see the next section). Qualitatively, here we have stronger quadratic interactions at low frequency, which, unlike in [HIT16], prevent us from obtaining cubic bounds for the linearized equation (and thus, for differences of solutions).

We further remark that in [HIT16] we also provide a proof of an almost global result for small localized data for gravity waves. Our aim is to also provide a similar result in this context. However, the ideas in [HIT16] do not directly carry over to this case, due first, to the lack of a scaling symmetry, and secondly, to the lack of cubic estimates for the linearized equation. We hope to be able to address these issues in subsequent work.

We note that the periodic case is almost identical and is not discussed separately. The only difference in the analysis is in how the constant functions are treated. This is discussed in detail in the Appendix to [HIT16], and carries over to the present paper without any change.

1E. *Outline of the paper.* There are three key steps in our analysis, which eventually provide all the ingredients which are necessary in order to prove our main results. These are as follows:

(i) *The normal-form analysis.* The constant vorticity water wave equation has many quadratic interactions, yet we seek to prove small-data lifespan bounds as if the nonlinearities were cubic. At least formally the key to this is the normal-form analysis [Shatah 1985], which allows us to replace quadratic nonlinearities with cubic ones. While the normal-form transformation for gravity waves is quite straightforward, in the presence of constant vorticity, this is no longer the case.

Indeed, the normal form turns out to be unbounded both at low frequency and at high frequency. This computation is fairly involved, and is carried out in the next section. Its redeeming feature is that its outcome is also quite explicit.

The normal form we calculate here is not directly used in any of the estimates we derive later on. However, it is crucially used in order to construct modified energies with cubic estimates, which is the base of our *quasilinear modified energy method* [Hunter, Ifrim, Tataru and Wong 2015; 2016].[1] Even though the normal form is badly unbounded, it has enough of a "null structure", or antisymmetry, so that the cubic energy corrections it generates are all of bounded type.

(ii) *The analysis of the linearized equation in Section 3.* This is a critical part of any local well-posedness result for a quasilinear problem. The derivation of the equations is also interesting, as it clarifies the quasilinear structure and the roles played by the advection coefficient $\underline{b}$, and the frequency shift $\underline{a}$.

As for the gravity waves in [HIT16], we are able to prove that the linearized problem is well-posed in our base space $\dot{\mathcal{H}}_0$. Further, the bounds we prove are in terms of our control parameters $\underline{A}$ and $\underline{B}$, and not in terms of the full Sobolev norm of the solution.

Unlike in [HIT16], we are no longer able to prove cubic estimates for the linearized equation. This is due to the unbounded low-frequency part of the normal-form transformation, which loses its skew-adjoint structure after linearization. Because of this, we are able to use the bounds for the linearized equation in the proof of local well-posedness, but only partially in the proof of the cubic lifespan result.

(iii) *The cubic energy estimates in Section 4.* Since we already have the conserved Hamiltonian, which controls the $\dot{\mathcal{H}}_0$-norm of (W, Q), our task here is to successively provide bounds for $(\mathbf{W}, R)$ in the

[1] For earlier attempts to adapt normal forms to quasilinear problems we refer the reader to [Wu 2009; Germain and Masmoudi 2014; Ionescu and Pusateri 2015]. Other equally successful approaches are the *paradiagonalization method* in [Alazard and Delort 2015], and the flow method of [Hunter and Ifrim 2012].

$\dot{\mathcal{H}}_k$ spaces for $k = 0, 1, \dots$. In all cases, we control the evolution of these norms using our pointwise control parameters $\underline{A}$ and $\underline{B}$.

These bounds come in two flavors: (a) local bounds, which apply for large data, and are needed for the local well-posedness result, and (b) cubic long-time bounds for small data, which are used for the cubic lifespan result. The former are obtained largely by differentiating the equation, and then by applying the bounds for the linearized equation. The latter, however, requires computing cubic energy corrections, and there, we rely heavily on the normal form.

The crucial step is the one for the $\dot{\mathcal{H}}_1$-norm of $(\boldsymbol{W}, R)$ (i.e., $k = 1$), as this is the level where we have our well-posedness result; for this reason, we describe this case in detail. The case $k = 0$ is simpler since $(\boldsymbol{W}, R)$ solves the linearized equation, so we already have the local bound. The case $k \geq 2$ is discussed last, without explicitly computing the modified energy.

Once the bounds for the linearized equation and for the differentiated equation are established, the remaining arguments in the proofs of our main theorems are a fairly straightforward repetition of arguments in [HIT16]. We outline this in the last section of the paper.

Finally, the Appendix plays two roles. In Appendix B we outline the derivation of the constant-vorticity gravity wave equation in holomorphic coordinates. Last, but not least, in Appendix A we collect a number of bilinear Coifman–Meyer and nonlinear Moser-type estimates, some from [HIT16], and prove the ones which are new in this paper.

2. The normal-form transformation

The nonlinear evolution (1-5) contains quadratic terms, yet for our problem we seek to prove cubic lifespan bounds, as if the nonlinearity is at least cubic. In this section we consider the question of finding a normal transformation, whose aim is to replace the original variables (W, Q) with normal-form variables $(\widetilde{W}, \widetilde{Q})$ of the form

$$\begin{cases} \widetilde{W} &= W + \boldsymbol{P} W_{[2]}, \\ \widetilde{Q} &= Q + \boldsymbol{P} Q_{[2]}, \end{cases} \tag{2-1}$$

where $W_{[2]}$ and $Q_{[2]}$ are quadratic forms in (W, Q), so that the normal-form variables satisfy an equation with only cubic-and-higher terms,

$$\begin{cases} \widetilde{W}_t + \widetilde{Q}_\alpha = \text{cubic and higher}, \\ \widetilde{Q}_t - ig\widetilde{W} + ic\widetilde{Q} = \text{cubic and higher}. \end{cases} \tag{2-2}$$

We will indeed show that such a normal-form transformation exists; as it turns out, it is highly unbounded, both at low and at high frequencies. However, this is expected, and it does not cause any difficulties. This is because we do not use the normal form directly, but only as a tool to help us construct modified energies in our quasilinear modified energy method. As it turns, even though the normal-form transformation is unbounded, it has some favorable structure, so that the modified cubic energies are nevertheless bounded.

2A. ***The resonance analysis.*** We begin by examining whether our evolution has quadratic resonant interactions. Taking into account the possible complex conjugations and the dispersion relation (1-10), the quadratic resonant interactions are associated with three pairs of characteristic frequencies (τ_1, ξ_1), (τ_2, ξ_2) and (τ, ξ) so that $(\tau_1, \xi_1) + (\tau_2, \xi_2) = (\tau, \xi)$. Combining this with the dispersion relation we obtain $\tau_1^2 + \tau_2^2 = \tau^2$, which leads to $\tau_1 \tau_2 = 0$. Hence, we either have $(\tau_1, \xi_1) = (0, 0)$ or $(\tau_2, \xi_2) = (0, 0)$. Thus, resonant interactions occur only when either one of the inputs or the output is at frequency zero. In terms of the normal-form transformation, this indicates that at most we will have singularities when either input is at frequency zero, or the output is at frequency zero. We will see that, due to the form of the quadratic terms in the equation, the former scenario happens, but the latter does not.

2B. *The normal-form computation.* We begin with the quadratic and expansion in the equation (1-5), and then we compute the normal-form transformation which eliminates the quadratic terms from the equation. Starting with

$$F \approx Q_\alpha - Q_\alpha W_\alpha + \boldsymbol{P}[\bar{Q}_\alpha W_\alpha - Q_\alpha \overline{W}_\alpha] + \boldsymbol{P}[(Q_\alpha - \bar{Q}_\alpha)(4|\Re W_\alpha|^2 - |W_\alpha|^2)],$$

we compute the multilinear expansion

$$\begin{cases} W_t + Q_\alpha = G^{(2)} + G^{(3+)}, \\ Q_t - igW + icQ = K^{(2)} + K^{(3+)}, \end{cases} \tag{2-3}$$

where the quadratic terms $(G^{(2)}, K^{(2)})$ are given by

$$\begin{cases} G^{(2)} = -\boldsymbol{P}[\bar{Q}_\alpha W_\alpha - Q_\alpha \overline{W}_\alpha] - i\frac{c}{2}\boldsymbol{P}[W\overline{W}_\alpha + \overline{W}W_\alpha] + i\frac{c}{2}WW_\alpha, \\ K^{(2)} = -Q_\alpha^2 - \boldsymbol{P}[|Q_\alpha|^2] + i\frac{c}{2}WQ_\alpha + i\frac{c}{2}\boldsymbol{P}[W\bar{Q}_\alpha - \overline{W}Q_\alpha]. \end{cases}$$

The role of the normal-form transformation is to eliminate the quadratic terms $(G^{(2)}, K^{(2)})$ from the equation (2-3). We can divide the quadratic terms above into two classes:

(a) holomorphic, i.e., those which are the product of two holomorphic functions,

(b) mixed, i.e., those which are the (projected) product of one holomorphic function with another's conjugate.

Thus, we expect the normal form to have a similar structure,

$$W_{[2]} = W_{[2]}^h + W_{[2]}^a, \quad Q_{[2]} = Q_{[2]}^h + Q_{[2]}^a, \tag{2-4}$$

where, allowing for all possible combinations, the above components must have the form

$$\begin{aligned} W_{[2]}^h &= B^h(W, W) + C^h(Q, Q) + D^h(W, Q), \\ W_{[2]}^a &= B^a(W, \overline{W}) + C^a(Q, \bar{Q}) + D^a(W, \bar{Q}) + E^a(Q, \overline{W}), \\ Q_{[2]}^h &= F^h(W, W) + H^h(Q, Q) + A^h(W, Q), \\ Q_{[2]}^a &= F^a(W, \overline{W}) + H^a(Q, \bar{Q}) + A^a(W, \bar{Q}) + G^a(Q, \overline{W}). \end{aligned} \tag{2-5}$$

All the above expressions are translation-invariant bilinear forms, whose symbols we need to compute. Our main result is as follows:

Proposition 2.1. *The equation (2-3) admits a normal-form transformation (2-1) as in (2-4)–(2-5), where the symbols for the bilinear operators in (2-5) are given by (2-12), (2-13), and (2-16).*

For later use, i.e., for computing the modified energy cubic corrections, we also translate the symbols (2-12), (2-13), and (2-16) into the spatial description. Precisely, the holomorphic terms have the form

$$\begin{cases} B^h(W, W) = -W\mathbf{W} + i\dfrac{c^2}{2g}(\mathbf{W}\partial_\alpha^{-1}W + W^2) + \dfrac{c^4}{4g^2}W\partial_\alpha^{-1}W, \\ C^h(Q, Q) = -\dfrac{c^2}{4g^2}QQ_\alpha, \\ D^h(W, Q) = -\dfrac{c}{2g}(\mathbf{W}Q + WQ_\alpha) + i\dfrac{c^3}{4g}(WQ + \partial_\alpha^{-1}WQ_\alpha), \\ F^h(W, W) = i\dfrac{c}{4}W^2 + \dfrac{c^3}{4g}W\partial_\alpha^{-1}W, \\ H^h(Q, Q) = -\dfrac{c}{2g}QQ_\alpha, \\ A^h(W, Q) = -WQ_\alpha + i\dfrac{c^2}{2g}\partial_\alpha^{-1}WQ_\alpha + i\dfrac{c^2}{4g}WQ, \end{cases} \tag{2-6}$$

and the antiholomorphic counterparts are given by

$$\begin{cases} B^a(W, \bar{W}) = -\mathbf{W}\bar{W} - i\dfrac{c^2}{2g}\mathbf{W}\partial_\alpha^{-1}\bar{W} + i\dfrac{c^2}{4g}|W|^2 - \dfrac{c^4}{4g^2}W\partial_\alpha^{-1}\bar{W}, \\ C^a(Q, \bar{Q}) = -\dfrac{c^2}{4g^2}\bar{Q}Q_\alpha, \\ D^a(W, \bar{Q}) = -\dfrac{c}{2g}\mathbf{W}\bar{Q} + i\dfrac{c^3}{4g^2}W\bar{Q}, \\ G^a(Q, \bar{W}) = -Q_\alpha\bar{W} - i\dfrac{c^2}{2g}Q_\alpha\partial_\alpha^{-1}\bar{W}, \\ E^a(Q, \bar{W}) = -\dfrac{c}{2g}Q_\alpha\bar{W} - i\dfrac{c^3}{4g^2}Q_\alpha\partial_\alpha^{-1}\bar{W}, \\ H^a(Q, \bar{Q}) = -\dfrac{c}{2g}\bar{Q}Q_\alpha, \\ A^a(W, \bar{Q}) = i\dfrac{c^2}{4g}W\bar{Q}, \\ F^a(W, \bar{W}) = i\dfrac{c}{2}|W|^2 - \dfrac{c^3}{4g}W\partial_\alpha^{-1}\bar{W}. \end{cases} \tag{2-7}$$

We note that while for computational purposes it is convenient to separate the two components of the normal form, in order to see the antisymmetric structure of the (low frequency) unbounded part, one has to consider them together; see the computations in Section 4. Toward that goal, we rewrite the quadratic normal-form components in the following form:

$$
\begin{aligned}
W^{[2]} &= -(W+\overline{W})W_\alpha - \frac{c}{2g}[(Q+\bar{Q})W_\alpha + (W+\overline{W})Q_\alpha] \\
&\quad + i\frac{c^2}{2g}\big[(\partial^{-1}W-\partial^{-1}\overline{W})W_\alpha + W^2 + \tfrac{1}{2}|W|^2\big] - \frac{c^2}{4g^2}(Q+\bar{Q})Q_\alpha \\
&\quad + i\frac{c^3}{4g^2}[(Q+\bar{Q})W + (\partial^{-1}W-\partial^{-1}\overline{W})Q_\alpha] + \frac{c^4}{4g^2}(\partial^{-1}W-\partial^{-1}\overline{W})W, \qquad (2\text{-}8)\\
Q^{[2]} &= -(W+\overline{W})Q_\alpha - \frac{c}{2g}(Q+\bar{Q})Q_\alpha + i\frac{c}{4}(W^2+2|W|^2) \\
&\quad + i\frac{c^2}{2g}\big[(\partial^{-1}W-\partial^{-1}\overline{W})Q_\alpha + \tfrac{1}{2}(Q+\bar{Q})W\big] + \frac{c^3}{4g}(\partial^{-1}W-\partial^{-1}\overline{W})W.
\end{aligned}
$$

Proof of Proposition 2.1. A priori, computing the normal form, i.e., all symbols of the bilinear expressions above, might appear quite involved. However, there are several observations which bring this analysis to a more manageable level:

- The analysis for the holomorphic products, and for the mixed terms is completely separate.
- The system we obtain for the symbols has polynomial coefficients, so the solutions are rational functions.
- Counting c as one half of a derivative; the problem is homogeneous. Thus, organizing symbols based on the powers of c, each such term will have a specific homogeneity. Further, at each power of c, we will encounter only half the terms, as there is a half-derivative difference between the scaling of W and that of Q.
- The terms without c are already known from the gravity wave problem [HIT16].

Given the above considerations, the natural strategy is to split the analysis into the holomorphic and the mixed part, and in each of these cases to successively solve for increasing powers of c. The computation stops at c^4.

(i) *Holomorphic terms.* Here we seek a normal form for the system

$$
\begin{cases}
W_t + Q_\alpha = i\frac{c}{2}WW_\alpha,\\
Q_t - igW + icQ = -Q_\alpha^2 + i\frac{c}{2}WQ_\alpha + \text{cubic}.
\end{cases}
$$

By checking parity, our normal form must be

$$
\begin{cases}
\widetilde{W} = W + B^h(W,W) + C^h(Q,Q) + D^h(W,Q),\\
\widetilde{Q} = Q + F^h(W,W) + H^h(Q,Q) + A^h(W,Q),
\end{cases}
$$

where B^h, C^h, F^h and H^h are symmetric bilinear forms with symbols $B^h(\xi,\eta)$, $C^h(\xi,\eta)$, $F^h(\xi,\eta)$, $H^h(\xi,\eta)$ and A^h and D^H are arbitrary. We compute

$$
\begin{aligned}
\widetilde{W}_t + \widetilde{Q}_\alpha = {}& -2B^h(Q_\alpha,W) + 2igC^h(W,Q) - 2icC^h(Q,Q) - D^h(Q_\alpha,Q) + igD^h(W,W),\\
& - icD^h(W,Q) + \partial_\alpha(F^h(W,W) + H^h(Q,Q) + A^h(W,Q)) + i\frac{c}{2}WW_\alpha + \text{cubic},
\end{aligned}
$$

$$\begin{aligned}\widetilde{Q}_t - ig\widetilde{W} + ic\widetilde{Q} = &-2F^h(Q_\alpha, W) + 2igH^h(W, Q) - icH^h(Q, Q) - A^h(Q_\alpha, Q) + igA^h(W, W)\\ &- igB^h(W, W) - igC^h(Q, Q) - igD^h(W, Q) + icF^h(W, W) - Q_\alpha^2\\ &+ i\frac{c}{2}WQ_\alpha + \text{cubic}.\end{aligned}$$

We denote the two input frequencies by ξ and η, both of which are negative. Then, matching like terms, we obtain the following linear system for the symbols:

$$\begin{cases} 2\eta B^h - 2gC^h + cD^h - (\xi+\eta)A^h = 0,\\ 2cC^h + [\xi D^h]_{\text{sym}} - (\xi+\eta)H^h = 0,\\ g[D^h]_{\text{sym}} + (\xi+\eta)F^h = -i\frac{c}{4}(\xi+\eta),\\ 2\eta F^h - 2gH^h + gD^h = i\frac{c}{2}\eta,\\ cH^h + [\xi A^h]_{\text{sym}} + gC^h = -i\xi\eta,\\ g[A^h]_{\text{sym}} - gB^h + cF^h = 0,\end{cases}$$

where "sym" stands for symmetrization. Using the first equation in the system above helps us to determine the symmetrized symbol of A^h,

$$A^h = \frac{1}{\xi+\eta}[cD^h - 2gC^h + 2\eta B^h],$$

which implies

$$\begin{aligned}[A^h]_{\text{sym}} &= \frac{1}{\xi+\eta}\big[c[D^h]_{\text{sym}} - 2gC^h + (\eta+\xi)B^h\big],\\ [\xi A^h]_{\text{sym}} &= \frac{1}{\xi+\eta}\big[c[\xi D^h]_{\text{sym}} - g(\xi+\eta)C^h + 2\xi\eta B^h\big].\end{aligned} \tag{2-9}$$

The fourth equation provides an expression for the symmetrized symbol of D^h,

$$D^h = -\frac{2}{g}\eta F^h + 2H^h + i\frac{c}{2g}\eta,$$

which implies

$$\begin{aligned} g[D^h]_{\text{sym}} &= -(\xi+\eta)F^h + 2gH^h + i\frac{c}{4}(\xi+\eta),\\ [\xi D^h]_{\text{sym}} &= -\frac{2}{g}\xi\eta F^h + (\xi+\eta)H^h + i\frac{c}{2g}\xi\eta,\end{aligned} \tag{2-10}$$

Using (2-10) in (2-9) gives

$$\begin{aligned}[A^h]_{\text{sym}} &= \frac{1}{\xi+\eta}\left[-\frac{c}{g}(\xi+\eta)F^h + 2cH^h + i\frac{c^2}{4g}(\xi+\eta) - 2gC^h + (\eta+\xi)B^h\right],\\ [\xi A^h]_{\text{sym}} &= \frac{1}{\xi+\eta}\left[-\frac{2c}{g}\xi\eta F^h + (\xi+\eta)cH^h + i\frac{c^2}{2g}\xi\eta - g(\xi+\eta)C^h + 2\xi\eta B^h\right].\end{aligned} \tag{2-11}$$

Thus, we return to the following system

$$\begin{cases} 2cC^h + [\xi D^h]_{\text{sym}} - (\xi+\eta)H^h = 0,\\ g[D^h]_{\text{sym}} + (\xi+\eta)F^h = -i\frac{c}{4}(\xi+\eta),\\ cH^h + [\xi A^h]_{\text{sym}} + gC^h = -i\xi\eta,\\ g[A^h]_{\text{sym}} - gB^h + cF^h = 0,\end{cases}$$

where we substitute the corresponding values from (2-10), (2-11), and obtain

$$\begin{cases} 2cC^h - \dfrac{2}{g}\xi\eta F^h + (\xi+\eta)H^h + i\dfrac{c}{2g}\xi\eta - (\xi+\eta)H^h = 0, \\ -(\xi+\eta)F^h + 2gH^h + i\dfrac{c}{4}(\xi+\eta) + (\xi+\eta)F^h = -i\dfrac{c}{4}(\xi+\eta), \\ cH^h + \dfrac{1}{\xi+\eta}\Big[-\dfrac{2c}{g}\xi\eta F^h + (\xi+\eta)cH^h + i\dfrac{c^2}{2g}\xi\eta - g(\xi+\eta)C^h + 2\xi\eta B^h\Big] + gC^h = -i\xi\eta, \\ \dfrac{1}{\xi+\eta}\Big[-c(\xi+\eta)F^h + 2cgH^h + i\dfrac{c^2}{4}(\xi+\eta) - 2g^2C^h + (\eta+\xi)gB^h\Big] - gB^h + cF^h = 0, \end{cases}$$

so

$$\begin{cases} 2cC^h - \dfrac{2}{g}\xi\eta F^h = -i\dfrac{c}{2g}\xi\eta, \\ H^h = -i\dfrac{c}{4g}(\xi+\eta), \\ cH^h - \dfrac{c}{g}\dfrac{\xi\eta}{\xi+\eta}F^h + \dfrac{\xi\eta}{\xi+\eta}B^h = -i\dfrac{\xi\eta}{2} - i\dfrac{c^2}{4g}\dfrac{\xi\eta}{\xi+\eta}, \\ cH^h - gC^h = -i\dfrac{c^2}{8g}(\xi+\eta). \end{cases}$$

The solution is

$$\begin{aligned} B^h &= -\frac{i}{2}(\xi+\eta) + i\frac{c^2}{4g}\frac{(\xi+\eta)^2}{\xi\eta} - i\frac{c^4}{8g^2}\frac{\xi+\eta}{\xi\eta}, \\ C^h &= -i\frac{c^2}{8g^2}(\xi+\eta), \\ F^h &= i\frac{c}{4} - i\frac{c^3}{8g}\frac{\xi+\eta}{\xi\eta}, \\ H^h &= -i\frac{c}{4g}(\xi+\eta). \end{aligned} \tag{2-12}$$

It remains to find A^h and D^h, which we obtain from

$$\begin{cases} 2\eta B^h - 2gC^h + cD^h - (\xi+\eta)A^h = 0, \\ 2\eta F^h - 2gH^h + gD^h = i\dfrac{c}{2}\eta. \end{cases}$$

Thus,

$$\begin{aligned} A^h &= -i\eta + i\frac{c^2}{2g}\eta\xi^{-1} + i\frac{c^2}{4g}, \\ D^h &= i\frac{c^3}{4g^2}\frac{\xi+\eta}{\xi} - i\frac{c}{2g}(\xi+\eta). \end{aligned} \tag{2-13}$$

(ii) *Mixed terms*. Here we need a normal form for the system

$$\begin{cases} W_t + Q_\alpha = -\boldsymbol{P}[\bar{Q}_\alpha W_\alpha - Q_\alpha \overline{W}_\alpha] - i\dfrac{c}{2}\boldsymbol{P}[W\overline{W}_\alpha + \overline{W}W_\alpha] + \text{cubic}, \\ Q_t - igW + icQ = -\boldsymbol{P}[|Q_\alpha|^2] + i\dfrac{c}{2}\boldsymbol{P}[W\bar{Q}_\alpha - \overline{W}Q_\alpha] + \text{cubic}. \end{cases}$$

The general expression for our normal form is

$$\begin{cases} \widetilde{W} = W + B^a(W,\overline{W}) + C^a(Q,\bar{Q}) + D^a(W,\bar{Q}) + E^a(Q,\overline{W}), \\ \widetilde{Q} = Q + F^a(W,\overline{W}) + H^a(Q,\bar{Q}) + A^a(W,\bar{Q}) + G^a(Q,\overline{W}), \end{cases} \tag{2-14}$$

where no symmetry assumption is required. We compute

$$\begin{aligned}
\widetilde{W}_t + \widetilde{Q}_\alpha = &-B^a(Q_\alpha, \overline{W}) - B^a(W, \bar{Q}_\alpha) + igC^a(W, \bar{Q}) - igC^a(Q, \overline{W}) - D^a(Q_\alpha, \bar{Q}) \\
&-igD^a(W, \overline{W}) + icD^a(W, \bar{Q}) + igE^a(W, \overline{W}) - icE^a(Q, \overline{W}) - E^a(Q, \bar{Q}_\alpha) \\
&- \boldsymbol{P}[\bar{Q}_\alpha W_\alpha - Q_\alpha \overline{W}_\alpha] + \partial_\alpha[F^a(W, \overline{W}) + H^a(Q, \bar{Q}) + A^a(W, \bar{Q}) + G^a(Q, \overline{W})] \\
&- i\frac{c}{2}\boldsymbol{P}[W\overline{W}_\alpha + \overline{W}W_\alpha] + \text{cubic},
\end{aligned}$$

$$\begin{aligned}
\widetilde{Q}_t - ig\widetilde{W} + ic\widetilde{Q} = &-F^a(Q_\alpha, \overline{W}) - F^a(W, \bar{Q}_\alpha) + igH^a(W, \bar{Q}) - igH^a(Q, \overline{W}) - A^a(Q_\alpha, \bar{Q}) \\
&-igA^a(W, \overline{W}) + icA^a(W, \bar{Q}) - G^a(Q, \bar{Q}_\alpha) + igG^a(W, \overline{W}) - igB^a(W, \overline{W}) \\
&-igC^a(Q, \bar{Q}) - igD^a(W, \bar{Q}) - igE^a(Q, \overline{W}) + icF^a(W, \overline{W}) + icH^a(Q, \bar{Q}) \\
&+icA^a(W, \bar{Q}) - \boldsymbol{P}[|Q_\alpha|^2] + i\frac{c}{2}\boldsymbol{P}[W\bar{Q}_\alpha - \overline{W}Q_\alpha] + \text{cubic}.
\end{aligned}$$

Now we denote by ξ the frequency of the holomorphic input and by η the frequency of the conjugated input (both negative). Matching again like terms, it remains to solve the system

$$\begin{cases}
\xi B^a + gC^a + cE^a - (\xi - \eta)G^a = -i\xi\eta, \\
\eta B^a + gC^a + (\xi - \eta)A^a + cD^a = -i\xi\eta, \\
\xi D^a - \eta E^a - (\xi - \eta)H^a = 0, \\
gD^a - gE^a - (\xi - \eta)F^a = i\frac{c}{2}(\eta - \xi), \\
\xi F^a + gH^a + gE^a = -i\frac{c}{2}\xi, \\
\eta F^a + gH^a + 2cA^a - gD^a = i\frac{c}{2}\eta, \\
\xi A^a + gC^a - cH^a - \eta G^a = i\xi\eta, \\
gA^a + gB^a - cF^a - gG^a = 0.
\end{cases} \tag{2-15}$$

To avoid solving an 8×8 system we expand the result in terms of powers of c. A homogeneity analysis shows that B^a, C^a, G^a and A^a contain only even powers, while the other four symbols contain only odd powers. We solve for the coefficients of increasing powers of c, noting that at each stage we only need to solve a 4×4 system. The outcome of the first step (i.e., if $c = 0$) is already known from gravity waves [HIT16]. The computations are somewhat tedious, but quite elementary. We omit them and only write the final result:

$$\begin{aligned}
C^a(\xi, \eta) &= -i\frac{c^2}{4g^2}\xi, & B^a(\xi, \eta) &= -i\frac{c^4}{4g^2}\eta^{-1} + i\frac{c^2}{2g}\xi\eta^{-1} + i\frac{c^2}{4g} - i\xi, \\
D^a(\xi, \eta) &= i\frac{c^3}{4g^2} - i\frac{c}{2g}\xi, & E^a(\xi, \eta) &= i\frac{c^3}{4g^2}\xi\eta^{-1} - i\frac{c}{2g}\xi, \\
A^a(\xi, \eta) &= i\frac{c^2}{4g}, & F^a(\xi, \eta) &= -i\frac{c^3}{4g}\eta^{-1} + i\frac{c}{2}, \\
H^a(\xi, \eta) &= -i\frac{c}{2g}\xi, & G^a(\xi, \eta) &= i\frac{c^2}{2g}\xi\eta^{-1} - i\xi.
\end{aligned} \tag{2-16}$$

□

3. The linearized equation

In this section we derive the linearized water wave equations, and prove energy estimates for them. We recall that for the similar problem in [HIT16] we were able to prove quadratic energy estimates in $\dot{\mathcal{H}}_0$, which apply for the large-data problem, but also cubic energy estimates for the small-data problem. By contrast, here we are only able to prove the quadratic energy estimates. This suffices for the local well-posedness theory, but is not so useful in order to establish improved lifespan bounds. It appears unlikely that cubic energy estimates hold in $\dot{\mathcal{H}}_0$ for the linearized problem; in any case, we leave this question open.

3A. ***Computing the linearization.*** The solutions for the linearized water wave equation around a profile (W, Q) are denoted by (w, q). However, it will be more convenient to immediately switch to diagonal variables (w, r), where

$$r := q - Rw.$$

We first recall the equations,

$$\begin{cases} W_t + \underline{F}(W_\alpha + 1) + i\frac{c}{2}W = 0, \\ Q_t - igW + \underline{F}Q_\alpha + icQ + \boldsymbol{P}[|R|^2] - i\frac{c}{2}T_1 = 0, \end{cases} \tag{3-1}$$

where $\underline{F} = F - i\frac{c}{2}F_1$ with

$$F = \boldsymbol{P}\left[\frac{R}{1+\overline{\boldsymbol{W}}} - \frac{\bar{R}}{1+\boldsymbol{W}}\right], \quad F_1 = \boldsymbol{P}\left[\frac{W}{1+\overline{\boldsymbol{W}}} + \frac{\overline{W}}{1+\boldsymbol{W}}\right], \quad T_1 = \boldsymbol{P}[W\bar{R} - \overline{W}R].$$

The linearization of R is

$$\delta R = \frac{q_\alpha - Rw_\alpha}{1+\boldsymbol{W}} = \frac{r_\alpha + R_\alpha w}{1+\boldsymbol{W}},$$

and that of $|R|^2$ is

$$\delta|R|^2 = n + \bar{n}, \quad n := \bar{R}\delta R = \frac{\bar{R}(r_\alpha + R_\alpha w)}{1+\boldsymbol{W}}.$$

The linearizations of F, F_1 and T_1 can be expressed in the form

$$\delta F = \boldsymbol{P}[m - \bar{m}], \quad \delta F_1 = \boldsymbol{P}[m_1 + \bar{m}_1], \quad \delta T_1 = \boldsymbol{P}[m_2 - \bar{m}_2],$$

where the auxiliary variables m, m_1, m_2 correspond to differentiating F, F_1 and T_1 with respect to the holomorphic variables,

$$m := \frac{q_\alpha - Rw_\alpha}{J} + \frac{\bar{R}w_\alpha}{(1+\boldsymbol{W})^2} = \frac{r_\alpha + R_\alpha w}{J} + \frac{\bar{R}w_\alpha}{(1+\boldsymbol{W})^2},$$

$$m_1 := \frac{1}{1+\overline{\boldsymbol{W}}}w - \frac{\overline{W}}{(1+\boldsymbol{W})^2}w_\alpha, \quad m_2 := \bar{R}w - \frac{\overline{W}r_\alpha + \overline{W}R_\alpha w}{1+\boldsymbol{W}}.$$

Given all of the above, it follows that the linearized water wave equations take the form

$$\begin{cases} w_t + \underline{F} w_\alpha + (1+\boldsymbol{W})\Big(\boldsymbol{P}[m-\bar{m}] - i\frac{c}{2}\boldsymbol{P}[m_1+\bar{m}_1]\Big) + i\frac{c}{2}w = 0, \\ q_t + \underline{F} q_\alpha + Q_\alpha\Big(P[m-\bar{m}] - i\frac{c}{2}\boldsymbol{P}[m_1+\bar{m}_1]\Big) - igw + icq + \boldsymbol{P}[n+\bar{n}] - i\frac{c}{2}\boldsymbol{P}[m_2-\bar{m}_2] = 0. \end{cases}$$

Now we transition in both equations from $\underline{F}$ to the real advection coefficient $\underline{b}$ using the relations (1-16). We also move to the right all the terms which we expect to be perturbative. These are terms like $\bar{P}m$, $\bar{P}n$, $\bar{P}m_1$, $\bar{P}m_2$, which are lower-order since the differentiated holomorphic variables have to be lower-frequency. The same applies to their conjugates. Then, our equations are rewritten as

$$\begin{cases} (\partial_t + \underline{b}\partial_\alpha)w + \left[i\frac{c}{2}w + (1+\boldsymbol{W})\Big(m - i\frac{c}{2}m_1\Big) - \frac{\bar{R}w_\alpha}{1+\boldsymbol{W}} - i\frac{c}{2}\frac{\bar{W}w_\alpha}{1+\boldsymbol{W}}\right] = \underline{\mathcal{G}}_0 \\ (\partial_t + \underline{b}\partial_\alpha)q - igw + icq + \left[Q_\alpha\Big(m - i\frac{c}{2}m_1\Big) + n - i\frac{c}{2}m_2 - \frac{\bar{R}q_\alpha}{1+\boldsymbol{W}} - i\frac{c}{2}\frac{\bar{W}q_\alpha}{1+\boldsymbol{W}}\right] = \underline{\mathcal{K}}_0, \end{cases}$$

where

$$\begin{cases} \underline{\mathcal{G}}_0 = (1+\boldsymbol{W})\Big((\boldsymbol{P}\bar{m} + \bar{\boldsymbol{P}}m) + i\frac{c}{2}(\boldsymbol{P}\bar{m}_1 - \bar{\boldsymbol{P}}m_1)\Big), \\ \underline{\mathcal{K}}_0 = Q_\alpha\Big((\boldsymbol{P}\bar{m} + \bar{\boldsymbol{P}}m) + i\frac{c}{2}(\boldsymbol{P}\bar{m}_1 - \bar{\boldsymbol{P}}m_1)\Big) + (\bar{\boldsymbol{P}}n - \boldsymbol{P}\bar{n}) - i\frac{c}{2}(\boldsymbol{P}\bar{m}_2 + \bar{\boldsymbol{P}}m_2). \end{cases}$$

Taking advantage of algebraic cancellations in the square brackets above we are left with

$$\begin{cases} (\partial_t + \underline{b}\partial_\alpha)w + \frac{1}{1+\bar{\boldsymbol{W}}}\Big(r_\alpha + R_\alpha w + i\frac{c}{2}(\bar{\boldsymbol{W}} - \boldsymbol{W})w\Big) = \underline{\mathcal{G}}_0, \\ (\partial_t + \underline{b}\partial_\alpha)q - igw + icq + \frac{R}{1+\bar{\boldsymbol{W}}}\Big(r_\alpha + R_\alpha w + i\frac{c}{2}(\bar{\boldsymbol{W}} - \boldsymbol{W})w\Big) - i\frac{c}{2}(R+\bar{R})w = \underline{\mathcal{K}}_0. \end{cases}$$

Now we can switch from q to $r = q - Rw$ and obtain a diagonalized system, namely

$$\begin{cases} (\partial_t + \underline{b}\partial_\alpha)w + \frac{1}{1+\bar{\boldsymbol{W}}}\Big(r_\alpha + R_\alpha w + i\frac{c}{2}(\bar{\boldsymbol{W}} - \boldsymbol{W})w\Big) = \underline{\mathcal{G}}_0, \\ (\partial_t + \underline{b}\partial_\alpha)r + icr - i\Big(g + \frac{c}{2}(\bar{R} - R) + i(\partial_t + \underline{b}\partial_\alpha)R\Big)w = \underline{\mathcal{K}}_0 - R\underline{\mathcal{G}}_0. \end{cases}$$

For the expression $(\partial_t + \underline{b}\partial_\alpha)R$ in the second equation we use (1-19) and compute the coefficient of w as follows:

$$\begin{aligned} g + \frac{c}{2}(\bar{R} - R) - i(\partial_t + \underline{b}\partial_\alpha)R &= g + \frac{c}{2}(\bar{R} - R) + cR - \frac{g\boldsymbol{W} - a}{1+\boldsymbol{W}} - \frac{c}{2}\frac{R\boldsymbol{W} + \bar{R}\boldsymbol{W} + N}{1+\boldsymbol{W}} \\ &= \frac{g + a + (c/2)(R + \bar{R} - N)}{1+\boldsymbol{W}}. \end{aligned}$$

This motivates the definition of $\underline{a}$ in (1-12). With this notation, we write the final form of the linearized equations as

$$\begin{cases} (\partial_t + \underline{b}\partial_\alpha)w + \frac{r_\alpha}{1+\bar{\boldsymbol{W}}} + \frac{R_\alpha w}{1+\bar{\boldsymbol{W}}} = \underline{\mathcal{G}}(w, r), \\ (\partial_t + \underline{b}\partial_\alpha)r + icr - i\frac{g + \underline{a}}{1+\boldsymbol{W}}w = \underline{\mathcal{K}}(w, r), \end{cases} \tag{3-2}$$

where $\underline{\mathcal{G}}(w,r)$, and $\underline{\mathcal{K}}(w,r)$ are given by

$$\underline{\mathcal{G}}(w,r)=\mathcal{G}(w,r)-i\frac{c}{2}\mathcal{G}_1(w,r),\qquad \underline{\mathcal{K}}(w,r)=\mathcal{K}(w,r)-i\frac{c}{2}\mathcal{K}_1(w,r),$$

where

$$\mathcal{G}(w,r)=(1+\boldsymbol{W})(\boldsymbol{P}\bar{m}+\bar{\boldsymbol{P}}m),\qquad \mathcal{G}_1(w,r)=-(1+\boldsymbol{W})(\boldsymbol{P}\bar{m}_1-\bar{\boldsymbol{P}}m_1)+\frac{(\bar{\boldsymbol{W}}-\boldsymbol{W})w}{1+\bar{\boldsymbol{W}}},\tag{3-3}$$

$$\mathcal{K}(w,r)=\bar{\boldsymbol{P}}n-\boldsymbol{P}\bar{n},\qquad \mathcal{K}_1(w,r)=\boldsymbol{P}\bar{m}_2+\bar{\boldsymbol{P}}m_2.\tag{3-4}$$

If $c=0$, then these equations coincide with those in [HIT16]. The fact that $g+\underline{a}$ is real and positive is crucial for the well-posedness of the linearized system.

We remark that while (w,r) are holomorphic, it is not directly obvious that the above evolution preserves the space of holomorphic states. To remedy this, one can also project the linearized equations onto the space of holomorphic functions via the projection $\boldsymbol{P}$. Then we obtain the equations

$$\begin{cases}(\partial_t+\mathfrak{M}_{\underline{b}}\partial_\alpha)w+\boldsymbol{P}\left[\dfrac{1}{1+\bar{\boldsymbol{W}}}r_\alpha\right]+\boldsymbol{P}\left[\dfrac{R_\alpha}{1+\bar{\boldsymbol{W}}}w\right]=\boldsymbol{P}\underline{\mathcal{G}}(w,r),\\[2ex] (\partial_t+\mathfrak{M}_{\underline{b}}\partial_\alpha)r+icr-i\boldsymbol{P}\left[\dfrac{g+\underline{a}}{1+\boldsymbol{W}}w\right]=\boldsymbol{P}\underline{\mathcal{K}}(w,r).\end{cases}\tag{3-5}$$

Since the original set of equations (1-5) is fully holomorphic, it follows that the two sets of equations (3-2) and (3-5) are algebraically equivalent.

In order to investigate the possibility of cubic linearized energy estimates it is also of interest to separate the quadratic parts $\underline{\mathcal{G}}^2$ and $\underline{\mathcal{K}}^2$ of $\underline{\mathcal{G}}$ and $\underline{\mathcal{K}}$. The holomorphic quadratic parts of $\mathcal{G}$ and $\mathcal{K}$, which also appear in [HIT16], are given by

$$\boldsymbol{P}\mathcal{G}^{(2)}(w,r)=-P[\boldsymbol{W}\bar{r}_\alpha]+P[R\bar{w}_\alpha],\qquad \boldsymbol{P}\mathcal{K}^{(2)}(w,r)=-\boldsymbol{P}[R\bar{r}_\alpha].$$

Next we compute the similar decomposition for $\mathcal{G}_1^2$ and $\mathcal{K}_1^2$, since the rest was done in [HIT16]. These are split into quadratic and cubic-and-higher terms as shown below:

$$\mathcal{G}_1=\mathcal{G}_1^{(2)}+\mathcal{G}_1^{(3+)},\qquad \mathcal{K}_1=\mathcal{K}_1^{(2)}+\mathcal{K}_1^{(3+)}.$$

For the quadratic parts we have the holomorphic components

$$\begin{cases}\boldsymbol{P}\mathcal{G}_1^{(2)}(w,r)=\boldsymbol{P}[\boldsymbol{W}\bar{w}]+\boldsymbol{P}[W\bar{w}_\alpha]+\boldsymbol{P}[\bar{\boldsymbol{W}}w]-\boldsymbol{P}[\boldsymbol{W}w],\\ \boldsymbol{P}\mathcal{K}_1^{(2)}(w,r)=\boldsymbol{P}[W\bar{r}_\alpha]-\boldsymbol{P}[R\bar{w}],\end{cases}\tag{3-6}$$

and the antiholomorphic components

$$\begin{cases}\bar{\boldsymbol{P}}\mathcal{G}_1^{(2)}(w,r)=-\bar{\boldsymbol{P}}[\bar{\boldsymbol{W}}w]-\bar{\boldsymbol{P}}[\bar{W}w_\alpha]+\bar{\boldsymbol{P}}[\bar{\boldsymbol{W}}w],\\ \bar{\boldsymbol{P}}\mathcal{K}_1^{(2)}(w,r)=\bar{\boldsymbol{P}}[\bar{W}r_\alpha]-\bar{\boldsymbol{P}}[\bar{R}w].\end{cases}$$

The cubic terms have the form

$$\begin{cases}\mathcal{G}_1^{(3+)}(w,r)=-\boldsymbol{W}(\boldsymbol{P}\bar{m}_1-\bar{\boldsymbol{P}}m_1)-(\boldsymbol{P}\bar{m}_1^{(3+)}-\bar{\boldsymbol{P}}m_1^{(3+)})-\bar{Y}(\bar{\boldsymbol{W}}-\boldsymbol{W})w,\\ \mathcal{K}_1^{(3+)}(w,r)=\bar{\boldsymbol{P}}m_2^{(3+)}+\boldsymbol{P}\bar{m}_2^{(3+)}.\end{cases}$$

For the purpose of simplifying nonlinear estimates, it is convenient to express $\mathcal{G}_1^{(3)}$ and $\mathcal{K}_1^{(3)}$ in a polynomial fashion. This is done using the variable $Y = \boldsymbol{W}/(1+\boldsymbol{W})$. Then we have

$$\begin{aligned}\bar{\boldsymbol{P}}m_1 &= -\bar{\boldsymbol{P}}[\bar{Y}w + (Y^2 - 2Y + 1)\overline{W}w_\alpha],\\ \bar{\boldsymbol{P}}m^{(3+)} &= \bar{\boldsymbol{P}}[(2Y - Y^2)\overline{W}w_\alpha],\\ \bar{\boldsymbol{P}}m_2^{(3+)} &= \bar{\boldsymbol{P}}[\overline{W}Yr_\alpha - \overline{W}R_\alpha(1-Y)w].\end{aligned}$$

3B. *Quadratic estimates for large data.* Our goal here is to study the well-posedness of the system (3-5) in $L^2 \times \dot{H}^{1/2}$. We begin with a more general version of the system (3-5), namely

$$\begin{cases}(\partial_t + \mathfrak{M}_{\underline{b}}\partial_\alpha)w + \boldsymbol{P}\left[\dfrac{1}{1+\overline{\boldsymbol{W}}}r_\alpha\right] + \boldsymbol{P}\left[\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}w\right] = G,\\ (\partial_t + \mathfrak{M}_{\underline{b}}\partial_\alpha)r + icr - i\boldsymbol{P}\left[\dfrac{g+\underline{a}}{1+\boldsymbol{W}}w\right] = K,\end{cases} \tag{3-7}$$

and define the associated linear energy

$$E_{\text{lin}}^{(2)}(w,r) = \int_{\mathbb{R}} (g+\underline{a})|w|^2 + \Im(r\bar{r}_\alpha)\, d\alpha.$$

We note that the positivity of the energy is closely related to the Taylor sign condition (1-26). Therefore, when $\underline{a}$ is positive, we have

$$E_{\text{lin}}^{(2)}(w,r) \approx_A \mathcal{E}_0(w,r).$$

Our first result uses the control parameters $A_{-1/2}$, A and B defined in (1-22), (1-20), and(1-21), respectively in order to establish (nearly) cubic bounds for the system (3-7):

Proposition 3.1. *The linear equation (3-7) is well-posed in $\dot{\mathcal{H}}_0$, and the following estimate holds*:

$$\frac{d}{dt}E_{\text{lin}}^{(2)}(w,r) = 2\Re\int_{\mathbb{R}} (g+\underline{a})\bar{w}G - i\bar{r}_\alpha K + c^2\Im R|w|^2\, d\alpha + O_A(\underline{A}\underline{B})E_{\text{lin}}^{(2)}(w,r). \tag{3-8}$$

We will also need a weighted version of this:

Lemma 3.2. *Let f be a real function and (w,r) solutions to (3-7). Then for the difference*

$$I := \frac{d}{dt}\Re\int f((g+\underline{a})|w|^2 - i\bar{r}_\alpha r)\, d\alpha - \Re\int f((g+\underline{a})\bar{w}F - i\bar{r}_\alpha G) + (\partial_t + b\partial_\alpha)f((g+\underline{a})|w|^2 - i\bar{r}_\alpha r)\, d\alpha$$

we have the estimate

$$|I| \lesssim_A (\underline{B}\|f\|_{L^\infty} + \underline{A}\||D|^{1/2}f\|_{\text{BMO}})\,\|(w,r)\|^2_{L^2\times\dot{H}^{1/2}}. \tag{3-9}$$

Our main use for the result in Proposition 3.1 is to apply it to the linearized equation (3-5):

Proposition 3.3. *The linearized equation (3-5) is well-posed in $L^2 \times \dot{H}^{1/2}$, and the following estimate holds*:

$$\frac{d}{dt}E_{\text{lin}}^{(2)}(w,r) \lesssim_A (\underline{B} + c\underline{A})E_{\text{lin}}^{(2)}(w,r). \tag{3-10}$$

Proof of Proposition 3.1. A direct computation yields

$$\frac{d}{dt}\int (g+\underline{a})|w|^2\,d\alpha = 2\Re\int (g+\underline{a})\bar{w}(\partial_t+\mathfrak{M}_{\underline{b}}\partial_\alpha)w+\underline{a}\bar{w}[\underline{b},\boldsymbol{P}]w_\alpha\,d\alpha+\int [\underline{a}_t+((g+\underline{a})\underline{b})_\alpha]|w|^2\,d\alpha.$$

A similar computation shows that

$$\frac{d}{dt}\int \Im(r\partial_\alpha\bar{r})\,d\alpha = 2\Im\int (\partial_t+\mathfrak{M}_{\underline{b}}\partial_\alpha)r\partial_\alpha\bar{r}\,d\alpha.$$

Adding the two and using the equations (3-7), the quadratic $\Re(w\bar{r}_\alpha)$-term cancels modulo another commutator term, and we obtain

$$\frac{d}{dt}E^{(2)}_{\text{lin}}(w,r) = 2\Re\int (g+\underline{a})\bar{w}G - i\bar{r}_\alpha K\,d\alpha + c^2\int \Im R|w|^2\,d\alpha + \underline{\text{err}}_1, \tag{3-11}$$

where

$$\begin{aligned}\underline{\text{err}}_1 = &\int [\underline{a}_t+((g+\underline{a})\underline{b})_\alpha - c^2\Im R]|w|^2\,d\alpha - 2\Re\int (g+\underline{a})\frac{R_\alpha}{1+\overline{\boldsymbol{W}}}|w|^2\,d\alpha\\ &- 2\Re\int \underline{a}\bar{w}[\bar{Y},\boldsymbol{P}](r_\alpha+R_\alpha w)\,d\alpha - 2\Re\int \underline{a}\bar{w}[\boldsymbol{P},\underline{b}]w_\alpha\,d\alpha.\end{aligned}$$

Using the auxiliary function $\underline{M}$ in (1-17), we rewrite it as

$$\begin{aligned}\underline{\text{err}}_1 = &\int (\underline{a}_t+\underline{b}\underline{a}_\alpha - c^2\Im R)|w|^2 - (g+\underline{a})\Big(i\frac{c}{2}(\boldsymbol{W}-\overline{\boldsymbol{W}})+\underline{M}\Big)|w|^2\,d\alpha\\ &- 2\Re\int \underline{a}\bar{w}([\bar{Y},\boldsymbol{P}](r_\alpha+R_\alpha w)+[\boldsymbol{P},\underline{b}]w_\alpha)\,d\alpha.\end{aligned}$$

At this point we need to re-express the error in terms of a, a_1, b, b_1, M, and M_1. The terms which do not have any c-factors were already estimated in [HIT16], so we only need to worry about the remaining components. There is only one exception: the first term in $\underline{\text{err}}_1$, whose counterpart was estimated separately in [HIT16], and which we also estimate separately here. Thus, the error we want to bound is composed of err_1 (the same as in [HIT16]) and additional terms, which we call $\text{err}_{1,1}$:

$$\underline{\text{err}}_1 = \text{err}_1 + \text{err}_{1,1},$$

where $\text{err}_{1,1}$ is further separated into terms

$$\text{err}_{1,1} := \text{err}^1_{1,1} + \text{err}^2_{1,1} + \text{err}^3_{1,1} - \text{err}^4_{1,1} + \text{err}^5_{1,1} + \text{err}^6_{1,1}, \tag{3-12}$$

which will be estimated separately, and are listed below:

$$\begin{aligned}\text{err}^1_{1,1} &:= \int (a_t+\underline{b}a_\alpha)|w|^2\,d\alpha,\\ \text{err}^2_{1,1} &:= \frac{c}{2}\int (a_{1,t}+\underline{b}a_{1,\alpha} - ig(\boldsymbol{W}-\overline{\boldsymbol{W}}) - 2c\Im R)|w|^2\,d\alpha,\\ \text{err}^3_{1,1} &:= \frac{c}{2}\int (i(g+a)M_1 - a_1M - ia(\boldsymbol{W}-\overline{\boldsymbol{W}}))|w|^2\,d\alpha,\\ \text{err}^4_{1,1} &:= 2\Re\int i\frac{c}{2}a\bar{w}[\boldsymbol{P},b_1]w_\alpha\,d\alpha,\end{aligned}$$

$$\mathrm{err}^5_{1,1} := 2\Re \int \frac{c}{2} a_1 \bar{w}\{[\bar{Y}, \boldsymbol{P}](r_\alpha + R_\alpha w) + [\boldsymbol{P}, b] w_\alpha\}\, d\alpha,$$
$$\mathrm{err}^6_{1,1} := \Re \int i\frac{c^2}{4} a_1 \bar{w}(2[P, b_1] w_\alpha + (M_1 - \boldsymbol{W} + \overline{\boldsymbol{W}}) w)\, d\alpha.$$

To conclude the proof of (3-8) we want to show that

$$|\underline{\mathrm{err}}_1| \lesssim_A (A + cA_{-1/2})(B + cA) E^{(2)}_{\mathrm{lin}}(w, r). \tag{3-13}$$

From [HIT16] we have the bound

$$|\mathrm{err}_1| \lesssim_A AB E^{(2)}_{\mathrm{lin}}(w, r),$$

so it remains to establish (3-13) for each of the $\mathrm{err}_{1,1}$ components. The positive constant c has the role of a scaling parameter; therefore it makes sense to group terms accordingly.

The bound for $\mathrm{err}^1_{1,1}$. This follows by Proposition A.3 in Appendix A.

The bound for $\mathrm{err}^2_{1,1}$. This follows by Proposition A.4, which we included in Appendix A.

The bound for $\mathrm{err}^3_{1,1}$. Here we use the pointwise estimates

$$\|a\|_{L^\infty} \lesssim A^2, \quad \|a_1\|_{L^\infty} \lesssim_A A_{-1/2}, \quad \|M\|_{L^\infty} \lesssim_A AB, \quad \|M_1\|_{L^\infty} \lesssim_A A^2.$$

The first and the third are from [HIT16], while for the second and fourth we use Lemma A.2 and Proposition A.4. This yields

$$|\mathrm{err}^2_{1,1}| \lesssim c(A^2 + ABA_{-1/2} + A^3 + A^4) E^{(2)}_{\mathrm{lin}}(w, r).$$

The bound for $\mathrm{err}^4_{1,1}$. For the terms in $\mathrm{err}^3_{1,1}$ we use the pointwise bounds of a and $b_{1,\alpha}$ (recall that $b_{1,\alpha} = \boldsymbol{W} - \overline{\boldsymbol{W}} - M_1$), which were obtained in [HIT16, Proposition 8.6] and in Lemma A.2, respectively:

$$|a| \lesssim A^2, \quad \|b_{1,\alpha}\|_{L^\infty} \lesssim A + A^2.$$

To estimate the commutator in L^2 we use a Coifman–Meyer bound, see, e.g., Lemma 8.1 in [HIT16]:

$$\|[\boldsymbol{P}, b_1] w_\alpha\|_{L^2} \lesssim \|b_{1,\alpha}\|_{L^\infty} \|w\|_{L^2}.$$

Combining the results above leads to

$$|\mathrm{err}^3_{1,1}| \lesssim c(A^3 + A^4) E^{(2)}_{\mathrm{lin}}(w, r).$$

The bound for $\mathrm{err}^5_{1,1}$. After using the above pointwise bound on a_1 it remains to estimate the commutators in L^2 (as above). The only difficulty here is that we need to move half of a derivative from r_α onto Y. Such estimates were already considered in [HIT16],

$$\|[\bar{Y}, \boldsymbol{P}] r_\alpha\|_{L^2} \lesssim \||D|^{1/2} Y\|_{\mathrm{BMO}} \|r\|_{\dot{H}^{1/2}}, \quad \|[\boldsymbol{P}, b] w_\alpha\|_{L^2} \lesssim \|b_\alpha\|_{\mathrm{BMO}} \|w\|_{L^2},$$

and suffice due to the bounds for b, a_1 and Y in Lemmas 2.5 and 2.7 in [HIT16], and Proposition A.4. For the remaining term in $\mathrm{err}^4_{1,1}$ we write $[\bar{Y}, \boldsymbol{P}](R_\alpha w) = [\overline{\boldsymbol{P}}, \overline{\boldsymbol{P}}[\bar{Y} R_\alpha]] w$ and use the Coifman–Meyer lemma [1976] (also discussed in Appendix B in [HIT16]) to estimate

$$\|\bar{\boldsymbol{P}}[\bar{\boldsymbol{P}}[\bar{Y}R_\alpha]w]\|_{L^2} \lesssim \|w\|_{L^2}\,\|\bar{\boldsymbol{P}}[\bar{Y}R_\alpha]\|_{\mathrm{BMO}} \lesssim \|w\|_{L^2}\,\||D|^{1/2}Y\|_{\mathrm{BMO}}\,\||D|^{1/2}R\|_{\mathrm{BMO}},$$

where the bilinear bound in the second step follows after a bilinear Littlewood–Paley decomposition (again, see [HIT16]). Hence,

$$|\mathrm{err}^4_{1,1}| \lesssim_A cA_{-1/2}ABE^{(2)}_{\mathrm{lin}}(w,r).$$

The bound for $\mathrm{err}^6_{1,1}$. Here we use the pointwise bounds on a_1, M_1 and $b_{1,\alpha}$, along with the Coifman–Meyer commutator estimate.

This concludes the proof of (3-13). □

Proof of Lemma 3.2. We rewrite I in the form

$$I := D_1 + D_2 + D_3 + D_4,$$

where

$$D_1 = \int f(\partial_t + \underline{b}\partial_\alpha)\underline{a}|w|^2\,d\alpha, \quad D_3 = \int f\left(b_\alpha|w|^2 - 2\Re\left(\bar{w}\boldsymbol{P}\left[\frac{R_\alpha}{1+\bar{W}}w\right]\right)\right)d\alpha,$$

$$D_2 = \Re\int f\bar{\boldsymbol{P}}[\underline{b}w_\alpha]\bar{w}\,d\alpha, \qquad D_4 = \int if\left(\bar{r}_\alpha\boldsymbol{P}\left[\frac{(g+a)w}{1+W}\right] - (g+a)\bar{w}\boldsymbol{P}\left[\frac{r_\alpha}{1+W}\right]\right)d\alpha.$$

For the first term we use the pointwise bounds in Propositions A.3 and A.4.

For the second term we directly use a Coifman–Meyer estimate to bound the middle factor in L^2.

For the third term we use the pointwise bound on b_1, and then harmlessly replace b_α by $\boldsymbol{P}[R/(1+\bar{W})]_\alpha$. Then it remains to estimate in L^2 the difference

$$\boldsymbol{P}\left[\frac{R}{1+\bar{W}}\right]_\alpha w - \boldsymbol{P}\left[\frac{R_\alpha}{1+\bar{W}}w\right].$$

If w is the low-frequency factor in either term, then we only need its cofactor in BMO and we win. Else we can drop the first projection, and we are left with estimating the difference

$$\boldsymbol{P}\left[\frac{R}{1+\bar{W}}\right]_\alpha - \frac{R_\alpha}{1+\bar{W}}$$

in L^∞, which was done in [HIT16].

Finally, the last term cancels if we drop the projections. Hence we are left with estimating in L^2 the expression

$$\bar{\boldsymbol{P}}\left[\frac{r_\alpha}{1+W}\right],$$

which is done by a Coifman–Meyer estimate. In the first term we bound the expression

$$\bar{\boldsymbol{P}}\left[\frac{aw}{1+W}\right]$$

in L^2 and then move a half-derivative on f. □

Proof of Proposition 3.3. To estimate the terms involving $\mathcal{G}$ and $\mathcal{K}$ we separate the quadratic and cubic parts, but more importantly we group these expressions keeping track of the scaling parameter c. In our previous paper [HIT16] we have already established the bounds for the components without c, namely

$$\|\boldsymbol{P}\mathcal{G}^{(2)}(w,r)\|_{L^2}+\|\boldsymbol{P}\mathcal{K}^{(2)}(w,r)\|_{\dot H^{1/2}}\lesssim B(\|w\|_{L^2}+\|r\|_{\dot H^{1/2}}),$$

while the cubic-and-higher terms satisfy

$$\|\boldsymbol{P}\mathcal{G}^{(3+)}(w,r)\|_{L^2}+\|\boldsymbol{P}\mathcal{K}^{(3+)}(w,r)\|_{\dot H^{1/2}}\lesssim_A AB(\|w\|_{L^2}+\|r\|_{\dot H^{1/2}}).$$

Hence it suffices to estimate the terms with the c-factor, and show that the quadratic terms satisfy

$$\|\boldsymbol{P}\mathcal{G}_1^{(2)}(w,r)\|_{L^2}+\|\boldsymbol{P}\mathcal{K}_1^{(2)}(w,r)\|_{\dot H^{1/2}}\lesssim A(\|w\|_{L^2}+\|r\|_{\dot H^{1/2}}), \tag{3-14}$$

while the cubic-and-higher terms satisfy

$$\|\boldsymbol{P}\mathcal{G}_1^{(3+)}(w,r)\|_{L^2}+\|\boldsymbol{P}\mathcal{K}_1^{(3+)}(w,r)\|_{\dot H^{1/2}}\lesssim_A A^2(\|w\|_{L^2}+\|r\|_{\dot H^{1/2}}). \tag{3-15}$$

In order to obtain the estimates claimed in (3-14), (3-15) we use the Coifman–Meyer-type commutator estimates [1976] described in [HIT16, Appendix B, Lemma 8.1],

The bounds for all terms in $\boldsymbol{P}\mathcal{G}_1^{(2)}(w,r)$ are immediate, except for the second, where we use (B.10) in [HIT16] with $s=\frac{1}{2}$ and $\sigma=\frac{1}{2}$, to write

$$\|[\boldsymbol{P},W]\bar{w}_\alpha\|_{L^2}\lesssim\|\boldsymbol{W}\|_{L^\infty}\|w\|_{L^2}.$$

For $\boldsymbol{P}\mathcal{K}_1^{(2)}(w,r)$ we use again (B.10) in [HIT16] with $s=\frac{1}{2}$, and $\sigma=\frac{1}{2}$, and conclude that

$$\|[\boldsymbol{P},W]\bar{r}_\alpha\|_{\dot H^{1/2}}\lesssim\||D|^{1/2}W\|_{L^\infty}\|r\|_{\dot H^{1/2}},\qquad \|[\boldsymbol{P},R]\bar{w}\|_{\dot H^{1/2}}\lesssim\||D|^{1/2}R\|_{L^\infty}\|w\|_{L^2}.$$

Thus (3-14) follows.

For the cubic-and-higher parts of $\mathcal{G}_1$ and $\mathcal{K}_1$ we apply the same type of commutator estimates, as well as the BMO bounds in [Proposition 8.2, HIT16]. Precisely, in $\mathcal{G}_1^{(3+)}$ there are three nontrivial terms to be estimated in L^2, namely

$$\boldsymbol{P}[\bar{w}_\alpha(\bar{Y}^2-2\bar{Y})W],\qquad W\boldsymbol{P}[\bar{w}_\alpha(\bar{Y}^2-2\bar{Y}+1)W],\qquad \boldsymbol{P}[W\bar{\boldsymbol{P}}[w_\alpha(Y^2-2Y+1)\overline{W}]].$$

For the first two we use (B.12) and (B.14) from [HIT16] as follows:

$$\|\boldsymbol{P}[\bar{w}_\alpha(\bar{Y}^2-2\bar{Y})W]\|_{L^2}\lesssim\|w\|_{L^2}\|\partial_\alpha\boldsymbol{P}[(\bar{Y}^2-2\bar{Y})W]\|_{\mathrm{BMO}}\lesssim_A\|\boldsymbol{W}\|_{\mathrm{BMO}}\|Y\|_{L^\infty}\|w\|_{L^2}.$$

Similarly, for the remaining terms we have

$$\|\boldsymbol{P}[\bar{w}_\alpha(\bar{Y}^2-2\bar{Y}+1)W]\|_{L^2}\lesssim_A\|\boldsymbol{W}\|_{\mathrm{BMO}}\|w\|_{L^2}.$$

Finally, we estimate the cubic component of $\mathcal{K}_1$; we again use (B.12) and (B.15) from [HIT16], and obtain

$$\||D|^{1/2}\boldsymbol{P}[\bar{r}_\alpha\bar{Y}W]\|_{L^2}\lesssim\|r\|_{\dot H^{1/2}}\|\partial_\alpha\boldsymbol{P}[\bar{Y}W]\|_{\mathrm{BMO}}\lesssim_A\|r\|_{\dot H^{1/2}}\|Y\|_{L^\infty}\|\boldsymbol{W}\|_{L^\infty},$$

and also

$$\||D|^{1/2}\boldsymbol{P}[\bar{w}(1-\bar{Y})\bar{R}_\alpha W]\|_{L^2} \lesssim \|w(1-Y)\|_{L^2}\,\||D|^{1/2}\boldsymbol{P}[\bar{R}_\alpha W]\|_{\mathrm{BMO}} \lesssim_A \|w\|_{L^2}\,\||D|^{1/2}R\|_{\mathrm{BMO}}\,\|\boldsymbol{W}\|_{L^\infty}.$$

This concludes the proof of (3-15), and thus of the proposition. □

3C. ***Nearly cubic estimates for small data.*** Our goal here is to investigate the possibility of obtaining cubic estimates for the system (3-5) in $L^2\times\dot{H}^{1/2}$. Unlike in [HIT16], this is no longer possible. Instead, our more limited goal in this section is to identify a main portion of the linearized equation for which cubic estimates are valid, precisely up to c^2-terms. This will come in very handy later on in the proof of cubic estimates for the differentiated equation.

Our model problem this time is the following subset of (3-5):

$$\begin{cases}(\partial_t+\mathfrak{M}_{\underline{b}}\partial_\alpha)w+\boldsymbol{P}\left[\dfrac{1}{1+\overline{\boldsymbol{W}}}r_\alpha\right]+\boldsymbol{P}\left[\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}w\right]=-\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha]+\boldsymbol{P}[R\bar{w}_\alpha]+G,\\ (\partial_t+\mathfrak{M}_{\underline{b}}\partial_\alpha)r+icr-i\boldsymbol{P}\left[\dfrac{g+\underline{a}}{1+\boldsymbol{W}}w\right]=-\boldsymbol{P}[R\bar{r}_\alpha]+K.\end{cases}\tag{3-16}$$

In our previous work [HIT16], for the case $c=0$, we identified a cubic correction to $E^{(2)}_{\mathrm{lin}}(w,r)$ for which cubic estimates hold for solutions to the linearized equation, namely

$$E^{(3)}_{\mathrm{lin}}(w,r):=\int_{\mathbb{R}}(g+\underline{a})|w|^2+\Im(r\bar{r}_\alpha)+2\Im(\bar{R}wr_\alpha)-2\Re(\overline{\boldsymbol{W}}w^2)\,d\alpha.$$

Our next result asserts that in our case the time derivative of $E^{(3)}_{\mathrm{lin}}$ will be quartic at the leading order, but will have a cubic terms with a coefficient of c^2.

Proposition 3.4. *Assume that $\underline{A}$ is small. Then we have the energy equivalence*

$$E^{(3)}_{\mathrm{lin}}(w,r)\approx E^{(2)}_{\mathrm{lin}}(w,r).\tag{3-17}$$

Further, the linear equation (3-7) is well-posed in $\dot{\mathcal{H}}_0$, and the following estimate holds:

$$\frac{d}{dt}E^{(3)}_{\mathrm{lin}}(w,r)=2\Re\int_{\mathbb{R}}[(g+\underline{a})\bar{w}-i\bar{R}_\alpha r_\alpha-2\overline{\boldsymbol{W}}w]G-i[\bar{r}_\alpha-(\bar{R}w)_\alpha]K\,d\alpha + c^2\Im R|w|^2\,d\alpha+O_A(\underline{A}\,\underline{B})E^{(2)}_{\mathrm{lin}}(w,r).\tag{3-18}$$

Applying this to the linearized equation (3-5) we obtain:

Proposition 3.5. *The linearized equation (3-5) is well-posed in $L^2\times\dot{H}^{1/2}$, and the following estimate holds*:

$$\frac{d}{dt}E^{(3)}_{\mathrm{lin}}(w,r)=2\Re\int_{\mathbb{R}}c[gw\boldsymbol{P}\mathcal{G}^{(2)}_1(w,r)-i\bar{r}_\alpha\boldsymbol{P}\mathcal{K}^{(2)}_1(w,r)]-i\frac{c^2}{2}R|w|^2\,d\alpha+O_A(\underline{A}\,\underline{B})E^{(2)}_{\mathrm{lin}}(w,r),\tag{3-19}$$

where $\boldsymbol{P}\mathcal{G}^{(2)}_1(w,r)$, $\boldsymbol{P}\mathcal{K}^{(2)}_1(w,r)$ are as in (3-6).

Proof of Proposition 3.4 . The energy equivalence is immediate due to the estimates in [HIT16] for the added cubic terms, and the pointwise bounds on $\underline{a}$ in Lemmas A.3 and A.4.

To prove the estimate in (3-18) we compute the time derivative of the cubic component of the energy $E^{(3)}_{\mathrm{lin}}(w,r)$, using the projected equations for w and r and the unprojected equations for R and $\boldsymbol{W}$:

$$\frac{d}{dt}\left(\Im\int \bar{R}wr_\alpha\,d\alpha-\Re\int \overline{\boldsymbol{W}}w^2\,d\alpha\right)=\Im\int -ig\overline{\boldsymbol{W}}wr_\alpha-\bar{R}r_\alpha r_\alpha+ig\bar{R}ww_\alpha+\bar{R}r_\alpha G+\bar{R}wK_\alpha\,d\alpha \\ +\Re\int \bar{R}_\alpha w^2+2\overline{\boldsymbol{W}}wr_\alpha-2\overline{\boldsymbol{W}}wG\,d\alpha+\underline{\mathrm{err}}_2,$$

where

$$\begin{aligned}\underline{\mathrm{err}}_2=\Im\int\Bigg\{&\left(i\left(\frac{g\overline{\boldsymbol{W}}^2+\underline{a}}{1+\overline{\boldsymbol{W}}}\right)-\underline{b}\bar{R}_\alpha-i\,\frac{c}{2}\,\frac{\bar{R}\overline{\boldsymbol{W}}+R\overline{\boldsymbol{W}}+\bar{N}}{1+\overline{\boldsymbol{W}}}\right)wr_\alpha\\ &-\bar{R}w\partial_\alpha\left(\mathfrak{M}_{\underline{b}}r_\alpha-i\boldsymbol{P}\left[\frac{\underline{a}-\boldsymbol{W}}{1+\boldsymbol{W}}w\right]+\boldsymbol{P}[R\bar{r}_\alpha]\right)\\ &-\bar{R}r_\alpha\left(\mathfrak{M}_{\underline{b}}w_\alpha-\boldsymbol{P}\left[\frac{\overline{\boldsymbol{W}}}{1+\overline{\boldsymbol{W}}}r_\alpha\right]+\boldsymbol{P}\left[\frac{R_\alpha}{1+\overline{\boldsymbol{W}}}w\right]+\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha-R\bar{w}_\alpha]\right)\Bigg\}\,d\alpha\\ +\Re\int\Bigg\{&w^2\left(\underline{b}\overline{\boldsymbol{W}}_\alpha+\frac{\overline{\boldsymbol{W}}-\boldsymbol{W}}{1+\boldsymbol{W}}\bar{R}_\alpha-(1+\overline{\boldsymbol{W}})\underline{M}+i\,\frac{c}{2}\,\overline{\boldsymbol{W}}(\overline{\boldsymbol{W}}-\boldsymbol{W})\right)\\ &+2\overline{\boldsymbol{W}}w\left(\mathfrak{M}_{\underline{b}}w_\alpha-\boldsymbol{P}\left[\frac{\overline{\boldsymbol{W}}}{1+\overline{\boldsymbol{W}}}r_\alpha\right]+\boldsymbol{P}\left[\frac{R_\alpha}{1+\overline{\boldsymbol{W}}}w\right]+\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha-R\bar{w}_\alpha]\right)\Bigg\}\,d\alpha.\end{aligned}\tag{3-20}$$

We now separate (3-20) into two parts, namely err_2, which was already estimated in [HIT16], and $\mathrm{err}_{2,1}$:

$$\underline{\mathrm{err}}_2:=\mathrm{err}_2+i\,\frac{c}{2}\mathrm{err}_{2,1},$$

where

$$\begin{aligned}\mathrm{err}_{2,1}:=\Im\int&\left\{\left(b_1\bar{R}_\alpha-\frac{\bar{R}\overline{\boldsymbol{W}}+R\overline{\boldsymbol{W}}+\bar{N}}{1+\overline{\boldsymbol{W}}}\right)wr_\alpha+\bar{R}w\partial_\alpha\boldsymbol{P}[b_1r_\alpha]-\bar{R}w\boldsymbol{P}\left[\frac{a_1}{1+\boldsymbol{W}}w\right]+\bar{R}r_\alpha\boldsymbol{P}[b_1w_\alpha]\right\}d\alpha\\ +\Re\int&\{w^2[-b_1\overline{\boldsymbol{W}}_\alpha-(1+\overline{\boldsymbol{W}})\bar{M}_1+\overline{\boldsymbol{W}}(\overline{\boldsymbol{W}}-\boldsymbol{W})]-2\overline{\boldsymbol{W}}w\boldsymbol{P}[b_1w_\alpha]\}\,d\alpha.\end{aligned}\tag{3-21}$$

We still need to estimate this last error.

Adding $\underline{\mathrm{err}}_2$ to (3-18) (but applied to solutions to (3-16)) gives us

$$\begin{aligned}\frac{d}{dt}E^{(3)}_{\mathrm{lin}}(w,r)&=\frac{d}{dt}E^{(2)}_{\mathrm{lin}}(w,r)+2\frac{d}{dt}\left\{\left(\Im\int \bar{R}wr_\alpha\,d\alpha-\Re\int\overline{\boldsymbol{W}}w^2\,d\alpha\right)\right\}\\ &=2\Re\int_{\mathbb{R}}(g+\underline{a})\bar{w}\{G-\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha]+\boldsymbol{P}[R\bar{w}_\alpha]\}-i\bar{r}_\alpha\{K-\boldsymbol{P}[R\bar{r}_\alpha]\}\,d\alpha\\ &\quad+c^2\Im\int_{\mathbb{R}}R|w|^2\,d\alpha+\underline{\mathrm{err}}_1\\ &\quad+2\Im\int -ig\overline{\boldsymbol{W}}wr_\alpha-\bar{R}r_\alpha r_\alpha+ig\bar{R}ww_\alpha+\bar{R}r_\alpha G+\bar{R}wK_\alpha\,d\alpha\\ &\quad+2\Re\int\bar{R}_\alpha w^2+2\overline{\boldsymbol{W}}wr_\alpha-2\overline{\boldsymbol{W}}wG\,d\alpha+2\underline{\mathrm{err}}_2.\end{aligned}$$

Rewriting the above expression leads to

$$\begin{aligned}\frac{d}{dt}E^{(3)}_{\mathrm{lin}}(w,r) &= \frac{d}{dt}E^{(2)}_{\mathrm{lin}}(w,r)+2\frac{d}{dt}\left\{\left(\Im\int \bar{R}wr_\alpha\,d\alpha-\Re\int \overline{\boldsymbol{W}}w^2\,d\alpha\right)\right\}\\ &= 2\Re\int_{\mathbb{R}}(g+\underline{a})\bar{w}G-i\bar{r}_\alpha K\,d\alpha+c^2\Im\int_{\mathbb{R}}R|w|^2\,d\alpha+\underline{\mathrm{err}}_1\\ &\quad+2\Re\int_{\mathbb{R}}(g+\underline{a})\bar{w}\{\boldsymbol{P}[R\bar{w}_\alpha]-\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha]\}+i\bar{r}_\alpha\boldsymbol{P}[R\bar{r}_\alpha]\,d\alpha\\ &\quad+2\Im\int -ig\overline{\boldsymbol{W}}wr_\alpha-\bar{R}r_\alpha r_\alpha+ig\bar{R}ww_\alpha+\bar{R}r_\alpha G+\bar{R}wK_\alpha\,d\alpha\\ &\quad+2\Re\int \bar{R}_\alpha w^2+2\overline{\boldsymbol{W}}wr_\alpha-2\overline{\boldsymbol{W}}wG\,d\alpha+2\underline{\mathrm{err}}_2.\end{aligned}$$

We obtain

$$\begin{aligned}\frac{d}{dt}E^{(3)}_{\mathrm{lin}}(w,r) &= 2\Re\int_{\mathbb{R}}[(g+\underline{a})\bar{w}-i\bar{R}_\alpha r_\alpha-2\overline{\boldsymbol{W}}w]G-i[\bar{r}_\alpha-(\bar{R}w)_\alpha]K\,d\alpha\\ &\quad+c^2\Im\int_{\mathbb{R}}R|w|^2\,d\alpha+2\Re\int_{\mathbb{R}}\underline{a}\bar{w}\{\boldsymbol{P}[R\bar{w}_\alpha]-\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha]\}\,d\alpha+\underline{\mathrm{err}}_1+2\underline{\mathrm{err}}_2.\end{aligned}$$

We introduce

$$\underline{\mathrm{err}}_3 := 2\underline{\mathrm{err}}_2-2\Re\int_{\mathbb{R}}\underline{a}\bar{w}\{\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha]-\boldsymbol{P}[R\bar{w}_\alpha]\}\,d\alpha,$$

which implies

$$\frac{d}{dt}E^{(3)}_{\mathrm{lin}}(w,r)=2\Re\int_{\mathbb{R}}[(g+\underline{a})\bar{w}-i\bar{R}_\alpha r_\alpha-2\overline{\boldsymbol{W}}w]G-i[\bar{r}-(\bar{R}w)]_\alpha K\,d\alpha+c^2\Im\int_{\mathbb{R}}R|w|^2\,d\alpha+\underline{\mathrm{err}}_1+\underline{\mathrm{err}}_3.$$

Given the bound (3-13) for $\underline{\mathrm{err}}_1$, the proof of (3-18) is concluded if we show that

$$|\underline{\mathrm{err}}_3|\lesssim \underline{A}\,\underline{B}\,E^{(2)}_{\mathrm{lin}}(w,r). \tag{3-22}$$

Due to the pointwise bound for $\underline{a}$, proved in Appendix A, and the Coifman–Meyer-type L^2-bound for $\boldsymbol{P}[\boldsymbol{W}\bar{r}_\alpha]-\boldsymbol{P}[R\bar{w}_\alpha]$ from [HIT16], it suffices to estimate $\underline{\mathrm{err}}_2$, which in turn reduces to estimating $\mathrm{err}_{2,1}$,

$$|\mathrm{err}_{2,1}|\lesssim \underline{A}\,\underline{B}\,E^{(2)}_{\mathrm{lin}}(w,r).$$

For the remainder of the proof we separately estimate several types of terms in $\mathrm{err}_{2,1}$:

Terms involving b_1. Here, we use the bounds for b_1 from Lemma A.2:

$$\|b_{1,\alpha}\|_{L^\infty}\lesssim A+A^2,\qquad \||D|^{1/2}b\|_{L^\infty}\lesssim A_{-1/2}+A_{-1/2}A.$$

We first collect all the terms that are contained in the first integral of $\mathrm{err}_{2,1}$, and which include b_1:

$$I_1 := \int_{\mathbb{R}} b_1\bar{R}_\alpha wr_\alpha+\bar{R}w\partial_\alpha\boldsymbol{P}[b_1r_\alpha]+\bar{R}r_\alpha\boldsymbol{P}[b_1w_\alpha]\,d\alpha.$$

After integrating by parts, we cancel two of the terms in I_1 and obtain

$$I_1 = I_2+I_3,$$

where

$$I_2 := \int_{\mathbb{R}}\bar{R}_\alpha w\overline{\boldsymbol{P}}[b_1r_\alpha]\,d\alpha \quad\text{and}\quad I_3 := \int_{\mathbb{R}}\bar{R}r_\alpha\boldsymbol{P}[b_1w_\alpha]-\bar{R}w_\alpha\boldsymbol{P}[b_1r_\alpha]\,d\alpha.$$

The bounds for I_2 follow easily because it has a commutator structure:

$$I_2 := \int_{\mathbb{R}} \bar{R}_\alpha w \bar{\boldsymbol{P}}[b_1 r_\alpha]\, d\alpha = \int_{\mathbb{R}} \boldsymbol{P}[\bar{R}_\alpha w] \bar{\boldsymbol{P}}[b_1 r_\alpha]\, d\alpha = \int_{\mathbb{R}} [\boldsymbol{P}, w]\bar{R}_\alpha \cdot [\bar{\boldsymbol{P}}, b_1] r_\alpha \, d\alpha,$$

where we can estimate both factors in L^2 using Coifman–Meyer estimates,

$$\|[\boldsymbol{P}, w]\bar{R}_\alpha\|_{L^2} \lesssim \|w\|_{L^2} \|R_\alpha\|_{\mathrm{BMO}}, \quad \|[\bar{\boldsymbol{P}}, b_1] r_\alpha\|_{L^2} \lesssim \||D|^{1/2} b_1\|_{\mathrm{BMO}} \|r\|_{\dot{H}^{1/2}}.$$

The bound for I_3 follows from Lemma 8.9 in Appendix B of [HIT16].

We next collect all the terms that are contained in the second integral appearing in the expression of $\mathrm{err}_{2,1}$, and which include b_1:

$$I_4 := \int_{\mathbb{R}} -b_1 \bar{\boldsymbol{W}}_\alpha w^2 - 2\bar{\boldsymbol{W}} w \boldsymbol{P}[b_1 w_\alpha]\, d\alpha.$$

As before, we integrate by parts and rewrite the expression for I_4 as

$$I_4 := \int_{\mathbb{R}} b_{1,\alpha} \bar{\boldsymbol{W}} w^2 + 2b_1 \bar{\boldsymbol{W}} w w_\alpha - 2\bar{\boldsymbol{W}} w \boldsymbol{P}[b_1 w_\alpha]\, d\alpha = \int_{\mathbb{R}} b_{1,\alpha} \bar{\boldsymbol{W}} w^2 + 2\bar{\boldsymbol{W}} w \bar{\boldsymbol{P}}[b_1 w_\alpha]\, d\alpha.$$

The first integral on the right-hand side is easy to bound since we know that $b_{1,\alpha}$ is in L^∞:

$$\left| \int_{\mathbb{R}} b_{1,\alpha} \bar{\boldsymbol{W}} w^2 \, d\alpha \right| \lesssim \|b_{1,\alpha}\|_{L^\infty} \|\boldsymbol{W}\|_{L^\infty} \|w\|_{L^2}^2.$$

For the last integral in I_4 we use the Coifman–Meyer-type estimate (established first in [HIT16]) to obtain

$$\left| \int_{\mathbb{R}} 2\bar{\boldsymbol{W}} w \bar{\boldsymbol{P}}[b_1 w_\alpha]\, d\alpha \right| \lesssim \|\boldsymbol{W}\|_{L^\infty} \|w\|_{L^2} \|[\bar{\boldsymbol{P}}, b_1] w_\alpha\|_{L^2} \lesssim \|b_{1,\alpha}\|_{L^\infty} \|\boldsymbol{W}\|_{L^\infty} \|w\|_{L^2}^2.$$

Thus,

$$|I_4| \lesssim \|b_{1,\alpha}\|_{L^\infty} \|\boldsymbol{W}\|_{L^\infty} \|w\|_{L^2}^2.$$

Quadrilinear terms bounded via $L^2 \cdot L^2$ pairing. Some of the remaining terms in $\mathrm{err}_{2,1}$ have straightforward bounds:

$$|I_7| := \left| \int_{\mathbb{R}} (1+\bar{\boldsymbol{W}}) \bar{M}_1 w^2 \, d\alpha \right| \lesssim_A (1+A) A^2 \|w\|_{L^2}^2,$$

$$|I_8| := \left| \int_{\mathbb{R}} \bar{\boldsymbol{W}} (\bar{\boldsymbol{W}} - \boldsymbol{W}) w^2 \, d\alpha \right| \lesssim_A A^2 \|w\|_{L^2}^2,$$

but others require a little bit of work. This includes the following expressions:

$$I_5 := \int_{\mathbb{R}} \frac{\bar{R}\bar{\boldsymbol{W}} + R\bar{\boldsymbol{W}} + \bar{N}}{1+\bar{\boldsymbol{W}}} w r_\alpha \, d\alpha,$$

$$I_6 := \int_{\mathbb{R}} \bar{R} w \partial_\alpha \boldsymbol{P}\left[\frac{a_1}{1+\boldsymbol{W}} w \right] d\alpha = -\int_{\mathbb{R}} \partial_\alpha \bar{\boldsymbol{P}}[\bar{R} w] \boldsymbol{P}[a_1(1-Y) w]\, d\alpha.$$

To obtain the bound for I_6 we use the Cauchy–Schwarz inequality and Lemma 2.1 from [HIT16]

$$|I_6| \lesssim \|\partial_\alpha \bar{\boldsymbol{P}}[\bar{R} w]\|_{L^2} \|\boldsymbol{P}[a_1(1-Y) w]\|_{L^2} \lesssim \|R_\alpha\|_{\mathrm{BMO}} \|a_1\|_{L^\infty} (1+\|Y\|_{L^\infty}) \|w\|_{L^2}^2.$$

Finally, the bound for I_5 is also a consequence of commutator estimates; to see this we rewrite it as

$$I_5 := \int_{\mathbb{R}} f w r_\alpha \, d\alpha = \int_{\mathbb{R}} \bar{\boldsymbol{P}}[f r_\alpha] w \, d\alpha, \quad \text{where } f := \frac{\bar{R}\bar{\boldsymbol{W}} + R\bar{\boldsymbol{W}} + \bar{N}}{1+\bar{\boldsymbol{W}}}.$$

Thus,

$$|I_5| = \left| \int_{\mathbb{R}} \bar{\boldsymbol{P}}[f r_\alpha] w \, d\alpha \right| \lesssim \|[\bar{\boldsymbol{P}}, f] r_\alpha\|_{L^2} \|w\|_{L^2} \lesssim \||D|^{1/2} f\|_{\mathrm{BMO}} \|r\|_{\dot{H}^{1/2}} \|w\|_{L^2},$$

where

$$\||D|^{1/2} f\|_{\mathrm{BMO}} \lesssim \underline{A}\,\underline{B}. \qquad \square$$

Proof of Proposition 3.5 . To prove the bound in (3-19) it suffices to apply the estimate in (3-18) with

$$G = \boldsymbol{P}\underline{\mathcal{G}}, \quad K = \boldsymbol{P}\underline{\mathcal{K}}.$$

In fact, we only have to resume our work in finding the new terms introduced by the vorticity assumption, which are the components carrying the c when expanding the right-hand side of (3-5), $\boldsymbol{P}\mathcal{G}_1$ and $\boldsymbol{P}\mathcal{K}_1$:

$$\underline{\mathcal{G}} = \mathcal{G} - i\frac{c}{2}\mathcal{G}_1, \quad \underline{\mathcal{K}} = \mathcal{K} - i\frac{c}{2}\mathcal{K}_1.$$

The terms we want to single out are the c-cubic terms appearing in the cubic part of the energy $E^3(w, r)$, (3-8).

For this we need to recall the quadratic components of $\boldsymbol{P}\mathcal{G}_1$ and $\boldsymbol{P}\mathcal{K}_1$

$$\begin{aligned} \boldsymbol{P}\mathcal{G}_1^{(2)}(w, r) &= \boldsymbol{P}[\boldsymbol{W}\bar{w}] + \boldsymbol{P}[W\bar{w}_\alpha] + \boldsymbol{P}[\bar{\boldsymbol{W}} w] - \boldsymbol{P}[\boldsymbol{W} w], \\ \boldsymbol{P}\mathcal{K}_1^{(2)}(w, r) &= \boldsymbol{P}[W\bar{r}_\alpha] - \boldsymbol{P}[R\bar{w}], \end{aligned}$$

and the antiholomorphic components (which will not matter in this computation since we work with the projected equations)

$$\begin{aligned} \bar{\boldsymbol{P}}\mathcal{G}_1^{(2)}(w, r) &= -\bar{\boldsymbol{P}}[\bar{\boldsymbol{W}} w] - \bar{\boldsymbol{P}}[\bar{W} w_\alpha] + \bar{\boldsymbol{P}}[\bar{\boldsymbol{W}} w], \\ \bar{\boldsymbol{P}}\mathcal{K}_1^{(2)}(w, r) &= \bar{\boldsymbol{P}}[\bar{W} r_\alpha] - \bar{\boldsymbol{P}}[\bar{R} w]. \end{aligned}$$

Thus, we need to bound the terms

$$\int_{\mathbb{R}} g w \boldsymbol{P}\mathcal{G}_1^{(2)}(w, r) \, d\alpha, \quad \int_{\mathbb{R}} \bar{r}_\alpha \boldsymbol{P}\mathcal{K}_1^{(2)}(w, r) \, d\alpha, \quad \int_{\mathbb{R}} c^2 \Im R |w|^2 \, d\alpha.$$

These bounds are all obtained using Lemma 2.1 in [HIT16], for example, for the first integral we obtain

$$\left| \int_{\mathbb{R}} g w \{[\boldsymbol{P}, \boldsymbol{W}]\bar{w} + [\boldsymbol{P}, W]\bar{w}_\alpha - [\boldsymbol{P}, \boldsymbol{W}] w\} \, d\alpha \right| \lesssim A \|w\|_{L^2}^2.$$

For the second integral we need to move half of a derivative off of $\bar{r}_\alpha$:

$$\left| \int_{\mathbb{R}} \bar{r}_\alpha \{[\boldsymbol{P}, W]\bar{r}_\alpha - [\boldsymbol{P}, R]\bar{w}\} \, d\alpha \right| \lesssim B \|r\|_{\dot{H}^{1/2}}^2 + A \|r\|_{\dot{H}^{1/2}} \|w\|_{L^2}.$$

The last one, is trivial

$$\left| \int_{\mathbb{R}} R |w|^2 \, d\alpha \right| \lesssim A_{-1/2} \|w\|_{L^2}^2. \qquad \square$$

4. Higher-order energy estimates

The main goal of this section is to prove energy bounds for the differentiated equations. These are the main ingredient for the lifespan part of Theorem 1, as well as for the cubic result in Theorem 2. Precisely, we will establish two types of energy bounds for $(\boldsymbol{W}, R)$ and their higher derivatives. The first one is a quadratic bound which applies independently of the size of the initial data; this yields the last part of Theorem 1. The cubic energy bound applies in the small-data case, and yields the cubic lifespan bound in our small-data result in Theorem 2. The large-data result is as follows:

Proposition 4.1. *For any $n \geq 0$ there exists an energy functional $E^{n,(2)}$ with the following properties whenever the conditions (1-25) and (1-26) hold uniformly*:

(i) *Norm equivalence*:

$$E^{n,(2)}(\boldsymbol{W}, R) \approx_{\underline{A}} \mathcal{E}_0(\partial^n \boldsymbol{W}, \partial^n R).$$

(ii) *Quadratic energy estimates for solutions to (1-19)*:

$$\frac{d}{dt} E^{n,(2)}(\boldsymbol{W}, R) \lesssim_{\underline{A}} \underline{B}\mathcal{E}_0(\partial^n \boldsymbol{W}, \partial^n R). \tag{4-1}$$

The small-data result is as follows:

Proposition 4.2. *For any $n \geq 0$ there exists an energy functional $E^{n,(3)}$ which has the following properties as long as $\underline{A} \ll 1$*:

(i) *Norm equivalence*:

$$E^{n,(3)}(\boldsymbol{W}, R) = (1 + O(\underline{A}))\mathcal{E}_0(\partial^n \boldsymbol{W}, \partial^n R) + O(c^4 \underline{A})\mathcal{E}_0(\partial^{n-1} \boldsymbol{W}, \partial^{n-1} R).$$

(ii) *Cubic energy estimates*:

$$\frac{d}{dt} E^{n,(3)}(\boldsymbol{W}, R) \lesssim_{\underline{A}} \underline{B}\,\underline{A}\big(\mathcal{E}_0(\partial^n \boldsymbol{W}, \partial^n R) + c^4 \mathcal{E}_0(\partial^{n-1} \boldsymbol{W}, \partial^{n-1} R)\big). \tag{4-2}$$

Here if $n = 0$ then $\mathcal{E}_0(\partial^{-1} \boldsymbol{W}, \partial^{-1} R)$ is naturally replaced by $\mathcal{E}(W, Q)$.

The case $n = 0$ of Proposition 4.1 corresponds to $\dot{\mathcal{H}}_0$-bounds for $(\boldsymbol{W}_\alpha, R)$. But these functions solve the linearized equation, so the desired bounds are a consequence of Proposition 3.1. Hence, we will begin with the proof of the cubic bounds for the $n = 0$ case.

Next, we compute the differentiated equations, first for $n = 1$ and then for $n \geq 2$, and show that, up to a certain class of bounded errors, suitable modifications of $(\boldsymbol{W}^{(n)}, R^{(n)})$ solve a linear system which is quite similar to the linearized equation. There are two reasons why we separate the case $n = 1$. First, this corresponds to $\dot{\mathcal{H}}_0$-bounds for $(\boldsymbol{W}_\alpha, R_\alpha)$, which play a special role as it is our threshold for local well-posedness. Secondly, there is a subtle difference in the choice of the modifications alluded to above, due to the fact that certain terms which are different for $n \geq 2$ coincide at $n = 1$.

At this point, the bounds for the linearized equation already yield the large-data result. What remains is to establish the cubic small-data result, which is where we implement our quasilinear modified energy method, using the normal form calculated in Section 2.

As part of the argument we need to express various nonlinear expressions in terms of their homogeneous expansions. To describe the decomposition of a nonlinear analytic expression F into homogeneous components we will use the notation $\Lambda^k F$ to denote the component of F of homogeneity k. We similarly introduce the operators $\Lambda^{\leq k}$ and $\Lambda^{\geq k}$. We carefully note that all of our multilinear expansions are with respect to the diagonal variables W and R, and not with respect to (W, Q).

Compared to [HIT16] here we lack scaling, but we can still introduce a notion of order of a multilinear expression. We begin with single terms, for which we assign orders as follows:

- The order of $W^{(k)}$ is $k-1$.
- The order of $R^{(k)}$ is $k-\frac{1}{2}$.
- The order of c is $\frac{1}{2}$.

For a multilinear form involving products of such terms we define the total order as the sum of the orders of all factors.

While not all expressions arising in the $(\mathbf{W}^{(n)}, R^{(n)})$ are multilinear in (W, R), they can be still viewed as multilinear in (W, R) and undifferentiated Y. Since Y scales like $\mathbf{W}$, it is natural to assign to it the homogeneity zero. According to this definition, all terms in the $\mathbf{W}^{(n)}$-equation have order $n+\frac{1}{2}$, and all terms in the R-equation have order n. Moving on to integral multilinear forms, all n-th energies have order $2n$, and their time derivatives have order $2n+\frac{1}{2}$.

A second useful bookkeeping device will be needed when we deal with integral multilinear forms. There it makes a difference how derivatives and also complex conjugations are distributed among factors. To account for this we define the leading order of a multilinear form to be the largest sum of the orders of two factors with opposite conjugations. Since we only allow nonnegative orders, for the n-th order energy this is at most $2n$. According to our definition, all the terms in our n-th order energy will have order $2n$, and all terms in its time derivative will have order $2n+\frac{1}{2}$. We remark that the half-integers in the definition of the orders impose a parity constraint in the terms associated to each power of c.

If all factors in a multilinear form have nonnegative orders, this imposes a constraint on the order of each factor. Unfortunately this does not appear to be the case here, as our multilinear forms will also contain factors of W and R, which have negative order. This is quite inconvenient. Fortunately, there is a simple way to avoid negative orders altogether. Precisely, we will never consider such factors alone, but in combination with c; thus, the allowed factors will be cR and c^2W, both of which have order 0. We carefully note last remark applies only partially in the special case $n=0$.

4A. ***The case $n=0$.*** The goal of this subsection is to obtain cubic energy estimates in $\dot{\mathcal{H}}_0$ for the system for diagonal variables $(\mathbf{W}, R)$. For convenience we recall the system here:

$$\begin{cases} \mathbf{W}_t + \underline{b}\mathbf{W}_\alpha + \dfrac{(1+\mathbf{W})R_\alpha}{1+\bar{\mathbf{W}}} = \underline{\mathcal{G}}_0, \\ R_t + \underline{b}R_\alpha + icR - i\dfrac{g\mathbf{W}-a}{1+\mathbf{W}} = \underline{\mathcal{K}}_0, \end{cases}$$

where

$$\underline{\mathcal{G}}_0 = (1+\mathbf{W})\underline{M} + i\frac{c}{2}\mathbf{W}(\mathbf{W}-\bar{\mathbf{W}}), \quad \underline{\mathcal{K}}_0 = i\frac{c}{2}\frac{R\mathbf{W}+\bar{R}\mathbf{W}+N}{1+\mathbf{W}}.$$

We want to be able to apply the quadratic and cubic bounds for the "modified" model (3-16), respectively. For that, we rewrite the above systems as

$$\begin{cases} \boldsymbol{W}_t + \underline{b}\boldsymbol{W}_\alpha + \dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}} + \dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}\boldsymbol{W} = -\boldsymbol{P}[\bar{R}_\alpha \boldsymbol{W}] + \boldsymbol{P}[R\overline{\boldsymbol{W}}_\alpha] + \underline{G}_0, \\ R_t + \underline{b}R_\alpha + icR - i\dfrac{(g+\underline{a})\boldsymbol{W}}{1+\boldsymbol{W}} = -\boldsymbol{P}[R\bar{R}_\alpha] + \underline{K}_0, \end{cases} \tag{4-3}$$

where, the expressions for $\underline{G}_0$ and $\underline{K}_0$, for the purpose of this section, are

$$\begin{cases} \underline{G}_0 := \boldsymbol{W}(\boldsymbol{P}[R\overline{\boldsymbol{W}}_\alpha] - \boldsymbol{P}[\bar{R}_\alpha \boldsymbol{W}]) + i\dfrac{c}{2}(1+\boldsymbol{W})M_1 + i\dfrac{c}{2}\boldsymbol{W}(\boldsymbol{W}-\overline{\boldsymbol{W}}) \\ \qquad\quad + (1+\boldsymbol{W})\{\bar{\boldsymbol{P}}[\bar{R}Y_\alpha - R_\alpha \bar{Y}] - \boldsymbol{P}[R\partial_\alpha(\overline{\boldsymbol{W}}\bar{Y})] + \boldsymbol{P}[\bar{R}_\alpha \boldsymbol{W} Y]\}, \\ \underline{K}_0 := \bar{\boldsymbol{P}}[\bar{R}R_\alpha] - i\dfrac{c}{2}N. \end{cases}$$

In (4-3) we have identified the leading part of the equation. We want to interpret the terms $(\underline{G}_0, \underline{K}_0)$ on the right as mostly perturbative, but also pay attention to the holomorphic quadratic part, given by

$$\boldsymbol{P}G_0^{(2)} = i\frac{c}{2}(\boldsymbol{P}[W\bar{Y}]_\alpha + \boldsymbol{W}^2 - \boldsymbol{P}[\boldsymbol{W}\overline{\boldsymbol{W}}]), \quad \boldsymbol{P}K_0^{(2)} = i\frac{c}{2}\boldsymbol{P}[W\bar{R}_\alpha - \overline{\boldsymbol{W}}R]. \tag{4-4}$$

Precisely, we first claim that the quadratic and cubic parts of $(\underline{G}_0, \underline{K}_0)$ satisfy the bounds

$$\|(\underline{G}_0^{(2)}, \underline{K}_0^{(2)})\|_{\dot{\mathcal{H}}_0} \lesssim_A \underline{B}\boldsymbol{N}_0, \tag{4-5}$$

$$\|(\underline{G}_0^{(3)}, \underline{K}_0^{(3)})\|_{\dot{\mathcal{H}}_0} \lesssim_A A\underline{B}\boldsymbol{N}_0, \tag{4-6}$$

respectively, where

$$\boldsymbol{N}_0 = \|(\boldsymbol{W}, R)\|_{L^2\times\dot{H}^{1/2}}.$$

The bounds for the components of M_1 and N are discussed in Lemma A.2. The $\boldsymbol{W}$ prefactors in $\underline{G}_0$ are harmless, as they are bounded by A in L^∞. For the remaining terms it suffices to use Coifman–Meyer-type estimates discussed in Appendix A. For instance we have

$$\|\boldsymbol{P}[R\overline{\boldsymbol{W}}_\alpha]\|_{L^2} \lesssim \|[\boldsymbol{P}, R]\overline{\boldsymbol{W}}_\alpha\|_{L^2} \lesssim \||D|^{1/2}\boldsymbol{W}\|_{\mathrm{BMO}}\|R\|_{\dot{H}^{1/2}} \lesssim_A B\boldsymbol{N}_0,$$

and all other terms in $\underline{G}_0$ are similar. Finally, for the first term in $\underline{K}_0$ we have

$$\|\boldsymbol{P}[\bar{R}R_\alpha]\|_{\dot{H}^{1/2}} \lesssim \||D|^{1/2}R\|_{L^2}\|R_\alpha\|_{\mathrm{BMO}} \lesssim B\boldsymbol{N}_0.$$

We are now ready to look at the cubic energies. We start by constructing the cubic normal-form energy by selecting the quadratic and cubic terms from the corresponding linear energy for the normal-form variables. Precisely, we have

$$\mathcal{E}_0(\widetilde{W}_\alpha, \widetilde{Q}_\alpha) = \mathcal{E}_0(\boldsymbol{W}, Q_\alpha) + 2\int_{\mathbb{R}} \Re\overline{\boldsymbol{W}}\partial_\alpha W^{[2]} - 2\Im\bar{Q}_{\alpha\alpha}\partial_\alpha Q^{[2]}\, d\alpha + \text{quartic}.$$

In the first term we substitute $Q_\alpha = R(1+\boldsymbol{W})$. In the integral we use the expressions (2-8), integrate by parts to eliminate the $\partial^{-1}W$- and Q-factors, and then replace Q_α with R. Separating the outcome of these computations into a leading part and a lower-order part, we write it in the form

$$E^{(3)}_{\mathrm{NF}} := E^{(3)}_{\mathrm{NF,high}} + E^{(3)}_{\mathrm{NF,low}},$$

where

$$E^{3}_{\mathrm{NF,high}}(\boldsymbol{W}, R) := \int_{\mathbb{R}} (g + c\Re R)|\boldsymbol{W}|^2 + \Im(R\bar{R}_\alpha) + 2\Im(\bar{R}\boldsymbol{W}R_\alpha) - 2\Re(\bar{\boldsymbol{W}}\boldsymbol{W}^2)\, d\alpha,$$

and

$$\begin{aligned} E^{(3)}_{\mathrm{NF,low}}(\boldsymbol{W}, R) := &- c\int_{\mathbb{R}} 2\Re R\{|\boldsymbol{W}|^2 - \Im(\bar{R}_\alpha R)\} - \boldsymbol{W}\bar{\boldsymbol{W}}R_\alpha + \boldsymbol{W}^2\bar{R}\, d\alpha \\ &- \frac{c^2}{g}\int_{\mathbb{R}} \tfrac{5}{2}\Im W\{|\boldsymbol{W}|^2 - \Im(\bar{R}_\alpha R)\} - \tfrac{1}{2}\bar{W}\boldsymbol{W}^2 - \tfrac{1}{2}\boldsymbol{W}\bar{R}^2\, d\alpha \\ &- \frac{3c^3}{2g}\Im\int_{\mathbb{R}} RW\bar{\boldsymbol{W}}\, d\alpha - \frac{3c^4}{2g}\Re\int_{\mathbb{R}} \boldsymbol{W}|W|^2\, d\alpha. \end{aligned}$$

For the leading-order part $E^{(3)}_{\mathrm{NF,high}}(\boldsymbol{W}, R)$ we consider the appropriate quasilinear correction

$$E^{(3)}_{\mathrm{high}}(\boldsymbol{W}, R) := \int_{\mathbb{R}} (g + \underline{a})|\boldsymbol{W}|^2 + \Im(R\bar{R}_\alpha) + 2\Im(\bar{R}\boldsymbol{W}R_\alpha) - 2\Re(\bar{\boldsymbol{W}}\boldsymbol{W}^2)\, d\alpha,$$

and the remainder $E^{3}_{\mathrm{NF,low}}(\boldsymbol{W}, R)$ remains unchanged. Hence, we define the quasilinear cubic energy

$$E^{0,(3)} := E^{3}_{\mathrm{high}} + E^{3}_{\mathrm{NF,low}}.$$

It remains to show that this energy has all the right properties in Proposition 4.2.

We begin with the energy equivalence. For the leading part this has already been done in the context of the linearized equation (see Proposition 3.4),

$$E^{(3)}_{\mathrm{high}}(\boldsymbol{W}, R) \approx (1 + O(A))E^{(2)}_{\mathrm{lin}}(\boldsymbol{W}, R).$$

So it remains to show that

$$E^{(3)}_{\mathrm{NF,low}}(\boldsymbol{W}, R) \approx O(\underline{A})(\mathcal{E}_0(\boldsymbol{W}, R) + c^4\mathcal{E}(W, Q)).$$

The bound is straightforward for all terms not containing R_α. So we now consider those. For the first one we have

$$\left|\int_{\mathbb{R}} R_\alpha |R|^2\, d\alpha\right| \lesssim \|R\|_{\dot{H}^{1/2}} \||D|^{1/2}(R\Re R)\|_{L^2} \lesssim \|R\|^2_{\dot{H}^{1/2}} \|R\|_{L^\infty} \lesssim A_{-1/2}\boldsymbol{N}_0^2,$$

which suffices. For the next term we have

$$\left|\int_{\mathbb{R}} R_\alpha \bar{\boldsymbol{W}}\boldsymbol{W}\, d\alpha\right| \lesssim \|R\|_{\dot{H}^{1/2}} \||D|^{1/2}\boldsymbol{P}[W\bar{\boldsymbol{W}}]\|_{L^2} \lesssim \|R\|_{\dot{H}^{1/2}} \||D|^{1/2}W\|_{\mathrm{BMO}} \|\boldsymbol{W}\|_{L^2} \lesssim A_{-1/2}\boldsymbol{N}_0^2.$$

Finally,

$$\|R_\alpha \bar{R}W\|_{L^2} \lesssim \|R\|_{\dot{H}^{1/2}} \||D|^{1/2}(\bar{R}W)\|_{L^2} \lesssim (\||D|^{1/2}R\|_{\mathrm{BMO}} \|W\|_{L^2} + \|R\|_{L^\infty} \||D|^{1/2}W\|_{L^2}) \|R\|_{\dot{H}^{1/2}}.$$

Now we consider the time derivative of modified quasilinear energy $E^{0,(3)}$ in order to prove the bound (4-2). By construction we know that

$$\Lambda^{\leq 3} \frac{d}{dt} E^{0,(3)} = 0.$$

We note that this is an algebraic property which follows from the normal-form-based construction even though the normal form itself is unbounded. Therefore it remains to estimate

$$\Lambda^{\geq 4} \frac{d}{dt} E^3 = \Lambda^{\geq 4} \frac{d}{dt} E^3_{\text{NF,high}} + \Lambda^{\geq 4} \frac{d}{dt} E^3_{\text{NF,low}}.$$

Due to (4-5) and (4-6) the estimate for the first term on the right-hand side follows directly from the bound (3-18) in Proposition 3.4 for the leading part of the linearized equation. Hence, it remains to consider the last term.

$E^3_{\text{NF,low}}(\boldsymbol{W}, R)$ is a trilinear form of order zero and leading order zero. We compute its time derivative using the relation

$$\frac{d}{dt} \int f_1 f_2 f_3 \, d\alpha = \int (\partial_t + \underline{b}\partial_\alpha) f_1 f_2 f_3 + f_1 (\partial_t + \underline{b}\partial_\alpha) f_2 f_3 + f_1 f_2 (\partial_t + \underline{b}\partial_\alpha) f_3 - \underline{b}_\alpha f_1 f_2 f_3 \, d\alpha. \quad (4\text{-}7)$$

Then its time derivative will be a multilinear form of order $\frac{1}{2}$, and also of leading order $\frac{1}{2}$. By inspection, we see that in this time derivative we can associate each W with a c^2-factor and each R with a c-factor, so that all each of the factors in all of the multilinear monomials have degree at least zero. Then, each multilinear monomial in $\Lambda^{\geq 4}\big(\frac{d}{dt} E^3_{\text{NF,low}}\big)$ contains exactly one factor of order $\frac{1}{2}$, and the rest are all factors of order zero. The factor of order $\frac{1}{2}$ can be either R_α or c, and the factors of order zero could be $\boldsymbol{W}$, Y, $c^2 W$, or cR. We have two cases to consider:

(a) If the factor of order $\frac{1}{2}$ is c, then we simply bound two of the remaining factors in L^2 and the others in L^∞.

(b) If the factor of order $\frac{1}{2}$ is R_α, then we use the $\dot{H}^{1/2}$-bound for R, and we are left with estimating an expression of the form $|D|^{1/2}(f_1 \cdots f_k)$, where $k \geq 3$, and each f_k has order zero. For this we use Lemma A.1 to estimate

$$\||D|^{1/2}(f_1 \cdots f_k)\|_{L^2} \lesssim \sum_j \||D|^{1/2} f_j\|_{L^\infty} \Big\| \prod_{l \neq j} f_l \Big\|_{L^2},$$

and conclude as above.

4B. ***The differentiated equations for n = 1.*** We begin by differentiating (1-19) in order to obtain a system for $(\boldsymbol{W}_\alpha, R_\alpha)$:

$$\begin{cases} \boldsymbol{W}_{\alpha t} + \underline{b}\boldsymbol{W}_{\alpha\alpha} + \dfrac{[(1+\boldsymbol{W})R_\alpha]_\alpha}{1+\bar{\boldsymbol{W}}} = (\underline{M} - \underline{b}_\alpha)\boldsymbol{W}_\alpha + (1+\boldsymbol{W})(R_\alpha \bar{Y}_\alpha + \underline{M}_\alpha) + i\dfrac{c}{2}[\boldsymbol{W}(\boldsymbol{W} - \bar{\boldsymbol{W}})]_\alpha, \\ R_{\alpha t} + \underline{b}R_{\alpha\alpha} + icR_\alpha - i\left(\dfrac{(g+a)\boldsymbol{W}_\alpha}{(1+\boldsymbol{W})^2} - \dfrac{a_\alpha}{1+\boldsymbol{W}}\right) = -\underline{b}_\alpha R_\alpha + i\dfrac{c}{2}\left(\dfrac{\boldsymbol{W}(R+\bar{R}) + N}{1+\boldsymbol{W}}\right)_\alpha. \end{cases}$$

We can expand the last term in the second equation, putting together all terms which involve $\boldsymbol{W}_\alpha$. The reason for this is that the $\boldsymbol{W}_\alpha$-terms will be unbounded in a suitable sense, and need to be treated as part of the leading-order linear operator. This will lead us to the coefficient a_1, which will be moved to the left, completing the coupling coefficient a to $\underline{a}$. We have

$$\left(\frac{\mathbf{W}(R+\bar{R})+N}{1+\mathbf{W}}\right)_\alpha = \frac{\mathbf{W}_\alpha}{(1+\mathbf{W})^2}\{(R+\bar{R})-N\} + \frac{\mathbf{W}(R_\alpha+\bar{R}_\alpha)+N_\alpha}{1+\mathbf{W}} = \frac{a_1\mathbf{W}_\alpha}{(1+\mathbf{W})^2} + \frac{\mathbf{W}(R_\alpha+\bar{R}_\alpha)+N_\alpha}{1+\mathbf{W}}.$$

Then our system becomes

$$\begin{cases} \mathbf{W}_{\alpha t} + \underline{b}\mathbf{W}_{\alpha\alpha} + \dfrac{[(1+\mathbf{W})R_\alpha]_\alpha}{1+\overline{\mathbf{W}}} = (\underline{M}-\underline{b}_\alpha)\mathbf{W}_\alpha + (1+\mathbf{W})(R_\alpha\bar{Y}_\alpha + \underline{M}_\alpha) + i\dfrac{c}{2}[\mathbf{W}(\mathbf{W}-\overline{\mathbf{W}})]_\alpha, \\ R_{t\alpha} + \underline{b}R_{\alpha\alpha} + icR_\alpha - i\dfrac{(g+\underline{a})\mathbf{W}_\alpha}{(1+\mathbf{W})^2} = -\underline{b}_\alpha R_\alpha - i\dfrac{a_\alpha}{1+\mathbf{W}} + i\dfrac{c}{2}\dfrac{\mathbf{W}(R_\alpha+\bar{R}_\alpha)+N_\alpha}{1+\mathbf{W}}. \end{cases}$$

In order to better compare this with the linearized system we introduce the modified variable $\mathbf{R} := R_\alpha(1+\mathbf{W})$ to further obtain

$$\begin{cases} \mathbf{W}_{\alpha t} + \underline{b}\mathbf{W}_{\alpha\alpha} + \dfrac{\mathbf{R}_\alpha}{1+\overline{\mathbf{W}}} = (\underline{M}-\underline{b}_\alpha)\mathbf{W}_\alpha + \mathbf{R}\bar{Y}_\alpha + (1+\mathbf{W})\underline{M}_\alpha + i\dfrac{c}{2}[\mathbf{W}(\mathbf{W}-\overline{\mathbf{W}})]_\alpha, \\ \mathbf{R}_t + \underline{b}\mathbf{R}_\alpha + ic\mathbf{R} - i\dfrac{(g+\underline{a})\mathbf{W}_\alpha}{1+\mathbf{W}} = \left(\underline{M} - \underline{b}_\alpha - \dfrac{R_\alpha}{1+\overline{\mathbf{W}}} + i\dfrac{c}{2}(\mathbf{W}-\overline{\mathbf{W}})\right)\mathbf{R} - ia_\alpha + i\dfrac{c}{2}[\overline{\mathbf{W}}R_\alpha + \mathbf{W}\bar{R}_\alpha + N_\alpha]. \end{cases}$$

Expanding the $\underline{b}_\alpha$-terms via (1-18) this yields

$$\begin{cases} \mathbf{W}_{\alpha t} + \underline{b}\mathbf{W}_{\alpha\alpha} + \dfrac{\mathbf{R}_\alpha}{1+\overline{\mathbf{W}}} + \dfrac{R_\alpha}{1+\overline{\mathbf{W}}}\mathbf{W}_\alpha = \underline{G}_1, \\ \mathbf{R}_t + \underline{b}\mathbf{R}_\alpha + ic\mathbf{R} - i\dfrac{(g+a)\mathbf{W}_\alpha}{1+\mathbf{W}} = \underline{K}_1, \end{cases} \tag{4-8}$$

where on the right we have placed all terms which should be thought of as "bounded". Precisely, $(\underline{G}_1, \underline{K}_1)$ have the form

$$\underline{G}_1 := G_1 - i\frac{c}{2}G_{1,1}, \quad \underline{K}_1 := K_1 - i\frac{c}{2}K_{1,1},$$

where the leading components have the same form as in [HIT16],

$$\begin{cases} G_1 = \mathbf{R}\bar{Y}_\alpha - \dfrac{\bar{R}_\alpha}{1+\mathbf{W}}\mathbf{W}_\alpha + 2M\mathbf{W}_\alpha + (1+\mathbf{W})M_\alpha, \\ K_1 = -2\left(\dfrac{\bar{R}_\alpha}{1+\mathbf{W}} + \dfrac{R_\alpha}{1+\overline{\mathbf{W}}}\right)\mathbf{R} + 2M\mathbf{R} + (R_\alpha\bar{R}_\alpha - ia_\alpha), \end{cases}$$

while the extra terms containing the vorticity c are

$$\begin{cases} G_{1,1} = 2M_1\mathbf{W}_\alpha + (1+\mathbf{W})M_{1,\alpha} - 2\mathbf{W}_\alpha(\mathbf{W}-\overline{\mathbf{W}}) - W(\mathbf{W}_\alpha - \overline{\mathbf{W}}_\alpha), \\ K_{1,1} = 2M_1\mathbf{R} - 2\mathbf{R}(\mathbf{W}-\overline{\mathbf{W}}) - \mathbf{P}[2\mathbf{W}\bar{R}_\alpha + W\bar{R}_{\alpha\alpha} - \overline{\mathbf{W}}_\alpha R] - \overline{\mathbf{P}}[2\overline{\mathbf{W}}R_\alpha + \overline{W}R_{\alpha\alpha} - \mathbf{W}_\alpha\bar{R}]. \end{cases}$$

Here on the left we have again the leading part of the linearized equation. Following the same approach as in [HIT16], we will interpret the terms on the right as mostly perturbative, but also keep track of their quadratic part. Thus, for bookkeeping purposes, we introduce two types of error terms, denoted $\text{err}(L^2)$

and $\mathrm{err}(\dot{H}^{1/2})$, which correspond to the two equations. The bounds for these errors are in terms of the control variables $\underline{A}$, $\underline{B}$, as well as the L^2-type norm

$$\boldsymbol{N}_1 = \|(\boldsymbol{W}_\alpha, R_\alpha)\|_{L^2 \times \dot{H}^{1/2}}.$$

In [HIT16], by $\mathrm{err}(L^2)$ we denote terms G_1 which satisfy the estimates

$$\|PG_1\|_{L^2} \lesssim_A AB\boldsymbol{N}_1,$$

as well as either one of the following two:

$$\|\bar{P}G_1\|_{L^2} \lesssim_A B\boldsymbol{N}_1 \quad \text{or} \quad \|\bar{P}G_1\|_{\dot{H}^{-1/2}} \lesssim_A A\boldsymbol{N}_1.$$

Here, in order to manage the new c-terms, we also include in $\mathrm{err}(L^2)$ the terms in $G_{1,1}$ for which

$$\|PG_{1,1}\|_{L^2} \lesssim_A A^2\boldsymbol{N}_1,$$

and either of the following two holds:

$$\|\bar{P}G_{1,1}\|_{L^2} \lesssim_A A\boldsymbol{N}_1 \quad \text{or} \quad \|\bar{P}G_{1,1}\|_{\dot{H}^{-1/2}} \lesssim_A A_{-1/2}\boldsymbol{N}_1.$$

Similarly, by $\mathrm{err}(\dot{H}^{1/2})$ we have denoted in [HIT16] the terms K_1 which satisfy the estimates

$$\|PK_1\|_{\dot{H}^{1/2}} \lesssim_A AB\boldsymbol{N}_1, \quad \|PK_1\|_{L^2} \lesssim_A A^2\boldsymbol{N}_1,$$

and

$$\|\bar{P}K_1\|_{L^2} \lesssim_A A\boldsymbol{N}_1.$$

To that, for the c-terms we add expressions $K_{1,1}$ which satisfy the estimates

$$\|PK_{1,1}\|_{\dot{H}^{1/2}} \lesssim_A A^2\boldsymbol{N}_1, \quad \|PK_{2,1}\|_{L^2} \lesssim_A AA_{-1/2}\boldsymbol{N}_1,$$

and

$$\text{either} \quad \|\bar{P}K_{1,1}\|_{L^2} \lesssim_A A_{-1/2}\boldsymbol{N}_1 \quad \text{or} \quad \|\bar{P}K_{1,1}\|_{\dot{H}^{1/2}} \lesssim_A A\boldsymbol{N}_1.$$

We will rely on [HIT16] for the analysis of the expressions G_1 and K_1, and handle just the new entries, i.e., the new terms accompanied by the vorticity factor c.

We recall that the use of the more relaxed quadratic control on the antiholomorphic terms, as opposed to the cubic control on the holomorphic terms, is motivated by the fact that the equations will eventually get projected on the holomorphic space, so the antiholomorphic components will have less of an impact. A key property of the space of errors is contained in Lemma 3.3 in [HIT16], which we will refer to as *the multiplicative bounds lemma*. For convenience we recall it here:

Lemma 4.3. *Let Φ be a function which satisfies*

$$\|\Phi\|_{L^\infty} \lesssim A, \quad \||D|^{1/2}\Phi\|_{\mathrm{BMO}} \lesssim B. \tag{4-9}$$

Then, we have the multiplicative bounds

$$\Phi\cdot \mathrm{err}(L^2)=\mathrm{err}(L^2),\qquad \Phi\cdot \mathrm{err}(\dot H^{1/2})=\mathrm{err}(\dot H^{1/2}), \tag{4-10}$$

$$\Phi\cdot P\,\mathrm{err}(L^2)=A\,\mathrm{err}(L^2),\quad \Phi\cdot P\,\mathrm{err}(\dot H^{1/2})=A\,\mathrm{err}(\dot H^{1/2}). \tag{4-11}$$

We now expand some of the terms in the above system. For this we will use the following bounds for M_1 (see Lemma A.2):

$$\|M_1\|_{L^\infty}\lesssim A^2,\quad \|M_1\|_{\dot H^1}\lesssim A\boldsymbol{N}_1. \tag{4-12}$$

First we note that

$$M_1\boldsymbol{W}_\alpha=\mathrm{err}(L^2),\quad M_1\boldsymbol{R}=\mathrm{err}(\dot H^{1/2}). \tag{4-13}$$

The first is straightforward in view of pointwise bound for M_1. For the second, by the multiplicative bounds lemma, we can replace $M_1\boldsymbol{R}$ by M_1R_α. After a Littlewood–Paley decomposition, the $\dot H^{1/2}$ estimate for M_1R_α is a consequence of the pointwise bound in (4-12) for low-high and balanced interactions, and of the $\dot H^{1/2}$-bound in (4-12) combined with Lemma 8.1 from Appendix B in [HIT16], provided that we also move half of a derivative from R_α onto M_1 for the high-low interactions, arriving at

$$\|M_1R_\alpha\|_{\dot H^{1/2}}\lesssim \|M_1\|_{L^\infty}\|R_\alpha\|_{\dot H^{1/2}}+\||D|^{1/2}R\|_{L^\infty}\|M_1\|_{\dot H^1}\lesssim A^2\boldsymbol{N}_1.$$

Next we consider $(1+\boldsymbol{W})M_{1,\alpha}$, for which we claim that

$$\begin{aligned} M_{1,\alpha}&=\boldsymbol{W}_\alpha\overline{\boldsymbol{W}}+2\boldsymbol{W}\overline{\boldsymbol{W}}_\alpha+\boldsymbol{P}[W\boldsymbol{W}_{\alpha\alpha}]+\mathrm{err}(L^2),\\ \boldsymbol{P}[W\overline{\boldsymbol{W}}_{\alpha\alpha}+\overline{\boldsymbol{W}}_\alpha\boldsymbol{W}]&=A^{-1}\,\mathrm{err}(L^2).\end{aligned} \tag{4-14}$$

By Lemma 4.3, this shows that

$$(1+\boldsymbol{W})M_{1,\alpha}=\boldsymbol{W}_\alpha\overline{\boldsymbol{W}}+2\boldsymbol{W}\overline{\boldsymbol{W}}_\alpha+\boldsymbol{P}[W\boldsymbol{W}_{\alpha\alpha}]+\mathrm{err}(L^2).$$

To prove (4-14) we discard cubic terms to rewrite

$$\begin{aligned} M_{1,\alpha}&=\boldsymbol{W}_\alpha\bar Y+2\boldsymbol{W}\bar Y_\alpha+\boldsymbol{P}[W\bar Y_{\alpha\alpha}]-\overline{\boldsymbol{P}}[\boldsymbol{W}_\alpha\bar Y+2\boldsymbol{W}\bar Y_\alpha+\overline{W}_\alpha Y+2\overline{\boldsymbol{W}}Y_\alpha+\overline{W}Y_{\alpha\alpha}]\\ &=\boldsymbol{W}_\alpha\overline{\boldsymbol{W}}+2\boldsymbol{W}\overline{\boldsymbol{W}}_\alpha+\boldsymbol{P}[W\overline{\boldsymbol{W}}_{\alpha\alpha}]-\boldsymbol{P}[f]-\overline{\boldsymbol{P}}[g]+\mathrm{err}(L^2),\end{aligned}$$

where

$$f=W(\overline{\boldsymbol{W}}\bar Y)_{\alpha\alpha},\quad g=\boldsymbol{W}_\alpha\bar Y+2\boldsymbol{W}\bar Y_\alpha+\overline{W}_\alpha Y+2\overline{\boldsymbol{W}}Y_\alpha+\overline{W}Y_{\alpha\alpha}.$$

Finally, for f and g we have L^2-bounds

$$\|\boldsymbol{P}f\|_{L^2}\lesssim A^2\boldsymbol{N}_1,\quad \|\overline{\boldsymbol{P}}g\|_{L^2}\lesssim A\boldsymbol{N}_1,$$

which follow from a commutator-type bound

$$\|P[W\bar\Phi_{\alpha\alpha}]\|_{L^2}\lesssim\|W_\alpha\|_{\mathrm{BMO}}\|\Phi_\alpha\|_{L^2} \tag{4-15}$$

derived from Lemma 8.1 in [HIT16].

The last term in $K_{1,1}$ is antiholomorphic, and also easily placed in $\mathrm{err}(\dot H^{1/2})$ by Coifman–Meyer-type bounds.

Taking into account all of the above expansions, it follows that our system can be rewritten in the form

$$\begin{cases} \boldsymbol{W}_{\alpha t}+\underline{b}\boldsymbol{W}_{\alpha\alpha}+\dfrac{\boldsymbol{R}_\alpha}{1+\overline{\boldsymbol{W}}}+\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}\boldsymbol{W}_\alpha=\operatorname{Princ}(G_1)-i\dfrac{c}{2}\operatorname{Princ}(G_{1,1})+\operatorname{err}(L^2),\\ \boldsymbol{R}_t+\underline{b}\boldsymbol{R}_\alpha+ic\boldsymbol{R}-i\dfrac{(g+\underline{a})\boldsymbol{W}_\alpha}{1+\boldsymbol{W}}=\operatorname{Princ}(K_1)-i\dfrac{c}{2}\operatorname{Princ}(K_{1,1})+\operatorname{err}(\dot{H}^{1/2}),\end{cases}$$

where $\operatorname{Princ}(G)$ refers to the terms in G that cannot be treated as error; they are quadratic and higher-order terms. We list their expressions below:

$$\begin{aligned}\operatorname{Princ}(G_1)&:=2\boldsymbol{R}\bar{Y}_\alpha-\frac{2\bar{R}_\alpha\boldsymbol{W}_\alpha}{1+\boldsymbol{W}}+P[R\overline{\boldsymbol{W}}_{\alpha\alpha}-\bar{R}_{\alpha\alpha}\boldsymbol{W}],\\ \operatorname{Princ}(G_{1,1})&:=-3\boldsymbol{W}\boldsymbol{W}_\alpha+3\overline{\boldsymbol{W}}\boldsymbol{W}_\alpha+3\boldsymbol{W}\overline{\boldsymbol{W}}_\alpha+P[W\overline{\boldsymbol{W}}_{\alpha\alpha}],\end{aligned}\tag{4-16}$$

and

$$\begin{aligned}\operatorname{Princ}(K_1)&:=-2\left(\frac{\bar{R}_\alpha}{1+\boldsymbol{W}}+\frac{R_\alpha}{1+\overline{\boldsymbol{W}}}\right)\boldsymbol{R}-P[\bar{R}_{\alpha\alpha}R],\\ \operatorname{Princ}(K_{1,1})&:=-2\boldsymbol{R}(\boldsymbol{W}-\overline{\boldsymbol{W}})-\boldsymbol{P}[2\boldsymbol{W}\bar{R}_\alpha+W\bar{R}_{\alpha\alpha}-\overline{\boldsymbol{W}}_\alpha R].\end{aligned}\tag{4-17}$$

One might wish to compare this system with the linearized system which was studied before. However, of the terms on the right, i.e., those in $\operatorname{Princ}(G_1)$ and $\operatorname{Princ}(K_1)$, cannot be all bounded in $L^2\times\dot{H}^{1/2}$, even after applying the projection operator $\boldsymbol{P}$. Precisely, the terms on the right which cannot be bounded directly in $L^2\times\dot{H}^{1/2}$ are

$$-\frac{2\bar{R}_\alpha\boldsymbol{W}_\alpha}{1+\boldsymbol{W}}\quad\text{and}\quad-2\left(\frac{\bar{R}_\alpha}{1+\boldsymbol{W}}+\frac{R_\alpha}{1+\overline{\boldsymbol{W}}}\right)\boldsymbol{R}.$$

There are no such problematic terms in $\operatorname{Princ}(G_{1,1})$ or in $\operatorname{Princ}(K_{1,1})$.

As in [HIT16], we eliminate these terms by conjugation with respect to a real exponential weight $e^{2\phi}$, where $\phi=-2\Re\log(1+\boldsymbol{W})$. Then

$$\phi_\alpha=-2\Re\frac{\boldsymbol{W}_\alpha}{1+\boldsymbol{W}},\quad(\partial_t+\underline{b}\partial_\alpha)\phi=2\Re\frac{R_\alpha}{1+\overline{\boldsymbol{W}}}-2\underline{M}-ic(\boldsymbol{W}-\overline{\boldsymbol{W}})\Re\frac{\boldsymbol{W}}{1+\boldsymbol{W}}.$$

We denote the weighted variables by

$$\mathfrak{w}=e^{2\phi}\boldsymbol{W}_\alpha,\quad\mathfrak{r}=e^{2\phi}\boldsymbol{R}.$$

Using (4-13) and Lemma 4.3 it follows that $\underline{M}w=\operatorname{err}(L^2)$ and $\underline{M}r=\operatorname{err}(\dot{H}^{1/2})$. Similarly we have

$$c\boldsymbol{W}_\alpha(\boldsymbol{W}-\overline{\boldsymbol{W}})\Re Y=\operatorname{err}(L^2),\quad c\boldsymbol{R}(\boldsymbol{W}-\overline{\boldsymbol{W}})\Re Y=\operatorname{err}(\dot{H}^{1/2}).$$

Then the only significant effect of the conjugation is to eliminate the above problem terms, and we are left with

$$\begin{cases}\mathfrak{w}_t+\underline{b}\mathfrak{w}_\alpha+\dfrac{\mathfrak{r}_\alpha}{1+\overline{\boldsymbol{W}}}+\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}\mathfrak{w}=\boldsymbol{P}[R\overline{\boldsymbol{W}}_{\alpha\alpha}-\bar{R}_{\alpha\alpha}W]-i\dfrac{c}{2}\operatorname{Princ}(G_{1,1})+\operatorname{err}(L^2),\\ \mathfrak{r}_t+\underline{b}\mathfrak{r}_\alpha+ic\mathfrak{r}-i\dfrac{(g+\underline{a})\mathfrak{w}}{1+\boldsymbol{W}}=-\boldsymbol{P}[\bar{R}_{\alpha\alpha}R]-i\dfrac{c}{2}\operatorname{Princ}(K_{1,1})+\operatorname{err}(\dot{H}^{1/2}).\end{cases}$$

Finally, as in [HIT16], we can also harmlessly substitute $\mathfrak{w}$ and $\mathfrak{r}$ into the leading error terms on the right to rewrite the above equation as

$$\begin{cases} \mathfrak{w}_t + \underline{b}\mathfrak{w}_\alpha + \dfrac{\mathfrak{r}_\alpha}{1+\overline{\boldsymbol{W}}} + \dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}\mathfrak{w} = \boldsymbol{P}[R\bar{\mathfrak{w}}_\alpha - \bar{\mathfrak{r}}_\alpha \boldsymbol{W}] - i\dfrac{c}{2}\operatorname{Princ}(G_{1,1}) + \operatorname{err}(L^2), \\ \mathfrak{r}_t + \underline{b}\mathfrak{r}_\alpha + ic\mathfrak{r} - i\dfrac{(g+\underline{a})\mathfrak{w}}{1+\boldsymbol{W}} = -\boldsymbol{P}[\bar{\mathfrak{r}}_\alpha R] - i\dfrac{c}{2}\operatorname{Princ}(K_{1,1}) + \operatorname{err}(\dot{H}^{1/2}). \end{cases}$$

One downside to the conjugation is that the new variables $(\mathfrak{w}, \mathfrak{r})$ are no longer holomorphic. To remedy this we project onto the holomorphic space to write a system for the variables $(w, r) = (\boldsymbol{P}\mathfrak{w}, \boldsymbol{P}\mathfrak{r})$. At this point one may legitimately be concerned that restricting to the holomorphic part might remove a good portion of our variables. However, this is not the case, as one can verify the claim by reviewing the discussion in Lemma 3.4 in [HIT16]. Following again [HIT16], after estimating some Coifman–Meyer-type commutators, the system for (w, r) is

$$\begin{cases} w_t + \mathfrak{M}_{\underline{b}} w_\alpha + \boldsymbol{P}\left[\dfrac{r_\alpha}{1+\overline{\boldsymbol{W}}}\right] + \boldsymbol{P}\left[\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}w\right] = G, \\ r_t + \mathfrak{M}_{\underline{b}} r_\alpha + icr - i\boldsymbol{P}\left[\dfrac{(g+\underline{a})w}{1+\boldsymbol{W}}\right] = K, \end{cases} \tag{4-18}$$

where

$$\begin{cases} G := \boldsymbol{P}[R\bar{w}_\alpha - \bar{r}_\alpha \boldsymbol{W}] - i\dfrac{c}{2}\boldsymbol{P}[\operatorname{Princ}(G_{1,1})] + \operatorname{err}(L^2), \\ K := -\boldsymbol{P}[\bar{r}_\alpha R] - i\dfrac{c}{2}\boldsymbol{P}[\operatorname{Princ}(K_{1,1})] + \operatorname{err}(\dot{H}^{1/2}). \end{cases}$$

This is our main system for the (slightly renormalized) differentiated variables $(\boldsymbol{W}_\alpha, R_\alpha)$. In order to use it we need to properly relate (w, r) to $(\boldsymbol{W}_\alpha, R_\alpha)$, and to estimate the terms in G and K. This is done in the following lemma.

Lemma 4.4. (a) *The energy of* (w, r) *above is equivalent to that of* $(\boldsymbol{W}_\alpha, R_\alpha)$,

$$\|(w, r)\|_{L^2\times\dot{H}^{1/2}} \approx_{\underline{A}} \|(\boldsymbol{W}_\alpha, R_\alpha)\|_{L^2\times\dot{H}^{1/2}} = \boldsymbol{N}_1, \tag{4-19}$$

and the difference is estimated by

$$\|(w, r) - (\boldsymbol{W}_\alpha, R_\alpha)\|_{L^2\times\dot{H}^{1/2}} \lesssim_A \underline{A}\boldsymbol{N}_1. \tag{4-20}$$

(b) *The error terms on the right in* (4-18) *are bounded*,

$$\|(\boldsymbol{P}[R\bar{w}_\alpha - \boldsymbol{W}\bar{r}_\alpha], \boldsymbol{P}[R\bar{r}_\alpha])\|_{L^2\times\dot{H}^{1/2}} \lesssim_A B\boldsymbol{N}_1, \quad \|P[R\bar{r}_\alpha]\|_{L^2} \lesssim A\boldsymbol{N}_1, \tag{4-21}$$

and

$$\|(\boldsymbol{P}\operatorname{Princ}(G_{1,1}), \boldsymbol{P}\operatorname{Princ}(K_{1,1}))\|_{L^2\times\dot{H}^{1/2}} \lesssim_A A\boldsymbol{N}_1, \quad \|\boldsymbol{P}\operatorname{Princ}(K_{1,1})\|_{L^2} \lesssim_A A_{-1/2}\boldsymbol{N}_1. \tag{4-22}$$

Proof. Part (a) and the first estimate in part (b) are from [HIT16], while part (c) follows either directly, for some terms, or by Coifman–Meyer estimates for the rest. □

Given the above lemma, the conclusion of Proposition 4.1 for $n = 1$ follows from the energy estimates for the linearized equation, namely (3-8) in Proposition 3.1; further, if $n = 1$ then we can take

$$E^{n,(2)}(\boldsymbol{W}, R) = E^{(2)}_{\text{lin}}(w, r).$$

4C. ***The differentiated equations for n ≥ 2.*** The first step is to derive a set of equations for $(\boldsymbol{W}^{(n)}, R^{(n)})$. We start again with the differentiated equations (1-19) and differentiate n times.

Compared with the case $n = 1$, we obtain many more terms. To separate them into leading order and lower order, we call lower-order terms any terms which do not involve $\boldsymbol{W}^{(n)}$, $R^{(n)}$ or derivatives thereof. In the computation below we take care to separate all the leading-order terms. Toward that end we define again the notion of *error term*. Unlike in the case $n = 1$, here we also include lower-order quadratic terms into the error. As before, we describe the error bounds in terms of the parameters $\underline{A}$, $\underline{B}$ and

$$\boldsymbol{N}_n = \|(\boldsymbol{W}^{(n)}, R^{(n)})\|_{L^2 \times \dot{H}^{1/2}}. \tag{4-23}$$

The acceptable errors in the $\boldsymbol{W}^{(n)}$-equation are denoted by $\text{err}(L^2)$ and are of two types, $\text{err}(L^2)^{[2]}$ and $\text{err}(L^2)$. The lower-order quadratic holomorphic terms are placed in $\text{err}(L^2)^{[2]}$, which is defined to be a linear combination of expressions of the form

$$\boldsymbol{P}[\boldsymbol{W}^{(j)} R^{(n+1-j)}], \quad \boldsymbol{P}[\overline{\boldsymbol{W}}^{(j)} R^{(n+1-j)}], \quad \boldsymbol{P}[\boldsymbol{W}^{(j)} \bar{R}^{(n+1-j)}], \quad 2 \le j \le n-1,$$

as well as terms involving the vorticity c, namely

$$\begin{aligned} &c\boldsymbol{P}[\boldsymbol{W}^{(j)} \boldsymbol{W}^{(n-j)}], && c\boldsymbol{P}[\overline{\boldsymbol{W}}^{(j)} \boldsymbol{W}^{(n-j)}], && 1 \le j \le n-1, \\ &c\boldsymbol{P}[R^{(j)} \bar{R}^{(n+1-j)}], && c\boldsymbol{P}[R^{(j)} R^{(n+1-j)}], && 2 \le j \le n-1. \end{aligned}$$

By interpolation and Hölder's inequality, terms G in $\text{err}(L^2)^{[2]}$ satisfy the bound

$$\|G\|_{L^2} \lesssim \underline{B} \boldsymbol{N}_n.$$

By $\text{err}(L^2)^{[3]}$ we denote terms G which satisfy the same estimates as $\text{err}(L^2)$ in the case $n = 1$, but with $\boldsymbol{N}_1$ replaced by $\boldsymbol{N}_n$.

The acceptable errors in the $R^{(n)}$-equation are denoted by $\text{err}(\dot{H}^{1/2})$ and are also of two types, $\text{err}(\dot{H}^{1/2})^{[2]}$ and $\text{err}(\dot{H}^{1/2})^{[3]}$. The first, $\text{err}(\dot{H}^{1/2})^{[2]}$, consists of holomorphic quadratic lower-order terms of the form

$$\begin{aligned} &\boldsymbol{P}[R^{(j)} R^{(n+1-j)}], && \boldsymbol{P}[\bar{R}^{(j)} R^{(n+1-j)}], && 2 \le j \le n-1, \\ &\boldsymbol{P}[\boldsymbol{W}^{(j)} \boldsymbol{W}^{(n+1-j)}], && \boldsymbol{P}[\overline{\boldsymbol{W}}^{(j)} \boldsymbol{W}^{(n-j)}], && 1 \le j \le n-1, \end{aligned}$$

as well as the c-terms

$$c\boldsymbol{P}[\boldsymbol{W}^{(j)} R^{(n-j)}], \quad c\boldsymbol{P}[\overline{\boldsymbol{W}}^{(j)} R^{(n-j)}], \quad c\boldsymbol{P}[\boldsymbol{W}^{(j)} \bar{R}^{(n-j)}], \quad 1 \le j \le n-1.$$

By interpolation and Hölder's inequality, terms K in $\text{err}(\dot{H}^{1/2})^{[2]}$ satisfy the bound

$$\|K\|_{\dot{H}^{1/2}} \lesssim \underline{B} \boldsymbol{N}_n, \quad \|K\|_{L^2} \lesssim A \boldsymbol{N}_n.$$

By $\operatorname{err}(\dot H^{1/2})^{[3]}$ we denote terms K which satisfy the same estimates as $\operatorname{err}(\dot H^{1/2})$ in the case $n=1$, but with $\boldsymbol{N}_1$ replaced by $\boldsymbol{N}_n$.

We now proceed to differentiate n times the equation (1-19). Our task is simplified due to [HIT16], where this analysis has already been carried out for the terms without c. Hence, we only concentrate here on the c-terms.

In addition, we remark that, as all terms in the $\boldsymbol{W}^{(n)}$-equation have the same homogeneity, whenever all the Sobolev exponents are within the lower-order range, we are guaranteed to get the correct L^2 estimate after interpolation and Hölder's inequality. The same applies to all cubic terms with at most n derivatives on any single factor. The same observation applies to all the lower-order terms in the $R^{(n)}$-equation, as well as to all cubic terms containing $R^{(n)}$. However, the terms containing $\boldsymbol{W}^{(n)}$ are unbounded and belong to the principal part of the $R^{(n)}$-equation. Because of these considerations, the computation below is largely of an algebraic nature.

We begin with the $\boldsymbol{W}$-equation. For the b_1-term we have

$$\begin{aligned}\partial^n(b_1\boldsymbol{W}_\alpha) &= b_1\boldsymbol{W}_\alpha^{(n)} + nb_{1,\alpha}\boldsymbol{W}^{(n)} + b_1^{(n)}\boldsymbol{W}_\alpha + \operatorname{err}(L^2)\\ &= b_1\boldsymbol{W}_\alpha^{(n)} + n(\boldsymbol{W}-\overline{\boldsymbol{W}})\boldsymbol{W}^{(n)} + \operatorname{err}(L^2).\end{aligned}$$

Here at the last step we use the simple observation that the linear part of the expression $b_1^{(n)}\boldsymbol{W}_\alpha$ contributes only lower-order terms, and the rest contributes cubic terms, which can be estimated either directly or using Coifman–Meyer bounds.

Next we consider

$$\begin{aligned}\partial^n((1+\boldsymbol{W})M_1) &= (1+\boldsymbol{W})\partial^{n+1}(\boldsymbol{P}[W\bar Y]-\bar{\boldsymbol{P}}[\overline{W}Y]) + \operatorname{err}(L^2)\\ &= \boldsymbol{W}^{(n)}\boldsymbol{W} + (n+1)\boldsymbol{W}\overline{\boldsymbol{W}}^{(n)} + \boldsymbol{P}[W\overline{\boldsymbol{W}}^{(n+1)}] + \operatorname{err}(L^2).\end{aligned}$$

Here all cubic terms involving $\overline{\boldsymbol{W}}^{(n+1)}$ can be directly bounded using Coifman–Meyer estimates. Finally, for the last c-term in the $\boldsymbol{W}^{(n)}$-equation we have

$$\partial^n(\boldsymbol{W}(\boldsymbol{W}-\overline{\boldsymbol{W}})) = 2\boldsymbol{W}^{(n)}\boldsymbol{W} - \boldsymbol{W}^{(n)}\overline{\boldsymbol{W}} - \boldsymbol{W}\overline{\boldsymbol{W}}^{(n)} + \operatorname{err}(L^2).$$

Next we turn our attention to the c-terms in the $R^{(n)}$-equation, We begin with

$$\begin{aligned}\partial^{n-1}(b_1R_\alpha) &= b_1R_\alpha^{(n)} + nb_{1,\alpha}R^{(n)} + b^{(n)}R_\alpha + \operatorname{err}(\dot H^{1/2})\\ &= b_1R_\alpha^{(n)} + n(\boldsymbol{W}-\overline{\boldsymbol{W}})R^{(n)} + \operatorname{err}(\dot H^{1/2}),\end{aligned}$$

where again the quadratic terms in $b^{(n)}R_\alpha$ are lower-order, and the cubic terms with $\overline{\boldsymbol{W}}^{(n)}$ are bounded as error terms via Coifman–Meyer estimates.

For the next term in the R-equation we write

$$\partial^n\frac{\boldsymbol{W}(R+\bar R)}{1+\boldsymbol{W}} = \boldsymbol{W}(R^{(n)}+\bar R^{(n)}) + \frac{\boldsymbol{W}^{(n)}(R+\bar R)}{1+\boldsymbol{W}} + \operatorname{err}(\dot H^{1/2}),$$

where the cubic $R^{(n)}$-terms are estimated directly as error terms. Finally, we need to consider

$$\partial^n \frac{N}{1+\boldsymbol{W}} = \partial^n\left(\frac{\boldsymbol{P}[W\bar{R}_\alpha - \overline{\boldsymbol{W}}R] + \bar{\boldsymbol{P}}[\overline{W}R_\alpha - \boldsymbol{W}\bar{R}]}{1+\boldsymbol{W}}\right)$$
$$= -\frac{N\boldsymbol{W}^{(n)}}{(1+\boldsymbol{W})^2} + \boldsymbol{P}[W\bar{R}_\alpha^{(n)} + n\boldsymbol{W}\bar{R}^{(n)} - \overline{\boldsymbol{W}}^{(n)}R - \overline{\boldsymbol{W}}R^{(n)}] + \text{err}(\dot{H}^{1/2}).$$

Combining the above computations for the c-terms in (1-19) with the prior computations for the non-c-terms in [HIT16] we obtain the differentiated system

$$\begin{cases} \boldsymbol{W}_t^{(n)} + \underline{b}\boldsymbol{W}_\alpha^{(n)} + \dfrac{((1+\boldsymbol{W})R^{(n)})_\alpha}{1+\overline{\boldsymbol{W}}} + \dfrac{R_\alpha}{1+\boldsymbol{W}}\boldsymbol{W}^{(n)} = \underline{G}_n + \text{err}(L^2), \\ R_t^{(n)} + \underline{b}R_\alpha^{(n)} + icR^{(n)} - i\left(\dfrac{(g+\underline{a})\boldsymbol{W}^{(n)}}{(1+\boldsymbol{W})^2}\right) = \underline{K}_n + \text{err}(\dot{H}^{1/2}), \end{cases}$$

where

$$\underline{G}_n = G_n - i\frac{c}{2}G_{n,1}, \quad \underline{K}_n = K_n - i\frac{c}{2}K_{n,1}.$$

From [HIT16] we have

$$\begin{cases} G_n = -n\dfrac{\bar{R}_\alpha}{1+\boldsymbol{W}}\boldsymbol{W}^{(n-1)} - (n-1)\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}}\boldsymbol{W}^{(n-1)} + P[R\overline{\boldsymbol{W}}_\alpha^{(n-1)} - \boldsymbol{W}\bar{R}_\alpha^{(n-1)}] \\ \qquad + R^{(n-1)}(n\overline{\boldsymbol{W}}_\alpha - (n-1)\boldsymbol{W}_\alpha) + n(R_\alpha\overline{\boldsymbol{W}}^{(n-1)} - \boldsymbol{W}_\alpha\bar{R}^{(n-1)}), \\ K_n = -n\left(\dfrac{R_\alpha}{1+\overline{\boldsymbol{W}}} + \dfrac{\bar{R}_\alpha}{1+\boldsymbol{W}}\right)R^{(n-1)} - (P[R\bar{R}_\alpha^{(n-1)}] - nR_\alpha\bar{R}^{(n-1)}), \end{cases}$$

and from the above computations,

$$\begin{cases} G_{n,1} = (n+2)[\boldsymbol{W}\overline{\boldsymbol{W}}^{(n)} - (\boldsymbol{W}-\overline{\boldsymbol{W}})\boldsymbol{W}^{(n)}] + \boldsymbol{P}[W\overline{\boldsymbol{W}}^{(n+1)}], \\ K_{n,1} = -(n+1)[(\boldsymbol{W}-\overline{\boldsymbol{W}})R^{(n)} + \boldsymbol{W}\bar{R}^{(n)}] + \boldsymbol{P}[R\overline{\boldsymbol{W}}^{(n)} - W\bar{R}_\alpha^{(n)}]. \end{cases}$$

To bring this equation to a form closer to the linearized equation we follow the lead of [HIT16] and perform several algebraic holomorphic substitutions for the $R^{(n)}$-variable, beginning with

$$\boldsymbol{R} = (1+\boldsymbol{W})R^{(n)},$$

and followed by

$$\widetilde{\boldsymbol{R}} = \boldsymbol{R} - R_\alpha\boldsymbol{W}^{(n-1)} + (2n+1)\boldsymbol{W}_\alpha R^{(n-1)}.$$

Finally, we conclude with the exponential conjugation by $e^{n\phi}$ where we use the same ϕ as before, namely $\phi = -2\Re\log(1+\boldsymbol{W})$, in order to eliminate the unbounded terms on the right. At the end we obtain an equation for $(\mathfrak{w} := e^{n\phi}\boldsymbol{W}^{(n-1)}, \mathfrak{r} := e^{n\phi}\widetilde{\boldsymbol{R}})$ where the leading part is exactly as in the linearized equation. We do not repeat this computation, as it primarily affects the part of the equation without c, which is fully described in [HIT16]. All the additional c-contributions are either cubic and bounded or quadratic and lower-order, so they are placed in the error. The outcome is the equation

$$\begin{cases} \mathfrak{w}_t + \underline{b}\mathfrak{w}_\alpha + \dfrac{\mathfrak{r}_\alpha}{1+\overline{\boldsymbol{W}}} + \dfrac{R_\alpha}{1+\boldsymbol{W}}\mathfrak{w} = \boldsymbol{P}[R\overline{\boldsymbol{W}}_\alpha^{(n)} - \boldsymbol{W}\bar{R}_\alpha^{(n)}] + (n+1)(R_\alpha\bar{\mathfrak{w}} - \boldsymbol{W}_\alpha\bar{\mathfrak{r}}) - i\,\dfrac{c}{2}\,G_{n,1} + \mathrm{err}(L^2), \\ \mathfrak{r}_t + \underline{b}\mathfrak{r}_\alpha + ic\mathfrak{r} - i\left(\dfrac{(g+\underline{a})\mathfrak{w}}{1+\boldsymbol{W}}\right) = -\boldsymbol{P}[R\bar{R}_\alpha^{(n)}] - (n+1)R_\alpha\bar{\mathfrak{r}} - i\,\dfrac{c}{2}\,K_{n,1} + \mathrm{err}(\dot{H}^{1/2}). \end{cases}$$

As $(\mathfrak{w}, \mathfrak{r})$ are no longer holomorphic, we project and work with the projected variables $(w, r) = (\boldsymbol{P}\mathfrak{w}, \boldsymbol{P}\mathfrak{r})$. After some additional commutator estimates, which are identical to those in the $n = 1$ case, we finally obtain

$$\begin{cases} w_t + \mathfrak{M}_{\underline{b}} w_\alpha + \boldsymbol{P}\left[\dfrac{r_\alpha}{1+\overline{\boldsymbol{W}}}\right] + \boldsymbol{P}\left[\dfrac{R_\alpha w}{1+\boldsymbol{W}}\right] = \boldsymbol{P}[R\bar{w}_\alpha - \boldsymbol{W}\bar{r}_\alpha] + (n+1)\boldsymbol{P}[R_\alpha\bar{w} - \boldsymbol{W}_\alpha\bar{r}] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -i\,\dfrac{c}{2}\,\boldsymbol{P}G_{n,1} + \mathrm{err}(L^2), \\ r_t + \mathfrak{M}_{\underline{b}} r_\alpha + icr - i\boldsymbol{P}\left[\dfrac{(g+\underline{a})w}{1+\boldsymbol{W}}\right] = -\boldsymbol{P}[R\bar{r}_\alpha] - (n+1)\boldsymbol{P}[R_\alpha\bar{r}] - i\,\dfrac{c}{2}\,\boldsymbol{P}K_{n,1} + \mathrm{err}(\dot{H}^{1/2}). \end{cases} \tag{4-24}$$

This is our main system for the (slightly modified) differentiated variables $(\boldsymbol{W}^{(n)}, R^{(n)})$. In order to use it we again need to relate (w, r) to $(\boldsymbol{W}^{(n)}, R^{(n)})$, and also to estimate the terms in $G_{n,1}$ and $K_{n,1}$. This is done in the following.

Lemma 4.5. (a) *The energy of* (w, r) *above is equivalent to that of* $(\boldsymbol{W}^{(n)}, R^{(n)})$,

$$\|(w, r)\|_{L^2\times\dot{H}^{1/2}} \approx_{\underline{A}} \|(\boldsymbol{W}^{(n)}, R^{(n)})\|_{L^2\times\dot{H}^{1/2}} = \boldsymbol{N}_n, \tag{4-25}$$

and the difference is estimated by

$$\|(w, r) - (\boldsymbol{W}^{(n)}, R^{(n)})\|_{L^2\times\dot{H}^{1/2}} \lesssim_A \underline{A}\boldsymbol{N}_n. \tag{4-26}$$

(b) *The error terms on the right in* (4-18) *are bounded*,

$$\|(\boldsymbol{P}[R\bar{w}_\alpha - \boldsymbol{W}\bar{r}_\alpha], \boldsymbol{P}[R\bar{r}_\alpha])\|_{L^2\times\dot{H}^{1/2}} \lesssim_A B\boldsymbol{N}_n, \quad \|\boldsymbol{P}[R\bar{r}_\alpha]\|_{L^2} \lesssim A\boldsymbol{N}_n, \tag{4-27}$$

$$\|(\boldsymbol{P}[R_\alpha\bar{w} - \boldsymbol{W}_\alpha\bar{r}], \boldsymbol{P}[R_\alpha\bar{r}])\|_{L^2\times\dot{H}^{1/2}} \lesssim_A B\boldsymbol{N}_n, \quad \|\boldsymbol{P}[R_\alpha\bar{r}]\|_{L^2} \lesssim_A A\boldsymbol{N}_n, \tag{4-28}$$

and

$$\|(\boldsymbol{P}G_{1,n}, \boldsymbol{P}K_{1,n})\|_{L^2\times\dot{H}^{1/2}} \lesssim_A A, \quad \|\boldsymbol{P}K_{1,n}\|_{L^2} \lesssim_A A_{-1/2}\boldsymbol{N}_n. \tag{4-29}$$

Proof. Part (a) and the first two bounds in part (b) are from [HIT16], while the last bound in part (b) follows either directly, for some terms, or by Coifman–Meyer estimates, for the rest. □

Given the above Lemma 4.4, the $n \geq 2$ case of the result in Proposition 4.1 is a direct consequence of our quadratic estimates for the linearized equation in Proposition 3.1.

Comparing the linear (w, r)-equation obtained above for the case $n \geq 2$ to the corresponding equation previously obtained for the case $n = 1$, we note that here we have two extra terms, namely the ones estimated in (4-28). On one hand this is good, because the bound (4-28) is no longer true if $n = 1$. But on the other hand, it means that we will no longer be able to use directly the cubic energy $E^{(3)}_{\mathrm{lin}}(w, r)$

introduced in the context of the linearized equation as the full high-frequency part of our cubic energy. To account for these extra terms, we will add a further correction to the energy $E^{(3)}_{\mathrm{lin}}(w, r)$, and define

$$E^{n,(3)}_{\mathrm{high}}(w,r) := \int (1+a)|w|^2 + \Im(\bar{r} r_\alpha) + 2n\Im(R_\alpha \bar{w}\bar{r}) + 2(\Im[\bar{R} w r_\alpha] - \Re[\overline{W}_\alpha w^2])\, d\alpha. \tag{4-30}$$

Then we will prove a counterpart to the small-data (nearly) cubic energy estimates in Proposition 4.2. Precisely, we claim that the evolution of this energy is governed by the following.

Lemma 4.6. *Let (w, r) be defined as above. Then*:

(a) *Assuming that $A \ll 1$, we have*

$$E^{n,(3)}_{\mathrm{high}}(w,r) \approx \mathcal{E}_0(w,r) \approx \boldsymbol{N}_n. \tag{4-31}$$

(b) *The solutions (w, r) of (4-24) satisfy*

$$\frac{d}{dt} E^{n,(3)}_{\mathrm{high}}(w,r) = 2\int -\Re\Big(\bar{w}\cdot\Big(i\frac{c}{2}G_{n,1} - \mathrm{err}(L^2)^{[2]}\Big)\Big) + \Im\Big(\bar{r}_\alpha\cdot\Big(i\frac{c}{2}K_{n,1} - \mathrm{err}(\dot{H}^{1/2})^{[2]}\Big)\Big)\, d\alpha + \int \frac{c^2}{2}\Im R|w|^2\, d\alpha + O_A(A\underline{B}\boldsymbol{N}_n). \tag{4-32}$$

Further, the same relation holds if $(\bar{w}, \bar{r})$ on the right are replaced by $(\overline{\boldsymbol{W}}^{(n)}, \bar{R}^{(n)})$.

We note that in the proof of this lemma we crucially use the fact that (w, r) are the ones associated to the differentiated equation, and not arbitrary functions in the energy space. This is also the reason why this lemma is presented here, rather than in Section 3.

Proof. Part (a) was already proved in [HIT16], so we proceed to part (b). Here, we begin with the cubic linearized energy, $E^{(3)}_{\mathrm{lin}}$. According to the bound (3-18) in Proposition 3.4, we have

$$\begin{aligned}\frac{d}{dt} E^{(3)}_{\mathrm{lin}}(w,r) = \int & 2\Re\Big(\Big(n\boldsymbol{P}[R_\alpha\bar{w} - \boldsymbol{W}_\alpha\bar{r}] - i\frac{c}{2}G_{n,1} + P\,\mathrm{err}(L^2)\Big)\cdot(\bar{w} - \overline{\boldsymbol{P}}[\bar{R}r_\alpha] - \overline{\boldsymbol{P}}[\overline{W}_\alpha w])\Big)\\ &- 2\Im\Big(\Big(-n\boldsymbol{P}[R_\alpha\bar{r}] - i\frac{c}{2}K_{n,1} + \boldsymbol{P}\,\mathrm{err}(\dot{H}^{1/2})\Big)\cdot(\bar{r}_\alpha + \overline{\boldsymbol{P}}[\bar{R}w]_\alpha)\Big)\, d\alpha\\ &+ O_A(\underline{A}\underline{B}\|(w,r)\|^2_{L^2\times\dot{H}^{1/2}}).\end{aligned}$$

By the Coifman–Meyer-type estimates in Lemma A.1 the following bound holds:

$$\|\bar{P}[\bar{R}r_\alpha]\|_{L^2} + \|\bar{P}[\overline{W}_\alpha w]\|_{L^2} + \|\bar{P}[\bar{R}w]\|_{\dot{H}^{1/2}} \lesssim A\|(w,r)\|_{L^2\times\dot{H}^{1/2}}. \tag{4-33}$$

Combining this with (4-21) and with the bounds for $G_{n,1}$, $K_{n,1}$ and the error terms we get

$$\begin{aligned}\frac{d}{dt} E^{(3)}_{\mathrm{lin}}(w,r) \le \int & 2\Re\Big(\Big(n\boldsymbol{P}[R_\alpha\bar{w} - \boldsymbol{W}_\alpha\bar{r}] - i\frac{c}{2}G_{n,1} + \boldsymbol{P}\,\mathrm{err}(L^2)^{[2]}\Big)\cdot\bar{w}\Big)\\ &- 2\Im\Big(\Big(-n\boldsymbol{P}[R_\alpha\bar{r}] - i\frac{c}{2}K_{n,1} + \boldsymbol{P}\,\mathrm{err}(\dot{H}^{1/2})^{[2]}\Big)\cdot\bar{r}_\alpha\Big)\, d\alpha\\ &+ O_A(\underline{A}\underline{B})\|(w,r)\|^2_{L^2\times\dot{H}^{1/2}},\end{aligned}$$

where the output from all error terms which are cubic-and-higher is all included in the last term on the right-hand side.

It remains to consider the contribution of the extra term in $E^{n,(3)}_{\text{high}}$ and show that

$$\frac{d}{dt}\int \Im(R_\alpha \bar{w}\bar{r})\,d\alpha = \int \Re((R_\alpha \bar{w} - \boldsymbol{W}_\alpha \bar{r})\bar{w}) + \Im(R_\alpha \bar{r}\bar{r}_\alpha)\,d\alpha + O_A(\underline{A}\underline{B})\|(w,r)\|^2_{L^2\times \dot{H}^{1/2}}. \tag{4-34}$$

Denote by G_n and K_n, respectively, the two right-hand sides in (4-24). By the definition of error terms and by (4-21)–(4-22) they satisfy the bounds

$$\|(G_n, K_n)\|_{L^2\times \dot{H}^{1/2}} \lesssim_A \underline{B}\boldsymbol{N}_n, \qquad \|K_n\|_{L^2} \lesssim_A \underline{A}\boldsymbol{N}_n.$$

Then their contribution in the above time derivative is estimated as

$$\left|\int \Im(R_\alpha \overline{\boldsymbol{P}}\,\overline{G}_n \bar{r} + R_\alpha \bar{w}\overline{\boldsymbol{P}}\,\overline{K}_n)\,d\alpha\right| = \left|\int \Im(R_\alpha \overline{\boldsymbol{P}}\,\overline{F}_n \bar{r} + P[R_\alpha \bar{w}]\overline{\boldsymbol{P}}\,\overline{K}_n)\,d\alpha\right| \lesssim_A \underline{A}\underline{B}\boldsymbol{N}_n$$

by using Hölder's inequality for the first term and the Coifman–Meyer commutator estimate in Lemma A.1 for the second.

We now consider the contributions due to the terms on the left of (4-24). These were already estimated in [HIT16] when $c = 0$, so we only need to estimate the c-terms, i.e., the contributions of b_1 and a_1.

The contributions of the b_1-terms are collected together in the real part of the expression

$$\begin{aligned} I &= \frac{c}{2}\int \partial_\alpha(b_1 R_\alpha)\bar{w}\bar{r} + R_\alpha \overline{\boldsymbol{P}}(b_1 \bar{w}_\alpha)\bar{r} + R_\alpha \bar{w}\overline{\boldsymbol{P}}(b_1 \bar{r}_\alpha)\,d\alpha \\ &= \frac{c}{2}\int R_\alpha\big([b_1, \boldsymbol{P}](\bar{w}_\alpha)\bar{r} + \bar{w}[b_1, \boldsymbol{P}](\bar{r}_\alpha)\big)\,d\alpha. \end{aligned}$$

Since $\|b_{1,\alpha}\|_{\mathrm{BMO}} \lesssim A$, we can bound this using Lemma A.1, and then use Hölder's inequality for all terms.

The contribution of a_1 is

$$\frac{c}{2}\Re\int R_\alpha \bar{w} P\left[\frac{a_1 \bar{w}}{1+\boldsymbol{W}}\right] d\alpha,$$

for which it suffices to use the pointwise bound for a_1 in Proposition A.4 and the classical Coifman–Meyer bound.

Finally, we consider the remaining contribution of the c-terms in the time derivative of R_α, for which we use the equations (1-19)

$$\frac{c}{2}\Re\int \bar{w}\bar{r}\partial_\alpha \frac{R\boldsymbol{W} + \bar{R}\boldsymbol{W} + N}{1+\boldsymbol{W}}\,d\alpha.$$

But this is a quartic expression for which we only need Sobolev embeddings and interpolation. □

4D. *Cubic small-data energy estimates: $n \geq 1$.* In this section we construct an n-th order energy functional $E^{n,(3)}$ with cubic estimates, for $n \geq 1$. While in essence the argument does not depend on n, there are nevertheless a few small differences between the cases $n = 1$ and $n \geq 2$. These differences will be pointed out at the appropriate places.

One ingredient in the construction of $E^{n,(3)}$ is the high-frequency (nearly) cubic energy $E^{n,(3)}_{\text{high}}$ in Lemma 4.6 ($n \geq 2$), and $E^{(3)}_{\text{lin}}$ in Proposition 3.4 ($n = 1$). However, this does not suffice, as the right-hand side of the energy relation (4-32) still contains lower-order cubic terms. To remedy this, here we use normal forms in order to add a lower-order correction to $E^{n,(3)}_{\text{high}}$, which removes the above-mentioned cubic terms. For this we follow the method introduced in [HIT16], though the computations here are significantly more involved.

We first provide an outline of the argument, and then return to each step in detail. The three main steps are as follows:

(1) *Construct the normal-form energy*. The normal-form variables $\widetilde{W}$, $\widetilde{Q}$ solve a system of the form

$$\begin{cases} \widetilde{W}_t + \widetilde{Q}_\alpha = \text{cubic and higher}, \\ \widetilde{Q}_t - i\widetilde{W} = \text{cubic and higher}. \end{cases} \tag{4-35}$$

Thus, the associated linear energies satisfy

$$\frac{d}{dt}\mathcal{E}_0(\widetilde{W}^{(n+1)}, \widetilde{Q}^{(n+1)}) = \text{quartic and higher}.$$

Here the quartic part of the left-hand side is highly unbounded, and it is not uniquely determined, as one can add arbitrary cubic terms to the normal form. Fortunately, it is also irrelevant for the above relation.

Thus, we eliminate it, and we define the normal-form energy as

$$E^n_{\text{NF}} = (\text{quadratic + cubic})[\mathcal{E}_0(\widetilde{W}^{(n+1)}, \widetilde{Q}^{(n+1)})].$$

Here, we carefully choose the quadratic and cubic terms in the expansion of $\mathcal{E}_0(\widetilde{W}^{(n)}, \widetilde{Q}^{(n)})$ in terms of the diagonal variables $(\boldsymbol{W}, R)$, rather than (W, Q).

The expression E^n_{NF} has only quadratic and cubic terms in $(\boldsymbol{W}, R)$, and still satisfies

$$\frac{d}{dt}E^n_{\text{NF}} = \text{unbounded quartic and higher}.$$

Its disadvantage, due to the fact that our problem is quasilinear, is that the terms on the right are highly unbounded.

(2) *Construct the quasilinear modified energy*. Here we construct *the quasilinear modified energy* $E^{n,(3)}$ using the normal-form energy E^n_{NF} and the high frequency modified energy $E^{n,(3)}_{\text{high}}(w, r)$. The first one has quartic estimates, which are unbounded. The second one has good (quartic) high-frequency estimates, but with cubic lower-order errors,

$$\frac{d}{dt}E^{n,(3)}_{\text{high}}(w, r) = \text{bounded quartic and higher} + \text{lower-order cubic}.$$

We seek to combine the two into a quasilinear modified energy $E^{n,(3)}$ which satisfies

$$\frac{d}{dt}E^{n,(3)} = \text{bounded quartic and higher}.$$

To achieve this, it is natural to separate E^n_{NF} into a leading part and a lower-order part,

$$E^n_{\text{NF}} = E^n_{\text{NF,high}} + E^n_{\text{NF,low}},$$

so that the leading part coincides with the high-frequency energy up to quartic terms,

$$E^{n,(3)}_{\text{high}} = E^n_{\text{NF,high}} + \text{quartic and higher},$$

and then define

$$E^{n,(3)} = E^{n,(3)}_{\text{high}} + E^{n}_{\text{NF,low}}.$$

This was the argument in [HIT16]; here it is slightly more complicated, as additional terms in $E^n_{\text{NF,low}}$ need to be treated as leading-order, and thus corrected in a quasilinear fashion.

(3) *Show that $E^{n,(3)}$ is a good quasilinear cubic energy*. That is, prove that the bound (4-2) holds. Here, by construction, we know that

$$\frac{d}{dt}E^{n,(3)} = \text{quartic and higher;}$$

therefore we can write

$$\frac{d}{dt}E^{n,(3)} = \text{quartic and higher}\Big(\frac{d}{dt}E^{n,(3)}_{\text{high}}(w,r)\Big) + \text{quartic and higher}\Big(\frac{d}{dt}E^{n}_{\text{NF,low}}\Big).$$

The favorable bound for the first part is already a direct consequence of Lemma 4.6, so it remains to use the equations in order to bound the second term. But this is technically simple, and the argument is more a matter of bookkeeping.

Now we proceed to implement the above strategy.

Step 1. The first step described above is implemented in the following proposition:

Proposition 4.7. *There exists a modified normal-form energy $E^n_{\text{NF}} = E^n_{\text{NF}}(W,R)$ with the following properties*:

(A) *E^n_{NF} has the form*

$$E^n_{\text{NF}} = E^n_{\text{NF,high}} + E^{n,c}_{\text{NF,high}} + E^n_{\text{NF,low}},$$

where the three components are defined as follows:

$$\begin{aligned}E^n_{\text{NF,high}} = &\int (1-4(n+1)\Re \boldsymbol{W})(g|\boldsymbol{W}^{(n)}|^2 + \Im[\bar{R}^{(n+1)}R^{(n)}])\,d\alpha,\\ &+2\int \Im[\bar{R}\boldsymbol{W}^{(n)}R^{(n+1)}] - g\Re[\overline{\boldsymbol{W}}(\boldsymbol{W}^{(n)})^2] + (n+1)\Im[R_\alpha \overline{\boldsymbol{W}}^{(n)}\bar{R}^{(n)}]\,d\alpha\\ &+\int c\Re R|\boldsymbol{W}^{(n)}|^2 + 2\Im(\boldsymbol{W}\bar{R}^{(n+1)}R^{(n)})\,d\alpha,\end{aligned} \tag{4-36}$$

where the last term in the second integral is dropped if $n=1$, and

$$E^{n,c}_{\text{NF,high}} = -\int \big[c(2n+3)\Re R + c^2\big(2n+\tfrac{5}{2}\big)\Im W\big](|\boldsymbol{W}^{(n)}|^2 - i\bar{R}^{(n+1)}R^{(n)})\,d\alpha. \tag{4-37}$$

Finally, $E^n_{\text{NF,low}}$ is a trilinear integral form of order $2n$, with the following restrictions:

(i) *The leading order for the terms without c is at most $2n-1$.*

(ii) *The leading order for the terms with c is at most $2n-\frac{1}{2}$.*

(iii) *The highest power of c is 4.*

(iv) *At most one factor of the form c^2W or cR is present.*

(B) *The energy functional* E^n_{NF} *is cubically accurate*,

$$\Lambda^{\leq 3}\frac{d}{dt}E^n_{\mathrm{NF}} = 0. \tag{4-38}$$

(C) *Its components satisfy the following estimates*:

$$\begin{aligned} E^n_{\mathrm{NF,high}} &= [1+O(A)]\mathcal{E}_0(\mathbf{W}^{(n)}, R^{(n)}),\\ E^{n,c}_{\mathrm{NF,high}} &= O(\underline{A})\mathcal{E}_0(\mathbf{W}^{(n)}, R^{(n)}),\\ E^n_{\mathrm{NF,low}} &= O(\underline{A})(\mathcal{E}_0(\mathbf{W}^{(n)}, R^{(n)}) + c^4\mathcal{E}_0(\mathbf{W}^{(n-1)}, R^{(n-1)})). \end{aligned} \tag{4-39}$$

Proof. Here we use the algebraic expression for the normal-form transformation, which is given below. We call lower-order terms any terms that can be included in $E^n_{\mathrm{NF,low}}$. Up to quartic terms we seek to have, at least formally, the relation

$$\begin{aligned} E^n_{\mathrm{NF}} &= \|(\widetilde{W}^{(n+1)}, \widetilde{Q}^{(n+1)})\|^2_{L^2\times\dot{H}^{1/2}} + \text{quartic}\\ &= \mathcal{E}_0(W^{(n+1)}, Q^{(n+1)}) - 2\Re\int \overline{W}^{(n+1)}\partial^{n+1}W^{[2]} - i\,\bar{Q}^{(n+1)}\partial^{n+1}Q^{[2]}\,d\alpha + \text{quartic}. \end{aligned} \tag{4-40}$$

In the first term we introduce the diagonal variable R by using $Q_\alpha = R(1+\mathbf{W})$, which allows us to write it as

$$I_{\mathrm{main}} = \mathcal{E}_0(\mathbf{W}^{(n)}, R^{(n)}) + 2\Im\int \bar{R}^{(n+1)}\partial^n(R\mathbf{W})\,d\alpha + \text{quartic}.$$

Differentiating we obtain its principal part, containing all terms with leading order above $2n-1$:

$$I_{\mathrm{main,high}} = \mathcal{E}_0(\mathbf{W}^{(n)}, R^{(n)}) + 2\Im\int \mathbf{W}\bar{R}^{(n+1)}R^{(n)} + R\bar{R}^{(n+1)}\mathbf{W}^{(n)} - (n+2)R_\alpha\bar{R}^{(n)}\mathbf{W}^{(n)}\,d\alpha.$$

Here the last term on the right no longer appears if $n=1$, which accounts for the similar modification in the proposition.

In the integral in (4-40), on the other hand, we have two concerns. First, we want to eliminate its unbounded part of low frequency, i.e., all terms with inverse derivatives of W, as well as all terms with undifferentiated Q; we remark that once this is done, the switch to diagonal variables is achieved simply by replacing Q_α with R. Secondly, we want to keep track of its highest-order terms, i.e., those terms which are at least energy strength (i.e., $\mathbf{W}^{(n)}\overline{\mathbf{W}}^{(n)}$, or $\bar{R}^{(n+1)}R^{(n)}$, and also $\bar{R}^{(n)}\mathbf{W}^{(n)}$ for terms without c). First we recall the expression for the normal form, see (2-8):

$$\begin{aligned} W^{[2]} &= -(W+\overline{W})W_\alpha - \frac{c}{2g}[(Q+\bar{Q})W_\alpha + (W+\overline{W})Q_\alpha]\\ &\quad + i\frac{c^2}{2g}\big[(\partial^{-1}W - \partial^{-1}\overline{W})W_\alpha + W^2 + \tfrac{1}{2}|W|^2\big] - \frac{c^2}{4g^2}(Q+\bar{Q})Q_\alpha\\ &\qquad + i\frac{c^3}{4g^2}[(Q+\bar{Q})W + (\partial^{-1}W - \partial^{-1}\overline{W})Q_\alpha] + \frac{c^4}{4g^2}(\partial^{-1}W - \partial^{-1}\overline{W})W, \end{aligned}$$

$$\begin{aligned} Q^{[2]} &= -(W+\overline{W})Q_\alpha - \frac{c}{2g}(Q+\bar{Q})Q_\alpha + i\frac{c}{4}(W^2 + 2|W|^2)\\ &\quad + i\frac{c^2}{2g}\big[(\partial^{-1}W - \partial^{-1}\overline{W})Q_\alpha + \tfrac{1}{2}(Q+\bar{Q})W\big] + \frac{c^3}{4g}(\partial^{-1}W - \partial^{-1}\overline{W})W. \end{aligned}$$

Next, we successively consider the terms in the integral in (4-40) organized by the power of c.

Terms with c^0. These are

$$I_0 = 2\Re \int -\overline{\boldsymbol{W}}^{(n)} \partial^{n+1}[(W+\overline{W})W_\alpha] + i\,\bar{Q}^{(n+2)} \partial^{n+1}[(W+\overline{W})Q_\alpha]\, d\alpha.$$

Here there are no low-frequency issues. Its high-frequency part, on the other hand, has the form

$$\begin{aligned} I_{0,\text{high}} = 2\Re \int & -(2n+2)\Re \boldsymbol{W}(|\boldsymbol{W}^{(n)}|^2 - i\,\bar{Q}^{(n+2)} Q^{(n+1)}) \\ & + i\,Q_\alpha \bar{Q}^{(n+2)}(W^{(n+1)} + \overline{W}^{(n+1)}) - (n+2) Q_\alpha \bar{Q}^{(n+1)} \boldsymbol{W}^{(n)}\, d\alpha. \end{aligned}$$

Terms with c. These are

$$\begin{aligned} I_1 = c\Re \int & -\overline{\boldsymbol{W}}^{(n)} \partial^{n+1}(\boldsymbol{W}(Q+\bar{Q}) + Q_\alpha(W+\overline{W})) \\ & + \bar{Q}^{(n+2)} \partial^{n+1}\big(\tfrac{1}{2}W^2 + |W|^2\big) + \frac{i}{g} \bar{Q}^{(n+2)} \partial^{n+1}(Q Q_\alpha + \bar{Q} Q_\alpha)\, d\alpha. \end{aligned}$$

We first verify that the terms with undifferentiated Q disappear when integrating by parts. These are

$$\begin{aligned} c\Re \int -2\Re Q\, \overline{\boldsymbol{W}}^{(n)} \boldsymbol{W}^{(n+1)}\, d\alpha &= c \int \Re Q_\alpha |W^{(n)}|^2\, d\alpha, \\ \frac{c}{g}\Re \int i \Re Q\, \bar{Q}^{(n+2)} Q^{(n+2)}\, d\alpha &= 0. \end{aligned}$$

Secondly, we compute the high-frequency component. Taking into account the above integration by parts, this is

$$I_{1,\text{high}} = c \int -(2n+2)|\boldsymbol{W}^{(n)}|^2 \Re Q_\alpha + \frac{i}{g}(2n+3)\Re Q_\alpha \bar{Q}^{(n+2)} Q^{(n+1)}\, d\alpha.$$

Terms with c^2. These are

$$\begin{aligned} I_2 = c^2 \Re \int & i\,\overline{\boldsymbol{W}}^{(n)} \partial^{n+1}\big(\boldsymbol{W}(\partial^{-1}W - \partial^{-1}\overline{W}) + W^2 + \tfrac{1}{2}|W|^2\big) - \frac{1}{2g} \overline{\boldsymbol{W}}^{(n)} \partial^{n+1}(Q Q_\alpha + \bar{Q} Q_\alpha) \\ & + \frac{1}{g} \bar{Q}^{(n+2)} \partial^{n+1}(Q_\alpha(\partial^{-1}W - \partial^{-1}\overline{W})) + \frac{1}{2g} \bar{Q}^{(n+2)} \partial^{n+1}(W Q + W \bar{Q})\, d\alpha. \end{aligned}$$

For the low-frequency analysis we compute

$$c^2 \Re \int -i\,\overline{\boldsymbol{W}}^{(n)} \boldsymbol{W}^{(n+1)}(\partial^{-1}W - \partial^{-1}\overline{W})\, d\alpha = -c^2 \int |\boldsymbol{W}^{(n)}|^2 \Im W\, d\alpha,$$

while the remaining $\partial^{-1}W$-terms and the Q-terms directly cancel.

Now we look at the high-frequency part. This is

$$I_{2,\text{high}} = c^2 \int -\big(2n + \tfrac{5}{2}\big)\Im W(|\boldsymbol{W}^{(n)}|^2 - i\,\bar{Q}^{(n+2)} Q^{(n+1)}) - (2n+4)(Q_\alpha + \bar{Q}_\alpha)\overline{\boldsymbol{W}}^{(n)} Q^{(n+1)}\, d\alpha.$$

Terms with c^3. These are

$$I_3 = -\frac{c^3}{2g}\Im \int \overline{\boldsymbol{W}}^{(n)} \partial^{n+1}[W(Q+\bar{Q}) + (\partial^{-1}W - \partial^{-1}\overline{W})Q_\alpha] - \bar{Q}^{(n+2)} \partial^{n+1}[(\partial^{-1}W - \partial^{-1}\overline{W})W]\, d\alpha.$$

For the low-frequency analysis we compute

$$\Re \int i\overline{\boldsymbol{W}}^{(n)}(\partial^{-1}W - \partial^{-1}\overline{W})Q^{(n+2)} - i\bar{Q}^{(n+2)}(\partial^{-1}W - \partial^{-1}\overline{W})\boldsymbol{W}^{(n)}\,d\alpha = 0,$$

$$\Re \int i\overline{\boldsymbol{W}}^{(n)}\boldsymbol{W}^{(n)}(Q + \bar{Q})\,d\alpha = 0,$$

while for the high-frequency analysis, taking the above into account, we are left with

$$I_{3,\text{high}} = \frac{c^3}{2g}\int i(2n+4)(W - \overline{W})\overline{\boldsymbol{W}}^{(n)}Q^{(n+1)}\,d\alpha.$$

Terms with c^4. This is

$$I_4 = \frac{c^4}{2g^2}\Re \int \boldsymbol{W}^{(n)}\partial^{(n+1)}[(\partial^{-1}W - \partial^{-1}\overline{W})W]\,d\alpha,$$

which exhibits cancellation at low frequency and has no high-frequency component.

The normal-form energy E^n_{NF} is obtained by replacing Q_α with R in the all the above terms, after eliminating the Q and $\partial^{-1}W$. The expressions of $E^n_{\text{NF,high}}$ and $E^{n,c}_{\text{NF,high}}$ are simply obtained by adding up all the high-frequency contributions above.

Since E^n_{NF} is obtained from the normal form, the relation (4-38) immediately follows. We note that establishing this property is a purely algebraic computation, which does not require the direct use of the normal form (which is unbounded).

Step 2. We begin with the quasilinear energy $E^{n,(3)}_{\text{high}}(w, r)$, which was defined in (4-30) if $n \geq 2$, and is set to be equal to $E^{(3)}_{\text{lin}}(w, r)$ if $n = 1$. We first compare it with the cubic energy $E^n_{\text{NF,high}}$. Precisely, we have

$$\Lambda^{\leq 3}E^{(3)}_{\text{lin}}(w, r) = \Lambda^{\leq 3}E^n_{\text{NF,high}}, \quad n = 1, \tag{4-41}$$

$$\Lambda^{\leq 3}E^{n,(3)}_{\text{high}}(w, r) = \Lambda^{\leq 3}E^n_{\text{NF,high}}, \quad n \geq 2. \tag{4-42}$$

This computation was already done in [HIT16] and is not repeated here. The only difference is in the c-term in $E^n_{\text{NF,high}}$, which arises due to the a_1 in $E^{(3)}_{\text{lin}}(w, r)$ and $E^{n,(3)}_{\text{high}}(w, r)$.

Next, we consider the term $E^{n,c}_{\text{NF,high}}$. This contains the leading-order energy density, so we have to treat it in a quasilinear manner. Precisely, we replace it by

$$E^{n,(3)}_{\text{high},c}(w, r) = -\int \bigl[c(2n+3)\Re R + c^2\bigl(2n + \tfrac{5}{2}\bigr)\Im W\bigr]((g + \underline{a})|w|^2 - i\bar{r}_\alpha r)\,d\alpha,$$

which admits a straightforward bound

$$|E^{n,(3)}_{\text{high},c}(w, r)| \lesssim cA_{-1/2}\boldsymbol{N}_n + c^2A_{-1}\boldsymbol{N}_n. \tag{4-43}$$

Now we are in a position to define our quasilinear modified cubic energy

$$E^{n,(3)}(\boldsymbol{W}, R) = E^{n,(3)}_{\text{high}}(w, r) + E^{n,(3)}_{\text{high},c}(w, r) + E^n_{\text{NF,low}}(\boldsymbol{W}, R). \tag{4-44}$$

By construction this satisfies

$$\Lambda^{<3} E^{n,(3)}(W, R) = \Lambda^{<3} E^n_{\text{NF}}(W, R). \tag{4-45}$$

Step 3. Now we proceed to prove Proposition 4.2. The norm equivalence is already known from (4-31) and (4-39), so we still need the energy estimate. For this we use (4-45) and (4-38) to see that the cubic terms vanish,

$$\Lambda^{\leq 3} \frac{d}{dt} E^{n,(3)} = \Lambda^{\leq 3} \frac{d}{dt} E^n_{\text{NF}} = 0.$$

Hence, it remains to look at the quartic-and-higher terms. For this we can split the terms,

$$\Lambda^{\geq 4} \frac{d}{dt} E^{n,(3)} = \Lambda^{\geq 4} \frac{d}{dt} E^{n,(3)}(w, r) + \Lambda^{\geq 4} \frac{d}{dt} E^{n,(3)}_{\text{high},c}(w, r) + \Lambda^{\geq 4} \frac{d}{dt} E^n_{\text{NF,low}}.$$

The bound for the first term on the right is a direct consequence of Proposition 3.1 for $n = 1$, and Lemma 4.6 for $n \geq 2$. From here on, the cases $n = 1$ and $n \geq 2$ are identical.

For the middle term $\Lambda^{\geq 4} \frac{d}{dt} E^{n,(3)}_{\text{high},c}(w, r)$ we will use Lemma 3.2, with either $f = W$ or $f = R$, and (w, r) as in this section. We claim that this yields

$$\left| \frac{d}{dt} E^{n,(3)}_{\text{high},c}(w, r) \right| \lesssim_{\underline{A}} \underline{B} N_n,$$

which by homogeneity considerations yields

$$\left| \Lambda^{\geq 4} \frac{d}{dt} E^{n,(3)}_{\text{high},c}(w, r) \right| \lesssim_{\underline{A}} \underline{A}\underline{B} N_n,$$

as desired.

(a) If $f = W$ then the bounds

$$\|f\|_{L^\infty} \lesssim_A A_{-1}, \quad \|D^{1/2} f\|_{L^\infty} \lesssim A_{-1/2}$$

allow us to estimate the difference I in the Lemma 3.2. The remaining terms

$$c^2 \int \Lambda^{\geq 2}(\partial_t + \underline{b}\partial_\alpha) W((g + \underline{a})|w|^2 - i\bar{r}_\alpha r)\, d\alpha \quad \text{and} \quad c^2 \int f((g + \underline{a})\bar{w} F - i\bar{r}_\alpha G)\, d\alpha$$

are c-times forms which are at least quartic, with order $2n + \frac{1}{2}$ and leading order $2n$ (this is because we have exactly one R-factor). Hence we can bound them using Hölder's inequality and interpolation by $cA\underline{A}N_n$.

(b) If $f = R$ then the bounds

$$\|f\|_{L^\infty} \lesssim_A A_{-1/2}, \quad \|D^{1/2} f\|_{L^\infty} \lesssim A$$

allow us to estimate the difference D in the lemma. The remaining terms are exactly as above, still bounded by $cA\underline{A}N_n$.

Finally, for the lower-order terms $\Lambda^{\geq 4} \frac{d}{dt} E^n_{\text{NF,low}}$ it suffices to have the following property:

Lemma 4.8. *For any term I in $E^n_{\text{NF,low}}$ we have*

$$\left| \Lambda^{\geq 4} \frac{d}{dt} I \right| \lesssim \underline{A}\underline{B}(N_n + c^4 N_{n-1}). \tag{4-46}$$

Proof. The terms without c in this computation were already estimated in [HIT16], so we need to consider only c-terms. A simple order analysis for these terms leads to the following properties:

(a) Their order is $2n + \frac{1}{2}$.

(b) Their leading order is at most $2n$.

(c) The highest power of c is 4.

Here part (b) is a obtained by using the relation (4-7) to compute time derivatives. This guarantees that the leading order does not increase by a full unit.

We also take a closer look at the c^2W- and cR-factors that can arise. In the trilinear form $E^n_{\mathrm{NF,low}}$ they can arise just once, so the question is what happens if we differentiate with respect to time. If W and R are differentiated in time then at most we obtain another W-factor, respectively an R- or cW-factor, so this pattern is preserved. If instead we differentiate a higher-order $W^{(k)}$- or $R^{(k)}$-factor using the equations (1-19), we can further produce undifferentiated W- and R-factors, as follows:

(i) Arising from cM_1 and its derivatives. There cW will appear in combinations of the form $\boldsymbol{P}[W\bar{Y}^{(k)}]$ and its conjugate. However, the frequency localization enforced by $\boldsymbol{P}$ guarantees that W is the higher-frequency factor, which, for estimates, ensures that we can freely move derivatives from $\bar{Y}$ to W.

(ii) Arising from derivatives of cN, where, for the same reasons as above, we can disregard the W-factors, but we can still produce a cR-factor.

(iii) Arising from b_1 and its derivatives. In the case of derivatives of b_1 the discussion concerning W-factors is again identical to case (i), so we can neglect those. Hence the only potential W-factor can arise from undifferentiated b_1. However, these are avoided due to the use of (4-7), where the transport is fully included in the time differentiation and the undifferentiated advection coefficient $\underline{b}$ never appears.

To summarize the above discussion, in the time derivative of $E^n_{\mathrm{NF,low}}$ we can have at most one c^2W-factor and at most two c^2W, cR-factors. A further simplification arises from the constraint (b) above. Precisely, there we can integrate by parts to rebalance derivatives in such a way that either

(a) both leading order terms have order at most n, or

(b) we have exactly the product $R^{(n+1)}\bar{R}^{(n)}$ as part of our multilinear expression.

In case (a), our estimates follow directly by Hölder's inequality and interpolation. In case (b), the remaining factors must have order zero, except for one which has order $\frac{1}{2}$. There we consider two scenarios.

(b1) If the factors $R^{(n+1)}$ and $\bar{R}^{(n)}$ are not frequency balanced, then another factor has the highest frequency and we can rebalance derivatives and use again Hölder's inequality and interpolation.

(b2) If the factors $R^{(n+1)}$ and $\bar{R}^{(n)}$ are frequency balanced, then all we need is to bound the remaining factors in L^∞. The only factor of order $\frac{1}{2}$ which we do not control in L^∞ is R_α, so Hölder's inequality and interpolation work unless our multilinear expression exactly contains

$$R^{(n+1)}\bar{R}^{(n)}R_\alpha,$$

and possibly other zero-order factors. Backtracking, the only way to produce such a term is by differentiating a cubic expression which has a leading order exactly $2n - \frac{1}{2}$ as well as an R_α-factor. Then this expression must be exactly

$$\mathbf{W}^{(n)} \bar{R}^{(n)} R_\alpha,$$

which cannot contain any c-factors so it is not within our purview here. ☐

Thus we have completed the proof of Proposition 4.7. ☐

5. Proofs of the main theorems

The results in Theorems 1 and 2 are a consequence of the estimates for the linearized equation in Section 3, and of the energy estimates for the higher Sobolev norms of the solutions in Section 4. The arguments here closely match their counterparts in our previous gravity waves paper [HIT16]. Because of this, we only provide an outline here, and refer the reader to [HIT16] for more details.

Outline of proof of Theorem 1. The steps of the proof are as follows:

Step 1: Existence of regular solutions. A standard approach here is to obtain solutions as limits of solutions to frequency truncated equations. As always, this truncation needs to happen in a symmetric way, so that uniform energy estimates survive. In addition, a specific difficulty we encounter in water wave evolutions is the fact that the evolution we are trying to truncate is degenerate hyperbolic, a condition which might not survive in a naive direct truncation procedure. In [HIT16] we bypass this difficulty by solving directly the differentiated system for the diagonal variables $(\mathbf{W}, R)$. In our case this is not entirely possible, as the system for $(\mathbf{W}, R)$ is not fully self-contained. Precisely, it contains W as a part of $\underline{b}$. Fortunately, this does not cause extra difficulties, as we can make the frequency truncation consistent with differentiation in the W-equation, and thus be able to freely use W in the $(\mathbf{W}, R)$-system. We note that formally $\mathbf{W}$ also appears undifferentiated in M_1 and N, but the commutator structure of those expressions implies that they actually can be rewritten (and estimated) in terms of $\mathbf{W}$ instead.

Precisely, our main truncated system for $(\mathbf{W}, R)$ has the form

$$\begin{cases} \mathbf{W}_t + \mathbf{P}_{<n}\underline{b}_n \mathbf{P}_{<n}\mathbf{W}_\alpha + \mathbf{P}_{<n}\dfrac{(1+\mathbf{P}_{<n}\mathbf{W})\mathbf{P}_{<n}R_\alpha}{1+\overline{\mathbf{P}_{<n}\mathbf{W}}} = \mathbf{P}_{<n}G(\mathbf{W}_{<n}, R_{<n}), \\ R_t + \mathbf{P}_{<n}\underline{b}_n \mathbf{P}_{<n}R_\alpha + ic\mathbf{P}_{<n}R - i\mathbf{P}_{<n}\dfrac{g\mathbf{P}_{<n}\mathbf{W} - a_n}{1+\mathbf{P}_{<n}\mathbf{W}} = \mathbf{P}_{<n}K(\mathbf{W}_{<n}, R_{<n}), \end{cases} \tag{5-1}$$

where G and K represent the right-hand-side terms in (1-19), and

$$\underline{b}_n = \underline{b}(W_{<n}, R_{<n}), \quad a_n = a(R_{<n}).$$

Here n is a dyadic-frequency parameter, and the multiplier $\mathbf{P}_{<n}$ selects the frequencies less than 2^n. The expression for $\mathbf{W}_t$ in the first equation above retains the structure from the original equations, i.e., it is an exact derivative. Thus we can recover the undifferentiated variable W in a way consistent with the $\mathbf{W}$-equation above, simply by integrating in time the relation

$$W_t + \mathbf{P}_{<n}[\underline{F}_n(1+\mathbf{P}_{<n}\mathbf{W})] + i\frac{c}{2}\mathbf{P}_{<n}W = 0.$$

The above set of equations is consistent, and for each dyadic frequency scale n it generates an ODE in the space $\mathcal{H}_k,\ k \geq 2$ for $(\boldsymbol{W}, R)$, with the additional property that W in L^2. A priori, the lifespan for this system depends on n. However, the same type of estimates used for the linearized system yield uniform bounds and a uniform lifespan depending only on the Sobolev norm of the data, and not on n. Then the regular solutions are obtained as weak limits of these approximate solutions as $n \to \infty$.

Step 2: Uniqueness of regular solutions ($(\boldsymbol{W}, R) \in \dot{\mathcal{H}}_2$). This is achieved in a more standard fashion. Given two solutions (W_1, Q_1) and (W_2, Q_2) we subtract the equations and do energy estimates for the difference $(\boldsymbol{W}_1 - \boldsymbol{W}_2, R_1 - R_2)$ in $\dot{\mathcal{H}}_0$, as well as simpler integrated bounds for $W_1 - W_2$ in L^2. Then close the argument and prove uniqueness via Gronwall's inequality.

Step 3: $\dot{\mathcal{H}}_1$-bounds for $(\boldsymbol{W}, R)$. Here we use the uniform bounds for the $\dot{\mathcal{H}}_2$-norm of $(\boldsymbol{W}, R)$ which depend only on the $\dot{\mathcal{H}}_1$-norm of $(\boldsymbol{W}, R)$ (via the control parameters $\underline{A}$ and $\underline{B}$), as in Proposition 4.1, to conclude that the regular solutions can be continued up to a time which depends only on the $\dot{\mathcal{H}}_1$-norm of the data.

Step 4: Construction of rough solutions, $(\boldsymbol{W}, R) \in \dot{\mathcal{H}}_1$. We regularize the data, $(W_{<k}(0), Q_{<k}(0))$, by truncating at frequencies $< 2^k$. The corresponding solutions will be regular, with a uniform lifespan bound. Thinking of k as a continuous parameter, we associate the k-derivative of the solutions $(W_{<k}, Q_{<k})$ to solutions for the linearized equation around $(W_{<k}, Q_{<k})$. Using both the energy bounds for the solutions $(\boldsymbol{W}_{<k}, R_{<k})$ in $\dot{\mathcal{H}}_0$ and $\dot{\mathcal{H}}_1$, and the $\dot{\mathcal{H}}_0$-bounds for the linearized equation, we show that these solutions inherit not only uniform bounds in $\dot{\mathcal{H}}_1$, but also uniform frequency envelope bounds in terms of the initial data frequency envelope. This suffices in order to prove strong convergence of $(\boldsymbol{W}_{<k}, R_{<k})$ in $\dot{\mathcal{H}}_1$, and of $(W_{<k}, Q_{<k})$ in $\dot{\mathcal{H}}_0$.

Step 5: Weak Lipschitz dependence for rough solutions. Here we show that for rough solutions, i.e., $(W, Q) \in \dot{\mathcal{H}}_0$ and $(\boldsymbol{W}, R) \in \dot{\mathcal{H}}_1$, we have Lipschitz dependence on the initial data in the $\dot{\mathcal{H}}_0$ topology for both $(W, Q) \in \dot{\mathcal{H}}_0$ and $(\boldsymbol{W}, R)$. This is a direct consequence of the $\dot{\mathcal{H}}_0$-bounds for the linearized equation.

Step 6: Continuous dependence on the data for rough solutions. This is a standard consequence of the frequency envelope bounds in Step 4 and the weak Lipschitz bounds in Step 5.

Step 7: Continuation of solutions. Here we show that the solutions extend with uniform bounds for as long as our control norms $\underline{A}$, $\underline{B}$ remain bounded. This is a consequence of the above local well-posedness in $\dot{\mathcal{H}}_1$ and the energy estimates in Proposition 4.1. □

Outline of proof of Theorem 2. This is an easy consequence of the $\dot{\mathcal{H}}_1$ well-posedness and the $\dot{\mathcal{H}}_1$ uniform bounds in Proposition 4.2, as the control norms $\underline{A}$ and $\underline{B}$ can be estimated in terms if the $\dot{\mathcal{H}}_1$-norm of $(\boldsymbol{W}, R)$ and the $\dot{\mathcal{H}}_0$-norm of (W, Q). □

Appendix A: Water-waves-related estimates

In this section we gather a number of bilinear and multilinear estimates which are used throughout the paper. Some are from [HIT16] and are just recalled here. The rest are connected to the new structure induced by the vorticity. We begin with some commutator bounds from [HIT16].

Lemma A.1. *The following commutator estimates hold*:

$$\||D|^s[\boldsymbol{P}, R]|D|^\sigma w\|_{L^2} \lesssim \||D|^{\sigma+s} R\|_{\mathrm{BMO}} \|w\|_{L^2}, \quad \sigma \geq 0,\ s \geq 0, \tag{A-1}$$

$$\||D|^s[\boldsymbol{P}, R]|D|^\sigma w\|_{L^2} \lesssim \||D|^{\sigma+s} R\|_{L^2} \|w\|_{\mathrm{BMO}}, \quad \sigma > 0,\ s \geq 0. \tag{A-2}$$

We remark that later this is applied to functions which are holomorphic/antiholomorphic, but that no such assumption is made above. Next, we have several bilinear estimates:

Lemma A.2. *The functions N and M_1 satisfy the pointwise bounds*

$$\|N\|_{L^\infty} \lesssim_A AA_{-1/2}, \quad \|M_1\|_{L^\infty} \lesssim_A A^2, \tag{A-3}$$

as well as the Sobolev bounds

$$\|N\|_{\dot H^{n+1/2}} \lesssim_A A\boldsymbol{N}_n, \quad \|M_1\|_{\dot H^n} \lesssim_A A\boldsymbol{N}_n.$$

Proof. We begin with the bounds for N, where N was defined in (1-13) as

$$N = \boldsymbol{P}[W\bar R_\alpha - \overline{\boldsymbol{W}} R] + \bar{\boldsymbol{P}}[\bar W R_\alpha - \boldsymbol{W}\bar R].$$

For the pointwise bound we claim that

$$\|N\|_{L^\infty} \lesssim \|W\|_{B_2^{3/4,\infty}} \|R\|_{B_2^{1/4,\infty}}. \tag{A-4}$$

This suffices since each of the right-hand-side factors is bounded by $\sqrt{AA_{-1/2}}$ by interpolation. To achieve this we observe that we can alternatively write N in the form

$$N = \bar{\boldsymbol{P}}[\bar W R_\alpha - \boldsymbol{W}\bar R] + \boldsymbol{P}[W\bar R_\alpha - \overline{\boldsymbol{W}} R] = \partial_\alpha(\bar{\boldsymbol{P}}[\bar W R] + \boldsymbol{P}[W\bar R]) - (\overline{\boldsymbol{W}} R + \boldsymbol{W}\bar R).$$

We apply a bilinear Littlewood–Paley decomposition and use the first expression above for the high-low interactions, and the second for the high-high interactions, to write $N = N_1 + N_2$, where

$$N_1 = \sum_k [\bar W_k R_{<k,\alpha} - W_{<k,\alpha}\bar R_k] + [W_k \bar R_{<k,\alpha} - \bar W_{<k,\alpha} R_k],$$

$$N_2 = \sum_k \partial_\alpha(\bar{\boldsymbol{P}}[\bar W_k R_k] + \boldsymbol{P}[W_k\bar R_k]) - (\bar W_{k,\alpha} R_k + W_{k,\alpha}\bar R_k).$$

We estimate the terms in N_1 separately; we show the argument for the first term:

$$\Big\|\sum_k \bar W_k R_{<k,\alpha}\Big\|_{L^\infty} \lesssim \sum_{j\leq k} 2^{(3/4)(j-k)} \|W_j\|_{L^\infty} \|R_k\|_{L^\infty} \lesssim \|W\|_{B_2^{3/4,\infty}} \|R\|_{B_2^{1/4,\infty}}.$$

For the first term in N_2 we note that the multiplier $\partial_\alpha \boldsymbol{P}_{<k+4}\boldsymbol{P}$ has an $O(2^k)$ L^∞-bound. Hence, we can estimate

$$\|N_2\|_{L^\infty} \lesssim \sum_k 2^k \|W_k\|_{L^\infty} \|R_k\|_{L^\infty} \lesssim \|W\|_{B_2^{3/4,\infty}} \|R\|_{B_2^{1/4,\infty}}.$$

For simplicity, we only prove the $\dot H^{1/2}$-bound for N. The rest can be done in a very similar way. To obtain the bound we apply a Coifman–Meyer-type commutator estimate:

$$\|N\|_{\dot H^{1/2}} \lesssim \||D|^{1/2}[\bar{\boldsymbol{P}}, \bar W]R_\alpha\|_{L^2} + \||D|^{1/2}[\bar{\boldsymbol{P}}, \bar R]\boldsymbol{W}\|_{L^2} + \||D|^{1/2}[\boldsymbol{P}, W]\bar R_\alpha\|_{L^2} + \||D|^{1/2}[\boldsymbol{P}, \boldsymbol{W}]R\|_{L^2}.$$

Each commutator can be bounded by $\lesssim AN_0$. For pointwise and Sobolev bounds of M_1 we refer to Lemma 2.8 (see the Appendix in [HIT16]); the exact same approach applies in the current case. □

Essential in the article are the bounds for a, which we established in [HIT16]:

Proposition A.3. *Assume that $R \in \dot H^{1/2} \cap \dot H^{3/2}$. Then the real frequency-shift a is nonnegative and satisfies the* BMO *bound*

$$\|a\|_{\mathrm{BMO}} \lesssim \|R\|^2_{\mathrm{BMO}^{1/2}}, \tag{A-5}$$

and the uniform bound

$$\|a\|_{L^\infty} \lesssim \|R\|^2_{\dot B^{1/2}_{\infty,2}}. \tag{A-6}$$

Moreover,

$$\||D|^{1/2}a\|_{\mathrm{BMO}} \lesssim \|R_\alpha\|_{\mathrm{BMO}}\,\||D|^{1/2}R\|_{L^\infty}, \qquad \|a\|_{B^{1/2,\infty}_2} \lesssim \|R_\alpha\|_{B^{1/2,\infty}_2}\,\||D|^{1/2}R\|_{L^\infty}, \tag{A-7}$$

$$\|(\partial_t + b\partial_\alpha)a\|_{L^\infty} \lesssim AB, \tag{A-8}$$

$$\|a\|_{\dot H^s} \lesssim \|R\|_{\dot H^{s+1/2}}\,\|R\|_{\mathrm{BMO}^{1/2}}, \qquad s > 0. \tag{A-9}$$

Here we need to supplement this with bounds for a_1. One notable difference between the two is that a_1 has a linear component, whereas a is purely quadratic. For various estimates we need to separate the two components of a_1:

Proposition A.4. *Assume that $R \in \dot H^{1/2} \cap \dot H^{3/2} \cap L^\infty$. Then*

$$\|a_1\|_{L^\infty} \lesssim_A A_{-1/2}(1 + A).$$

Moreover, the following estimate holds:

$$\|(\partial_t + \underline{b}\partial_\alpha)a_1 + 2g\Im \boldsymbol{W} - 2c\Im R\|_{L^\infty} \lesssim AB + cA^2.$$

Proof. We first recall the expression for a_1:

$$a_1 = R + \bar R - N.$$

Using the equation for R, we only need to prove the pointwise bound for $(\partial_t + \underline{b}\partial_\alpha)N$. We begin with the following computation:

$$(\partial_t + \underline{b}\partial_\alpha)a_1 + 2g\Im \boldsymbol{W} - 2c\Im R = -2a\Im Y - c\Im \frac{R\boldsymbol{W} + \bar R \boldsymbol{W} + N}{1 + \boldsymbol{W}} - (\partial_t + \underline{b}\partial_\alpha)N.$$

Based on the previously established pointwise bounds for a and N we can estimate all but the last term. This is considerably more delicate. However, as seen in the proof of Lemma A.2, the expression for N exhibits exactly the same cancellation structure for "high × high → low" interactions as we have in the bilinear expression for a. Hence, the argument in the proof of (A-8) in [HIT16] immediately adapts here. □

Appendix B: Holomorphic coordinates

Our goal here is to introduce the holomorphic coordinates and the holomorphic set of variables (W, Q) describing the free surface, respectively the holomorphic velocity potential restricted to the free surface, and to derive the evolution equations for (W, Q).

We proceed as in [Hunter, Ifrim and Tataru 2016; Ifrim and Tataru 2017]. Let $\mathbb{H}$ be the lower half-space, with complex coordinates denoted by $\alpha + i\beta$. Let $\mathcal{F} : \mathbb{H} \to \Omega(t)$ to be the conformal transformation that maps the α-axis into $\Gamma(t)$, which is unique up to α-translations. The x- and y-coordinates are given by

$$x = x(\alpha, \beta, t), \quad y = y(\alpha, \beta, t),$$

and satisfy the Cauchy–Riemann equations

$$x_\alpha = y_\beta, \quad x_\beta = -y_\alpha.$$

We fix the conformal map by assuming that

$$x(\alpha, \beta, t) + iy(\alpha, \beta, t) - \alpha + i\beta \to 0, \quad \alpha, \beta \to \infty.$$

To switch between the two sets of coordinates we will use

$$\begin{aligned} \partial_\alpha &= x_\alpha \partial_x + y_\alpha \partial_y, & \partial_\beta &= x_\beta \partial_x + y_\beta \partial_y, \\ \partial_x &= \frac{1}{j}(x_\alpha \partial_\alpha + x_\beta \partial_\beta), & \partial_y &= \frac{1}{j}(y_\alpha \partial_\alpha + y_\beta \partial_\beta), \end{aligned}$$

where

$$j = x_\alpha y_\beta - x_\beta y_\alpha = x_\alpha^2 + x_\beta^2 = y_\alpha^2 + y_\beta^2.$$

We also need a similar relation for the time derivative. Assume we have

$$\mathbb{H} \xrightarrow{\mathcal{F}} \Omega(t) \xrightarrow{f} \mathbb{C}$$

for an arbitrary function f. Let

$$g(\alpha, \beta, t) = f(x(\alpha, \beta, t), y(\alpha, \beta, t), t).$$

Then

$$g_t = f_t + x_t f_x + f_y y_t.$$

So

$$\begin{aligned} f_t &= g_t - x_t f_x - f_y y_t \\ &= g_t - x_t \frac{1}{j}(x_\alpha g_\alpha + x_\beta g_\beta) - y_t \frac{1}{j}(y_\alpha g_\alpha + y_\beta g_\beta) \\ &= g_t - \frac{1}{j}(x_t x_\alpha + y_t y_\alpha) g_\alpha - \frac{1}{j}(x_t x_\beta + y_t y_\beta) g_\beta. \end{aligned} \tag{B-1}$$

Define ψ to be the (harmonic) composition of $\mathcal{F}$ to the velocity potential φ,

$$\psi = \varphi \circ \mathcal{F} : \mathbb{H} \to \mathbb{R}, \quad \psi(\alpha, \beta, t) = \varphi(x(\alpha, \beta, t), y(\alpha, \beta, t), t).$$

The velocity components (u, v) can now be expressed in terms of the velocity potential ψ by

$$u = \frac{1}{j}(x_\alpha \psi_\alpha + x_\beta \psi_\beta) + cy, \quad v = \frac{1}{j}(y_\alpha \psi_\alpha + y_\beta \psi_\beta). \tag{B-2}$$

It follows that

$$u^2 + v^2 = \varphi_x^2 + \varphi_y^2 = \frac{1}{j}(\psi_\alpha^2 + \psi_{\beta^2}).$$

Boundary values. Setting $\beta = 0$ gives the boundary values of the holomorphic functions defined in the lower half-plane. In particular, we introduce the notation

$$x(\alpha, 0, t) =: X(\alpha, t), \quad y(\alpha, 0, t) =: Y(\alpha, t),$$

so that $\alpha \to X + iY$ parametrizes $\Gamma(t)$. The function $(x - \alpha) + i(y - \beta)$ is holomorphic in $\mathbb{H}$ and decays at infinity, which implies that on the boundary we have

$$\begin{cases} Y = H(X - \alpha), \\ X = \alpha + HY. \end{cases}$$

Also set

$$z(\alpha, \beta, t) = x(\alpha, \beta, t) + iy(\alpha, \beta, t), \quad Z(\alpha, t) = z(\alpha, 0, t).$$

Then our "holomorphic" variable W, which describes the surface $\Gamma(t)$, will be

$$W = Z - \alpha.$$

As z is holomorphic in $\mathbb{H}$, so is $1/z_\alpha - 1$; further, it decays at infinity. Its boundary values on the real axis are given by

$$\frac{1}{Z_\alpha - 1} = \left(\frac{X_\alpha}{J} - 1\right) - i\frac{Y_\alpha}{J},$$

which leads to the relations

$$\begin{cases} \dfrac{X_\alpha}{J} - 1 = -H\left[\dfrac{Y_\alpha}{J}\right], \\ \dfrac{Y_\alpha}{J} = H\left[\dfrac{X_\alpha}{J} - 1\right]. \end{cases} \tag{B-3}$$

We also introduce the notation $\Psi(\alpha, t) := \psi(\alpha, \beta, t)$ for the real velocity potential restricted to $\Gamma(t)$ and expressed in holomorphic coordinates, and at the same time define $\Theta(\alpha, t)$ by

$$\begin{cases} \Psi = H\Theta, \\ \Theta = -H\Psi. \end{cases}$$

Up to a constant, this is the trace of the stream function θ on the free boundary. Since ψ is harmonic in the lower half-plane, we have

$$\psi_\beta|_{\beta=0} - H\Psi_\alpha = -\Theta_\alpha.$$

Our holomorphic velocity potential will be the function

$$Q = \Psi + i\Theta.$$

Further we need to focus on the two boundary conditions: kinematic and dynamic.

The kinematic boundary condition. The kinematic boundary condition states that the normal component of the velocity of the boundary is equal to the normal component of the fluid velocity, meaning that

$$(X_t, Y_t)\cdot(-Y_\alpha, X_\alpha) = (u, v)\cdot(-Y_\alpha, X_\alpha),$$

where $(-Y_\alpha, X_\alpha)$ is a normal to $\Gamma(t)$. Expanding the expression above and using (B-2) we can re-express the kinematic boundary condition in holomorphic coordinates:

$$X_\alpha Y_t - Y_\alpha X_t = H\Psi_\alpha - cYY_\alpha = -\Theta_\alpha - cYY_\alpha. \tag{B-4}$$

The goal now is to obtain a second equations for X_t and Y_t, and then solve for an explicit form of those boundary values. Divide (B-4) by J,

$$\left(\frac{X_\alpha}{J} - 1\right)Y_t - \frac{Y_\alpha}{J}X_t + Y_t = -\frac{\Theta_\alpha}{J} - c\frac{YY_\alpha}{J},$$

and use (B-3) to obtain

$$-H\left[\frac{Y_\alpha}{J}\right]Y_t - \frac{Y_\alpha}{J}H[Y_t] + Y_t = -\frac{\Theta_\alpha}{J} - c\frac{YY_\alpha}{J},$$

which further simplifies to

$$\frac{Y_\alpha}{J}Y_t - H\left[\frac{Y_\alpha}{J}\right]H[Y_t] + X_t = -H\left[\frac{\Theta_\alpha}{J}\right] - cH\left[\frac{YY_\alpha}{J}\right].$$

Thus, a second equation for (X_t, Y_t) is

$$X_\alpha X_t + Y_\alpha Y_t = -JH\left[\frac{\Theta_\alpha}{J}\right] - cJH\left[\frac{YY_\alpha}{J}\right]. \tag{B-5}$$

From (B-4) and (B-5) we have

$$\begin{cases} X_t = -H\left[\dfrac{\Theta_\alpha}{J}\right]X_\alpha - cH\left[\dfrac{YY_\alpha}{J}\right]X_\alpha + \dfrac{\Theta_\alpha}{J}Y_\alpha + c\dfrac{YY_\alpha}{J}Y_\alpha, \\ Y_t = -H\left[\dfrac{\Theta_\alpha}{J}\right]Y_\alpha - cH\left[\dfrac{YY_\alpha}{J}\right]Y_\alpha - \dfrac{\Theta_\alpha}{J}X_\alpha - c\dfrac{YY_\alpha}{J}X_\alpha. \end{cases} \tag{B-6}$$

The dynamic boundary condition. We have already determined the spatial form of the dynamic boundary condition in (1-4). From (B-1) and the kinematic boundary conditions (B-4)–(B-5) it follows that, on the boundary, φ_t is given as

$$\varphi_t|_{\beta=0} = \Psi_t - \frac{1}{J}(X_\alpha X_t + Y_\alpha Y_t)\Psi_\alpha + \frac{1}{J}(X_\alpha Y_t - Y_\alpha X_t)\Theta_\alpha.$$

Substituting X_t and Y_t from (B-6) yields

$$\varphi_t|_{\beta=0} = \Psi_t + H\left[\frac{\Theta_\alpha}{J}\right]\Psi_\alpha + cH\left[\frac{YY_\alpha}{J}\right]\Psi_\alpha - \frac{1}{J}\Theta_\alpha^2 - \frac{1}{J}cYY_\alpha\Theta_\alpha. \tag{B-7}$$

We express all the terms on the right in (1-4) in terms of the traces on the boundary of the corresponding functions. Doing so, after some simplifications we arrive at the following equation:

$$\Psi_t + H\left[\frac{\Theta_\alpha}{J}\right]\Psi_\alpha - \frac{\Theta_\alpha^2}{J} + \frac{1}{2J}(\Psi_\alpha^2 + \Theta_\alpha^2) + gY + cH\left[\frac{YY_\alpha}{J}\right]\Psi_\alpha - c\Theta + c\frac{Y}{J}X_\alpha\Psi_\alpha = 0. \tag{B-8}$$

The real form of the equations. The equations (B-6) and (B-8) provide us with a system describing the evolution of the free boundary and the velocity potential restricted to the free boundary, as follows:

$$\begin{cases} Y_t = -H\left[\dfrac{\Theta_\alpha}{J}\right]Y_\alpha - cH\left[\dfrac{YY_\alpha}{J}\right]Y_\alpha - \dfrac{\Theta_\alpha}{J}X_\alpha - c\dfrac{YY_\alpha}{J}X_\alpha, \\ \Psi_t = -H\left[\dfrac{\Theta_\alpha}{J}\right]\Psi_\alpha + \dfrac{\Theta_\alpha^2}{J} - \dfrac{1}{2J}(\Psi_\alpha^2 + \Theta_\alpha^2) - gY - cH\left[\dfrac{YY_\alpha}{J}\right]\Psi_\alpha + c\Theta - c\dfrac{Y}{J}X_\alpha\Psi_\alpha. \end{cases}$$

Here X and Θ are dependent variables.

The Hamiltonian associated to the system is

$$\mathcal{E}(Y, \Psi) = \frac{1}{2}\int\left\{\Psi|\partial_\alpha|\Psi + gY^2X_\alpha + c\Psi_\alpha Y^2 + \frac{1}{3}c^2Y^3X_\alpha\right\}d\alpha,$$

where $|\partial_\alpha| = H\partial_\alpha$. Thus

$$\begin{cases} \dfrac{\delta\mathcal{E}}{\delta Y} = gYX_\alpha + g\tfrac{1}{2}|\partial_\alpha|(Y^2) + c\Psi_\alpha Y + \tfrac{1}{2}c^2Y^2X_\alpha + \tfrac{1}{6}c^2|\partial_\alpha|(Y^3), \\ \dfrac{\delta\mathcal{E}}{\delta\Psi} = |\partial_\alpha|\Psi - cYY_\alpha. \end{cases}$$

We can write the above equations for (Y, Ψ) in a skew-symmetric form

$$\begin{pmatrix} Y \\ \Psi \end{pmatrix}_t = \begin{pmatrix} 0 & \boldsymbol{A} \\ -\boldsymbol{A}^* & \boldsymbol{B} \end{pmatrix}\begin{pmatrix} \delta\mathcal{E}/\delta Y \\ \delta\mathcal{E}/\delta\Psi \end{pmatrix}, \tag{B-9}$$

where

$$\boldsymbol{A} = \frac{X_\alpha}{J} + Y_\alpha H\frac{1}{J}, \qquad \boldsymbol{B} = \Psi_\alpha H\frac{1}{J} + \frac{1}{J}H\Psi_\alpha - c\partial_\alpha^{-1},$$
$$\boldsymbol{A}^* = \frac{X_\alpha}{J} - \frac{1}{J}HY_\alpha, \qquad \boldsymbol{B}^* = -\boldsymbol{B}.$$

This form immediately implies that $\mathcal{E}$ is conserved along the flow. There are several Hamiltonian symmetries, which correspond to conservation laws of the system in accordance with Noether's principle. The horizontal translation invariance

$$Y(\alpha, t) \to Y(\alpha + a, t), \qquad \Psi(\alpha, t) \to \Psi(\alpha + a, t)$$

is generated by the functional $\mathcal{P}$ which we will derive below. This functional corresponds to total horizontal momentum, and it is given by the system

$$\begin{pmatrix} Y \\ \Psi \end{pmatrix}_\alpha = \begin{pmatrix} 0 & \boldsymbol{A} \\ -\boldsymbol{A}^* & \boldsymbol{B} \end{pmatrix}\begin{pmatrix} \delta\mathcal{P}/\delta Y \\ \delta\mathcal{P}/\delta\Psi \end{pmatrix}, \tag{B-10}$$

where

$$\mathcal{P}(Y,\Psi)=\int\left\{\Psi Y_\alpha-\frac{c}{2}Y^2X_\alpha\right\}d\alpha.$$

The variational derivatives of $\mathcal{P}$ are

$$\begin{aligned}\delta\mathcal{P}/\delta Y&=-\Psi_\alpha-cYX_\alpha-\frac{c}{2}|\partial_\alpha|(Y^2),\\ \delta\mathcal{P}/\delta\Psi&=Y_\alpha.\end{aligned}$$

Thus, our conserved energies are

$$\begin{aligned}\mathcal{E}(Y,\Psi)&=\frac{1}{2}\int\left\{\Psi|\partial_\alpha|\Psi+gY^2X_\alpha+c\Psi_\alpha Y^2+\frac{c^2}{3}Y^3X_\alpha\right\}d\alpha,\\ \mathcal{P}(Y,\Psi)&=\int\left\{\Psi Y_\alpha-\frac{c}{2}Y^2X_\alpha\right\}d\alpha,\end{aligned}\tag{B-11}$$

where $|\partial_\alpha|=H\partial_\alpha$.

The complex form of the equations. Recall that our holomorphic variables are

$$Z=X+iY,\qquad Q=\Psi+i\Theta.$$

We also introduce the notation

$$F=H\left[\frac{\Theta_\alpha}{J}\right]+i\frac{\Theta_\alpha}{J}=\boldsymbol{P}\left[\frac{Q_\alpha-\bar{Q}_\alpha}{J}\right],\qquad J=|Z_\alpha|^2,\tag{B-12}$$

noting that F is also the boundary value of a function which is holomorphic in the lower half-plane.

Using these notations, a straightforward computation allows us to re-express the kinematic boundary conditions (B-6) in a single equation for the motion of the boundary

$$Z_t+FZ_\alpha+2ic\boldsymbol{P}\left[\frac{YY_\alpha}{J}\right]Z_\alpha=0,$$

which is equivalent to

$$Z_t+FZ_\alpha-i\frac{c}{4}\boldsymbol{P}\left[\frac{\partial_\alpha(Z-\bar{Z})^2}{J}\right]Z_\alpha=0.\tag{B-13}$$

Next we rewrite the dynamic boundary condition. For this we apply the operator $2\boldsymbol{P}=\boldsymbol{I}-iH$ in the equation (B-8). We do not repeat the computations in [HIT16] for the case $c=0$. Instead, we use them directly to obtain, from (B-8), the equation

$$Q_t-ig(Z-\alpha)+FQ_\alpha+icQ+\boldsymbol{P}\left[\frac{|Q_\alpha|^2}{J}\right]+cK=0,\tag{B-14}$$

where

$$K=(\boldsymbol{I}-iH)\left[H\left[\frac{YY_\alpha}{J}\right]\Psi_\alpha+\frac{Y}{J}X_\alpha\Psi_\alpha\right].$$

To simplify K we use the relation $H(fg - HfHg) = fHg + Hfg$ to rewrite it as

$$\begin{aligned} K &= H\left[\frac{YY_\alpha}{J}\right]\Psi_\alpha - iH\left[\frac{YY_\alpha}{J}\Theta_\alpha\right] + i\frac{YY_\alpha}{J}\Psi_\alpha + iH\left[\frac{YY_\alpha}{J}\right]\Theta_\alpha + (\boldsymbol{I} - iH)\left[\frac{Y}{J}X_\alpha\Psi_\alpha\right] \\ &= i(\boldsymbol{I} - iH)\left[\frac{YY_\alpha}{J}\right]Q_\alpha + (\boldsymbol{I} - iH)\left[\frac{Y}{J}(X_\alpha\Psi_\alpha + Y_\alpha\Theta_\alpha)\right] \\ &= \boldsymbol{P}\left[\frac{Y}{\bar{Z}_\alpha} - \frac{Y}{Z_\alpha}\right]Q_\alpha + \boldsymbol{P}\left[\frac{YQ_\alpha}{Z_\alpha} + \frac{Y\bar{Q}_\alpha}{\bar{Z}_\alpha}\right]. \end{aligned}$$

Further, we write $Y = -i((Z-\alpha) - (\bar{Z}-\alpha))$, eliminate the projected antiholomorphic terms and remove the projection in front of holomorphic terms to obtain

$$K = -\frac{i}{2}\boldsymbol{P}\left[\frac{Z-\alpha}{\bar{Z}_\alpha} + \frac{\bar{Z}-\alpha}{Z_\alpha}\right]Q_\alpha - \frac{i}{2}\boldsymbol{P}\left[\frac{(Z-\alpha)\bar{Q}_\alpha}{\bar{Z}_\alpha} - \frac{(\bar{Z}-\alpha)Q_\alpha}{Z_\alpha}\right]. \tag{B-15}$$

Thus our set of holomorphic equations consists of (B-13) and (B-14)–(B-15).

For the last step we use the relation $Z = W + \alpha$ to replace Z by W. For the last expression in (B-13) we have

$$\begin{aligned} \boldsymbol{P}\left[\frac{\partial_\alpha(Z-\bar{Z})^2}{J}\right] &= 2\boldsymbol{P}\left[\frac{Z-\alpha}{\bar{Z}_\alpha} - \frac{Z-\alpha}{Z_\alpha} + \frac{\bar{Z}-\alpha}{Z_\alpha}\right] \\ &= 2\boldsymbol{P}\left[\frac{W}{1+\bar{W}_\alpha} - \frac{W}{1+W_\alpha} + \frac{\bar{W}}{1+W_\alpha}\right] \\ &= 2F_1 - \frac{2W}{1+W_\alpha}, \end{aligned}$$

where

$$F_1 = \boldsymbol{P}\left[\frac{W}{1+\bar{W}_\alpha} + \frac{\bar{W}}{1+W_\alpha}\right].$$

For K on the other hand we have

$$K = -\frac{i}{2}F_1 Q_\alpha - \frac{i}{2}\boldsymbol{P}\left[\frac{W\bar{Q}_\alpha}{1+\bar{W}_\alpha} - \frac{\bar{W}Q_\alpha}{1+W_\alpha}\right].$$

Hence, setting

$$\underline{F} = F - i\frac{c}{2}F_1,$$

our equations become

$$\begin{cases} W_t + \underline{F}(W_\alpha + 1) + i\frac{c}{2}W = 0, \\ Q_t + \underline{F}Q_\alpha + icQ - igW + \boldsymbol{P}\left[\frac{|Q_\alpha|^2}{J}\right] + i\frac{c}{2}\left\{\boldsymbol{P}\left[\frac{W\bar{Q}_\alpha}{1+\bar{W}_\alpha}\right] - \boldsymbol{P}\left[\frac{\bar{W}Q_\alpha}{1+W_\alpha}\right]\right\} = 0. \end{cases} \tag{B-16}$$

We can also re-express the Hamiltonian and horizontal momentum in terms of the holomorphic variables (W, Q). This gives

$$\mathcal{E}(W, Q) = \Re\int g|W|^2(1+W_\alpha) - iQ\bar{Q}_\alpha + cQ_\alpha(\Im W)^2 - \frac{c^3}{2i}|W|^2W(1+W_\alpha)\,d\alpha. \tag{B-17}$$

A second conserved quantity is the horizontal momentum,

$$\mathcal{P} = \int \left\{ \frac{1}{i}(\bar{Q} W_\alpha - Q \overline{W}_\alpha) - c|W|^2 + \frac{c}{2}(W^2 \overline{W}_\alpha + \overline{W}^2 W_\alpha) \right\} d\alpha. \tag{B-18}$$

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