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GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE RADIAL, DEFOCUSING, CUBIC WAVE EQUATION WITH INITIAL DATA IN A CRITICAL BESOV SPACE

BENJAMIN DODSON

We prove that the cubic wave equation is globally well-posed and scattering for radial initial data lying in $B^2_{1,1} \times B^1_{1,1}$. This space of functions is a scale-invariant subspace of $\dot{H}^{1/2} \times \dot{H}^{-1/2}$.

1. Introduction

The three-dimensional cubic nonlinear wave equation,

$$u_{tt} - \Delta u = -u^3 = F(u), \qquad u(0,x) = u_0, \quad u_t(0,x) = u_1, \quad x \in \mathbb{R}^3, \tag{1-1}$$

has been a topic of recent interest in the study of dispersive partial differential equations. This is due to the fact that the Hamiltonian for (1-1),

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2\,dx + \frac{1}{2}\int u_t(t,x)^2\,dx + \frac{1}{4}\int u(t,x)^4\,dx = E(u(0)), \tag{1-2}$$

does not control the critical Sobolev norm.

A solution to (1-1) obeys the scaling symmetry that if $u(t,x)$ solves (1-1), then for any $\lambda > 0$

$$\lambda u(\lambda t, \lambda x) \tag{1-3}$$

also solves (1-1) with initial data $(\lambda u_0(\lambda x), \lambda^2 u_1(\lambda x))$. It is a general rule that, for any dimension $d \geq 1$,

$$\|u_0\|_{\dot{H}^{(d-2)/2}(\mathbb{R}^d)} = \|\lambda u_0(\lambda x)\|_{\dot{H}^{(d-2)/2}(\mathbb{R}^d)}, \quad \|u_1\|_{\dot{H}^{(d-4)/2}(\mathbb{R}^d)} = \|\lambda^2 u_1(\lambda x)\|_{\dot{H}^{(d-4)/2}(\mathbb{R}^d)}. \tag{1-4}$$

Thus in three dimensions (1-1) is called $\dot{H}^{1/2}$-critical.

Local well-posedness theory for (1-1) in L^2-based Sobolev spaces is completely determined by the critical $s_c = \frac{1}{2}$.

Negatively, using the arguments found in [Christ et al. 2003; Lindblad and Sogge 1995], one can show that the initial value problem (1-1) fails to be even locally well-posed for data lying in spaces less regular than $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, that is, any space $\dot{H}^s \times \dot{H}^{s-1}$, $s < \frac{1}{2}$.

Positively:

Lemma 1.1. *Equation* (1-1) *is locally well-posed in* $\dot{H}^s \times \dot{H}^{s-1}$ *for any* $s \geq \frac{1}{2}$.

Proof. See [Lindblad and Sogge 1995]. ☐

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Local well-posedness is defined in the usual way.

Definition (locally well-posed). The initial value problem (1-1) is said to be locally well-posed if there exists an open interval $I \subset \mathbb{R}$ containing 0 such that:

(1) A unique solution $u \in L_t^\infty \dot{H}^s(I \times \mathbb{R}^3) \cap L_{t,\mathrm{loc}}^4 L_x^4(I \times \mathbb{R}^3)$, $u_t \in L_t^\infty \dot{H}^{s-1}(I \times \mathbb{R}^3)$ exists.

(2) u is continuous in time, $u \in C(I; \dot{H}^s(\mathbb{R}^3))$, $u_t \in C(I; \dot{H}^{s-1}(\mathbb{R}^3))$.

(3) u depends continuously on the initial data. That is, for any compact $J \subset I$, if $\|u_0 - u_0^*\|_{\dot{H}^s} < \epsilon$ and $\|u_1 - u_1^*\|_{\dot{H}^{s-1}} < \epsilon$ for some $\epsilon < \epsilon_0(J) > 0$ sufficiently small, then

$$\|u^* - u\|_{L_{t,x}^4(J \times \mathbb{R}^3)} + \|u^* - u\|_{L_t^\infty \dot{H}^s(J \times \mathbb{R}^3)} + \|u_t^* - u_t\|_{L_t^\infty \dot{H}^{s-1}(J \times \mathbb{R}^3)} \leq \delta(\epsilon), \tag{1-5}$$

where u is the unique solution with initial data (u_0, u_1) and u^* is the solution with initial data (u_0^*, u_1^*) and $\delta(\epsilon)$ is a continuous function of ϵ with $\delta(0) = 0$.

The defocusing, energy-critical nonlinear wave equation, obtained from (1-1) either by changing $-u^3$ to $-u^5$ or by changing from three dimensions to four has now been completely worked out. For initial data in the energy class (which occurs for $u_0 \in \dot{H}^1$ and $u_1 \in L^2$) a priori bounds on scattering norms and concentration compactness properties of solutions have been established in [Struwe 1988; Grillakis 1990; Ginibre et al. 1992; Shatah and Struwe 1993; Bahouri and Shatah 1998; Bahouri and Gérard 1999; Nakanishi 1999; Tao 2006b].

Remark. The focusing case (obtained by changing the sign of the nonlinearity) is considerably more complicated. Focusing problems are not addressed at all in this paper, and so the interested reader is referred to [Kenig 2015].

For the radial version of (1-1) in three dimensions,

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + u^3 = u_{tt} - \frac{1}{r}\partial_{rr}(ru) + u^3 = 0, \qquad u(0,r) = u_0(r), \quad u_t(0,r) = u_1(r), \tag{1-6}$$

the lack of control of the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ norm is the only obstacle to proving global well-posedness and scattering for (1-6). Indeed:

Theorem 1.2. *Suppose $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ and $u_1 \in \dot{H}^{-1/2}(\mathbb{R}^3)$ are radial functions, and u solves* (1-1) *on a maximal interval $0 \in I \subset \mathbb{R}$ with*

$$\sup_{t \in I} \|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|u_t(t)\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} < \infty. \tag{1-7}$$

Then $I = \mathbb{R}$ and the solution u scatters both forward and backward in time.

Proof. See [Dodson and Lawrie 2015]. □

Scattering is also defined in the usual way.

Definition (scattering). A solution to (1-1) is said to scatter forward in time if there exist some $u_0^+ \in \dot{H}^{1/2}$, $u_1^+ \in \dot{H}^{-1/2}$ such that

$$\lim_{t \to \infty} \|u(t) - S(t)(u_0^+, u_1^+)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|u_t(t) - \partial_t S(t)(u_0^+, u_1^+)\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} = 0, \tag{1-8}$$

where $u(t) = S(t)(u_0, u_1)$ is the solution to the linear wave equation

$$u_{tt} - \Delta u = 0, \qquad u(0,x) = u_0, \quad u_t(0,x) = u_1. \tag{1-9}$$

A solution to (1-1) is said to scatter backward in time if there exist $u_0^- \in \dot{H}^{1/2}$, $u_1^- \in \dot{H}^{-1/2}$ such that

$$\lim_{t \to -\infty} \|u(t) - S(t)(u_0^-, u_1^-)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|u_t(t) - \partial_t S(t)(u_0^-, u_1^-)\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} = 0. \tag{1-10}$$

A solution which scatters both forward and backward in time is called scattering.

In this paper, (1-1) is proved to be globally well-posed and scattering for initial data lying in a critical space. The proof of global well-posedness is fairly general, and could be applied to a broad range of nonlinearities. The proof of scattering utilizes hyperbolic coordinates and relies on the fact that the cubic exponent $3 = (d+3)/(d-1)$ is the conformal exponent in three dimensions.

Hyperbolic coordinates were utilized by Tataru [2001] to prove weighted Strichartz estimates that extended previous results of [Georgiev et al. 1997]. Miao et al. [2018] recently proved a result similar to Theorem 1.4 for the five-dimensional problem, also for the conformal exponent. The conformal exponent is $\dot{H}^{1/2}$-critical, and it is straightforward to prove that the energy of a solution to (1-1) in hyperbolic coordinates scales like the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ norm.

Recently, Shen [2017], also working in hyperbolic coordinates, was able to prove a scattering result for data lying in a weighted energy space. Later, Dodson [2016] combined the result of Shen [2017] with the I-method, proving:

Theorem 1.3. *Suppose there exists a positive constant $\epsilon > 0$ such that*

$$\|u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbb{R}^3)} + \||x|^{2\epsilon} u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbb{R}^3)} \le A < \infty, \tag{1-11}$$

$$\|u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} + \||x|^{2\epsilon} u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} \le A < \infty. \tag{1-12}$$

Then (1-1) *has a global solution and there exists some $C(A,\epsilon) < \infty$ such that*

$$\int_{\mathbb{R}} \int (u(t,x))^4 \, dx \, dt \le C(A,\epsilon), \tag{1-13}$$

which proves that u scatters both forward and backward in time.

Remark. A straightforward application of the Strichartz estimates of [Ginibre and Velo 1995; Strichartz 1977] shows that

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty \tag{1-14}$$

is equivalent to scattering.

Note that conditions (1-11) and (1-12) fall just short of lying in the critical Sobolev space $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, and are not quite invariant under the scaling (1-3). In this paper we will study the radial, nonlinear wave equation in three dimensions,

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + u^3 = 0, \qquad u_0 \in B^2_{1,1}, \quad u_1 \in B^1_{1,1}. \tag{1-15}$$

The Besov spaces $B^s_{q,r}$ will be defined in the next section. By the Sobolev embedding theorem, this space is a subspace of $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, and the norm is invariant under (1-3).

We believe that this is the first result in which large-data scattering was proved for initial data in a scale-invariant space for which the norm was not controlled by a conserved quantity.

Theorem 1.4. *The initial value problem* (1-1) *is globally well-posed and scattering for* $u_0 \in B^2_{1,1}(\mathbb{R}^3)$, *radial*, *and* $u_1 \in B^1_{1,1}(\mathbb{R}^3)$, *radial. Moreover*,

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \leq C(\|u_0\|_{B^2_{1,1}}, \|u_1\|_{B^1_{1,1}}). \tag{1-16}$$

The proof of this theorem utilizes the fact that the free solution with such initial data is only singular at the origin in space and time, $t = 0$ and $x = 0$. Thus, using a Gronwall-type inequality, the local solution to (1-1) can be extended to a global solution that is the sum of a solution to the free wave equation combined with a finite energy term. A Morawetz estimate in hyperbolic coordinates then proves scattering.

The proof of Theorem 1.4 will occupy the remainder of this paper. In Section 2 we will begin by defining the Besov spaces and recalling basic Strichartz estimates. Then in Section 3 the local theory of (1-1) will be discussed. Global well-posedness will then be proved in Section 4. In Section 5 we will switch to hyperbolic coordinates to prove scattering. Finally in Section 6 we will use a profile decomposition to show that the bounds obtained for any $u_0 \in B^2_{1,1}$, $u_1 \in B^1_{1,1}$ depend only on size.

2. Besov spaces and linear estimates

We now present some harmonic analysis estimates that will be used in this paper. None of these results are new.

Theorem 2.1 (Hardy–Littlewood–Sobolev inequality). *For any* $0 < s < 1$, *if*

$$\frac{1}{q} = \frac{1}{p} + s - 1,$$

then

$$\left\| \frac{1}{|t|^s} * F(t) \right\|_{L^q(\mathbb{R})} \lesssim_s \|F\|_{L^q(\mathbb{R})}. \tag{2-1}$$

Definition (Littlewood–Paley decomposition). Let $\phi \in C_0^\infty(\mathbb{R}^3)$ be a radial, decreasing function supported on $|x| \leq 2$ and $\phi(x) = 1$ for $|x| \leq 1$. Then for any $j \in \mathbb{Z}$ let P_j be the Fourier multiplier

$$P_j f = \mathcal{F}^{-1}\big((\phi(2^{-j}\xi) - \phi(2^{-j+1}\xi))\hat{f}(\xi)\big), \tag{2-2}$$

where

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} f(x)\, dx, \tag{2-3}$$

$$\mathcal{F}^{-1}g = (2\pi)^{-d/2} \int e^{ix\cdot\xi} g(\xi)\, d\xi. \tag{2-4}$$

Then for any Schwartz function f,

$$f = \sum_{j\in\mathbb{Z}} P_j f. \tag{2-5}$$

Let $K_j(x)$ be the kernel of the Littlewood–Paley multiplier P_j. By direct computation using stationary phase estimates, for any N,

$$|K_j(x)| \lesssim_{d,N} \frac{2^{jd}}{(1+2^j|x|)^N}. \tag{2-6}$$

This implies K_j has an L^1 norm that is uniformly bounded in j, so for any $1 \leq p \leq \infty$,

$$\|P_j f\|_{L^p(\mathbb{R}^d)} \lesssim_d \|f\|_{L^p(\mathbb{R}^d)}. \tag{2-7}$$

A direct computation also gives Bernstein's inequality

$$\|P_j f\|_{L^p(\mathbb{R}^d)} \lesssim_d 2^{-j}\|\nabla f\|_{L^p(\mathbb{R}^d)}, \tag{2-8}$$

along with the Sobolev embedding estimate, for $1 \leq p \leq q \leq \infty$,

$$\|P_j f\|_{L^q(\mathbb{R}^d)} \lesssim_d 2^{jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}. \tag{2-9}$$

The Littlewood–Paley decomposition is foundational to the definition of Besov spaces.

Definition (Besov spaces). Suppose $1 \leq p \leq \infty$, $1 \leq r \leq \infty$, and $s \geq 0$. Then

$$\|f\|_{B^s_{r,p}(\mathbb{R}^d)} = \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|P_j f\|^r_{L^p(\mathbb{R}^d)}\right)^{1/r}. \tag{2-10}$$

The Besov space $B^s_{r,p}$ is then the completion of the Schwartz space under this norm. $B^s_{r,p}$ is a Banach space under this topology.

Remark. Observe that for any $s \in \mathbb{R}$ we have $B^s_{2,2}(\mathbb{R}^d) = \dot{H}^s(\mathbb{R}^d)$.

The Besov spaces are well-behaved with respect to multiplying by smooth cutoff functions.

Lemma 2.2. *Suppose $\chi(x) \in C_0^\infty(\mathbb{R}^3)$. Then*

$$\|\chi(x)u\|_{B^{1/2}_{1,2}(\mathbb{R}^3)} \lesssim \|u\|_{B^{1/2}_{1,2}(\mathbb{R}^3)}, \tag{2-11}$$

$$\|\chi(x)u\|_{B^{-1/2}_{1,2}(\mathbb{R}^3)} \lesssim \|u\|_{B^{-1/2}_{1,2}(\mathbb{R}^3)}. \tag{2-12}$$

Also if $\chi(x) = 1$ on $|x| \leq 1$ then if $u_0 \in B^2_{1,1}$ and $u_1 \in B^1_{1,1}$, we have

$$\lim_{R\to\infty}\left\|\left(1-\chi\left(\frac{x}{R}\right)\right)u_0\right\|_{B^{1/2}_{1,2}(\mathbb{R}^3)} + \left\|\left(1-\chi\left(\frac{x}{R}\right)\right)u_1\right\|_{B^{-1/2}_{1,2}(\mathbb{R}^3)} = 0. \tag{2-13}$$

Proof. Split $P_j(\chi f)$ as

$$P_j(\chi f) = \chi(P_j f) + [P_j, \chi]f. \tag{2-14}$$

By Hölder's inequality,

$$\sum_j 2^{j/2}\|\chi(P_j f)\|_{L^2} \lesssim \sum_j 2^{j/2}\|P_j f\|_{L^2}, \tag{2-15}$$

so it only remains to compute

$$\sum_j 2^{j/2}\|[P_j, \chi]f\|_{L^2}. \tag{2-16}$$

By (2-6) and the fundamental theorem of calculus,

$$\int K_j(x-y)[\chi(y)f(y)-\chi(x)f(y)] \lesssim \sum_k \int |K_j(x-y)||x-y||P_k f(y)|\,dy, \tag{2-17}$$

and therefore by Bernstein's inequality and (2-6),

$$\sum_{j\geq 0} 2^{j/2}\|P_j(\chi P_{\geq 0} f)\|_{L^2} \lesssim \sum_{j\geq 0} 2^{-j/2}\sum_{k\geq 0} 2^{-k/2}\|P_k f\|_{\dot H^{1/2}} \lesssim \sum_k 2^{k/2}\|P_k f\|_{\dot H^{1/2}} \lesssim \|f\|_{B^{1/2}_{1,2}}. \tag{2-18}$$

Also by Bernstein's inequality, the Sobolev embedding theorem, and Hölder's inequality,

$$\sum_{j\geq 0} 2^{j/2}\|P_j(\chi P_{\leq 0} f)\|_{L^2} \lesssim \|\nabla(\chi(P_{\leq 0} f))\|_{L^2} \lesssim \|f\|_{B^{1/2}_{1,2}}. \tag{2-19}$$

Next, by Hölder's inequality in space,

$$\|P_{\leq 0}(\chi(P_{\leq 0} f))\|_{B^{1/2}_{1,2}} \lesssim \|P_{\leq 0} f\|_{L^6} \lesssim \|f\|_{B^{1/2}_{1,2}}. \tag{2-20}$$

Finally,

$$\|P_{\leq 0}(\chi(P_{\geq 0} f))\|_{B^{1/2}_{1,2}} \lesssim \sum_{k\geq 0} 2^{-k/2}\|P_k f\|_{\dot H^{1/2}} \lesssim \|f\|_{B^{1/2}_{1,2}}. \tag{2-21}$$

Combining (2-18)–(2-21), we have proved

$$\|\chi f\|_{B^{1/2}_{1,2}(\mathbb{R}^3)} \lesssim \|f\|_{B^{1/2}_{1,2}(\mathbb{R}^3)}. \tag{2-22}$$

Also observe that (2-18)–(2-21) imply

$$\|\chi f\|_{B^{1/2}_{\infty,2}} \lesssim \|f\|_{B^{1/2}_{\infty,2}}, \tag{2-23}$$

and therefore by duality

$$\|\chi g\|_{B^{-1/2}_{1,2}} \lesssim \|g\|_{B^{-1/2}_{1,2}}. \tag{2-24}$$

To prove (2-13) observe that $B^{1/2}_{1,2} \times B^{-1/2}_{1,2}$ is invariant under the scaling (1-3); that is,

$$\begin{aligned}\left\|\left(1-\chi\left(\frac{x}{R}\right)\right)u_0\right\|_{B^{1/2}_{1,2}} &+ \left\|\left(1-\chi\left(\frac{x}{R}\right)\right)u_1\right\|_{B^{-1/2}_{1,2}} \\ &= R\|(1-\chi(x))u_0(Rx)\|_{B^{1/2}_{1,2}} + R^2\|(1-\chi(x))u_1(Rx)\|_{B^{-1/2}_{1,2}}.\end{aligned} \tag{2-25}$$

The dominated convergence theorem, (2-19), and (2-20) imply

$$\lim_{R\to\infty} R\|(1-\chi(x))P_{\leq 0}(u_0(Rx))\|_{B^{1/2}_{1,2}} + R^2\|(1-\chi(x))P_{\leq 0}(u_1(Rx))\|_{B^{-1/2}_{1,2}} = 0. \tag{2-26}$$

Meanwhile, (2-18), (2-21), and the dominated convergence theorem imply

$$\begin{aligned}&\lim_{R\to\infty} R\|(1-\chi(x))P_{\geq 0}(u_0(Rx))\|_{B^{1/2}_{1,2}} + R^2\|(1-\chi(x))P_{\geq 0}(u_1(Rx))\|_{B^{-1/2}_{1,2}} \\ &= \lim_{R\to\infty} R\sum_{j\geq 0} 2^{j/2}\|(1-\chi(x))P_j(u_0(Rx))\|_{L^2} + R^2\sum_{j\geq 0} 2^{-j/2}\|(1-\chi(x))P_j(u_1(Rx))\|_{L^2} = 0,\end{aligned} \tag{2-27}$$

completing the proof. □

Theorem 2.3 (radial Sobolev embedding theorem). *For any* j,

$$\| |x| P_j f \|_{L^\infty(\mathbb{R}^3)} \lesssim \| P_j f \|_{\dot{H}^{1/2}(\mathbb{R}^3)}. \tag{2-28}$$

Proof. Since f is radial, assume without loss of generality $x = (0, 0, |x|)$. Writing ξ in polar coordinates, $\xi = (r \cos\varphi \cos\theta, r \sin\varphi \cos\theta, r \sin\theta)$, $x \cdot \xi = |x| r \sin\theta$, so by the Fourier inversion formula,

$$f(x) = (2\pi)^{-1/2} \int_0^\infty \hat{f}(r) r^2 \int_{-\pi/2}^{\pi/2} e^{i|x|r\sin\theta} \cos\theta \, d\theta \, dr. \tag{2-29}$$

Then by a change of variables,

$$\begin{aligned} f(x) &= (2\pi)^{-1/2} \int_0^\infty r^2 \hat{f}(r) \int_{-1}^1 e^{i|x|ru} \, du \, dr \\ &= (2\pi)^{-1/2} \frac{1}{i|x|} \int_0^\infty \hat{f}(r) r [e^{i|x|r} - e^{-i|x|r}] \, dr. \end{aligned} \tag{2-30}$$

Replacing f by $P_j f$,

$$i|x| P_j f(x) = (2\pi)^{-1/2} \int_0^\infty \widehat{P_j f}(r) r [e^{i|x|r} - e^{-i|x|r}] \, dr. \tag{2-31}$$

By Plancherel's theorem,

$$\int_0^\infty |\widehat{P_j f}(r)|^2 r^3 \, dr \sim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2, \tag{2-32}$$

so by Hölder's inequality and the support of P_j,

$$(2\text{-}31) \lesssim \|P_j f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2, \tag{2-33}$$

completing the proof. □

Now observe that the solution to the free wave equation

$$u_{tt} - \Delta u = 0, \qquad u(0, x) = f(x), \quad u_t(0, x) = g(x), \tag{2-34}$$

is given by the Fourier multiplier

$$u(t, x) = \mathcal{F}^{-1}(\cos(t|\xi|) \hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi)) = S(t)(f, g). \tag{2-35}$$

Then the solution to

$$u_{tt} - \Delta u = F, \qquad u(0, x) = f(x), \quad u_t(0, x) = g(x), \tag{2-36}$$

is given by

$$S(t)(f, g) + \int_0^t S(t - \tau)(0, F) \, d\tau. \tag{2-37}$$

Remark. Sometimes, if $u = S(t)(f, g)$ it is convenient to write

$$(u(t), \partial_t u(t)) = S(t)(f, g). \tag{2-38}$$

By standard stationary phase calculations:

Theorem 2.4 (dispersive estimate).

$$\|S(t)(f,g)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t}[\|\nabla^2 f\|_{L^1(\mathbb{R}^3)} + \|\nabla g\|_{L^1(\mathbb{R}^3)}]. \tag{2-39}$$

This has been proved in many textbooks. See for example [Evans 2010].

The dispersive estimates can be used to prove Strichartz estimates.

Theorem 2.5. *Let $I \subset \mathbb{R}$, $t_0 \in I$, be an interval and let u solve the linear wave equation*

$$u_{tt} - \Delta u = F, \qquad u(t_0) = u_0, \quad u_t(t_0) = u_1. \tag{2-40}$$

Then we have the estimates

$$\begin{aligned}\|u\|_{L_t^p L_x^q(I\times\mathbb{R}^3)} + \|u\|_{L_t^\infty \dot{H}^s(I\times\mathbb{R}^3)} + \|u_t\|_{L_t^\infty \dot{H}^{s-1}(I\times\mathbb{R}^3)} \\ \lesssim_{p,q,s,\tilde{p},\tilde{q}} \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I\times\mathbb{R}^3)},\end{aligned} \tag{2-41}$$

whenever $s \geq 0$, $2 \leq p, \tilde{p} \leq \infty$, $2 \leq q, \tilde{q} < \infty$, and

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leq \frac{1}{2}. \tag{2-42}$$

Proof. See for example [Tao 2006a]. □

Remark. This theorem can easily be combined with the Christ–Kiselev lemma, see [Smith and Sogge 2000], and the fact that $|\nabla|$ commutes with the operator $(\partial_{tt} - \Delta)$ to prove many additional estimates.

Lemma 2.6 (perturbation lemma). *Let $I \subset \mathbb{R}$ be a time interval. Let $t_0 \in I$, $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$ and M, A, A' be positive constants. Let $\tilde{u}$ solve the equation*

$$(\partial_{tt} - \Delta)\tilde{u} = F(\tilde{u}) = e \tag{2-43}$$

on $I \times \mathbb{R}^3$, and also suppose $\sup_{t\in I} \|(\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \leq A$, $\|\tilde{u}\|_{L^4_{t,x}(I\times\mathbb{R}^3)} \leq M$,

$$\|(u_0 - \tilde{u}(t_0), u_1 - \partial_t \tilde{u}(t_0))\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \leq A', \tag{2-44}$$

and

$$\|e\|_{L^{4/3}_{t,x}(I\times\mathbb{R}^3)} + \|S(t-t_0)(u_0 - \tilde{u}(t_0), u_1 - \partial_t \tilde{u}(t_0))\|_{L^4_{t,x}(I\times\mathbb{R}^3)} \leq \epsilon. \tag{2-45}$$

Then there exists $\epsilon_0(M, A, A')$ such that if $0 < \epsilon < \epsilon_0$ then there exists a solution to (1-1) *on I with $(u(t_0), \partial_t u(t_0)) = (u_0, u_1)$, $\|u\|_{L^4_{t,x}(I\times\mathbb{R}^3)} \leq C(M, A, A')$, and for all $t \in I$,*

$$\|(u(t), \partial_t u(t)) - (\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \leq C(A, A', M)(A' + \epsilon). \tag{2-46}$$

Proof. The method of proof is by now fairly well known. See for example Lemma 2.20 of [Kenig and Merle 2008]. □

3. Local theory

By the dominated convergence theorem, for any $u_0 \in B^2_{1,1},\ u_1 \in B^1_{1,1}$, and $\delta > 0$ there exists some $j_0(\delta) < \infty$ such that

$$\sum_{j \geq j_0} 2^{2j} \|P_j u_0\|_{L^1(\mathbb{R}^3)} + \sum_{j \geq j_0} 2^j \|P_j u_1\|_{L^1(\mathbb{R}^3)} < \delta. \tag{3-1}$$

Then by the rescaling (1-3) with $\lambda = 2^{-j_0}$, if $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$ and $u_{1,\lambda}(x) = \lambda^2 u_1(\lambda x)$,

$$\sum_{j \geq 0} 2^{2j} \|P_j u_{0,\lambda}\|_{L^1(\mathbb{R}^3)} + \sum_{j \geq 0} 2^j \|P_j u_{1,\lambda}\|_{L^1(\mathbb{R}^3)} < \delta. \tag{3-2}$$

To simplify notation let u_0 and u_1 refer to the $u_{0,\lambda}$ and $u_{1,\lambda}$ such that (3-2) holds.

Lemma 3.1. *Fix $\epsilon_0 > 0$ small. There exists some $\delta(\epsilon, \|u_0\|_{B^2_{1,1}}, \|u_1\|_{B^1_{1,1}}) > 0$ such that if* (3-2) *holds then*

$$\|u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \epsilon, \tag{3-3}$$

$$\|u\|_{L^\infty_t B^{1/2}_{1,2}([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{3-4}$$

Proof. Assume that (3-2) holds for some $\delta \ll \epsilon_0$. By the Sobolev embedding theorem and the definition of Besov spaces (see page 1027),

$$\|S(t)(P_{\leq 0} u_0, P_{\leq 0} u_1)\|_{L^4_x(\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}, \tag{3-5}$$

while by Theorem 2.5, (3-1), and (3-2),

$$\|S(t)(P_{\geq 0} u_0, P_{\geq 0} u_1)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \delta. \tag{3-6}$$

Taking $\delta > 0$ sufficiently small, (3-5) and (3-6) imply

$$\|S(t)(u_0, u_1)\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \epsilon_0. \tag{3-7}$$

Then by the contraction mapping principle and Theorem 2.5,

$$\|u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \|S(t)(u_0, u_1)\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} + \|u\|^3_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)}, \tag{3-8}$$

which when $\epsilon_0 > 0$ is sufficiently small implies

$$\|u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \epsilon_0. \tag{3-9}$$

Next observe that by Theorem 2.5 we also have

$$\||\nabla|^{1/4} u\|_{L^8_t L^{8/3}_x([-\delta,\delta]\times\mathbb{R}^3)} + \||\nabla|^{-1/4} u\|_{L^{8/3}_t L^8_x([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \epsilon_0 \tag{3-10}$$

and

$$\|P_j u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} + \|P_j u\|_{L^\infty_t \dot{H}^{1/2}([-\delta,\delta]\times\mathbb{R}^3)}$$
$$\lesssim \|P_j u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|P_j u_1\|_{\dot{H}^{-1/2}(\mathbb{R}^3)}$$
$$+2^{-j/2} \sum_{j_1 \le j_2 \le j-5} \sum_{j-3 \le j_3 \le j+3} \|P_{j_1}\|_{L^{8/3}_t L^8_x([-\delta,\delta]\times\mathbb{R}^3)} \|P_{j_2} u\|_{L^{8/3}_t L^8_x([-\delta,\delta]\times\mathbb{R}^3)} \|P_{j_3} u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)}$$
$$+2^{j/4} \sum_{j-5 \le j_1 \le j_2 \le j_3} \|P_{j_1} u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} \|P_{j_2} u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} \|P_{j_3} u\|_{L^8_t L^{8/3}_x([-\delta,\delta]\times\mathbb{R}^3)}. \quad (3\text{-}11)$$

Then by (3-9) and (3-10),

$$\sum_j \|P_j u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)} + \|P_j u\|_{L^\infty_t \dot{H}^{1/2}([-\delta,\delta]\times\mathbb{R}^3)}$$
$$\lesssim \sum_j \|P_j u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|P_j u_1\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} + \epsilon_0^2 \sum_j \|P_j u\|_{L^4_{t,x}([-\delta,\delta]\times\mathbb{R}^3)}, \quad (3\text{-}12)$$

which also implies

$$\|u\|_{L^\infty_t B^{1/2}_{1,2}([-\delta,\delta]\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}, \quad (3\text{-}13)$$

completing the proof. □

Next suppose $\chi(x)$ is a smooth function that is supported on $|x| \le 1$ and is equal to 1 on $|x| \le \frac{1}{2}$. By Lemma 2.2 there exists some $R(u_0, u_1, \epsilon)$ such that

$$\left\| \left(1 - \chi\left(\frac{x}{R}\right)\right) u_0 \right\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \left\| \left(1 - \chi\left(\frac{x}{R}\right)\right) u_1 \right\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \le \epsilon. \quad (3\text{-}14)$$

Remark. Notice that R depends on u_0 and u_1, not just their size. This dependence will be removed upon making a profile decomposition.

Again applying the scaling symmetry (1-3), this time with $\lambda = 2R$, setting

$$u_{0,\lambda}(x) = \lambda u_0(\lambda x), \quad u_{1,\lambda}(x) = \lambda^2 u_1(\lambda x), \quad (3\text{-}15)$$

and letting u_λ denote the solution to (1-1) with initial data $(u_{0,\lambda}, u_{1,\lambda})$ yields

$$\|P_{>2R} u_{0,\lambda}\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|P_{>2R} u_{1,\lambda}\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \le \epsilon, \quad (3\text{-}16)$$

$$\|(1-\chi(2x)) u_{0,\lambda}\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|(1-\chi(2x)) u_{1,\lambda}\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \le \epsilon, \quad (3\text{-}17)$$

$$\|u_\lambda\|_{L^4_{t,x}([-\frac{\delta}{2R}, \frac{\delta}{2R}]\times\mathbb{R}^3)} \lesssim \epsilon_0, \quad (3\text{-}18)$$

and finally

$$\|u_\lambda\|_{L^\infty_t B^{1/2}_{1,2}([-\frac{\delta}{2R}, \frac{\delta}{2R}]\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}. \quad (3\text{-}19)$$

The next step is to show that this local solution has a singularity that is isolated in a suitable sense. We will once again abuse notation and use u in place of u_λ in (3-18).

Observe that the dispersive estimates imply that the linear wave equation $u_{tt} - \Delta u = 0$ with initial data (u_0, u_1) lies in L^∞ when $t > 0$. Indeed, by (2-39),

$$\begin{aligned}\|S(t)(u_0,u_1)\|_{L^\infty} &\lesssim \frac{1}{t}\sum_j [2^{2j}\|P_j u_0\|_{L^1(\mathbb{R}^3)} + 2^j\|P_j u_1\|_{L^1(\mathbb{R}^3)}]\\ &\lesssim \frac{1}{t}[\|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}]. \end{aligned}\tag{3-20}$$

Interpolating (3-20) with Bernstein's inequality, for any j,

$$\|S(t)(P_j u_0, P_j u_1)\|_{L^6(\mathbb{R}^3)} \lesssim \frac{2^{-j/6}}{t^{2/3}}[2^{2j}\|P_j u_0\|_{L^1(\mathbb{R}^3)} + 2^j\|P_j u_1\|_{L^1(\mathbb{R}^3)}],\tag{3-21}$$

while by the Sobolev embedding theorem $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$\|S(t)P_j(u_0,u_1)\|_{L^6(\mathbb{R}^3)} \lesssim 2^{j/2}[2^{2j}\|P_j u_0\|_{L^1(\mathbb{R}^3)} + 2^j\|P_j u_1\|_{L^1(\mathbb{R}^3)}].\tag{3-22}$$

Thus by direct computation

$$\sup_{t>0} t^{1/2}\|S(t)(u_0,u_1)\|_{L^6(\mathbb{R}^3)} + \|S(t)(u_0,u_1)\|_{L^2_t L^6_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}.\tag{3-23}$$

Lemma 3.2. *If $\delta > 0$ is given by the local result in Lemma 3.1 for some $\epsilon_0 > 0$, then*

$$\sup_{-\frac{\delta}{2R}<t<\frac{\delta}{2R}} t^{1/2}\|u\|_{L^6_x(\mathbb{R}^3)} + \|u\|_{L^2_t L^6_x([-\frac{\delta}{2R},\frac{\delta}{2R}]\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}.\tag{3-24}$$

Proof. By the dispersive estimates (Theorem 2.4), the Hardy–Littlewood–Sobolev inequality, and interpolation

$$\begin{aligned}\Big\|\int_0^t S(t-\tau)F(u(\tau))\,d\tau\Big\|_{L^2_t L^6_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)} &\lesssim \||\nabla|^{1/3}F(u)\|_{L^{6/5}_{t,x}([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\\ &\lesssim \||\nabla|^{1/2}u\|^{2/3}_{L^\infty_t L^2_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\|u\|^{4/3}_{L^4_{t,x}([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\|u\|_{L^2_t L^6_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\\ &\lesssim \epsilon_0^{4/3}\|u\|_{L^2_t L^6_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}.\end{aligned}\tag{3-25}$$

Combining (3-25) with (3-23) proves

$$\|u\|_{L^2_t L^6_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}.\tag{3-26}$$

Next let $c > 0$ be a small constant to be determined later. Again by Theorem 2.4, the Hardy–Littlewood–Sobolev inequality, and interpolation,

$$\begin{aligned}\sup_{t\in[0,\frac{\delta}{2R}]} t^{1/2}\Big\|\int_0^{(1-c)t} S(t-\tau)F(u(\tau))\,d\tau\Big\|_{L^6(\mathbb{R}^3)} &\lesssim \frac{1}{c^{1/2}}\||\nabla|^{1/3}F(u)\|_{L^{6/5}_{t,x}([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\\ &\lesssim \frac{1}{c^{1/2}}\||\nabla|^{1/2}u\|^{2/3}_{L^\infty_t L^2_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\|u\|^{4/3}_{L^4_{t,x}([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\|u\|_{L^2_t L^6_x([0,\frac{\delta}{2R}]\times\mathbb{R}^3)}\\ &\lesssim \frac{\epsilon_0^{4/3}}{c^{1/2}}(\|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}).\end{aligned}\tag{3-27}$$

Also for any $t \in [0, \delta/(2R)]$, by Theorem 2.4,

$$t^{1/2} \left\| \int_{(1-c)t}^{t} S(t-\tau) F(u(\tau))\, d\tau \right\|_{L^6(\mathbb{R}^3)} \lesssim \Big(\sup_{t \in [0, \frac{\delta}{2R}]} t^{1/2} \|u(t)\|_{L^6(\mathbb{R}^3)} \Big)^{5/3} \| |\nabla|^{1/2} u \|^{2/3}_{L_t^\infty L_x^2([0,\frac{\delta}{2R}] \times \mathbb{R}^3)} \times \|u\|^{2/3}_{L_t^\infty L_x^3([0,\frac{\delta}{2R}]\times\mathbb{R}^3)} \cdot \int_{(1-c)t}^{t} \frac{1}{(t-\tau)^{2/3}} \frac{1}{t^{1/3}}\, d\tau \lesssim c^{1/3} \Big(\sup_{t \in [0, \frac{\delta}{2R}]} t^{1/2} \|u(t)\|_{L^6(\mathbb{R}^3)} \Big)^{5/3}. \tag{3-28}$$

Therefore, choosing $c > 0$ small and $\epsilon_0(c) > 0$ small,

$$\|u\|_{L_t^2 L_x^6([0,\frac{\delta}{2R}]\times\mathbb{R}^3)} + \sup_{0<t<\delta} t^{1/2} \|u(t)\|_{L^6} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}. \tag{3-29}$$

Then by time reversal symmetry the proof of Lemma 3.2 is complete. □

Next, we show that a local solution may be written as a sum of a term with bounded energy and a term with good dispersive properties. To simplify notation let $\delta_1 = \delta/(2R)$. By energy inequalities, Strichartz estimates (Theorem 2.5), and Lemma 3.2,

$$\left\| \int_{\frac{\delta_1}{10}}^{\delta_1} S(t-\tau) F(u(\tau))\, d\tau \right\|_{\dot H^1 \times L^2(\mathbb{R}^3)} \lesssim \|u\|^3_{L_t^3 L_x^6([\frac{\delta_1}{10},\delta_1]\times\mathbb{R}^3)} \lesssim \frac{1}{\delta_1^{1/2}}. \tag{3-30}$$

Next, by the radial Sobolev embedding theorem (Theorem 2.3) and (3-13), if $\chi \in C_0^\infty(\mathbb{R}^3)$ is supported on $|x| \le 1$, $\chi(x) = 1$ on $|x| \le \frac{1}{2}$, then

$$\left\| \left(1 - \chi\left(\frac{10x}{\delta_1}\right)\right) F(u) \right\|_{L_t^1 L_x^2([0,\frac{\delta_1}{10}]\times\mathbb{R}^3)} \lesssim \delta_1^{1/2} \left\| \left(1 - \chi\left(\frac{10x}{\delta_1}\right)\right) u \right\|_{L^\infty_{t,x}([0,\frac{\delta_1}{10}]\times\mathbb{R}^3)} \|u\|^2_{L^4_{t,x}([0,\frac{\delta_1}{10}]\times\mathbb{R}^3)} \lesssim \frac{1}{\delta_1^{1/2}}. \tag{3-31}$$

Now for $t > \delta_1$ let

$$v(t) = S(t) \chi\left(\frac{10x}{\delta_1}\right)(u_0, u_1) + \int_0^{\frac{\delta_1}{10}} S(t-\tau) \chi\left(\frac{10x}{\delta_1}\right) F(u(\tau))\, d\tau. \tag{3-32}$$

Combining Lemma 2.2 with

$$\|[P_j, \chi] F(u)\|_{L_t^1 \dot H^{-1/2}([-\frac{\delta_1}{10},\frac{\delta_1}{10}]\times\mathbb{R}^3)} \lesssim 2^{-j} \delta_1^{-1} \|F(u)\|_{L_t^1 L_x^{3/2}([-\frac{\delta_1}{10},\frac{\delta_1}{10}]\times\mathbb{R}^3)}, \tag{3-33}$$

Lemma 3.2,

$$\left\| P_{\le 0} \chi\left(\frac{10x}{\delta_1}\right) F(u) \right\|_{\dot H^{-1/2}(\mathbb{R}^3)} \lesssim \|u\|^3_{L_x^3(\mathbb{R}^3)}, \tag{3-34}$$

(3-10)–(3-13), the sharp Huygens principle, which implies v is supported on $\{(x,t) : ||x|-t| \leq \frac{1}{2}\delta_1\}$, and the radial Sobolev embedding theorem (Theorem 2.3), we have

$$\|v(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t}[\|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)}]. \tag{3-35}$$

This implies good properties of $S(t-\delta_1)(v(\delta_1), v_t(\delta_1))$.

Lemma 3.3. *Let $w(\delta_1)+v(\delta_1)=u(\delta_1)$ and let*

$$w(\delta_1) = S(\delta_1)\Big(1-\chi\Big(\frac{10x}{\delta_1}\Big)\Big)(u_0,u_1) + \int_0^{\frac{\delta_1}{10}} S(\delta_1-\tau)\Big(1-\chi\Big(\frac{10x}{\delta_1}\Big)\Big)F(u(\tau))\,d\tau + \int_{\frac{\delta_1}{10}}^{\delta_1} S(\delta_1-\tau)F(u(\tau))\,d\tau. \tag{3-36}$$

Then

$$\|w(\delta_1)\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} \lesssim \delta_1^{-1/2}. \tag{3-37}$$

Proof. By (3-31) and (3-32) it only remains to compute

$$\Big\|\Big(1-\chi\Big(\frac{10x}{\delta_1}\Big)\Big)u_0\Big\|_{\dot{H}^1(\mathbb{R}^3)} + \Big\|\Big(1-\chi\Big(\frac{10x}{\delta_1}\Big)\Big)u_1\Big\|_{L^2(\mathbb{R}^3)}. \tag{3-38}$$

First,

$$|u_1(0,r)| \lesssim \int_r^\infty |\partial_r u_1(0,s)|\,ds \lesssim \frac{1}{r^2}, \tag{3-39}$$

so

$$\int_{\frac{\delta_1}{10}}^\infty |u_1(r,0)|^2 r^2\,dr \lesssim \int_{\frac{\delta_1}{10}}^\infty |u_1(r,0)|\,dr \lesssim \frac{1}{\delta_1}. \tag{3-40}$$

Next, for any j, Bernstein's inequality implies $\|P_j u_0\|_{L^\infty} \lesssim 2^{3j}\|P_j u_0\|_{L^1}$, so

$$\int_0^{2^{-j}} \frac{r^2}{r}|\partial_r(P_j u_0)|\,dr \lesssim 2^{2j}\|P_j u_0\|_{L^1(\mathbb{R}^3)}, \tag{3-41}$$

while by Bernstein's inequality

$$\int_{2^{-j}}^\infty \frac{r^2}{r}|\partial_r(P_j u_0)|\,dr \lesssim 2^{2j}\|P_j u_0\|_{L^1(\mathbb{R}^3)}. \tag{3-42}$$

Thus $\|(1/r)\partial_r u_0\|_{L^1(\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)}$, and since u_0 is radially symmetric $\Delta u_0 = (\partial_{rr} + (2/r)\partial_r)u_0$, so $\|\partial_{rr}u_0\|_{L^1(\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)}$. By the fundamental theorem of calculus,

$$|u_r(r)| \leq \int_r^\infty |u_{rr}(s)|\,ds \lesssim \frac{1}{r^2}. \tag{3-43}$$

Therefore,

$$\int_{\frac{\delta_1}{10}}^\infty |u_r(r)|^2 r^2\,dr \lesssim \int_{\frac{\delta_1}{10}}^\infty |u_r(r)|\,dr \lesssim \frac{1}{\delta_1}, \tag{3-44}$$

and

$$\int_0^\infty (u_r(r))^2 r^3\,dr \le \int_0^\infty \Big(\int_r^\infty |u_{rr}(s)|\,ds\Big) r\,dr \lesssim \int_0^\infty |u_{rr}(s)| s^2\,ds < \infty, \tag{3-45}$$

completing the proof. □

4. Proof of global well-posedness

In this section we extend local well-posedness to global well-posedness, proving:

Theorem 4.1. *Equation* (1-1) *is globally well-posed*, *and for any compact interval* $J \subset \mathbb{R}$,

$$\|u\|_{L^4_{t,x}(J\times\mathbb{R}^3)} < \infty. \tag{4-1}$$

Proof. By time reversal symmetry, to prove this it suffices to show that the local well-posedness result of Lemma 3.1 can be extended to all times $t > \delta_1$. Throughout the proof the implicit constant depends on δ_1 and $\|u_0\|_{B^2_{1,1}} + \|u_1\|_{B^1_{1,1}}$.

For $t > \delta_1$ let

$$u(t) = w(t) + v(t), \tag{4-2}$$

where $v(t)$ is given by (3-32) and w solves

$$w_{tt} - \Delta w = -u^3. \tag{4-3}$$

By Strichartz estimates, (2-11), and (3-9),

$$\|v\|_{L^4_{t,x}([\delta_1,\infty)\times\mathbb{R}^3)} \lesssim \|u_0\|_{B^2_{1,1}} + \|u_1\|_{B^1_{1,1}}. \tag{4-4}$$

Thus to prove (4-1) it suffices to prove that $w \in L^4_x$ for all $t \in [\delta_1, \infty)$.

Copying (1-2) let $E(w(t))$ be the energy of w,

$$E(w(t)) = \frac{1}{2}\int |\nabla w(t,x)|^2\,dx + \frac{1}{2}\int (w_t(t,x))^2\,dx + \frac{1}{4}\int (w(t,x))^4\,dx. \tag{4-5}$$

By (3-19), (3-37), and the Sobolev embedding theorem, $w \in L^3 \cap L^6$, so

$$E(w(\delta_1)) \lesssim 1. \tag{4-6}$$

Next,

$$\begin{aligned}\frac{d}{dt}E(w(t)) &= \int ((w(t,x))^3 - (u(t,x))^3) w_t(t,x)\,dx \\ &= -\int w_t(t,x)[(v(t,x))^3 + 3v(t,x)^2 w(t,x) + 3v(t,x)w(t,x)^2]\,dx. \end{aligned}\tag{4-7}$$

It suffices to show that (3-35) and (4-4) give good bounds on the growth of $E(w(t))$. Indeed,

$$\int w_t(t,x)(w(t,x))^2 v(t,x)\,dx \lesssim \|v(t)\|_{L^\infty(\mathbb{R}^3)} \|w(t)\|^2_{L^4(\mathbb{R}^3)} \|w_t(t)\|_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{t} E(w(t)), \tag{4-8}$$

and

$$\int w_t(t,x)(v(t,x))^3\,dx \lesssim \|w_t(t)\|_{L^2(\mathbb{R}^3)}\|v(t)\|_{L^\infty(\mathbb{R}^3)}\|v(t)\|_{L^4(\mathbb{R}^3)}^2$$
$$\lesssim \frac{1}{t}E(w(t))^{1/2}\|v(t)\|_{L^4(\mathbb{R}^3)}^2. \tag{4-9}$$

Finally

$$\int w_t(t,x)v(t,x)^2 w(t,x)\,dx \lesssim \|w_t(t)\|_{L^2(\mathbb{R}^3)}\|w(t)\|_{L^4(\mathbb{R}^3)}\|v(t)\|_{L^4(\mathbb{R}^3)}\|v(t)\|_{L^\infty(\mathbb{R}^3)}$$
$$\lesssim \frac{1}{t}E(w(t))^{3/4}\|v(t)\|_{L^4(\mathbb{R}^3)}, \tag{4-10}$$

so by interpolation

$$\frac{d}{dt}E(w(t)) \lesssim \frac{1}{t}E(w(t)) + \frac{1}{t}\|v(t)\|_{L^4(\mathbb{R}^3)}^4. \tag{4-11}$$

Then by Gronwall's inequality and time reversal symmetry there exist constants $C_1(\|u_0\|_{B^2_{1,1}}, \|u_1\|_{B^1_{1,1}}, \delta_1)$ and $C_2(\|u_0\|_{B^2_{1,1}}, \|u_1\|_{B^1_{1,1}}, \delta_1)$ such that

$$E(w(t)) \lesssim C_1(1+|t|)^{C_2}. \tag{4-12}$$

This combined with (4-2) implies that u is global. □

5. Hyperbolic coordinates

Having shown that the solution to (1-1) is globally well-posed, the next step is to show that the solution scatters. It is possible to prove this by utilizing hyperbolic coordinates and the fact that by Theorem 4.1, $u(t,x)$ is well-defined for all $(t,x) \in \mathbb{R}^{1+3}$.

Theorem 5.1. *The global solution given in Theorem 4.1 scatters both forward and backward in time.*

Proof. By time reversal symmetry and (1-14), it suffices to show that

$$\int_0^\infty\int_0^\infty u(t,r)^4 r^2\,dr\,dt < \infty. \tag{5-1}$$

To begin setting up hyperbolic coordinates, translate $t_0 = 0$ in time to $t_0 = 1-\delta_1$. Let $\tilde{u}$ refer to the time-translated solution in Theorem 4.1,

$$\tilde{u}(t,r) = u(t-(1-\delta_1), r). \tag{5-2}$$

Once again, we make an abuse of notation and let u refer to $\tilde{u}$. Inequality (4-1) implies that after time translation

$$\int_0^1\int_0^\infty u(t,r)^4 r^2\,dr\,dt < \infty. \tag{5-3}$$

Next, by small-data arguments, see for example [Lindblad and Sogge 1995], the solution to (1-1) with initial data given by (3-14), has finite $L^4_{t,x}$ norm. Finite propagation speed and (3-14) imply

$$\int_1^\infty\int_{r>\frac{1}{2}+t} u(t,r)^4 r^2\,dr\,dt \lesssim \epsilon_0. \tag{5-4}$$

Therefore, it only remains to prove

$$\int_1^\infty \int_{r \leq \frac{1}{2} + t} u(t,r)^4 r^2 \, dr \, dt < \infty. \tag{5-5}$$

Such x and t fall within the domain, which may be described by hyperbolic coordinates. Let

$$\tilde{w}(\tau, s) = \frac{e^\tau \sinh s}{s} w(e^\tau \cosh s, e^\tau \sinh s), \tag{5-6}$$

$$\tilde{v}(\tau, s) = \frac{e^\tau \sinh s}{s} v(e^\tau \cosh s, e^\tau \sinh s), \tag{5-7}$$

$$\tilde{u}(\tau, s) = \frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s). \tag{5-8}$$

Now by a change of variables

$$\begin{aligned} \int_0^\infty \int_0^\infty \tilde{u}(\tau,s)^4 s^2 \left(\frac{s}{\sinh s}\right)^2 ds \, d\tau &= \int_0^\infty \int_0^\infty \frac{e^{4\tau} (\sinh s)^4}{s^4} \left(\frac{s^4}{(\sinh s)^2}\right) u(e^\tau \cosh s, e^\tau \sinh s)^4 \, ds \, d\tau \\ &= \iint_{t^2 - r^2 \geq 1} u(t,r)^4 r^2 \, dr \, dt. \end{aligned} \tag{5-9}$$

The analogous estimate also holds for v and w. Since (4-1) implies

$$\int_1^2 \int u(t,r)^4 r^2 \, dr \, dt < \infty, \tag{5-10}$$

proving (5-5) is equivalent to proving

$$\int_0^\infty \int_0^\infty \tilde{u}(\tau,s)^4 \left(\frac{s}{\sinh s}\right)^2 s^2 \, ds \, d\tau < \infty. \tag{5-11}$$

Also, since v is a solution to the linear wave equation with data in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, it suffices to prove

$$\int_0^\infty \int_0^\infty \tilde{w}(\tau,s)^4 \left(\frac{s}{\sinh s}\right)^2 s^2 \, ds \, d\tau < \infty. \tag{5-12}$$

Direct computation and (4-3) imply that $\tilde{w}$ solves

$$\partial_{\tau\tau} \tilde{w} - \partial_{ss} \tilde{w} - \frac{2}{s} \partial_s \tilde{w} = -\left(\frac{s}{\sinh s}\right)^2 \tilde{u}^3. \tag{5-13}$$

Moreover, $\tilde{w}$ has bounded hyperbolic energy

$$\frac{1}{2} \int_0^\infty \tilde{w}_s(\tau,s)^2 s^2 \, ds + \frac{1}{2} \int_0^\infty \tilde{w}_\tau(\tau,s)^2 s^2 \, ds + \frac{1}{4} \int_0^\infty \left(\frac{s}{\sinh s}\right)^2 \tilde{w}(\tau,s)^4 s^2 \, ds. \tag{5-14}$$

Indeed:

Lemma 5.2. *There exists some* $0 < \tau < \delta_1$ *such that*

$$E(\tilde{w}(\tau)) < \infty. \tag{5-15}$$

Proof. To prove (5-15) it suffices to show

$$\frac{1}{\delta_1}\int_0^{\delta_1} E(\tilde{w}(\tau))\,d\tau < \infty. \tag{5-16}$$

By the hyperbolic Pythagorean theorem,

$$e^{\tau}(\cosh(s)-\sinh(s)) = e^{\tau}\frac{\cosh^2(s)-\sinh^2(s)}{\cosh(s)+\sinh(s)} = \frac{e^{\tau}}{\cosh(s)+\sinh(s)}. \tag{5-17}$$

Therefore, for any fixed $\tau > 0$,

$$\lim_{s\nearrow\infty} e^{\tau}(\cosh(s)-\sinh(s)) = 0. \tag{5-18}$$

Combining (3-32) with the fact that $t_0 = 0$ was translated to $t_0 = 1-\delta_1$, for any $t \geq 1$,

$$v(t) = S(t-1+\delta_1)\chi\left(\frac{10x}{\delta_1}\right)(u_0,u_1) + \int_0^{\frac{\delta_1}{10}} S(t-1+\delta_1-\tau)\chi\left(\frac{10x}{\delta_1}\right)F(u(\tau))\,d\tau. \tag{5-19}$$

Therefore, by finite propagation speed there exists some s_0 such that, for any $0 < \tau < \delta_1$,

$$\int_{s_0}^{\infty} s^2\tilde{w}_s(\tau,s)^2\,ds + \int_{s_0}^{\infty} s^2\tilde{w}_{\tau}(\tau,s)\,ds = \int_{s_0}^{\infty} s^2\tilde{u}_s(\tau,s)^2\,ds + \int_{s_0}^{\infty} s^2\tilde{u}_{\tau}(\tau,s)\,ds. \tag{5-20}$$

Now, if u is a radial solution to (1-1), then by (5-8),

$$\begin{aligned} s\tilde{u}(\tau,s) &= e^{\tau}(\sinh s)u(e^{\tau}\cosh s, e^{\tau}\sinh s) \\ &= \frac{1}{2}(e^{\tau+s}-(1-\delta_1))u_0(e^{\tau+s}-(1-\delta_1)) + \frac{1}{2}(1-\delta_1-e^{\tau-s})u_0(1-\delta_1-e^{\tau-s}) \\ &\quad + \frac{1}{2}\int_{e^{\tau-s}+(1-\delta_1)}^{e^{\tau+s}-(1-\delta_1)} u_1(r)r\,dr + \frac{1}{2}\int_{1-\delta_1}^{e^{\tau}\cosh s}\int_{-e^{\tau-s}+t}^{e^{\tau+s}-t} ru^3(t,r)\,dr\,dt. \end{aligned} \tag{5-21}$$

Now by direct computation,

$$\begin{aligned} \partial_{\tau+s}(s\tilde{u}(\tau,s))|_{\tau=0} &= \frac{e^s}{2}u_0(e^s-(1-\delta_1)) + \frac{e^s}{2}(e^s-(1-\delta_1))u_0'(e^s-(1-\delta_1)) \\ &\quad + \frac{e^s}{2}(e^s-(1-\delta_1))u_1(e^s-(1-\delta_1)) + \frac{e^s}{2}\int_{1-\delta_1}^{\cosh s}(e^s-t)u^3(t,e^s-t)\,dt, \end{aligned} \tag{5-22}$$

and

$$\begin{aligned} &\partial_{\tau-s}(s\tilde{u}(\tau,s))|_{\tau=0} \\ &\quad = \frac{e^{-s}}{2}u_0((1-\delta_1)-e^{-s}) + \frac{e^{-s}}{2}((1-\delta_1)-e^{-s})u_0'((1-\delta_1)-e^{-s}) \\ &\qquad + \frac{e^{-s}}{2}((1-\delta_1)-e^{-s})u_1((1-\delta_1)-e^{-s}) + \frac{e^{-s}}{2}\int_{1-\delta_1}^{e^{\tau}\cosh s}(t-e^{-s})u^3(t,t-e^{-s})\,dt. \end{aligned} \tag{5-23}$$

First, making a change of variables and using (3-39)–(3-45),

$$\int_{s_0}^{\infty} e^{2s} u_0(e^s-(1-\delta_1))^2\,ds \lesssim \int_0^{\infty} u_0(r)^2 r\,dr < \infty, \tag{5-24}$$

$$\int_{s_0}^{\infty} e^{2s}(e^s-(1-\delta_1))^2 u_0'(e^s-(1-\delta_1))^2\,ds \lesssim \int_0^{\infty} (\partial_r u_0(r))^2 r^3\,dr < \infty, \tag{5-25}$$

$$\int_{s_0}^{\infty} e^{2s}(e^s-(1-\delta_1))^2 u_1(e^s-(1-\delta_1))^2\,ds \lesssim \int_0^{\infty} r^3 u_1(r)^2\,dr < \infty. \tag{5-26}$$

Also, by (3-39)–(3-45), the fact that $e^{\tau}(\cosh s-\sinh s) \le \frac{1}{4}$ when $s \ge s_0$ and $0<\tau<\delta_1$, $|ru(t,r)| \lesssim 1$ for $t-r \le \frac{1}{2}$, (5-3), and (5-4), we have

$$\begin{aligned}\int_{s_0}^{\infty} e^{2s}\biggl(\int_{1-\delta_1}^{\cosh s} (e^s-t)u^3(t,e^s-t)\,dt\biggr)^2 ds &\lesssim \int_{s_0}^{\infty}\int_{1-\delta_1}^{\cosh s} e^{3s}(e^s-t)^2 u^6(t,e^s-t)\,dt\,ds \\ &\lesssim \int_{s_0}^{\infty}\int_{1-\delta_1}^{\cosh(s)} e^{3s} u^4(t,e^s-t)\,dt\,ds < \infty.\end{aligned} \tag{5-27}$$

Additionally,

$$\int_{s_0}^{\infty} e^{-2s} u_0((1-\delta_1)-e^{-s})^2\,ds \lesssim \int_{s_0}^{\infty} e^{-2s}\,ds < \infty, \tag{5-28}$$

$$\int_{s_0}^{\infty} e^{-2s}((1-\delta_1)-e^{-s})^2 (u_0'((1-\delta_1)-e^{-s}))^2\,ds \lesssim \int_{s_0}^{\infty} e^{-2s}\,ds < \infty, \tag{5-29}$$

$$\int_{s_0}^{\infty} e^{-2s}((1-\delta_1)-e^{-s})^2 u_1((1-\delta_1)-e^{-s})^2\,ds \lesssim \int_{s_0}^{\infty} e^{-2s}\,ds < \infty. \tag{5-30}$$

Also by the fact that $|u(t,r)|r$ is uniformly bounded for $t-r \le \frac{1}{2}$,

$$\int_{s_0}^{\infty} e^{-2s}\biggl(\int_{1-\delta_1}^{e^{\tau}\cosh s} (t-e^{-s})u^3(t,t-e^{-s})\,dt\biggr)^2 ds \lesssim \int_{s_0}^{\infty} e^{-2s}\,ds < \infty. \tag{5-31}$$

In fact the above computations could be made for any $0<\tau<\delta_1$ with some uniform s_0. Now then, integrating by parts,

$$\int_{s_0}^{\infty} \partial_s(s\tilde{w}(\tau,s))^2\,ds = \int_{s_0}^{\infty} s^2 \tilde{w}_s(\tau,s)^2\,ds - s_0\tilde{w}(\tau,s_0)^2. \tag{5-32}$$

Remark. It is straightforward to verify that by (5-18),

$$\lim_{s\nearrow\infty} s|\tilde{w}(\tau,s)|^2 = 0, \tag{5-33}$$

so (5-32) is well-defined. Therefore,

$$\sup_{0<\tau<\delta_1} \int_{s_0}^{\infty} \tilde{w}_{\tau}(s,\tau)^2 s^2\,ds + \int_{s_0}^{\infty} \tilde{w}_{\tau}(s,\tau)^2 s^2\,ds < \infty. \tag{5-34}$$

So by (5-16) it suffices to show that

$$\int_0^{\delta_1}\int_0^{s_0} s^2\tilde{w}_s(\tau,s)^2\,ds\,d\tau+\int_0^{\delta_1}\int_0^{s_0} s^2\tilde{w}_\tau(\tau,s)\,ds\,d\tau<\infty. \tag{5-35}$$

This fact is an immediate consequence of (5-6), Theorem 4.1, and the fact that $e^\tau\sinh s$ and $e^\tau\cosh s$ are uniformly bounded when $s\le s_0$ and $\tau\le\delta_1$. Thus, for some $0<\tau_0<\delta_1$,

$$\int_0^{\infty} s^2\tilde{w}_s(\tau_0,s)^2\,ds+\int_0^{\infty} s^2\tilde{w}_\tau(\tau_0,s)\,ds<\infty. \tag{5-36}$$

An application of the Sobolev embedding theorem $\dot H^1\hookrightarrow L^6$ combined with the fact that $s/\sinh s\in L^1\cap L^\infty(s^2\,ds)$ completes the proof of Lemma 5.2. □

Next we compute

$$\frac{d}{d\tau}E(\tilde{w}(\tau))=\int \tilde{w}_\tau[\tilde{u}^3-\tilde{w}^3]\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds. \tag{5-37}$$

Because $v(t,r)$ is supported on $t-r=1+O(\delta_1)$, (5-17) implies $1/\sinh s\lesssim e^{-\tau}$ on the support of $\tilde{v}(\tau,s)$. Therefore, the radial Sobolev embedding theorem implies $\|s\tilde{v}(\tau,s)\|_{L^\infty}<\infty$, so

$$\|\tilde{w}_\tau(\tau)\|_{L^2}\Big\|\tilde{v}(\tau,s)^2\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^2}\Big\|\tilde{v}(\tau,s)\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^\infty}\lesssim e^{-\tau}E(\tilde{w}(\tau))^{1/2}\Big\|\tilde{v}(\tau,s)^2\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^2}. \tag{5-38}$$

Meanwhile,

$$\|\tilde{w}_\tau(\tau)\|_{L^2}\Big\|\tilde{v}(\tau,s)\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^\infty}\Big\|\tilde{w}(\tau,s)^2\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^2}\lesssim e^{-\tau}E(\tilde{w}(\tau)), \tag{5-39}$$

and

$$\begin{aligned}\|\tilde{w}_\tau(\tau)\|_{L^2}\Big\|\tilde{v}(\tau,s)\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^\infty}\Big\|\tilde{w}(\tau,s)\Big(\frac{s}{\sinh s}\Big)^{1/2}\Big\|_{L^4}\Big\|\tilde{v}(\tau,s)\Big(\frac{s}{\sinh s}\Big)^{1/2}\Big\|_{L^4}&\\ \lesssim e^{-\tau}E(\tilde{w}(\tau))^{3/4}\Big\|\tilde{v}(\tau,s)\Big(\frac{s}{\sinh s}\Big)^{1/2}\Big\|_{L^4}&.\end{aligned} \tag{5-40}$$

Now by this, $\|v\|_{L^4_{t,x}}<\infty$, (5-9), and Gronwall's inequality, we know $E(\tilde{w}(\tau))$ is uniformly bounded on $\mathbb{R}$.

Next we prove the Morawetz estimate.

Theorem 5.3.

$$\iint \tilde{w}(s,\tau)^4\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds\,d\tau<\infty. \tag{5-41}$$

Proof. We have

$$M(\tau)=\int \tilde{w}_\tau\Big(\frac{x}{|x|}\cdot\nabla\tilde{w}\Big)\,dx. \tag{5-42}$$

Then

$$\frac{d}{d\tau}M(\tau)=\int\Big(\frac{\cosh s}{\sinh s}\Big)\Big(\frac{s}{\sinh s}\Big)^2\tilde{w}^4 s^2\,ds+\iint\frac{x}{|x|}\cdot(\nabla\tilde{w})(\tilde{u}^3-\tilde{w}^3)s^2\,ds\,d\tau. \tag{5-43}$$

As in the bounded energy computations,

$$\|\nabla\tilde{w}\|_{L^2}\left\|\tilde{v}^2\left(\frac{s}{\sinh s}\right)\right\|_{L^2}\left\|\tilde{v}\left(\frac{s}{\sinh s}\right)\right\|_{L^\infty}\lesssim e^{-\tau}E(w(\tau))^{1/2}\left\|\tilde{v}^2\left(\frac{s}{\sinh s}\right)\right\|_{L^2}, \tag{5-44}$$

$$\|\nabla\tilde{w}\|_{L^2}\left\|\tilde{v}\left(\frac{s}{\sinh s}\right)\right\|_{L^\infty}\left\|\tilde{w}^2\left(\frac{s}{\sinh s}\right)\right\|_{L^2}\lesssim e^{-\tau}E(\tilde{w}), \tag{5-45}$$

and

$$\|\nabla\tilde{w}\|_{L^\infty}\left\|\tilde{v}\left(\frac{s}{\sinh s}\right)\right\|_{L^\infty}\left\|\tilde{w}\left(\frac{s}{\sinh s}\right)^{1/2}\right\|_{L^4}\left\|\tilde{v}\left(\frac{s}{\sinh s}\right)^{1/2}\right\|_{L^4} \lesssim e^{-\tau}E(\tilde{w})^{3/4}\left\|\tilde{v}\left(\frac{s}{\sinh s}\right)^{1/2}\right\|_{L^4}. \tag{5-46}$$

Therefore, by the fundamental theorem of calculus, the fact that the energy is uniformly bounded, and (5-43),

$$\int_0^\infty\int_0^\infty \tilde{w}(s,\tau)^4\left(\frac{s}{\sinh s}\right)^2 s^2\,ds\,d\tau<\infty, \tag{5-47}$$

completing the proof of Theorem 5.3. □

Since $(\cosh s/\sinh s)\geq 1$, Theorem 5.3 directly implies (5-12), which completes the proof of Theorem 5.1. □

Remark. Notice that Theorem 5.1 implies

$$\int_0^\infty\int_0^\infty u(t,r)^4 r^2\,dr\,dt\leq C(\|u_0\|_{B^2_{1,1}},\|u_1\|_{B^1_{1,1}},\delta_1)<\infty. \tag{5-48}$$

Thus Theorem 5.1 is not equivalent to Theorem 1.4. This $\delta_1>0$ depends on the support of u_0 and u_1 in space (3-14) and in frequency (3-2). To remove this requirement, it is necessary make a profile decomposition, the subject of the final section of this paper.

6. Profile decomposition

Now, to prove Theorem 1.4 from Theorem 5.1, it only remains to show that if (u_0^n,u_1^n) is a bounded sequence in $B^2_{1,1}\times B^1_{1,1}$ and $u^n(t)$ is the corresponding solution to (1-1) with initial data (u_0^n,u_1^n), then

$$\|u^n(t)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \tag{6-1}$$

is uniformly bounded. This may be accomplished by proving that (u_0^n,u_1^n) must converge to a maximizer in $B^2_{1,1}\times B^1_{1,1}$. An argument of this type was used in [Gérard 1998] to prove the existence of a maximizer of the Sobolev embedding, and for many other maximizer problems. See [Bahouri and Gérard 1999] for an early application of the profile decomposition to the nonlinear wave equation.

The intuition behind this argument may be summarized as follows. The uncertainty principle implies that when most of $B^2_{1,1}\times B^1_{1,1}$ lies below frequency 1, $R\gtrsim 1$, where R is defined in (3-14). On the other hand, if $R \gg 1$ and (u_0, u_1) is a radial function supported on the annulus $R \leq r \leq 2R$, then the $\dot{H}^{1/2}$ norm on balls of radius cR for some $c > 0$ small would actually be fairly small.

Combining the small-data arguments of [Lindblad and Sogge 1995] with finite propagation speed, one can show

$$\|u\|_{L^4_{t,x}([-cR,cR]\times\mathbb{R}^3)} \ll 1. \tag{6-2}$$

This provides substantial improvement over the frequency scale arguments used in the proof of Lemma 3.1.

Theorem 6.1 (profile decomposition). *Suppose that there is a uniformly bounded, radially symmetric sequence*

$$\|u_0^n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|u_1^n\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \leq C_0 < \infty. \tag{6-3}$$

Then there exists a subsequence, also denoted by $(u_0^n, u_1^n) \subset \dot{H}^{1/2} \times \dot{H}^{-1/2}$, such that for any $N < \infty$

$$S(t)(u_0^n, u_1^n) = \sum_{j=1}^{N} \Gamma_n^j S(t)(\phi_0^j, \phi_1^j) + S(t)(R_{0,n}^N, R_{1,n}^N), \tag{6-4}$$

with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S(t)(R_{0,n}^N, R_{1,n}^N)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} = 0. \tag{6-5}$$

$\Gamma_n^j = (\lambda_n^j, t_n^j)$ belongs to the group $(0,\infty)\times\mathbb{R}$, which acts by

$$\Gamma_n^j F(t,x) = \lambda_n^j F(\lambda_n^j(t - t_n^j), \lambda_n^j x). \tag{6-6}$$

The Γ_n^j are pairwise orthogonal; that is, for every $j \neq k$,

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + (\lambda_n^j)^{1/2}(\lambda_n^k)^{1/2}|t_n^j - t_n^k| = \infty. \tag{6-7}$$

Furthermore, for every $N \geq 1$,

$$\|(u_{0,n}, u_{1,n})\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} = \sum_{j=1}^{N} \|(\phi_0^j, \phi_0^k)\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} + \|(R_{0,n}^N, R_{1,n}^N)\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} + o_n(1). \tag{6-8}$$

Proof. Ramos [2012] proved this result for data which need not be radially symmetric. Such a result is substantially more difficult since it requires accounting for Lorentz transformations and translation in space.

Now take a uniformly bounded sequence

$$\|u_0^n\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1^n\|_{B^1_{1,1}(\mathbb{R}^3)} \leq C_0 < \infty \tag{6-9}$$

such that if $u^n(t)$ is the solution of (1-1) with initial data (u_0^n, u_1^n), then

$$\|u^n(t)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \nearrow \sup\{\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} : \|u_0\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1\|_{B^1_{1,1}(\mathbb{R}^3)} \leq C_0\}. \tag{6-10}$$

By the Sobolev embedding theorem,

$$\|u_0^n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|u_1^n\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \lesssim \|u_0^n\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1^n\|_{B^1_{1,1}(\mathbb{R}^3)} \leq C_0 < \infty, \tag{6-11}$$

which by Theorem 6.1 gives a profile decomposition

$$S(t)(u_0^n, u_1^n) = \sum_{j=1}^{N} S(t - t_n^j)\big(\lambda_n^j \phi_0^j(\lambda_n^j x), (\lambda_n^j)^2 \phi_1^j(\lambda_n^j x)\big) + S(t)(R_{0,n}^N, R_{1,n}^N). \tag{6-12}$$

In the course of proving Theorem 6.1, Ramos [2012] proved

$$S(\lambda_n^j t_n^j)\left(\frac{1}{\lambda_n^j} u_0^n\left(\frac{x}{\lambda_n^j}\right), \frac{1}{(\lambda_n^j)^2} u_1^n\left(\frac{x}{\lambda_n^j}\right)\right) \rightharpoonup \phi_0^j(x) \tag{6-13}$$

weakly in $\dot{H}^{1/2}(\mathbb{R}^3)$, and

$$\partial_t S(t + \lambda_n^j t_n^j)\left(\frac{1}{\lambda_n^j} u_0^n\left(\frac{x}{\lambda_n^j}\right), \frac{1}{(\lambda_n^j)^2} u_1^n\left(\frac{x}{\lambda_n^j}\right)\right)\bigg|_{t=0} \rightharpoonup \phi_1^j(x) \tag{6-14}$$

weakly in $\dot{H}^{-1/2}(\mathbb{R}^3)$. The fact that (u_0^n, u_1^n) is uniformly bounded in $B^2_{1,1} \times B^1_{1,1}$ prevents t_n^j from going off to $-\infty$ or $+\infty$.

Lemma 6.2. *For each j, t_n^j is uniformly bounded.*

Proof. The proof of this fact utilizes dispersive estimates and Lemma 4.1 from [Ramos 2012]:

Lemma 6.3.

$$(u_0^n, u_1^n) \rightharpoonup (\phi_0, \phi_1) \tag{6-15}$$

weakly in $\dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3)$ is equivalent to

$$S(t)(u_0^n, u_1^n) \rightharpoonup S(t)(\phi_0, \phi_1) \tag{6-16}$$

weakly in $L^4_{t,x}(\mathbb{R} \times \mathbb{R}^3)$.

Now observe that for any j,

$$S(t + \lambda_n^j t_n^j)\left(\frac{1}{\lambda_n^j} u_0^n(x), \frac{1}{(\lambda_n^j)^2} u_1^n(x)\right) \rightharpoonup S(t)(\phi_0^j, \phi_1^j) \tag{6-17}$$

weakly in $L^4_{t,x}$.

Now, by the dispersive estimates (Theorem 2.4), for any $l \in \mathbb{Z}$,

$$\begin{aligned}\left\| P_l S(t + \lambda_n^j t_n^j)\left(\frac{1}{\lambda_n^j} u_0^n(x), \frac{1}{(\lambda_n^j)^2} u_1^n(x)\right)\right\|_{L^\infty(\mathbb{R}^3)} \\ \lesssim \frac{1}{|t + \lambda_n^j t_n^j|}\left[2^{2l}\left\| P_l\left(\frac{1}{\lambda_n^j} u_0^n\left(\frac{x}{\lambda_n^j}\right)\right)\right\|_{L^1(\mathbb{R}^3)} + 2^l \left\| P_l\left(\frac{1}{(\lambda_n^j)^2} u_1^n\left(\frac{x}{\lambda_n^j}\right)\right)\right\|_{L^1(\mathbb{R}^3)}\right].\end{aligned} \tag{6-18}$$

Meanwhile, by Bernstein's inequality and the Sobolev embedding theorem

$$\Big\| P_l S(t+\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j} u_0^n(x), \frac{1}{(\lambda_n^j)^2} u_1^n(x)\Big)\Big\|_{L^2(\mathbb{R}^3)} \lesssim 2^{-l/2}\Big[2^{2l}\Big\| P_l\Big(\frac{1}{\lambda_n^j} u_0^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big\|_{L^1(\mathbb{R}^3)} + 2^l \Big\| P_l\Big(\frac{1}{(\lambda_n^j)^2} u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big\|_{L^1(\mathbb{R}^3)}\Big]. \tag{6-19}$$

Then by interpolation, for any $l \in \mathbb{Z}$,

$$\Big\| P_l S(t+\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j} u_0^n(x), \frac{1}{(\lambda_n^j)^2} u_1^n(x)\Big)\Big\|_{L^4_{t,x}(\{|t+\lambda_n^j t_n^j|>C2^{-l}\}\times\mathbb{R}^3)} \lesssim \frac{1}{C^{1/4}}\Big[2^{2l}\Big\| P_l\Big(\frac{1}{\lambda_n^j} u_0^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big\|_{L^1(\mathbb{R}^3)} + 2^l \Big\| P_l\Big(\frac{1}{(\lambda_n^j)^2} u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big\|_{L^1(\mathbb{R}^3)}\Big]. \tag{6-20}$$

Thus if $\limsup_{n\to\infty} \lambda_n^j |t_n^j| = \infty$, then

$$S(t+\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j} u_0^n(x), \frac{1}{(\lambda_n^j)^2} u_1^n(x)\Big) \rightharpoonup 0 \tag{6-21}$$

weakly in $L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)$. Therefore, $(\phi_0^j, \phi_1^j) = (0,0)$, completing the proof of Lemma 6.3. ☐

Thus $\lambda_n^j |t_n^j|$ is uniformly bounded, so after passing to a subsequence, $\lambda_n^j t_n^j$ converges to some $t_0^j \in \mathbb{R}$. Therefore,

$$S(\lambda_n^j t_n^j)(\phi_0^j, \phi_1^j) \to S(t_0^j)(\phi_0^j, \phi_1^j) \tag{6-22}$$

strongly in $\dot H^{1/2}(\mathbb{R}^3)\times \dot H^{-1/2}(\mathbb{R}^3)$. Absorbing the error into $(R_{0,n}^N, R_{1,n}^N)$ and taking

$$(\tilde\phi_0^j, \tilde\phi_1^j) = S(t_0^j)(\phi_0, \phi_1), \tag{6-23}$$

assume $t_n^j \equiv 0$. Therefore,

$$(u_0^n, u_1^n) = \sum_{j=1}^N \big(\lambda_n^j \phi_0^j(\lambda_n^j x), (\lambda_n^j)^2 \phi_1^j(\lambda_n^j x)\big) + (R_{0,n}^N, R_{1,n}^N), \tag{6-24}$$

and

$$\lim_{n\to\infty} \frac{\lambda_j^n}{\lambda_k^n} + \frac{\lambda_k^n}{\lambda_j^n} = \infty. \tag{6-25}$$

But then

$$\|u_0^n\|_{B^2_{1,1}(\mathbb{R}^3)} + \|u_1^n\|_{B^1_{1,1}(\mathbb{R}^3)} \le C_0 < \infty \tag{6-26}$$

combined with Lemma 6.3, (6-24), and (6-25) implies that for any j

$$\|\phi_0^j\|_{B^2_{1,1}(\mathbb{R}^3)} + \|\phi_1^j\|_{B^1_{1,1}(\mathbb{R}^3)} \le C_0. \tag{6-27}$$

Possibly reordering j, (6-8) implies that there exists $N_0(\epsilon, C_0)$ such that if $j \ge N_0(\epsilon)$,

$$\|(\phi_0^j, \phi_1^j)\|_{\dot H^{1/2}\times \dot H^{-1/2}} < \epsilon. \tag{6-28}$$

Now for each j let $v^j(t,x)$ be the solution of (1-1) with initial data (ϕ_0^j,ϕ_1^j). By the small-data arguments of [Lindblad and Sogge 1995], when $j \geq N_0(\epsilon)$,

$$\|v^j\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\phi_0^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|\phi_1^j\|_{\dot{H}^{-1/2}(\mathbb{R}^3)}. \tag{6-29}$$

Meanwhile, by Theorem 5.1 combined with (6-27), when $j \leq N_0(\epsilon)$,

$$\|v^j\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim_{j,C_0} 1. \tag{6-30}$$

Also by (6-25), for any $j \neq k$, the Lebesgue dominated convergence theorem implies

$$\lim_{n\to\infty} \iint |\lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x)|^2\, |\lambda_n^k v^k(\lambda_n^k t, \lambda_n^k x)|^2\, dx\, dt = 0. \tag{6-31}$$

Therefore,

$$\lim_{n\to\infty} \Big\| \sum_{1\leq j\leq N} \lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x)\Big\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \tag{6-32}$$

is uniformly bounded, independent of N. Also,

$$F\bigg(\sum_{j=1}^N \lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x)\bigg) - \sum_{j=1}^N F(\lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x)) = \sum_{1\leq j\neq k\leq N} O(|\lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x)|\, |\lambda_n^k v^k(\lambda_n^k t, \lambda_n^k x)|^2), \tag{6-33}$$

so by (6-30), (6-31), and (6-32),

$$\lim_{n\to\infty} \bigg\| F\bigg(\sum_{j=1}^N \lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x)\bigg) - \sum_{j=1}^N F(\lambda_n^j v^j(\lambda_n^j t, \lambda_n^j x))\bigg\|_{L^{4/3}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} = 0. \tag{6-34}$$

Therefore, by Lemma 2.6, the solution $u_N^n(t,x)$ to (1-1) with initial data

$$\sum_{j=1}^N (\lambda_n^j \phi_0^j(\lambda_n^j x), (\lambda_n^j)^2 \phi_1^j(\lambda_n^j x)) \tag{6-35}$$

has

$$\lim_{n\to\infty} \|u_N^n(t)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \tag{6-36}$$

bounded uniformly in N. By another application of Lemma 2.6 combined with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S(t)(R_{0,n}^N, R_{1,n}^N)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} = 0, \tag{6-37}$$

if $u^n(t)$ is the solution to (1-1) with initial data (u_0^n, u_1^n) satisfying (6-3), then

$$\|u^n(t)\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \tag{6-38}$$

is uniformly bounded. This proves Theorem 1.4. □

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