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Bennett, Carbery and Tao (2006) considered the k -linear restriction estimate in \mathbb{R}^{n+1} and established the near optimal $L^{2/(k-1)}$ estimate under transversality assumptions only. In 2017, we showed that the trilinear restriction estimate improves its range of exponents under some curvature assumptions. In this paper we establish almost sharp multilinear estimates for a class of hypersurfaces with curvature for $4 \leq k \leq n$. Together with previous results in the literature, this shows that curvature improves the range of exponents in the multilinear restriction estimate at all levels of lower multilinearity, that is, when $k \leq n$.

1. Introduction

For $n \geq 1$, let $U \subset \mathbb{R}^n$ be an open, bounded and connected neighborhood of the origin and let $\Sigma : U \rightarrow \mathbb{R}^{n+1}$ be a smooth parametrization of an n -dimensional submanifold of \mathbb{R}^{n+1} (hypersurface), which we denote by $S = \Sigma(U)$. To this parametrization of S we associate the operator \mathcal{E} defined by

$$\mathcal{E}f(x) = \int_U e^{ix \cdot \Sigma(\xi)} f(\xi) d\xi.$$

Given k smooth, compact hypersurfaces $S_i \subset \mathbb{R}^{n+1}$, $i = 1, \dots, k$, where $1 \leq k \leq n + 1$, the k -linear restriction estimate is the inequality

$$\left\| \prod_{i=1}^k \mathcal{E}_i f_i \right\|_{L^p(\mathbb{R}^{n+1})} \lesssim \prod_{i=1}^k \|f_i\|_{L^2(U_i)}. \tag{1-1}$$

In a more compact format this estimate is abbreviated as

$$\mathcal{R}^*(2 \times \dots \times 2 \rightarrow p).$$

The fundamental question regarding the above estimate is the value of the optimal p for which it holds true. Given that the estimate $\mathcal{R}^*(2 \times \dots \times 2 \rightarrow \infty)$ is trivial, the optimality is translated into the smallest p for which the estimate holds true. Bennett, Carbery and Tao [Bennett et al. 2006] clarified the role of transversality between the surfaces involved and established that, under a transversality condition between S_1, \dots, S_k , the optimal exponent is $p = 2/(k - 1)$; the actual result in that paper is near-optimal, and the optimal problem is currently open. The optimality can be easily revealed by taking the S_i to be transversal hyperplanes, in which case the estimate becomes the classical Loomis–Whitney inequality.

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It is also known, in some cases (precisely when $k \leq 2$), or expected, in most of the others, that curvature assumptions on the surfaces involved improve the range of exponents in (1-1), except for the case $k = n + 1$. In [Bejenaru 2017c] we formalized the following conjecture.

Conjecture 1.1. *Under appropriate transversality and curvature conditions on the surfaces S_i , the estimate $\mathcal{R}^*(2 \times \cdots \times 2 \rightarrow p)$ holds true for any $p \geq p(k) = 2(n + 1 + k)/(k(n + k - 1))$.*

The case $k = 1$ has been understood for a very long time. Without any curvature assumptions, the optimal exponent is $p = \infty$; once the surface has some nonvanishing principal curvatures, the exponent improves to $p = 2(l + 2)/l$, where l is the number of nonvanishing principal curvatures. The case of nonzero Gaussian curvature, corresponding to $l = n$, is the classical result due to Tomas and Stein; see [Stein 1993].

The case $k = 2$ without any curvature assumptions corresponds to the classical L^2 bilinear estimate, where the optimal estimate has been established. Once curvature assumptions are allowed, the best possible exponent in $\mathcal{R}^*(2 \times 2 \rightarrow p)$ is $p = (n + 3)/(n + 1)$ and it was conjectured in [Foschi and Klainerman 2000]. The problem was intensely studied; see [Bourgain 1995; Wolff 2001; Tao 2001; 2003; Tao and Vargas 2000a; Lee 2006; Lee and Vargas 2010; Bejenaru 2017b]. The problem is solved in the regime $p > (n + 3)/(n + 1)$ for general hypersurfaces with curvature; the end-point $p = (n + 3)/(n + 1)$ is solved only for cones; see [Tao 2001].

The case $k = n + 1$ is fairly well understood. We note that in this case, additional curvature assumptions have no effect on the optimality of p . It is conjectured that if the hypersurfaces $S_i \subset \mathbb{R}^{n+1}$ are transversal, then (1-1) holds true for $p \geq p_0 = 2/n$. If the S_i are transversal hyperplanes, (1-1) is the classical Loomis–Whitney inequality and its proof is elementary. Once the surfaces are allowed to have nonzero principal curvatures, things become far more complicated and the problem has been the subject of extensive research; see, e.g., [Bennett et al. 2006; Guth 2010]. Bennett, Carbery and Tao [Bennett et al. 2006] established a near-optimal version of (1-1), which is (1-1) with an additional R^ϵ factor when the estimate is made over balls of radius R in \mathbb{R}^{n+1} . The optimal result for (1-1), that is, without the ϵ -loss, is an open problem. In some cases one can use ϵ -removal techniques to derive the result without the ϵ -loss for $p > 2/n$; see [Bourgain and Guth 2011] for the case of surfaces with nonvanishing Gaussian curvature. The end-point for the multilinear Kakeya version of (1-1) (a slightly weaker statement than (1-1)) was established by Guth [2010] using tools from algebraic topology.

In the remaining cases, $3 \leq k \leq n$, the k -linear restriction theory has been addressed in [Bennett et al. 2006] only under transversality assumptions and the authors established the near-optimal result for $p \geq 2/(k - 1)$. The exponent $2/(k - 1)$ is sharp for generic surfaces, but it is not the optimal exponent once curvature assumptions are brought into the problem; indeed note that $p(k) < 2/(k - 1)$.

In [Bejenaru 2017c] we looked at the trilinear estimate (corresponding to $k = 3$) and proved the Conjecture 1.1 in the regime $p > p(3)$ for a particular class of surfaces: the double-conic ones. These surfaces have the nice property that they have the exact “amount” of curvature to obtain the estimate with the optimal exponent $p(3)$, and no more, in the sense that they are “flat” in the unnecessary directions.

In this paper we provide the equivalent result for $4 \leq k \leq n$ for $(k - 1)$ -conical surfaces. We note that passing from the case $k = 3$ to $k \geq 4$ requires not only additional technical ideas, but also conceptual ones.

We describe bellow the class of hypersurfaces for which we prove the [Conjecture 1.1](#). We start with the definition of a foliation. A $(k-1)$ -dimensional foliation of the (n -dimensional) hypersurface S is a decomposition of S into a union of connected disjoint sets $\{S_\alpha\}_{\alpha \in \mathcal{A}}$, called the leaves of the foliation, with the following property: every point in S has a neighborhood V and a local system of coordinates $x: V \subset S \rightarrow \mathbb{R}^n$ such that for each leaf S_α the coordinates of $V \cap S_\alpha$ are $\xi_k = \text{constant}, \dots, \xi_n = \text{constant}$.

We now formalize the conditions that we impose on our surfaces. As before, S_i , $i \in \{1, \dots, k\}$, are hypersurfaces with smooth parametrizations $\Sigma_i: U_i \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, where each U_i is an open, bounded and connected neighborhood of the origin (note that different U_i may belong to different hyperplanes identified with the same \mathbb{R}^n). In addition, we assume the following three hypotheses:

(i) (foliation) For each $i \in \{1, \dots, k\}$, the hypersurface S_i admits the foliation

$$S_i = \bigcup_{\alpha} S_{i,\alpha},$$

where, for each α , the leaf $S_{i,\alpha}$ is a flat submanifold of dimension $k-1$.

(ii) (the leaves are completely flat) If $S_{N_i(\zeta_i)}$ is the shape operator of S_i at $\zeta_i \in S_i$ with choice of normal $N_i(\zeta_i)$, we assume that for every $v \in T_{\zeta_i} S_{i,\alpha}$ (the tangent plane at $S_{i,\alpha}$ at the point $\zeta_i \in S_{i,\alpha}$) the following holds true:

$$S_{N_i(\zeta_i)} v = 0.$$

(iii) (transversality and curvature) There exists $\nu > 0$ such that for any $\zeta_i \in S_i$, $i \in \{1, \dots, k\}$, for any $l \in \{1, \dots, k\}$ and for any orthonormal basis $v_{k+1}, \dots, v_{n+1} \in (T_{\zeta_l} S_{l,\alpha})^\perp \subset T_{\zeta_l} S_l$ the following holds true:

$$\text{vol}(N_1(\zeta_1), \dots, N_k(\zeta_k), S_{N_l(\zeta_l)} v_{k+1}, \dots, S_{N_l(\zeta_l)} v_{n+1}) \geq \nu. \quad (1-2)$$

In (1-2) vol is the standard volume form of $n+1$ vectors in \mathbb{R}^{n+1} ; thus the condition quantifies the linear independence of the vectors $N_1(\zeta_1), \dots, N_k(\zeta_k), S_{N_l(\zeta_l)} v_{k+1}, \dots, S_{N_l(\zeta_l)} v_{n+1}$.

The condition (ii) says that the $S_{i,\alpha}$ are, in some sense, completely flat components of the S_i since, besides being subsets of affine planes of dimension $k-1$, the normal $N(\zeta)$ to S_i is constant as we vary ζ along $S_{i,\alpha}$ for fixed α .

The first thing to read in condition (iii) is the transversality condition between S_1, \dots, S_k due to the transversality between any choice on normals. The condition (iii) also says that the submanifolds transversal to the leaves carry the curvature assumptions, in the sense that their tangent space does not contain any eigenvectors of the shape operator. In addition, for each $i \in \{1, \dots, k\}$, we are guaranteed to have transversality between $N_1(\zeta_1), \dots, N_k(\zeta_k)$ and $S_{N_i}(T_{\zeta_i}(S_{l,\alpha}))^\perp$.

In fact (iii) is equivalent to the apparently weaker condition:

(iii') There exists $\nu > 0$ such that for any $\zeta_i \in S_i$, $i \in \{1, \dots, k\}$, for any $l \in \{1, \dots, k\}$ and for any unit vector $v \in (T_{\zeta_l} S_{l,\alpha})^\perp \subset T_{\zeta_l} S_l$ the following holds true:

$$\text{vol}(N_1(\zeta_1), \dots, N_k(\zeta_k), S_{N_l(\zeta_l)} v) \geq \nu. \quad (1-3)$$

Obviously here vol stands for the $(k+1)$ -dimensional volume of the parallelepiped determined by the vectors $N_1(\zeta_1), \dots, N_k(\zeta_k), S_{N_l(\zeta_l)} v$.

At this point we can state the main result of this paper.

Theorem 1.2. *Assume that S_1, \dots, S_k satisfy the conditions (i)–(iii) above. Given any p with*

$$p(k) = \frac{2(n+k+1)}{k(n+k-1)} < p \leq \infty,$$

the following holds true:

$$\left\| \prod_{i=1}^k \varepsilon_i f_i \right\|_{L^p(\mathbb{R}^{n+1})} \leq C(p) \prod_{i=1}^k \|f_i\|_{L^2(U_i)} \quad \text{for all } f_i \in L^2(U_i). \quad (1-4)$$

To the best of our knowledge this result is the first instance when the k -linear restriction estimate, with $4 \leq k \leq n$, is proved for the almost optimal exponent, that is, $p > p(k)$. However, very recently Guth [2016a] formulated a weaker version of Conjecture 1.1, which he proved in the case when the S_i are subsets of the paraboloid, and for the same range of parameters $p(k) < p \leq \infty$. The formulation of this weaker version is technical and we skip it here. Guth [2016a] used this weaker version to improve the range of the linear restriction theory. It is important to note that Guth employed polynomial partition methods to prove his result. The arguments we use in this paper are very different; see the details below.

The result in Theorem 1.2 and the corresponding one in [Bejenaru 2017c] show that the Conjecture 1.1 holds true at least in some model cases. We hope that this result will lead the way towards a complete resolution of the conjecture, which, in turn, should have important consequences. The multilinear theory discussed above has had major impact in other problems. We mention a few such examples: In harmonic analysis, the bilinear and $(n+1)$ -linear restriction theory were used to improve results in the context of the Schrödinger maximal function, see [Bourgain 2013; Lee 2003; Tao and Vargas 2000b; Du et al. 2017], the restriction conjecture, see [Tao 2003; Bourgain and Guth 2011; Guth 2016a; 2016b], and the decoupling conjecture, see [Bourgain and Demeter 2015; Bourgain et al. 2016]. In partial differential equations, the linear theory inspired the Strichartz estimates, see [Tao 2006], while the bilinear restriction theory is used in the context of more sophisticated techniques, such as the profile decomposition, see [Merle and Vega 1998], and concentration compactness methods, see [Kenig and Merle 2006].

Theorem 1.2 reveals the following geometric feature: the optimal k -linear restriction estimate discards the effect of $k-1$ curvatures; indeed, each S_i has precisely $k-1$ vanishing principal curvatures, and thus it relies only on $n+1-k$ principal curvatures being nonzero, although the actual statement has to be more rigorous. This geometric feature of the problem was conjectured by Bennett, Carbery and Tao [Bennett et al. 2006].

We continue with an overview of the paper and highlight some of the elements used in the proof of Theorem 1.2. The reader may look at the paper as split into two parts: Sections 2 through 4 and Sections 6 and 7, with Section 5 marking the transition between the two. In Sections 2 through 4 we adapt to our current setup the standard arguments that are similar to our previous works in the bilinear and trilinear setup, see [Bejenaru 2017b; 2017c]: overview of the geometry of the problem, wave packet theory, table construction and the induction-on-scales argument. All these ideas originate from [Tao 2001].

The second part of the paper, Sections 6 and 7, contains the novel ideas in this paper and they play a key role in establishing the improved estimate (4-4) in Section 4. We note that the equivalent results (to

those in Sections 6 and 7 here) in the bilinear and trilinear theory are much simpler, given the structure of the problem, and can be easily derived inside the body of the main argument. The results in Sections 6 and 7 hold in the context of general hypersurfaces; in particular they do not assume the foliation structure or curvature properties used in Theorem 1.2. We also think that these results are new in the literature and may be of independent interest.

The starting ideas originate in the prior work of the author on the multilinear restriction estimate in [Bejenaru 2017a]. In that paper we proved that the k -linear restriction estimate

$$\left\| \prod_{i=1}^k \mathcal{E}_i f_i \right\|_{L^{2/(k-1)}(B(0,r))} \leq C(\epsilon) r^\epsilon \prod_{i=1}^k \|f_i\|_{L^2(U_i)} \quad (1-5)$$

improves under appropriate localizations of one of the factors f_i . These localizations are precisely the ones carried by the wave packets appearing in the decomposition of one of the factors $\mathcal{E}_i f_i$, and one needs to obtain an appropriate estimate for such superpositions of wave packets. This was an easy task in the case of the trilinear estimate because the estimate is made in L^1 and the triangle inequality holds true. The triangle inequality fails to hold true in the spaces $L^{2/(k-1)}$ with $k \geq 4$; the way to deal with this aspect is to further refine the techniques developed in [Bejenaru 2017a] and derive good “off-diagonal”-type estimates, which in turn give the desired estimate with the correct localization gain; see Theorem 6.1. A further localization to cubes is needed for technical reasons; see Corollary 6.2. This analysis is carried out in Section 6.

In Section 7 we prove the estimate

$$\left\| \|\mathcal{E}_1 f_1\|_{L^2(S(q))} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)} \right\|_{l_q^{2/(k-1)}} \leq C(\epsilon) r^{\frac{k}{2}} r^\epsilon \prod_{i=1}^k \|f_i\|_{L^2(U_i)}. \quad (1-6)$$

Here q are cubes of size r and the $l_q^{2/(k-1)}$ is taken over such cubes contained in a larger cube of size r^2 ; $S(q) = S + q$, where S is a surface with some “good” properties. This estimate has the character of a k -linear restriction estimate, although it is more complex due to the factor $\|\mathcal{E}_1 f_1\|_{L^2(S(q))}$. If S were a point (that is, of dimension zero), then the above estimate is similar to the k -linear restriction estimate; however, the surface we encounter has the maximal dimension that allows (1-6) to hold true. Another interesting aspect is that the maximal dimension of S saturates the estimate (1-6) in the following sense: while for $k < n + 1$ (1-5) improves under appropriate localizations of some f_i , (1-6) does not, just as the $(n+1)$ -linear restriction estimate does not improve under localizations.

We identified (1-6) as the necessary ingredient to closing the improved estimate (4-4) in Section 4. We note that in the bilinear theory the result used is the one above with $k = 1$ and that means that the term $\prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)}$ does not appear. In this case the estimate (1-6) corresponds to an energy estimate for a free wave across hypersurfaces that are transversal to its directions of propagation; this is a classical tool in PDE. In the trilinear theory the estimate (1-6) is used for $k = 2$; thus it is an l^2 -type estimate that can be dealt with in a direct manner, by using wave packet decompositions for both free waves and some analysis on their interaction. It is in the quadrilinear (or higher) case that the true character of (1-6) comes to light. The analysis of the estimate (1-6) is carried out in Section 7.

1A. Notation. We start by clarifying the role of various constants that appear in the argument. N is a large integer that depends only on the dimension. C is a large constant that may change from line to line, and may depend on N , but not on c and C_0 introduced below. C is used in the definition of the following relations: $A \lesssim B$, meaning $A \leq CB$, $A \ll B$, meaning $A \leq C^{-1}B$, and $A \approx B$, meaning $A \lesssim B \wedge B \lesssim A$. For a given number $r \geq 0$, by $A = O(r)$ we mean that $A \approx r$. C_0 is a constant that is independent of any other constant and its role is to reduce the size of cubes in the inductive argument. We could set $C_0 = 4$ throughout the argument, but we keep it this way so that its role in the argument is not lost. Finally, $c \ll 1$ is a very small variable meant to make expressions $\ll 1$ and most estimates will be stated to hold in a range of c .

We use the standard notation $(\xi_1, \dots, \bar{\xi}_i, \dots, \xi_l) := (\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_l)$.

By powers of type $R^{\alpha+}$ we mean $R^{\alpha+\epsilon}$ for arbitrary $\epsilon > 0$. Practically they should be seen as $R^{\alpha+\epsilon}$ for arbitrary $0 < \epsilon \lesssim 1$. The estimates where such powers occur will obviously depend on ϵ .

By $B(x, R)$ we denote the ball centered at x with radius R in the underlying space (most of the time it will be \mathbb{R}^n or \mathbb{R}^{n+1}).

Let $\eta_0 : \mathbb{R}^n \rightarrow [0, +\infty)$ be a Schwartz function, normalized in L^1 , that is, $\|\eta_0\|_{L^1} = 1$, and with Fourier transform supported in the unit ball. Given some $r > 0$ we define $\eta_r(x) = r^{-n}\eta_0(r^{-1}x)$ and note that $\hat{\eta}_r$ is supported in $B(0, r)$. We will abuse notation and use the same η_0 for functions with the same properties, but with a different base space, such as $\eta_0 : \mathbb{R}^{n+1} \rightarrow [0, +\infty)$.

A disk $D \subset \mathbb{R}^{n+1}$ has the form

$$D = D(x_D, t_D; r_D) = \{(x, t_D) \in \mathbb{R}^{n+1} : |x - x_D| \leq r_D\}$$

for some $(x_D, t_D) \in \mathbb{R}^{n+1}$ and $r_D > 0$. We define the associated smooth cut-off

$$\tilde{\chi}_D(x, t) = \left(1 + \frac{|x - x_D|}{r_D}\right)^{-N}.$$

A cube $Q \subset \mathbb{R}^{n+1}$ of size R has the standard definition

$$Q = \{(x, t) \in \mathbb{R}^{n+1} : \|(x - x_Q, t - t_Q)\|_{l^\infty} \leq \frac{1}{2}R\},$$

where $c_Q = (x_Q, t_Q)$ is the center of the cube. Given a constant $\alpha > 0$ we define αQ to be the dilation by α of Q around its center; that is, $\alpha Q = \{(x, t) \in \mathbb{R}^{n+1} : \|(x - x_Q, t - t_Q)\|_{l^\infty} \leq \alpha \cdot \frac{1}{2}R\}$.

Given a cube $q \subset \mathbb{R}^{n+1}$ of size r we will use two functions that are highly concentrated in q . One is built with the help of η_0 (as mentioned earlier, we abuse notation here as we should be using the corresponding $\eta_0 : \mathbb{R}^{n+1} \rightarrow [0, +\infty)$ with similar properties):

$$\chi_q(x) = \eta_0\left(\frac{x - c(q)}{r}\right).$$

This localization function has nice properties on the Fourier side. The other localization function is

$$\tilde{\chi}_q(x) = \left(1 + \left|\frac{x - c(q)}{r}\right|\right)^{-N}$$

for some large N . This localization has better properties on the physical side.

We recall the standard estimate for superpositions of functions in L^p for $p \leq 1$

$$\left\| \sum_{\alpha} f_{\alpha} \right\|_{L^p}^p \leq \sum_{\alpha} \|f_{\alpha}\|_{L^p}^p, \tag{1-7}$$

as well as the estimate for sequences

$$\|a_i \cdot b_i\|_{L_i^{2/(k-1)}} \lesssim \|a_i\|_{L_i^2} \|b_i\|_{L_i^{2/(k-2)}}. \tag{1-8}$$

2. Geometry of the surfaces and consequences

We start this section by simplifying the setup. The surfaces are bounded, and therefore we can always break them into smaller (and similar) pieces, as we do to accommodate the additional hypotheses described below.

First note that we can assume each S_i to be of graph type: there is a smooth map $\varphi_i : U_i \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that $S = \{\Sigma_i(\xi) = (\xi, \varphi_i(\xi)) : \xi \in U_i\}$. Here the U_i are open and connected with compact closure. It is less important that the graphs are of type $\zeta_{n+1} = \varphi_i(\zeta_1, \dots, \zeta_n)$ (we can also have $\zeta_k = \varphi_i(\zeta_1, \dots, \zeta_k, \dots, \zeta_{n+1})$), although we can accommodate this by a rotation of coordinates. Then each flat leaf $S_{i,\alpha}$ corresponds to a flat leaf $U_{i,\alpha}$, in the sense that $\Sigma_i(U_{i,\alpha}) = S_{i,\alpha}$; this is indeed the case since projections onto hyperplanes along a vector transversal to S_i take $(k-1)$ -dimensional affine planes to $(k-1)$ -dimensional affine planes.

We can find a system of coordinates $x_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that parametrizes each leaf $U_{i,\alpha}$ into a new flat leaf $\tilde{U}_{i,\alpha}$ characterized by $\xi_k = \text{constant}, \dots, \xi_n = \text{constant}$. Finally, we assume that each U_i has small enough diameter.

Next, we derive a key geometric consequence of our setup. Given a surface S_i we define $\mathcal{N}_i := \{N_i(\zeta_i) : \zeta_i \in S_i\}$ to be the set of normals at S_i . By $\text{dspan } \mathcal{N}_i$ we denote the following subset of the classical span of \mathcal{N}_i :

$$\text{dspan } \mathcal{N}_i := \{\alpha N_{\alpha} + \beta N_{\beta} : N_{\alpha}, N_{\beta} \in \mathcal{N}_i, \alpha, \beta \in \mathbb{R}\}.$$

Note that $\text{dspan } \mathcal{N}_i$ is the set of linear combinations of two vectors in \mathcal{N}_i ; it is not a linear subspace.

Given a set of indexes $I \subset \{1, 2, \dots, k\}$ we also define

$$d\mathcal{N}_I := \{\alpha N_{\alpha} + \beta N_{\beta} : N_{\alpha} \in \mathcal{N}_i, N_{\beta} \in \mathcal{N}_j, i, j \in I, i \neq j, \alpha, \beta \in \mathbb{R}\}.$$

With this notation in place, we claim the following result.

Lemma 2.1. *Assume $S_i, i = 1, \dots, k$, satisfy the conditions (i)–(iii). Let $I = \{3, \dots, k\}$. Then for any $N \in \text{dspan } \mathcal{N}_1, N_2 \in \mathcal{N}_2$ and $\tilde{N} \in d\mathcal{N}_I$, the following holds true:*

$$\text{vol}(N, N_2, \tilde{N}) \gtrsim |N| \cdot |N_2| \cdot |\tilde{N}|. \tag{2-1}$$

The above statement is symmetric as we can switch the particular role each $S_i, i = 1, \dots, k$, plays in the above estimate.

Proof. The proof is similar to the one provided in [Bejenaru 2017c]. We write $N = \alpha N_\alpha + \beta N_\beta$ for some $N_\alpha \neq N_\beta$ and consider $\gamma : [0, t_0] \rightarrow S_1$, a smooth curve with the property that $N_1(\gamma(0)) = N_\alpha$ and $N_1(\gamma(t_0)) = N_\beta$. We also assume that $|\gamma'(t)| = 1$ on $[0, t_0]$ and that $0 \leq t_0 \ll 1$; this is possible because we assumed U_1 to be of small diameter. In addition, if α_0 is such that $\gamma(0) \in S_{1,\alpha_0}$, we can assume that $\gamma'(0) \in (T_{\gamma(0)}S_{1,\alpha_0})^\perp$. Then we have

$$\begin{aligned} N_1(\gamma(t_0)) &= N_1(\gamma(0)) + \int_0^{t_0} S_{N_1(\gamma(s))} \gamma'(s) ds \\ &= N_1(\gamma(0)) + t_0 S_{N_1(\gamma(0))} \gamma'(0) + O(t_0^2). \end{aligned}$$

We then continue with

$$\begin{aligned} N &= \alpha N_1(\gamma(0)) + \beta(N_1(\gamma(0)) + t_0 S_{N_1(\gamma(0))} \gamma'(0) + O(t_0^2)) \\ &= (\alpha + \beta) N_1(\gamma(0)) + \beta t_0 S_{N_1(\gamma(0))} \gamma'(0) + \beta O(t_0^2). \end{aligned}$$

The two vectors $N_1(\gamma(0))$ and $S_{N_1(\gamma(0))} \gamma'(0)$ are transversal; thus $|N| \approx |\alpha + \beta| + t_0 |\beta| |S_{N_1(\gamma(0))} \gamma'(0)|$ (here we use that $t_0 \ll 1$), and also

$$\begin{aligned} \text{vol}(N, N_2, \tilde{N}) &\approx \text{vol}((\alpha + \beta) N_1(\gamma(0)) + \beta t_0 S_{N_1(\gamma(0))} \gamma'(0), N_2, \tilde{N}) \\ &\gtrsim |(\alpha + \beta) N_1(\gamma(0)) + \beta t_0 S_{N_1(\gamma(0))} \gamma'(0)| \cdot |\tilde{N}| \approx |N| \cdot |\tilde{N}|, \end{aligned}$$

where we have used the following consequence of (1-2):

$$\text{vol}(N_1(\gamma(0)), S_{N_1(\gamma(0))} v, N_2, \tilde{N}) \gtrsim |\tilde{N}|,$$

which holds true for any unit vector $v \in (T_{\gamma(0)}S_{1,\alpha_0})^\perp \subset T_{\gamma(0)}S_1$ and any vector $\tilde{N} \in dN_I$. □

Using an argument similar to that above, one can easily establish the dispersive estimate

$$|N_i(\zeta_1) - N_i(\zeta_2)| \approx d(S_{i,\alpha_1}, S_{i,\alpha_2}), \tag{2-2}$$

where $S_{i,\alpha_1}, S_{i,\alpha_2}$ are the leaves to which ζ_1, ζ_2 belong, respectively. Here the distance between S_{i,α_1} and S_{i,α_2} can be defined either by using geodesics inside the hypersurface S_i (using the induced metric from the ambient space \mathbb{R}^{n+1}) or, equivalently, by using the classical distance between sets in \mathbb{R}^{n+1} .

3. Free waves, wave packets and tables on cubes

In this section we collect some of the preparatory ingredients that are needed in the proof of our main result. The setup described here originated in the work of Tao [2001] on the bilinear restriction estimate. All of the results here have been discussed in our previous works; see [Bejenaru 2017b; 2017c]. We do not repeat some of the proofs as they are similar to those found in these three mentioned papers.

3A. Rephrasing the problem in terms of free waves. We reformulate our problem in terms of free waves, this being motivated by the use of wave packets in the proof of Theorem 1.2. Once the wave packet decomposition is made and its properties are clear, the formalization of the problem as an evolution equation can be forgotten.

Assume we are given a surface S with a graph-type parametrization $\zeta_{n+1} = \varphi(\xi)$, where $\xi = (\zeta_1, \dots, \zeta_n)$. We rename the variable ζ_{n+1} by τ ; thus the equation of S becomes $\tau = \varphi(\xi)$. We parametrize the physical space by $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. We make the choice that τ is the Fourier variable corresponding to t , while ξ is the Fourier variable corresponding to x . In what follows we use the convention that \hat{f} denotes the Fourier transform of f with respect to the x -variable.

We define the free wave $\phi = \mathcal{E}f$ as

$$\phi(x, t) = \mathcal{E}f(x, t) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t\varphi(\xi))} f(\xi) d\xi.$$

Note that $\phi(0) = \check{f}$ and $\hat{\phi}(\xi, t) = e^{it\varphi(\xi)} \hat{\phi}(\xi, 0)$. We define the mass of a free wave by $M(\phi(t)) := \|\phi(t)\|_{L^2}^2$ and note that it is time-independent:

$$M(\phi(t)) := \|\phi(t)\|_{L^2}^2 = \|\hat{\phi}(t)\|_{L^2}^2 = \|\hat{\phi}(0)\|_{L^2}^2 = \|\phi(0)\|_{L^2}^2 = M(\phi(0)).$$

The proof of (1-4) relies on estimating $\prod_{i=1}^k \mathcal{E}_i f_i$ on cubes on the physical side and seeing how this behaves as the size of the cubes goes to infinity by using an inductive-type argument with respect to the size of the cubes. Before we formalize this strategy, we note that at every stage of the inductive argument we relocalize functions both on the physical and frequency spaces, and, as a consequence, we need to quantify the new support on the frequency side. This will be done by using the margin of a function.

We assume we are given a reference set V inside which we want to keep all functions supported. If f is supported in $U \subset V$ we define the margin of f relative to V by

$$\text{margin}(f) := \text{dist}(\text{supp}(f), V^c).$$

In terms of free waves $\phi = \mathcal{E}f$, the margin is defined by

$$\text{margin}(\phi(t)) := \text{dist}(\text{supp}_\xi(\hat{\phi}(t)), V^c) = \text{dist}(\text{supp}(f), V^c),$$

where we have used that the Fourier support of $\hat{\phi}(t)$ is time-independent and that $\hat{\phi}(0) = \check{f}$. In other words, the margin of a free wave is time-independent.

In practice, we work with k different types of free waves, $\phi_i = \mathcal{E}_i f_i$, $i = 1, \dots, k$. They are assumed to be graphs with different phase functions φ_i and with potentially different ambient domains, that is, the U_i are subsets of different subspaces isomorphic to \mathbb{R}^n (for instance the U_i can be subsets of the hyperplanes $\xi_i = 0$). The above construction changes only by choosing τ to be the coordinate in the direction normal to the ambient hyperplane to which U_i belongs, while ξ are the coordinates in the ambient hyperplane. Obviously, the margin of each ϕ_i is then defined with respect to some V_i in the same ambient hyperplane. When choosing the reference sets V_i we need to impose that the conditions (i)–(iii) hold true on $\Sigma_i(V_i)$ as well.

Next, we prepare the elements that are needed for the induction-on-scale argument. Given that the estimate is trivial for $p = \infty$, it suffices to focus on the result above in the cases $p(k) < p \leq 2/(k-1)$ and this is what we will do. Note that the exponent $2/(k-1)$ is precisely the one for which the k -linear restriction theory is expected to hold true without any curvature assumptions.

Definition 3.1. Let $p(k) \leq p \leq 2/(k - 1)$. Given $R \geq C_0$ we define $A_p(R)$ to be the best constant for which the estimate

$$\left\| \prod_{i=1}^k \phi_i \right\|_{L^p(Q_R)} \leq A_p(R) \prod_{i=1}^k M(\phi_i)^{\frac{1}{2}} \tag{3-1}$$

holds true for all cubes Q_R of size R and $\phi_i = \mathcal{E}_i f_i$ obeying the margin requirement

$$\text{margin}^i(\phi_i) \geq M - R^{-\frac{1}{4}}, \quad i = 1, \dots, k. \tag{3-2}$$

The goal is to obtain a uniform estimate on $A_p(R)$ with respect to R . In the absence of the margin requirement above, $A_p(R)$ would be an increasing function. However, since the argument needs to tolerate the margin relaxation, we also define

$$\bar{A}_p(R) := \sup_{1 \leq r \leq R} A_p(r)$$

and the new $\bar{A}_p(R)$ is obviously increasing with respect to R .

Then (1-4), and, as a consequence, the main result of this paper, [Theorem 1.2](#), follow from the next result.

Proposition 3.2. *Assume $0 < \epsilon < 1$. If $R \gg 2^{2C_0}$ and $R^{-(1/4)+} \ll c \ll 1$, there exists $C(\epsilon)$ such that the following holds true:*

$$A_p(R) \leq (1 + cC) \left((1 + cC)^p (\bar{A}_p(\frac{1}{2}R))^p + (C(\epsilon)c^{-C} R^{\frac{n+k+1}{2}(\frac{1}{p} - \frac{k}{2} \cdot \frac{n+k-1}{n+k+1}) + \epsilon})^p \right)^{\frac{1}{p}}. \tag{3-3}$$

Deriving (1-4) from (3-3) is standard; see the corresponding argument in the trilinear case in [\[Bejenaru 2017c\]](#). Thus we reduce the proof of [Theorem 1.2](#) to proving (3-3).

3B. Tables on cubes. Let $Q \subset \mathbb{R}^{n+1}$ be a cube of radius R . Given $j \in \mathbb{N}$ we split Q into $2^{(n+1)j}$ cubes of size $2^{-j}R$ and denote this family by $\mathcal{Q}_j(Q)$; thus we have $Q = \bigcup_{q \in \mathcal{Q}_j(Q)} q$. If $j \in \mathbb{N}$ and $0 \leq c \ll 1$, we define the (c, j) -interior $I^{c,j}(Q)$ of Q by

$$I^{c,j}(Q) := \bigcup_{q \in \mathcal{Q}_j(Q)} (1 - c)q. \tag{3-4}$$

Given $j \in \mathbb{N}$ we define a table Φ on Q to be a vector $\Phi = (\Phi^{(q)})_{q \in \mathcal{Q}_j(Q)}$ and define its mass by

$$M(\Phi) = \sum_{q \in \mathcal{Q}_j(Q)} M(\Phi^{(q)}).$$

We define the margin of a table as the minimum margin of its components:

$$\text{margin}(\Phi) = \min_{q \in \mathcal{Q}_j(Q)} \text{margin}(\Phi^{(q)}).$$

We recall from [\[Bejenaru 2017c\]](#) the following result, which originated in [\[Tao 2001\]](#):

Lemma 3.3. *Assume $0 < p < \infty$, $R \gg 1$, $0 < c \ll 1$ and f is smooth. Given a cube $Q_R \subset \mathbb{R}^{n+1}$ of size R , there exists a cube Q of size $2R$ contained in $4Q_R$ such that*

$$\|f\|_{L^p(Q_R)} \leq (1 + cC)\|f\|_{L^p(I^{c,j}(Q))}. \quad (3-5)$$

3C. Wave packets. In this section we formalize the wave packet construction for $(k-1)$ -conical surfaces. We assume that S is of $(k-1)$ -conic type and has the graph-type parametrization $\Sigma : U \rightarrow S$, where $\Sigma(\xi) = (\xi, \varphi(\xi))$, with foliations $U = \bigcup_{\alpha} U_{\alpha}$, $S = \bigcup_{\alpha} S_{\alpha}$, $\Sigma(U_{\alpha}) = S_{\alpha}$.

For the foliation $U = \bigcup_{\alpha} U_{\alpha}$, we choose a system of coordinates $\mathbf{x} : U \rightarrow \tilde{U}$ such that for each leaf U_{α} the coordinates of U_{α} are $\xi_k = \text{constant}, \dots, \xi_n = \text{constant}$. Let $\tilde{U}' = \pi(\tilde{U})$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k+1}$ is the projection $\pi(\xi_1, \dots, \xi_n) = (\xi_k, \dots, \xi_n)$. Let $\tilde{\mathcal{L}}$ be a maximal r^{-1} -separated subset of $\tilde{U}' \subset \mathbb{R}^{n-k+1}$. For each $\tilde{\xi} \in \tilde{\mathcal{L}}$, we have $\mathbf{x}^{-1}(\cdot, \tilde{\xi})$ is a leaf; that is, $\mathbf{x}^{-1}(\cdot, \tilde{\xi}) = U_{\alpha}$ for some α . In each such leaf we pick ξ_T and define \mathcal{L} to be the set obtained this way. It is not important which $\xi_T \in \mathbf{x}^{-1}(\cdot, \tilde{\xi})$ is chosen, since from condition (ii) it follows that, for $\xi \in U_{\alpha}$, the normal $N(\Sigma(\xi))$ to S is constant as ξ varies inside the leaf U_{α} . We denote by $U(\xi_T)$ the leaf U_{α} to which ξ_T belongs and by $S(\xi_T) = \Sigma(U(\xi_T))$, the corresponding leaf on S . We note that $d(U(\xi_{T_1}), U(\xi_{T_2})) \approx d(\tilde{\xi}_1, \tilde{\xi}_2)$, which combined with (2-2) gives

$$|N(\Sigma(\xi_{T_1})) - N(\Sigma(\xi_{T_2}))| \approx d(U(\xi_{T_1}), U(\xi_{T_2})) \approx d(\tilde{\xi}_1, \tilde{\xi}_2). \quad (3-6)$$

Let L be the lattice $L = c^{-2}r\mathbb{Z}^n$. With $x_T \in L$, $\xi_T \in \mathcal{L}$, we define the tube

$$T = T(x_T, \xi_T) := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_T + t\nabla\varphi(\xi_T)| \leq c^{-2}r\}$$

and denote by \mathcal{T} the set of such tubes. One notices that T is the $c^{-2}r$ neighborhood of the line passing through $(x_T, 0)$ in the direction $N(\Sigma(\xi_T))$.

Associated to a tube $T \in \mathcal{T}$, we define the cut-off $\tilde{\chi}_T$ on \mathbb{R}^{n+1} by

$$\tilde{\chi}_T(x, t) = \tilde{\chi}_{D(x_T - t\nabla\varphi(\xi_T), t; c^{-2}r)}(x).$$

We are ready to state the main result of this section.

Lemma 3.4. *Let Q be a cube of radius $R \gg 1$, let c be such that $R^{-(1/4)+} \ll c \lesssim 1$ and let $J \in \mathbb{N}$ be such that $r = 2^{-J}R \approx R^{1/2}$. Let $\phi = \mathcal{E}f$ be a free wave with $\text{margin}(\phi) > 0$. For each $T \in \mathcal{T}$ there is a free wave ϕ_T that is localized in a neighborhood of size $CR^{-1/2}$ of the leaf $S(\xi_T)$ and obeys $\text{margin}(\phi_T) \geq \text{margin}(f) - CR^{-1/2}$. The map $f \rightarrow \phi_T$ is linear and*

$$\phi = \sum_{T \in \mathcal{T}} \phi_T. \quad (3-7)$$

If $\text{dist}(T, Q) \geq 4R$ then

$$\|\phi_T\|_{L^{\infty}(Q)} \lesssim c^{-C} \text{dist}(T, Q)^{-N} M(\phi)^{\frac{1}{2}}. \quad (3-8)$$

The estimates

$$\sum_T \sup_{q \in Q_J(Q)} \tilde{\chi}_T(x_q, t_q)^{-N} \|\phi_T\|_{L^2(q)}^2 \lesssim c^{-C} r M(\phi) \quad (3-9)$$

and

$$\sum_{q_0} M\left(\sum_T m_{q_0,T} \phi_T\right) \leq (1 + cC)M(\phi), \tag{3-10}$$

hold true provided that the coefficients $m_{q_0,T} \geq 0$ satisfy

$$\sum_{q_0} m_{q_0,T} = 1 \quad \text{for all } T \in \mathcal{T}. \tag{3-11}$$

This type of wave packet decomposition was introduced in [Tao 2001] in the context of a bilinear restriction estimate for conical hypersurfaces (1-conical in our language). The strength of this result lies in the use of the small parameter c and the tight mass estimate (3-10). In the case $c \approx 1$, the above decomposition is the standard wave packet decomposition.

In the case of double-conical surfaces the analogous result was proved in [Bejenaru 2017c]. The argument for Lemma 3.4 is entirely similar to the results just mentioned and we will not duplicate it here.

In the case $c \approx 1$, we will use the following variation of (3-9). Fix $N \in \mathbb{N}$; then for each tube $T \in \mathcal{T}$, there are coefficients $c_{N,T}$ such that

$$\sup_{q \in \mathcal{Q}_J(Q)} \tilde{\chi}_T(x_q, t_q)^{-\frac{N}{2}} \|\phi_T\|_{L^2(q)} \lesssim r^{\frac{1}{2}} \cdot c_N(T). \tag{3-12}$$

with the property that

$$\sum_{T \in \mathcal{T}} c_N(T)^2 \lesssim M(\phi). \tag{3-13}$$

4. Table construction and the induction argument

This section contains the main argument for the proof of Theorem 1.2. In Proposition 4.1 we construct tables on cubes, which is a way of reorganizing the information on one term, say ϕ_1 , at smaller scales based on information from one of the other interacting terms, $\phi_i, i = 2, \dots, k$. This type of argument is inspired by the work on the conic surfaces of Tao [2001]. Based on this table construction, we will prove the inductive bound claimed in Proposition 3.2.

Proposition 4.1. *Let Q be a cube of size $R \gg 2^{2C_0}$. Assume $\phi_i = \mathcal{E}_i f_i, i = 1, \dots, k$, have positive margin. Then there is a table $\Phi_1 = \Phi_c(\phi_1, \phi_2, Q)$ with depth C_0 such that the following properties hold true:*

$$\phi_1 = \sum_{q \in \mathcal{Q}_{C_0}(Q)} \Phi_1^{(q)}, \tag{4-1}$$

$$\text{margin}(\Phi) \geq \text{margin}(\phi) - CR^{-\frac{1}{2}}, \tag{4-2}$$

$$M(\Phi) \leq (1 + cC)M(\phi), \tag{4-3}$$

and for any $q', q'' \in \mathcal{Q}_{C_0}(Q), q' \neq q''$,

$$\left\| \Phi_1^{(q')} \prod_{i=2}^k \phi_i \right\|_{L^{2/(k-1)}((1-c)q'')} \lesssim c^{-C} R^{-\frac{n-k+1}{4}} \prod_{i=1}^k M^{\frac{1}{2}}(\phi_i). \tag{4-4}$$

Remark 1. The above result is stated for scalars ϕ_1, \dots, ϕ_k , but it holds for vector versions as well. Most important is that we can construct $\Phi_1 = \Phi_c(\phi_1, \Phi_2, Q)$, where Φ_2 is a vector-free wave and all its scalar components satisfy similar properties to the ϕ_2 above.

Remark 2. We note that $\Phi_1 = \Phi_c(\phi_1, \phi_2, Q)$ means that the table Φ_1 is constructed from ϕ_1 , which is natural in light of (4-1), and ϕ_2 . But it does not depend on ϕ_3, \dots, ϕ_k . Obviously, we could have constructed it from ϕ_1 and ϕ_3 (or any other ϕ_k), ending with a different object.

In the proof below we use the results in Sections 6 and 7 in a crucial way. The reason we provide those results in later sections is that, at first reading, it is instructive to get the main points and the motivation for the results in Sections 6 and 7 before the argument becomes too technical.

Proof. There are several scales involved in this argument. The large scale is the size R of the cube Q . The coarse scale is $2^{-C_0} R \gg R^{1/2}$, where this is the size of the smaller cubes in $\mathcal{Q}_{C_0}(Q)$ and the subject of the claims in the proposition. Then there is the fine scale $r = 2^{-j} R$ chosen such that $r \approx R^{1/2}$. Notice that r is the proper scale for wave packets corresponding to time scales R and also that their scale is $c^{-2} r \ll 2^{-C_0} R$, the last one being the scale of cubes in $\mathcal{Q}_{C_0}(Q)$.

We use Lemma 3.4 with $J = j$ to construct the wave packet decomposition for ϕ_1 :

$$\phi_1 = \sum_{T_1 \in \mathcal{T}_1} \phi_{1, T_1}.$$

For any $q_0 \in \mathcal{Q}_{C_0}(Q)$ and $T_1 \in \mathcal{T}_1$ we define

$$m_{q_0, T_1} := \|\tilde{\chi}_{T_1} \phi_2\|_{L^2(q_0)}^2,$$

$$m_{T_1} := \sum_{q_0 \in \mathcal{Q}_{C_0}(Q)} m_{q_0, T_1}.$$

Based on this we define

$$\Phi_1^{(q_0)} := \sum_{T_1} \frac{m_{q_0, T_1}}{m_{T_1}} \phi_{1, T_1}. \quad (4-5)$$

By combining the definitions above with the decomposition property (3-7), we obtain

$$\phi_1 = \sum_{q_0 \in \mathcal{Q}_{C_0}(Q)} \Phi_1^{(q_0)},$$

thus justifying (4-1).

The margin estimate (4-2) follows from the margin estimate on tubes provided by Lemma 3.4. The coefficients m_{q_0, T_1} satisfy (3-11); thus the estimate (4-3) follows from (3-10).

All that is left to prove is (4-4), which is equivalent to

$$\left(\sum_{\substack{q \in \mathcal{Q}_j(Q) \\ d(q, q_0) \geq cR}} \left\| \Phi_1^{(q_0)} \prod_{i=2}^k \phi_i \right\|_{L^{2/(k-1)}(q)}^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \lesssim c^{-C} r^{-\frac{n-k+1}{2}} \prod_{i=1}^k M(\phi_i). \quad (4-6)$$

Note that the cubes q are selected at the finer scale dictated by the size of cubes in $\mathcal{Q}_j(Q)$. In the definition of $\Phi_1^{(q_0)}$, see (4-5), we have the full family \mathcal{T}_1 . In the above estimate, we estimate the output

inside q ; thus, in light of (3-8), the terms ϕ_{T_1} with $T_1 \cap q \neq \emptyset$ are the ones that really matter. Indeed, if we split $\Phi_1^{(q_0)}$ as

$$\Phi_1^{(q_0)} := \sum_{T_1 \cap q \neq \emptyset} \frac{m_{q_0, T_1}}{m_{T_1}} \phi_{1, T_1} + \sum_{k \in \mathbb{N}} \sum_{d(T_1, q) \approx 2^k c^{-2} r} \frac{m_{q_0, T_1}}{m_{T_1}} \phi_{1, T_1},$$

we can use (1-7) to reduce the problem to estimating each term in the first sum above. Indeed, in light of (3-9), the contributions of terms from the second sum come with additional decay 2^{-kN} , which, for N large enough, can be easily estimated. Thus it suffices to prove the estimate (4-6) with $\Phi_1^{(q_0)}$ replaced by the first sum above.

For fixed q , it is a straightforward exercise to check that the Setup in Section 6 is satisfied: Simply let $J = \{T_1 \in \mathcal{T}_1 : T_1 \cap q \neq \emptyset\}$ and let $\phi_{T_1} = \mathcal{E}_{1, T_1} f_{1, T_1}$. Thus we can invoke (6-4) to obtain

$$\left\| \left(\sum_{T_1 \cap q \neq \emptyset} \frac{m_{q_0, T_1}}{m_{T_1}} \phi_{T_1} \right) \prod_{i=2}^k \phi_i \right\|_{L^{2/(k-1)}(q)} \lesssim C(\epsilon) r^{-\frac{n+1}{2}} r^\epsilon \sum_{T_1 \cap q \neq \emptyset} \frac{m_{q_0, T_1}}{m_{T_1}} \|\tilde{\chi}_q \phi_{1, T_1}\|_{L^2} \prod_{i=2}^k \|\tilde{\chi}_q \phi_i\|_{L^2}.$$

Since $m_{q_0, T_1} \leq m_{T_1}$,

$$\frac{m_{q_0, T_1}}{m_{T_1}} \leq \frac{m_{q_0, T_1}^{1/2}}{m_{T_1}^{1/2}},$$

from this we obtain

$$\sum_{T_1 \cap q \neq \emptyset} \frac{m_{q_0, T_1}}{m_{T_1}} \|\phi_{1, T_1} \tilde{\chi}_q\|_{L^2} \lesssim \left(\sum_{T_1 \cap q \neq \emptyset} \frac{\|\phi_{1, T_1} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \right)^{\frac{1}{2}} \left(\sum_{T_1 \cap q \neq \emptyset} m_{q_0, T} \tilde{\chi}_{T_1}(x_q, t_q) \right)^{\frac{1}{2}}.$$

Next we claim the estimate

$$\sum_{T_1 \in \mathcal{T}_1} m_{q_0, T_1} \tilde{\chi}_{T_1}(x_q, t_q) \lesssim \|\tilde{\chi}_{S(q)} \phi_2\|_{L^2}^2. \tag{4-7}$$

Using the definition of m_{q_0, T_1} we identify the function

$$\tilde{\chi}_{S(q)} = \left(\sum_{T_1 \in \mathcal{T}_1} \tilde{\chi}(x_q, t_q) \tilde{\chi}_{T_1} \right) \chi_{q_0}$$

which makes (4-7) hold true. Here the surface $S(q)$ is the translation by $c(q)$ of the neighborhood of size r of the cone of normals at S_1 , which we denote by $\mathcal{CN}_1 := \{\alpha N_1(\zeta) : \zeta \in S_1, \alpha \in \mathbb{R}\}$. It is important to note that we do not consider the whole cone but only the part with $cR \leq \alpha \lesssim R$. Note that $\tilde{\chi}_{S(q)}$ has the following decay property:

$$\tilde{\chi}_{S(q)}(x, t) \lesssim c^{-4} \left(1 + \frac{d((x, t), S(q))}{c^{-2} r} \right)^{-N}.$$

This is a consequence of the fact that the tubes T_1 passing thorough q separate inside q_0 and of the separation between q and q_0 , which is quantified by $d(q, q_0) \gtrsim cR$. Quantitatively speaking, given a point in q_0 close to $S(q)$, there are $\lesssim c^{-4}$ tubes T_1 passing through the point and q ; this follows from the dispersion estimate (3-6) and the geometry of the family of tubes \mathcal{T}_1 .

We define

$$A(q) = \left(\sum_{T_1 \cap q \neq \emptyset} \frac{\|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \right)^{\frac{1}{2}}, \quad B(q) = \|\tilde{\chi}_q \phi_2\|_{L^2},$$

$$C(q) = \|\tilde{\chi}_{S(q)} \phi_2\|, \quad D(q) = \prod_{i=3}^k \|\tilde{\chi}_q \phi_i\|_{L^2}.$$

To conclude the proof of (4-6), it suffices to show

$$\left(\sum_{\substack{q \in \mathcal{Q}_j(\mathcal{Q}) \\ d(q, q_0) \gtrsim cR}} A(q)^{\frac{2}{k-1}} B(q)^{\frac{2}{k-1}} C(q)^{\frac{2}{k-1}} D(q)^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \lesssim r^{\frac{k}{2}} r^\epsilon \prod_{i=1}^k M(\phi_i)^{\frac{1}{2}}.$$

This will be a consequence of the two inequalities

$$\left(\sum_{\substack{q \in \mathcal{Q}_j(\mathcal{Q}) \\ d(q, q_0) \gtrsim cR}} A(q)^2 B(q)^2 \right)^{\frac{1}{2}} \lesssim r^{\frac{1}{2}} M(\phi_1)^{\frac{1}{2}}, \quad (4-8)$$

$$\left(\sum_{\substack{q \in \mathcal{Q}_j(\mathcal{Q}) \\ d(q, q_0) \gtrsim cR}} C(q)^{\frac{2}{k-2}} D(q)^{\frac{2}{k-2}} \right)^{\frac{k-2}{2}} \lesssim r^{\frac{k-1}{2}} r^\epsilon \prod_{i=2}^k M(\phi_i)^{\frac{1}{2}}. \quad (4-9)$$

The proof of (4-8) is similar to the one we used in the bilinear and trilinear theory; see [Bejenaru 2017b; 2017c]. By rearranging the sum, it suffices to show

$$\sum_{T_1} \sum_{q \cap T_1 \neq \emptyset} \frac{\|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}^2 \|\phi_2 \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim r M(\phi_1).$$

The inner sum is estimated as

$$\sum_{q \cap T_1 \neq \emptyset} \frac{\|\phi_2 \tilde{\chi}_q\|_{L^2}^2}{m_{T_1} \tilde{\chi}_{T_1}(x_q, t_q)} \lesssim \frac{\|\phi_2 \tilde{\chi}_{T_1}\|_{L^2}^2}{m_{T_1}} \lesssim 1,$$

and the outer one is estimated by

$$\sum_{T_1} \sup_q \|\phi_{1,T_1} \tilde{\chi}_q\|_{L^2}^2 \lesssim r \sum_{T_1} M(\phi_{1,T_1}) \lesssim r M(\phi_1),$$

which is obvious given the size of q in the x_1 -direction is $\approx r$ and the mass of ϕ_{1,T_1} is constant across slices in space with $x_1 = \text{constant}$.

In proving (4-9), we can take advantage of the fast decay of $\tilde{\chi}_q$ away from q and of $\tilde{\chi}_{S(q)}$ away from $S(q)$, and at the cost of picking factors of type c^{-C} , it suffices to show

$$\left\| \|\phi_2\|_{L^2(S(q))} \prod_{i=3}^k \|\phi_i\|_{L^2(q)} \right\|_{l_q^{2/(k-2)}} \lesssim r^{\frac{k-1}{2}} r^\epsilon \prod_{i=2}^k M(\phi_i). \quad (4-10)$$

The $l_q^{2/(k-2)}$ norm is computed over the set of $q \in \mathcal{Q}_j(Q)$, the set of cubes of size r contained in the larger cube of size r^2 . This estimate is the subject of [Theorem 7.1](#) in [Section 7](#). The statement of [Theorem 7.1](#) requires S to have certain properties in relation to the other surfaces S_2, \dots, S_k ; see (P1), (P2) at the beginning of [Section 7](#). The fact that S satisfies these properties follows from [Lemma 2.1](#). \square

Proof of Proposition 3.2. This is entirely similar to the argument used in [[Bejenaru 2017b](#); [2017c](#)], see the corresponding proofs there. \square

We have finished the proof of our main result [Theorem 1.2](#). Obviously we owe a justification for some estimates used in the body of the proof of [Proposition 4.1](#) and this what will be covered in the next two sections of the paper.

5. The second part: the multilinear estimate revised

We have arrived at the middle point in this article. In the first half, [Sections 1](#) through [4](#) we proved the main result, [Theorem 1.2](#). In the second part, and [Sections 6](#) and [7](#), we provide some of the supporting details used in the proof of [Theorem 1.2](#). However we think that these are not just technical results, and they may be of independent interest.

We point out a major difference between the hypotheses used in the two parts. For [Theorem 1.2](#) we assume the particular foliation structure and curvature condition described by conditions (i)–(iii). In the second part, [Sections 6](#) and [7](#), we provide results in a general setup which we describe below.

We are given k smooth hypersurfaces $S_i = \Sigma_i(U_i)$ with smooth parametrizations Σ_i . These should be seen as new surfaces, different than the ones for which [Theorem 1.2](#) states a result. The most important difference is that the S_i , $i = 1, \dots, k$, used here are generic; in other words they are not assumed to have a foliation structure, nor curvature properties as the surfaces in our main result, [Theorem 1.2](#).

We assume the transversality condition: there exists $\nu > 0$ such that

$$\text{vol}(N_1(\zeta_1), \dots, N_k(\zeta_k)) \geq \nu \tag{5-1}$$

for all choices $\zeta_i \in \Sigma_i(U_i)$. Here by $\text{vol}(N_1(\zeta_1), \dots, N_k(\zeta_k))$ we mean the volume of the k -dimensional parallelepiped spanned by the vectors $N_1(\zeta_1), \dots, N_k(\zeta_k)$.

Each of these (parametrizations of) hypersurfaces generates the corresponding \mathcal{E}_i operator

$$\mathcal{E}_i f(x) = \int_{U_i} e^{ix \cdot \Sigma_i(\xi)} f(\xi) d\xi.$$

6. The multilinear estimate: localization and superposition

In this section we provide the proof of a localized version of the multilinear estimate. The motivation comes from the argument in the previous section. The proofs build on the ideas introduced in [[Bejenaru 2017a](#)] and later refined in [[Bejenaru 2017b](#)].

We work under the setup described in [Section 5](#). Given unit vectors N_{k+1}, \dots, N_{n+1} , we introduce the following transversality condition: there exists $\nu > 0$ such that

$$|\det(N_1(\zeta_1), \dots, N_k(\zeta_k), N_{k+1}, \dots, N_{n+1})| \geq \nu \quad (6-1)$$

for all choices $\zeta_i \in \Sigma_i(U_i)$.

Assume $\Sigma_1(\text{supp } f_1) \subset B(\mathcal{H}_1, \mu)$, where $B(\mathcal{H}_1, \mu)$ is the neighborhood of size μ of the k -dimensional affine subspace \mathcal{H}_1 . Assume that $|N_1(\zeta_1) - \pi_{\mathcal{H}_1} N_1(\zeta_1)| \lesssim \mu$ for all $\zeta_1 \in \Sigma_1(\text{supp } f_1)$, where $\pi_{\mathcal{H}_1} : \mathbb{R}^{n+1} \rightarrow \mathcal{H}_1$ is the projection onto \mathcal{H}_1 . In addition assume that if N_i , $i = k+1, \dots, n+1$, is a basis of the normal space \mathcal{H}_1^\perp to \mathcal{H}_1 , then $N_1(\zeta_1), \dots, N_k(\zeta_k), N_{k+1}, \dots, N_{n+1}$ are transversal in the sense of [\(6-1\)](#). Under these hypotheses we proved in [\[Bejenaru 2017b, Theorem 1.3\]](#) that

$$\left\| \prod_{i=1}^k \mathcal{E}_i f_i \right\|_{L^{2/(k-1)}(B(0,r))} \leq C(\epsilon) \mu^{\frac{n-k+1}{2}} r^\epsilon \prod_{i=1}^k \|f_i\|_{L^2(U_i)}. \quad (6-2)$$

The multilinear estimate [\(6-2\)](#) is a statement about the product of some functions in $L^{2/(k-1)}$. It is very natural to ask how this estimate behaves with respect to superpositions of one factor, that is, replacing f_1 by $\sum_\alpha f_{1,\alpha}$. If $2/(k-1) \geq 1$, then the triangle inequality holds true in $L^{2/(k-1)}$ and the answer is simple: in a sublinear fashion. If $2/(k-1) < 1$, the triangle inequality fails in $L^{2/(k-1)}$ and the sublinearity cannot be argued in the same way. However,

$$\begin{aligned} \left\| \mathcal{E}_1 \left(\sum_\alpha f_{1,\alpha} \right) \prod_{i=2}^k \mathcal{E}_i f_i \right\|_{L^{2/(k-1)}(B(0,R))} &\leq C(\epsilon) R^\epsilon \left\| \sum_\alpha f_{1,\alpha} \right\|_{L^2} \prod_{i=2}^k \|f_i\|_{L^2(U_i)} \\ &\leq C(\epsilon) R^\epsilon \sum_\alpha \|f_{1,\alpha}\|_{L^2} \prod_{i=2}^k \|f_i\|_{L^2(U_i)} \end{aligned}$$

and this indicates again sublinear behavior with respect to superpositions of one input. In the above the set of indexes α is taken to be of finite cardinality (to avoid unnecessary distractions) and the key point is that the estimate is independent of the cardinality of this set.

The main question is whether the sublinearity aspect of the estimate holds true for the refinement [\(6-2\)](#) of the multilinear estimate. An a posteriori argument as above fails to give the optimal result when each term $f_{1,\alpha}$ has good localization properties, but $\sum_\alpha f_{1,\alpha}$ does not have such localization properties.

Setup. We are given J , a finite set, and open, bounded and connected sets $U_{1,\alpha} \subset \mathcal{H}_{1,\alpha}$ for all $\alpha \in J$, where $\mathcal{H}_{1,\alpha}$ are affine hyperplanes. For each $\alpha \in J$ we assume the following: there are k -dimensional hyperplanes $\mathcal{H}'_{1,\alpha}$ with the property that $S_{1,\alpha} = \Sigma_{1,\alpha}(U_{1,\alpha}) \subset B(\mathcal{H}'_{1,\alpha}, \mu)$, where $B(\mathcal{H}'_{1,\alpha}, \mu)$ is the neighborhood of size μ of $\mathcal{H}'_{1,\alpha}$. The following property holds: $|N_1(\zeta_1) - \pi_{\mathcal{H}'_{1,\alpha}} N_1(\zeta_1)| \lesssim \mu$ for all $\zeta_1 \in S_{1,\alpha}$, where $\pi_{\mathcal{H}'_{1,\alpha}} : \mathbb{R}^{n+1} \rightarrow \mathcal{H}'_{1,\alpha}$ is the projection onto $\mathcal{H}'_{1,\alpha}$. Let $\tilde{\mathcal{H}}_{1,\alpha} = \mathcal{H}_{1,\alpha} \cap \mathcal{H}'_{1,\alpha}$ be the $(k-1)$ -dimensional affine subspace $\tilde{\mathcal{H}}_{1,\alpha} \subset \mathcal{H}_{1,\alpha}$; we also assume that $U_{1,\alpha} \subset B(\tilde{\mathcal{H}}_{1,\alpha}, \mu)$.

We assume that $S_{1,\alpha} \subset S_1 = \Sigma_1(U_1)$ for all $\alpha \in J$, and S_1 satisfies the following global property: there is an orthonormal set of vectors N_i , $i = k, \dots, n+1$, such that [\(6-1\)](#) is satisfied.

For each $\alpha \in J$, we assume that if N_i , $i = k+1, \dots, n+1$, is a basis of the normal space $\tilde{\mathcal{H}}_{1,\alpha}^\perp \subset \mathcal{H}_{1,\alpha}$, then $N_1(\zeta_1), \dots, N_k(\zeta_k), N_{k+1}, \dots, N_{n+1}$ are transversal in the sense of [\(6-1\)](#).

For each $\alpha \in J$ we define

$$\mathcal{E}_{1,\alpha} f(x) = \int_{U_{1,\alpha}} e^{ix \cdot \Sigma_{1,\alpha}(\xi)} f(\xi) d\xi.$$

Without restricting the generality of the problem, we can assume that $S_{1,\alpha}$ are of graph type; that is, $\Sigma_{1,\alpha}(\xi^\alpha) = (\xi^\alpha, \varphi_{1,\alpha}(\xi^\alpha))$, where ξ^α is the coordinate in $\mathcal{H}_{1,\alpha}$. In addition, for each α , we pick and fix some $\eta_{1,\alpha} \in U_{1,\alpha}$.

The next result states how the multilinear estimate behaves with respect to superposition of localized functions.

Theorem 6.1. *We assume the Setup above. Let $\mu, R > 0$ be such that $R \leq \mu^{-1}$. Then for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true:*

$$\left\| \left(\sum_{\alpha} \mathcal{E}_{1,\alpha} f_{1,\alpha} \right) \prod_{i=2}^k \mathcal{E}_i f_i \right\|_{L^{2/(k-1)}(B(0,R))} \leq C(\epsilon) \mu^{\frac{n+1-k}{2}} R^\epsilon \left(\sum_{\alpha} \|f_{1,\alpha}\|_{L^2(U_{1,\alpha})} \right) \prod_{i=2}^k \|f_i\|_{L^2(U_i)}. \tag{6-3}$$

In Section 4 we used the following consequence of the above theorem.

Corollary 6.2. *We assume the Setup above. Assume that $\mu \approx r^{-1}$ and q is a cube of size $\approx r$. Then for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true:*

$$\left\| \left(\sum_{\alpha} \mathcal{E}_{1,\alpha} f_{1,\alpha} \right) \prod_{i=2}^k \mathcal{E}_i f_i \right\|_{L^{2/(k-1)}(q)} \leq C(\epsilon) r^{-\frac{n+1}{2}} r^\epsilon \left(\sum_{\alpha} \|\tilde{\chi}_q \mathcal{E}_{1,\alpha} f_{1,\alpha}\|_{L^2} \right) \prod_{i=2}^k \|\tilde{\chi}_q \mathcal{E}_i f_i\|_{L^2}. \tag{6-4}$$

We note that the apparent gain of a factor of $r^{-k/2}$ in this corollary over the result in Theorem 6.1 has to do with replacing $\|f_i\|_{L^2(U_i)}$ by $\|\tilde{\chi}_q \mathcal{E}_i f_i\|_{L^2}$.

The result of the corollary is not an immediate consequence of Theorem 6.1; but it follows easily from the arguments used in the proof of Theorem 6.1.

The plan is the following: we introduce some notation specific to this section and then we proceed with the proof of the above two results.

6A. Notation. Assume $\mathcal{H}_1 \subset \mathbb{R}^{n+1}$ is a hyperplane passing through the origin. Let N_1 be its normal and let $\pi_{N_1} : \mathbb{R}^{n+1} \rightarrow \mathcal{H}_1$ be the associated projection along the normal N_1 . We denote by $\mathcal{F}_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ the Fourier transform and by \mathcal{F}_1^{-1} the inverse Fourier transform. We denote the variables in \mathbb{R}^{n+1} by $x = (x_1, x')$, where x_1 is the coordinate along N_1 and x' is the coordinate along \mathcal{H}_1 . We denote by ξ' the Fourier variable corresponding to x' . For $f : U_1 \subset \mathcal{H}_1 \rightarrow \mathbb{C}$, $f \in L^2(U_1)$, the operator \mathcal{E}_1 takes the form

$$\mathcal{E}_1 f(x) = \int_{U_1} e^{i(x' \xi' + x_1 \varphi_1(\xi'))} f(\xi') d\xi'. \tag{6-5}$$

We define the differential operator $\nabla_{\varphi_1}(D'/i)$ to be the operator with symbol $\nabla_{\varphi_1}(\xi')$. The following commutator estimate holds true:

$$\left(x' - x'_0 - x_1 \nabla_{\varphi_1} \left(\frac{D'}{i} \right) \right)^N \mathcal{E}_1 f = \mathcal{E}_1 (\mathcal{F}_1((x' - x'_0)^N \mathcal{F}_1^{-1} f)) \quad \text{for all } N \in \mathbb{N}. \tag{6-6}$$

This is a direct computation using (6-5) and it suffices to check it for $N = 1$. The role of (6-6) will be to quantify localization properties of $\mathcal{F}_1^{-1} f$ on hyperplanes with $x_1 = \text{constant}$.

We take $\mathcal{H}_i, i = 1, \dots, k$, to be reference hyperplanes that are used in defining $\mathcal{E}_i f_i, i = 1, \dots, k$. Their normals are denoted by $N_i, i = 1, \dots, k$, respectively. Note that since $S_{1,\alpha} \subset S_1$ for all $\alpha \in J$, it follows that N_1 is transversal to all $\mathcal{H}_{1,\alpha}$. We then pick unit vectors N_{k+1}, \dots, N_{n+1} such that (6-1) is satisfied.

We construct $\mathcal{L} := \{z_1 N_1 + \dots + z_{n+1} N_{n+1} : (z_1, \dots, z_{n+1}) \in \mathbb{Z}^{n+1}\}$ to be the oblique lattice in \mathbb{R}^{n+1} generated by the unit vectors N_1, \dots, N_{n+1} . In each $\mathcal{H}_i, i = 2, \dots, k$, we construct the induced lattice $\mathcal{L}(\mathcal{H}_i) = \pi_{N_i}(\mathcal{L})$; this is a lattice since the projection is taken along a direction of the original lattice \mathcal{L} .

Given $r > 0$ we define $\mathcal{C}(r)$ to be the set of parallelepipeds of size r in \mathbb{R}^{n+1} relative to the lattice \mathcal{L} ; a parallelepiped in $\mathcal{C}(r)$ has the form

$$q(\mathbf{j}) := \left[r \left(j_1 - \frac{1}{2} \right), r \left(j_1 + \frac{1}{2} \right) \right] \times \dots \times \left[r \left(j_{n+1} - \frac{1}{2} \right), r \left(j_{n+1} + \frac{1}{2} \right) \right],$$

where $\mathbf{j} = (j_1, \dots, j_{n+1}) \in \mathbb{Z}^{n+1}$. For such a parallelepiped we define $c(q) = r \mathbf{j} = (r j_1, \dots, r j_{n+1}) \in r \mathcal{L}$ to be its center. For each $i = 2, \dots, k$, we let $\mathcal{C}\mathcal{H}_i(r) = \pi_{N_i} \mathcal{C}(r)$ be the set of parallelepipeds of size r in the hyperplane \mathcal{H}_i . Given two parallelepipeds $q, q' \in \mathcal{C}(r)$ or $\mathcal{C}\mathcal{H}_i(r)$ we define $d(q, q')$ to be the distance between them when considered as subsets of the underlying space, which we take to be \mathbb{R}^{n+1} or \mathcal{H}_i .

For each $i \in \{2, \dots, k\}, r > 0$, we define the linear operator $\mathcal{T}_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ to be the operator that takes $\mathcal{L}(\mathcal{H}_i)$ to the standard lattice \mathbb{Z}^n in \mathcal{H}_i . Then for each $q \in \mathcal{C}\mathcal{H}_i(r)$, define $\chi_q : \mathcal{H}_i \rightarrow \mathbb{R}$ by

$$\chi_q(x) = \eta_0 \left(\mathcal{T}_i \left(\frac{x - c(q)}{r} \right) \right).$$

Notice that $\mathcal{F}_i \chi_q$ has Fourier support in the ball of radius $\lesssim r^{-1}$. By the Poisson summation formula and properties of η_0 ,

$$\sum_{q \in \mathcal{C}\mathcal{H}_i(r)} \chi_q = 1. \tag{6-7}$$

Using the properties of χ_q , a direct exercise shows that for each $N \in \mathbb{N}$, the following holds true:

$$\sum_{q \in \mathcal{C}\mathcal{H}_i(r)} \left\| \left\langle \frac{x - c(q)}{r} \right\rangle^N \chi_q g \right\|_{L^2}^2 \lesssim_N \|g\|_{L^2}^2 \tag{6-8}$$

for any $g \in L^2(\mathcal{H}_i)$. Here, the variable x is the argument of g and belongs to \mathcal{H}_i .

Next we turn our attention to similar objects corresponding to the more complex family indexed by $\alpha \in J$. Given $\tilde{\mathcal{H}}_{1,\alpha} \subset \mathcal{H}_{1,\alpha}$ a subspace of dimension $k - 1$, we let $\tilde{\pi}_\alpha : \mathcal{H}_{1,\alpha} \rightarrow \tilde{\mathcal{H}}_{1,\alpha}$ be the orthogonal projection onto $\tilde{\mathcal{H}}_{1,\alpha}$. We denote by $(\tilde{\mathcal{H}}_{1,\alpha})^\perp$ the normal subspace to $\tilde{\mathcal{H}}_{1,\alpha}$ in $\mathcal{H}_{1,\alpha}$.

We let $\pi_{N_{1,\alpha}} : \mathbb{R}^{n+1} \rightarrow \mathcal{H}_{1,\alpha}$ be the projection onto $\mathcal{H}_{1,\alpha}$ and $\tilde{\pi}_{1,\alpha} := \tilde{\pi}_\alpha \circ \pi_{N_{1,\alpha}} : \mathbb{R}^{n+1} \rightarrow \tilde{\mathcal{H}}_{1,\alpha}$ be the projection onto $\tilde{\mathcal{H}}_{1,\alpha}$. We define the lattices $\mathcal{L}(\mathcal{H}_{1,\alpha}) = \mathbb{Z}^n$ inside $\mathcal{H}_{1,\alpha}$ and $\mathcal{L}(\tilde{\mathcal{H}}_{1,\alpha}) = \mathbb{Z}^{k-1}$ inside $\tilde{\mathcal{H}}_{1,\alpha}$ with respect to orthonormal basis in each case. They are constructed such that $\tilde{\pi}_\alpha(\mathcal{L}(\mathcal{H}_{1,\alpha})) = \mathcal{L}(\tilde{\mathcal{H}}_{1,\alpha})$; this holds true if the orthonormal basis in $\tilde{\mathcal{H}}_{1,\alpha}$ is a subset of the orthonormal basis in $\mathcal{H}_{1,\alpha}$.

Inside the subspace $\mathcal{H}_{1,\alpha}$ we construct $\mathcal{C}_{1,\alpha}(r)$ to be the set of cubes of size r centered at points from the lattice $r\mathcal{L}(\mathcal{H}_{1,\alpha})$ and sides parallel to the directions of the lattice. Inside the subspace $\tilde{\mathcal{H}}_{1,\alpha}$ we construct $\tilde{\mathcal{C}}_{1,\alpha}(r)$ be the set of cubes of size r centered at points from the lattice $r\mathcal{L}(\tilde{\mathcal{H}}_{1,\alpha})$ with sides parallel to the directions of the lattice. Therefore $\tilde{\mathcal{C}}_{1,\alpha}(r) = \tilde{\pi}^\alpha \mathcal{C}_{1,\alpha}(r)$. Then we define $\mathfrak{S}_{1,\alpha}(r)$ to be the set of infinite cubical strips $\mathfrak{s} = q \times (\tilde{\mathcal{H}}_{1,\alpha})^\perp \subset \mathcal{H}_{1,\alpha}$, where $q \in \tilde{\mathcal{C}}_{1,\alpha}(r)$. We denote by $c(\mathfrak{s}) := c(q) \subset r\mathcal{L}(\tilde{\mathcal{H}}_{1,\alpha})$ the center of the strip. We note that given $q_1, q_2 \in \mathcal{C}_{1,\alpha}(r)$, they belong to the same cubical strip in $\mathfrak{S}_{1,\alpha}(r)$ if and only if $\tilde{\pi}_\alpha q_1 = \tilde{\pi}_\alpha q_2$. For $q \in \mathcal{C}_{1,\alpha}(r)$, we let $\mathfrak{s}(\tilde{\pi}_\alpha q)$ be the infinite cubical strip it belongs to as a subset in $\mathfrak{S}_{1,\alpha}(r)$. Given a strip $\mathfrak{s} \in \mathfrak{S}_{1,\alpha}(r)$ we define $\chi_\mathfrak{s} : \mathcal{H}_{1,\alpha} \rightarrow \mathbb{R}$

$$\chi_\mathfrak{s}(x) = \eta_0 \left(\frac{\tilde{\pi}_{1,\alpha}(x) - c(\mathfrak{s})}{r} \right),$$

where, by abusing notation, $\eta_0 : \mathbb{R}^{k-1} \rightarrow \mathbb{R}$ is entirely similar to the η_0 introduced in Section 6A, except that it acts on \mathbb{R}^{k-1} instead of \mathbb{R}^n . A key property of $\chi_\mathfrak{s}$ is that it is constant in directions from the subspace $(\tilde{\mathcal{H}}_{1,\alpha})^\perp$.

One unpleasant feature of the above construction is that the lattice \mathcal{L} does not project exactly into the lattices $\mathcal{L}(\tilde{\mathcal{H}}_{1,\alpha})$ via $\tilde{\pi}_{1,\alpha}$; similarly $\mathcal{C}_\alpha(r)$ does not project well into $\tilde{\mathcal{C}}_{1,\alpha}(r)$ via $\tilde{\pi}_{1,\alpha}$. This is an inherent feature of the fact that there are too many subspaces $\tilde{\mathcal{H}}_{1,\alpha}$. As a consequence, given $q \in \mathcal{C}(r)$, it is not necessarily true that $\tilde{\pi}_{1,\alpha}(q) \in \tilde{\mathcal{C}}_{1,\alpha}(r)$; however $\tilde{\pi}_{1,\alpha}(q)$ intersects a finite number of $q' \in \tilde{\mathcal{C}}_{1,\alpha}(r)$. Abusing notation, we define

$$\mathfrak{s}^\alpha(\tilde{\pi}_{1,\alpha}(q)) = \bigcup_{\substack{q' \in \tilde{\mathcal{C}}_{1,\alpha}(r) \\ q' \cap \tilde{\pi}_{1,\alpha}(q) \neq \emptyset}} \mathfrak{s}^\alpha(q'),$$

the strip generated by the projection of q onto $\tilde{\mathcal{H}}_{1,\alpha}$.

Recalling that $\mathcal{L} := \{z_1 N_1 + \dots + z_{n+1} N_{n+1} : (z_1, \dots, z_{n+1}) \in \mathbb{Z}^{n+1}\}$, we denote the coordinates of a point in the lattice by (z_1, \dots, z_{n+1}) and define

$$\|g\|_{l^2_{z_1, z_{k+1}, \dots, z_{n+1}} l^2_{z_2, \dots, z_k}(\mathcal{L})} = \sup_{z_1, z_{k+1}, \dots, z_{n+1}} \|g(z_1, \cdot, z_{k+1}, \dots, z_{n+1})\|_{l^2_{z_2, \dots, z_k}},$$

where \cdot stands for the variables z_2, \dots, z_k with respect to which l^2 is computed.

With this notation in place we have the following result:

Lemma 6.3. *Assume $g_1 \in l^2_{z_1, z_{k+1}, \dots, z_{n+1}} l^2_{z_2, \dots, z_k}(\mathcal{L})$ and $g_i \in l^2(\mathcal{L}(\mathcal{H}_i))$, $i = 2, \dots, k$. Then the following holds true:*

$$\left\| g_1(z) \prod_{i=2}^k g_i(\pi_{N_i}(z)) \right\|_{l^2/(k-1)(\mathcal{L})} \lesssim \|g_1\|_{l^2_{z_1, z_{k+1}, \dots, z_{n+1}} l^2_{z_2, \dots, z_k}(\mathcal{L})} \prod_{i=2}^k \|g_i\|_{l^2(\mathcal{L}(\mathcal{H}_i))}. \tag{6-9}$$

Proof. The function $g_i \circ \pi_{N_i}$ is independent of the z_i -variable; therefore it holds true that $g_i \circ \pi_{N_i} \in l^2_{z_1, z_{k+1}, \dots, z_{n+1}} l^2_{z_2, \dots, z_{i-1}} l^2_{z_i} l^2_{z_{i+1}, \dots, z_k}$ and

$$\|g_i \circ \pi_{N_i}\|_{l^2_{z_1, z_{k+1}, \dots, z_{n+1}} l^2_{z_2, \dots, z_{i-1}} l^2_{z_i} l^2_{z_{i+1}, \dots, z_k}} \leq \|g_i\|_{l^2(\mathcal{L}(\mathcal{H}_i))}.$$

where then norms $l_{z_1, z_{k+1}, \dots, z_{n+1}}^2 l_{z_2, \dots, z_{i-1}}^2 l_{z_i}^\infty l_{z_{i+1}, \dots, z_k}^2$ are defined in the standard fashion. Then the result is a direct consequence of the Hölder inequality in its discrete version. \square

6B. Proofs of the main results.

Proof of Theorem 6.1. The argument is based on an induction on scales. Given a $0 < \delta \ll 1$, we break the surfaces into smaller pieces of diameter $\lesssim \delta$. A result on the smaller scales is converted to a result at the original scale at the cost of a large power of δ^{-1} , which is absorbed into $C(\epsilon)$. Thus, the focus will be on providing a result in the context of surfaces with diameter less than δ .

We run an induction with respect to the size of the cube where estimates are made. We show that passing from an estimate on cubes of size R to an estimate on cubes of size $\delta^{-1}R$ can be done by accumulating constants that are independent of δ and R . In implementing this approach, we use a phase-space approach that alters the support of $f_{1,\alpha}, f_2, \dots, f_k$ by a factor $\approx R^{-1/2}$ where $R \geq \delta^{-2}$. This is fine with f_2, \dots, f_k but not with $f_{1,\alpha}$, $\alpha \in J$, given that their support in some directions is $\mu \ll \delta$. This will require extra care.

We work under the hypothesis that $U_i \subset B_i(0; \delta)$, $i = 2, \dots, k$, where $B_i(0; \delta)$ is the ball in the hyperplane \mathcal{H}_i . For a function $f_i : \mathcal{H}_i \rightarrow \mathbb{C}$, its margin is defined as

$$\text{margin}^i(f_i) := \text{dist}(\text{supp}(f_i), B_i(0; 2\delta)^c), \quad i = 2, \dots, k, \quad (6-10)$$

where supp is the support of f_i .

We work under the hypothesis that $U_{1,\alpha} \subset B'(0; \delta) \times B''(0; \mu)$, where $B'(0; \delta)$ is the ball in the hyperplane $\tilde{\mathcal{H}}_{1,\alpha}$ centered at the origin and of diameter δ and $B''(0; \mu)$ is the ball in the hyperplane $(\tilde{\mathcal{H}}_{1,\alpha})^\perp$ centered at the origin and of diameter μ . Accordingly, we split the coordinates in $\mathcal{H}_{1,\alpha}$ as $\xi^\alpha = (\xi'^\alpha, \xi''^\alpha)$, where ξ'^α is the coordinate in $\tilde{\mathcal{H}}_{1,\alpha}$ and ξ''^α is the coordinate in $(\tilde{\mathcal{H}}_{1,\alpha})^\perp$. Given a function $f : \mathcal{H}_{1,\alpha} \rightarrow \mathbb{R}$, its margin is defined by

$$\text{margin}^{1,\alpha}(f) := \inf_{\xi''^\alpha} \text{dist}(\text{supp}_{\xi'^\alpha}(f(\cdot, \xi''^\alpha)), B'(0; 2\delta)^c), \quad (6-11)$$

where $\text{supp}_{\xi'^\alpha}$ is the support of f in the ξ'^α -variable. On the physical side we denote by x'^α, x''^α the dual variables to $\xi'^\alpha, \xi''^\alpha$, respectively. We complete the system of coordinates to $(\xi_1^\alpha, \xi'^\alpha, \xi''^\alpha)$ and $(x_1^\alpha, x'^\alpha, x''^\alpha)$, where ξ_1^α is the coordinate in the direction of $N_{1,\alpha}$, the normal to $\mathcal{H}_{1,\alpha}$, and x_1^α is the dual coordinate.

Our induction aims at quantifying the behavior of $A(R)$ defined below.

Definition 6.4. Given $R \geq \delta^{-2}$ we define $A(R)$ to be the best constant for which the estimate

$$\left\| \left(\sum_\alpha |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right) \prod_{i=2}^k \mathcal{E}_i f_i \right\|_{L^{2/(k-1)}(Q)} \leq A(R) \prod_{i=1}^k \|f_i\|_{L^2} \quad (6-12)$$

holds true for all parallelepipeds $Q \in \mathcal{C}(R)$, with f_i obeying the margin requirement

$$\text{margin}^i(f_i) \geq \delta - R^{-\frac{1}{2}}, \quad i = 2, \dots, k, \quad \text{margin}^{1,\alpha}(f_{1,\alpha}) \geq \delta - R^{-\frac{1}{2}} \quad \text{for all } \alpha \in J, \quad (6-13)$$

and $f_{1,\alpha}$ is supported in $B(\tilde{\mathcal{H}}_{1,\alpha}; \mu) \subset \mathcal{H}_{1,\alpha}$ for all $\alpha \in J$.

Note that in (6-12) we use absolute values. This indicates that we do not use any cancellation properties between the components $\mathcal{E}_{1,\alpha} f_{1,\alpha}$. However, using the stronger statement with the absolute values plays a crucial role in carrying out the induction argument.

We start with the parallelepiped Q of size $\delta^{-1}R$ centered at the origin. To keep notation compact we define

$$H = \prod_{i=2}^k \mathcal{E}_i f_i, \quad G = \prod_{i=2}^k \|f_i\|_{L^2}.$$

For each $q \in \mathcal{C}(R) \cap Q$, the induction hypothesis is

$$\left\| \left(\sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right) \cdot H \right\|_{L^{2/(k-1)}(q)} \leq A(R) \left(\sum_{\alpha} \|f_{1,\alpha}\|_{L^2(U_{1,\alpha})} \right) G. \tag{6-14}$$

We claim the following strengthening of (6-14):

$$\begin{aligned} & \left\| \left(\sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right) \cdot H \right\|_{L^{2/(k-1)}(q)} \\ & \lesssim_N A(R) \prod_{i=2}^k \left(\sum_{q' \in \mathcal{C}\mathcal{H}_i(R)} \left\langle \frac{d(\pi_{N_i} q, q')}{R} \right\rangle^{-2N-n^2} \left\| \left\langle \frac{x-c(q')}{R} \right\rangle^N \chi_{q' \mathcal{F}_i^{-1}} f_i \right\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \cdot \sum_{\alpha} \left(\sum_{\mathfrak{s}^{\alpha} \in \mathfrak{S}_{1,\alpha}(R)} \left\langle \frac{d(\tilde{\pi}_{1,\alpha}(q), \mathfrak{s}^{\alpha})}{R} \right\rangle^{-2N-2k} \left\| \left\langle \frac{x',\alpha - c(\mathfrak{s}^{\alpha})}{R} \right\rangle^N \chi_{\mathfrak{s}^{\alpha} \mathcal{F}_{1,\alpha}^{-1}} f_{1,\alpha} \right\|_{L^2}^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{6-15}$$

Similar improvements were provided in [Bejenaru 2017a]; in particular the improvement for the terms f_i , $i = 2, \dots, k$, was established, as claimed above (it can also be derived along similar, but simpler, lines as those in the arguments we provide below for the $f_{1,\alpha}$ terms). The improvement for $f_{1,\alpha}$ was also provided in that paper in the case when there is only one function $f_{1,\alpha}$, that is, J contains one element only. Here we provide an argument for general finite sets J and note that the cardinality of J does not impact $A(R)$.

Therefore, in justifying (6-15) we focus on the improvement for the $f_{1,\alpha}$ terms only. Given $q \in \mathcal{C}(R) \cap Q$ and $d \in \mathbb{N}$, let

$$A^{\alpha}(q, d) = \left\{ \mathfrak{s} \in \mathfrak{S}_{1,\alpha}(r) : \left\langle \frac{d(\tilde{\pi}_{1,\alpha}(q), \mathfrak{s})}{R} \right\rangle \approx d \right\}.$$

We can modify the sets such that each strip \mathfrak{s} belongs to only one $A^{\alpha}(q, d)$.

From (6-6) we obtain the identity

$$\begin{aligned} & \sum_{\alpha} \sum_{\mathfrak{s}^{\alpha} \in A^{\alpha}(q,d)} \left| \left(x',\alpha - c(\mathfrak{s}^{\alpha}) - x_1^{\alpha} \nabla_{\xi',\alpha} \varphi_1 \left(\frac{D^{\alpha}}{i} \right) \right) \mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{\mathfrak{s}^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha} \right| \\ & = \sum_{\alpha} \sum_{\mathfrak{s}^{\alpha} \in A^{\alpha}(q,d)} |\mathcal{E}_{1,\alpha}^{\alpha} \mathcal{F}_{1,\alpha}(x',\alpha - c(\mathfrak{s}^{\alpha})) \chi_{\mathfrak{s}^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}|, \end{aligned} \tag{6-16}$$

where the differential operator $\nabla_{\xi'}\varphi_1(D^\alpha/i)$ has symbol $\nabla_{\xi'}\varphi_1(\xi^\alpha)$. We have the following sequence of estimates:

$$\begin{aligned} & \left\| \sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} |(x'^{\alpha} - c(\mathfrak{s}^\alpha) - x_1^\alpha \nabla_{\xi'}\varphi_1(\eta_{1,\alpha}))\mathcal{E}_{1,\alpha}\mathcal{F}_{1,\alpha}\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}| \cdot H \right\|_{L^{2/(k-1)}(q)} \\ & \leq \left\| \sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} |(x'^{\alpha} - c(\mathfrak{s}^\alpha) - x_1^\alpha \nabla_{\xi'}\varphi_1(\xi^\alpha))\mathcal{E}_{1,\alpha}\mathcal{F}_{1,\alpha}\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}| \cdot H \right\|_{L^{2/(k-1)}(q)} \\ & \quad + \left\| x_1^\alpha (\nabla_{\xi'}\varphi_1(\eta_{1,\alpha}) - \nabla_{\xi'}\varphi_1(\xi^\alpha))\mathcal{E}_{1,\alpha}\mathcal{F}_{1,\alpha}\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha} \cdot H \right\|_{L^{2/(k-1)}(q)}. \end{aligned}$$

We invoke (6-16) and continue with

$$\begin{aligned} & = \left\| \sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} |\mathcal{E}_{1,\alpha}\mathcal{F}_{1,\alpha}(x'^{\alpha} - c(\mathfrak{s}^\alpha))\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}| \cdot H \right\|_{L^{2/(k-1)}(q)} \\ & \quad + \left\| \sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} x_1^\alpha \mathcal{E}_{1,\alpha}\mathcal{F}_{1,\alpha}(\nabla_{\xi'}\varphi_1(\eta_{1,\alpha}) - \nabla_{\xi'}\varphi_1(\xi^\alpha))\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha} \right\|_{L^{2/(k-1)}(q)}. \end{aligned}$$

We apply the induction hypothesis, and use that inside Q we have $|x_1^\alpha| \lesssim \delta^{-1}R$ for all $\alpha \in J$, to further continue with

$$\begin{aligned} & \leq A(R) \left(\sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} \|(x'^{\alpha} - c(\mathfrak{s}^\alpha))\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}\|_{L^2} \right) G \\ & \quad + A(R)\delta^{-1}R \left(\sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} \|(\nabla_{\xi'}\varphi_1(\eta_{1,\alpha}) - \nabla_{\xi'}\varphi_1(\xi^\alpha))\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}\|_{L^2} \right) G \\ & \lesssim A(R) \left(\sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} \|(x'^{\alpha} - c(\mathfrak{s}^\alpha))\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}\|_{L^2} + R\|\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}\|_{L^2} \right) G \\ & \lesssim RA(R) \left(\sum_{\alpha} \sum_{\mathfrak{s}^\alpha \in A^\alpha(q,d)} \left\| \left\langle \frac{x'^{\alpha} - c(\mathfrak{s}^\alpha)}{R} \right\rangle \chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha} \right\|_{L^2} \right) G. \end{aligned}$$

Note that it is in the above use of the induction estimate for $\mathcal{E}_{1,\alpha}\mathcal{F}_{1,\alpha}(x'^{\alpha} - c(\mathfrak{s}^\alpha))\chi_{\mathfrak{s}^\alpha}\mathcal{F}_{1,\alpha}^{-1}f_{1,\alpha}$ that we need to tolerate the relaxed support of $f_{1,\alpha}$. The margin of $f_{1,\alpha}$ is $\geq \delta - (\delta^{-1}R)^{-1/2} = \delta - \delta^{1/2}R^{-1/2}$ and it is affected by the convolution $\mathcal{F}_{1,\alpha}((x'^{\alpha} - c(\mathfrak{s}^\alpha))\chi_{\mathfrak{s}^\alpha})$ by a factor of at most CR^{-1} , which is smaller than $\frac{1}{2}\delta^{1/2}R^{-1/2}$ provided that δ is small relative to C^{-1} . Hence the new margin is $\geq \delta - \frac{1}{2}\delta^{1/2}R^{-1/2} \geq \delta - R^{-1/2}$, which is the required margin for using the induction hypothesis on cubes of size R .

We claim that for any $\mathfrak{s}^\alpha \in A^\alpha(q,d)$

$$\left\| \left\langle \frac{x'^{\alpha} - c(\mathfrak{s}^\alpha) - x_1^\alpha \nabla_{\xi'}\varphi_1(\eta_{1,\alpha})}{R} \right\rangle \right\|_{L^\infty(q)} \approx \left\langle \frac{d(\tilde{\pi}_{1,\alpha}(q), \mathfrak{s}^\alpha)}{R} \right\rangle \approx d$$

uniformly in α . This statement is invariant to rotations of coordinates, therefore we can assume that $\nabla_{\xi^\alpha}\varphi_1(\eta_{1,\alpha}) = 0$ and moreover that $x'^{\alpha} = (x_2, \dots, x_k)$ and $x''^{\alpha} = (x_{k+1}, \dots, x_{n+1})$. This way, $\tilde{\pi}_{1,\alpha}(x) = (0, x_2, \dots, x_k, 0, \dots, 0)$ and the statement is obvious.

From the above we obtain that, for d large,

$$dR \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |\mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}| \lesssim \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |(x',\alpha - c(s^{\alpha}) - x_1^{\alpha} \nabla_{\xi'} \varphi_1(\eta_{1,\alpha})) \mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}|.$$

Combining all the above estimates gives

$$\begin{aligned} dR \left\| \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |\mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}| \cdot H \right\|_{L^{2/(k-1)}(q)} \\ \lesssim \left\| \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |((x',\alpha) - c(s^{\alpha}) - x_1^{\alpha} \nabla_{\xi'} \varphi_1(\eta_{1,\alpha})) \mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}| \cdot H \right\|_{L^{2/(k-1)}(q)} \\ \lesssim RA(R) \left(\sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} \left\| \left\langle \frac{x',\alpha - c(s^{\alpha})}{R} \right\rangle \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha} \right\|_{L^2} \right) G. \end{aligned}$$

From this we conclude with (after more iterations of the same argument)

$$\begin{aligned} \left\| \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |\mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}| \cdot H \right\|_{L^{2/(k-1)}(q)} \\ \lesssim d^{-N} A(R) \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} \left\| \left\langle \frac{x',\alpha - c(s^{\alpha})}{R} \right\rangle^N \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha} \right\|_{L^2} \cdot G. \end{aligned}$$

Note, that while the argument above assumed d is large, this last inequality holds for all d , since it is trivial for d small. The summation over d is done in the usual manner:

$$\begin{aligned} \left\| \sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| H \right\|_{L^{2/(k-1)}(q)}^{\frac{2}{k-1}} \\ = \left\| \sum_d \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |\mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}| H \right\|_{L^{2/(k-1)}(q)}^{\frac{2}{k-1}} \\ \lesssim \sum_d \left\| \sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} |\mathcal{E}_{1,\alpha} \mathcal{F}_{1,\alpha} \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha}| H \right\|_{L^{2/(k-1)}(q)}^{\frac{2}{k-1}} \\ \lesssim (A(R))^{\frac{2}{k-1}} \sum_d d^{-N \cdot \frac{2}{k-1}} \left(\sum_{\alpha} \sum_{s^{\alpha} \in A^{\alpha}(q,d)} \left\| \left\langle \frac{x',\alpha - c(s^{\alpha})}{R} \right\rangle^N \chi_{s^{\alpha}} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha} \right\|_{L^2} \right)^{\frac{2}{k-1}} G^{\frac{2}{k-1}}. \end{aligned}$$

Using (1-8) together with the straightforward estimate

$$\|d^{-\frac{k}{2}}\|_{l_{\mathbb{N}}^{2/(k-1)}} \lesssim 1,$$

and we can continue the sequence of inequalities we started above

$$\begin{aligned}
& \left\| \sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| H \right\|_{L^{2/(k-1)}(q)}^{\frac{2}{k-1}} \\
& \lesssim (A(R))^{\frac{2}{k-1}} \left\| d^{-(N-\frac{k}{2})} \sum_{\alpha} \sum_{\mathfrak{s}^{\alpha} \in A^{\alpha}(q,d)} \left\| \left\langle \frac{x',\alpha - c(\mathfrak{s}^{\alpha})}{R} \right\rangle^N \chi_{\mathfrak{s}^{\alpha}} \mathcal{F}_1^{-1} f_{1,\alpha} \right\|_{L^2} \right\|_{l_d^2}^{\frac{2}{k-1}} G^{\frac{2}{k-1}} \\
& \lesssim (A(R))^{\frac{2}{k-1}} \left(\sum_{\alpha} \left\| d^{-(N-\frac{k}{2})} \sum_{\mathfrak{s}^{\alpha} \in A^{\alpha}(q,d)} \left\| \left\langle \frac{x',\alpha - c(\mathfrak{s}^{\alpha})}{R} \right\rangle^N \chi_{\mathfrak{s}^{\alpha}} \mathcal{F}_1^{-1} f_{1,\alpha} \right\|_{L^2} \right\|_{l_d^2} \right)^{\frac{2}{k-1}} G^{\frac{2}{k-1}} \\
& \lesssim (A(R))^{\frac{2}{k-1}} \left(\sum_{\alpha} \left(\sum_{\mathfrak{s}^{\alpha}} \left\langle \frac{d(\tilde{\pi}_{1,\alpha}(q), \mathfrak{s}^{\alpha})}{R} \right\rangle^{-(2N-2k)} \left\| \left\langle \frac{x',\alpha - c(\mathfrak{s}^{\alpha})}{R} \right\rangle^N \chi_{\mathfrak{s}^{\alpha}} \mathcal{F}_1^{-1} f_{1,\alpha} \right\|_{L^2}^2 \right)^{\frac{1}{2}} \right)^{\frac{2}{k-1}} G^{\frac{2}{k-1}}.
\end{aligned}$$

In passing to the last line we used that the cardinality of $A^{\alpha}(q, d)$ is $\approx \langle d \rangle^{k-1}$ in order to bound the $l_{\mathfrak{s}^{\alpha} \in A^{\alpha}(q,d)}^1$ -norm of the summand by the $l_{\mathfrak{s}^{\alpha} \in A^{\alpha}(q,d)}^2$ of the same quantity.

We are done with the justification of (6-15) and continue with the final step in the induction on scales. We define the functions $g_i : \mathcal{L}(\mathcal{H}_i) \rightarrow \mathbb{R}$ for $i = 2, \dots, k$ by

$$g_i(\mathbf{j}) = \left(\sum_{q' \in \mathcal{C}\mathcal{H}_i(R)} \left\langle \frac{d(q(\mathbf{j}), q')}{R} \right\rangle^{-(2N-n^2)} \left\| \left\langle \frac{x' - c(q')}{R} \right\rangle^N \chi_{q'} \mathcal{F}_i^{-1} f_i \right\|_{L^2}^2 \right)^{\frac{1}{2}}$$

for $\mathbf{j} \in \mathcal{L}(\mathcal{H}_i)$, and $g_1 : \mathcal{L} \rightarrow \mathbb{R}$ by

$$g_1(\mathbf{j}) = \sum_{\alpha} \left(\sum_{\mathfrak{s}^{\alpha}} \left\langle \frac{d(\tilde{\pi}_{1,\alpha}(q(\mathbf{j})), \mathfrak{s}^{\alpha})}{R} \right\rangle^{-(2N-2k)} \left\| \left\langle \frac{x',\alpha - c(\mathfrak{s}^{\alpha})}{R} \right\rangle^N \chi_{\mathfrak{s}^{\alpha}} \mathcal{F}_1^{-1} f_{1,\alpha} \right\|_{L^2}^2 \right)^{\frac{1}{2}}$$

for $\mathbf{j} \in \mathcal{L}$. Using (6-8), it is obvious that, provided N is large enough (in terms of n only), the following holds true:

$$\|g_i\|_{l^2(\mathcal{L}(\mathcal{H}_i))} \lesssim \|f_i\|_{L^2}, \quad i = 2, \dots, k.$$

We also claim that

$$\|g_1\|_{l_{z_1, z_{k+1}, \dots, z_{n+1}}^{\infty} l_{z_2, \dots, z_k}^2(\mathcal{L})} \lesssim \sum_{\alpha} \|f_{1,\alpha}\|_{L^2}. \quad (6-17)$$

This is a consequence of the following geometrical observation: Say $\mathbf{j} = \sum_{i=1}^{n+1} z_i N_i$, where $z_i \in \mathbb{Z}$. We fix $z_1, z_{k+1}, \dots, z_{n+1}$ and note that as we vary z_2, \dots, z_k , the $\tilde{\pi}_{1,\alpha}(q(\mathbf{j}))$ are almost disjoint and, most importantly, the strips they generate, $\mathfrak{s}^{\alpha}(\tilde{\pi}_{1,\alpha} q(\mathbf{j})) \subset \mathfrak{S}_{1,\alpha}(R)$, are almost disjoint for each $\alpha \in J$ (given a point in $\mathcal{H}_{1,\alpha}$ there are finitely many \mathbf{j} such that the point belongs to $\mathfrak{s}^{\alpha}(\tilde{\pi}_{1,\alpha} q(\mathbf{j}))$). This is due to the fact that the projections $\tilde{\pi}_{1,\alpha}$ onto the affine subspace $\tilde{\mathcal{H}}_{1,\alpha}$ are taken along directions that are transversal to N_2, \dots, N_k and the infinite sides of the strips are in directions that are transversal to N_2, \dots, N_k . Using this geometric observation, (6-17) follows from the equivalent of (6-8) for strips.

Then we apply (6-9) to conclude with

$$\left\| \left(\sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right) \cdot H \right\|_{L^{2/(k-1)}(\mathcal{Q})} \lesssim A(R) \prod_{i=1}^k \|f_i\|_{L^2}.$$

Thus we obtain

$$A(\delta^{-1}R) \leq CA(R)$$

for a constant C that is independent of δ and R . Iterating this gives $A(\delta^{-N}r) \leq C^N A(r)$. Therefore $\max_{r \in [0, \delta^{-2}]} A(\delta^{-N}r) \leq C^N \max_{r \in [0, \delta^{-2}]} A(r) = C^N C(\delta) \mu^{(n+1-k)/2}$ is obtained from the uniform pointwise bound

$$\begin{aligned} \left\| \left(\sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right) \prod_{i=2}^k \mathcal{E}_i f_i \right\|_{L^\infty} &\lesssim \left\| \sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right\|_{L^\infty} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^\infty} \\ &\lesssim \mu^{\frac{n+1-k}{2}} \left(\sum_{\alpha} \|f_{1,\alpha}\|_{L^2} \right) \prod_{i=2}^k \|f_i\|_{L^2}, \end{aligned} \tag{6-18}$$

which is integrated over arbitrary cubes of size $\leq \delta^{-2}$. Note that we have used the support properties of $f_{1,\alpha}$ to obtain the improved bound.

For $R \in [\delta^{-N}, \delta^{-N-1}]$, the above implies

$$A(R) \leq C^N C(\delta) \mu^{\frac{n+1-k}{2}} \leq R^\epsilon C(\delta) \mu^{\frac{n+1-k}{2}}$$

provided that $C^N \leq \delta^{-N\epsilon}$. Therefore choosing $\delta = C^{-1/\epsilon}$ leads to the desired result. □

Proof of Corollary 6.2. In each \mathcal{H}_i , $i = 1, \dots, k$, $y_i \in \mathbb{R}$, we define $\mathcal{H}_i + y_i N_i$ to be the translation of \mathcal{H}_i by $y_i N_i$. Also $\mathcal{CH}_i(r) + y_i N_i$ is the corresponding translation of $\mathcal{CH}_i(r)$ by $y_i N_i$.

Given any vector $y \in \mathbb{R}^{n+1}$ with $|y_i - c_i(q)| \leq r$, $i = 1, \dots, k$, and $y_i = c_i(q)$, $k + 1 \leq i \leq n + 1$, we claim

$$\begin{aligned} &\left\| \left(\sum_{\alpha} |\mathcal{E}_{1,\alpha} f_{1,\alpha}| \right) \cdot H \right\|_{L^{2/(k-1)}(q)} \\ &\lesssim_N C(\epsilon) r^\epsilon \mu^{\frac{n-k+1}{2}} \prod_{i=2}^k \left(\sum_{q' \in \mathcal{CH}_i(r) + y_i N_i} \left\langle \frac{d(\pi_{N_i} q, q')}{r} \right\rangle^{-(2N-n^2)} \left\| \left\langle \frac{x-c(q')}{r} \right\rangle^N \chi_{q'} \mathcal{F}_i^{-1} f_i \right\|_{L^2(\mathcal{H}_i + y_i N_i)}^2 \right)^{\frac{1}{2}} \\ &\quad \cdot \sum_{\alpha} \left(\sum_{q' \in \mathcal{C}_{1,\alpha}(r) + y_1 N_1} \left\langle \frac{d(\tilde{\pi}_{1,\alpha}(q), q')}{r} \right\rangle^{-(2N-2k)} \left\| \left\langle \frac{x^\alpha - c(q')}{r} \right\rangle^N \chi_{q'} \mathcal{F}_{1,\alpha}^{-1} f_{1,\alpha} \right\|_{L^2(\mathcal{H}_{1,\alpha} + y_1 N_1)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

It suffices to prove this estimate for $y = 0$, in which case it is very similar to (6-15). Except that, for the $f_{1,\alpha}$ terms we do not use strips, but cubes. This should be a reason for concern, as the use of strips was necessary to keep the localization of the $f_{1,\alpha}$ at scale μ intact throughout the induction process. However, given that $\mu \approx r^{-1}$, the multiplication with $\chi_{q'}$ alters the localization by a factor of $r^{-1} \approx \mu$. A similar argument to the one used in the proof of (6-15) gives the above estimate.

Next we average the above estimate over the values of (y_1, \dots, y_k) satisfying $|y_i - c_i(q)| \leq r$ (keeping $y_i = c_i(q)$, $i \geq k+1$) to obtain

$$\begin{aligned} \left\| \prod_{i=1}^k \mathcal{E}_i f_i \right\|_{L^1(q)} &\lesssim C(\epsilon) r^\epsilon (r^{-1})^{\frac{n-k+1}{2}} r^{-\frac{k}{2}} \\ &\cdot \prod_{i=1}^k \left(\int_{|y_i| \leq r} \sum_{q' \in \mathcal{C}\mathcal{H}_i(r) + y_i N_i} \left\langle \frac{d(\pi_{N_i} q, q')}{r} \right\rangle^{-N} \left\| \left\langle \frac{x-c(q')}{r} \right\rangle^N \chi_{q'} \mathcal{E}_i f_i \right\|_{L^2(\mathcal{H}_i + y_i N_i)}^2 \right)^{\frac{1}{2}} \\ &\lesssim C(\epsilon) r^\epsilon r^{-\frac{n+1}{2}} \prod_{i=1}^k \|\tilde{\chi}_q \mathcal{E}_i f_i\|_{L^2}, \end{aligned}$$

which finishes the proof. \square

7. A new multilinear estimate

In this section we address (4-10), the last supporting detail in the proof of Proposition 4.1. As described in Section 5, we are given k smooth hypersurfaces $S_i = \Sigma_i(U_i)$ with smooth parametrizations Σ_i obeying (5-1). These hypersurfaces can be thought of as living in the frequency space and generate the operators \mathcal{E}_i . In addition we are given another smooth surface S of dimension $n - k + 1$, which should be thought of as living in the physical space, with the following properties:

(P1) S is uniformly transversal to $N_1(\zeta_1), \dots, N_k(\zeta_k)$ for all choices $\zeta_i \in S_i$. There exists $\nu > 0$ such that, for any $\zeta_i \in S_i$, $i = 1, \dots, k$, for any $y \in S$ and for any orthonormal basis v_{k+1}, \dots, v_{n+1} of $T_y S$, the following holds true:

$$\text{vol}(N_1(\zeta_1), \dots, N_k(\zeta_k), v_{k+1}, \dots, v_{n+1}) \geq \nu.$$

(P2) There exists $\nu > 0$ such that for any $P_1, P_2 \in S$, for any $\zeta_1 \in S_1$, for any $\zeta_i \in S_i$, $\zeta_j \in S_j$, $2 \leq i < j \leq k$ and for any $\alpha_i, \alpha_j \in \mathbb{R}$, the following holds true:

$$\text{vol}(\overrightarrow{P_1 P_2}, N_1(\zeta_1), \overrightarrow{v}) \geq \nu |\overrightarrow{P_1 P_2}| \cdot |\overrightarrow{v}|, \quad (7-1)$$

where $\overrightarrow{v} = \alpha_i N_i(\zeta_i) - \alpha_j N_j(\zeta_j)$.

As already mentioned in Section 5, in this section we make no curvature assumptions on S_i . However, we note that property (P2) follows from curvature properties similar to those used in Theorem 1.2; in other words the curvature properties have been encoded in the structure of S .

Given $r > 0$, we recall that $\mathcal{C}(r)$ is the set of unit cubes in \mathbb{R}^{n+1} with centers in the lattice $r\mathbb{Z}^{n+1}$. With S as above and for each $q \in \mathcal{C}(r)$, we define

$$S(q) = q + S \cap B(0, r^2).$$

Here $S \cap B(0, r^2)$ should be understood as follows: we cut the surface S at scale $\approx r^2$, and whether this is performed in a ball or cube, centered at the origin or somewhere else, is unimportant. The reason for doing this comes from the use of wave packets and their scales.

More generally, given a subset $A \subset \mathcal{C}(r)$, we define

$$S(A) = \bigcup_{q \in A} S(q).$$

The main result of this section is the following theorem.

Theorem 7.1. *Assume that S_i , $i = 1, \dots, k$, and S are as above. Then for any $\epsilon > 0$, there is $C(\epsilon)$ such that the following holds true:*

$$\left(\sum_{q \in \mathcal{C}(r) \cap B(0, r^2)} \left(\|\mathcal{E}_1 f_1\|_{L^2(S(q))} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)} \right)^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \leq C(\epsilon) r^{\frac{k}{2}} r^\epsilon \prod_{i=1}^k \|f_i\|_{L^2(U_i)}. \quad (7-2)$$

The above result has a multilinear flavor to it. The factor $r^{k/2}$ has to appear because we consider the mass of $\mathcal{E}_i f_i$ in neighborhoods of size r of hypersurfaces across which we would have good energy estimates; see the proof of the theorem for details. Otherwise (7-2) is similar to a multilinear restriction estimate, see (6-2) (with $\mu = 1$), except that now, one of the objects $\mathcal{E}_1 f_1$ is measured in a more complex fashion.

The complexity of this estimate does not stem from the fact that we collect energy from various spatial regions; indeed if v_i are arbitrary vectors, then an estimate of the type

$$\left\| \prod_{i=1}^k \|\mathcal{E}_i f_i\|_{L^2(q+v_i)} \right\|_{l_q^{2/(k-1)}} \lesssim r^{\frac{k}{2}} r^\epsilon \prod_{i=1}^k \|f_i\|_{L^2(U_i)}$$

is similar to the one with $v_i = 0$, which in turn is similar to (6-2) (with $\mu = 1$).

The complexity has to do with the factor $\|\mathcal{E}_1 f_1\|_{L^2(U_i)} \| \cdot \|_{L^2(q+v_1)}$ being replaced with $\|\mathcal{E}_1 f_1\|_{L^2(q+S)}$, that is, with collecting the energy of $\mathcal{E}_1 f_1$ not only across a cube $q + v_1$, but across a thickened surface $q + S$. It is the dimensionality of the surface S being $n - k + 1$ versus that of v_1 being 0 that changes the character of the estimate. Another feature to point out is the following: the classical multilinear estimate improves under certain localization properties of the support of the interacting functions (see the μ -factor in (6-2)); (7-2) does not improve under such localizations.

In [Bejenaru 2017b] we provided an energy estimate of the type

$$\|\mathcal{E}_1 f_1\|_{L^2(\tilde{S}+q)} \lesssim r^{\frac{1}{2}} \|f_1\|_{L^2(U_1)}, \quad (7-3)$$

where \tilde{S} is a hypersurface (i.e., of codimension 1) that is transversal to the propagation directions of $\mathcal{E}_1 f_1$, that is, to any $N_1(\zeta_1)$ with $\zeta_1 \in S_1$.

The starting point of the arguments in this section is a refinement of (7-3) in terms of wave packets. We use the result of Lemma 3.4 with $c = 1$ and $R = 4r^2$ to obtain the wave packet decomposition

$$\mathcal{E}_1 f_1 = \sum_{T_1 \in \mathcal{T}_1} \phi_{T_1}.$$

We also recall the definition of $c_N(T_1)$ from (3-12) and their property (3-13).

Lemma 7.2. *There exists $N \in \mathbb{N}$ such that for any $q \in \mathcal{C}(r)$ centered inside $B(0, r^2)$, the following holds true:*

$$\|\mathcal{E}_1 f_1\|_{L^2(S(q))} \lesssim r^{\frac{1}{2}} \left(\sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(q))}{r} \right\rangle^{-N} c_{2N}(T_1)^2 \right)^{\frac{1}{2}}. \quad (7-4)$$

Proof. For every $q \in \mathcal{C}(r)$ whose center lies inside $B(0, r^2)$, it is obvious that $S(q) \subset B(0, 4r^2)$. We write

$$\begin{aligned} \|\mathcal{E}_1 f_1\|_{L^2(S(q))}^2 &\lesssim \sum_{q' \cap S(q) \neq \emptyset} \|\mathcal{E}_1 f_1\|_{L^2(q')}^2 \\ &\lesssim \sum_{q' \cap S(q) \neq \emptyset} \sum_{T_1 \in \mathcal{T}_1} \|\phi_{T_1}\|_{L^2(q')}^2 = \sum_{T_1 \in \mathcal{T}_1} \sum_{q' \cap S(q) \neq \emptyset} \|\phi_{T_1}\|_{L^2(q')}^2 \\ &\lesssim \sum_{T_1 \in \mathcal{T}_1} \sum_{q' \cap S(q) \neq \emptyset} \tilde{\chi}_{T_1}(x_{q'}, t_{q'})^N \tilde{\chi}_{T_1}(x_{q'}, t_{q'})^{-N} \|\phi_{T_1}\|_{L^2(q')}^2 \\ &\lesssim \sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(q))}{r} \right\rangle^{-N} r c_{2N}(T_1)^2. \end{aligned}$$

In justifying the last line we used the following two estimates: the obvious estimate

$$\text{supp}_{q' \cap S(q) \neq \emptyset} \tilde{\chi}_{T_1}(x_{q'}, t_{q'})^N \lesssim \left\langle \frac{d(T_1, S(q))}{r} \right\rangle^{-N},$$

as well as

$$\sum_{q' \cap S(q) \neq \emptyset} \tilde{\chi}_{T_1}(x_{q'}, t_{q'})^{-N} \|\phi_{T_1}\|_{L^2(q')}^2 \lesssim r c_{2N}(T_1)^2. \quad (7-5)$$

We justify (7-5) as follows: From (3-12) we obtain

$$\text{supp}_{q'} \tilde{\chi}_T(x_{q'}, t_{q'})^{-2N} \|\phi_{T_1}\|_{L^2(q')}^2 \lesssim r c_{2N}(T_1)^2.$$

Then (7-5) follows from

$$\sum_{q' \cap S(q) \neq \emptyset} \tilde{\chi}_{T_1}(x_{q'}, t_{q'})^N \lesssim 1.$$

But, choosing N large enough, this is a direct consequence of the transversality between T_1 and $S(q)$. \square

Proof of Theorem 7.1. As we already explained in the proof of Theorem 6.1, it suffices to establish the result under the following assumption: given some $0 < \delta \ll 1$, the diameter of U_i is $\leq \delta$.

The setup is also similar to the one in Section 6. We pick $\zeta_i^0 \in \Sigma_i$, let $N_i = N_i(\zeta_i^0)$ be the normal to Σ_i and let \mathcal{H}_i be the transversal hyperplane passing through the origin with normal $N_i(\zeta_i^0)$. Using a smooth change of coordinates, we can assume that $U_i \subset \mathcal{H}_i$ and that

$$\mathcal{E}_i f_i = \int_{U_i} e^{i(x' \xi' + x_i \varphi_i(\xi'))} f_i(\xi') d\xi', \quad (7-6)$$

where $x = (x_i, x')$, x_i is the coordinate in the direction of N_i and x' are the coordinates in the directions from \mathcal{H}_i . Since the diameter of U_i is $\lesssim \delta$, it follows that $|\nabla \varphi_i(x) - \nabla \varphi_i(y)| \lesssim \delta$ for any $x, y \in U_i$. Using

the normals N_i we construct all entities described in Section 6A as well as the margin of a function $f : \mathcal{H}_i \rightarrow \mathbb{C}$ as defined in (6-10).

We complete the system of vectors by choosing N_{k+1}, \dots, N_{n+1} such that (6-1) is satisfied. We then construct the lattice $\mathcal{L} := \{z_1 N_1 + \dots + z_{n+1} N_{n+1} : (z_1, \dots, z_{n+1}) \in \mathbb{Z}^{n+1}\}$ and for a given $r > 0$ we let $\mathcal{C}(r)$ be the set of parallelepipeds of size r in \mathbb{R}^{n+1} relative to the lattice \mathcal{L} . The lattice \mathcal{L} and the set of parallelepipeds $\mathcal{C}(r)$ obtained this way are ‘‘oblique’’. Thus this set is different than the one claimed in (7-2), which is built on the standard orthonormal basis. However, passing from results in terms of an oblique lattice to the ones in the standard basis is easy: it can be done by changing coordinates, or by direct estimates.

Our induction aims at quantifying the behavior of $A(R)$ defined below.

Definition 7.3. Given $r \leq R \lesssim r^2$ we define $A(R)$ to be the best constant for which the estimate

$$\left(\sum_{q \in \mathcal{C}(r) \cap Q} \left(\|\mathcal{E}_1 f_1\|_{L^2(S(q))} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)} \right)^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \leq A(R) r^{\frac{k}{2}} \left(\sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(Q))}{R} \right\rangle^{-N} c_{2N}(T_1)^2 \right)^{\frac{1}{2}} \prod_{i=2}^k \|f_i\|_{L^2(U_i)} \quad (7-7)$$

holds true for all parallelepipeds $Q \in \mathcal{C}(R)$, and all $f_i \in L^2(U_i)$, $i = 2, \dots, k$, obeying the margin requirement

$$\text{margin}^i(f_i) \geq \delta - R^{-\frac{1}{2}}. \quad (7-8)$$

Note that in the above definition $f_1 \in L^2(U_1)$ is a fixed element, unlike f_2, \dots, f_k , which can vary.

The above estimate holds true for $R = r$ with $A(r) \approx 1$; indeed, it follows from (7-4) and the obvious estimate

$$\|\mathcal{E}_i f_i\|_{L^2(q)} \lesssim r^{\frac{1}{2}} \|f_i\|_{L^2(U_i)}.$$

Note also that we limit the range of the argument to $R \lesssim r^2$. This is important so as to be able to use the wave packet described above.

Next, we proceed with the induction step. We provide an estimate inside any cube $Q \in \mathcal{C}(\delta^{-1} R)$ based on prior information on estimates inside cubes $Q \in \mathcal{C}(R) \cap Q$. Without restricting the generality of the argument, we assume that Q is centered at the origin and recall that each $Q \in \mathcal{C}(R) \cap Q$ has its center in \mathcal{L} . When such a Q is projected using π_{N_i} onto \mathcal{H}_i one obtains $\pi_{N_i} Q \in \mathcal{C}\mathcal{H}_i$. We let Q_0 be the cube in $\mathcal{C}(R)$ centered at the origin.

We strengthen the induction hypothesis (7-7) to

$$\left(\sum_{q \in \mathcal{C}(r) \cap Q} \left(\|\mathcal{E}_1 f_1\|_{L^2(S(q))} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)} \right)^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \lesssim A(R) \left(\sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(Q))}{R} \right\rangle^{-N} c_{2N}(T_1)^2 \right)^{\frac{1}{2}} \cdot \prod_{i=2}^k \left(\sum_{Q' \in \mathcal{C}\mathcal{H}_i(R)} \left\langle \frac{d(\pi_{N_i} Q, Q')}{R} \right\rangle^{-(N-2n^2)} \left\| \left\langle \frac{x - c(Q')}{R} \right\rangle^N \chi_{Q'} \mathcal{F}_i^{-1} f_i \right\|_{L^2}^2 \right)^{\frac{1}{2}}. \quad (7-9)$$

The improvement for the terms $\mathcal{E}_i f_i$ with $i \geq 2$ is standard by now, see (6-15) and the references to [Bejenaru 2017a]. Using (7-9) we conclude the argument using the discrete Loomis–Whitney inequality in (6-9). For $i = 2, \dots, n$, we define the functions $g_i : \mathcal{L}(\mathcal{H}_i) \rightarrow \mathbb{R}$ by

$$g_i(\mathbf{j}) = \left(\sum_{Q' \in \mathcal{C}\mathcal{H}_i(R)} \left\langle \frac{d(Q(\mathbf{j}), Q')}{R} \right\rangle^{-(N-2n^2)} \left\| \left\langle \frac{x' - c(Q')}{R} \right\rangle^N \chi_{q' \mathcal{F}_i^{-1} f_i} \right\|_{L^2}^2 \right)^{\frac{1}{2}}, \quad \mathbf{j} \in \mathcal{L}(\mathcal{H}_i),$$

where we recall that $Q(\mathbf{j}) \in \mathcal{C}\mathcal{H}_i(R)$ is the cube centered at $R\mathbf{j}$.

From (6-8), it is easy to see that for N large enough (depending only on n), $g_i \in l^2(\mathbb{Z}^n)$, $i = 2, \dots, k$, with

$$\|g_i\|_{l^2(\mathcal{L}(\mathcal{H}_i))} \lesssim \|f_i\|_{L^2}.$$

For $i = 1$ and $\mathbf{j} \in \mathcal{L}$, we recall that $Q(\mathbf{j}) = Q_0 + R\mathbf{j} \in \mathcal{C}(R)$ is the cube centered at $R\mathbf{j}$, and define

$$g_1(\mathbf{j}) = \left(\sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(Q(\mathbf{j})))}{R} \right\rangle^{-N} c_{2N}(T_1)^2 \right)^{\frac{1}{2}}.$$

We claim that $g_1 \in l_{j_1, j_{k+1}, \dots, j_{n+1}}^\infty l_{j_2, \dots, j_k}^2(D)$, where $D = \{\mathbf{j} \in \mathcal{L} : \|\mathbf{j}\|_{l^\infty} \leq \delta^{-1}\}$ is the domain of interest, together with the estimate

$$\|g_1\|_{l_{j_1, j_{k+1}, \dots, j_{n+1}}^\infty l_{j_2, \dots, j_k}^2(D)} \lesssim \left(\sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(Q))}{\delta^{-1}R} \right\rangle^{-N} c_{2N}(T_1)^2 \right)^{\frac{1}{2}}. \quad (7-10)$$

We assume for a moment (7-10) to be true. Using (7-9), we invoke (6-9) and the above estimates on g_i to obtain

$$\begin{aligned} & \left(\sum_{q \in \mathcal{C}(r) \cap Q} \left(\|\mathcal{E}_1 f_1\|_{L^2(S(q))} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)} \right)^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \\ &= \left(\sum_{Q \in \mathcal{C}(R) \cap Q} \sum_{q \in \mathcal{C}(r) \cap Q} \left(\|\mathcal{E}_1 f_1\|_{L^2(S(q))} \prod_{i=2}^k \|\mathcal{E}_i f_i\|_{L^2(q)} \right)^{\frac{2}{k-1}} \right)^{\frac{k-1}{2}} \\ &\lesssim A(R) r^{\frac{k}{2}} \left(\sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(Q))}{R} \right\rangle^{-N} c_{2N}(T_1)^2 \right)^{\frac{1}{2}} \prod_{i=2}^k \|f_i\|_{L^2(U_i)}. \end{aligned}$$

Thus we establish that

$$A(\delta^{-1}R) \lesssim A(R).$$

This implies (7-2) in a standard fashion, see for instance [Bejenaru 2017a], and concludes our proof.

We owe an argument for the claim (7-10). We fix $j_1, j_{k+1}, \dots, j_{n+1}$ with

$$\max\{|j_1|, |j_{k+1}|, \dots, |j_{n+1}|\} \leq \delta^{-1}.$$

Then (7-10) is a consequence of the following estimate:

$$\sum_{\substack{j_2, \dots, j_k \\ |j_i| \leq \delta^{-1}}} \sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(Q(\mathbf{j})))}{R} \right\rangle^{-N} c_{2N}(T_1)^2 \lesssim \sum_{T_1 \in \mathcal{T}_1} \left\langle \frac{d(T_1, S(\delta^{-1}Q_0))}{\delta^{-1}R} \right\rangle^{-N} c_{2N}(T_1)^2,$$

which in turn follows from

$$\sum_{\substack{j_2, \dots, j_k \\ |j_j| \leq \delta^{-1}}} \left\langle \frac{d(T_1, S(Q(\mathbf{j})))}{R} \right\rangle^{-N} \lesssim \left\langle \frac{d(T_1, S(\delta^{-1}Q_0))}{\delta^{-1}R} \right\rangle^{-N}. \tag{7-11}$$

The estimate (7-11) is easily derived from the following claim: given any $d \in \mathbb{N}$, there are $\lesssim d^{k-1}$ values of $\mathbf{j} \in D$ such that $d(T_1, S(Q(\mathbf{j}))) \leq dR$.

Thus, the last thing we need to do is to establish the claim above. Let $\mathbf{j}_1, \mathbf{j}_2 \in D$ be such that $d(T_1, S(Q(\mathbf{j}_1))), d(T_1, S(Q(\mathbf{j}_2))) \approx dR$. Let L_1 be the center line of T_1 ; it has direction $N_1 = N_1(\zeta_1)$ for some $\zeta_1 \in S_1$. Using the fact that $R \geq r$, we conclude that there are points $P_1, P_2 \in T_1, \tilde{P}_1 \in S(Q(\mathbf{j}_1)), \tilde{P}_2 \in S(Q(\mathbf{j}_2))$ with the following properties:

- $P_1, P_2 \in L_1$.
- $\tilde{P}_1 \in S + R\mathbf{j}_1, \tilde{P}_2 \in S + R\mathbf{j}_2$.
- $d(P_1, \tilde{P}_1), d(P_2, \tilde{P}_2) \lesssim dR$.

From the vector identity

$$\overrightarrow{\tilde{P}_1 \tilde{P}_2} = \overrightarrow{\tilde{P}_1 P_1} + \overrightarrow{P_1 P_2} + \overrightarrow{P_2 \tilde{P}_2}$$

and the above properties, we obtain

$$|\overrightarrow{\tilde{P}_1 \tilde{P}_2} - \overrightarrow{P_1 P_2}| \lesssim dR.$$

On the other hand, $\tilde{P}_1 = Q_1 + R\mathbf{j}_1, \tilde{P}_2 = Q_2 + R\mathbf{j}_2$ for some $Q_1, Q_2 \in S$, and therefore

$$\overrightarrow{\tilde{P}_1 \tilde{P}_2} - \overrightarrow{P_1 P_2} = \overrightarrow{Q_1 Q_2} + R(\mathbf{j}_1 - \mathbf{j}_2) + \alpha N_1$$

for some $\alpha \in \mathbb{R}$. Now we bring in the transversality considerations, see (7-1), to conclude that

$$dR \gtrsim |\overrightarrow{Q_1 Q_2} + R(\mathbf{j}_1 - \mathbf{j}_2) + \alpha N_1| \gtrsim R|\mathbf{j}_1 - \mathbf{j}_2|;$$

here we use the structure of the lattice \mathcal{L} to infer that $\mathbf{j}_1 - \mathbf{j}_2 = \alpha_i N_i(\zeta_i) - \alpha_j N_j(\zeta_j)$ for some $i, j \in \{2, \dots, k\}$ and some $\alpha_i, \alpha_j \in \mathbb{R}$.

Thus $d \gtrsim |\mathbf{j}_1 - \mathbf{j}_2|$, and, as a consequence, there are about d^{k-1} values of \mathbf{j} with the property that $d(T_1, S(Q(\mathbf{j}))) \leq dR$. □

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