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CLASSIFICATION OF POSITIVE SINGULAR SOLUTIONS TO A NONLINEAR BIHARMONIC EQUATION WITH CRITICAL EXPONENT

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For $n \geq 5$, we consider positive solutions u of the biharmonic equation

$$\Delta^2 u = u^{(n+4)/(n-4)} \quad \text{on } \mathbb{R}^n \setminus \{0\},$$

with a nonremovable singularity at the origin. We show that $|x|^{(n-4)/2} u$ is a periodic function of $\ln |x|$ and we classify all periodic functions obtained in this way. This result is relevant for the description of the asymptotic behavior of local solutions near singularities and for the Q-curvature problem in conformal geometry.

1. Introduction and main results

In this paper we are interested in positive solutions u of the equation

$$\Delta^2 u = u^{(n+4)/(n-4)} \quad \text{in } \mathbb{R}^n \setminus \{0\} \tag{1}$$

for $n \geq 5$. As we will explain later in more detail, this equation serves on one hand as a model problem for higher-order equations with critical nonlinearity and on the other hand has a concrete meaning in the Q-curvature problem in conformal geometry. It is well known that the absence of the maximum principle for equations involving the bi-Laplacian poses great challenges both on a conceptual and on a technical level. Nevertheless we succeed here in proving a classification result for positive solutions of (1) which is completely analogous to its second-order counterpart.

We will work throughout with classical solutions of (1), that is, $u \in C^4(\mathbb{R}^n \setminus \{0\})$. Because of the regularity theory in [Uhlenbeck and Viaclovsky 2000] (which extends that in [Chang et al. 1999] to $n \geq 5$) this is not a restriction.

In the fundamental work [Lin 1998] it was shown that all solutions u with a *removable singularity* at the origin (so that (1) holds in all of $\mathbb{R}^n$) are given by

$$u(x) = c_n \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{(n-4)/2}, \quad c_n = \big((n-4)(n-2)n(n+2) \big)^{(n-4)/8}, \tag{2}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Solutions of the closely related equation $\Delta^2 u = |u|^{8/(n-4)} u$ in $\mathbb{R}^n$ are, in particular, given by optimizers of the Sobolev inequality

$$\int_{\mathbb{R}^n} (\Delta u)^2 \, \mathrm{d}x \geq \mathcal{S}_n \left(\int_{\mathbb{R}^n} |u|^{2n/(n-4)} \, \mathrm{d}x \right)^{(n-4)/n}.$$

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These optimizers were classified in [Lieb 1983] in an equivalent dual formulation and are again given by constant multiples of the functions in (2). For a classification of positive solutions with removable singularities of the four-dimensional analogue of (1) we refer to [Chang and Yang 1997; Lin 1998] and for the higher-order case to [Wei and Xu 1999; Martinazzi 2009].

In this paper we will be concerned with solutions u of (1) with *nonremovable singularities*. It was also shown by Lin [1998] that such solutions are necessarily radial. We pass to logarithmic coordinates (in this context also known as *Emden–Fowler coordinates*) and write

$$u(x)=|x|^{-(n-4)/2}v(\ln|x|).$$

By a short computation we find that (1) for u is equivalent to the following ordinary differential equation for v:

$$v^{(4)}-\frac{n(n-4)+8}{2}v''+\frac{n^2(n-4)^2}{16}v-|v|^{8/(n-4)}v=0 \quad \text{in } \mathbb{R}. \tag{3}$$

Note that positive solutions u of (1) correspond to positive solutions v of (3) and so $|v|^{8/(n-4)}v = v^{(n+4)/(n-4)}$. For some of our results, however, we also need to consider not necessarily positive functions v, and for such functions (3) is the relevant extension. We set

$$a_0=\left(\frac{n(n-4)}{4}\right)^{(n-4)/4}.$$

Our first main result classifies all positive periodic solutions of (3) and describes their shape.

Theorem 1. (i) *Let $v\in C^4(\mathbb{R})$ be a solution of* (3). *Then* $\inf_{\mathbb{R}}|v|\le a_0$, *with equality if and only if v is a nonzero constant.*

(ii) *Let $a\in(0,a_0)$. Then there is a unique* (*up to translations*) *bounded solution $v\in C^4(\mathbb{R})$ of* (3) *with minimal value a. This solution is periodic*, *has a unique local maximum and minimum per period and is symmetric with respect to its local extrema.*

To state our second main result, we denote by v_a the unique solution to (3) obtained from Theorem 1 by requiring that $v_a(0)=\min_{\mathbb{R}} v_a = a$. Also, denote by L_a the minimal period of v_a. For the constant solution $v_{a_0}\equiv a_0$, we set $L_{a_0}=0$.

The following theorem provides a classification of positive solutions u of (1) with nonremovable singularities in terms of a two-parameter family.

Theorem 2. *Let $u\in C^4(\mathbb{R}^n\setminus\{0\})$ be a positive solution of* (1) *whose singularity at the origin is nonremovable. Then there are $a\in(0,a_0]$ and $L\in[0,L_a]$ such that*

$$u(x)=|x|^{-(n-4)/2}v_a(\log|x|+L),$$

where v_a is the solution of (3) *introduced after Theorem 1. Moreover*, $\partial u/\partial|x|<0$ *for all $x\in\mathbb{R}^n\setminus\{0\}$.*

This theorem answers an open question raised in [Guo et al. 2017a] and shows, in particular, that the positivity of the scalar curvature in their conjecture is not necessary.

It is easy to see that as $a \to 0$ one has $L_a \to \infty$ and $v_a(t + L_a/2) \to c_n(2\cosh t)^{-(n-4)/2}$. Undoing the logarithmic change of variables we therefore recover the nonsingular solution (2) in the limit $a \to 0$.

We believe that Theorems 1 and 2 will have several applications. Firstly, they should be a key step in describing the asymptotic behavior near the origin of positive solutions u of $\Delta^2 u = u^{(n+4)/(n-4)}$ in a punctured ball $\{0 < |x| < \rho\}$. This would be the fourth-order analogue of a celebrated result of Caffarelli, Gidas and Spruck [Caffarelli et al. 1989]; see also [Korevaar et al. 1999]. Secondly, we believe that our theorems will prove useful in the construction of constant Q-curvature metrics with isolated singularities in the spirit of the classical works [Schoen 1988; Mazzeo and Pacard 1999] for the scalar curvature; see [Baraket and Rebhi 2002; Guo et al. 2017b] for results in this direction in the fourth-order case. For an introduction to the Q-curvature problem see, for instance, [Hang and Yang 2016].

We end this introduction by comparing the statements and proofs of Theorems 1 and 2 with their second-order counterpart, which concerns positive solutions u of

$$-\Delta u = u^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n \setminus \{0\} \tag{4}$$

for $n \geq 3$. A famous result of Caffarelli, Gidas and Spruck [Caffarelli et al. 1989] says that if this equation is valid on all of $\mathbb{R}^n$, then

$$u(x) = c_n' \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{(n-2)/2}, \qquad c_n' = (n(n-2))^{(n-2)/4},$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Moreover, they show that if u is a positive solution of (4) with a nonremovable singularity, then u is radial. Using this information, Schoen [1989] observed that all solutions can be classified by standard phase-plane analysis. Indeed, setting

$$u(x) = |x|^{-(n-2)/2} v(\ln |x|)$$

one obtains

$$-v'' + \frac{(n-2)^2}{4} v - v^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbb{R}$$

and the positive solutions of this equation are given by the constant $((n-2)/2)^{(n-2)/2}$, by the homoclinic solution $c_n'(2\cosh(t+T))^{(n-2)/2}$ and by periodic solutions uniquely parametrized, up to translations, by their minimal value in $(0, ((n-2)/2)^{(n-2)/2})$. Moreover, these periodic solutions have a unique local maximum and minimum per period and are symmetric with respect to their local extrema.

Thus, our Theorems 1 and 2 provide exactly the same conclusions as in the second-order case. Their proofs, however, are considerably more difficult, because the phase "plane" in the fourth-order case is four-dimensional. Moreover, solutions to fourth-order equations show, in general, a much richer and typically more erratic behavior than solutions to second-order equations; see, e.g., the introduction of the textbook [Peletier and Troy 2001] for examples. To emphasize the structure of our equation we abbreviate

$$A = \frac{n(n-4)+8}{2}, \qquad B = \frac{n^2(n-4)^2}{16}, \qquad p = \frac{n+4}{n-4}, \tag{5}$$

and

$$f(v) = |v|^{p-1}v - Bv \tag{6}$$

and rewrite (3) as

$$v^{(4)} - Av'' - f(v) = 0 \quad \text{in } \mathbb{R}. \tag{7}$$

Of fundamental importance for us is that the coefficients A and B in (3) satisfy the inequalities

$$A > 0 \quad \text{and} \quad 4B < A^2. \tag{8}$$

These inequalities guarantee that the characteristic equation $\xi^4 - A\xi^2 + B = 0$ associated to the linearization of (7) around the zero solution has four distinct, real solutions. The picture that has emerged from the analysis of fourth-order equations is that under this structural assumption the solution set is better behaved than that of general fourth-order equations and resembles in some sense the solution set of second-order equations; see, e.g., [Peletier and Troy 2001; van den Berg 2000; Buffoni et al. 1996]. The reason is that certain techniques are available which are reminiscent of the maximum principle. Technically, this better, second-order-like behavior can be proved for *bounded* solutions of the equation and for such solutions there are certain substitutes for two-dimensional phase-plane arguments (see, in particular, Propositions 4 and 6). Parts of our analysis will rely on results of van den Berg [2000] for bounded solutions, which in turn rely on results of Buffoni, Champneys and Toland [Buffoni et al. 1996]. Our crucial new ingredient, however, which does not appear in these works, is that *global solutions are necessarily bounded* (Lemma 11). We emphasize that boundedness is a nonlocal property and breaks the local character of the ODE analysis.

Most of our results (except for the explicit expression of the homoclinic solution) hold, mutatis mutandis, for any equation of the form (7) with f given by (6), where $p > 1$ is arbitrary and A and B are arbitrary subject to (8).

2. Classification of global ODE solutions

In this section we will classify all solutions v of (7) which are defined on all of $\mathbb{R}$. Positivity will not play a role here.

We begin with some preliminary remarks, which we will use several times below. The function $v \mapsto f(v)$ in (6) has exactly three zeros, namely, at 0 and at $\pm B^{1/(p-1)} = \pm a_0$. These correspond to exactly three constant solutions. Moreover, if $v(t)$ is a solution to (7), then so are the functions

- $v(-t)$ (because (7) contains only even-order derivatives),
- $-v(t)$ (because f is odd) and
- $v(t+T)$ for any $T \in \mathbb{R}$ (because (7) is autonomous).

We now state the main result of this section.

Proposition 3. *Let $v \in C^4(\mathbb{R})$ be a solution of* (7). *Then one of the following three alternatives holds*:

(a) $v \equiv \pm B^{1/(p-1)}$, *or* $v \equiv 0$.

(b) $v(t) = \pm c_n (2\cosh(t-T))^{-(n-4)/2}$ *for some* $T \in \mathbb{R}$ *with* c_n *from* (2).

(c) v *is periodic, has a unique local maximum and minimum per period and is symmetric with respect to its local extrema.*

For the proof of this proposition we will need two results, taken from [van den Berg 2000], which quantify the intuition that the set of bounded solutions to the fourth-order equation (7) behaves in some respects similarly to the set of solutions of a second-order equation. As we pointed out in the introduction, for this it is crucial that the relation $4B < A^2$ is satisfied. The first result is that every *bounded* entire solution v is uniquely determined by only *two* (instead of four) initial values.

Proposition 4 [van den Berg 2000, Theorem 1]. *Let* $v, w \in C^4(\mathbb{R})$ *be bounded solutions of* (7) *and suppose that* $v(0) = w(0)$ *and* $v'(0) = w'(0)$. *Then* $v \equiv w$.

Since this result is of crucial importance for us, we give a (slightly more direct) proof with our notation in the Appendix. Proposition 4 has the following consequence.

Corollary 5. *Let* $v \in C^4(\mathbb{R})$ *be a bounded solution of* (7):

(i) *Suppose that* $v'(t_0) = 0$ *for some* $t_0 \in \mathbb{R}$. *Then* v *is symmetric with respect to* t_0; *i.e., for all* $t \in \mathbb{R}$, $v(t_0 + t) = v(t_0 - t)$.

(ii) *Suppose that* $v(t_0) = 0$ *for some* $t_0 \in \mathbb{R}$. *Then* v *is antisymmetric with respect to* t_0; *i.e., for all* $t \in \mathbb{R}$, $v(t_0 - t) = -v(t_0 + t)$.

Proof. (i) Since (7) is autonomous, we may assume $t_0 = 0$. Moreover, if v is a solution, then so is $w(t) := v(-t)$. Thus $v(0) = w(0)$ and, by assumption, $v'(0) = w'(0) = 0$. Proposition 4 gives $v \equiv w$.

(ii) Again, we may assume $t_0 = 0$. Moreover, if v solves (7), then so does $w(t) := -v(-t)$. Since $v(0) = w(0)$ and $v'(0) = w'(0)$, we conclude by Proposition 4 that $v \equiv w$. □

In order to state the second result from [van den Berg 2000] that we need, we introduce

$$F(v) = \int_0^v f(s)\,\mathrm{d}s = \frac{|v|^{p+1}}{p+1} - \tfrac{1}{2} B v^2$$

as well as the following quantity, also referred to as the *energy*:

$$\mathcal{E}_v(t) = -v'''(t)v'(t) + \tfrac{1}{2}(v''(t))^2 + \tfrac{1}{2} A (v'(t))^2 + F(v(t)).$$

Using (7) one easily finds that for every solution v of (7)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_v(t) = 0;$$

that is, the *energy is conserved*. We emphasize that this conservation is a local property and valid on the maximal interval of existence and does not require any a priori boundedness assumptions like Proposition 4 and the following Proposition 6 and Lemma 7.

The second result says that, as in the second-order case, the energy is a parameter which orders bounded solutions in the (v, v')-phase plane.

Proposition 6 [van den Berg 2000, Theorem 2]. *Let $v, w \in C^4(\mathbb{R})$ be bounded solutions of* (7) *with $v(0) = w(0)$ and either $v'(0) > w'(0) \geq 0$ or $v'(0) < w'(0) \leq 0$. Then $\mathcal{E}_v > \mathcal{E}_w$.*

For the proof we refer to [van den Berg 2000]. The assumption there is satisfied since $4B < A^2$. (Note that no a priori bound on the solutions is necessary for our f.)

Next, we state two lemmas concerning the asymptotic behavior of solutions at infinity.

Lemma 7 [van den Berg 2000, Lemma 4]. *Let $v \in C^4(\mathbb{R})$ be a bounded solution of* (7). *If v is eventually monotone for $t \to \infty$, then*

$$\lim_{t\to\infty} v(t) \in \{0, \pm B^{1/(p-1)}\} \qquad \text{and} \qquad \lim_{t\to\infty} v^{(k)}(t) = 0 \quad \text{for } k = 1, 2, 3.$$

Similarly, if v is eventually monotone for $t \to -\infty$, then

$$\lim_{t\to-\infty} v(t) \in \{0, \pm B^{1/(p-1)}\} \qquad \text{and} \qquad \lim_{t\to-\infty} v^{(k)}(t) = 0 \quad \text{for } k = 1, 2, 3.$$

The following lemma from [Guo et al. 2017a] shows that (7) does not have a solution which tends to either plus or minus infinity at infinity; that is, solutions that blow up do so in finite time.

Lemma 8 [Guo et al. 2017a, Lemma 2.1]. *Let $v \in C^4(\mathbb{R})$ be a solution of* (7). *If $a_+ := \lim_{t\to\infty} v(t) \in \mathbb{R} \cup \{\pm\infty\}$ exists, then $a_+ \in \mathbb{R}$. Similarly, if $a_- := \lim_{t\to-\infty} v(t) \in \mathbb{R} \cup \{\pm\infty\}$ exists, then $a_- \in \mathbb{R}$.*

This lemma is proved in [Guo et al. 2017a] for *positive* solutions. An inspection of the proof shows, however, that this positivity is not needed.

We now use the above results to show uniqueness, up to translations, of the positive homoclinic solution. A similar result for $p = 2$ appears in [Amick and Toland 1992] with a different proof.

Lemma 9. *Let $v, w \in C^4(\mathbb{R})$ be positive solutions of* (7) *with $\lim_{|t|\to\infty} v(t) = \lim_{|t|\to\infty} w(t) = 0$ and $v'(0) = w'(0) = 0$. Then $v \equiv w$.*

Proof. Let us first prove that 0 is the only zero of v' and w'. Indeed, if v' had another zero at, say, $t_0 > 0$, then by repeated application of Corollary 5 (note that by assumption, v is bounded) we deduce that v must be periodic of period $2t_0$. In particular $0 < v(0) = v(2kt_0)$ for all $k \in \mathbb{N}$, which contradicts the assumption that $v(t) \to 0$ as $t \to \infty$. The argument for w is analogous. Hence we must have

$$v'(t) < 0 \quad \text{and} \quad w'(t) < 0 \quad \text{for all } t > 0. \tag{9}$$

Next, by Lemma 7 and by energy conservation,

$$\mathcal{E}_v = \lim_{t\to\infty} \mathcal{E}_v(t) = F(0) = 0 \quad \text{and} \quad \mathcal{E}_w = \lim_{t\to\infty} \mathcal{E}_w(t) = F(0) = 0. \tag{10}$$

If $v(0) = w(0)$, we are done by Proposition 4.

To complete the proof, let us suppose for contradiction that $v(0) > w(0)$. We claim that this implies $v > w$ everywhere. Indeed, otherwise there is $t_0 > 0$ such that $v > w$ on $[0, t_0)$ and $v(t_0) = w(t_0)$. Then by (9) we infer that $v'(t_0) \leq w'(t_0) < 0$. If $v'(t_0) = w'(t_0)$, then Proposition 4 implies $v \equiv w$, contradicting $v(0) > w(0)$. If $v'(t_0) < w'(t_0) < 0$, then Proposition 6 implies $\mathcal{E}_v > \mathcal{E}_w$, which contradicts (10). Hence $v > w$ everywhere.

We can now derive the desired contradiction. For every $R > 0$, we have, using integration by parts and the fact that v and w satisfy (7),

$$\begin{aligned} 0 &= \int_{-R}^{R} w(v^{(4)} - Av'' - f(v)) \\ &= b(R) + \int_{-R}^{R} v(w^{(4)} - Aw'' - f(w)) + \int_{-R}^{R} wv(w^{p-1} - v^{p-1}) \\ &= b(R) + \int_{-R}^{R} wv(w^{p-1} - v^{p-1}). \end{aligned}$$

Here, $b(R)$ contains all the boundary terms coming from the integrations by parts. By Lemma 7 we have $b(R) \to 0$ as $R \to \infty$. But since $\int_{-R}^{R} wv(w^{p-1} - v^{p-1})$ is a negative and strictly decreasing function of R, we obtain a contradiction by choosing R large enough. □

For the concrete values of A, B and p in (5) one can compute the homoclinic solution explicitly. We emphasize that this is the only place in the proof of Proposition 3 where the precise form of A, B and p enters.

Corollary 10. *Suppose that v is a positive solution of* (3) *with* $\lim_{|t|\to\infty} v(t) = 0$. *Then there is $T \in \mathbb{R}$ such that*

$$v(t) = c_n(2\cosh(t - T))^{-(n-4)/2}, \quad t \in \mathbb{R},$$

with c_n from (2).

Proof. A straightforward calculation shows that $w(t) = c_n(2\cosh(t))^{-(n-4)/2}$ solves (3). From the assumptions on v it follows that v has a global maximum at some $T \in \mathbb{R}$. Since $v'(T) = 0$, we can apply Lemma 9 to deduce that $v(\,\cdot\, + T) = w$. □

The following lemma is one of the key new results in this paper.

Lemma 11. *Let $v \in C^4(\mathbb{R})$ be a solution of* (7). *Then v is bounded.*

Proof. By replacing $v(t)$ by $v(-t)$, we only need to show that v is bounded on $[0, \infty)$. We consider the set $Z_+ = \{t \geq 0 : v'(t) = 0\}$.

If Z_+ is bounded (in particular, if it is empty), then v is monotone for large t and thus admits a limit a_+ as $t \to \infty$. By Lemma 8, a_+ is finite and therefore v is bounded on $[0, \infty)$.

We now assume that Z_+ is unbounded. Since $F(u) \to \infty$ as $|u| \to \infty$, there is an $R > |v(0)|$ such that $F(u) > \mathcal{E}_v$ for all $|u| \geq R$. We claim that $|v| < R$ on $[0, \infty)$ which, in particular, implies that v is bounded on $[0, \infty)$. Indeed, by contradiction assume that $M_R := \{t \geq 0 : |v(t)| \geq R\}$ is nonempty and define $t^* := \inf M_R$. Since $|v(0)| < R$, we must have $t^* > 0$ and $|v(t^*)| = R$. Replacing $v(t)$ by $-v(t)$ if necessary (which does not change the set Z_+), we may assume that $v(t^*) = R$. Then also $v'(t^*) \geq 0$. Since Z_+ is unbounded, the set $Z_+ \cap [t^*, \infty)$ is nonempty and we can set $T := \inf(Z_+ \cap [t^*, \infty))$. Then $v'(T) = 0$ and $v' \geq 0$ on $[t^*, T]$ by continuity of v'. Thus $v(T) \geq v(t^*) = R$, and we deduce that

$$\mathcal{E}_v(T) = \tfrac{1}{2} v''(T)^2 + F(v(T)) \geq F(v(T)) > \mathcal{E}_v,$$

a contradiction to energy conservation. □

We are now ready to prove the main result of this section.

Proof of Proposition 3. Let $v \in C^4(\mathbb{R})$ be a solution to (7) and set

$$Z := \{t \in \mathbb{R} : v'(t) = 0\}.$$

We distinguish several cases:

Suppose first that $Z = \varnothing$, so v is strictly monotone. We will show that this case cannot occur. Up to replacing $v(t)$ by $v(-t)$, we may assume that v is strictly increasing, and so both limits $a_\pm = \lim_{t\to\pm\infty} v(t)$ exist in $\mathbb{R} \cup \{\pm\infty\}$. By Lemma 8 both limits are finite. By Lemma 7, we are reduced to studying three cases, each of which will lead to a contradiction via an energy argument.

If $a_- = 0$ and $a_+ = B^{1/(p-1)}$, then using Lemma 7 we get $\lim_{t\to-\infty} \mathcal{E}_v(t) = F(0) = 0$, while $\lim_{t\to+\infty} \mathcal{E}_v(t) = F(B^{1/(p-1)}) < 0$, a contradiction to energy conservation. Analogously, a contradiction is obtained if $a_- = -B^{1/(p-1)}$ and $a_+ = 0$.

It remains to consider the case $a_- = -B^{1/(p-1)}$ and $a_+ = B^{1/(p-1)}$. Then as above, by Lemma 7,

$$\lim_{|t|\to\infty} \mathcal{E}_v(t) = F(B^{1/(p-1)}) < 0. \tag{11}$$

On the other hand, by [van den Berg 2000, Corollary 6], the inequality

$$\mathcal{E}_v(t) \geq \tfrac{1}{2} v''(t)^2 + F(v(t)) \tag{12}$$

holds for all $t \in \mathbb{R}$. But now evaluating the energy at t_0 such that $v(t_0) = 0$ gives, together with (12), that $\mathcal{E}_v(t_0) \geq \frac{1}{2} v''(t_0)^2 + F(0) \geq 0$, in contradiction to (11) and energy conservation. Altogether, we have shown that the case $Z = \varnothing$ cannot occur.

If $|Z| = 1$, we may assume, up to a translation, that $Z = \{0\}$. Then v is strictly monotone on $(-\infty, 0)$ and $(0, \infty)$, and so both limits $a_\pm = \lim_{t\to\pm\infty} v(t)$ exist in $\mathbb{R} \cup \{\pm\infty\}$. By Lemma 8 these limits are finite, so v is bounded and, by Corollary 5, even. Therefore $a_+ = a_-$. By Lemma 7, only three cases can occur: $a_+ = a_- = 0$ or $a_+ = a_- = \pm B^{1/(p-1)}$. In the first case, monotonicity implies that either $v > 0$ or $v < 0$, and we conclude that $v(t) = \pm c_n (2\cosh(t))^{-(n-4)/2}$ by Corollary 10.

As for the other cases, let us assume without loss of generality that $a_+ = a_- = B^{1/(p-1)}$ (otherwise replace v by $-v$). We derive a contradiction as follows. Since v is strictly monotone on $[0, \infty)$, $v(0) \neq B^{1/(p-1)}$, and from $\mathcal{E}_v(0) = \frac{1}{2} v''(0)^2 + F(v(0)) \geq F(B^{1/(p-1)})$ we infer that $v(0) = -B^{1/(p-1)}$ (since F attains its global minimal value only at $\pm B^{1/(p-1)}$). Hence v changes sign; i.e., there is $t_0 \in \mathbb{R}$ such that $v(t_0) = 0$. By Corollary 5, v is antisymmetric with respect to t_0. But this is a contradiction to the fact that both a_+ and a_- are positive. Altogether we have thus shown that if $|Z| = 1$, then $v(t) = \pm c_n (2\cosh(t))^{-(n-4)/2}$.

Finally, let us consider the case where $|Z| \geq 2$. By continuity of v', we see that unless v is constant (and hence $v \equiv \pm B^{1/(p-1)}$ or $v \equiv 0$), the closed set Z cannot be dense; i.e., there are real numbers $c < d$ such that $v'(c) = v'(d) = 0$ and $v' \neq 0$ on (c, d). By Lemma 11, v is bounded and therefore we can use Corollary 5 as in the first part of the proof of Lemma 9 to conclude that v must be periodic of period $2(d - c)$. Moreover, since v is strictly monotone on (c, d), there is only one maximum and minimum per period interval, and these are strict. The symmetry with respect to the extrema follows at once from Corollary 5. □

We end this section with one more result that will be needed in the proof of Theorem 2.

Lemma 12. *Let $v \in C^4(\mathbb{R})$ be a positive solution of* (3). *Then*

$$v' < \sqrt{\tfrac{1}{2}A - \sqrt{\left(\tfrac{1}{2}A\right)^2 - B}}\; v.$$

For our values of A and B we have

$$\sqrt{\tfrac{1}{2}A - \sqrt{\left(\tfrac{1}{2}A\right)^2 - B}} = \tfrac{1}{2}(n-4),$$

but the lemma is true for general A and B satisfying (8).

Proof. Because of (8) we can introduce the two positive numbers

$$\lambda = \tfrac{1}{2}A - \sqrt{\left(\tfrac{1}{2}A\right)^2 - B} \quad \text{and} \quad \mu = \tfrac{1}{2}A + \sqrt{\left(\tfrac{1}{2}A\right)^2 - B}. \tag{13}$$

Using $\lambda + \mu = A$ and $\lambda\mu = B$ we can write (3) in terms of the auxiliary function

$$\phi(t) := v''(t) - \lambda v(t)$$

as

$$\phi'' - \mu\phi = v^p. \tag{14}$$

According to Proposition 3, ϕ attains its maximum on $\mathbb{R}$. Since $v > 0$, the maximum principle implies that $\phi < 0$.

The function $w := v'/v$ satisfies

$$w' = -w^2 + \lambda + \frac{\phi}{v}. \tag{15}$$

According to Proposition 3 there is a $t_0 \in \mathbb{R}$ with $v'(t_0) = 0$, and therefore also $w(t_0) = 0$. We shall show now that $M := \{t > t_0 : \ w(t) \geq \sqrt{\lambda}\}$ is empty, which yields the claimed inequality.

Suppose by contradiction that $M \neq \varnothing$ and let $t_1 := \inf M$. It is easy to see that $t_1 > t_0$. Then certainly $w'(t_1) \geq 0$. On the other hand, since $w(t_1) = \sqrt{\lambda}$, (15) implies

$$w'(t_1) = \frac{\phi(t_1)}{v(t_1)} < 0,$$

where the inequality comes from $\phi < 0$ and $v > 0$. This is a contradiction. □

3. Proofs of the main results

3.1. ***Proof of Theorem 1.*** We begin with the proof of part (i) of Theorem 1. Let $v \in C^4(\mathbb{R})$ be a solution of (7). By Proposition 3, the only case where

$$\inf_{\mathbb{R}} |v| \leq B^{1/(p-1)} \tag{16}$$

may fail to hold is when v is periodic. In this case, v possesses a local minimum at, say, $t_0 \in \mathbb{R}$. Note that if v has a zero then (16) is automatically fulfilled, so we may assume that v has a fixed sign and, up to replacing v by $-v$, we may assume that $v > 0$. But by [Guo et al. 2017a, Lemma 2.6], either v is constant (and hence $v \equiv B^{1/(p-1)}$) or $v(t_0) < B^{1/(p-1)}$, so that (16) holds with strict inequality.

We turn now to proving part (ii) of Theorem 1. We proceed via a shooting argument. The value $a \in (0, B^{1/(p-1)})$ will be considered to be fixed throughout the following argument.

For $\beta \geq 0$, we denote by v_β the unique solution of (7) with the initial values

$$v(0) = a, \quad v'(0) = 0, \quad v''(0) = \beta, \quad v'''(0) = 0, \tag{17}$$

and by $T_\beta \in (0, \infty]$ its maximal forward time of existence. Also, let $b := -\min_{v \in \mathbb{R}_+} f(v)$.

Suppose that $\beta > b/A =: \beta_0$. Then we see from

$$v_\beta^{(4)} = A v_\beta'' + f(v_\beta) \tag{18}$$

and (17) that $v_\beta^{(4)} > 0$ initially. Thus, v_β'' increases initially, and since the right-hand side of (18) is positive initially, it is easy to see that it will stay positive on $[0, T_\beta)$. Thus, $v_\beta^{(4)} > 0$ on $[0, T_\beta)$, which implies that v_β and its first three derivatives all keep increasing on $[0, T_\beta)$. Thus, if $T_\beta = \infty$, then v_β is unbounded. On the other hand, if $T_\beta < \infty$, then $v_\beta(t) \to \infty$ as $t \to T_\beta$ (since f is locally Lipschitz). To summarize, v_β increases monotonically on $[0, T_\beta)$ and diverges to $+\infty$ as $t \to T_\beta$ for $\beta \geq \beta_0$.

So we can restrict our search to $\beta \in [0, \beta_0]$. However, for all $\beta \leq \beta_0$, we have the uniform energy bound

$$\mathcal{E}_{v_\beta}(0) = \tfrac{1}{2}\beta^2 + F(a) \leq \tfrac{1}{2}\beta_0^2 + F(a).$$

Since $F(v) \to \infty$ as $v \to \infty$, there is an $R > 0$ such that $F(v) > \frac{1}{2}\beta_0^2 + F(a)$ for all $v > R$. This implies that whenever $\beta \leq \beta_0$ and $v_\beta(t_0) > R$, we must have $v_\beta'(t_0) \neq 0$, for otherwise

$$\mathcal{E}_{v_\beta}(t_0) = \tfrac{1}{2} v_\beta''(t_0)^2 + F(v_\beta(t_0)) \geq F(v_\beta(t_0)) > \tfrac{1}{2}\beta_0^2 + F(a),$$

which contradicts the upper bound on $\mathcal{E}_{v_\beta}(0)$ and energy conservation. In particular, a v_β which enters the interval (R, ∞) cannot leave it again, and hence is certainly not the periodic solution we are looking for.

On the other hand, if $\beta = 0$, we see from (18) that $v_0^{(4)}(0) = f(a) < 0$, and hence $v_0(t)$ and $v_0''(t)$ are strictly decreasing on some small interval $t \in (0, \sigma)$. Since $f(v) < 0$ for $v \in (0, a)$, we deduce from (18) that $v_0^{(k)}(t),\ k = 1, 2, 3$, stay strictly negative until $v_0(t)$ reaches a negative value. Hence, if $\beta = 0$, there must be t_0 such that $v_0(t_0) < 0$.

All of the previous considerations lead us to defining the shooting sets

$$S := \{\beta \geq 0 : v_\beta(t) < 0 \text{ for some } t \in (0, T_\beta)\},$$
$$T := \{\beta \geq 0 : v_\beta(t) > R \text{ for some } t \in (0, T_\beta) \text{ and } v_\beta > 0 \text{ on } [0, t]\}.$$

Clearly, S and T are open in $[0, \infty)$ because of the continuous dependence of the solution on the initial conditions. Moreover, S and T are disjoint because, as we observed above, once a solution v_β enters the interval (R, ∞), it stays there. We also already argued above that $0 \in S$ and $(\beta_0, \infty) \subset T$; i.e., both $S \neq \varnothing$ and $T \neq \varnothing$.

Since our shooting parameter interval $[0, \infty)$ is connected, we deduce that $S \cup T \neq [0, \infty)$. Hence there must be $\beta^* > 0$ and a corresponding solution $v^* := v_{\beta^*}$ such that $0 \leq v^* \leq R$. In particular, v^* is bounded. This and the fact that f is locally Lipschitz imply that $T_{\beta^*} = \infty$. By even reflection, we obtain a solution defined on all of $\mathbb{R}$, which we still refer to as v^*. Since $\beta^* > 0$, we know v^* has a strict local minimum in 0. By the classification of solutions from Proposition 3, v^* must be periodic. Moreover, it has a unique local maximum and minimum per period and is symmetric with respect to its extrema.

The uniqueness of v^* up to translations follows from Proposition 4. □

3.2. ***Proof of Theorem 2.*** By [Lin 1998, Theorem 4.2], the positivity of u and the nonremovability of the singularity in 0 imply that u is radially symmetric. Since the function v defined by $u(x) = |x|^{-(n-4)/2} v(\ln|x|)$ satisfies (3), we are in a position to apply the classification result from Proposition 3 and we claim that v is either the constant $B^{1/(p-1)} = a_0$ or periodic. Indeed, the only case that remains to be excluded is that $v(t) = c_n(2\cosh(t-T))^{-(n-4)/2}$. But in this case, it is clear that $v(t) \sim c_n e^{t(n-4)/2}$ as $t \to -\infty$ and hence the singularity of u would be removable, contradicting the assumptions. Thus, either v is constant or periodic.

Let $a := \inf v$. Then, by the first part of Theorem 1, $a \in (0, a_0]$, and $a = a_0$ if and only if $v \equiv a_0$. Moreover, for $a < a_0$ the function v is periodic with minimal value a. Therefore, by the second part of Theorem 1, $v(t) = v_a(t+L)$ for some $L \in \mathbb{R}$.

Finally, a simple computation shows that the inequality $\partial u/\partial|x| < 0$ is equivalent to $v' < \frac{1}{2}(n-4)v$, which follows from Lemma 12. □

Appendix: Proof of Proposition 4

In this appendix, we give the proof of Proposition 4, following and simplifying [van den Berg 2000].

Let v and w be bounded solutions of (7) which satisfy $v(0) = w(0)$ and $v'(0) = w'(0)$. We can assume without loss that $v''(0) \geq w''(0)$ (otherwise exchange v and w). We may assume furthermore (up to replacing $v(t)$ and $w(t)$ by $v(-t)$ and $w(-t)$) that $v'''(0) \geq w'''(0)$.

Suppose, by contradiction, that $v \not\equiv w$. Then by uniqueness of ODE solutions, $v^{(k)}(0) \neq w^{(k)}(0)$ for $k = 2$ or $k = 3$. In both cases, we deduce from our hypotheses on the initial conditions that

$$v(t) > w(t) \quad \text{on } (0, \sigma)$$

for some sufficiently small $\sigma > 0$.

With the positive numbers λ and μ from (13) we define the auxiliary functions

$$\phi(t) := v''(t) - \lambda v(t) \quad \text{and} \quad \psi(t) := w''(t) - \lambda w(t).$$

Then by the hypotheses, we have

$$(\phi - \psi)(0) \geq 0 \quad \text{and} \quad (\phi - \psi)'(0) \geq 0. \tag{19}$$

As in (14), equation (7) for v and w implies

$$(\phi - \psi)''(t) - \mu(\phi - \psi)(t) = |v(t)|^{p-1} v(t) - |w(t)|^{p-1} w(t) \quad \text{for all } t \in \mathbb{R}.$$

Since $v(t) > w(t)$ on $(0, \sigma)$ and since the function $u \mapsto |u|^{p-1}u$ is strictly increasing on $\mathbb{R}$, this implies

$$(\phi - \psi)''(t) - \mu(\phi - \psi)(t) > 0 \quad \text{for all } t \in (0, \sigma). \tag{20}$$

The inequalities (19) and (20) and the fact that $\mu > 0$ easily imply $(\phi - \psi)(t) \geq 0$ for $t \in (0, \sigma)$, or equivalently,

$$(v - w)''(t) \geq \lambda(v - w)(t) > 0 \quad \text{for all } t \in (0, \sigma). \tag{21}$$

Since $(v-w)'(0) \geq 0$ by the hypotheses of the lemma and since $\lambda > 0$, we see from (21) that $(v-w)'(t) > 0$ for all $t \in (0, \sigma)$. Hence $v - w$ is strictly increasing on $(0, \sigma)$ and since $\sigma > 0$ was arbitrary with the property that $v - w > 0$ on $(0, \sigma)$, we infer that $v - w$ remains strictly positive for all times.

Repeating the above arguments for the interval $(0, \infty)$ instead of $(0, \sigma)$, we see from (21) that $(v - w)'$ is positive and strictly increasing on $(0, \infty)$. This of course contradicts the boundedness of $v - w$. This proves that in fact we must have $v \equiv w$, concluding the proof of Proposition 4.

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