ANALYSIS & PDE

Volume 12 No. 4 2019

JONAS HIRSCH

NONEXISTENCE OF WENTE'S L^∞ ESTIMATE FOR THE NEUMANN PROBLEM

msp

ANALYSIS AND PDE
Vol. 12, No. 4, 2019
dx.doi.org/10.2140/apde.2019.12.1049

msp

NONEXISTENCE OF WENTE'S L^∞ ESTIMATE FOR THE NEUMANN PROBLEM

JONAS HIRSCH

We provide a counterexample of Wente's inequality in the context of Neumann boundary conditions. We will also show that Wente's estimate fails for general boundary conditions of Robin type.

1. Introduction

Wente's L^∞ estimate is a fundamental example of a "gain" of regularity due to the special structure of Jacobian determinants. It concerns the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases} \tag{1-1}$$

for the specific choice of $f = \det(\nabla V)$ with $V \in H^1(D, \mathbb{R}^2)$. Wente's theorem states:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be the disc and $f \in \mathcal{H}^1(D)$. Then if u is the unique solution in $W_0^{1,1}(\Omega, \mathbb{R})$ to (1-1), we have the estimate*

$$\|u\|_{L^\infty(D)} + \|\nabla u\|_{L^2(D)} \leq C \|\nabla V\|^2_{L^2(D)}.$$

The proof can be found in the original article [Wente 1971]. Later on it was proved that Wente's inequality holds true under the slightly weaker assumption that $f \in \mathcal{H}^1(D)$, where $\mathcal{H}^1(D)$ is the local Hardy space; see [Semmes 1994, Definition 1.90]. Proofs can be found for instance in [Hélein 2002; Topping 1997]. This estimate found many applications; an incomplete list includes [Rivière 2008; Colding and Minicozzi 2008; Lamm and Lin 2013].

It is natural to ask whether a similar estimate holds true for the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{1}{2\pi} \displaystyle\int_D f & \text{on } \partial D, \end{cases} \tag{1-2}$$

again for the specific choice of $f = \det(\nabla V)$ with $V \in H^1(D, \mathbb{R}^2)$.

The aim of this note is to show that Wente's L^∞ estimate fails for the Neumann problem.

MSC2010: primary 35J05; secondary 35J25.
Keywords: compensated compactness, Jacobian determinants.

Theorem 1.2. *There exists a sequence* $V_n = (a_n, b_n) \in C^\infty(\bar{D}, \mathbb{R}^2)$, $\|\nabla V_n\|_{L^{2,1}(D)} \le C$, $\int \det(\nabla V_n) = 0$ *for all n with the property that if* $u_n \in W^{1,1}(D)$ *are the solutions to* (1-2) *with* $f_n = \det(\nabla V_n)$ *one has*

$$\left\| u_n - \fint_D u_n \right\|_{L^\infty(D)}, \|\nabla u_n\|_{L^2(D)} \to +\infty \quad \text{as } n \to \infty.$$

Additionally we can extend the above example to more general boundary conditions. Namely we have the following:

Theorem 1.3. *Let* $E \subset \partial D$ *be a nonempty union of open intervals, with* $0 < \mathcal{H}^1(E) < 2\pi$ *and* $\alpha, \beta, \gamma \in \mathbb{R}$ *given, with* $\alpha > 0$, $\gamma \ge 0$. *There exists a sequence* $V_n = (a_n, b_n) \in C^\infty(\bar{D}, \mathbb{R}^2)$, *with* $\|\nabla V_n\|_{L^{2,1}(D)} < C$, *with the property that if* $u_n \in W^{1,1}(D)$ *is the solution to*

$$\begin{cases} -\Delta u_n = \det(\nabla V_n) & \text{in } D, \\ \alpha \dfrac{\partial u_n}{\partial \nu} + \beta \dfrac{\partial u_n}{\partial \tau} + \gamma u_n = 0 & \text{on } E, \\ u = 0 & \text{on } \partial D \setminus E, \end{cases} \tag{1-3}$$

one has

$$\|\nabla u_n\|_{L^2(D)} \to \infty \quad \text{as } n \to \infty.$$

The paper is organized as follows. In Section 3 we collect some known results and a priori estimates. In Section 4 we give the proof of Theorem 1.2 and in Section 5 its extension to mixed Robin boundary conditions.

While finishing this paper the author became aware that a similar example has been found independently by Francesca Da Lio and Francesco Palmurella [2017].

2. Some remarks on the conformal invariance of the problem

Let $m : U \to D$ be a smooth conformal map from a domain U with Lipschitz continuous boundary to the disc (i.e., up to conjugation m corresponds to holomorphic map on U). If u is a solution of the Dirichlet problem (1-1) then $u \circ m$ is a solution of

$$\begin{cases} -\Delta(u \circ m) = \left(\tfrac{1}{2}|\nabla m|^2\right) f \circ m & \text{in } U, \\ u \circ m = 0 & \text{on } \partial U. \end{cases}$$

In particular in the case $f = \det(\nabla V)$ we have $\left(\tfrac{1}{2}|\nabla m|^2\right) f \circ m = \det(\nabla(V \circ m))$. Additionally one notes that Wente's estimate in Theorem 1.1 is as well conformally invariant since for any function w one has

$$\|w \circ m\|_{L^\infty(U)} = \|w\|_{L^\infty(D)}, \quad \|\nabla(w \circ m)\|_{L^2(U)} = \|\nabla w\|_{L^2(D)}.$$

In the case of the Neumann problem one has to be a bit more careful. If u is a solution to (1-2) then $u \circ m$ solves

$$\begin{cases} -\Delta(u \circ m) = \left(\tfrac{1}{2}|\nabla m|^2\right) f \circ m & \text{in } U, \\ \dfrac{\partial(u \circ m)}{\partial \nu} = \left(\tfrac{1}{2}|\nabla m|^2\right)^{1/2} \dfrac{1}{2\pi} \displaystyle\int_D f & \text{on } \partial U. \end{cases}$$

Although we have

$$\frac{1}{2\pi}\int_D f = \frac{1}{2\pi}\int_U \big(\tfrac{1}{2}|\nabla m|^2\big) f\circ m,$$

the problem is only conformally "invariant" if $\int_D f = 0$ since $|\nabla m| = 1$ on ∂U implies that m is a rigid motion. Furthermore one should note that even in the case $\int f = 0$, in general one has

$$⨍_U u\circ m \neq ⨍_D u.$$

Nonetheless we can forget about the additional condition $\int_D \det(\nabla V_n) = 0$ in the proofs of Theorems 1.2 and 1.3 by the following procedure. Consider a sequence V_n as stated, but not necessarily satisfying $\alpha_n := \int \det(\nabla V_n) = 0$, that is compactly supported in some ball $B_{r_0}(p)$ for some $0 < r_0 < \frac{1}{4}$ and $p \in \partial D$. Let us fix two smooth functions $\hat{a}, \hat{b}$ supported in $B_{2r_0}(p) \setminus B_{r_0}(p)$ satisfying

$$\int_D d\hat{a}\wedge d\hat{b} = 1.$$

For instance take $\hat{a} = \varphi_1(z)$ and $\hat{b} = \varphi_2(z)\theta$, where φ_i are two bump functions such that $\operatorname{spt}(\varphi_1) \subset \{\varphi_2 = 1\}$,

$$\int_D d\hat{a}\wedge d\hat{b} = \int_{\partial D} \hat{a}\nabla_\theta \hat{b} = \int_{\partial D}\varphi_1 = 1.$$

Let $\hat{u}$ be the smooth unique solution to (1-2) with $⨍_D \hat{u} = 1$, $f = \det(\nabla\widehat{V})$ and $\widehat{V} = (\hat{a}, \hat{b})$. Since $|\alpha_n| \le \frac{1}{2}\|\nabla V_n\|^2_{L^2(D)}$ and $\operatorname{spt}(V_n)\cap\operatorname{spt}(\widehat{V}) = \varnothing$ for all n we can pass to $\tilde{u}_n = u_n - \alpha_n\hat{u}$, which solves the Neumann problem (1-2) with right-hand side

$$\det(\nabla V_n - \alpha\nabla\widehat{V}) = \det(\nabla V_n) - \alpha\det(\nabla\widehat{V}).$$

Since $\int_D \det(\nabla V_n - \alpha\nabla\widehat{V}) = 0$ we have $\partial\tilde{u}_n/\partial\nu = 0$ on ∂D. By the uniform boundedness of α_n we still have

$$\Big\|\tilde{u}_n - ⨍_D \tilde{u}_n\Big\|_{L^\infty(D)}, \|\nabla\tilde{u}_n\|_{L^2(D)} \to +\infty \quad \text{as } n\to\infty$$

and we obtain the full strength of the theorems.

3. Some known results

Classical solutions to (1-1) and (1-2) have to be understood in the distributional sense.

Definition 3.1. A function u is called a solution of the Dirichlet problem if $u \in W_0^{1,1}(D,\mathbb{R})$ and

$$\int_D \nabla u\cdot\nabla\psi - f\psi = 0 \quad \text{for all } \psi \in C_0^1(D). \tag{3-1}$$

A function u is called a solution of the Neumann problem if $u \in W^{1,1}(D,\mathbb{R})$ and

$$\frac{1}{2\pi}\int_D f\int_{\partial D}\psi = \int_D \nabla u\cdot\nabla\psi - f\psi \quad \text{for } \psi\in C_0^\infty(\mathbb{R}^2) \text{ for all } \psi \in C^1(\overline{D}). \tag{3-2}$$

The Green's functions for both problems are explicit. For the Dirichlet problem it is

$$G_D(x,y)=\frac{1}{2\pi}\ln(|x-y|)-\frac{1}{2\pi}\ln(|y|\,|x-y^*|),\quad \text{with } y^*=\frac{y}{|y|^2}, \tag{3-3}$$

and for Neumann problem it is

$$G_N(x,y)=\frac{1}{2\pi}\ln(|x-y|)+\frac{1}{2\pi}\ln(|y|\,|x-y^*|)-\tfrac{1}{4}|x|^2-\tfrac{1}{4}|y|^2. \tag{3-4}$$

Using G_N one has the representation formula

$$u(y)-\int_D u=-\int_{\partial D} G_N(x,y)\frac{\partial u}{\partial \nu}+\int_D G(x,y)\Delta u \quad \text{for } u\in C^2(\bar{D}).$$

In terms of existence and uniqueness one has:

Lemma 3.2. *For every $f\in L^1(D)$ there exists a solution u_D/u_N to the Dirichlet/ Neumann problem in the sense of Definition 3.1. Furthermore the solutions belong to $W^{1,p}(D,\mathbb{R})$ for every $p<2$, are unique (up to constant in the Neumann problem) and satisfy the estimate*

$$\|Du\|_{L^p(D)}\le C_p\|f\|_{L^1(D)}. \tag{3-5}$$

Proof. There are several proofs in the literature treating the case of uniqueness and a priori estimates; see for instance [Littman et al. 1963; Ancona 2009, Appendix A]. In our case existence and the a priori estimate (3-5) can be obtained by using the Green's functions G_D, G_N. Uniqueness for the Dirichlet problem can be obtained by antisymmetric reflection: Let u be a distributional solution of (3-1) with $f=0$. One checks that

$$\hat{u}(x):=\begin{cases} u(x) & \text{for } x\in D,\\ -u(x^*) & \text{for } x\notin D \text{ with } x^*=x/|x|^2\end{cases}$$

solves

$$\int_{\mathbb{R}^2}\nabla\hat{u}\cdot\nabla\psi=\int_D \nabla u\cdot\nabla(\psi(x)-\psi(x^*)) \quad \text{for all } \psi\in C^1_c(\mathbb{R}^2).$$

But since $\psi(x)-\psi(x^*)\in C^{0,1}_0(D)$ we deduce that $\hat{u}$ is harmonic and therefore smooth in $\mathbb{R}^2$. Now the maximum principle applies since u takes the boundary values in the strong sense.

Similarly we deduce the uniqueness in the Neumann problem using the symmetric reflection: Let v be a distributional solution of (3-2) with $f=0$. As before one checks that

$$\hat{v}(x):=\begin{cases} v(x) & \text{for } x\in D,\\ v(x^*) & \text{for } x\notin D\end{cases}$$

solves

$$\int_{\mathbb{R}^2}\nabla\hat{v}\cdot\nabla\psi=\int_D \nabla v\cdot\nabla(\psi(x)+\psi(x^*)) \quad \text{for all } \psi\in C^1_c(\mathbb{R}^2).$$

But since $\psi(x)+\psi(x^*)\in C^{0,1}(\bar{D})$ we deduce that $\hat{v}$ is harmonic and therefore smooth in $\mathbb{R}^2$. Now the maximum principle implies that $v=\text{constant}$. □

4. Proof of Theorem 1.2

In the following we will always identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, i.e., $i = e_2$.

Proof of Theorem 1.2. The main step of the proof consists in the following claim: For every $r_0 > 0$ there exists a sequence $(a_n, b_n) \in C^\infty(\bar{D}, \mathbb{R}^2)$ with the properties that

$$\operatorname{spt}(a_n) \cup \operatorname{spt}(b_n) \subset B_{r_0}(-e_2), \tag{4-1a}$$

$$a_n, b_n \rightharpoonup 0 \quad \text{in } H^1(D), \tag{4-1b}$$

$$\|a_n\|_{L^\infty(D)} + \|\nabla a_n\|_{L^{2,1}(D)}, \qquad \|b_n\|_{L^\infty(D)} + \|\nabla b_n\|_{L^{2,1}(D)} \le C, \tag{4-1c}$$

$$\|da_n \wedge db_n\|_{H^{-1}(D)} \to \infty \quad \text{as } n \to \infty. \tag{4-2}$$

Given such a sequence we can conclude the theorem. Let u_n be the unique solution to the Dirichlet problem (1-1) with right-hand side $f_n = da_n \wedge db_n$ and h_n be the unique harmonic function satisfying

$$\frac{\partial h_n}{\partial \nu} = \frac{\partial u_n}{\partial \nu} - \frac{1}{2\pi} \int_{\partial D} \frac{\partial u_n}{\partial \nu} \quad \text{on } \partial D.$$

Such a harmonic function exists since

$$\int_{\partial D} \left(\frac{\partial u_n}{\partial \nu} - \frac{1}{2\pi} \int_{\partial D} \frac{\partial u_n}{\partial \nu} \right) = 0.$$

It is straightforward to check that

$$v_n := u_n - h_n$$

is the unique solution to the Neumann problem (1-2). Observe that v_n is a Cauchy sequence in $W^{1,p}(D)$ for all $p < 2$ converging to $v \in W^{1,p}(D)$, the unique solution of (1-2) with $f = da \wedge db$. By Wente's theorem we have

$$\|\nabla v_n\|_{L^2(D)} \ge \|\nabla h_n\|_{L^2(D)} - \|\nabla u_n\|_{L^2(D)} \ge \|\nabla h_n\|_{L^2(D)} - C\|\nabla a_n\|_{L^2(D)} \|\nabla b_n\|_{L^2(D)}.$$

The theorem follows by showing that

$$\|\nabla h_n\|_{L^2(D)} \to \infty. \tag{4-3}$$

To do so we will use the Dirichlet-to-Neumann map in the following formulation: Let

$$X_0 := \left\{ h \in H^1(D) : \Delta h = 0 \text{ in } D \text{ and } ⨍_D h = 0 \right\},$$
$$Y_0 := \left\{ u \in H^1(D) : ⨍_D u = 0 \right\}.$$

Endowed with the L^2 inner product $\langle u, v \rangle = \int_D \nabla u \cdot \nabla v$, we obtain Hilbert spaces satisfying $X_0 \subset Y_0$. If we set $Z_0^* := \{ l \in Y_0^* : l(\psi) = 0 \text{ for all } \psi \in H_0^1(D) \cap Y_0 \}$ then classical results concerning Dirichlet-to-Neumann operators imply that the operator

$$A : X_0 \to Z_0^*, \quad \text{with } Ah := \frac{\partial h}{\partial \nu},$$

is continuous and onto; i.e., it has a continuous inverse A^{-1}.

Next we identify

$$\frac{\partial u_n}{\partial \nu} - \frac{1}{2\pi}\int_{\partial D} \frac{\partial u_n}{\partial \nu}$$

with a linear functional $l_n \in Y_0^*$; i.e.,

$$l_n(\psi) := \int_{\partial D}\Big(\frac{\partial u_n}{\partial \nu} - \frac{1}{2\pi}\int_{\partial D} \frac{\partial u_n}{\partial \nu}\Big)\psi.$$

We will show that they are elements of Z_0^* with the property that $\|l_n\|_{H^{-1}(D)} \to +\infty$. The normal derivative of a solution $u \in W^{1,1}(D)$ to the Dirichlet problem (1-1), with $f \in L^1(D)$, is given in the sense of distributions by

$$\int_{\partial D} \frac{\partial u}{\partial \nu}\psi := \int_D \nabla u \cdot \nabla \psi - f\psi \quad \text{for } \psi \in C^1(\bar{D}). \tag{4-4}$$

The distribution is supported on ∂D since given $\psi_1, \psi_2 \in C^\infty(\bar{D})$ with $\psi_1 = \psi_2$ on ∂D we have $\varphi = \psi_1 - \psi_2 \in C_0^1(\bar{D})$ with $\varphi = 0$ on ∂D and so by (3-1) we have

$$\int_{\partial D} \frac{\partial u}{\partial \nu}\varphi = \int_D \nabla u \cdot \nabla \varphi - f\varphi = 0.$$

By density of $C_c^\infty(D)$ in $H_0^1(D)$ we conclude $l_n(\psi) = 0$ for all $\psi \in H_0^1(D)$. Furthermore it is straightforward to check that l_n vanishes on the constant functions and hence l_n is a well-defined element of Y_0^*, since $l_n(\psi) = l_n(\psi - ⨍\psi)$. Thus we conclude that $l_n \in Z_0^*$ for all n. The first part of (4-4) and the second part in the definition of l_n are uniformly bounded by Wente's theorem (Theorem 1.1) because

$$\int_D \nabla u_n \cdot \nabla \psi \le \|\nabla u_n\|_{L^2(D)} \, \|\nabla \psi\|_{L^2(D)}$$
$$\left|\frac{1}{2\pi}\int_{\partial D} \frac{\partial u_n}{\partial \nu}\right| = \left|\frac{1}{2\pi}\int_D f_n\right| \le \frac{1}{2\pi}\|\nabla a_n\|_{L^2(D)} \, \|\nabla b_n\|_{L^2(D)}.$$

Hence $\|l_n\|_{H^{-1}(D)} \to \infty$ by (4-2). Since $h_n = A^{-1}(l_n)$ and A^{-1} is continuous, we conclude (4-3).

It remains to construct the sequence (a_n, b_n) with the properties (4-1)–(4-2). Performing a translation we can consider the translated disc $D' := D + i$; i.e., $D' \subset H := \mathbb{C} \cap \{y \ge 0\} = \{re^{i\theta} : 0 < \theta < \pi\}$. Furthermore one readily checks that if $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h then we have pointwise

$$d\Re(h) \wedge d\Im(h) = |h'(z)|^2 dx \wedge dy \quad \text{and} \quad |d\Re(h)|^2 = |d\Im(h)|^2 = |h'(z)|^2. \tag{4-5}$$

We will construct our contradicting sequence (a_n, b_n) as the real and imaginary parts of a sequence of holomorphic functions h_n on H multiplied by a truncation function φ.

Indeed consider the family of Möbius transformations of the complex plane $\mathbb{C}$

$$m_\epsilon(z) := \frac{z - i\epsilon}{z + i\epsilon}.$$

We observe that m_ϵ maps the upper half-space H onto the disc D for every $\epsilon > 0$. Furthermore one readily calculates

$$m'_\epsilon(z) = \frac{2i\epsilon}{(z+i\epsilon)^2}, \quad m_\epsilon^{-1}(z) = i\epsilon\frac{z+1}{1-z}. \tag{4-6}$$

We note that for every $\delta > 0$ one has $m'_\epsilon(z) \to 0$ and $m_\epsilon(z) \to 1$ uniformly on $\mathbb{C}\setminus D_\delta$ for $\epsilon \to 0$. Furthermore $m_\epsilon^{-1}(z) \to 0$ uniformly on $\mathbb{C}\setminus D_\delta(1)$. Thus we can conclude that $l_\epsilon := |m'_\epsilon(z)|^2\, dx\wedge dy \to \pi\delta_0$ in the sense of distributions; i.e., given $\psi \in C_c^0(\mathbb{C})$ arbitrary one has

$$\int_H \psi(z)|m'_\epsilon(z)|^2\, dx\wedge dy = \int_D \psi\circ m_\epsilon^{-1}(z)\, dx\wedge dy \to \psi(0)\pi.$$

Furthermore we conclude that if φ is any cutoff function with $\varphi = 1$ in a neighborhood of 0 we still have $l_\epsilon\lfloor\varphi \to \pi\delta_0$. Since $\pi\delta_0 \notin H^{-1}(H)$ we conclude that $\|l_\epsilon\lfloor\varphi\|_{H^{-1}(D)} \to \infty$ as $\epsilon \to 0$. Fixing a sequence $\epsilon_n \to 0$, we have

$$a_n := \varphi\,\Re(m_{\epsilon_n} - 1) \quad\text{and}\quad b_n := \varphi\,\Im(m_{\epsilon_n} - 1)$$

satisfy $a_n, b_n \in C^\infty(H)$ and $a_n, b_n \to 0$ uniformly in C^1 on $\overline{H}\setminus D_\delta$ for any $\delta > 0$. Hence for an appropriate choice of φ the first two parts of (4-1) follow.

We calculate

$$da_n\wedge db_n = l_\epsilon\lfloor\varphi^2 + \varphi d\varphi\wedge\big(\Re(m_{\epsilon_n})d\Im(m_{\epsilon_n}) - \Im(m_{\epsilon_n})d\Re(m_{\epsilon_n})\big) = l_\epsilon\lfloor\varphi^2 + \varphi d\varphi\wedge w_\epsilon.$$

Since we have $\operatorname{spt}(d\varphi) \subset \mathbb{C}\setminus D_\delta$ for some $\delta > 0$ and $|w_\epsilon| \to 0$ uniformly on $\mathbb{C}\setminus D_\delta$ we conclude that $\|\varphi d\varphi\wedge w_\epsilon\|_{H^{-1}} \to 0$ as $n \to \infty$. Hence $da_n\wedge db_n \to \pi\delta_0$ in the sense of distributions and therefore $\|da_n\wedge db_n\|_{H^{-1}(H)} \to \infty$ as $n \to \infty$; i.e., (4-2) holds.

It remains to show that $|da_n|$, $|db_n|$ are uniformly bounded in $L^{2,1}$. By (4-6) we have

$$\{z\in H : |m'_\epsilon(z)| \ge t\} = B_{r(t)}(-i\epsilon)\cap H, \quad\text{with } \frac{2\epsilon}{r(t)^2} = t$$

and $|m'_\epsilon|(z) \le 2/\epsilon$ for all $z \in H$. Hence we may estimate

$$\mu(t) := |\{z\in H : |m'_\epsilon(z)| \ge t\}| \le \pi r(t)^2 = \frac{2\epsilon}{t}\pi.$$

Recall that the $L^{2,1}$ norm can be written as

$$\|f\|_{L^{2,1}(H)} = 2\int_0^\infty \mu_f(t)^{1/2}\, dt.$$

Here $\mu_f(t) = |\{z\in H : |f(z)| > t\}|$ is the distribution function; see [Grafakos 2014, Proposition 1.4.9]. Using the estimates above we obtain

$$\||m'_\epsilon|\|_{L^{2,1}(H)} \le 2\sqrt{2\pi\epsilon}\int_0^{2/\epsilon}\frac{1}{\sqrt{t}}\, dt \le 8\sqrt{\pi},$$

which is uniformly bounded in ϵ, proving the last part of (4-1). □

Remark 4.1. Observe that if the solution to the Neumann problem is not in $H^1(D)$ then it can neither be in L^∞ nor in $W^{2,1}(D)$. Indeed $u \in W^{2,1}(D)$ would imply $u \in L^\infty$ since $W^{2,1}(D)$ embeds in L^∞ in two dimensions; see for instance Theorem 3.3.10 combined with Theorem 3.3.4 in [Hélein 2002]. If u were in $L^\infty(D)$ then we could take $u_\epsilon \in C^\infty(\bar{D})$ with $u_\epsilon \to u$ in $W^{1,1}(D)$ and uniformly bounded in $L^\infty(D)$. Testing (3-2) with u_ϵ would give

$$\int_D \nabla u \cdot \nabla u_\epsilon = \int_D f u_\epsilon + \frac{1}{2\pi} \int_D f \int_{\partial D} u_\epsilon \leq 2\|f\|_{L^1} \|u_\epsilon\|_{L^\infty}.$$

The right-hand side is bounded independent of ϵ so we conclude that $u \in H^1(D)$, a contradiction.

By using more or less an abstract functional analytic argument we are able to obtain the following corollary. Its proof is presented in the Appendix.

Corollary 4.2. *There exists $a, b \in H^1(D)$ with the additional properties $a, b \in L^\infty(D)$ and $da, db \in L^{2,1}(D)$ such that if $u \in W^{1,1}(D)$ denotes the solution to the Neumann problem (1-2) with $f = da \wedge db$ then $u \notin H^1(D)$.*

5. More general boundary conditions

Our construction of the counterexample relies mainly on the continuity of the Dirichlet-to-Neumann map D_0. The extension to more general boundary conditions of Robin type follows by finding a replacement of the Dirichlet-to-Neumann map. The replacement is constructed as follows:

$$X := \{h \in H^1(D) : \Delta h = 0 \text{ in } D \text{ and } h = 0 \text{ on } \partial D \setminus E\},$$
$$Y := \{u \in H^1(D) : u = 0 \text{ on } \partial D \setminus E\}.$$

Since by assumption $\mathcal{H}^1(\partial D \setminus E) > 0$ we can endow X, Y with the norm $\|u\| = \|\nabla u\|_{L^2(D)}$. Finally we define the closed subset $Z^* \subset Y^*$ by

$$Z^* := \{l \in Y^* : l(u) = 0 \text{ for all } u \in H^1_0(D)\}.$$

Obviously one has the inclusion $X \subset Y$ and $Z^* \subset Y^*$.

Lemma 5.1. *The operator $B : X \to Z^*$ defined by*

$$\langle Bh, \psi \rangle = \int_{\partial D} \left(\alpha \frac{\partial h}{\partial \nu} + \beta \frac{\partial h}{\partial \tau} + \gamma h \right) \psi := \alpha \int_D \nabla h \cdot \nabla \psi + \beta \int_{\partial D} \frac{\partial h}{\partial \tau} \psi + \gamma \int_{\partial D} h \psi$$

is continuous and onto, with continuous inverse $B^{-1} : Z^* \to X$.

Proof. Instead of B itself we consider the family of operators $B_s : X \to Z^*$ for $s \in [0, 1]$. B_s is defined as B with $s\beta, s\gamma$ replacing β, γ. Since h is harmonic in D we have $\langle B_s h, \psi \rangle = 0$ for all $\psi \in H^1_0(D)$ by density of $C^\infty_c(D)$ in $H^1_0(D)$. Furthermore we have the estimate

$$\begin{aligned}\langle B_s h, \psi \rangle &\leq \alpha \|\nabla h\|_{L^2(D)} + |s\beta| \left\| \frac{\partial h}{\partial \tau} \right\|_{H^{-1/2}\partial D} \|\psi\|_{H^{1/2}\partial D} + s\gamma \|h\|_{L^2(\partial D)} \|\psi\|_{L^2(\partial D)} \\ &\leq (\alpha + C|\beta| + C\gamma) \|\nabla h\|_{L^2(D)} \|\nabla \psi\|_{L^2(D)}.\end{aligned}$$

In the last line we used that for harmonic functions we have

$$\left\|\frac{\partial h}{\partial \tau}\right\|_{H^{-1/2}(\partial D)} = \left\|\frac{\partial h}{\partial \nu}\right\|_{H^{-1/2}(\partial D)} = \|\nabla h\|_{L^2(D)}$$

and the trace theorem for Sobolev functions.

This shows that B_s is a family of uniformly bounded operators taking values in Z^*. Since $X \subset Y$ we have the lower bound

$$\begin{aligned}\langle B_s h, h\rangle &= \alpha \int_D \nabla h \cdot \nabla h + s\beta \frac{1}{2} \int_{\partial D} \frac{\partial h^2}{\partial \tau} + s\gamma \int_{\partial D} h^2 \\ &= \alpha \int_D \nabla h \cdot \nabla h + s\gamma \int_{\partial D} h^2 \geq \alpha \|\nabla h\|^2_{L^2(D)}.\end{aligned}$$

Finally since $B_s = (1-s)B_0 + sB$, the *method of continuity*, see, e.g., [Gilbarg and Trudinger 1998, Theorem 5.2], applies and $B = B_1$ is onto if and only if B_0 is onto. By construction we have $B_0 h = \alpha(\partial h/\partial \nu)$, the classical normal derivative on E, which is known to be onto by the Dirichlet-to-Neumann map. □

Now we are able to complete the proof of the theorem.

Proof of Theorem 1.3. The construction is now essentially the same as in the proof of Theorem 1.2. After a rotation we may assume that $-i = -e_2 \in E$. Fix $r_0 > 0$ such that $\partial D \cap B_{r_0}(-i) \subset E$. Let $a_n, b_n, u_n \in C^\infty(\bar{D})$ be the sequences constructed in the proof of Theorem 1.2. By the choice of $r_0 > 0$ we have ensured that

$$\operatorname{spt}(a_n) \cup \operatorname{spt}(b_n) \subset B_{r_0}(-i).$$

Observe that

$$l_n := \alpha \frac{\partial u_n}{\partial \nu} + \beta \frac{\partial u_n}{\partial \tau} + \gamma u_n \in Z^*$$

because

$$\langle B u_n, \psi\rangle = \alpha \int_{\partial D} \frac{\partial u_n}{\partial \nu} \psi = \alpha \int_D \nabla u_n \cdot \nabla \psi - \alpha \int_D da_n \wedge db_n \psi$$

and the discussion below (4-4) applies. Furthermore we have

$$\|l_n\|_{Z^*} \geq \alpha \|da_n \wedge db_n\|_{H^{-1}(D)} - \alpha \|\nabla u_n\|_{L^2(D)}.$$

By Wente's theorem (Theorem 1.1), $\|\nabla u_n\|_{L^2(D)}$ is uniformly bounded and so the application of Lemma 5.1 gives for $h_n := B^{-1}(l_n)$ that

$$\|\nabla h_n\|_{L^2(D)} \to \infty \quad \text{as } n \to \infty.$$

We conclude by observing that $v_n := u_n - h_n$ satisfies the boundary value problem (1-3) because $u_n = h_n = 0$ on $\partial D \setminus E$ and

$$\begin{cases} -\Delta v_n = -\Delta u_n = da_n \wedge db_n & \text{in } D, \\ \alpha \dfrac{\partial v_n}{\partial \nu} + \beta \dfrac{\partial v_n}{\partial \tau} + \gamma v_n = l_n - B(h_n) = 0 & \text{on } E. \end{cases}$$

The blow-up of the H^1 norm now follows since

$$\|\nabla v_n\|_{L^2(D)} \geq \|\nabla h_n\|_{L^2(D)} - \|\nabla u_n\|_{L^2(D)} \to \infty.$$

□

As before we obtain as a consequence of Theorem 1.3 the following:

Corollary 5.2. *There exists $a, b \in H^1(D)$ with the additional properties $a, b \in L^\infty(D)$ and $da, db \in L^{2,1}(D)$ such that if $u \in W^{1,1}(D)$ denotes the solution to the problem (1-3) with $f = da \wedge db$ then $u \notin H^1(D)$.*

Its combined proof with Corollary 4.2 can be found in the Appendix.

Appendix: Abstract functional analytic argument

Now we want to present the abstract functional analytic argument that leads to Corollaries 4.2 and 5.2. We will first proof an "easier" version where every embedding of the involved spaces is linear. Thereafter we show how the same idea translates to our setting.

Lemma A.1. *Consider Banach spaces $E_1 \subset E_2$ and $F_1 \subset F_2$ such that the inclusion $\subset$ corresponds to a continuous embedding. Let $A : E_2 \to F_2$ be a continuous linear operator. Suppose that F_1 is a Hilbert space and there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ with the properties that*

(a) *$Ax_n \in F_1$ and $\|x_n\|_{E_1} \leq 1$ for all $n \in \mathbb{N}$;*

(b) *$\limsup_{n\to\infty} \|Ax_n\|_{F_1} = \infty$;*

(c) *$f \in F_1 \mapsto \langle Ax_n, f\rangle$ extends to a linear functional l_n on F_2 for each n.*

Then there exists $x \in E_1$ such that $Ax \in F_2 \setminus F_1$ in the sense that there is a sequence $l_n \in F_2^*$ with $\|l_n\|_{F_1^*} \leq 1$ but

$$l_n(Ax) \to \infty.$$

Proof. Passing to a subsequence we may assume that the lim sup in (b) is actually a limit.

In a first step we show by induction that there exists $\{y_1, \dots, y_n\} \in E_1$ with the properties

(i) $\|y_i\|_{E_1} \leq 1$ for all i;

(ii) $\langle Ay_i, Ay_j\rangle = 0$ if $i \neq j$;

(iii) $\|Ay_i\|_{F_1} \geq 2^{2i}$ for all i.

By (b) there exists $m_1 \in \mathbb{N}$ such that $\|Ax_{m_1}\| \geq 4$. Hence we may set $y_1 := x_{m_1}$.

Now suppose $\{y_1, \dots, y_n\}$ have been chosen. We define the linear continuous operator $P_n : F_1 \to F_1$ by

$$P_n := \sum_{i=1}^{n} \frac{Ay_i \otimes Ay_i}{\|Ay_i\|^2}.$$

It is obvious that $P_n = P_n^t$ and (ii) implies that $P_n^2 = P_n$; i.e., P_n is the orthogonal projection onto the finite-dimensional space $V_n := \operatorname{span}\{Ay_1, \dots, Ay_n\}$. Hence $(P_n A) : E_1 \to V_n$ is a continuous linear operator onto a finite-dimensional vector space. Let $(P_n A)^{-1} : V_n \to \operatorname{span}\{y_1, \dots, y_n\}$ denote the inverse of the operator $(P_n A)$ restricted to the finite-dimensional space $\operatorname{span}\{y_1, \dots, y_n\}$. We may define now the operator

$$Q_n : E_1 \to E_1, \quad Q_n := (P_n A)^{-1} \circ (P_n A).$$

We note that Q_n is continuous and $Q_n^2 = Q_n$; hence Q_n is a projection operator. As a direct consequence we have as well that $(I - Q_n)$ is a continuous projection operator; here I denotes the identity map on E_2.

By construction we have

$$P_n\, A\, (I - Q_n) = 0. \tag{A-1}$$

The range of Q_n is finite and $(A Q_n)$ is a continuous operator and therefore

$$\limsup_{m\to\infty} \|(A Q_n) x_m\|_{F_1} < \infty.$$

Hence we have

$$\lim_{m\to\infty} \|A(I - Q_n)x_m\|_{F_1} \geq \lim_{m\to\infty} \|A x_m\|_{F_1} - \limsup_{m\to\infty} \|(A Q_n)x_m\|_{F_1} = \infty.$$

Thus there exists $m_{n+1} \in \mathbb{N}$ such that

$$\|A(I - Q_n)x_{m_{n+1}}\|_{F_1} > 2^{2(n+1)} \|I - Q_n\|.$$

We define $y_{n+1} = (I - Q_n)x_{m_{n+1}} / \|I - Q_n\|$. Clearly we have $\|y_{n+1}\|_{E_1} \leq 1$ and (iii) holds by the choice of m_{n+1}. Finally (ii) follows using that P_n is a orthogonal projection, that Q_n is a projection and (A-1):

$$\langle A y_i, A y_{n+1}\rangle = \langle P_n A y_i, A(I - Q_n) y_{n+1}\rangle = \langle P_n A y_i, (P_n A(I - Q_n)) y_{n+1}\rangle = 0.$$

Having the sequence $\{y_i\}_{i\in\mathbb{N}}$ at our disposal we obtain x as follows: For each n we define the elements $z_n \in E_1$ and $f_n \in F_1$ by

$$z_n := \sum_{i=1}^{n} 2^{-i} y_i \quad \text{and} \quad f_n := \sum_{i=1}^{n} 2^{-i} \frac{A y_i}{\|A y_i\|_{F_1}}.$$

Since E_1, F_1 are Banach spaces we have that their limits exist: $z = \lim_{n\to\infty} z_n = \sum_{i=1}^{\infty} 2^{-i} y_i \in E_1$ and

$$f = \lim_{n\to\infty} f_n = \sum_{i=1}^{\infty} 2^{-i} \frac{A y_i}{\|A y_i\|_{F_1}}.$$

Assumption (c) implies that for each $i \in \mathbb{N}$ the map

$$f \in F_1 \mapsto \left\langle \frac{A y_i}{\|A y_i\|_{F_1}}, f \right\rangle$$

extends to a continuous linear functional $l_i \in F_1^*$. Therefore the continuous linear functional $L_n := \sum_{i=1}^{n} 2^{-i} l_i$ has the desired properties using (i)–(iii) since

$$\begin{aligned} L_n(Az) &= \lim_{m\to\infty} L_n(A z_m) = \lim_{m\to\infty} \langle f_n, A z_m\rangle \\ &= \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} 2^{-i-j} \left\langle \frac{A y_i}{\|A y_i\|_{F_1}}, A y_j \right\rangle = \sum_{i=1}^{n} 2^{-2i} \|A y_i\|_{F_1} \geq n. \end{aligned}$$

□

Observe that we could directly apply the above result with the following choice of spaces: let $E_1 = \mathcal{H}^1_{\text{loc}}(D)$ be the local Hardy space of the disk, $E_2 = L^1(D)$, $F_1 = \{f \in H^1(D) : ⨍_D f = 0\}$ and $F_2 = W^{1,1}(D)$. But this would not give single elements $a, b \in H^1(D)$ as stated in the Corollaries 4.2 and 5.2.

Proof of Corollaries 4.2 and 5.2. We introduce the space

$$X := \left\{h \in H^1(D) : ⨍_D h = 0 \text{ and } dh \in L^{2,1}(D)\right\}.$$

It becomes a complete Banach space with respect to the norm $\|h\|_X := \|dh\|_{L^{2,1}}$. Furthermore as suggested before we set $E_2 := L^1(D),\ F_1 := H^1(D),\ F_2 = W^{1,1}(D)$. Observe that we have a "bilinear" linear embedding of $X \times X \hookrightarrow E_2$ by $(h, k) \mapsto dh \wedge dk$ with $\|dh \wedge dk\|_{L^1} \leq \|dh\|_{L^{2,1}} \|dk\|_{L^{2,1}}$.

The construction of (a, b) out of the contradicting sequence is the same in the case of a Neumann or Robin-type boundary condition. Hence we will give a simultaneous proof for both. We denote by $A : L^1(D) \to W^{1,1}(D)$ the solution operator to problem (1-2) or problem (1-3). Recall that by classical elliptic theory there is a constant $C_A > 0$ such that $\|Ax\|_{H^1} \leq C_A \|x\|_{L^2}$.

Let $(a_n, b_n) \in C^\infty(\bar{D}, \mathbb{R}^2)$ be the corresponding contradicting sequence of Theorem 1.2 or Theorem 1.3. Without loss of generality we may assume that $⨍ a_n = 0 = ⨍ b_n$ for all n; hence $a_n, b_n \in X$. From now on we do not have to distinguish the cases anymore.

We will now proceed approximately as in Lemma A.1. By induction we show the existence of a sequence $\{y_1, y_2, \ldots, y_n\} \in L^1(D) \cap C^\infty(\bar{D})$ with the properties

(i) $\|y_i\|_{L^1} \leq 1$ for all i;

(ii) $\langle Ay_i, Ay_j\rangle = 0$ if $i \neq j$;

(iii) $\|Ay_i\|_{F_1} \geq 2^{3i}$ for all i.

Simultaneously we will construct a sequence of tuples $(h_i, k_i) \in X \cap C^\infty(\bar{D}) \times X \cap C^\infty(\bar{D}),\ i = 1, \ldots, n$, such that

(1) $\|h_i\|_{L^\infty} + \|dh_i\|_{L^{2,1}} + \|k_i\|_{L^\infty} + \|dk_i\|_{L^{2,1}} \leq 1$;

(2) $dh_i \wedge dk_i = y_i + R_i$ with $\|R_i\|_{L^2} \leq 1$;

(3) $\|dh_i\|_{L^2} + \|dk_i\|_{L^2} \leq \left(1 + \sum_{j<i} \|dh_j\|_{L^\infty} + \|dk_j\|_{L^\infty}\right)^{-1}$.

We start the induction by choosing (a_1, b_1) in the contradicting sequence such that $\|A(da_1 \wedge db_1)\| > 2^2$. We set $y_1 = da_1 \wedge db_1$ and $(h_1, k_1) = (a_1, b_1)$. All properties are clearly satisfied ($R_1 = 0$).

Now suppose that we have chosen $y_i, (h_i, k_i)$ for $i = 1, \ldots, n$. We want to construct y_{n+1} and the tuple (h_{n+1}, k_{n+1}). As in the previous lemma we define the projection operators

$$P_n := \sum_{i=1}^{n} \frac{Ay_i \otimes Ay_i}{\|Ay_i\|^2}, \qquad Q_n := (P_n A)^{-1}(P_n A).$$

Here $(P_n A)^{-1}$ denotes as before the inverse of $(P_n A)$ if restricted to the space $\operatorname{span}\{y_1, \ldots, y_n\}$. Hence for all $x \in L^1(D)$ we have $Q_n x = \sum_{i=1}^n \alpha_i y_i$ and the existence of a constant $C_n > 0$ such that $\sum_{i=1}^n |\alpha_i| \leq C_n$ for all $x \in L^1(D)$ with $\|x\|_{L^1} \leq 1$. Furthermore due to the properties of the contradicting sequence, there exists $m \in \mathbb{N}$ such that

$$\|A(I - Q_n) da_m \wedge db_m\|_{H^1} \geq 2^{3(n+1)} C_n^2 \left(n + 3 + \sum_{j \leq n} \|dh_j\|_{L^\infty} + \|dk_j\|_{L^\infty}\right)^2.$$

Let $Q_n da_m \wedge db_m = \sum_{i=1}^n \alpha_i y_i$, and define the elements

$$\tilde{y}_{n+1} := (I - Q_n)da_m \wedge db_m, \quad \tilde{h}_{n+1} := a_m - \sum_{i=1}^n \alpha_i h_i, \quad \tilde{k}_{n+1} := b_m + \sum_{i=1}^n k_i.$$

We calculate

$$\begin{aligned} d\tilde{h}_{n+1} \wedge d\tilde{k}_{n+1} &= da_m \wedge db_m - \sum_{i=1}^n (\alpha_i dh_i \wedge dk_i) \\ &\quad + \underbrace{d\Big(-\sum_{i=1}^n \alpha_i h_i\Big) \wedge db_m}_{(I)} + \underbrace{da_m \wedge d\Big(\sum_{i=1}^n k_i\Big)}_{(II)} - \underbrace{\sum_{i<j} (\alpha_i dh_i \wedge dk_j + \alpha_j dh_j \wedge dk_i)}_{(III)} \\ &\overset{(2)}{=} da_m \wedge db_m - \sum_{i=1}^n \alpha_i y_i - \sum_{i=1}^n \alpha_i R_i + (I) + (II) + (III). \end{aligned}$$

We estimate the size of the remainder terms in $L^2(D)$: Due to (2), we have $\left\| \sum_{i=1}^n \alpha_i R_i \right\|_{L^2} \leq C_n$. The terms (I), (II) are similarly estimated by

$$\left\| d\Big(-\sum_{i=1}^n \alpha_i h_i\Big) \wedge db_m \right\|_{L^2} \leq \Big(\sum_{i=1}^n |\alpha_i| \|dh_i\|_{L^\infty}\Big) \|db_m\|_{L^2},$$

$$\left\| da_m \wedge d\Big(\sum_{i=1}^n k_i\Big) \right\|_{L^2} \leq \Big(\sum_{i=1}^n \|dk_i\|_{L^\infty}\Big) \|da_m\|_{L^2}.$$

Adding both we obtain $\|(I)\|_{L^2} + \|(II)\|_{L^2} \leq C_n \big(1 + \sum_{j \leq n} \|dh_j\|_{L^\infty} + \|dk_j\|_{L^\infty}\big)$. The last term can be estimated using only property (3) by

$$\begin{aligned} \|(III)\|_{L^2} &\leq \sum_{i=1}^n |\alpha_i| \, \|dh_i\|_{L^2} \Big(\sum_{j<i} \|dk_j\|_{L^\infty}\Big) + \|dk_i\|_{L^2} \Big(\sum_{j<i} |\alpha_j| \, \|dh_j\|_{L^\infty}\Big) \\ &\leq \Big(\sum_{i=1}^n |\alpha_i|\Big) + \sup_{j \leq n} |\alpha_j| n \leq (n+1) C_n. \end{aligned}$$

Hence we found that $\|\widetilde{R}_{n+1}\|_{L^2} \leq C_n \big(n + 3 + \sum_{j \leq n} \|dh_j\|_{L^\infty} + \|dk_j\|_{L^\infty}\big)$, where $\widetilde{R}_{n+1} = -\sum_{i=1}^n \alpha_i R_i + (I) + (II) + (III)$ and

$$d\tilde{h}_{n+1} \wedge d\tilde{k}_{n+1} = (I - Q_n) da_m \wedge db_m + \widetilde{R}_{n+1} = \tilde{y}_{n+1} + \widetilde{R}_{n+1}.$$

The desired functions are now simply

$$y_{n+1} = \frac{\tilde{y}_{n+1}}{\lambda_n}, \quad h_{n+1} = \frac{\tilde{h}_{n+1}}{\lambda_n}, \quad k_{n+1} = \frac{\tilde{k}_{n+1}}{\lambda_n}, \quad \text{with } \lambda_n = C_n \Big(n + 3 + \sum_{j \leq n} \|dh_j\|_{L^\infty} + \|dk_j\|_{L^\infty}\Big).$$

Having established the existence of the sequences y_i, h_i, k_i with the claimed properties we construct $a, b \in X$ and a sequence $f_n \in H^1(D) = F_1$ very much as in the proof of Lemma A.1: Due to (1) and the fact that X is a complete Banach space we can define elements

$$a := \sum_{i=1}^{\infty} 2^{-i} h_i, \quad b := \sum_{i=1}^{\infty} 2^{-i} k_i.$$

Furthermore for each $n \in \mathbb{N}$ let

$$f_n := \sum_{i=1}^{n} 2^{-i} \frac{A y_i}{\|A y_i\|_{H^1}}.$$

Observe that f_n is a finite sum of C^1-functions; hence it is C^1 and can therefore be considered as an element of $(L^1)^* = L^\infty$. It remains to check that $\lim_{n\to\infty} \int_D f_n A(da \wedge db) = +\infty$. We have

$$A(da \wedge db) = \lim_{m\to\infty} \sum_{i=1}^{m} 2^{-2i} A(dh_i \wedge dk_i) + \sum_{i<j}^{m} 2^{-i-j} A(dh_i \wedge dk_j + dh_j \wedge dk_i).$$

Using (2) we estimate

$$\left\langle \frac{A y_k}{\|A y_k\|_{H^1}}, A(dh_i \wedge dk_i) \right\rangle = \left\langle \frac{A y_k}{\|A y_k\|_{H^1}}, A y_i + A R_i \right\rangle \geq \delta_{ki} \|A y_i\|_{H^1} - C_A \|R_i\|_{L^2} \geq \delta_{ki} \|A y_i\|_{H^1} - C_A.$$

Hence

$$\sum_{i=1}^{m} 2^{-2i} \left\langle \frac{A y_k}{\|A y_k\|_{H^1}}, A(dh_i \wedge dk_i) \right\rangle \geq 2^{-2k} \|A y_k\|_{H^1} - \lim_{m\to\infty} \sum_{i=1}^{m} 2^{-2i} C_A \geq 2^k - C_A.$$

Using (3) we get

$$\begin{aligned} \sum_{i<j}^{m} 2^{-i-j} \|A(dh_i \wedge dk_j + dh_j \wedge dk_i)\|_{H^1} &\leq C_A \sum_{i<j}^{m} 2^{-i-j} (\|dh_i\|_{L^2} \|dk_j\|_{L^\infty} + \|dh_j\|_{L^\infty} \|dk_i\|_{L^2}) \\ &\leq C_A \sum_{i=1}^{m} 2^{-i} 2 \leq 2C_A. \end{aligned}$$

Finally combining both we obtain

$$\left\langle \frac{A y_k}{\|A y_k\|_{H^1}}, A(da \wedge db) \right\rangle \geq 2^k - 3C_A.$$

This completes the estimate since

$$\int_D f_n A(da \wedge db) = \sum_{k=1}^{n} 2^{-k} \left\langle \frac{A y_k}{\|A y_k\|_{H^1}}, A(da \wedge db) \right\rangle \geq n - 3C_A. \qquad \square$$

Acknowledgments

First of all the author thanks Vincent Millot for proposing the consideration of Möbius transformations and in this way improving the result. Furthermore he thanks Guido De Philippis for various useful discussions and he wants to thank the anonymous referees for carefully reading an earlier version of this manuscript, and for their valuable comments and suggestions. The author is supported by the MIUR SIR-grant Geometric Variational Problems (RBSI14RVEZ).

References

[Ancona 2009] A. Ancona, "Elliptic operators, conormal derivatives and positive parts of functions", *J. Funct. Anal.* **257**:7 (2009), 2124–2158. MR Zbl

[Colding and Minicozzi 2008] T. H. Colding and W. P. Minicozzi, II, "Width and finite extinction time of Ricci flow", *Geom. Topol.* **12**:5 (2008), 2537–2586. MR Zbl

[Da Lio and Palmurella 2017] F. Da Lio and F. Palmurella, "Remarks on Neumann boundary problems involving Jacobians", *Comm. Partial Differential Equations* **42**:10 (2017), 1497–1509. MR Zbl

[Gilbarg and Trudinger 1998] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften **224**, Springer, 1998.

[Grafakos 2014] L. Grafakos, *Classical Fourier analysis*, 3rd ed., Graduate Texts in Mathematics **249**, Springer, 2014. MR Zbl

[Hélein 2002] F. Hélein, *Harmonic maps, conservation laws and moving frames*, 2nd ed., Cambridge Tracts in Mathematics **150**, Cambridge University Press, 2002. MR Zbl

[Lamm and Lin 2013] T. Lamm and L. Lin, "Estimates for the energy density of critical points of a class of conformally invariant variational problems", *Adv. Calc. Var.* **6**:4 (2013), 391–413. MR Zbl

[Littman et al. 1963] W. Littman, G. Stampacchia, and H. F. Weinberger, "Regular points for elliptic equations with discontinuous coefficients", *Ann. Scuola Norm. Sup. Pisa* (3) **17** (1963), 43–77. MR Zbl

[Rivière 2008] T. Rivière, "Analysis aspects of Willmore surfaces", *Invent. Math.* **174**:1 (2008), 1–45. MR Zbl

[Semmes 1994] S. Semmes, "A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller", *Comm. Partial Differential Equations* **19**:1-2 (1994), 277–319. MR Zbl

[Topping 1997] P. Topping, "The optimal constant in Wente's L^∞ estimate", *Comment. Math. Helv.* **72**:2 (1997), 316–328. MR Zbl

[Wente 1971] H. C. Wente, "An existence theorem for surfaces of constant mean curvature", *Bull. Amer. Math. Soc.* **77** (1971), 200–202. MR Zbl

Received 19 Jul 2017. Revised 15 Apr 2018. Accepted 23 Jul 2018.

JONAS HIRSCH: jonas.hirsch@sissa.it
Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2019 is US $310/year for the electronic version, and $520/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 12 No. 4 2019