ANALYSIS & PDE

Volume 12 No. 5 2019

msp

Analysis & PDE

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1149

ON THE LUZIN N-PROPERTY AND THE UNCERTAINTY PRINCIPLE FOR SOBOLEV MAPPINGS

ADELE FERONE, MIKHAIL V. KOROBKOV AND ALBA ROVIELLO

We say that a mapping $v:\mathbb{R}^n\to\mathbb{R}^d$ satisfies the (τ,σ)-N-property if $\mathcal{H}^\sigma(v(E))=0$ whenever $\mathcal{H}^\tau(E)=0$, where $\mathcal{H}^\tau$ means the Hausdorff measure. We prove that every mapping v of Sobolev class $W^k_p(\mathbb{R}^n,\mathbb{R}^d)$ with $kp>n$ satisfies the (τ,σ)-N-property for every $0<\tau\neq\tau_*:=n-(k-1)p$ with

$$\sigma=\sigma(\tau):=\begin{cases}\tau & \text{if } \tau>\tau_*,\\ p\tau/(kp-n+\tau) & \text{if } 0<\tau<\tau_*.\end{cases}$$

We prove also that for $k>1$ and for the critical value $\tau=\tau_*$ the corresponding (τ,σ)-N-property fails in general. Nevertheless, this (τ,σ)-N-property holds for $\tau=\tau_*$ if we assume in addition that the highest derivatives $\nabla^k v$ belong to the Lorentz space $L_{p,1}(\mathbb{R}^n)$ instead of L_p.

We extend these results to the case of fractional Sobolev spaces as well. Also, we establish some Fubini-type theorems for N-Nproperties and discuss their applications to the Morse–Sard theorem and its recent extensions.

1. Introduction

The classical Luzin N-property means that for a mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ one has meas $f(E)=0$ whenever meas $E=0$. (Here meas E is the usual n-dimensional Lebesgue measure.)

This property plays a crucial role in classical real analysis and differentiation theory [Saks 1937]. It is very useful also in elasticity theory and in geometrical analysis, especially in the theory of quasiconformal mappings and, more generally, in the theory of mappings with bounded distortions, i.e., mappings $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ of Sobolev class $W^1_n(\mathbb{R}^n)$ such that $\|f'(x)\|^n\le K\det f'(x)$ almost everywhere with some constant $K\in[1,+\infty)$. The notion of mappings with bounded distortion was introduced by Yu. G. Reshetnyak; see, e.g., his classical books [Reshetnyak 1989; 1994; Goldshtein and Reshetnyak 1990]. He proved that they satisfy the N-property and this was very helpful in his subsequent proofs of other basic topological properties of such mappings (openness, discreteness and etc.). Further this MBD theory was successfully developed by many mathematicians in both analytical and geometrical directions, and many interesting and deep results were obtained; see the monographs [Rickman 1993; Iwaniec and Martin 2001], for example.

The notion of mappings with bounded distortion leads to the theory of more general mappings with finite distortion (i.e., when K in the definition above depends on x and is not assumed to be uniformly bounded; see, e.g., the pioneering paper [Vodop'yanov and Goldshtein 1976], where the monotonicity,

MSC2010: primary 46E35, 58C25; secondary 26B35, 46E30.
Keywords: Sobolev–Lorentz mappings, fractional Sobolev classes, Luzin N-property, Morse–Sard theorem, Hausdorff measure.

continuity and N-property of such mappings from the class W^1_n were established). This theory has been intensively developed in the last decades (see, e.g., the book [Hencl and Koskela 2014] for an overview), and studying the N-property constitutes one of the most important directions [Kauhanen et al. 2001; D'Onofrio et al. 2016].

Note that the belonging of a mapping to the Sobolev class $W^1_n(\mathbb{R}^n,\mathbb{R}^n)$ is crucial for N-properties. Indeed, every mapping of class $W^1_p(\mathbb{R}^n,\mathbb{R}^n)$ with $p>n$ is continuous and supports the N-property (it is a simple consequence of the Morrey inequality). But even if a mapping $f\in W^1_n(\mathbb{R}^n,\mathbb{R}^n)$ is continuous (which is not guaranteed in general), it may not have the N-property. On the other hand, the N-property holds for functions of the class $W^1_n(\mathbb{R}^n,\mathbb{R}^n)$ under some additional assumptions on its topological features, namely, for homeomorphic and open mappings [Reshetnyak 1987] (see also [Roskovec 2018]) and for quasimonotone[1] mappings [Vodop'yanov and Goldshtein 1976; Malý and Martio 1995].

The results above are very delicate and sharp: indeed, for any $p<n$ there are homeomorphisms $f\in W^1_p(\mathbb{R}^n,\mathbb{R}^n)$ without the N-property. This phenomenon was discovered by S. P. Ponomarev [1971]. In recent years his construction has been very refined and an example was constructed of a Sobolev homeomorphism with zero Jacobian a.e. which belongs simultaneously to all the classes $W^1_p(\mathbb{R}^n,\mathbb{R}^n)$ with $p<n$ [Hencl 2011; Černý 2011] — of course, this "strange" homeomorphism certainly fails to have the N-property.[2]

In the positive direction, it was proved in [Kauhanen et al. 1999], see also [Romanov 2008], that every mapping of the Sobolev–Lorentz class $W^1_{n,1}(\mathbb{R}^n,\mathbb{R}^n)$ (i.e., its distributional derivatives belong to the Lorentz space $L_{n,1}$; see Section 2 for the exact definitions) satisfies the N-property. Note that this space $W^1_{n,1}(\mathbb{R}^n,\mathbb{R}^n)$ is limiting in a natural sense between classes W^1_n and W^1_p with $p>n$.

Another direction is to study the N-properties with respect to Hausdorff (instead of Lebesgue) measures. One of the most elegant results was achieved for the class of plane quasiconformal mappings.

The famous area distortion theorem of K. Astala [1994] implies the following dimension distortion result: if $f:\mathbb{C}\to\mathbb{C}$ is a K-quasiconformal mapping (i.e., it is a plane homeomorphic mapping with K-bounded distortion) and E is a compact set of Hausdorff dimension $t\in(0,2)$, then the image $f(E)$ has Hausdorff dimension at most $t'=2Kt/(2+(K-1)t)$. This estimate is sharp; however, it leaves open the endpoint case: does $\mathcal{H}^t(E)=0$ imply $\mathcal{H}^{t'}(f(E))=0$? The remarkable paper [Lacey et al. 2010] gives an affirmative answer to Astala's conjecture (see also [Astala et al. 2013], where the further implication $H^t(E)<\infty\Rightarrow\mathcal{H}^{t'}(f(E))<\infty$ was considered).

Let us go to results which are closer to the present paper. It is more natural to discuss the topic in the scale of fractional Sobolev spaces, i.e., for (*Bessel*)-*potential space* $\mathscr{L}^\alpha_p$ with $\alpha>0$. Recall that a function $v:\mathbb{R}^n\to\mathbb{R}^d$ belongs to the space $\mathscr{L}^\alpha_p$ if it is a convolution of the Bessel kernel K_α with a

[1] Some of these results were generalised for the more delicate case of Carnot groups and manifolds; see, e.g., [Vodop'yanov 2003].

[2] Moreover, even the examples of bi-Sobolev homeomorphisms of class $W^1_p(\mathbb{R}^n,\mathbb{R}^n)$, $p<n-1$, with zero Jacobian a.e. were constructed recently; see, e.g., [D'Onofrio et al. 2014; Černý 2015]. Such homeomorphisms are impossible in the Sobolev class $W^1_{n-1}(\mathbb{R}^n,\mathbb{R}^n)$. Furthermore, Hencl and Vejnar [2016] constructed an example of a Sobolev homeomorphism $f\in W^1_1((0,1)^n,\mathbb{R}^n)$ such that the Jacobian $\det f'(x)$ changes its sign on sets of positive measure.

function $g \in L_p(\mathbb{R}^n)$, where $\widehat{K}_\alpha(\xi) = (1 + 4\pi^2\xi^2)^{-\alpha/2}$. It is well known that

$$\mathscr{L}^\alpha_p(\mathbb{R}^n) = W^\alpha_p(\mathbb{R}^n) \quad \text{if } \alpha \in \mathbb{N} \text{ and } 1 < p < \infty.$$

Recently H. Hencl and P. Honzík proved, in particular, the following assertion:

Theorem 1.1 [Hencl and Honzík 2015]. *Let $n, d \in \mathbb{N}$, $\alpha > 0$, $p > 1$, $\alpha p > n$, and $0 < \tau \le n$. Suppose that a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the* (*fractional*) *Sobolev class $\mathscr{L}^\alpha_p$. Then for any set $E \subset \mathbb{R}^n$ with Hausdorff dimension $\dim_H E \le \tau$ the inequality $\dim_H f(E) \le \sigma(\tau)$ holds, where*

$$\sigma(\tau) := \begin{cases} \tau & \text{if } \tau \ge \tau_* := n - (\alpha - 1)p, \\ p\tau/(\alpha p - n + \tau) & \text{if } 0 < \tau < \tau_*. \end{cases} \tag{1-1}$$

But as above (see the discussion around the Astala theorem), this result raises a natural question. What happens in the limiting case, i.e., is it true that $\mathcal{H}^\tau(E) = 0$ implies $\mathcal{H}^{\sigma(\tau)}(f(E)) = 0$? Of course, such an N-property is much more precise and stronger than the assertion of Theorem 1.1.

Six years ago G. Alberti [2012] announced the validity of the following result, obtained in collaboration with M. Csörnyei, E. D'Aniello and B. Kirchheim.

Theorem 1.2. *Let $k, n, d \in \mathbb{N}$, $p > 1$, $kp > n$, and $0 < \tau \le n$. Suppose that a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the Sobolev class W^k_p and $\tau \ne \tau_* = n - (k-1)p$. Then f has the (τ, σ)-N-property, where the value $\sigma = \sigma(\tau)$ is defined in* (1-1).

Here for convenience we use the following notation: a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ is said to satisfy the (τ, σ)-N-property if $\mathcal{H}^\sigma(f(E)) = 0$ whenever $\mathcal{H}^\tau(E) = 0$, $E \subset \mathbb{R}^n$.

We remark that in [Alberti 2012] the limiting case $\tau = \tau_* > 0$ is left as *an open question*. Further, as far as we know, proofs of the results announced have not been published (it was written in [Alberti 2012] that the work was still "in progress").

In the present paper we extend the assertion above to the case of fractional Sobolev spaces and also we cover the critical case $\tau = \tau_*$ as well.

Theorem 1.3. *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p > n$, and $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that $0 < \tau \le n$. Then the following assertions hold*:

(i) *If $\tau \ne \tau_* = n - (\alpha - 1)p$, then v has the (τ, σ)-N-property, where the value $\sigma = \sigma(\tau)$ is defined in* (1-1).

(ii) *If $\alpha > 1$ and $\tau = \tau_* > 0$, then $\sigma(\tau) = \tau_*$ and the mapping v in general has **no** (τ_*, τ_*)-N-property; i.e., it could be that $\mathcal{H}^{\tau_*}(v(E)) > 0$ for some $E \subset \mathbb{R}^n$ with $\mathcal{H}^{\tau_*}(E) = 0$.*

Remark 1.4. We stress that there is no "competition" with Alberti, Csörnyei, D'Aniello and Kirchheim concerning Theorems 1.2–1.3. When we published our first paper on the topic [Bourgain et al. 2013], those authors contacted us and it was agreed that mutual citations would be provided (and indeed appeared in [Alberti 2012; Bourgain et al. 2013]). Similarly, when the present paper was finished, we contacted one of those authors. They told us that after [Alberti 2012] they had some further progress, especially for $\tau = \tau_*$. We came to an agreement that each research group could publish their results with independent proofs, respecting each other's activity in the subject.

Remark 1.5. If $\alpha = 1$ and $p > n$, then $\tau_* = n$ and $\mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d) = W_p^1(\mathbb{R}^n, \mathbb{R}^d)$, and the validity of the (τ, σ)-N-property for all $\tau \in (0, n]$ and for all mappings of these spaces is a simple corollary of the classical Morrey inequality [Malý and Martio 1995].

Theorem 1.3 omits the limiting cases $\alpha p = n$ and $\tau = \tau_*$. It is possible to cover these cases as well using the Lorentz norms. Namely, denote by $\mathscr{L}_{p,1}^\alpha(\mathbb{R}^n, \mathbb{R}^d)$ the space of functions which can be represented as a convolution of the Bessel potential K_α with a function g from the Lorentz space $L_{p,1}$ (see the definition of these spaces in Section 2); that is,

$$\|v\|_{\mathscr{L}_{p,1}^\alpha} := \|g\|_{L_{p,1}}.$$

Theorem 1.6. *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p \geq n$, and $0 < \tau \leq n$. Suppose that $v \in \mathscr{L}_{p,1}^\alpha(\mathbb{R}^n, \mathbb{R}^d)$. Then v is a continuous function satisfying the (τ, σ)-N-property, where again the value $\sigma = \sigma(\tau)$ is defined in* (1-1) (*i.e., the limiting case $\tau = \tau_*$ is* ***included***).

Remark 1.7. In the case $\alpha = k \in \mathbb{N}$, $kp = n$, $p \geq 1$, we have $\tau_* = p$ and the validity of the (τ, σ)-N-property for mappings of the corresponding Sobolev–Lorentz space $W_{p,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ was proved in [Bourgain et al. 2015; Korobkov and Kristensen 2018].

1A. ***The counterexample for the limiting case $\tau = \tau_*$ in Theorem 1.3(ii).*** Suppose again that

$$n > (\alpha - 1)p > n - p.$$

Let us demonstrate that the positive assertion in Theorem 1.3(i) is very sharp: it fails in general for the limiting case

$$\tau = \tau_* = n - (\alpha - 1)p.$$

Take

$$n = 4, \quad \alpha = 2, \quad p = 3.$$

Then by definition

$$\tau_* = 1.$$

So we have to construct a function from the Sobolev space $\mathscr{L}_3^2(\mathbb{R}^4) = W_3^2(\mathbb{R}^4)$ which does not have the N-property with respect to $\mathcal{H}^1$-measure. Consider the restrictions (traces) of functions from $W_3^4(\mathbb{R}^4)$ to the real line. It is well known that the space of these traces coincides exactly with the Besov space $B_{3,3}^1(\mathbb{R})$; see, e.g., [Jonsson and Wallin 1984, Chapter 1, Theorem 4 on p. 20]. Consider the function of one real variable

$$f_\sigma(x) = e^{-x^2} \sum_{m=1}^{\infty} 5^{-m} m^{-\sigma} \cos(5^m x),$$

where

$$\tfrac{1}{3} < \sigma < \tfrac{1}{2}.$$

It is known that $f_\sigma \in B_{3,3}^1(\mathbb{R})$ under the assumptions above; see, e.g., §6.8 in Chapter V of [Stein 1970]. Nevertheless, the following result holds.

Theorem 1.8. *The function $f_\sigma : \mathbb{R} \to \mathbb{R}$ from above does not have the* (1,1)-*N-property* (*with respect to $\mathcal{H}^1$-measure*).

This result is a direct consequence of the following two classical facts:

Theorem 1.9 [Saks 1937, Chapter IX, Theorem 7.7]. *If a continuous function $f:\mathbb{R}\to\mathbb{R}$ has the N-property, then it is differentiable on a set of positive measure.*

Theorem 1.10 [Zygmund 1959, Chapter V, §6, p. 206]. *The continuous function*

$$f(x)=\sum_{m=1}^{\infty} b^{-m}\varepsilon_m\cos(b^m x),$$

with $b>1$ and $\varepsilon_m\to 0$, $\sum_{m=1}^{\infty}\varepsilon_m^2=\infty$, is not differentiable almost everywhere.

Note that the functions f_σ, f from Theorems 1.8 and 1.10 are the typical examples of so-called lacunary Fourier series.

From Theorem 1.8 it follows that there exists a function $v\in W_3^2(\mathbb{R}^4)$ whose restriction to the real line coincides with f_σ; i.e., v does *not* have the (1,1)-N-property. The construction of the counterexample is finished.

1B. ***Fubini-type theorems for N-properties.*** The N-properties formulated above have an important application in the recent extension of the Morse–Sard theorem to Sobolev spaces (see [Ferone et al. 2017] and also Section 1C below). Here we need the following notion.

For a pair numbers $\tau,\sigma>0$ we will say that a continuous function $v:\mathbb{R}^n\to\mathbb{R}^d$ satisfies the (τ,σ)-N_*-property if for every $q\in[0,\sigma]$ and for any set $E\subset\mathbb{R}^n$ with $H^\tau(E)=0$ we have

$$\mathcal{H}^{\tau(1-\frac{q}{\sigma})}(E\cap v^{-1}(y))=0\quad\text{for }\mathcal{H}^q\text{-a.a. }y\in\mathbb{R}^d. \tag{1-2}$$

This implies, in particular, the usual (τ,σ)-N-property

$$\mathcal{H}^\sigma(v(E))=0\quad\text{whenever }\mathcal{H}^\tau(E)=0.$$

(Indeed, it is sufficient to take $q=\sigma$ in (1-2).) In other words, the (τ,σ)-N_*-property is stronger than the usual (τ,σ) one.

The N_*-property can be considered as a Fubini-type theorem for the usual N-property. Now we can strengthen our previous results in the following way.

Theorem 1.11. *Let $\alpha>0$, $1<p<\infty$, $\alpha p>n$, and $v\in\mathscr{L}_p^\alpha(\mathbb{R}^n,\mathbb{R}^d)$. Suppose that $0<\tau\le n$. Then:*

(i) *If $\tau\ne\tau_*=n-(\alpha-1)p$, then v has the (τ,σ)-N_*-property, where the value $\sigma=\sigma(\tau)$ is defined in* (1-1).

(ii) *If $\alpha>1$ and $\tau=\tau_*$, then $\sigma(\tau)=\tau_*$ and the mapping v in general has no (τ_*,τ_*)-N-property; i.e., it could be that $\mathcal{H}^{\tau_*}(v(E))>0$ for some $E\subset\mathbb{R}^n$ with $\mathcal{H}^{\tau_*}(E)=0$.*

Remark 1.12. If $\alpha=1$ and $p>n$, then $\tau_*=n$ and $\mathscr{L}_p^\alpha(\mathbb{R}^n,\mathbb{R}^d)=W_p^1(\mathbb{R}^n,\mathbb{R}^d)$, and the validity of the (τ,σ)-N_*-property for all $\tau\in(0,n]$ and for all mappings of these spaces is a simple corollary of the classical Morrey inequality and Theorem 4.1 below.

Of course, Theorem 1.11 omits the limiting cases $\alpha p=n$ and $\tau=\tau_*$. Again, it is possible to cover these cases as well using the Lorentz norms.

Theorem 1.13. *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p \geq n$, and $0 < \tau \leq n$. Suppose that $v \in \mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then v is a continuous function satisfying the (τ, σ)-N_*-property, where again the value $\sigma = \sigma(\tau)$ is defined in* (1-1).

Remark 1.14. In the case $\alpha = k \in \mathbb{N}$, $kp = n$, $p \geq 1$, we have $\tau_* = p$ and the validity of the (τ, σ)-N_*-property for mappings of the corresponding Sobolev–Lorentz space $W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ was proved in [Bourgain et al. 2015; Hajłasz et al. 2017].

1C. ***Application to the Morse–Sard and Dubovitskiĭ–Federer theorems.*** The classical Morse–Sard theorem claims that for a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^m$ of class C^k the measure of the set of critical values $v(Z_{v,m})$ is zero under the condition $k > \max(n-m, 0)$. Here the critical set, or m-critical set, is defined as $Z_{v,m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla v(x) < m\}$. Further Dubovitskiĭ [1957; 1967] and Federer [1969, Theorem 3.4.3] independently found some elegant extensions of this theorem to the case of other (e.g., lower) smoothness assumptions. They also established the sharpness of their results within the C^k category.

Recently the following *bridge theorem*, which includes all the results above as particular cases, was found.

We say that a mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the class $C^{k,\alpha}$ for some integer positive k and $0 < \alpha \leq 1$ if there exists a constant $L \geq 0$ such that

$$|\nabla^k v(x) - \nabla^k v(y)| \leq L|x-y|^{\alpha} \quad \text{for all } x, y \in \mathbb{R}^n.$$

To simplify the notation, let us make the following agreement: for $\alpha = 0$ we identify $C^{k,\alpha}$ with usual spaces of C^k-smooth mappings. The following theorem was obtained in [Ferone et al. 2017].

Theorem 1.15. *Let $m \in \{1, \ldots, n\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha \leq 1$, and $v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any $q \in (m-1, \infty)$ the equality*

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d$$

holds, where

$$\mu_q = n - m + 1 - (k+\alpha)(q - m + 1),$$

and $Z_{v,m}$ denotes the set of m-critical points of v, that is, $Z_{v,m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla v(x) \leq m-1\}$.

Here and in the following we interpret $\mathcal{H}^{\beta}$ as the counting measure when $\beta \leq 0$. Let us note that for the classical C^k-case, i.e., when $\alpha = 0$, the behaviour of the function μ_q is very natural:

$$\begin{aligned}
\mu_q &= 0 && \text{for } q = q_\circ = m - 1 + (n-m+1)/k && \text{(Dubovitskiĭ–Federer theorem, 1967)},\\
\mu_q &< 0 && \text{for } q > q_\circ && \text{(Dubovitskiĭ–Federer theorem, 1967)},\\
\mu_q &= n - m - k + 1 && \text{for } q = m && \text{(Dubovitskiĭ theorem, 1957)},\\
\mu_q &= n - m + 1 && \text{for } q = m - 1. &&
\end{aligned}$$

The last value cannot be improved in view of the trivial example of a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^d$ of rank $m-1$.

Thus, Theorem 1.15 contains all the previous theorems (Morse–Sard, Dubovitskiĭ–Federer, and even the Bates theorem [1993] for $C^{k,1}$-Lipschitz functions) as particular cases.

Intuitively, the sense of this bridge theorem is very close to *Heisenberg's uncertainty principle* in theoretical physics: the more precise the information we receive on the measure of the image of the critical set, the less precisely the preimages are described, and vice versa.

The following analog of the bridge theorem, Theorem 1.15, was obtained for the Sobolev and fractional Sobolev cases (items (i)–(ii) and items (iii)–(iv), respectively).

Theorem 1.16 [Hajłasz et al. 2017; Ferone et al. 2017]. *Let* $m \in \{1, \dots, n\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha < 1$, $p \geq 1$ *and let* $v : \mathbb{R}^n \to \mathbb{R}^d$ *be a mapping for which one of the following cases holds*:

(i) $\alpha = 0$, $kp > n$, *and* $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$.

(ii) $\alpha = 0$, $kp = n$, *and* $v \in W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

(iii) $0 < \alpha < 1$, $p > 1$, $(k+\alpha)p > n$, *and* $v \in \mathscr{L}^{k+\alpha}_p(\mathbb{R}^n, \mathbb{R}^d)$.

(iv) $0 < \alpha < 1$, $p > 1$, $(k+\alpha)p = n$, *and* $v \in \mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

Then for any $q \in (m-1, \infty)$ *the equality*

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \textit{for } \mathcal{H}^q\textit{-a.a. } y \in \mathbb{R}^d$$

holds, *where again*

$$\mu_q = n - m + 1 - (k+\alpha)(q - m + 1),$$

and $Z_{v,m}$ *denotes the set of* m*-critical points of* v, *that is*, $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \operatorname{rank} \nabla v(x) \leq m-1\}$.

Here A_v means the set of nondifferentiability points for v. Recall, that by approximation results [Swanson 2002; Korobkov and Kristensen 2018] under the conditions of Theorem 1.16 the equalities

$$\mathcal{H}^{\tau}(A_v) = 0 \quad \text{for all } \tau > \tau_* := n - (k+\alpha-1)p \quad \text{in cases (i), (iii),}$$
$$\mathcal{H}^{\tau_*}(A_v) = \mathcal{H}^{p}(A_v) = 0 \quad \tau_* := n - (k+\alpha-1)p = p \quad \text{in cases (ii), (iv)}$$

are valid (in particular, $A_v = \varnothing$ if $(k+\alpha-1)p > n$). Our purpose is to prove that the impact of the "bad" set A_v is negligible in the bridge Dubovitskiĭ–Federer theorem (Theorem 1.16), i.e., that the following statement holds:

Theorem 1.17. *Let the conditions of Theorem 1.16 be fulfilled for a function* $v : \mathbb{R}^n \to \mathbb{R}^d$. *Then*

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \textit{for } \mathcal{H}^q\textit{-a.a. } y \in \mathbb{R}^d$$

for any $q > m - 1$.

Remark 1.18. Since $\mu_q \leq 0$ for $q \geq q_\circ = m - 1 + \frac{n-m+1}{k+\alpha}$, the assertions of Theorems 1.16–1.17 are equivalent to the equality $0 = \mathcal{H}^q[v(A_v \cup Z_{v,m})]$ for $q \geq q_\circ$, so it is sufficient to check the assertions of Theorems 1.16–1.17 for $q \in (m-1, q_\circ]$ only.

Finally, let us comment briefly that the merge ideas for the proofs are from our previous papers [Bourgain et al. 2015; Korobkov and Kristensen 2014; 2018; Hajłasz et al. 2017]. In particular, the papers [Bourgain et al. 2013; 2015] by one of the authors with J. Bourgain and J. Kristensen contain many of the key ideas that allow us to consider nondifferentiable Sobolev mappings. For the implementation of these ideas one relies on estimates for the Hardy–Littlewood maximal function in terms of Choquet-type integrals with respect to Hausdorff capacity. In order to take full advantage of the Lorentz context we exploit the recent estimates from [Korobkov and Kristensen 2018] (recalled in Theorem 2.11 below); see also [Adams 1988] for the case $p = 1$.

2. Preliminaries

By an *n-dimensional interval* we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If Q is an n-dimensional cubic interval then we write $\ell(Q)$ for its side-length.

For a subset S of $\mathbb{R}^n$ we write $\mathscr{L}^n(S)$ for its outer Lebesgue measure (sometimes we use the symbol $\operatorname{meas} S$ for the same object). The m-dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the m-dimensional Hausdorff content by $\mathcal{H}^m_\infty$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathcal{H}^m(S) = \lim_{t\searrow 0} \mathcal{H}^m_t(S) = \sup_{t>0} \mathcal{H}^m_t(S),$$

where for each $0 < t \le \infty$,

$$\mathcal{H}^m_t(S) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} S_i)^m : \operatorname{diam} S_i \le t,\ S \subset \bigcup_{i=1}^{\infty} S_i \right\}.$$

It is well known that $\mathcal{H}^n(S) = \mathcal{H}^n_\infty(S) \sim \mathscr{L}^n(S)$ for sets $S \subset \mathbb{R}^n$ ("$\sim$" means, here and in the following, that these values have upper and lower bounds with positive constants independent of the set S).

By $L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, we will denote the usual Lebesgue space equipped with the norm $\|\cdot\|_{L_p}$. The notation $\|f\|_{L_p(E)}$ means $\|1_E \cdot f\|_{L_p}$, where 1_E is the indicator function of E.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,\mathrm{loc}}(\Omega)$, then the precise representative w^* is defined for *all* $x \in \Omega$ by

$$w^*(x) = \begin{cases} \lim_{r\searrow 0} ⨍_{B(x,r)} w(z)\,\mathrm{d}z & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases}$$

where the dashed integral as usual denotes the integral mean,

$$⨍_{B(x,r)} w(z)\,\mathrm{d}z = \frac{1}{\mathscr{L}^n(B(x,r))} \int_{B(x,r)} w(z)\,\mathrm{d}z,$$

and $B(x,r) = \{y : |y-x| < r\}$ is the open ball of radius r centred at x. Henceforth we omit special notation for the precise representative, writing simply $w^* = w$.

For $0 \le \beta < n$, the *fractional maximal function* of w of order β is given by

$$M_\beta w(x) = \sup_{r>0} r^\beta ⨍_{B(x,r)} |w(z)|\,\mathrm{d}z. \tag{2-1}$$

When $\beta = 0$, M_0 reduces to the usual Hardy-Littlewood maximal operator M.

The Sobolev space $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$-valued functions $f \in \mathrm{L}_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \le k$ belong to $\mathrm{L}_p(\mathbb{R}^n)$; for detailed definitions and differentiability properties of such functions, see, e.g., [Evans and Gariepy 1992; Mazya 1985; Ziemer 1989; Dorronsoro 1989]. Denote by $\nabla^k f$ the vector-valued function consisting of all k-th order partial derivatives of f arranged in some fixed order. However, for the case of first order derivatives $k = 1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x, that is, the $d \times n$ matrix whose r-th row is the vector of partial derivatives of the r-th coordinate function.

We use the norm

$$\|f\|_{\mathrm{W}^k_p} = \|f\|_{\mathrm{L}_p} + \|\nabla f\|_{\mathrm{L}_p} + \cdots + \|\nabla^k f\|_{\mathrm{L}_p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s$ $(s \in \mathbb{N})$ will be the usual euclidean norms.

If $k < n$, then it is well known that functions from Sobolev spaces $\mathrm{W}^k_p(\mathbb{R}^n)$ are continuous for $p > n/k$ and can be discontinuous for $p \le p_\circ = n/k$ [Mazya 1985; Ziemer 1989]. The Sobolev–Lorentz space $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n) \subset \mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space. Among other things, functions that are locally in $\mathrm{W}^k_{p_\circ,1}$ on $\mathbb{R}^n$ are in particular continuous.

Here we only mentioned the Lorentz space $\mathrm{L}_{p,1}$, and in this case one may rewrite the norm as follows [Malý 2003, Proposition 3.6]:

$$\|f\|_{L_{p,1}} = \int_0^{+\infty} [\mathscr{L}^n(\{x \in \mathbb{R}^n : |f(x)| > t\})]^{\frac{1}{p}}\, \mathrm{d}t.$$

As for Lebesgue norm we set $\|f\|_{L_{p,1}(E)} := \|1_E \cdot f\|_{L_{p,1}}$. Of course, we have the inequality

$$\|f\|_{L_p} \le \|f\|_{L_{p,1}}. \tag{2-2}$$

Moreover, recall that by properties of Lorentz spaces, the standard estimate

$$\|Mf\|_{L_{p,q}} \le C\|f\|_{L_{p,q}} \tag{2-3}$$

holds for $1 < p < \infty$ [Malý 2003, Theorem 4.4].

Denote by $\mathrm{W}^k_{p,1}(\mathbb{R}^n)$ the space of all functions $v \in \mathrm{W}^k_p(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla^k v\|_{\mathrm{L}_{p,1}}$ is finite.

2A. *On potential spaces $\mathscr{L}^\alpha_p$.* In this paper we deal with the (*Bessel*) *potential spaces* $\mathscr{L}^\alpha_p$ with $\alpha > 0$. Recall that a function $v : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the space $\mathscr{L}^\alpha_p$ if it is a convolution of the Bessel kernel K_α with a function $g \in L_p(\mathbb{R}^n)$:

$$v = \mathscr{G}_\alpha(g) := K_\alpha * g,$$

where $\widehat{K}_\alpha(\xi) = (1 + 4\pi^2\xi^2)^{-\alpha/2}$. In particular,

$$\|v\|_{\mathscr{L}^\alpha_p} := \|g\|_{L_p}.$$

It is well known that

$$\mathscr{L}^\alpha_p(\mathbb{R}^n) = W^\alpha_p(\mathbb{R}^n) \quad \text{if } \alpha \in \mathbb{N} \text{ and } 1 < p < \infty, \tag{2-4}$$

and we use the agreement that $\mathscr{L}^\alpha_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ when $\alpha = 0$. Moreover, the following well-known result holds:

Theorem 2.1 [Stein 1970, Lemma 3, p. 136]. *Let $\alpha \geq 1$ and $1 < p < \infty$. Then $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n)$ if and only if $v \in \mathscr{L}_p^{\alpha-1}(\mathbb{R}^n)$ and $\partial v/\partial x_j \in \mathscr{L}_p^{\alpha-1}(\mathbb{R}^n)$ for every $j = 1, \dots, n$.*

The following technical bounds will be used on several occasions (for convenience, we prove them in the Appendix).

Lemma 2.2. *Let $\alpha > 1$, $n + p > \alpha p > n$, and $p > 1$. Suppose that $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n)$; i.e., $v = \mathscr{G}_\alpha(g)$ for some $g \in L_p(\mathbb{R}^n)$. Then for every n-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $r = \ell(Q) \leq 1$ the estimate*

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_p(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n-1}} \int_Q I_{\alpha-1}|g|(y)\,\mathrm{d}y\right] \tag{2-5}$$

holds, where the constant C depends on n, p, d, α only, and

$$I_\beta f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-\beta}}\,\mathrm{d}y$$

is the Riesz potential of order β.

Sometimes it is not convenient to work with the Riesz potential, and we need also the following variant of the estimates above.

Lemma 2.3. *Let $\alpha > 0$, $n + p > \alpha p > n$, and $p > 1$. Suppose that $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n)$; i.e., $v = \mathscr{G}_\alpha(g)$ for some $g \in L_p(\mathbb{R}^n)$. Fix arbitrary $\theta > 0$ such that $\alpha + \theta \geq 1$. Then for every n-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $r = \ell(Q) \leq 1$ the estimate*

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_p(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n+\theta-1}} \int_Q M_{\alpha-1+\theta}\, g(y)\,\mathrm{d}y\right] \tag{2-6}$$

holds, where the constant C depends on n, p, d, α, θ only.

For reader's convenience, we prove Lemmas 2.2–2.3 in the Appendix.

2B. *On Lorentz potential spaces $\mathscr{L}_{p,1}^\alpha$.* To cover some other limiting cases, denote by $\mathscr{L}_{p,1}^\alpha(\mathbb{R}^n, \mathbb{R}^d)$ the space of functions which can be represented as a convolution of the Bessel potential K_α with a function g from the Lorentz space $L_{p,1}$; that is,

$$\|v\|_{\mathscr{L}_{p,1}^\alpha} := \|g\|_{L_{p,1}}.$$

Because of inequality (2-2), we have the evident inclusion

$$\mathscr{L}_{p,1}^\alpha(\mathbb{R}^n) \subset \mathscr{L}_p^\alpha(\mathbb{R}^n).$$

Since these spaces are not so common, let us discuss briefly some of their properties. We need some technical facts concerning the Lorentz spaces.

Lemma 2.4 [Rakotondratsimba 1998]. *Let $1 < p < \infty$. Then for any $j = 1, \dots, n$ the Riesz transform $\mathscr{R}_j$ is continuous from $L_{p,1}(\mathbb{R}^n)$ to $L_{p,1}(\mathbb{R}^n)$.*

Lemma 2.5 [Schep 1995]. *Let $1 < p < \infty$ and μ be a finite Borel measure on $\mathbb{R}^n$. Then the convolution transform $f \mapsto f * \mu$ is continuous in the space $L_{p,1}(\mathbb{R}^n)$ and in $\mathscr{L}_p^\alpha(\mathbb{R}^n)$ for all $\alpha > 0$.*

Using these facts and repeating almost word for word the arguments from [Stein 1970, §3.3–3.4], one can obtain the following very natural results.

Theorem 2.6 (cf. [Stein 1970, Lemma 3, p. 136]). *Let $\alpha \geq 1$ and $1 < p < \infty$. Then $f \in \mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n)$ if and only if $f \in \mathscr{L}^{\alpha-1}_{p,1}(\mathbb{R}^n)$ and $\partial f/\partial x_j \in \mathscr{L}^{\alpha-1}_{p,1}(\mathbb{R}^n)$ for every $j = 1, \dots, n$.*

Corollary 2.7. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then $\mathscr{L}^{k}_{p,1}(\mathbb{R}^n) = W^k_{p,1}(\mathbb{R}^n)$, where $W^k_{p,1}(\mathbb{R}^n)$ is the space of functions such that all its distributional partial derivatives of order $\leq k$ belong to $L_{p,1}(\mathbb{R}^n)$.*

Note that the space $W^k_{p,1}(\mathbb{R}^n)$ admits an even simpler (but equivalent) description: it consists of functions f from the usual Sobolev space $W^k_p(\mathbb{R}^n)$ satisfying the additional condition $\nabla^k f \in L_{p,1}(\mathbb{R}^n)$ (i.e., this condition is on the highest derivatives only); see, e.g., [Malý 2003].

As before, we need some standard estimates.

Lemma 2.8. *Let $\alpha > 0$, $n + p \geq \alpha p \geq n$, and $p > 1$. Suppose that $v \in \mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n)$; i.e., $v = \mathscr{G}_\alpha(g)$ for some $g \in L_{p,1}(\mathbb{R}^n)$. Then the function v is continuous and for every n-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $r = \ell(Q) \leq 1$ the estimate*

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_{p,1}(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n+\theta-1}} \int_Q M_{\alpha-1+\theta}\, g(y)\,\mathrm{d}y\right] \tag{2-7}$$

holds for arbitrary (fixed) parameter $\theta > 0$ such that $\alpha + \theta \geq 1$ (here the constant C again depends on n, p, d, α, θ only). Furthermore, if $\alpha > 1$, then

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_{p,1}(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n-1}} \int_Q I_{\alpha-1}|g|(y)\,\mathrm{d}y\right]. \tag{2-8}$$

For the reader's convenience, we prove Lemma 2.8 in the Appendix.

2C. ***On Choquet-type integrals.*** Let $\mathscr{M}^\beta$ be the space of all nonnegative Borel measures μ on $\mathbb{R}^n$ such that

$$|||\mu|||_\beta = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta} \mu(I) < \infty,$$

where the supremum is taken over all n-dimensional cubic intervals $I \subset \mathbb{R}^n$ and $\ell(I)$ denotes the side length of I.

Recall the following classical theorem proved by D. R. Adams.

Theorem 2.9 (see [Mazya 1985, §1.4.1] or [Adams 1973]). *Let $\alpha > 0$, $n - \alpha p > 0$, $s > p > 1$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g \in L_p(\mathbb{R}^n)$ the estimate*

$$\int |I_\alpha g|^s \,\mathrm{d}\mu \leq C\, |||\mu|||_\beta \cdot \|g\|^s_{L_p} \tag{2-9}$$

holds with $\beta = \frac{s}{p}(n - \alpha p)$, where C depends on n, p, s, α only.

The estimate (2-9) fails for the limiting case $s = p$. Namely, there exist functions $g \in L_p(\mathbb{R}^n)$ such that $|I_\alpha g|(x) = +\infty$ on some set of positive $(n-\alpha p)$-Hausdorff measure. Nevertheless, there are two ways to cover this limiting case $s = p$. The first way is to use the maximal function M_α instead of the Riesz potential on the left-hand side of (2-9).

Theorem 2.10 [Adams 1998, Theorem 7, p. 28]. *Let $\alpha > 0,\ n-\alpha p > 0,\ s \ge p > 1$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g \in L_p(\mathbb{R}^n)$ the estimate*

$$\int |M_\alpha g|^s \, \mathrm{d}\mu \le C \, |||\mu|||_\beta \cdot \|g\|_{L_p}^s \tag{2-10}$$

holds with $\beta = (s/p)(n-\alpha p)$, where C depends on $n,\ p,\ s,\ \alpha$ only.

The second way is to use the Lorentz norm instead of the Lebesgue norm on the right-hand side of (2-9):

Theorem 2.11 [Korobkov and Kristensen 2018, Theorem 0.2]. *Let $\alpha > 0,\ n-\alpha p > 0$, and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g \in L_p(\mathbb{R}^n)$ the estimate*

$$\int |I_\alpha g|^p \, \mathrm{d}\mu \le C \, |||\mu|||_\beta \cdot \|g\|_{L_{p,1}}^p$$

holds with $\beta = n-\alpha p$, where C depends on $n,\ p,\ \alpha$ only.

2D. ***On Fubini-type theorems for N-properties.*** Recall that by the usual Fubini theorem if a set $E \subset \mathbb{R}^2$ has zero plane measure, then for $\mathcal{H}^1$-almost all straight lines L parallel to the coordinate axes we have $\mathcal{H}^1(L \cap E) = 0$. The next result can be considered as a Fubini-type theorem for the N-property.

Theorem 2.12 [Hajłasz et al. 2017, Theorem 5.3]. *Let $\mu \ge 0,\ q > 0$, and $v : \mathbb{R}^n \to \mathbb{R}^d$ be a continuous function. For a set $E \subset \mathbb{R}^n$ define the set function*

$$\Phi(E) = \inf_{E \subset \bigcup_j D_j} \sum_j (\operatorname{diam} D_j)^\mu [\operatorname{diam} v(D_j)]^q,$$

where the infimum is taken over all countable families of compact sets $\{D_j\}_{j\in\mathbb{N}}$ such that $E \subset \bigcup_j D_j$. Then $\Phi(\cdot)$ is countably subadditive and we have the implication

$$\Phi(E) = 0 \quad \Longrightarrow \quad \big[\mathcal{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \textit{for } \mathcal{H}^q\textit{-a.a. } y \in \mathbb{R}^d\big].$$

2E. ***On local properties of considered potential spaces.*** Let $\mathcal{B}$ be some family of continuous functions defined on $\mathbb{R}^n$. For a set $\Omega \subset \mathbb{R}^n$ define the space $\mathcal{B}_{\mathrm{loc}}(\Omega)$ in the following standard way:

$$\mathcal{B}_{\mathrm{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} : \text{for any compact set } E \subset \Omega, \text{there exists } g \in \mathcal{B} \text{ such that } f(x) = g(x) \text{ for all } x \in E\}.$$

For simplicity put $\mathcal{B}_{\mathrm{loc}} = \mathcal{B}_{\mathrm{loc}}(\mathbb{R}^n)$.

It is easy to see that for $\alpha > 0$ and $q > s > p > 1$ with $\alpha p \ge n$ the following inclusions hold:

$$\mathscr{L}^\alpha_{q,\mathrm{loc}} \subset \mathscr{L}^\alpha_{s,\mathrm{loc}} \subset \mathscr{L}^\alpha_{p,1,\mathrm{loc}}.$$

Since the N-properties have a local nature, this means that if we prove some N- (or N_*-) property for $\mathscr{L}^\alpha_p$, then the same N-property will be valid for the spaces $\mathscr{L}^\alpha_{p,1}$ and $\mathscr{L}^\alpha_q$ for all $q > p$. Similarly, if we prove some N- (or N_*-) property for $\mathscr{L}^\alpha_{p,1}$, then the same N-property will be valid for the spaces $\mathscr{L}^\alpha_q$ with $q > p$, etc.

3. Proofs of the N-properties (Theorems 1.3, 1.6)

In this section we will prove Theorems 1.3 and 1.6. For each theorem, we will consider different cases. The most interesting case is when $\alpha p < n + p$, which implies that $\tau_* > 0$: in such a situation we will consider the supercritical case $\tau > \tau_* > 0$ and the subcritical case $0 < \tau < \tau_*$ (see, respectively, Sections 3A and 3B below). The case $\alpha p \geq n + p$ is contained in Section 3C.

In the proofs we will consider a particular family of intervals to cover a given set, whose properties are more suitable for our aims. Below *a dyadic interval* means a closed cube in $\mathbb{R}^n$ of the form $[k_1/2^l, (k_1+1)/2^l] \times \cdots \times [k_n/2^l, (k_n+1)/2^l]$, where k_i, l are integers. Define

$$\Lambda^s(E) = \inf\left\{ \sum_{i=1}^{\infty} \ell(Q_i)^s : E \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ dyadic} \right\}.$$

It is well known that $\Lambda^s(E) \sim \mathcal{H}^s(E)$ for all subset $E \subset \mathbb{R}^n$; in particular, Λ^s and $\mathcal{H}^s$ have the same null sets.

Let $\{Q_j\}_{j\in\mathbb{N}}$ be a family of n-dimensional dyadic intervals. For a given parameter $\tau > 0$ we say that the family $\{Q_j\}$ is *regular* if $\sum \ell(Q_j)^\tau < \infty$ and for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^\tau \geq \sum_{j:Q_j \subset Q} \ell(Q_j)^\tau \tag{3-1}$$

holds. Since dyadic intervals are either nonoverlapping or contained in one another, (3-1) implies that any regular family $\{Q_j\}$ must in particular consist of nonoverlapping intervals. Moreover, the following result holds.

Lemma 3.1 [Bourgain et al. 2015, Lemma 2.3]. *Let $\{J_i\}$ be a family of n-dimensional dyadic intervals with $\sum_i \ell(J_i)^\tau < \infty$. Then there exists a regular family $\{Q_j\}$ of n-dimensional dyadic intervals such that $\bigcup_i J_i \subset \bigcup_j Q_j$ and*

$$\sum_j \ell(Q_j)^\tau \leq \sum_i \ell(J_i)^\tau.$$

3A. ***Proof of Theorem 1.3: the supercritical case $\tau > \tau_* > 0$.*** Fix the parameters $n \in \mathbb{N}$, $\alpha > 0$, $p > 1$ such that

$$\alpha p > n, \quad \tau_* = n - (\alpha - 1)p > 0, \tag{3-2}$$

and take

$$\tau \in (\tau_*, n]. \tag{3-3}$$

Fix also a mapping $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. If $\alpha = 1$, then $v \in W^1_p(\mathbb{R}^n)$ with $p > n$ and $\tau = n$, and the result is well known. So we restrict our attention to the nontrivial case $\alpha > 1$, $\tau < n$.

Now let $\{Q_i\}_{i\in\mathbb{N}}$ be a regular family of n-dimensional dyadic intervals. Consider the corresponding measure μ defined as

$$\int f \, d\mu := \sum_i \frac{1}{\ell(Q_i)^{n-\tau}} \int_{Q_i} f(y)\, dy. \tag{3-4}$$

As usual, for a measurable set $E \subset \mathbb{R}^n$ put $\mu(E) = \int 1_E \, d\mu$, where 1_E is an indicator function of E.

Lemma 3.2 [Korobkov and Kristensen 2014, Lemma 2.4]. *For any regular family $\{Q_i\}_{i\in\mathbb{N}}$ of n-dimensional dyadic intervals the corresponding measure μ defined by* (3-4) *satisfies*

$$\mu(Q) \le \ell(Q)^\tau$$

for any dyadic cube $Q \subset \mathbb{R}^n$.

From this fact and from the Adams Theorem 2.9, we immediately obtain:

Lemma 3.3. *Let $g \in L_p(\mathbb{R}^n)$. Then for any regular family $\{Q_i\}$ of n-dimensional dyadic intervals the estimate*

$$\sum_i \frac{1}{\ell(Q_i)^{n-\tau}} \int_{Q_i} (I_{\alpha-1}|g|)^s \, dy \le C \|g\|^s_{L_p} \tag{3-5}$$

holds, where $s := (\tau/\tau_*)p > p$ and C does not depend on g.

Now we are ready to formulate the key step of the proof.

Lemma 3.4. *Under the assumptions above, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_i\}$ of n-dimensional dyadic intervals if*

$$\sum_i \ell(Q_i)^\tau < \delta,$$

then

$$\sum_i [\operatorname{diam} v(Q_i)]^\tau < \varepsilon.$$

Proof. Since $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$, by the definition of this space, it is easy to see that for any $\tilde{\varepsilon} > 0$ there exists a representation

$$v = v_1 + v_2,$$

where $v_i \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$, $v_1 \in C^\infty(\mathbb{R}^n)$,

$$\|\nabla v_1\|_{L_\infty(\mathbb{R}^n)} < \infty,$$

and

$$v_2 = \mathscr{G}_\alpha(g) \quad \text{with } \|g\|_{L_p} < \tilde{\varepsilon}. \tag{3-6}$$

This means, in particular, that

$$|\nabla v_1(x)| < K \quad \text{for all } x \in \mathbb{R}^n, \tag{3-7}$$

for some $K = K(\tilde{\varepsilon}, v) \in \mathbb{R}$. Take any regular family $\{Q_i\}$ of n-dimensional dyadic intervals such that

$$\sum_i \ell(Q_i)^\tau < \delta \tag{3-8}$$

(the exact value of δ will be specified below). Put $r_i = \ell(Q_i)$. Then by Lemma 2.2

$$\sum_i [\operatorname{diam} v(Q_i)]^\tau \le C(S_1 + S_2 + S_3),$$

where

$$S_1=\sum_i[\operatorname{diam} v_1(Q_i)]^\tau \overset{(3\text{-}7)-(3\text{-}8)}{\le} n^{\tau/2}K^\tau\delta,$$
$$S_2=\sum_i \|Mg\|^\tau_{L_p(Q_i)} r_i^{\tau(\alpha-\frac{n}{p})},$$
$$S_3=\sum_i\biggl(\frac{1}{r_i^{n-1}}\int_{Q_i} I_{\alpha-1}|g|(y)\,\mathrm{d}y\biggr)^\tau.$$

Let us estimate S_2. Since $\alpha-\frac{n}{p}<1$ by (3-2), we can apply the Hölder inequality to obtain

$$S_2\le\biggl(\sum_i\|Mg\|^{\tau\frac{p}{n-p(\alpha-1)}}_{L_p(Q_i)}\biggr)^{\frac{n}{p}-\alpha+1}\cdot\biggl(\sum_i r_i^\tau\biggr)^{\alpha-\frac{n}{p}} \overset{(3\text{-}8)}{\le} \biggl(\sum_i\|Mg\|^{\tau\frac{p}{n-p(\alpha-1)}}_{L_p(Q_i)}\biggr)^{\frac{n}{p}-\alpha+1}\cdot\delta^{\alpha-\frac{n}{p}}$$
$$\overset{(3\text{-}2)}{=}\biggl(\sum_i\|Mg\|^{p\frac{\tau}{\tau_*}}_{L_p(Q_i)}\biggr)^{\frac{\tau_*}{p}}\cdot\delta^{\alpha-\frac{n}{p}} \overset{(3\text{-}3)}{\le} \|Mg\|^\tau_{L_p(\cup_i Q_i)}\cdot\delta^{\alpha-\frac{n}{p}} \overset{(3\text{-}6)}{\le} C\,\tilde{\varepsilon}^\tau\cdot\delta^{\alpha-\frac{n}{p}};$$

here C is the constant from the the Hardy–Littlewood maximal inequality. Similarly, taking $s=(\tau/\tau_*)p$ and applying twice the Hölder inequality in S_3 (the first time for the integrals, and the second time for sums), we obtain

$$S_3\le\sum_i\biggl(\int_{Q_i}(I_{\alpha-1}|g|)^s\,\mathrm{d}y\biggr)^{\frac{\tau_*}{p}}\cdot r_i^{n(\tau-\frac{\tau_*}{p})}\cdot r_i^{(1-n)\tau}=\sum_i\biggl(\frac{1}{r_i^{n-\tau}}\int_{Q_i}(I_{\alpha-1}|g|)^s\,\mathrm{d}y\biggr)^{\frac{\tau_*}{p}}\cdot r_i^{(1-\frac{\tau_*}{p})\tau}$$
$$\overset{\text{Hölder}}{\le}\biggl(\sum_i\frac{1}{r_i^{n-\tau}}\int_{Q_i}(I_{\alpha-1}|g|)^s\,\mathrm{d}y\biggr)^{\frac{\tau_*}{p}}\cdot\biggl(\sum_i r_i^\tau\biggr)^{1-\frac{\tau_*}{p}} \overset{(3\text{-}5),\,(3\text{-}6),\,(3\text{-}8)}{\le} C\,\tilde{\varepsilon}^\tau\cdot\delta^{1-\frac{\tau_*}{p}}.$$

So taking δ sufficiently small so that $K^\tau\delta<\frac{1}{2}\varepsilon$ is small, we have $S_1+S_2+S_3<\varepsilon$ as required, and Lemma 3.4 is proved. □

Finally, if E is a set such that $\mathcal{H}^\tau(E)=0$, then also $\Lambda^\tau(E)=0$, and this lemma together with Lemma 3.1 implies the validity of the assertion Theorem 1.3(i) for the supercritical case $\tau>\tau_*>0$.

3B. ***Proof of Theorem 1.3: the subcritical case $0<\tau<\tau_*$.*** Now fix the parameters $n\in\mathbb{N}$, $\alpha>0$, $p>1$ such that

$$\alpha p>n,\quad \tau_*=n-(\alpha-1)p>0, \tag{3-9}$$

and take

$$\tau\in(0,\tau_*),\quad \sigma=\frac{p\tau}{\alpha p-n+\tau}.$$

Evidently, by this definition

$$\sigma>\tau. \tag{3-10}$$

Fix also a mapping $v\in\mathscr{L}^\alpha_p(\mathbb{R}^n,\mathbb{R}^d)$. Take an additional parameter θ such that

$$(\alpha-1+\theta)>0\quad\text{and}\quad n-(\alpha-1+\theta)p>0.$$

From Lemma 3.2 and the Adams theorem 2.10, taking $s = p$, we immediately obtain:

Lemma 3.5. *Let $g \in L_p(\mathbb{R}^n)$. Then for any τ-regular family $\{Q_i\}$ of n-dimensional dyadic intervals the estimate*

$$\sum_i \frac{1}{\ell(Q_i)^{n-\tau_\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p \, dy \le C \|g\|_{L_p}^p \tag{3-11}$$

holds, where $\tau_\theta = n - (\alpha - 1 + \theta) p$ and C does not depend on g.

As in the previous case, the proof of Theorem 1.3 in the case $0 < \tau < \tau^*$ will be complete once we establish the following result.

Lemma 3.6. *Under above assumptions, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_i\}$ of n-dimensional dyadic intervals if*

$$\sum_i \ell(Q_i)^\tau < \delta,$$

then

$$\sum_i [\operatorname{diam} v(Q_i)]^\sigma < \varepsilon.$$

Proof. Again, since $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d)$, by the definition of this space, for any $\tilde{\varepsilon} > 0$ there exists a representation

$$v = v_1 + v_2,$$

where $v_i \in \mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d)$, $v_1 \in C^\infty(\mathbb{R}^n)$,

$$\|\nabla v_1\|_{L_\infty(\mathbb{R}^n)} < \infty,$$

and

$$v_2 = \mathscr{G}_\alpha(g) \quad \text{with } \|g\|_{L_p} < \tilde{\varepsilon}. \tag{3-12}$$

This means, in particular, that

$$|\nabla v_1(x)| < K \quad \text{for all } x \in \mathbb{R}^n, \tag{3-13}$$

for some $K = K(\tilde{\varepsilon}, v) \in \mathbb{R}$. Take any regular family $\{Q_i\}$ of n-dimensional dyadic intervals such that

$$\sum_i \ell(Q_i)^\tau < \delta < 1 \tag{3-14}$$

(the exact value of δ will be specified below). Put $r_i = \ell(Q_i)$. Then by Lemma 2.3

$$\sum_i [\operatorname{diam} v(Q_i)]^\sigma \le C(S_1 + S_2 + S_3),$$

where

$$S_1 = \sum_i [\operatorname{diam} v_1(Q_i)]^\sigma \overset{(3\text{-}10),(3\text{-}13)-(3\text{-}14)}{\le} CK^\sigma \delta,$$

$$S_2 = \sum_i \|Mg\|^\sigma_{L_p(Q_i)} r_i^{\sigma(\alpha-\frac{n}{p})},$$

$$S_3 = \sum_i \biggl(\frac{1}{r_i^{n-1+\theta}} \int_{Q_i} M_{\alpha-1+\theta} g(y)\,\mathrm{d}y\biggr)^\sigma.$$

Let us estimate S_2. Since by assumptions (3-9) the inequality $\sigma < p$ holds and

$$\frac{p-\sigma}{p} = \frac{\alpha p - n}{\alpha p - n + \tau}, \quad \sigma\frac{p}{p-\sigma} = \frac{\tau}{\alpha-(n/p)} \tag{3-15}$$

we can apply the Hölder inequality to obtain

$$S_2 \le \biggl(\sum_i \|Mg\|^p_{L_p(Q_i)}\biggr)^{\frac{\sigma}{p}} \cdot \biggl(\sum_i r_i^{\sigma(\alpha-\frac{n}{p})\frac{p}{p-\sigma}}\biggr)^{\frac{p-\sigma}{p}}$$
$$= (\|Mg\|^p_{L_p(\cup_i Q_i)})^{\frac{\sigma}{p}} \cdot \biggl(\sum_i r_i^\tau\biggr)^{\frac{p-\sigma}{p}} \overset{(3\text{-}14),\,(3\text{-}12)}{\le} C\,\tilde\varepsilon^\sigma \delta^{1-\frac{\sigma}{p}}.$$

Similarly, applying twice the Hölder inequality in S_3 (the first time for the integrals, and the second time for sums), we obtain

$$S_3 \le \sum_i \biggl(\int_{Q_i} (M_{\alpha-1+\theta} g)^p\,\mathrm{d}y\biggr)^{\frac{\sigma}{p}} \cdot r_i^{n\frac{p-1}{p}\sigma} \cdot r_i^{(1-n-\theta)\sigma} = \sum_i \biggl(\frac{1}{r_i^{n-\tau\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p\,\mathrm{d}y\biggr)^{\frac{\sigma}{p}} \cdot r_i^{(\alpha-\frac{n}{p})\sigma}$$
$$\overset{\text{Hölder}}{\le} \biggl(\sum_i \frac{1}{r_i^{n-\tau\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p\,\mathrm{d}y\biggr)^{\frac{\sigma}{p}} \cdot \biggl(\sum_i r_i^{(\alpha-\frac{n}{p})\sigma\frac{p}{p-\sigma}}\biggr)^{1-\frac{\sigma}{p}}$$
$$\overset{(3\text{-}15)}{=} \biggl(\sum_i \frac{1}{r_i^{n-\tau\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p\,\mathrm{d}y\biggr)^{\frac{\sigma}{p}} \cdot \biggl(\sum_i r_i^\tau\biggr)^{1-\frac{\sigma}{p}} \overset{(3\text{-}11),(3\text{-}12),(3\text{-}14)}{\le} C\,\tilde\varepsilon^\sigma \cdot \delta^{1-\frac{\sigma}{p}}.$$

So taking δ sufficiently small so that $K^\tau\delta < \frac{1}{2}\varepsilon$ is small, we have $S_1 + S_2 + S_3 < \varepsilon$ as required, and the lemma is proved. □

Finally, we conclude exactly as in the previous case.

3C. ***Proof of Theorem 1.3: the supercritical case $\tau_* \le 0 < \tau$.*** Consider now the case $\alpha p > n$ and $\tau_* = n - (\alpha-1)p \le 0$. If $(\alpha-1)p > n$, then every function $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$ is locally Lipschitz (even C^1) and the result is trivial. Suppose now $(\alpha-1)p = n$. Under these assumptions, let $\tau > 0$ and $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Take a number $1 < \tilde p < p$ such that $\alpha\tilde p > n$ and $\tau > \tau^*_* = n - (\alpha-1)\tilde p > 0$. Then we have $v \in \mathscr{L}^\alpha_{\tilde p,\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^d)$ (see Section 2E). Therefore, by the previous case $\tau > \tilde\tau_* > 0$, the mapping v has the (τ,τ)-N-property. □

3D. ***Proof of Theorem 1.6.*** The proof of Theorem 1.6 is very similar to that of Theorem 1.3: the main differences concern the limiting cases $\alpha p = n$ or $\tau = \tau^*$.

Case I: $\alpha p > n$ and $\tau \neq \tau^*$. The required assertion follows immediately from Theorem 1.3 and from the inclusion $\mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n) \subset \mathscr{L}^{\alpha}_{p}(\mathbb{R}^n)$ (this inclusion follows from the definitions of these space and from the relation $L_{p,1}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$).

Case II: $\alpha p = n$ and $\tau > \tau_* > 0$. The required assertion can be proved by repeating almost word for word the same arguments as in the supercritical case in Theorem 1.3 with the following evident modifications: now one has to apply the estimate (2-8) (which covers the case $\alpha p = n$) instead of previous estimate (2-5), and, in addition, one needs the following analog of the additivity property for the Lorentz norms:

$$\sum_i \|f\|^p_{\mathrm{L}_{p,1}(Q_i)} \leq \|f\|^p_{\mathrm{L}_{p,1}(\cup_i Q_i)}$$

for any family of disjoint cubes [Malý 2003, Lemma 3.10].

Case III: $\alpha p \geq n$ and $\tau = \tau^*$. The required assertion can be proved by repeating almost word for word the same arguments as in the supercritical case in Theorem 1.3 with the following evident modifications: now $\tau = \tau_*$ (this simplifies the calculations a little bit) and one has to apply Theorem 2.11 (which covers the case $s = p$) and the estimate (2-8) instead of Theorem 2.9 (where $s > p$) and the inequality (2-5), respectively.

Case IV: $\alpha p = n$ and $0 < \tau < \tau^*$. By a direct calculation, we get $\sigma(\tau) \equiv p$ for any $\tau \in (0, \tau_*]$, and the result follows from the above-considered critical case $\tau = \tau_*$.

Thus Theorems 1.3 and 1.6 are proved completely.

Remark 3.7. Really, we have proved that under the assumptions of Theorems 1.3 and 1.6, for every fixed function $v : \mathbb{R}^n \to \mathbb{R}^d$ from the considered potential spaces and for the corresponding pair (τ, σ) the following assertion holds: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every τ-regular family of cubes $Q_i \subset \mathbb{R}^n$ if $\sum_i \ell(Q_i)^\tau < \delta$, then $\sum_i [\operatorname{diam} v(Q_i)]^\sigma < \varepsilon$.

4. Proof of "Fubini-type" N_*-properties

Here we have to prove Theorems 1.11 and 1.13. We need the following general fact.

Theorem 4.1. *Let $\tau \in (0, n]$, $\sigma > 0$, and let $v : \mathbb{R}^n \to \mathbb{R}^d$ be a continuous function. Suppose that for any $E \subset \mathbb{R}^n$ with $\mathcal{H}^\tau(E) = 0$ and for every $\varepsilon > 0$ there exists a family of compact sets $\{D_i\}_{i \in \mathbb{N}}$ such that*

$$E \subset \bigcup_i D_i \quad \textit{and} \quad \sum_i [\operatorname{diam} D_i]^\tau < \varepsilon \quad \textit{and} \quad \sum_i [\operatorname{diam} v(D_i)]^\sigma < \varepsilon. \tag{4-1}$$

Then v has the (τ, σ)-N_*-property; i.e., for every $q \in [0, \sigma]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ we have

$$\mathcal{H}^{\tau\left(1-\frac{q}{\sigma}\right)}(E \cap v^{-1}(y)) = 0 \quad \textit{for } \mathcal{H}^q\textit{-a.a. } y \in \mathbb{R}^d. \tag{4-2}$$

Proof. Let the assumptions of the theorem be fulfilled. Fix $q \in [0, \sigma]$. If $q = 0$ or $q = \sigma$, then the required assertion (4-2) follows trivially from these assumptions. Suppose now that

$$0 < q < \sigma.$$

Fix an arbitrary $\varepsilon > 0$ and take the corresponding sequence of compact sets D_i satisfying (4-1). Put $\mu = \tau\bigl(1 - \frac{q}{\sigma}\bigr) < \tau$. Then

$$\begin{aligned}\sum_i (\operatorname{diam} D_i)^{\mu} [\operatorname{diam} v(D_i)]^q &\overset{\text{Hölder}}{\leq} \Bigl(\sum_i [\operatorname{diam} D_i]^{\mu\frac{\sigma}{\sigma-q}}\Bigr)^{1-\frac{q}{\sigma}} \cdot \Bigl(\sum_i [\operatorname{diam} v(D_i)]^{\sigma}\Bigr)^{\frac{q}{\sigma}} \\ &= \Bigl(\sum_i [\operatorname{diam} D_i]^{\tau}\Bigr)^{1-\frac{q}{\sigma}} \Bigl(\sum_i [\operatorname{diam} v(D_i)]^{\sigma}\Bigr)^{\frac{q}{\sigma}} \overset{(4\text{-}1)}{<} \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, now the required assertion follows immediately from Theorem 2.12. □

The theorem just proved and Remark 3.7 clearly imply the assertions of Theorems 1.11 and 1.13.

4A. ***Proof of Theorem 1.17.*** Fix a mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ for which the assumptions of Theorem 1.16 are fulfilled. We have to prove that

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d, \tag{4-3}$$

for any $q > m - 1$, where $\mu_q = n - m + 1 - (k + \alpha)(q - m + 1)$ and A_v is the set of nondifferentiability points of v. Recall that, by approximation results [Swanson 2002; Korobkov and Kristensen 2018], under the conditions of Theorem 1.16 the equalities

$$\mathcal{H}^{\tau}(A_v) = 0 \quad \text{for all } \tau > \tau_* := n - (k + \alpha - 1)p \quad \text{in cases (i), (iii),} \tag{4-4}$$

$$\mathcal{H}^{\tau_*}(A_v) = \mathcal{H}^{p}(A_v) = 0 \quad \tau_* := n - (k + \alpha - 1)p = p \quad \text{in cases (ii), (iv)} \tag{4-5}$$

are valid.

Because of Remark 1.18 we can assume without loss of generality that $q \in (m - 1, q_\circ]$. Then for all cases (i)–(iv) we have

$$\begin{aligned}\Bigl(\frac{n}{k+\alpha} \leq p\Bigr) &\implies \Bigl(q - m + 1 \leq q_\circ - m + 1 = \frac{n - m + 1}{k + \alpha} \leq p\Bigr) \\ &\implies \mu_q = n - m + 1 - (k + \alpha)(q - m + 1) \\ &\qquad = n - (k + \alpha - 1)(q - m + 1) - q \geq n - (k + \alpha - 1)p - q = \tau_* - q.\end{aligned}$$

In other words,

$$\mu_q \geq \tau_* - q, \tag{4-6}$$

where the equality holds if and only if

$$k = 1, \quad \alpha = 0, \quad \mu_q = n - q = \tau_* - q \tag{4-7}$$

or

$$m = 1, \quad (k+\alpha)p = n, \quad q = p = \tau_*, \quad \mu_q = 0. \tag{4-8}$$

Below for convenience we consider the cases (i)–(iv) of Theorem 1.16 separately.

Case I: $\alpha = 0,\ kp > n,\ p \geq 1,\ v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$. This case splits into the following three subcases.

Case Ia: $k = 1,\ p > n,\ \tau_* = n,\ \mu_q = n - q$. Then the required assertion (4-3) follows immediately from the equality $\mathcal{H}^n(A_v) = 0$ and from Remark 1.12.

Case Ib: $\tau_* < 0$ or $\tau_* = 0,\ k = n+1,\ p = 1$. Then the set A_v is empty (since functions of the space $W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ are C^1-smooth), and there is nothing to prove.

Case Ic: $\tau_* \geq 0,\ p > 1,\ k > 1,\ kp > n$. Then by (4-4) we have

$$\text{for all } \tau > \tau_*, \quad \mathcal{H}^\tau(A_v) = 0. \tag{4-9}$$

Further, by Theorem 1.11 the function v has the (τ, τ)-N_*-property for every $\tau > \tau_*$. This implies, in particular, by virtue of (4-9), that for every $\tau > \tau_*$ and for every $q \in [0, \tau]$ the equality

$$\mathcal{H}^{\tau - q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{4-10}$$

holds. Fix $q \in (m-1, q_\circ]$ and take $\tau = q + \mu_q$. Since by construction $\mu_q \geq 0$, we have $\tau \geq q$. Moreover, by (4-6)–(4-8) we have $\tau > \tau_*$. The last two inequalities together with (4-10) imply

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d.$$

So the required assertion is proved for this case.

Case II: $\alpha = 0,\ kp = n,\ p \geq 1,\ v \in W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$. In this case by definition

$$\tau_* := n - (k-1)p = p,$$

and, by (4-5) we have

$$\mathcal{H}^p(A_v) = 0. \tag{4-11}$$

Further, by [Hajłasz et al. 2017, Theorem 2.3] the function v has the (τ, τ)-N_*-property for every $\tau \geq p$. This implies, in particular, by virtue of (4-11), that for every $\tau \geq p$ and for every $q \in [0, \tau]$ the equality

$$\mathcal{H}^{\tau - q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{4-12}$$

holds. Fix $q \in (m-1, q_\circ]$ and take $\tau = q + \mu_q$. Since by construction $\mu_q \geq 0$, we have $\tau \geq q$. Moreover, by (4-6)–(4-8) we have $\tau \geq \tau_* = p$. The last two inequalities together with (4-12) imply

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d.$$

So the required assertion is proved for this case.

Case III: $0 < \alpha < 1,\ (k+\alpha)p > n,\ p > 1,\ v \in \mathscr{L}^{k+\alpha}_p(\mathbb{R}^n, \mathbb{R}^d)$. If $\tau_* = n - (k+\alpha-1)p < 0$, then $A_v = \varnothing$ and there is nothing to prove. Suppose now that $\tau_* \geq 0$. We obtain from Theorem 1.11 that v has the (τ,τ)-N_*-property for every $\tau > \tau_* := n-(\alpha-1)p$. This implies, in particular, by virtue of (4-4), that for every $\tau > \tau_*$ and for every $q \in [0,\tau]$ the equality

$$\mathcal{H}^{\tau-q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{4-13}$$

holds. Fix $q \in (m-1, q_\circ]$ and take $\tau = q+\mu_q$. Since by construction $\mu_q \ge 0$, we have $\tau \ge q$. Moreover, by (4-6)–(4-8) we have $\tau > \tau_*$. The last two inequalities together with (4-13) imply

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d.$$

So the required assertion is proved for this case.

Case IV: $0<\alpha<1,\ (k+\alpha)p = n,\ p>1,\ v \in \mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n,\mathbb{R}^d)$. In this case by definition

$$\tau_* := n-(k-1)p = p,$$

and, by (4-5) we have

$$\mathcal{H}^p(A_v) = 0. \tag{4-14}$$

Further, by Theorem 1.13 the function v has the (τ,τ)-N_*-property for every $\tau \ge p$. This implies, in particular, by virtue of (4-14), that for every $\tau \ge p$ and for every $q \in [0,\tau]$ the equality

$$\mathcal{H}^{\tau-q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{4-15}$$

holds. Fix $q \in (m-1, q_\circ]$ and take $\tau = q+\mu_q$. Since by construction $\mu_q \ge 0$, we have $\tau \ge q$. Moreover, by (4-6)–(4-8) we have $\tau \ge \tau_* = p$. The last two inequalities together with (4-15) imply

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d.$$

So the required assertion is proved for this case, which is the last one.

Thus Theorem 1.17 is proved completely. □

Appendix

We prove the technical estimates of Lemmas 2.2, 2.3 and 2.8. Fix a cube $Q \subset \mathbb{R}^n$ of size $r = \ell(Q) \le 1$. Recall that by $2Q$ we denote the double cube with the same centre as Q of size $\ell(2Q) = 2\ell(Q)$. We need some general elementary estimates.

Lemma A.1. *For any measurable function $g : \mathbb{R}^n \to \mathbb{R}_+$ and for every $x \in Q$ the inequality*

$$\int_{2Q} \frac{g(y)}{|x-y|^{n-\alpha}}\,dy \le C \int_Q \frac{Mg(y)}{|x-y|^{n-\alpha}}\,dy \tag{A-1}$$

holds.

Here C denotes some universal constant that does not depend on g, Q, etc.

Proof. Fix $x \in Q$. Define $r_0 = \frac{7}{2}\sqrt{n}r$. In particular, $2Q \subset B\big(x, \frac{1}{2}r_0\big)$.

Now put $r_j = 2^{-j} r_0$ and $B_j = B(x, r_j) \setminus B(x, r_{j+1}),\ j \in \mathbb{N}$. Clearly,

$$2Q = \bigcup_{j \in \mathbb{N}} (2Q \cap B_j) \tag{A-2}$$

and

$$\operatorname{meas}(Q \cap B_j) \geq C r_j^n \quad \text{for all } j \in \mathbb{N} \tag{A-3}$$

(here and henceforth we denote by C general constants depending on the parameters n, p, d, α only).

Since $|x - y| \sim r_j$ for $y \in B_j$, by the definition of the maximal function, it is easy to see that the estimate

$$\int_{2Q\cap B_j} \frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \leq C r_j^{\alpha} M g(z) \quad \text{for all } z \in Q \cap B_j$$

holds for all $j \in \mathbb{N}$. Integrating this inequality with respect to $z \in Q \cap B_j$ and using (A-3), we have

$$\int_{2Q\cap B_j} \frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \leq C r_j^{\alpha-n} \int_{Q\cap B_j} M g(z)\,\mathrm{d}z. \tag{A-4}$$

Since $|x - z| \sim r_j$ for $z \in Q \cap B_j$, the last inequality implies

$$\int_{2Q\cap B_j} \frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \leq C \int_{Q\cap B_j} \frac{M g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y. \tag{A-5}$$

Then summing these inequalities for all $j \in \mathbb{N}$ and taking into account (A-2), we obtain the required estimate (A-1). □

Henceforth, fix $p > 1$, $\alpha > 0$ with $n + p \geq \alpha p \geq n$ (in particular, $\alpha < n + 1$), and a function $v(x) = \mathcal{G}_\alpha(x) = \int_{\mathbb{R}^n} g(y)\, K_\alpha(x-y)\,\mathrm{d}y$ with some $g \in L_p(\mathbb{R}^n)$.

Split our function v into a sum

$$v = v_1 + v_2, \tag{A-6}$$

where

$$v_1 := \int_{\mathbb{R}^n} g_1(y)\, K_\alpha(x-y)\,\mathrm{d}y, \qquad v_2 := \int_{\mathbb{R}^n} g_2(y)\, K_\alpha(x-y)\,\mathrm{d}y,$$

and

$$g_1 := g \cdot 1_{2Q}, \qquad g_2 := g \cdot 1_{\mathbb{R}^n \setminus 2Q}.$$

Lemma A.2. *If $n + p > \alpha p > n$, we have*

$$\operatorname{diam} v_1(Q) \leq C\, \|Mg\|_{L_p(Q)}\, r^{\alpha - \frac{n}{p}}. \tag{A-7}$$

Proof. If $0 < \alpha < n$, then $K_\alpha(x) < c_\alpha\, |x|^{\alpha-n}$ (see [Adams and Hedberg 1996, page 10], for example), and from Lemma A.1 we have

$$|v_1(x)| \leq C \int_Q \frac{Mg(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \quad \text{for all } x \in Q,$$

so the required estimate (A-7) follows immediately from the Hölder inequality.

If $n \le \alpha < n+1$, then

$$\left|\nabla K_\alpha(x)\right| \le C\,|x|^{\alpha-n-1}$$

(see [Adams and Hedberg 1996, page 13], for example), and by Lemma A.1 we have

$$|\nabla v_1(x)| \le C \int_Q \frac{Mg(y)}{|x-y|^{n-\alpha+1}}\,\mathrm{d}y \quad \text{for all } x \in Q. \tag{A-8}$$

Then by the Hardy–Littlewood–Sobolev inequality for Riesz potentials we have

$$\|\nabla v_1\|_{L_q(Q)} \le C\,\|Mg\|_{L_p(Q)},$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha-1}{n}.$$

It is easy to see that $q > n$, then by the Morrey inequality

$$\operatorname{diam} v_1(Q) \le C\,\|\nabla v_1\|_{L_q(Q)}\, r^{1-\frac{n}{q}} \le C_1\,\|Mg\|_{L_p(Q)} r^{\alpha-\frac{n}{p}}$$

as required. □

We need a modification of lemma above to the case of Lorentz spaces.

Lemma A.3. *If $n+p \ge \alpha p \ge n$, we have*

$$\operatorname{diam} v_1(Q) \le C\,\|Mg\|_{L_{p,1}(Q)}\, r^{\alpha-\frac{n}{p}}.$$

Proof. We have to repeat the previous arguments using the following facts for Lorentz norms: the generalized Hölder inequality

$$\int_Q \frac{f(y)}{|y-x|^{n-\alpha}}\,\mathrm{d}y \le \|f\|_{L_{p,1}} \cdot \left\| \frac{1_Q}{|\cdot - x|^{n-\alpha}} \right\|_{L_{\frac{p}{p-1},\infty}} = C\,\|f\|_{L_{p,1}} r^{\alpha-\frac{n}{p}}$$

for $n > \alpha \ge \frac{n}{p}$ [Malý 2003, Theorem 3.7], and the generalized Hardy–Littlewood–Sobolev inequality for Riesz potentials

$$\|I_\beta f\|_{L_{q,1}(Q)} \le C\,\|f\|_{L_{p,1}(Q)} \qquad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$$

if $\beta p \le n$ [Bennett and Sharpley 1988, Theorem IV.4.18], and the generalized Morrey inequality

$$\operatorname{diam} v_1(Q) \le C\,\|\nabla v_1\|_{L_{q,1}(Q)}\, r^{1-\frac{n}{q}}$$

for $q \ge n$ (see, e.g., [Korobkov and Kristensen 2014, Lemma 1.3]). □

Now we have to estimate the term v_2.

Lemma A.4. *For an arbitrary positive parameter $\theta \ge 1-\alpha$ the inequality*

$$\operatorname{diam}\bigl[v_2(Q)\bigr] \le C\, r^{1-\theta-n} \int_Q M_{\alpha+\theta-1} g(y)\,\mathrm{d}y \tag{A-9}$$

holds, where we recall that $r = \ell(Q)$.

Proof. Without loss of generality suppose that Q is centred at the origin. Since

$$C_1|y| \le |y-x| \le C_2|y| \quad \text{for all } x \in Q, \text{ for all } y \in \mathbb{R}^n \setminus 2Q, \tag{A-10}$$

and $K'_\alpha(\rho) \le C\rho^{\alpha-1-n}$ for $0 < \alpha < n+1$, it is easy to deduce that

$$\begin{aligned} \operatorname{diam} v_2(Q) &\le \sup_{x_1,x_2\in Q} \int_{\mathbb{R}^n\setminus 2Q} |g(y)|[K_\alpha(x_1-y)-K_\alpha(x_2-y)]\,\mathrm{d}y \\ &\le C\, r \int_{\mathbb{R}^n\setminus 2Q} \frac{|g(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y. \end{aligned} \tag{A-11}$$

Fix $\theta > 0$ such that

$$\alpha + \theta - 1 \ge 0. \tag{A-12}$$

Put $r_0 = \frac{1}{2}r,\ r_j = 2^j r_0$, and consider a sequence of sets $B_j = B(0, r_{j+1}) \setminus B(0, r_j)$. By construction,

$$\mathbb{R}^n \setminus 2Q \subset \bigcup_{j\in\mathbb{N}} B_j \tag{A-13}$$

and

$$\int_{B_j} \frac{|g_2(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y \le C r_j^{-\theta} r_j^{\alpha+\theta-1} ⨍_{B_j} |g_2(y)|\,\mathrm{d}y \le C r_j^{-\theta} M_{\alpha+\theta-1} g_2(0), \tag{A-14}$$

where we recall that $g_2 := g \cdot 1_{\mathbb{R}^n\setminus 2Q}$. Therefore, by summing over j and using (A-13) and the elementary formula for geometric progressions, we obtain

$$\int_{\mathbb{R}^n\setminus 2Q} \frac{|g_2(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y \le C M_{\alpha+\theta-1} g_2(0) \sum_{j=1}^{\infty} r_j^{-\theta} \le C r^{-\theta} M_{\alpha+\theta-1} g_2(0), \tag{A-15}$$

It is easy to check (using the assumption that $g_2 \equiv 0$ on $2Q$) that $M_{\alpha+\theta-1} g_2(0) \le C M_{\alpha+\theta-1} g_2(z)$ for every $z \in Q$. Therefore,

$$M_{\alpha+\theta-1} g_2(0) \le C ⨍_Q M_{\alpha+\theta-1} g_2(z)\,\mathrm{d}z; \tag{A-16}$$

thus

$$\int_{\mathbb{R}^n\setminus 2Q} \frac{|g_2(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y \le C r^{-\theta-n} \int_Q M_{\alpha+\theta-1} g_2(z)\,\mathrm{d}z. \tag{A-17}$$

Finally we obtain from (A-11) that

$$\operatorname{diam}[v(Q)] \le C r^{1-\theta-n} \int_Q M_{\alpha+\theta-1} g_2(z)\,\mathrm{d}z \tag{A-18}$$

as required. □

The next result is established using the same arguments, with some evident simplifications.

Lemma A.5. *If, in addition to the assumptions above, we have* $\alpha > 1$, *then the estimate*

$$\operatorname{diam} v_2(Q) \le C\, r^{1-n} \int_Q I_{\alpha-1}|g|(y)\,\mathrm{d}y \tag{A-19}$$

holds, where we recall that $I_{\alpha-1}|g|$ *is the corresponding Riesz potential of the function* $|g|$.

Lemmas A.2–A.5 clearly imply the assertions of Lemmas 2.2, 2.3 and 2.8.

Acknowledgments

Korobkov was partially supported by the Ministry of Education and Science of the Russian Federation (Project number 1.8126.2017/8.9). The main part of the paper was written during a visit of Korobkov to the University of Campania "Luigi Vanvitelli" in 2017, and he is very thankful for the hospitality. Also the authors are very grateful to the academician Sergei Konyagin and to the professor Sergey Vodop'yanov for the fruitful discussion concerning the lacunary Fourier series and Besov spaces.

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Received 26 Jun 2017. Revised 12 Jul 2018. Accepted 12 Aug 2018.

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ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1177

UNSTABLE NORMALIZED STANDING WAVES FOR THE SPACE PERIODIC NLS

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For the stationary nonlinear Schrödinger equation $-\Delta u + V(x)u - f(u) = \lambda u$ with periodic potential V we study the existence and stability properties of multibump solutions with prescribed L^2-norm. To this end we introduce a new nondegeneracy condition and develop new superposition techniques which allow us to match the L^2-constraint. In this way we obtain the existence of infinitely many geometrically distinct solutions to the stationary problem. We then calculate the Morse index of these solutions with respect to the restriction of the underlying energy functional to the associated L^2-sphere, and we show their orbital instability with respect to the Schrödinger flow. Our results apply in both, the mass-subcritical and the mass-supercritical regime.

1. Introduction

Suppose that $N \in \mathbb{N}$ and consider the stationary nonlinear Schrödinger equation with prescribed L^2-norm

$$-\Delta u + V(x)u - f(u) = \lambda u, \quad u \in H^1(\mathbb{R}^N), \quad |u|_2^2 = \alpha, \tag{P_α}$$

which we will call the *constrained* equation. Here $|\cdot|_2$ denotes the standard L^2-norm, $V \in L^\infty(\mathbb{R}^N)$ is periodic in all coordinates, f is a superlinear nonlinearity of class C^1 with Sobolev-subcritical growth, $\alpha > 0$ is given, u is the unknown weak solution and $\lambda \in \mathbb{R}$ is an unknown parameter. Solutions to (P_α) are standing wave solutions for the time-dependent Schrödinger equation modeling a Bose–Einstein condensate in a periodic optical lattice [Aftalion and Helffer 2009; Morsch and Oberthaler 2006; Baizakov et al. 2003; Efremidis and Christodoulides 2003; Fleischer et al. 2003; Louis et al. 2003; Ostrovskaya and Kivshar 2003; Hilligsøe et al. 2002; Dalfovo et al. 1999]. In this model α is proportional to the total number of atoms in the condensate.

Set

$$\Sigma_\alpha := \{u \in H^1(\mathbb{R}^N) : |u|_2^2 = \alpha\} \tag{1-1}$$

for $\alpha > 0$. Define the functional $\Phi\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) - \int_{\mathbb{R}^N} F(u), \tag{1-2}$$

Ackermann was supported by CONACYT grant 237661, UNAM-DGAPA-PAPIIT grant IN100718 and the program UNAM-DGAPA-PASPA (Mexico).
MSC2010: primary 35J91, 35Q55; secondary 35J20.
Keywords: nonlinear Schrödinger equation, periodic potential, standing wave solution, orbitally unstable solution, multibump construction, prescribed norm.

where we set $F(s) := \int_0^s f$. Then the pair (u, λ) is a weak solution of (P_α) if and only if u is a critical point of the restriction of Φ to Σ_α with Lagrange multiplier λ.

Not assuming periodicity of V but instead $\sup_{\mathbb{R}^N} V = \lim_{|x|\to\infty} V(x)$, the existence of a minimizer of Φ on Σ_α in the mass-subcritical case was shown under additional assumptions on the growth of the nonlinearity f by Lions [1984]; see also [Jeanjean and Squassina 2011] for a different approach. For constant V, solutions of (P_α) are constructed in the mass-supercritical case in [Bartsch and Soave 2017; Bartsch and de Valeriola 2013; Jeanjean 1997]; here the corresponding critical points of $\Phi|_{\Sigma_\alpha}$ are not local minimizers. In [Bellazzini et al. 2017; Bellazzini and Jeanjean 2016; Fukuizumi and Ohta 2003; Fukuizumi 2001] local minimizers are found in the mass-supercritical case under spatially constraining potentials.

The structure of the solution set of the constrained equation is rather poorly understood up to now in the case where $V \in L^\infty(\mathbb{R}^N)$ is not constant, but 1-periodic in all coordinates. In contrast, a large amount of information is available for the *free* equation

$$-\Delta u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N),$$

where essentially the parameter λ is fixed but the L^2-norm is not prescribed anymore. Of particular interest for us are the results on the existence of so-called *multibump solutions*. In [Arioli et al. 2009; Kryszewski and Szulkin 2009; Ackermann 2006; 1996; Ackermann and Weth 2005; Rabinowitz 1997; Spradlin 1995; Alama and Li 1992; Coti Zelati and Rabinowitz 1992], an infinite number of solutions are built using nonlinear superposition of translates of a special solution which satisfies a nondegeneracy condition of some form.

The main goal of the present work is to apply nonlinear superposition techniques to the constrained problem with periodic V to obtain an infinity of L^2-normalized solutions in the form of multibump solutions. We succeed in doing this, but have to impose a stricter nondegeneracy condition than in the case of the free equation which nevertheless is fulfilled in many situations. This provides, as far as we know, the first result on multibump solutions for the constrained problem, and also the first multiplicity result in the case of a nonconstant periodic potential V. We also compute the Morse index of these normalized multibump solutions with respect to the restricted functional $\Phi|_{\Sigma_\alpha}$, and we will use the Morse index information to derive orbital instability of the multibump solutions.

To state our results, we need the following hypotheses. We consider, as usual, the critical Sobolev exponent defined by $2^* := 2N/(N-2)$ in the case $N \geq 3$ and $2^* := \infty$ in the case $N = 1, 2$:

(H1) $V \in L^\infty(\mathbb{R}^N)$.

(H2) V is 1-periodic in all coordinates.

(H3) $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$,

$$\lim_{s\to\infty} \frac{f'(s)}{|s|^{2^*-2}} = 0$$

if $N \geq 3$, and there is $p > 2$ such that

$$\lim_{s\to\infty} \frac{f'(s)}{|s|^{p-2}} = 0$$

if $N = 1$ or $N = 2$.

Throughout this paper we assume (H1) and (H3). It is well known that Φ is well-defined by (1-2) and of class C^2. The standard example for a function satisfying (H3) is $f(s) := |s|^{p-2}s$ with $p \in (2, 2^*)$. In the following, we let $H^{-1}(\mathbb{R}^N)$ denote the topological dual of $H^1(\mathbb{R}^N)$. For our main result, we need the notion of a *fully nondegenerate* critical point of $\Phi|_{\Sigma_\alpha}$.

Definition 1.1. Assume (H1) and (H3). For $\alpha > 0$, a critical point $u \in H^1(\mathbb{R}^N)$ of $\Phi|_{\Sigma_\alpha}$ with Lagrangian multiplier λ will be called *fully nondegenerate* if for every $g \in H^{-1}(\mathbb{R}^N)$ there exists a unique weak solution $z_g \in H^1(\mathbb{R}^N)$ of the linearized equation

$$-\Delta z_g + [V-\lambda]z_g - f'(u)z_g = g \quad \text{in } \mathbb{R}^N, \tag{1-3}$$

and if in the case $g = u$ we have $\int_{\mathbb{R}^N} uz_u \neq 0$. Here, as usual, we regard $H^1(\mathbb{R}^N)$ as a subspace of $H^{-1}(\mathbb{R}^N)$, so $u \in H^{-1}(\mathbb{R}^N)$.

As we shall see in Section 2 below, the full nondegeneracy of a critical point $u \in H^1(\mathbb{R}^N)$ of $\Phi|_{\Sigma_\alpha}$ with Lagrangian multiplier λ implies the nondegeneracy of the Hessian of $\Phi|_{\Sigma_\alpha}$ at u. By definition, this Hessian is the bilinear form

$$(v, w) \mapsto \int_{\mathbb{R}^N} (\nabla v \nabla w + [V-\lambda]vw - f'(u)vw) \tag{1-4}$$

defined on the tangent space

$$T_u\Sigma_\alpha = \{v \in H^1(\mathbb{R}^N) : (v, u)_2 = 0\};$$

see Definition 2.5 below. Here $(\,\cdot\,,\cdot\,)_2$ denotes the standard scalar product in $L^2(\mathbb{R}^N)$. We also need to fix the following elementary notation. If $n \in \mathbb{N}$ and $a = (a^1, a^2, \dots, a^n) \in (\mathbb{Z}^N)^n$ is a tuple of n elements from $\mathbb{Z}^N$, define

$$d(a) := \min_{i\neq j}|a^i - a^j|.$$

Moreover, for $b \in \mathbb{R}^N$ we denote by $\mathcal{T}_b$ the associated translation operator; i.e., for $u\colon \mathbb{R}^N \to \mathbb{R}$ the function $\mathcal{T}_b u\colon \mathbb{R}^N \to \mathbb{R}$ is given by

$$\mathcal{T}_b u(x) := u(x-b) \quad \text{for } x \in \mathbb{R}^N.$$

Our first main result is the following.

Theorem 1.2 (multibump solutions). *Assume* (H1)–(H3) *and fix* $\alpha > 0$, $n \in \mathbb{N}$, $n \geq 2$. *Moreover, suppose that* $\bar{u}$ *is a fully nondegenerate critical point of* $\Phi|_{\Sigma_{\alpha/n}}$ *with Lagrangian multiplier* $\bar{\lambda}$. *Then for every* $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that for every* $a \in (\mathbb{Z}^N)^n$ *with* $d(a) \geq R_\varepsilon$ *there is a critical point* u_a *of* $\Phi|_{\Sigma_\alpha}$ *with Lagrange multiplier* λ_a *such that*

$$\Big\| u_a - \sum_{i=1}^n \mathcal{T}_{a^i}\bar{u} \Big\|_{H^1(\mathbb{R}^N)} \leq \varepsilon \quad \textit{and} \quad |\lambda_a - \bar{\lambda}| \leq \varepsilon.$$

If ε *is chosen small enough then* u_a *is unique. Moreover, if* $\bar{u}$ *is a positive function and* $f(\bar{u}) \geq 0$ *on* $\mathbb{R}^N$, $f(\bar{u}) \not\equiv 0$, *then* u_a *is positive as well.*

The proof of Theorem 1.2 is based on a general shadowing lemma, a simple consequence of Banach's fixed point theorem, applied to approximate zeros of the gradient of the extended Lagrangian G_α for the constrained variational problem on Σ_α. If $\bar{u}$ is a nondegenerate local minimum of Φ on $\Sigma_{\alpha/n}$ then it is easy to see that the sum $\tilde{u}$ of n translates of $\bar{u}$ is an approximate zero of ∇G_α if these translates are far enough apart from each other. The shadowing lemma implies that to obtain a zero of ∇G_α near $\tilde{u}$ it is sufficient to prove that $\mathrm{D}^2 G_\alpha(\tilde{u})$ is invertible and that the norm of its inverse is bounded appropriately. This step is the main difficulty and requires the assumption of full nondegeneracy of $\bar{u}$.

Our next result is concerned with the Morse index of the solutions u_a given in Theorem 1.2 with respect to the functional $\Phi|_{\Sigma_\alpha}$. For this we recall that the Morse index $m(u)$ of a critical point u of $\Phi|_{\Sigma_\alpha}$ with Lagrangian multiplier λ is defined as the maximal dimension of a subspace $W \subset T_u\Sigma_\alpha$ such that the quadratic form in (1-4) is negative definite on W. If such a maximal dimension does not exist, one sets $m(u) = \infty$. We also introduce the following additional assumption:

(H4) $f(s)/|s|$ is nondecreasing in $\mathbb{R}$ and $f(s)s > 0$ for all $s \neq 0$.

Theorem 1.3. *Assume* (H1)–(H3), *fix* $\alpha > 0$, $n \in \mathbb{N}$, $n \geq 2$, *and suppose that* $\bar{u}$ *is a fully nondegenerate critical point of* $\Phi|_{\Sigma_{\alpha/n}}$ *with Lagrangian multiplier* $\bar{\lambda}$ *and finite Morse index* $m(\bar{u})$. *Moreover*, *let* $z_{\bar{u}}$ *be given as in Definition 1.1 with* $u = \bar{u}$. *Then the critical points* u_a *found in Theorem 1.2 have*, *for small* ε, *the following Morse index* $m(u_a)$ *with respect to* $\Phi|_{\Sigma_\alpha}$:

$$m(u_a) = \begin{cases} n(m(\bar{u}) + 1) - 1 & \text{if } (\bar{u}, z_{\bar{u}})_2 < 0, \\ nm(\bar{u}) & \text{if } (\bar{u}, z_{\bar{u}})_2 > 0. \end{cases}$$

If moreover (H4) *holds true*, *then* $m(u_a) > 0$.

The key role of the sign of the scalar product $(\bar{u}, z_{\bar{u}})_2$ in this theorem is not surprising since it is closely related to variational properties of the underlying critical point $\bar{u}$. More precisely, we shall see in Lemma 2.6 below that it determines the relationship between the Morse index of $\bar{u}$ with respect to $\Phi|_{\Sigma_{\alpha/n}}$ and its *free* Morse index with respect to the functional $u \mapsto \Phi(u) - \bar{\lambda}|u|_2^2$ on $H^1(\mathbb{R}^N)$.

We now consider the special case where (H4) holds true and $\bar{u}$ is a nondegenerate local minimum of $\Phi|_{\Sigma_{\alpha/n}}$. By a nondegenerate local minimum we mean a critical point $\bar{u}$ of $\Phi|_{\Sigma_{\alpha/n}}$ with Lagrangian multiplier $\bar{\lambda}$ such that the quadratic form in (1-4) is positive definite on $T_u\Sigma_{\alpha/n}$. In this case, we shall see in Section 2 below that $\bar{u}$ is fully nondegenerate, and we will deduce the following corollary from Theorems 1.2 and 1.3 in Section 4.

Corollary 1.4. *Assume* (H1)–(H4) *and fix* $\alpha > 0$, $n \in \mathbb{N}$, $n \geq 2$. *Moreover*, *suppose that* $\bar{u}$ *is a nondegenerate local minimum of* $\Phi|_{\Sigma_{\alpha/n}}$ *with Lagrangian multiplier* $\bar{\lambda}$. *Then for every* $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that for every* $a \in (\mathbb{Z}^N)^n$ *with* $d(a) \geq R_\varepsilon$ *there is a critical point* u_a *of* $\Phi|_{\Sigma_\alpha}$ *with Lagrange multiplier* λ_a *such that*

$$\Big\| u_a - \sum_{i=1}^n \mathcal{T}_{a^i}\bar{u} \Big\|_{H^1(\mathbb{R}^N)} \leq \varepsilon \quad \text{and} \quad |\lambda_a - \bar{\lambda}| \leq \varepsilon.$$

If ε *is chosen small enough then* u_a *is unique. Moreover*, u_a *does not change sign and has Morse index* $m(u_a) = n - 1$ *with respect to* $\Phi|_{\Sigma_\alpha}$.

Next we present an example where the nondegeneracy hypotheses of the previous theorems can be verified. For this we make the following assumptions:

(H5) $V \in C^2(\mathbb{R}^N)$ is 1-periodic in all coordinates, positive, and has a nondegenerate critical point at some point $x_0 \in \mathbb{R}^N$.

(H6) $f(s) = |s|^{p-2}s$ for some $p \in (2, 2^*)\backslash\{2+4/N\}$.

We then consider the constrained singularly perturbed equation

$$-\varepsilon^2\Delta u + V(x)u - |u|^{p-2}u = \lambda u, \quad u \in H^1(\mathbb{R}^N), \quad |u|_2^2 = \alpha, \tag{$P_{\alpha,\varepsilon}$}$$

in the semiclassical limit $\varepsilon \to 0$. Its weak solutions correspond, for each $\varepsilon > 0$, to critical points and Lagrange multipliers of the restriction of the functional

$$\Phi_\varepsilon\colon H^1(\mathbb{R}^N) \to \mathbb{R}, \quad \Phi_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N}(\varepsilon^2|\nabla u|^2 + Vu^2) - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p,$$

to Σ_α. We also consider the related free problem

$$-\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{F_ε}$$

whose weak solutions coincide with critical points of Φ_ε for every $\varepsilon > 0$. It is well known, see [Grossi 2002], that there exists a locally unique curve of solutions of (F_ε) that concentrate near x_0 as $\varepsilon \to 0$. For our purposes we need to show additional properties of these solutions.

Theorem 1.5. *Assume* (H5) *and* (H6). *Then there exist* $\varepsilon_0 > 0$ *and a continuous map* $(0, \varepsilon_0) \to H^1(\mathbb{R}^N)$, $\varepsilon \to \bar{u}_\varepsilon$, *such that the following properties hold true*:

(i) *For each* $\varepsilon \in (0, \varepsilon_0)$ *the function* $\bar{u}_\varepsilon$ *is a positive solution of* (F_ε).

(ii) *As* $\varepsilon \to 0$, *the functions* $x \mapsto \bar{u}_\varepsilon$ *concentrate near* x_0 *in the sense that the functions* $x \mapsto \bar{u}_\varepsilon(x_0 + \varepsilon x)$ *converge in* $H^1(\mathbb{R}^N)$ *to the unique radial positive solution* $u_0 \in H^1(\mathbb{R}^N)$ *of the equation*

$$-\Delta u_0 + V(x_0)u_0 = u_0^{p-1}$$

in $\mathbb{R}^N$.

(iii) $|\bar{u}_\varepsilon|_2^2 \to 0$ *as* $\varepsilon \to 0$.

(iv) *For each* $\varepsilon \in (0, \varepsilon_0)$ *the function* $\bar{u}_\varepsilon$ *is a fully nondegenerate critical point of the restriction of* Φ_ε *to* $\Sigma_{|\bar{u}_\varepsilon|_2^2}$ *with Morse index*

$$m(\bar{u}_\varepsilon) = \begin{cases} m_V & \text{if } 2 < p < 2 + \dfrac{4}{N}, \\ m_V + 1 & \text{if } 2 + \dfrac{4}{N} < p < 2^*. \end{cases} \tag{1-5}$$

Here m_V *denotes the number of negative eigenvalues of the Hessian of* V *at* x_0.

We emphasize that properties (i)–(ii) were already proved in [Grossi 2002], and that (iii) follows from (ii) by a simple change of variable. For our purposes, the property (iv) is of key importance. We shall also see in Section 5 below that, for $\varepsilon \in (0, \varepsilon_0)$,

$$(\bar{u}_\varepsilon, z_{\bar{u}_\varepsilon})_2 < 0 \quad \text{if } 2 < p < 2 + \frac{4}{N} \qquad \text{and} \qquad (\bar{u}_\varepsilon, z_{\bar{u}_\varepsilon})_2 > 0 \quad \text{if } 2 + \frac{4}{N} < p < 2^*, \tag{1-6}$$

where $z_{\bar{u}_\varepsilon}$ is given as in Definition 1.1 corresponding to $u = \bar{u}_\varepsilon$. Since the solutions $\bar{u}_\varepsilon$ in Theorem 1.5 depend continuously on ε and $|\bar{u}_\varepsilon|_2^2 \to 0$ as $\varepsilon \to 0$, we can find, for every $\alpha > 0$ and large enough $n \in \mathbb{N}$, a number $\varepsilon_n \in (0, \varepsilon_0)$ such that $|\bar{u}_{\varepsilon_n}|_2^2 = \alpha/n$. The combination of Theorems 1.2, 1.3 and 1.5 with (1-6) therefore yields the following corollary.

Corollary 1.6. *Assume* (H5) *and* (H6). *Then for every $\alpha > 0$ there exist $n_\alpha \in \mathbb{N}$ and a sequence $\varepsilon_n \to 0$ such that for every $n \ge n_\alpha$ the problem P_{α,ε_n} has infinitely many geometrically distinct positive solutions. More precisely, for every $n \in \mathbb{N}$ with $n \ge n_\alpha$, and every $\delta > 0$ there exists $R_{\delta,n} > 0$ such that for every $a \in (\mathbb{Z}^N)^n$ with $d(a) \ge R_{\delta,n}$ there is a critical point u_a of $\Phi_{\varepsilon_n}|_{\Sigma_\alpha}$ with Lagrange multiplier λ_a such that*

$$\Big\| u_a - \sum_{i=1}^{n} \mathcal{T}_{a^i} \bar{u}_{\varepsilon_n} \Big\|_{H^1(\mathbb{R}^N)} \le \delta \quad \text{and} \quad |\lambda_a| \le \delta.$$

If δ is chosen small enough then u_a is unique. Moreover, u_a is a positive function, and its Morse index with respect to $\Phi|_{\Sigma_\alpha}$ is given by

$$m(u_a) = \begin{cases} n(m_V + 1) - 1 & \text{if } 2 < p < 2 + \dfrac{4}{N}, \\ n(m_V + 1) & \text{if } 2 + \dfrac{4}{N} < p < 2^*, \end{cases}$$

where m_V denotes the number of negative eigenvalues of the Hessian of V at x_0.

Our next result is concerned with the orbital instability of the normalized multibump solutions we have constructed in the previous theorems. For this we focus on odd nonlinearities f in (P_α) satisfying (H3) and therefore assume

(H7) the function f is odd.

We also assume (H1) and (H3), so Φ in (1-2) is a well-defined C^2-functional. If $\varphi \in \Sigma_\alpha$ is a critical point of $\Phi|_{\Sigma_\alpha}$ with Lagrangian multiplier λ, then the function

$$u_\varphi \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \quad u_\varphi(t, x) = \varphi(x) e^{i\lambda t}, \tag{1-7}$$

is a solution of the time-dependent nonlinear Schrödinger equation

$$-i u_t = -\Delta u + V(x) u - g(|u|^2) u, \tag{1-8}$$

where g is defined by $f(t) = g(|t|^2)t$. Solutions of this special type are usually called solitary wave solutions. The solution u_φ is called *orbitally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $u \colon [0, t_0) \to H^1(\mathbb{R}^N, \mathbb{C})$ of (1-8) with $\|u(0, \cdot) - \varphi\|_{H^1} < \delta$ can be extended to a solution $[0, \infty) \to H^1(\mathbb{R}^N, \mathbb{C})$ which satisfies

$$\sup_{0<t<\infty} \inf_{s\in\mathbb{R}} \|u(t, \cdot) - u_\varphi(s, \cdot)\|_{H^1} < \varepsilon.$$

Otherwise, u_φ is called *orbitally unstable*. We then have the following result.

Theorem 1.7. *Assume* (H1), (H3), *and* (H7), *and suppose that $\varphi \in \Sigma_\alpha$ is a positive function which is a critical point of $\Phi|_{\Sigma_\alpha}$ with positive Morse index and Lagrangian multiplier $\lambda < \inf \sigma_{\mathrm{ess}}(-\Delta + V)$. Then the corresponding solitary wave solution u_φ of* (1-8) *is orbitally unstable.*

Here and in the following, $\sigma_{\text{ess}}(-\Delta + V)$ denotes the essential spectrum of the Schrödinger operator $-\Delta + V$. We note that Theorem 1.7 neither requires periodicity of V, nor does it require the assumption on the oddness of a certain difference of numbers of eigenvalues in the seminal instability result in [Grillakis et al. 1990, p. 309]. Theorem 1.7 applies to the normalized multibump solutions constructed in Theorem 1.2 and Corollaries 1.4 and 1.6 in the case where the nonlinearity satisfies (H4) and (H7). In these cases, the extra assumption $\lambda < \inf \sigma_{\text{ess}}(-\Delta + V)$ follows from Lemma 2.9 below and the fact that the Lagrangian multipliers of the multibump solutions are arbitrarily close to the multiplier of the initial solution.

There are many results on the orbital stability and instability of the standing waves generated by solutions to (P_α); see [Ianni and Le Coz 2009; Stuart 2008; Hilligsøe et al. 2002; Grillakis et al. 1987; Cazenave and Lions 1982]. However, none of these results covers the situation addressed in Theorem 1.7.

The paper is organized as follows. In Section 2 we collect some preliminary notions and observations. In particular, here we explain our new notions of fully nondegenerate restricted critical point and of the free Morse index. In Section 3 we then prove Theorem 1.2. In Section 4 we derive a general result on the Morse index of normalized multibump solutions which gives rise to Theorem 1.3. At the end of this section, we also complete the proof of Corollary 1.4. In Section 5, we analyze the singular perturbed equation (F_ε) and we prove Theorem 1.5. In Section 6, we then prove the orbital instability result given in Theorem 1.7. Finally, in the Appendix we provide a computation of the free Morse index of the solutions u_ε considered in Theorem 1.5. This computation is partly contained in [Lin and Wei 2008, proof of Theorem 2.5], but some details have been omitted there. We therefore provide a somewhat different argument in detail for the convenience of the reader.

We finally remark that the main results of our paper can be extended to more general nonlinearities. In particular, Theorem 1.2 has an abstract proof that extends to nonlinearities that also depend on x, 1-periodically in every coordinate. This proof also extends to nonlocal nonlinearities with convolution terms as in [Ianni and Le Coz 2009]. This follows from Brézis–Lieb-type splitting properties for these nonlinearities that were proved in [Ackermann 2006].

Notation. In the remainder of the paper, we write $|\cdot|_p$ for the standard $L^p(\mathbb{R}^N)$-norm, $1 \le p \le \infty$. We also use the notation $(\,\cdot\,,\cdot\,)_2$ for the standard $L^2(\mathbb{R}^N)$-scalar product. For the sake of brevity, we write L^2 in place of $L^2(\mathbb{R}^N)$ and H^k in place of $H^k(\mathbb{R}^N)$ for $k \in \mathbb{N}$. By (H1), $-\Delta + V$ is a self adjoint operator in L^2 with domain H^2. Since we assume (H1) throughout the paper and λ is a free parameter in (P_α), we may assume without loss of generality that $\gamma := \min \sigma(-\Delta + V) > 0$, where $\sigma(-\Delta + V)$ stands for the spectrum of $-\Delta + V$. Then H^1 is the form domain (the energy space) of $-\Delta + V$, and we may endow H^1 with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + Vuv), \quad u, v \in H^1. \tag{1-9}$$

The norm $\|\cdot\|$ induced by $\langle\,\cdot\,,\cdot\,\rangle$ is equivalent to the standard norm on H^1. It will be convenient to define $S := (-\Delta + V)^{-1}$; then we have

$$\langle u, v\rangle = (S^{-1/2}u, S^{-1/2}v)_2 \quad \text{for } u, v \in H^1. \tag{1-10}$$

We point out that, for a subspace $Z \subset H^1$, the notation $Z^\perp$ always refers to the orthogonal complement of Z in H^1 with respect to the scalar product $\langle \cdot, \cdot \rangle$.

We recall that the spectrum $\sigma(-\Delta + V)$ is purely essential if (H2) is assumed. In this case, it also follows that all powers of S are equivariant with respect to the action of $\mathbb{Z}^N$. Hence

$$\langle \mathcal{T}_a v, \mathcal{T}_a w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in H^1, \text{ for all } a \in \mathbb{Z}^N.$$

For any two normed spaces X, Y the space of bounded linear operators from X in Y is denoted by $\mathcal{L}(X, Y)$, and we write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

For a C^1-functional Θ defined on H^1, we let $\mathrm{d}\Theta \colon H^1 \to (H^1)^*$ denote the derivative of Θ and $\nabla\Theta \colon H^1 \to H^1$ the gradient with respect to the scalar product $\langle \cdot, \cdot \rangle$ defined in (1-9). Moreover, if Θ is of class C^2, then $\mathrm{d}^2\Theta(u) \colon H^1 \times H^1 \to \mathbb{R}$ denotes the Hessian of Θ at a point $u \in H^1$, whereas $\mathrm{D}^2\Theta(u) \in \mathcal{L}(H^1)$ stands for the derivative of the gradient of Θ at u. We then have

$$\langle \mathrm{D}^2\Theta(u)v, w \rangle = \mathrm{d}^2\Theta(u)[v, w] \quad \text{for } v, w \in H^1.$$

2. Some preliminary abstract results and notions

We now state some abstract results which will be used in Section 3 in the proof of Theorem 1.2. We start with a standard corollary of Banach's fixed point theorem, which is sometimes referred to as a *shadowing lemma*.

Lemma 2.1. *Let $(E, \|\cdot\|)$ be a Banach space, let $h \colon E \to E$ be continuously differentiable with derivative $\mathrm{d}h \colon E \to \mathcal{L}(E)$, and let $v_0 \in E$, $\delta > 0$, $q \in (0, 1)$ satisfy the following*:

(i) *$T := \mathrm{d}h(v_0) \in \mathcal{L}(E)$ is an isomorphism.*

(ii) *$\|h(v_0)\| < \delta(1-q)/\|T^{-1}\|_{\mathcal{L}(E)}$.*

(iii) *$\|\mathrm{d}h(y) - T\|_{\mathcal{L}(E)} \le q/\|T^{-1}\|_{\mathcal{L}(E)}$ for $y \in B_\delta(v_0)$.*

Then h has a unique zero in $B_\delta(v_0)$.

The proof of this lemma is standard by showing that the map $y \mapsto y - T^{-1}h(y)$ defines a q-contraction on $\overline{B_\delta(v_0)}$. Applying Banach's fixed point theorem to this map gives rise to a unique zero of h in $\overline{B_\delta(v_0)}$, and it easily follows from the above assumptions that this zero is contained in $B_\delta(v_0)$.

We will use the following immediate corollary of Lemma 2.1.

Corollary 2.2. *Let $(E, \|\cdot\|)$ be a Banach space, let $h \colon E \to E$ be differentiable and such that its derivative $\mathrm{d}h \colon E \to \mathcal{L}(E)$ is uniformly continuous on bounded subsets of E. Moreover, let $(v_k)_k$ be a bounded sequence in E such that*

(i) *$h(v_k) \to 0$ as $k \to \infty$;*

(ii) *$\mathrm{d}h(v_k) \in \mathcal{L}(E)$ is an isomorphism for $k \in \mathbb{N}$, and $\sup_{k\in\mathbb{N}} \|\mathrm{d}h(v_k)^{-1}\|_{\mathcal{L}(E)} < \infty$.*

Then there exist $k_0 \in \mathbb{N}$ and $u_k \in E$, $k \ge k_0$, with

$$h(u_k) = 0 \quad \text{for } k \ge k_0 \tag{2-1}$$

and

$$\|u_k - v_k\| \to 0 \quad \text{as } k \to \infty. \tag{2-2}$$

Moreover, the sequence $(u_k)_k$ *is uniquely determined by properties* (2-1), (2-2) *for large* k.

In the remainder of this section, we collect some preliminary results and notions related to the functional Φ defined in (1-2) and its restrictions to spheres with respect to the $L^2(\mathbb{R}^N)$-norm. Recall that we are assuming conditions (H1) and (H3). We define

$$\Psi(u) := \int_{\mathbb{R}^N} F(u),$$

so

$$\Phi(u) = \tfrac{1}{2}\|u\|^2 - \Psi(u).$$

Following [Ackermann 2006] we say that a map $g\colon X \to Y$ of Banach spaces X and Y *BL-splits* if $g(x_n) - g(x_n - x^*) \to g(x^*)$ in Y if $x_n \rightharpoonup x^*$ in X. For example, by [Ackermann 2006, Remark 3.3] the maps $\|\cdot\|^2$ and $|\cdot|_2^2$ BL-split. The next result about BL-splitting maps is less obvious:

Lemma 2.3. *The maps* Ψ, $\nabla\Psi$ *and* $\mathrm{D}^2\Psi$ *BL-split, and they are uniformly continuous on bounded subsets of* H^1.

Before we give the proof we fix some $p \in (2, 2^*)$ if $N \geq 3$ and we use p given in (H3) if $N = 1, 2$. Using (H3) it is easy to construct, for every $\varepsilon > 0$, functions $f_{i,\varepsilon} \in C^1(\mathbb{R})$, $i = 1, 2, 3$, and a constant $C_\varepsilon > 0$ such that

$$f = \sum_{i=1}^{3} f_{i,\varepsilon} \tag{2-3}$$

and such that

$$|f'_{1,\varepsilon}(s)| \leq \varepsilon, \quad |f'_{2,\varepsilon}(s)| \leq C_\varepsilon |s|^{p-2}, \quad \text{and} \quad |f'_{3,\varepsilon}(s)| \leq \varepsilon |s|^{2^*-2} \quad \text{for all } s \in \mathbb{R}. \tag{2-4}$$

If $N = 1, 2$ we simply choose $f_{3,\varepsilon} \equiv 0$ and ignore all terms that contain 2^*.

Proof of Lemma 2.3. We only prove this in the case $N \geq 3$; the other cases are treated similarly. Consider $(u_n) \subseteq H^1$ such that $u_n \rightharpoonup u$. Then (u_n) is bounded in H^1 and therefore also in L^q for $q \in [2, 2^*]$. For fixed $\varepsilon > 0$ we have

$$|f'_{2,\varepsilon}(u_n) - f'_{2,\varepsilon}(u_n - u) - f'_{2,\varepsilon}(u)|_{p/(p-2)} \to 0$$

by [Ackermann 2016, Theorem 1.3]. On the other hand, there are varying constants $C > 0$, independent of ε, such that

$$|f'_{1,\varepsilon}(u_n) - f'_{1,\varepsilon}(u_n - u) - f'_{1,\varepsilon}(u)|_\infty \leq C\varepsilon,$$
$$|f'_{3,\varepsilon}(u_n) - f'_{3,\varepsilon}(u_n - u) - f'_{3,\varepsilon}(u)|_{2^*/(2^*-2)} \leq C\varepsilon$$

for all n. For all $v, w \in H^1$ with $\|v\| = \|w\| = 1$ it follows that

$$\begin{aligned}
&\big|\big\langle\big(\mathrm{D}^2\Psi(u_n) - \mathrm{D}^2\Psi(u_n - u) - \mathrm{D}^2\Psi(u)\big)v, w\big\rangle\big| \\
&\qquad \leq C\varepsilon|v|_2|w|_2 + |f'_{2,\varepsilon}(u_n) - f'_{2,\varepsilon}(u_n - u) - f'_{2,\varepsilon}(u)|_{p/(p-2)}|v|_p|w|_p + C\varepsilon|v|_{2^*}|w|_{2^*} \\
&\qquad \leq C(\varepsilon + o(1))
\end{aligned}$$

and hence $\limsup_{n\to\infty}\|D^2\Psi(u_n)-D^2\Psi(u_n-u)-D^2\Psi(u)\|_{\mathcal{L}(H^1)}\le C\varepsilon$. Letting $\varepsilon\to 0$ we obtain the claim for $D^2\Psi$. The proof for the uniform continuity of $D^2\Psi$ on bounded subsets of H^1 is similar. One treats the maps $\nabla\Psi$ and Ψ analogously. □

We shall need the following simple consequence of (H4).

Lemma 2.4. *If conditions* (H1) *and* (H3)–(H4) *hold true and* $u\in H^1\backslash\{0\}$ *satisfies* $\nabla\Phi(u)=\lambda Su$ *for some* $\lambda\in\mathbb{R}$, *then*

$$\langle (D^2\Phi(u)-\lambda S)u,u\rangle<0.$$

Proof. By (H3) and (H4), the map $s\mapsto f'(s)s^2-f(s)s$ is nonnegative in $\mathbb{R}$, and it is positive on a nonempty open subset of $(-\varepsilon,\varepsilon)\backslash\{0\}$ for every $\varepsilon>0$. Moreover, since $u\in H^1$ is a weak solution of

$$-\Delta u+[V(x)-\lambda]u=f(u)\quad \text{in } \mathbb{R}^N$$

by assumption, standard elliptic regularity shows that u is continuous and that $u(x)\to 0$ as $|x|\to\infty$. Consequently, we have

$$\begin{aligned}\langle D^2\Phi(u)u,u\rangle-\lambda\langle Su,u\rangle&=\langle D^2\Phi(u)u,u\rangle-\langle\nabla\Phi(u),u\rangle\\&=\langle\nabla\Psi(u),u\rangle-\langle D^2\Psi(u)u,u\rangle=\int_{\mathbb{R}^N}(f(u)u-f'(u)u^2)<0,\end{aligned}$$

as claimed. □

As before, for $\alpha>0$, we consider the sphere $\Sigma_\alpha\subset H^1$ as defined in (1-1), and we let $J_\alpha:\Sigma_\alpha\to\mathbb{R}$ denote the restriction of Φ to Σ_α. We note that, for $u\in\Sigma_\alpha$, the tangent space of Σ_α at u is given by

$$T_u\Sigma_\alpha=\{v\in H^1:(v,u)_2=0\}=\{v\in H^1:\langle v,Su\rangle=0\}\subset H^1, \tag{2-5}$$

where latter equality follows from (1-10). If u is a critical point of J_α, we have

$$\nabla\Phi(u)=\lambda Su \tag{2-6}$$

for some $\lambda\in\mathbb{R}$, the corresponding Lagrange multiplier. Moreover, the Hessian $d^2J_\alpha(u)$ is a well-defined quadratic form on $T_u\Sigma_\alpha$ given by

$$d^2J_\alpha(u)[v,w]=\langle D^2\Phi(u)v,w\rangle-\lambda\langle Sv,w\rangle\quad\text{for } v,w\in T_u\Sigma_\alpha. \tag{2-7}$$

For the general definition of the Hessian of C^2-functionals on Banach manifolds at critical points, see, e.g., [Palais 1963, p. 307]. To see (2-7), one may argue with local coordinates for Σ_α at u, as is done, e.g., in [Edwards 1994, Theorem 8.9] in the finite-dimensional case. Alternatively, to prove (2-7) we may consider smooth vector fields $\tilde v,\tilde w$ on Σ_α with $\tilde v(u)=v,\ \tilde w(u)=w$, and we extend $\tilde v,\tilde w$ arbitrarily as smooth vector fields $\tilde v,\tilde w: H^1\to H^1$. Using (2-6), we then have

$$\begin{aligned}d^2J_\alpha(u)[v,w]&=\partial_{\tilde v}\partial_{\tilde w}\Phi(u)=\partial_{\tilde v}|_u\langle\nabla\Phi,w\rangle=\langle D^2\Phi(u)v,w\rangle+\langle\nabla\Phi(u),d\tilde w(u)v\rangle\\&=\langle D^2\Phi(u)v,w\rangle+\lambda(u,d\tilde w(u)v)_2=\langle D^2\Phi(u)v,w\rangle-\lambda(v,w)_2,\end{aligned}$$

where the last equality follows from the fact that the function $u_*\mapsto h(u_*):=(u_*,\tilde w(u_*))_2$ vanishes on Σ_α and therefore $0=\partial_{\tilde v}h(u)=(v,w)_2+(u,dw(u)v)_2$.

We need the following definitions.

Definition 2.5. Let $u \in H^1$ be a critical point of J_α with Lagrange multiplier λ. Put $\Lambda := T_u \Sigma_\alpha$ and let $P \in \mathcal{L}(H^1, \Lambda)$ denote the $\langle \cdot\,, \cdot \rangle$-orthogonal projection onto Λ. Moreover, put $B := \mathrm{D}^2\Phi(u) - \lambda S$.

(a) The *Morse index* $m(u) \in \mathbb{N} \cup \{0, \infty\}$ of u with respect to J_α is defined as

$$m(u) := \sup\{\dim Z : Z \text{ subspace of } \Lambda \text{ with } \langle Bv, v\rangle < 0 \text{ for all } v \in Z\backslash\{0\}\}.$$

(b) The *free Morse index* $m_{\mathrm{f}}(u) \in \mathbb{N} \cup \{0, \infty\}$ of u is defined as

$$m_{\mathrm{f}}(u) := \sup\{\dim Z : Z \text{ subspace of } H^1 \text{ with } \langle Bv, v\rangle < 0 \text{ for all } v \in Z\backslash\{0\}\}.$$

(c) We call u a *nondegenerate* critical point of J_α if $PB|_\Lambda$ is an isomorphism of Λ.

(d) We call u *freely nondegenerate* if B is an isomorphism of H^1. In this case we put

$$z_u := B^{-1}Su \in H^1.$$

For a critical point $u \in H^1$ of J_α, it is clear that

$$m_{\mathrm{f}}(u) = m(u) \quad \text{or} \quad m_{\mathrm{f}}(u) = m(u) + 1. \tag{2-8}$$

In the case where u is freely nondegenerate, the scalar product $(z_u, u)_2$ determines whether u is nondegenerate and which case occurs in (2-8). More precisely, we have the following simple but important lemma.

Lemma 2.6. *Let $u \in H^1$ be a freely nondegenerate critical point of J_α with Lagrange multiplier λ:*

(a) *u is nondegenerate if and only if $(z_u, u)_2 \neq 0$.*

(b) *If $m(u)$ is finite and $(z_u, u)_2 > 0$, then $m_{\mathrm{f}}(u) = m(u)$.*

(c) *If $m(u)$ is finite and $(z_u, u)_2 < 0$, then $m_{\mathrm{f}}(u) = m(u) + 1$.*

Proof. In the following, we let $\mathcal{N}(L)$ denote the kernel and $\mathcal{R}(L)$ denote the range of a linear operator L. Moreover, we let B, P and Λ be as in Definition 2.5.

(a): By definition, we have $z_u = B^{-1}Su \in \mathcal{N}(PB) \setminus \{0\}$. Moreover, we have $\dim \mathcal{N}(PB) = 1$ since $B\colon H^1 \to H^1$ is an isomorphism. Consequently,

$$\mathcal{N}(PB) = \mathrm{span}(z_u) \quad \text{and} \quad \mathcal{R}(PB) = \Lambda.$$

Now, again by definition, u is nondegenerate if and only if $PB|_\Lambda\colon \Lambda \to \Lambda$ is an isomorphism, and this holds true if and only if $H^1 = \mathrm{span}(z_u) \oplus \Lambda$. By (2-5), the latter property is equivalent to $(z_u, u)_2 \neq 0$.

(b) and (c): Since $\operatorname{codim} \Lambda = 1$ and $z_u \notin \Lambda$, there are, for every $\phi \in H^1$, unique elements $\mu \in \mathbb{R}$ and $w \in \Lambda$ such that

$$\phi = \mu z_u + w. \tag{2-9}$$

Recall that $\operatorname{span}(Su) = \mathcal{N}(P) = \Lambda^{\perp}$. We therefore have the representation

$$\begin{aligned}\langle B\phi, \phi\rangle &= \mu^2\langle Bz_u, z_u\rangle + 2\mu\langle Bz_u, w\rangle + \langle Bw, w\rangle\\ &= \mu^2\langle Su, z_u\rangle + 2\mu\langle Su, w\rangle + \langle Bw, w\rangle\\ &= \mu^2(z_u, u)_2 + \langle Bw, w\rangle. \end{aligned} \tag{2-10}$$

To see (b), recall that the definition of $m(u)$ implies the existence of a subspace $Z \subset \Lambda$ of codimension $m(u)$ in Λ such that $\langle B\phi, \phi\rangle \geq 0$ for all $\phi \in Z$. Since $z_u \notin \Lambda$, the space $\widetilde{Z} := \operatorname{span}(z_u) \oplus Z$ has at most codimension $m(u)$ in H^1. Moreover, in the representation (2-9) for $\phi \in \widetilde{Z}$ we find $w \in Z$. Therefore, (2-10) yields $\langle B\phi, \phi\rangle \geq \langle Bw, w\rangle \geq 0$. This implies $m_{\mathrm{f}}(u) \leq m(u)$, and thus equality follows by (2-8).

To see (c), let $Z \subset \Lambda$ be an $m(u)$-dimensional subspace such that $\langle Bw, w\rangle < 0$ for all $w \in Z\backslash\{0\}$. Put $\widetilde{Z} := \operatorname{span}(z_u) \oplus Z$. Then $\dim \widetilde{Z} = m(u) + 1$, and for the representation (2-9) for $\phi \in \widetilde{Z}\backslash\{0\}$ we find $w \in Z$. Then (2-10) implies $\langle B\phi, \phi\rangle < 0$ since either $\mu \neq 0$ or $w \in Z\backslash\{0\}$. Consequently, $m_{\mathrm{f}}(u) \geq m(u) + 1$, and thus equality follows by (2-8). □

Parts (b) and (c) of Lemma 2.6 can also be derived from [Maddocks 1985, (2.7) of Theorem 2]. For the convenience of the reader we gave a simple direct proof.

Definition 2.7. A critical point $u \in H^1$ of J_α will be called *fully nondegenerate* if u is freely nondegenerate and the equivalent properties in Lemma 2.6(a) hold true.

Definition 2.7 is consistent with Definition 1.1, as the function $z_u = B^{-1}Su$ defined in Definition 2.5 is uniquely determined as the weak solution of (1-3) with $g = u$.

In the next lemma, we show that nondegenerate local minima of J_α are fully nondegenerate critical points.

Lemma 2.8. *Suppose that* (H4) *holds true, and let $u \in H^1$ be a nondegenerate critical point of J_α with $m(u) = 0$ (i.e., u is a nondegenerate local minimum of J_α). Then u is fully nondegenerate, and either u or $-u$ is a positive function.*

Proof. We continue using the notation from the proof of Lemma 2.6. Since u is nondegenerate, we have $\Lambda = \mathcal{R}(PB|_\Lambda)$ and therefore $H^1 = \mathcal{N}(P) + \mathcal{R}(B|_\Lambda)$. This implies $\operatorname{codim}\mathcal{R}(B) \leq \operatorname{codim}\mathcal{R}(B|_\Lambda) \leq 1$ and hence that $\mathcal{R}(B)$ is closed. Since $PB|_\Lambda$ is injective, $\mathcal{N}(B) \cap \Lambda = \{0\}$ and hence $\dim\mathcal{N}(B) \leq 1$. If $\dim\mathcal{N}(B) = 1$ were true, then we would have $H^1 = \mathcal{N}(B) \oplus \Lambda$. Since the quadratic form $\langle B\,\cdot\,, \cdot\,\rangle$ is positive definite on Λ, it would be positive semidefinite on H^1, in contradiction with Lemma 2.4. Therefore $\mathcal{N}(B) = \{0\}$ and B, being symmetric with closed range, is an isomorphism. Hence u is freely nondegenerate, and thus it is also fully nondegenerate.

Next, we suppose by contradiction that u changes sign. A variant of the proof of Lemma 2.4 then shows that the quadratic form $\langle B\,\cdot\,, \cdot\,\rangle$ is negative definite on the two-dimensional subspace $\operatorname{span}(u^+, u^-) \subset H^1$, where $u^\pm := \max\{0, \pm u\}$ denotes the positive, respectively negative, part of u. Since this space has a nontrivial intersection with Λ, we thus obtain a contradiction to the assumption $m(u) = 0$. □

Next we add an observation for the case where u is a fully nondegenerate critical point of J_α and a *positive* function.

Lemma 2.9. *Let $u \in H^1$ be a fully nondegenerate critical point of J_α with Lagrangian multiplier λ such that u is a positive function and $f(u) \geq 0$ on $\mathbb{R}^N$, $f(u) \not\equiv 0$. Then we have*

$$\lambda < \inf \sigma(-\Delta + V). \tag{2-11}$$

Proof. Since u is freely nondegenerate, we see that

$$\lambda \notin \sigma(-\Delta + V - f'(u)). \tag{2-12}$$

Moreover, $u(x) \to 0$ as $|x| \to \infty$ by standard elliptic estimates, and the same is true for the functions $x \mapsto f'(u(x))$, $x \mapsto f(u(x))/u(x)$. Consequently, by (2-12) and Theorem 14.6 and the proof of Theorem 14.9 in [Hislop and Sigal 1996], we have for $L_0 := -\Delta + V$ and $L := -\Delta + V - f(u)/u$ that

$$\lambda \notin \sigma_{\text{ess}}(-\Delta + V - f'(u)) = \sigma_{\text{ess}}(L_0) = \sigma_{\text{ess}}(L),$$

where σ_{ess} denotes the essential spectrum. Since u is an eigenfunction of the Schrödinger operator L corresponding to the eigenvalue λ, it follows that λ is isolated in $\sigma(L)$. Since moreover u is positive, it is then easy to see that $\lambda = \inf \sigma(L)$, and that λ is a simple eigenvalue. On the other hand, the assumption $f(u)/u \geq 0$ implies

$$\inf \sigma(L_0) \geq \inf \sigma(L) = \lambda.$$

If $\lambda = \inf \sigma(L_0)$ were true, we could obtain from $\lambda \notin \sigma_{\text{ess}}(L_0)$ that λ is also an isolated eigenvalue of L_0 with a positive eigenfunction v. But then, since $f(u) \not\equiv 0$ by assumption,

$$\lambda = \frac{\int_{\mathbb{R}^N} (|\nabla v|^2 + V v^2)}{\int_{\mathbb{R}^N} v^2} > \frac{\int_{\mathbb{R}^N} (|\nabla v|^2 + (V - f(u)/u) v^2)}{\int_{\mathbb{R}^N} v^2} \geq \lambda,$$

a contradiction. Hence $\lambda < \inf \sigma(L_0)$. □

We close this section by introducing the *extended Lagrangian*

$$G_\alpha \colon H^1 \times \mathbb{R} \to \mathbb{R}, \quad G_\alpha(u, \lambda) := \Phi(u) - \tfrac{1}{2}\lambda(|u|_2^2 - \alpha) = \Phi(u) - \tfrac{1}{2}\lambda(\langle Su, u\rangle - \alpha).$$

By definition, $u \in H^1$ is a critical point of J_α with Lagrange multiplier λ if and only if (u, λ) is a critical point of G_α. We endow $H^1 \times \mathbb{R}$ with the natural scalar product

$$\langle (u, s), (v, t)\rangle := \langle u, v\rangle + st.$$

The respective gradient of G_α is

$$\nabla G_\alpha \colon H^1 \times \mathbb{R} \to H^1 \times \mathbb{R}, \quad \nabla G_\alpha(u, \lambda) = \big(\nabla \Phi(u) - \lambda S u, -\tfrac{1}{2}(|u|_2^2 - \alpha)\big). \tag{2-13}$$

Moreover, we have

$$\mathrm{D}^2 G_\alpha(u, \lambda)[(v, \mu)] = (\mathrm{D}^2 \Phi(u) v - \lambda S v - \mu S u, -\langle Su, v\rangle). \tag{2-14}$$

The operator $\mathrm{D}^2 G_\alpha(u, \lambda)$ is known in the literature as the *bordered Hessian* of Φ at (u, λ). It has been used extensively in finite-dimensional settings to discern local extrema of restricted functionals; see, e.g., [Greenberg et al. 2000; Shutler 1995; Hassell and Rees 1993; Hughes 1991; Spring 1985; Baxley and Moorhouse 1984]. We will use it only in Section 3 below for a gluing procedure respecting an L^2-constraint.

Although we do not need this property in the present paper, we note that a critical point $u \in H^1$ of J_α is nondegenerate if and only if $D^2 G_\alpha(u, \lambda)$ is an isomorphism of $H^1 \times \mathbb{R}$. The proof is straightforward.

3. Gluing bumps with L^2-constraint

This section is devoted to the proof of Theorem 1.2, which we reformulate in the following way for matters of convenience. We continue to use the notation introduced in Section 2.

Theorem 3.1. *Assume* (H1)–(H3) *and fix* $\alpha > 0$. *Given* $n \in \mathbb{N}$, $n \geq 2$, *suppose that* $\bar{u}$ *is a fully nondegenerate critical point of* $J_{\alpha/n}$ *with Lagrange multiplier* $\bar{\lambda}$. *Let also* $(a_k) \subseteq (\mathbb{Z}^N)^n$ *be a sequence such that* $d(a_k) \to \infty$ *as* $k \to \infty$. *Then there exists* $k_0 \in \mathbb{N}$ *such that for* $k \geq k_0$ *there exist critical points* u_k *of* J_α *with Lagrange multiplier* λ_k. *Moreover, we have*

$$\|u_k - v_k\| \to 0 \quad \textit{and} \quad |\lambda_k - \bar{\lambda}| \to 0 \quad \textit{as } k \to \infty, \quad \textit{where } v_k := \sum_{i=1}^{n} \mathcal{T}_{a_k^i} \bar{u} \in H^1, \tag{3-1}$$

and the sequence $(u_k)_k$ *is uniquely determined by these properties for large* k. *Furthermore, if* $\bar{u}$ *is a positive function and* $f(\bar{u}) \geq 0$ *on* $\mathbb{R}^N$, $f(\bar{u}) \not\equiv 0$, *then* u_k *is positive as well for large* k.

The remainder of this section is devoted to the proof of this theorem. Let $\alpha > 0$, $n \geq 2$, and $\bar{u}$, $\bar{\lambda}$ be as in the statement of the theorem. Since $\bar{u}$ is nondegenerate and freely nondegenerate, Definition 2.5 and Definition 2.7 imply

$$B := D^2\Phi(\bar{u}) - \bar{\lambda} S \in \mathcal{L}(H^1) \text{ is an isomorphism} \tag{3-2}$$

and

$$\text{there exists } z_{\bar{u}} \in H^1 \text{ with } (z_{\bar{u}}, \bar{u})_2 \neq 0 \text{ and } B z_{\bar{u}} = S\bar{u}. \tag{3-3}$$

Let $(a_k) \subseteq (\mathbb{Z}^N)^n$ be a sequence such that $d(a_k) \to \infty$ as $k \to \infty$, and let $v_k \in H^1$ be given as in (3-1) for $k \in \mathbb{N}$. For simplicity we assume that

$$a_k^1 = 0 \quad \text{for all } k \in \mathbb{N}. \tag{3-4}$$

We wish to prove that

$$\nabla G_\alpha(v_k, \bar{\lambda}) \to 0 \quad \text{as } k \to \infty \tag{3-5}$$

and that

$$\begin{gathered} D^2 G_\alpha(v_k, \bar{\lambda}) \in \mathcal{L}(H^1 \times \mathbb{R}) \text{ is invertible for large } k, \\ \text{and the norm of the inverse remains bounded as } k \to \infty. \end{gathered} \tag{3-6}$$

Once these assertions are proved, we may apply Corollary 2.2 with $h := \nabla G_\alpha$ to find, for k large, critical points u_k of J_α with Lagrange multiplier λ_k such that (3-1) holds true. Here we use the fact that the sequence $(v_k)_k$ is bounded in H^1 and that $D^2\Phi$ is uniformly continuous on bounded subsets of H^1.

By the BL-splitting properties, (2-13) implies

$$\left\| \nabla G_\alpha(v_k, \bar{\lambda}) - \sum_{i=1}^{n} \nabla G_{\alpha/n}(\mathcal{T}_{a_k^i} \bar{u}, \bar{\lambda}) \right\|_{\mathcal{L}(H^1 \times \mathbb{R})} \to 0.$$

Since $\|\nabla G_{\alpha/n}(\mathcal{T}_{a_k^i}\bar{u},\bar{\lambda})\|_{\mathcal{L}(H^1\times\mathbb{R})} = \|\nabla G_{\alpha/n}(\bar{u},\bar{\lambda})\|_{\mathcal{L}(H^1\times\mathbb{R})} = 0$ for $i = 1, 2, \dots, n$ and every k, (3-5) follows.

We now turn to the (more difficult) proof of (3-6). For this we consider the operators

$$B_k := \mathrm{D}^2\Phi(v_k) - \bar{\lambda}S \in \mathcal{L}(H^1) \quad \text{for } k \in \mathbb{N}.$$

and we claim that

$$\mathcal{T}_{-a_k^i} B_k \mathcal{T}_{a_k^i} w \to Bw \quad \text{in } H^1 \text{ for } w \in H^1,\ i = 1, 2, \dots, n. \tag{3-7}$$

To see this, we recall that $\mathrm{D}^2\Psi$ BL-splits and that therefore

$$\mathrm{D}^2\Psi(v_k) = \sum_{j=1}^{n} \mathrm{D}^2\Psi(\mathcal{T}_{a_k^j}\bar{u}) + o(1) \quad \text{in } \mathcal{L}(H^1), \tag{3-8}$$

which implies

$$B_k = I - \bar{\lambda}S - \mathrm{D}^2\Psi(v_k) = I - \bar{\lambda}S - \sum_{j=1}^{n} \mathrm{D}^2\Psi(\mathcal{T}_{a_k^j}\bar{u}) + o(1) \quad \text{in } \mathcal{L}(H^1). \tag{3-9}$$

It is easy to see that

$$\mathcal{T}_{-a_k^i}\mathrm{D}^2\Psi(\mathcal{T}_{a_k^i}\bar{u})\mathcal{T}_{a_k^i} = \mathrm{D}^2\Psi(\bar{u}) \quad \text{for } k \in \mathbb{N} \text{ and } i = 1, \dots, n. \tag{3-10}$$

Moreover, if $i \neq j$, then for $w \in H^1$ we have

$$\mathrm{D}^2\Psi(\mathcal{T}_{a_k^j}\bar{u})\mathcal{T}_{a_k^i}w = \mathcal{T}_{a_k^j}\mathcal{T}_{-a_k^j}\mathrm{D}^2\Psi(\mathcal{T}_{a_k^j}\bar{u})\mathcal{T}_{a_k^j}\mathcal{T}_{a_k^i-a_k^j}w = \mathcal{T}_{a_k^j}\mathrm{D}^2\Psi(\bar{u})\mathcal{T}_{a_k^i-a_k^j}w \to 0 \tag{3-11}$$

in H^1, since $\mathcal{T}_{a_k^i-a_k^j}w \rightharpoonup 0$ and $\mathrm{D}^2\Psi(\bar{u}) \in \mathcal{L}(H^1)$ is a compact operator. Combining (3-9)–(3-11) and recalling that S commutes with $\mathcal{T}_{a_k^i}$, we find that

$$\begin{aligned}\mathcal{T}_{-a_k^i}B_k\mathcal{T}_{a_k^i}w &= (I - \bar{\lambda}S)w - \sum_{j=1}^{n}\mathcal{T}_{-a_k^i}\mathrm{D}^2\Psi(\mathcal{T}_{a_k^j}\bar{u})\mathcal{T}_{a_k^i}w + o(1)\\ &= (I - \bar{\lambda}S)w - \mathrm{D}^2\Psi(\bar{u})w + o(1) = Bw + o(1) \quad \text{as } k \to \infty\end{aligned}$$

for $w \in H^1$ and $i = 1, \dots, n$, as claimed in (3-7).

We note that (3-7) implies

$$\mathcal{T}_{-a_k^i}B_k\mathcal{T}_{a_k^j}w = \mathcal{T}_{a_k^j-a_k^i}\mathcal{T}_{-a_k^j}B_k\mathcal{T}_{a_k^j}w = \mathcal{T}_{a_k^j-a_k^i}Bw + o(1) \rightharpoonup 0 \quad \text{in } H^1 \tag{3-12}$$

for $w \in H^1$ and $i \neq j$. We now prove (3-6) by contradiction. Supposing that (3-6) does not hold true, we find, after passing to a subsequence, that there are $w_k \in H^1$ and $\mu_k \in \mathbb{R}$ such that $\|w_k\|^2 + \mu_k^2 = 1$ and $\mathrm{D}^2G_\alpha(v_k,\bar{\lambda})[(w_k,\mu_k)] \to 0$. By (2-14) this implies

$$B_k w_k - \mu_k S v_k \to 0 \quad \text{in } H^1, \tag{3-13}$$

$$(v_k, w_k)_2 \to 0 \quad \text{in } \mathbb{R}. \tag{3-14}$$

Define for $i = 1, 2, \dots, n$, possibly after passing to a subsequence, the functions

$$w^i := \operatorname*{w-lim}_{k\to\infty}\mathcal{T}_{-a_k^i}w_k \in H^1$$

and $\mu := \lim_{k\to\infty} \mu_k$. Let $z_{\bar u} \in H^1$ be given as in (3-3). Forming the H^1-scalar product of (3-13) with $\mathcal{T}_{a_k^i} z_{\bar u}$ and using (3-7) together with the fact that $\mathcal{T}_{-a_k^i} v_k \rightharpoonup \bar u$ in H^1, we obtain

$$\begin{aligned} o(1) &= \langle B_k w_k, \mathcal{T}_{a_k^i} z_{\bar u}\rangle - \mu_k \langle S v_k, \mathcal{T}_{a_k^i} z_{\bar u}\rangle = \langle w_k, B_k \mathcal{T}_{a_k^i} z_{\bar u}\rangle - \mu_k (v_k, \mathcal{T}_{a_k^i} z_{\bar u})_2 \\ &= \langle \mathcal{T}_{-a_k^i} w_k, \mathcal{T}_{-a_k^i} B_k \mathcal{T}_{a_k^i} z_{\bar u}\rangle - \mu_k (\mathcal{T}_{-a_k^i} v_k, z_{\bar u})_2 = \langle w^i, B z_{\bar u}\rangle - \mu(\bar u, z_{\bar u})_2 + o(1) \\ &= \langle w^i, S\bar u\rangle - \mu(\bar u, z_{\bar u})_2 + o(1) = (w^i, \bar u)_2 - \mu(\bar u, z_{\bar u})_2 + o(1) \end{aligned}$$

for $i = 1, \dots, n$. Hence

$$(w^i, \bar u)_2 = \mu(\bar u, z_{\bar u})_2 \quad \text{for } i = 1, \dots, n.$$

By (3-14) we thus have

$$0 = \lim_{k\to\infty} (v_k, w_k)_2 = \lim_{k\to\infty} \sum_{i=1}^n (\mathcal{T}_{a_k^i} \bar u, w_k)_2 = \lim_{k\to\infty} \sum_{i=1}^n (\bar u, \mathcal{T}_{-a_k^i} w_k)_2 = \sum_{i=1}^n (\bar u, w^i)_2 = n\mu(\bar u, z_{\bar u})_2.$$

Since $(\bar u, z_{\bar u})_2 \neq 0$, this gives $\mu = 0$. Hence (3-13) reduces to

$$B_k w_k \to 0 \quad \text{in } H^1 \text{ as } k \to \infty. \tag{3-15}$$

We now set

$$z_k := w_k - \sum_{j=1}^n \mathcal{T}_{a_k^j} w^j \quad \text{for } k \in \mathbb{N},$$

so

$$\mathcal{T}_{-a_k^i} z_k \rightharpoonup 0 \quad \text{for } i = 1, \dots, n. \tag{3-16}$$

By (3-7), (3-12) and (3-15) we have

$$\begin{aligned} 0 = \underset{k\to\infty}{\text{w-lim}}\, \mathcal{T}_{-a_k^i} B_k w_k &= \underset{k\to\infty}{\text{w-lim}} \Bigl[\sum_{j=1}^n \mathcal{T}_{-a_k^i} B_k \mathcal{T}_{a_k^j} w^j + \mathcal{T}_{-a_k^i} B_k z_k\Bigr] \\ &= B w^i + \underset{k\to\infty}{\text{w-lim}}\, \mathcal{T}_{-a_k^i} B_k z_k. \end{aligned} \tag{3-17}$$

Moreover,

$$D^2\Psi(\bar u) \mathcal{T}_{-a_k^i} z_k \to 0 \quad \text{in } H^1 \text{ for } i = 1, \dots, n \tag{3-18}$$

by (3-16) and since $D^2\Psi(\bar u) \in \mathcal{L}(H^1)$ is a compact operator, which by (3-10) implies

$$\mathcal{T}_{-a_k^i} D^2\Psi(\mathcal{T}_{a_k^j} \bar u) z_k = \mathcal{T}_{a_k^j - a_k^i} D^2\Psi(\bar u) \mathcal{T}_{-a_k^j} z_k \to 0 \quad \text{in } H^1 \tag{3-19}$$

for $i, j = 1, \dots, n$. Using (3-9) again, we obtain

$$\begin{aligned} \underset{k\to\infty}{\text{w-lim}}\, \mathcal{T}_{-a_k^i} B_k z_k &= \underset{k\to\infty}{\text{w-lim}} \Bigl(\mathcal{T}_{-a_k^i} (I - \bar\lambda S) z_k - \sum_{j=1}^n \mathcal{T}_{-a_k^i} D^2\Psi(\mathcal{T}_{a_k^j} \bar u) z_k\Bigr) \\ &= \underset{k\to\infty}{\text{w-lim}} (I - \bar\lambda S) \mathcal{T}_{-a_k^i} z_k = 0 \end{aligned}$$

for $i = 1, \dots, n$. Combining this with (3-17), we conclude that $Bw^i = 0$ for $i = 1, \dots, n$ and thus

$$w^i = 0 \quad \text{for } i = 1, \dots, n$$

by (3-2). We therefore have $w_k = z_k$ for all k. Recalling (3-15), (3-9), (3-4), and choosing $i = 1$ in (3-18) and (3-19), we find

$$o(1) = B_k w_k = B_k z_k = (I - \bar{\lambda} S) z_k - \sum_{j=1}^{n} \mathrm{D}^2 \Psi(\mathcal{T}_{a_k^j} \bar{u}) z_k + o(1) = (I - \bar{\lambda} S) z_k + o(1)$$
$$= (I - \bar{\lambda} S) z_k - \mathrm{D}^2 \Psi(\bar{u}) z_k + o(1) = B z_k + o(1) = B w_k + o(1),$$

and thus $w_k \to 0$ in H^1 by (3-2). Since $\mu = 0$, this contradicts our assumption that $\|w_k\|^2 + \mu_k^2 = 1$ for all k. This proves (3-6), as desired.

In the following we assume $N \geq 3$. The cases $N = 1, 2$ are proved similarly, ignoring those terms below that include the critical exponent 2^*.

As remarked above, applying Corollary 2.2 with $h := \nabla G_\alpha$ now yields, for k large, critical points u_k of J_α with Lagrange multiplier λ_k such that (3-1) holds true. To finish the proof of Theorem 3.1, we now assume that $\bar{u} \in H^1$ is positive with $f(\bar{u}) \geq 0$ in $\mathbb{R}^N$, $f(\bar{u}) \not\equiv 0$, and we show that u_k is also positive for k large. By Lemma 2.9 we then have $\bar{\lambda} < \inf \sigma(-\Delta + V) = \gamma$, so

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + [V - \bar{\lambda}]|v|^2) \geq (\gamma - \bar{\lambda}) \|v\|^2 \quad \text{for all } v \in H^1.$$

On the other hand, for fixed $\varepsilon \in (0, \gamma - \bar{\lambda})$ it easily follows from (H3), Sobolev embeddings, the representation (2-3), and (2-4), that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} f(v) v \leq \varepsilon \|v\|^2 + C \|v\|^p + \varepsilon \|v\|^{2^*} \quad \text{for } v \in H^1.$$

Moreover, since v_k is positive, (3-1) implies $u_k^- := \min\{u_k, 0\} \to 0$ in H^1 as $k \to \infty$. However, we have

$$0 = \int_{\mathbb{R}^N} (-\Delta u_k + [V - \lambda_k] u_k - f(u_k)) u_k^-$$
$$= \int_{\mathbb{R}^N} (|\nabla u_k^-|^2 + [V - \lambda_k]|u_k^-|^2) - \int_{\mathbb{R}^N} f(u_k^-) u_k^-$$

and therefore

$$(\gamma - \bar{\lambda}) \|u_k^-\|^2 \leq \int_{\mathbb{R}^N} (|\nabla u_k^-|^2 + [V - \bar{\lambda}]|u_k^-|^2)$$
$$= o(1) |u_k^-|_2^2 + \int_{\mathbb{R}^N} (|\nabla u_k^-|^2 + [V - \lambda_k]|u_k^-|^2)$$
$$= o(1) \|u_k^-\|^2 + \int_{\mathbb{R}^N} f(u_k^-) u_k^-$$
$$\leq (\varepsilon + o(1)) \|u_k^-\|^2 + C \|u_k^-\|^p + \varepsilon \|u_k^-\|^{2^*}.$$

By the choice of ε, this implies $u_k^- = 0$ for large k. Consequently, u_k is strictly positive on $\mathbb{R}^N$ for large k by the strong maximum principle. The proof of Theorem 3.1 is finished.

4. Morse index and nondegeneracy of normalized multibump solutions

In this section, we prove a general result on the nondegeneracy and the Morse index of normalized multibump solutions built from fully nondegenerate critical points of the restriction of Φ to $\Sigma_{\alpha/n}$. Moreover, we also complete the proof of Corollary 1.4 at the end of the section.

Recall, for $\alpha > 0$ and a critical point u of $J_\alpha = \Phi|_{\Sigma_\alpha}$, the definitions of the Morse index $m(u)$ and the free Morse index $m_{\mathrm{f}}(u)$ given in Definition 2.5. The following theorem is the main result of this section, and together with Lemma 2.6 it readily implies Theorem 1.3.

Theorem 4.1. *Assume* (H1)–(H3) *and fix* $\alpha > 0$. *Given* $n \in \mathbb{N}$, $n \geq 2$, *suppose that* $\bar{u}$ *is a fully nondegenerate critical point of* $J_{\alpha/n}$ *with Lagrange multiplier* $\bar{\lambda}$ *and finite Morse index* $m(\bar{u})$. *Furthermore, let* $(a_k) \subseteq (\mathbb{Z}^N)^n$ *be a sequence such that* $d(a_k) \to \infty$ *as* $k \to \infty$, *and such that the critical points* u_k *of* J_α *with Lagrange multiplier* λ_k *and with*

$$\|u_k - v_k\| \to 0 \quad \textit{and} \quad |\lambda_k - \bar{\lambda}| \to 0 \quad \textit{as } k \to \infty, \quad \textit{where } v_k := \sum_{i=1}^{n} \mathcal{T}_{a_k^i} \bar{u} \in H^1 \tag{4-1}$$

from Theorem 3.1 exist for all k. *Then, for* k *sufficiently large*, u_k *is a nondegenerate critical point of* J_α, $m(u_k) = n(m(\bar{u}) + 1) - 1$ *if* $(\bar{u}, z_{\bar{u}})_2 < 0$, *and* $m(u_k) = nm(\bar{u})$ *if* $(\bar{u}, z_{\bar{u}})_2 > 0$. *If* (H4) *holds true*, *then* $m(u_k) > 0$ *for large* k.

To prove this theorem, we set $B := \mathrm{D}^2\Phi(\bar{u}) - \bar{\lambda}S$ and $B_k := \mathrm{D}^2\Phi(v_k) - \bar{\lambda}S$, as in Section 3. Moreover, we consider the self adjoint operators

$$C_k := \mathrm{D}^2\Phi(u_k) - \lambda_k S \in \mathcal{L}(H^1)$$

for $k \in \mathbb{N}$. First we show that the constrained critical points u_k of Φ are freely nondegenerate and that

$$m_{\mathrm{f}}(u_k) = nm_{\mathrm{f}}(\bar{u}) \quad \text{for large } k.$$

To this end it is sufficient to prove the following.

Lemma 4.2. *It holds true that*

$$\limsup_{k\to\infty} \inf_{\substack{W \leqslant H^1 \\ \dim W = nm_{\mathrm{f}}(\bar{u})}} \sup_{\substack{w \in W \\ \|w\|=1}} \langle C_k w, w\rangle < 0, \tag{4-2}$$

$$\liminf_{k\to\infty} \inf_{\substack{W \leqslant H^1 \\ \dim W = nm_{\mathrm{f}}(\bar{u})+1}} \sup_{\substack{w \in W \\ \|w\|=1}} \langle C_k w, w\rangle > 0. \tag{4-3}$$

Proof. By (4-1) and since $\mathrm{D}^2\Phi\colon H^1 \to \mathcal{L}(H^1)$ is uniformly continuous on bounded subsets of H^1, the assertion follows once we have established the following estimates:

$$\limsup_{k\to\infty} \inf_{\substack{W \leqslant H^1 \\ \dim W = nm_{\mathrm{f}}(\bar{u})}} \sup_{\substack{w \in W \\ \|w\|=1}} \langle B_k w, w\rangle < 0, \tag{4-4}$$

$$\liminf_{k\to\infty} \inf_{\substack{W \leqslant H^1 \\ \dim W = nm_{\mathrm{f}}(\bar{u})+1}} \sup_{\substack{w \in W \\ \|w\|=1}} \langle B_k w, w\rangle > 0. \tag{4-5}$$

Let $Z \subset H^1$ denote the generalized eigenspace of the self-adjoint operator B in H^1 corresponding to its $m_{\mathrm{f}}(\bar{u})$ negative eigenvalues. Pick $\delta > 0$ such that $\langle Bw, w\rangle \le -\delta\|w\|^2$ for all $w \in Z$ and $\langle By, y\rangle \ge \delta\|y\|^2$ for all $y \in Z^{\perp}$. Put

$$Z_k := \sum_{i=1}^{n} \mathcal{T}_{a_k^i} Z \subset H^1 \quad \text{for } k \in \mathbb{N}.$$

Since $d(a_k) \to \infty$, the sum is direct and hence $\dim Z_k = n m_{\mathrm{f}}(\bar{u})$ for k sufficiently large. If $w_k \in Z_k$ satisfies $\|w_k\| = 1$ for all k, then it suffices to show

$$\limsup_{k\to\infty} \langle B_k w_k, w_k\rangle \le -\delta \tag{4-6}$$

along a subsequence to prove (4-4). We write

$$w_k = \sum_{i=1}^{n} \mathcal{T}_{a_k^i} \rho_k^i \quad \text{for } k \in \mathbb{N} \text{ with } \rho_k^i \in Z.$$

Since Z is finite-dimensional, we may pass to a subsequence such that $\rho_k^i \to \rho^i \in Z$ for $i = 1, \dots, n$ as $k \to \infty$. It is easy to see that then

$$1 = \|w_k\|^2 = \sum_{i=1}^{n} \|\rho^i\|^2 + o(1) \quad \text{as } k \to \infty.$$

Thus (3-7) and (3-12) imply

$$\begin{aligned}\langle B_k w_k, w_k\rangle &= \sum_{i,j=1}^{n} \langle B_k \mathcal{T}_{a_k^i} \rho_k^i, \mathcal{T}_{a_k^j} \rho_k^j\rangle = \sum_{i,j=1}^{n} \langle \mathcal{T}_{-a_k^j} B_k \mathcal{T}_{a_k^i} \rho^i, \rho^j\rangle + o(1) = \sum_{i=1}^{n} \langle B\rho^i, \rho^i\rangle + o(1) \\ &\le -\delta \sum_{i=1}^{n} \|\rho^i\|^2 + o(1) = -\delta + o(1),\end{aligned}$$

that is, (4-6).

If $y_k \in Z_k^{\perp}$ satisfies $\|y_k\| = 1$ for all k, then it suffices to show

$$\liminf_{k\to\infty} \langle B_k y_k, y_k\rangle \ge \delta \tag{4-7}$$

for a subsequence to prove (4-5). Passing to a subsequence, we may assume that

$$w^i := \operatorname*{w-lim}_{k\to\infty} \mathcal{T}_{-a_k^i} y_k$$

exists for $i = 1, \dots, n$. Let $v \in Z$. Since $\mathcal{T}_{a_k^i} v \in Z_k$, we infer that

$$0 = \langle \mathcal{T}_{a_k^i} v, y_k\rangle = \langle v, \mathcal{T}_{-a_k^i} y_k\rangle = \langle v, w^i\rangle + o(1) \quad \text{for } i = 1, \dots, n.$$

Consequently,

$$w^i \in Z^{\perp} \quad \text{for } i = 1, \dots, n. \tag{4-8}$$

We now set

$$z_k := y_k - \sum_{i=1}^{n} \mathcal{T}_{a_k^i} w^i \quad \text{for } k \in \mathbb{N},$$

noting that

$$\operatorname*{w-lim}_{k\to\infty} \mathcal{T}_{-a_k^i} z_k = 0 \quad \text{for } i = 1, \dots, n. \tag{4-9}$$

In particular, this implies

$$z_k \rightharpoonup 0 \quad \text{in } H^1 \tag{4-10}$$

by (3-4) which we may again assume without loss of generality. Using (3-7), (3-12), and (4-9) we obtain the splitting

$$\begin{aligned}
\langle B_k y_k, y_k\rangle &= \langle B_k z_k, z_k\rangle + 2\sum_{i=1}^{n} \langle B_k \mathcal{T}_{a_k^i} w^i, z_k\rangle + \sum_{i,j=1}^{n} \langle B_k \mathcal{T}_{a_k^i} w^i, \mathcal{T}_{a_k^j} w^j\rangle \\
&= \langle B_k z_k, z_k\rangle + 2\sum_{i=1}^{n} \langle \mathcal{T}_{-a_k^i} B_k \mathcal{T}_{a_k^i} w^i, \mathcal{T}_{-a_k^i} z_k\rangle + \sum_{i,j=1}^{n} \langle \mathcal{T}_{-a_k^j} B_k \mathcal{T}_{a_k^i} w^i, w^j\rangle \\
&= \langle B_k z_k, z_k\rangle + \sum_{i=1}^{n} \langle B w^i, w^i\rangle + o(1),
\end{aligned} \tag{4-11}$$

where

$$\begin{aligned}
\langle B_k z_k, z_k\rangle &= \|z_k\|^2 - \lambda |z_k|_2^2 - \langle D^2\Psi(v_k) z_k, z_k\rangle \\
&= \|z_k\|^2 - \lambda |z_k|_2^2 - \sum_{i=1}^{n} \langle D^2\Psi(\mathcal{T}_{a_k^i}\bar{u}) z_k, z_k\rangle + o(1) \\
&= \|z_k\|^2 - \lambda |z_k|_2^2 - \sum_{i=1}^{n} \langle D^2\Psi(\bar{u}) \mathcal{T}_{-a_k^i} z_k, \mathcal{T}_{-a_k^i} z_k\rangle + o(1) \\
&= \|z_k\|^2 - \lambda |z_k|_2^2 + o(1) \\
&= \|z_k\|^2 - \lambda |z_k|_2^2 - \langle D^2\Psi(\bar{u}) z_k, z_k\rangle + o(1) \\
&= \langle B z_k, z_k\rangle + o(1).
\end{aligned} \tag{4-12}$$

Here we have used (3-8), (3-10), (4-9), (4-10), and the compactness of the operator $D^2\Psi(\bar{u}) \in \mathcal{L}(H^1)$.

Let $P \in \mathcal{L}(\mathcal{H}^1)$ denote the $\langle\,\cdot\,,\cdot\,\rangle$-orthogonal projection on Z, and let $Q := I - P$. Since P has finite range, we see that

$$z_k - Q z_k = P z_k \to 0 \quad \text{in } H^1 \text{ as } k \to \infty. \tag{4-13}$$

Combining (4-8), (4-11), (4-12), and (4-13), we obtain

$$\begin{aligned}
\langle B_k y_k, y_k\rangle &= \langle B Q z_k, Q z_k\rangle + \sum_{i=1}^{n} \langle B w^i, w^i\rangle + o(1) \ge \delta\Big(\|Q z_k\|^2 + \sum_{i=1}^{n} \|w^i\|^2\Big) + o(1) \\
&= \delta\Big(\|z_k\|^2 + \sum_{i=1}^{n} \|w^i\|^2\Big) + o(1) = \delta \|y_k\|^2 + o(1) = \delta + o(1),
\end{aligned}$$

and hence (4-7). □

From Lemma 4.2 it follows that C_k is invertible for large k and that the norm of its inverse remains bounded as $k \to \infty$. We now recall the function $z_{u_k} = C_k^{-1} S u_k \in H^1$, which by Lemma 2.6 is of key importance to compute $m(u_k)$.

Lemma 4.3. *For $i = 1, \dots, n$ we have*

$$\mathcal{T}_{-a_k^i} z_{u_k} \rightharpoonup z_{\bar{u}} = B^{-1} S\bar{u} \quad \text{in } H^1 \text{ as } k \to \infty.$$

Proof. Let $\psi \in H^1$, and let $\varphi = B^{-1}\psi \in H^1$. Recalling that $\mathrm{D}^2\Phi \colon H^1 \to \mathcal{L}(H^1)$ is uniformly continuous on bounded subsets of H^1, we may deduce from (3-7) that

$$\mathcal{T}_{-a_k^i} C_k \mathcal{T}_{a_k^i} \varphi = \mathcal{T}_{-a_k^i} B_k \mathcal{T}_{a_k^i} \varphi + o(1) \to B\varphi = \psi \quad \text{in } H^1$$

as $k \to \infty$. Since moreover the sequence $(z_{u_k})_k$ is bounded in H^1 and $\mathcal{T}_{-a_k^i} u_k \rightharpoonup \bar{u}$ in H^1 as $k \to \infty$, we have

$$\begin{aligned}\langle z_{\bar{u}}, \psi\rangle &= \langle B^{-1}(S\bar{u}), \psi\rangle = \langle S\bar{u}, \varphi\rangle = \langle S(\mathcal{T}_{-a_k^i} u_k), \varphi\rangle + o(1) = \langle S u_k, \mathcal{T}_{a_k^i}\varphi\rangle + o(1)\\ &= \langle C_k z_{u_k}, \mathcal{T}_{a_k^i}\varphi\rangle + o(1) = \langle z_{u_k}, C_k \mathcal{T}_{a_k^i}\varphi\rangle + o(1) = \langle \mathcal{T}_{-a_k^i} z_{u_k}, \mathcal{T}_{-a_k^i} C_k \mathcal{T}_{a_k^i}\varphi\rangle + o(1)\\ &= \langle \mathcal{T}_{-a_k^i} z_{u_k}, \psi\rangle + o(1) \quad \text{as } k \to \infty. \end{aligned}$$

□

Proof of Theorem 4.1. With the help of Lemma 4.3, we compute

$$\begin{aligned}(u_k, z_{u_k})_2 = (v_k, z_{u_k})_2 + o(1) &= \sum_{i=1}^{n} (\mathcal{T}_{a_k^i}\bar{u}, z_{u_k})_2 + o(1)\\ &= \sum_{i=1}^{n} (\bar{u}, \mathcal{T}_{-a_k^i} z_{u_k})_2 + o(1) = n(\bar{u}, z_{\bar{u}})_2 + o(1).\end{aligned}$$

Since $(\bar{u}, z_{\bar{u}})_2 \neq 0$ as $\bar{u}$ is fully nondegenerate by assumption, we infer that $(u_k, z_{u_k})_2$ is also nonzero and has the same sign as $(\bar{u}, z_{\bar{u}})_2$ for large k. Moreover, u_k is freely nondegenerate by Lemma 4.2, so Lemma 2.6 yields that u_k is a fully nondegenerate critical point of $\Phi|_{\Sigma_\alpha}$ for large k. Its Morse index is, by the same token, $m(u_k) = m_{\mathrm{f}}(u_k) - 1 = n m_{\mathrm{f}}(\bar{u}) - 1 = n(m(\bar{u}) + 1) - 1$ if $(\bar{u}, z_{\bar{u}})_2 < 0$, and it is $m(u_k) = m_{\mathrm{f}}(u_k) = n m_{\mathrm{f}}(\bar{u}) = n m(\bar{u})$ if $(\bar{u}, z_{\bar{u}})_2 > 0$.

To show the last statement of the present theorem, suppose that (H4) is satisfied. Lemma 2.4 implies $\langle B\bar{u}, \bar{u}\rangle < 0$, that is, $m_{\mathrm{f}}(\bar{u}) > 0$. In any case it follows from the preceding calculations that $m(u_k) > 0$ for large k. This completes the proof of Theorem 4.1. □

Proof of Corollary 1.4. Let $\bar{u}$ be a nondegenerate local minimum of $J_{\alpha/n}$ with Lagrange multiplier $\bar{\lambda}$. Moreover, let $(a_k) \subseteq (\mathbb{Z}^N)^n$ be a sequence such that $d(a_k) \to \infty$ as $k \to \infty$. By Lemma 2.8, $\bar{u}$ is fully nondegenerate and, without loss of generality, a positive function. Thus, (H4) and Theorem 3.1 imply the existence of positive critical points u_k of J_α with Lagrange multiplier λ_k for large k and such that (4-1) holds true. Moreover, the sequence $(u_k)_k$ is uniquely determined by these properties. Since $m_{\mathrm{f}}(\bar{u}) > 0 = m(\bar{u})$ by (H4) and Lemma 2.4, Theorem 4.1 now implies u_k is nondegenerate with $m(u_k) = n - 1$ for large k. □

5. Proof of Theorem 1.5

In this section we wish to prove Theorem 1.5. For this we will assume hypotheses (H5) and (H6). Without loss of generality we may also assume for the nondegenerate critical point x_0 of V that

$$x_0 = 0 \quad \text{and} \quad V(x_0) = 1.$$

We are then concerned with positive solutions of the singularly perturbed equation

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-2}u, \quad u \in H^1, \tag{5-1}$$

where $p \in (2, 2^*)$. By [Grossi 2002, Theorem 1.1], there exists ε_0 and a family of positive single peak solutions $\bar{u}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0)$, of (5-1) which concentrates at $x_0 = 0$. This means that each $\bar{u}_\varepsilon$ has only one local maximum, and the rescaled functions

$$u_\varepsilon \in H^1, \quad u_\varepsilon(x) := \bar{u}_\varepsilon(\varepsilon x), \tag{5-2}$$

converge, as $\varepsilon \to 0$, in H^1 to the unique radial positive solution of the limit equation

$$-\Delta u_0 + u_0 = u_0^{p-1} \quad \text{in } \mathbb{R}^N. \tag{5-3}$$

Moreover, as follows from the uniqueness statement in [loc. cit., Theorem 1.1], this convergence property after rescaling determines the solutions $\bar{u}_\varepsilon$ uniquely for $\varepsilon > 0$ small. In addition, we can assume by [loc. cit., Theorem 6.2] that $\bar{u}_\varepsilon$ is nondegenerate; i.e., the linear operator

$$H^1 \mapsto H^1, \quad v \mapsto v - (p-1)(-\varepsilon^2\Delta + V)^{-1}\bar{u}_\varepsilon^{p-2}v, \quad \text{is an isomorphism} \tag{5-4}$$

for $\varepsilon \in (0, \varepsilon_0)$. Here, for $\varepsilon > 0$, the operator $-\varepsilon^2\Delta + V \in \mathcal{L}(H^1, H^{-1})$ is understood as the Hilbert space isomorphism $H^1 \to H^{-1}$ associated with the scalar product

$$(u, v) \mapsto \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \cdot \nabla v + Vuv)$$

on H^1 via Riesz's representation theorem. Since $0 < \min V \le \max V < \infty$, this scalar product is equivalent to the standard scalar product on H^1, which we denote by

$$\langle u, v \rangle_{H^1} := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv). \tag{5-5}$$

We also let $\|\cdot\|_{H^1}$ denote the associated norm.

Lemma 5.1. *The map* $(0, \varepsilon_0) \to H^1$, $\varepsilon \mapsto \bar{u}_\varepsilon$, *is continuous.*

Proof. For $\varepsilon > 0$, let $K(\varepsilon) := -\varepsilon^2\Delta + V \in \mathcal{L}(H^1, H^{-1})$. Then the map $K\colon (0, \infty) \to \mathcal{L}(H^1, H^{-1})$ is continuous. Moreover, since p is subcritical, the nonlinear superposition operator $H^1 \to H^{-1}$, $u \mapsto |u|^{p-2}u$, is of class C^1. Consequently, the map

$$h\colon (0, \infty) \times H^1 \to H^{-1}, \quad (\varepsilon, u) \mapsto K(\varepsilon)u - |u|^{p-2}u,$$

is continuous, and continuously differentiable in its second argument. Since $\bar{u}_\varepsilon$ is a weak solution of (5-1), we have $h(\varepsilon, u_\varepsilon) = 0$. Furthermore, the operator

$$h_u(\varepsilon, u_\varepsilon) = K(\varepsilon) - (p-1)|\bar{u}_\varepsilon|^{p-2} \in \mathcal{L}(H^1, H^{-1})$$

is an isomorphism as a consequence of (5-4). Hence the claim follows from the implicit function theorem; see, e.g., [Deimling 1985, Theorem 15.1]. □

Since the map $\varepsilon \mapsto \bar{u}_\varepsilon$ is continuous and

$$|\bar{u}_\varepsilon|_2^2 = \int_{\mathbb{R}^N} \bar{u}_\varepsilon^2 = \varepsilon^N \int_{\mathbb{R}^N} u_\varepsilon^2 = \varepsilon^N \int_{\mathbb{R}^N} u_0^2 + o(1) = o(1) \quad \text{as } \varepsilon \to 0,$$

the assertions (i)–(iii) of Theorem 1.5 are already verified. The remainder of this section is devoted to the proof of Theorem 1.5(iv).

For this we first note that the function $u_\varepsilon \in H^1$ defined in (5-2) satisfies the rescaled equation

$$-\Delta u_\varepsilon + V_\varepsilon(x) u_\varepsilon = |u_\varepsilon|^{p-2} u_\varepsilon, \quad u \in H^1, \tag{5-6}$$

with

$$V_\varepsilon \colon \mathbb{R}^N \to \mathbb{R}, \quad V_\varepsilon(x) = V(\varepsilon x). \tag{5-7}$$

Moreover, by (5-4), the linear operator

$$B^\varepsilon \in \mathcal{L}(H^1), \quad B^\varepsilon v = v - (p-1)(-\Delta + V_\varepsilon)^{-1} u_\varepsilon^{p-2} v, \quad \text{is an isomorphism} \tag{5-8}$$

for $\varepsilon \in (0, \varepsilon_0)$. We also note that the functions u_ε have uniform exponential decay; i.e., there exist constants $\alpha, C > 0$ such that

$$|u_\varepsilon(x)| \le C \mathrm{e}^{-\alpha |x|} \quad \text{for all } x \in \mathbb{R}^N,\ \varepsilon \in (0, \varepsilon_0); \tag{5-9}$$

see [Grossi 2002, Lemma 4.2(i)]. Moreover,

$$u_\varepsilon \to u_0 \quad \text{in } H^2(\mathbb{R}^N) \text{ and uniformly in } \mathbb{R}^N; \tag{5-10}$$

see [loc. cit., Theorem 4.1 and Lemma 4.2(ii)]. Note that u_ε satisfies [loc. cit., Equation (4.1)] with $c_{i,y,\varepsilon} = 0$ since it is a solution of (5-6).

We need to recall some properties of the unique radial positive solution u_0 of the limit equation (5-3) and therefore consider the functional

$$\Phi_0^* \colon H^1 \to \mathbb{R}, \quad \Phi_0^*(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p.$$

It is easy to see that $\mathrm{D}^2 \Phi_0^*(u_0) \in \mathcal{L}(H^1)$ has exactly one negative eigenvalue, the value $2 - p$, with corresponding eigenspace generated by u_0. Here, the symbol D^2 denotes the derivative of the gradient with respect to the scalar product $\langle \cdot, \cdot \rangle_{H^1}$.

Its kernel is spanned by the partial derivatives $\partial_1 u_0, \partial_2 u_0, \dots, \partial_N u_0$; see [Ni and Takagi 1993, Lemma 4.2(i)]. Letting $\widetilde{H}$ denote the $\langle \cdot, \cdot \rangle$-orthogonal complement of $\operatorname{span}(\partial_1 u_0, \partial_2 u_0, \dots, \partial_N u_0)$

in H^1, we therefore find that the operator

$$B^0 \in \mathcal{L}(H^1), \quad B^0 v = \mathrm{D}^2\Phi_0^*(u_0)v = v - (p-1)[\Delta+1]^{-1}{u_0}^{p-2},$$

restricts to an isomorphism $\widetilde{H} \to \widetilde{H}$. Moreover, $\widetilde{H}$ contains all radial functions, so in particular $u_* := [\Delta+1]^{-1}u_0 \in \widetilde{H}$. Consequently, there exists a unique $z_* \in \widetilde{H}$ with $B^0 z_* = u_*$.

Lemma 5.2. *We have*

$$(z_*, u_0)_2 = \Big(\frac{N}{4} - \frac{1}{p-2}\Big)|u_0|_2^2 = \frac{p-(2+4/N)}{4N(p-2)}|u_0|_2^2.$$

Proof. For $\lambda > 0$, consider the function

$$w_\lambda \in H^1, \quad w_\lambda(x) = \lambda^{1/(p-2)} u_0(\sqrt{\lambda}x) \quad \text{for } x \in \mathbb{R}^N,$$

which is the unique radial positive solution of

$$-\Delta w_\lambda + \lambda w_\lambda - w_\lambda^{p-1} = 0 \quad \text{in } \mathbb{R}^N, \tag{5-11}$$

so $w_1 = u_0$. Moreover, consider

$$\tilde{z} \in H^1, \quad \tilde{z}(x) = \frac{\partial}{\partial\lambda}\Big|_{\lambda=1} w_\lambda(x).$$

We claim that $z_* = -\tilde{z}$. Indeed, we have $B^0\tilde{z} = -u_*$ since differentiating (5-11) at $\lambda = 1$ yields

$$-\Delta\tilde{z} + \tilde{z} - (p-1)u_0^{p-2}\tilde{z} = -u_0 \quad \text{in } \mathbb{R}^N. \tag{5-12}$$

Moreover, $\tilde{z} \in \widetilde{H}$ since $\tilde{z}$ is a radial function. By the remarks above, this implies $z_* = -\tilde{z}$. We therefore compute

$$\begin{aligned}(z_*, u_0)_2 = -(\tilde{z}, u_0)_2 &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=1}|w_\lambda|_2^2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=1}\Big(\lambda^{2/(p-2)}\int_{\mathbb{R}^N} u_0^2(\sqrt{\lambda}x)\,\mathrm{d}x\Big)\\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=1}\lambda^{2/(p-2)-N/2}|u_0|_2^2 = \frac{1}{2}\Big(\frac{N}{2} - \frac{2}{p-2}\Big)|u_0|_2^2,\end{aligned}$$

as claimed. □

Next we collect some properties of the scaled potentials V_ε, $\varepsilon \in (0, \varepsilon_0)$, defined in (5-7). Note that these functions are uniformly bounded and satisfy

$$|V_\varepsilon(x) - 1| \le c\,\varepsilon^2|x|^2 \quad \text{for } x \in \mathbb{R}^N,\ \varepsilon \in (0,\varepsilon_0), \text{ with a constant } c > 0. \tag{5-13}$$

We also note that

$$\lim_{\varepsilon\to 0}\frac{\partial_i V_\varepsilon(x)}{\varepsilon^2} = \sum_{j=1}^N \partial_{ij}V(0)x_j \quad \text{locally uniformly in } x \in \mathbb{R}^N \tag{5-14}$$

for $i = 1, \dots, N$, so

$$|\partial_i V_\varepsilon(x)| \le c\,\varepsilon^2|x| \quad \text{for } x \in \mathbb{R}^N,\ \varepsilon \in (0,\varepsilon_0), \text{ with a constant } c > 0. \tag{5-15}$$

Next we consider

$$z_\varepsilon := [B^\varepsilon]^{-1}(-\Delta + V_\varepsilon)^{-1}u_\varepsilon \in H^1 \quad \text{for } \varepsilon \in (0, \varepsilon_0),$$

where B^ε is defined in (5-8). Hence z_ε is the unique weak solution of

$$-\Delta z_\varepsilon + V_\varepsilon(x)z_\varepsilon - (p-1)u_\varepsilon^{p-2}z_\varepsilon = u_\varepsilon \quad \text{in } \mathbb{R}^N. \tag{5-16}$$

We claim that

$$(z_\varepsilon, u_\varepsilon)_2 \to (z_*, u_0)_2 \quad \text{as } \varepsilon \to 0. \tag{5-17}$$

To prove this, we argue by contradiction and suppose that there exists $\delta > 0$ and a sequence $(\varepsilon_n)_n \in (0, \varepsilon_0)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$|(z_n, u_n)_2 - (z_*, w)_2| \geq \delta \quad \text{for all } n \in \mathbb{N}, \quad \text{where } z_n := z_{\varepsilon_n} \text{ and } u_n := u_{\varepsilon_n}. \tag{5-18}$$

We first claim that the sequence $(z_n)_n$ is bounded in H^1. Indeed, if not, we can pass to a subsequence such that $\|z_n\|_{H^1} > 0$ for all n and $\|z_n\|_{H^1} \to \infty$ as $n \to \infty$. We then consider $y_n := z_n/\|z_n\|_{H^1}$, and we may pass to a subsequence such that $y_n \rightharpoonup y$ in H^1. Since y_n is a weak solution of the equation

$$-\Delta y_n + V_{\varepsilon_n} y_n - (p-1)u_n^{p-2}y_n = \frac{u_n}{\|z_n\|_{H^1}} \quad \text{in } \mathbb{R}^N \text{ for every } n, \tag{5-19}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^N}[\nabla y \nabla v + yv - (p-1)u_0^{p-2}v] &= \lim_{n\to\infty}\int_{\mathbb{R}^N}[\nabla y_n \nabla v + V_{\varepsilon_n} y_n v - (p-1)u_n^{p-2}y_n v] \\ &= \lim_{n\to\infty}\frac{1}{\|z_n\|_{H^1}}\int_{\mathbb{R}^N} u_n v = 0 \quad \text{for every } v \in H^1.\end{aligned}$$

Consequently, $y \in H^1$ is a weak solution of $-\Delta y + y - (p-1)u_0^{p-2}y = 0$ in $\mathbb{R}^N$, which means that $B^0 y = 0$. Hence there exist $a_1, \dots, a_N \in \mathbb{R}$ with $y = \sum_{i=1}^N a_i\, \partial_i u_0$. Next we note that $\partial_i u_n$ solves the equation

$$-\Delta(\partial_i u_n) + V_\varepsilon\, \partial_i u_n + u_n\, \partial_i V_{\varepsilon_n} - (p-1)u_n^{p-2}\, \partial_i u_n = 0 \quad \text{for } i = 1, \dots, N.$$

Multiplying this equation with y_n and integrating over $\mathbb{R}^N$, we obtain by (5-19) that

$$\int_{\mathbb{R}^N} u_n y_n\, \partial_i V_{\varepsilon_n} = -\frac{1}{\|z_n\|_{H^1}}\int_{\mathbb{R}^N} u_n\, \partial_i u_n = 0 \quad \text{for all } n \in \mathbb{N}.$$

Dividing this equation by ε_n^2 and passing to the limit, we may then use (5-9), (5-14), (5-15) and Lebesgue's theorem to see that

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{1}{\varepsilon_n^2}\int_{\mathbb{R}^N} u_n\, y_n\, \partial_i V_{\varepsilon_n} = \sum_{j=1}^N \int_{\mathbb{R}^N} \partial_{ij} V(0) x_j u_0(x) y(x)\, \mathrm{d}x \\ &= \sum_{\ell,j=1}^N a_\ell\, \partial_{ij} V(0)\int_{\mathbb{R}^N} x_j u_0(x)\, \partial_\ell u_0(x)\, \mathrm{d}x = -\frac{|u_0|_2^2}{2}\sum_{j=1}^N a_j\, \partial_{ij} V(0) \quad \text{for } i = 1, \dots, N.\end{aligned}$$

Here we have integrated by parts in the last step. Since 0 is a nondegenerate critical point of V by assumption, we conclude that $a_j = 0$ for $j = 1, \dots, N$ and therefore $y = 0$. This implies in particular that (y_n^2) is bounded in $L^{p/2}$ and that $y_n^2 \to 0$ in $L^{p/2}_{\mathrm{loc}}$. Moreover, $u_n^{p-2} \to u_0^{p-2}$ in $L^{p/(p-2)}$. Testing (5-19) with y_n we obtain

$$\int_{\mathbb{R}^N} (|\nabla y_n|^2 + V_{\varepsilon_n} |y_n|^2) = (p-1) \int_{\mathbb{R}^N} u_n^{p-2} |y_n|^2 + \frac{1}{\|z_n\|_{H^1}} \int_{\mathbb{R}^N} u_n y_n \to 0$$

as $n \to \infty$ and therefore $\|y_n\|_{H^1} \to 0$ as $n \to \infty$, which is a contradiction. We thus conclude that the sequence $(z_n)_n$ is bounded. We may thus pass to a subsequence such that $z_n \rightharpoonup z$ in H^1. We then have by (5-16)

$$\begin{aligned}\int_{\mathbb{R}^N} [\nabla z \nabla v + zv - (p-1) u_0^{p-2} v] &= \lim_{n \to \infty} \int_{\mathbb{R}^N} [\nabla z_n \nabla v + V_{\varepsilon_n} z_n v - (p-1) u_n^{p-2} z_n v] \\ &= \lim_{n \to \infty} \int_{\mathbb{R}^N} u_n v = \int_{\mathbb{R}^N} u_0 v \quad \text{for every } v \in H^1.\end{aligned}$$

Consequently, $z \in H^1$ is a weak solution of $-\Delta z + z - (p-1) u_0^{p-2} z = u_0$ in $\mathbb{R}^N$, which means that $B^0 z = u_*$. As a consequence, $B^0(z - z_*) = 0$, which implies $z - z_* \in \operatorname{span}(\partial_1 u_0, \partial_2 u_0, \dots, \partial_N u_0)$ and therefore $(z - z_*, u_0)_2 = 0$. We thus conclude that

$$(z_n, u_n)_2 \to (z, u_0)_2 = (z_*, u_0)_2 \quad \text{as } n \to \infty,$$

contrary to (5-18). This shows (5-17), as claimed. Combining (5-17) with Lemma 5.2, we see that for fixed $p \in (2, 2^*) \setminus \{2 + 4/N\}$, we may take $\varepsilon_0 > 0$ smaller if necessary such that

$$(z_\varepsilon, u_\varepsilon)_2 < 0 \quad \text{if } 2 < p < 2 + \frac{4}{N} \qquad \text{and} \qquad (z_\varepsilon, u_\varepsilon)_2 > 0 \quad \text{if } 2 + \frac{4}{N} < p < 2^*. \tag{5-20}$$

Moreover, from (5-20) we immediately deduce (1-6) by rescaling. Since $\bar{u}_\varepsilon$ is a critical point of Φ_ε, it is also a critical point of $\Phi_\varepsilon|_{\Sigma_{|\bar{u}_\varepsilon|_2^2}}$ with Lagrange multiplier 0, which implies, together with (1-6) and Definition 1.1, that $\bar{u}_\varepsilon$ is a fully nondegenerate critical point of $\Phi_\varepsilon|_{\Sigma_{|\bar{u}_\varepsilon|_2^2}}$.

To conclude the proof of Theorem 1.5, it remains to compute the Morse index of $\bar{u}_\varepsilon$ for $\varepsilon > 0$ small. From (1-6) and Lemma 2.6, we deduce that

$$m(\bar{u}_\varepsilon) = m_{\mathrm{f}}(\bar{u}_\varepsilon) - 1 \quad \text{if } 2 < p < 2 + \frac{4}{N} \qquad \text{and} \qquad m(\bar{u}_\varepsilon) = m_{\mathrm{f}}(\bar{u}_\varepsilon) \quad \text{if } 2 + \frac{4}{N} < p < 2^*. \tag{5-21}$$

It therefore suffices to compute the free Morse index $m_{\mathrm{f}}(\bar{u}_\varepsilon)$, which by rescaling is the same as the free Morse index $m_{\mathrm{f}}(u_\varepsilon)$ with respect to the rescaled potential

$$\Phi_\varepsilon^* \colon H^1 \to \mathbb{R}, \quad \Phi_\varepsilon^*(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\varepsilon u^2) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p.$$

More precisely, the equalities in (1-5) follow from (5-21) once we have shown that

$$m_{\mathrm{f}}(u_\varepsilon) = m_V + 1 \quad \text{for all } p \in (2, 2^*) \text{ and } \varepsilon > 0 \text{ small}, \tag{5-22}$$

where m_V denotes the number of negative eigenvalues of the Hessian of V at x_0. The argument is partly contained in the proof of [Lin and Wei 2008, Theorem 2.5]. Nevertheless, since some details are omitted there, we give a complete proof of (5-22) in the Appendix. The proof of Theorem 1.5 is thus finished.

6. Orbital instability

This section is devoted to the proof of Theorem 1.7. To simplify the presentation we only give a proof for the case $N \geq 3$; the cases $N = 1, 2$ can be treated similarly, slightly modifying the arguments below.

Throughout this section, we consider the special case where the nonlinearity f is odd. We may therefore write it in the form $f(t) = g(|t|^2)t$, where $g \in C([0,\infty)) \cap C^1((0,\infty))$ satisfies $g(0) = 0$ and

$$\lim_{s\to\infty} \frac{g'(s)}{s^{2^*/2-2}} = 0.$$

Note that in this case we have

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(|u|^2) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V|u|^2\big) - \int_{\mathbb{R}^N} G(|u|^2)$$

for $u \in H^1$ with $G(t) = \frac{1}{2}\int_0^t g$ for $t \geq 0$. To prove the assertion on orbital instability given in Theorem 1.7, we apply an argument from [Esteban and Strauss 1994] with some modifications. We identify $\mathbb{C}$ with $\mathbb{R}^2$ and write the time-dependent nonlinear Schrödinger equation (1-8) as the following system in $\boldsymbol{u} = \binom{u_1}{u_2}$ with $u_1 = \operatorname{Re} u,\ u_2 = \operatorname{Im} u$:

$$\boldsymbol{u}_t = J(-\Delta \boldsymbol{u} + V(x)\boldsymbol{u} - g(u_1^2 + u_2^2)\boldsymbol{u}) \quad \text{with } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6-1}$$

In order to set up the functional analytic equation for this system, we denote the dual paring between H^{-1} and H^1 by $\langle\,\cdot\,,\cdot\,\rangle_*$. We put $\mathcal{H} := H^1 \times H^1$ and write $\mathcal{H}^* = H^{-1} \times H^{-1}$ for the topological dual of $\mathcal{H}$. Recalling that we are assuming $\min \sigma(-\Delta + V) > 0$, we use the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{H}} = \langle u_1, v_1\rangle + \langle u_2, v_2\rangle = \sum_{i=1}^{2} \int_{\mathbb{R}^N} \big(\nabla u_i \cdot \nabla v_i + V u_i v_i\big) \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{H},$$

and denote the induced norm by $\|\cdot\|_{\mathcal{H}}$. The dual pairing between $\mathcal{H}^*$ and $\mathcal{H}$ is given by

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{H}^*,\mathcal{H}} = \langle u_1, v_1\rangle_* + \langle u_2, v_2\rangle_* \quad \text{for } \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{H}^*,\ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{H}.$$

As usual in the context of Gelfand triples, we consider the continuous embedding $I\colon H^1 \hookrightarrow H^{-1}$ given by

$$\langle Iu, v\rangle_* := \int_{\mathbb{R}^N} uv \quad \text{for } u, v \in H^1.$$

The corresponding embedding $\mathcal{H} \hookrightarrow \mathcal{H}^*$ will also be denoted by I; i.e., we set

$$\langle I\boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{H}^*,\mathcal{H}} := \int_{\mathbb{R}^N} (u_1 v_1 + u_2 v_2) \quad \text{for } \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{H}.$$

With this notation, we write system (6-1) in the more abstract form of a Hamiltonian system. For this we consider the functionals

$$\widetilde{\Phi} \in C^2(\mathcal{H}, \mathbb{R}), \qquad \widetilde{\Phi}(\boldsymbol{u}) = \tfrac{1}{2}\|\boldsymbol{u}\|_{\mathcal{H}}^2 - \int_{\mathbb{R}^N} G(u_1^2 + u_2^2),$$

$$\widetilde{\Phi}_\lambda \in C^2(\mathcal{H}, \mathbb{R}), \quad \widetilde{\Phi}_\lambda(\boldsymbol{u}) = \Phi(\boldsymbol{u}) - \frac{\lambda}{2}\int_{\mathbb{R}^N} (u_1^2 + u_2^2).$$

With this notation, (6-1) can be written as

$$(I\boldsymbol{u})_t = J\mathbf{d}\widetilde{\Phi}(\boldsymbol{u}) \quad \text{in } \mathcal{H}^*,$$

where $\mathbf{d}\widetilde{\Phi}\colon \mathcal{H} \to \mathcal{H}^*$ denotes the derivative of $\widetilde{\Phi}$ and J is regarded as a matrix multiplication operator on $\mathcal{H}^* = H^{-1} \times H^{-1}$.

Now let $\varphi \in \Sigma_\alpha$ satisfy the assumptions of Theorem 1.7, and let $\lambda \in \mathbb{R}$ be the corresponding Lagrangian multiplier. Moreover, in the following, we let $\mathbf{d}^2\widetilde{\Phi}_\lambda(\boldsymbol{\psi}) \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$ denote the second derivative of $\widetilde{\Phi}_\lambda$ at $\boldsymbol{\psi} := \binom{\varphi}{0} \in \mathcal{H}$, which by direct computation is given as

$$\mathbf{d}^2\widetilde{\Phi}_\lambda(\boldsymbol{\psi}) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad \text{where } \begin{cases} L_1 w = -\Delta w + [V(x) - \lambda]w - f'(\varphi)w, \\ L_2 w = -\Delta w + [V(x) - \lambda]w - g(|\varphi|^2)w. \end{cases}$$

Note here that $f'(t) = g(|t|^2) + 2g'(|t|^2)t^2$, so by (H3) we have $L_i \in \mathcal{L}(H^1, H^{-1})$ for $i = 1, 2$. Similarly as noted in [Esteban and Strauss 1994, p. 187], the orbital instability of the solitary wave solution u_φ in (1-7) follows by the same argument as in the proof of [Grillakis et al. 1990, Theorem 6.2] once we have established the following.

Proposition 6.1. *The operator*

$$\boldsymbol{M} := J\mathbf{d}^2\widetilde{\Phi}_\lambda(\boldsymbol{\psi}) \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$$

has a positive real eigenvalue; *i.e.*, *there exists* $\rho > 0$ *and* $\boldsymbol{w} \in \mathcal{H}\backslash\{0\}$ *such that* $\boldsymbol{M}\boldsymbol{w} = \rho I\boldsymbol{w}$.

The remainder of this section is devoted to the proof of Proposition 6.1. We first note that

$$L_2\varphi = 0 \quad \text{in } H^{-1},$$

since φ is a critical point of $\Phi|_{\Sigma_\alpha}$ with Lagrangian multiplier λ. Moreover, since $\lambda < \inf\sigma_{\text{ess}}(-\Delta + V)$ by assumption, and since $g(|\varphi|^2)$ vanishes at infinity, Persson's theorem [Hislop and Sigal 1996, Theorem 14.11] implies

$$0 < \inf\sigma_{\text{ess}}(-\Delta + V - \lambda) = \inf\sigma_{\text{ess}}(L_2).$$

Since moreover φ is a positive eigenfunction of L_2 corresponding to the eigenvalue 0, it follows that $0 = \inf\sigma(L_2)$ is a simple isolated eigenvalue. Consequently, putting

$$\widetilde{\Lambda} := \{v \in H^{-1} : \langle v, \varphi\rangle_* = 0\} \subset H^{-1},$$

$$\Lambda := I^{-1}(\widetilde{\Lambda}) = \{v \in H^1 : \textstyle\int_{\mathbb{R}^N} v\varphi = 0\} \subset H^1,$$

we see that the quadratic form $v \mapsto \langle L_2 v, v\rangle_*$ is positive definite on Λ and that L_2 defines an isomorphism $\Lambda \mapsto \widetilde{\Lambda}$. From these properties, we deduce the following.

Lemma 6.2. *We have* $\langle IL_2^{-1}Iv, v\rangle_* > 0$ *for all* $v \in \Lambda\backslash\{0\}$.

Proof. Let $v \in \Lambda\backslash\{0\}$; then $Iv \in \widetilde{\Lambda}$ and by the remarks above there exists $\tilde{v} \in \Lambda\backslash\{0\}$ with $L_2\tilde{v} = Iv$. Consequently, we have

$$\langle IL_2^{-1}Iv, v\rangle_* = \langle I\tilde{v}, v\rangle_* = \langle Iv, \tilde{v}\rangle_* = \langle L_2\tilde{v}, \tilde{v}\rangle_* > 0,$$

by the positive definiteness of the quadratic form $\tilde{v} \mapsto \langle L_2\tilde{v}, \tilde{v}\rangle_*$ on Λ. □

The following lemma is the key step in the proof of Proposition 6.1. It resembles [Esteban and Strauss 1994, Lemma 2.2], but we need to prove it by a different (more general) argument since our setting does not satisfy the assumptions in that paper.

Lemma 6.3. *We have*

$$\mu := \inf_{v\in\Lambda\backslash\{0\}} \frac{\langle L_1 v, v\rangle_*}{\langle IL_2^{-1}Iv, v\rangle_*} \in (-\infty, 0).$$

Moreover, μ *is attained at some* $v \in \Lambda\backslash\{0\}$ *satisfying the equation*

$$L_1 v = \mu IL_2^{-1}Iv + I\beta\varphi \quad \text{in } H^{-1} \tag{6-2}$$

for some $\beta \in \mathbb{R}$.

Proof. Since φ has positive Morse index with respect to $\Phi|_{\Sigma_\alpha}$, there exists $v \in \Lambda\backslash\{0\}$ with $\langle L_1 v, v\rangle_* < 0$, which implies $\mu < 0$. In the following, we consider the spectral decomposition

$$\Lambda = V^- \oplus V^+$$

with the properties that $\dim V^- < \infty$ and

$$\langle L_1 v, v\rangle_* \le 0, \quad \langle L_1 w, w\rangle_* \ge \delta\|w\|^2, \quad \langle L_1 v, w\rangle_* = 0 \quad \text{for } v \in V^-, w \in V^+, \tag{6-3}$$

with some $\delta > 0$. The existence of such a decomposition follows from the fact that $\inf\sigma_{\text{ess}}(L_1) = \inf\sigma_{\text{ess}}(-\Delta + V - \lambda) > 0$. For $v \in \Lambda$, we now write $v = v^- + v^+$ with $v^- \in V^-$, $v^+ \in V^+$. Let $(v_n)_n \subset \Lambda\backslash\{0\}$ be a minimizing sequence for the quotient

$$v \mapsto q(v) := \frac{\langle L_1 v, v\rangle_*}{\langle IL_2^{-1}Iv, v\rangle_*}.$$

Since $\mu = \inf_{v\in\Lambda\backslash\{0\}} q(v) < 0$, we may assume that

$$\langle L_1 v_n, v_n\rangle_* = \langle L_1 v_n^-, v_n^-\rangle_* + \langle L_1 v_n^+, v_n^+\rangle_* < 0 \quad \text{for all } n \in \mathbb{N}. \tag{6-4}$$

Thus $v_n^- \ne 0$, and we may assume that $\|v_n^-\| = 1$ for all $n \in \mathbb{N}$. Since V^- is finite-dimensional, we may pass to a subsequence such that $v_n^- \to v_- \in V^-$ with $\|v_-\| = 1$. Then (6-3) and (6-4) imply

$$\delta \limsup_{n\to\infty} \|v_n^+\|^2 \le \limsup_{n\to\infty} \langle L_1 v_n^+, v_n^+\rangle_* \le -\lim_{n\to\infty} \langle L_1 v_n^-, v_n^-\rangle_* = -\langle L_1 v_-, v_-\rangle_*$$

and thus v_n^+ is bounded in H^1 as well. Hence $(v_n)_n \subset \Lambda$ is bounded in H^1, and we may thus pass to a subsequence such that

$$v_n^+ \rightharpoonup v_+, \quad v_n \rightharpoonup v := v_- + v_+ \in \Lambda \backslash \{0\},$$
$$\langle L_1 v_n, v_n\rangle_* \to \kappa_1 \le 0 \quad \text{and} \quad \langle I L_2^{-1} I v_n, v_n\rangle_* \to \kappa_2 \ge 0$$

as $n \to \infty$. By weak lower semicontinuity, we then have

$$\langle L_1 v_+, v_+\rangle_* \le \lim_{n\to\infty} \langle L_1 v_n^+, v_n^+\rangle_* = \kappa_1 - \langle L_1 v_-, v_-\rangle_*$$

and thus

$$\langle L_1 v, v\rangle_* \le \kappa_1 \le 0.$$

Consequently, since also

$$0 < \langle I L_2^{-1} I v, v\rangle_* \le \kappa_2$$

by Lemma 6.2 and weak lower semicontinuity, we find that

$$q(v) = \frac{\langle L_1 v, v\rangle_*}{\langle I L_2^{-1} I v, v\rangle_*} \le \frac{\langle L_1 v, v\rangle_*}{\kappa_2} \le \frac{\kappa_1}{\kappa_2} = \mu.$$

Hence v is a minimizer of q in $\Lambda \backslash \{0\}$, and therefore $q(v) = \mu > -\infty$. Moreover, v minimizes the functional

$$\Lambda \to \mathbb{R}, \quad w \mapsto \langle L_1 w - \mu I L_2^{-1} I w, w\rangle_*,$$

and therefore we have

$$\langle L_1 v - \mu I L_2^{-1} I v, w\rangle_* = 0 \quad \text{for all } w \in \Lambda.$$

This implies that there exists $\beta \in \mathbb{R}$ such that

$$\langle L_1 v - \mu I L_2^{-1} I v, w\rangle_* = \beta \int_{\mathbb{R}} \varphi w \quad \text{for all } w \in H^1,$$

i.e.,

$$L_1 v - \mu I L_2^{-1} I v = \beta I \varphi \quad \text{in } H^{-1},$$

which gives (6-2). □

Proof of Proposition 6.1 (*completed*). Let μ and v be as in Lemma 6.3, let $\rho = \sqrt{-\mu} > 0$, and consider

$$\boldsymbol{w} = \begin{pmatrix} v \\ -\rho L_2^{-1} I v + \rho^{-1}\beta\varphi \end{pmatrix} \in \mathcal{H} \backslash \{0\}.$$

Then we have

$$\boldsymbol{M}\boldsymbol{w} = \begin{pmatrix} 0 & -L_2 \\ L_1 & 0 \end{pmatrix} \boldsymbol{w} = \begin{pmatrix} \rho I v \\ \mu I L_2^{-1} I v + I\beta\varphi \end{pmatrix} = \rho I \boldsymbol{w},$$

so $\boldsymbol{w} \in \mathcal{H}$ is an eigenfunction of $\boldsymbol{M}$ corresponding to the eigenvalue $\rho > 0$. □

Appendix: Proof of (5-22)

In this section we compute the free Morse index of the rescaled single peak solutions u_ε of (5-6) studied in Section 5. More precisely, we will prove the equality (5-22) for $\varepsilon > 0$ small. We continue to use the notation from Section 5. Recall that since u_ε is a critical point of Φ_ε^* on $\Sigma_{|u_\varepsilon|_2^2}$ with Lagrange multiplier 0, the free Morse index coincides with the Morse index of u_ε as a critical point of Φ_ε^* in H^1. Recall moreover that u_ε has a unique local maximum point x_ε, where $x_\varepsilon \to 0$ as $\varepsilon \to 0$ by [Grossi 2002, Proposition 5.2]. Put

$$u_{0,\varepsilon} := u_0(\,\cdot\, - x_\varepsilon) = \mathcal{T}_{x_\varepsilon} u_0 \in H^1 \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

We first need the following refined convergence estimate:

$$\|u_{0,\varepsilon} - u_\varepsilon\|_{H^2} = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{A-1}$$

Suppose by contradiction that this is false; then along a sequence $(\varepsilon_n)_n \subset (0, \varepsilon_0)$ with $\varepsilon_n \to 0$ we have $d_n := \|u_{0,\varepsilon_n} - u_{\varepsilon_n}\|_{H^2} \ge n\varepsilon_n^2$ for all $n \in \mathbb{N}$. Put $w_n := (u_{0,\varepsilon_n} - u_{\varepsilon_n})/d_n$; then w_n is a weak solution of the equation

$$-\Delta w_n + w_n = \frac{1}{d_n}(u_{0,\varepsilon_n}^{p-1} - u_{\varepsilon_n}^{p-1} + (V_{\varepsilon_n} - 1)u_{\varepsilon_n}) = \tau_n w_n + \frac{V_{\varepsilon_n} - 1}{d_n} u_{\varepsilon_n}, \tag{A-2}$$

with

$$\tau_n(x) = (p-1)\int_0^1 [(1-s)u_{0,\varepsilon_n} + s u_{\varepsilon_n}]^{p-2}\, \mathrm{d}s.$$

We pass to a subsequence such that $w_n \rightharpoonup w$ in H^2. Since $\tau_n \to (p-1)u_0^{p-2}$ as $n \to \infty$ uniformly in $\mathbb{R}^N$ by (5-10), and since

$$\left| \frac{V_{\varepsilon_n} - 1}{d_n} u_{\varepsilon_n}(x) \right| \le \frac{c}{n} |x|^2 \mathrm{e}^{-\alpha|x|} \quad \text{for } x \in \mathbb{R}^N,\ n \in \mathbb{N} \text{ with constants } c, \alpha > 0 \tag{A-3}$$

by (5-9) and (5-13), we may pass to the limit in (A-2) to see that w is a (weak) solution of the equation

$$-\Delta w + w - (p-1)u_0^{p-2} w = 0.$$

Consequently, $w = \sum_{\ell=1}^N a_\ell\, \partial_\ell u_0$ with $\ell = 1, \dots, N$. However, since both u_{0,ε_n} and u_{ε_n} attain a maximum at x_{ε_n}, we infer from (A-2) and elliptic regularity that

$$0 = \lim_{n\to\infty} \partial_j w_n(x_{\varepsilon_n}) = \partial_j w(0) = \sum_{\ell=1}^N a_\ell\, \partial_{\ell j} u_0(0) \quad \text{for } j = 1, \dots, N.$$

It is well known that 0 is the only maximum point of u_0; see, e.g., [McLeod 1993, Lemma 1(b)]. Considering that $u_0(x) = U_0(|x|)$, where U_0 is the solution with initial values $U_0(0) = u_0(0)$ and $U_0'(0) = 0$ of the ordinary differential equation on $[0, \infty)$ corresponding to radial solutions of (5-3), and considering the uniqueness of solutions to that ODE, it is clear that 0 is a nondegenerate maximum point for u_0. Hence it follows that $a_1, \dots, a_N = 0$ and thus $w = 0$. This implies $w_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and thus

$$-\Delta w_n + w_n = o(1) \quad \text{in } L^2(\mathbb{R}^N)$$

by (A-2), (A-3), and since τ_n has exponential decay in x, uniformly in n. The boundedness of the inverse of $-\Delta+1$ on L^2 implies $\|w_n\|_{H^2}\to 0$, contrary to the definition of w_n. Hence (A-1) follows.

We now consider the uniformly bounded families of linear operators

$$A_\varepsilon := \mathrm{D}^2\Phi_\varepsilon^*(u_\varepsilon)\in\mathcal{L}(H^1),$$
$$C_\varepsilon := \mathcal{T}_{-x_\varepsilon}\circ A_\varepsilon\circ\mathcal{T}_{x_\varepsilon}\in\mathcal{L}(H^1),\quad \varepsilon\in(0,\varepsilon_0).$$

Here, as before, the symbol D^2 denotes the derivative of the gradient with respect to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{H^1}$. The quadratic form associated with A_ε is given by

$$\langle A_\varepsilon v, w\rangle_{H^1}=\int_{\mathbb{R}^N}(\nabla v\cdot\nabla w+[V_\varepsilon-(p-1)u_\varepsilon^{p-2}]vw)\quad\text{for } v,w\in H^1. \tag{A-4}$$

It is then clear that A_ε and C_ε share the same spectrum. We have

$$\lim_{\varepsilon\to 0}\|C_\varepsilon v-B^0v\|_{H^1}=\lim_{\varepsilon\to 0}\|A_\varepsilon v-B^0v\|_{H^1}=0\quad\text{for all } v\in H^1, \tag{A-5}$$

where, as before, $B^0=\mathrm{D}^2\Phi_0^*(u_0)\in\mathcal{L}(H^1)$, and the convergence is uniform on compact subsets of H^1. We claim that

$$\|C_\varepsilon\,\partial_i u_0\|_{H^1}=O(\varepsilon^2)\quad\text{for } i=1,\dots,N, \tag{A-6}$$

and that

$$\langle C_\varepsilon\,\partial_i u_0,\partial_j u_0\rangle_{H^1}=\tfrac{1}{2}\varepsilon^2\,\partial_{ij}V(0)|u_0|_2^2+o(\varepsilon^2)\quad\text{for } i,j=1,\dots,N \tag{A-7}$$

as $\varepsilon\to 0$. For this we recall that $\partial_i u_\varepsilon$ solves the equation

$$-\Delta(\partial_i u_\varepsilon)+V_\varepsilon\,\partial_j u_\varepsilon-(p-1)u_\varepsilon^{p-2}\,\partial_j u_\varepsilon=-u_\varepsilon\,\partial_j V_\varepsilon, \tag{A-8}$$

and therefore (5-9) and (5-14) yield

$$\begin{aligned}A_\varepsilon\,\partial_i u_\varepsilon&=(-\Delta+1)^{-1}(-\Delta(\partial_i u_\varepsilon)+V_\varepsilon\,\partial_i u_\varepsilon-(p-1)u_\varepsilon^{p-2}\,\partial_i u_\varepsilon)\\&=-(-\Delta+1)^{-1}u_\varepsilon\,\partial_j V_\varepsilon=O(\varepsilon^2)\quad\text{in } H^1.\end{aligned} \tag{A-9}$$

Combining this with (A-1), we find that

$$\|C_\varepsilon\,\partial_i u_0\|_{H^1}=\|A_\varepsilon\,\partial_i u_{0,\varepsilon}\|_{H^1}=\|A_\varepsilon\,\partial_i u_\varepsilon\|_{H^1}+O(\varepsilon^2)=O(\varepsilon^2),$$

as claimed in (A-6). To see (A-7), we note that

$$\begin{aligned}\langle C_\varepsilon\,\partial_i u_0,\partial_j u_0\rangle_{H^1}&=\langle A_\varepsilon\,\partial_i u_{0,\varepsilon},\partial_j u_{0,\varepsilon}\rangle_{H^1}\\&=\langle A_\varepsilon\,\partial_i u_\varepsilon,\partial_j u_\varepsilon\rangle_{H^1}+\langle A_\varepsilon\,\partial_i u_{0,\varepsilon},\partial_j(u_{0,\varepsilon}-u_\varepsilon)\rangle_{H^1}+\langle A_\varepsilon\,\partial_j u_\varepsilon,\partial_i(u_{0,\varepsilon}-u_\varepsilon)\rangle_{H^1},\end{aligned} \tag{A-10}$$

where, since $\partial_i u_{0,\varepsilon}$ satisfies $-\Delta\,\partial_i u_{0,\varepsilon}+\partial_i u_{0,\varepsilon}-(p-1)u_{0,\varepsilon}^{p-2}\,\partial_i u_{0,\varepsilon}=0$ in $\mathbb{R}^N$,

$$\langle A_\varepsilon\,\partial_i u_{0,\varepsilon},\partial_j(u_{0,\varepsilon}-u_\varepsilon)\rangle_{H^1}=\int_{\mathbb{R}^N}[V_\varepsilon-1+(p-1)(u_{0,\varepsilon}^{p-2}-u_\varepsilon^{p-2})]\,\partial_i u_{0,\varepsilon}\,\partial_j(u_{0,\varepsilon}-u_\varepsilon)=o(\varepsilon^2)$$

as $\varepsilon\to 0$. Here, in the last step, we used (A-1) together with the fact that

$$\|[V_\varepsilon-1+(p-1)(u_{0,\varepsilon}^{p-2}-u_\varepsilon^{p-2})]\,\partial_i u_{0,\varepsilon}\|_{L^2}\to 0\quad\text{as }\varepsilon\to 0.$$

Moreover,

$$|\langle A_\varepsilon\, \partial_j u_\varepsilon, \partial_i(u_{0,\varepsilon}-u_\varepsilon)\rangle_{H^1}| \le \|A_\varepsilon\, \partial_j u_\varepsilon\|_{H^1}\|\partial_i(u_{0,\varepsilon}-u_\varepsilon)\|_{H^1} \le O(\varepsilon^4)$$

by (A-1) and (A-9). Inserting these estimates in (A-10) and using (A-8) once more, together with (5-9), (5-10), and (5-14) we find that

$$\begin{aligned}\langle C_\varepsilon\, \partial_i u_0, \partial_j u_0\rangle_{H^1} &= \langle A_\varepsilon\, \partial_i u_\varepsilon, \partial_j u_\varepsilon\rangle_{H^1} + o(\varepsilon^2) = -\int_{\mathbb{R}^N} u_\varepsilon\, \partial_i V_\varepsilon\, \partial_j u_\varepsilon + o(\varepsilon^2)\\ &= -\varepsilon^2 \sum_{\ell=1}^{N} \partial_{i\ell} V(0) \int_{\mathbb{R}^N} x_\ell u_0\, \partial_j u_0\, \mathrm{d}x + o(\varepsilon^2) = \tfrac{1}{2}\varepsilon^2\, \partial_{ij} V(0)|u_0|_2^2 + o(\varepsilon^2).\end{aligned}$$

In the last step we have integrated by parts again. This yields (A-7).

To conclude the proof of (5-22), we now put $X = \operatorname{span}(u_0)$, $Y := \operatorname{span}(\partial_1 u_0, \dots, \partial_N u_0)$, and we let Z denote the $\langle\,\cdot\,,\cdot\,\rangle_{H^1}$-orthogonal complement of $X \oplus Y$ in H^1. We then have the $\langle\,\cdot\,,\cdot\,\rangle_{H^1}$-orthogonal decomposition $H^1 = X \oplus Y \oplus Z$, and we let $P_X, P_Y, P_Z \in \mathcal{L}(H^1)$ denote the corresponding orthogonal projections onto X, Y, and Z. It then follows from (A-6) that

$$\|C_\varepsilon P_Y\|_{\mathcal{L}(H^1)} = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{A-11}$$

Moreover, by the remarks before Lemma 5.2, there exists $0 < \delta < 1$ such that

$$\langle B^0 u_0, u_0\rangle_{H^1} \le -\delta \qquad \text{and} \qquad \langle B^0 w, w\rangle_{H^1} \ge \delta\|w\|_{H^1}^2 \quad \text{for all } w \in Z. \tag{A-12}$$

It then follows from (A-5) that

$$\langle C_\varepsilon u_0, u_0\rangle_{H^1} < -\tfrac{1}{2}\delta \quad \text{for } \varepsilon > 0 \text{ sufficiently small.} \tag{A-13}$$

We also claim that

$$\inf_{w\in Z, \|w\|_{H^1}=1} \langle C_\varepsilon w, w\rangle_{H^1} > \delta_+ := \tfrac{1}{2}\min\bigl\{\delta, \inf_{\mathbb{R}^N} V\bigr\} \quad \text{for } \varepsilon > 0 \text{ sufficiently small.} \tag{A-14}$$

Indeed, suppose by contradiction there exist $\varepsilon_n \in (0, \varepsilon_0)$ and $w_n \in Z$ with $\|w_n\|_{H^1} = 1$ for $n \in \mathbb{N}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\langle C_{\varepsilon_n} w_n, w_n\rangle_{H^1} \le \delta_+ \quad \text{as } n \to \infty. \tag{A-15}$$

Passing to a subsequence, we may then assume that $w_n \rightharpoonup w$ in H^1 with $w \in Z$. We put $\tilde w_n := \mathcal{T}_{x_{\varepsilon_n}} w_n = w_n(\,\cdot\, - x_{\varepsilon_n})$ for $n \in \mathbb{N}$; then also $\tilde w_n \rightharpoonup w$, and we may pass to a subsequence such that $\tilde w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\tilde w_n \to w$ pointwise a.e. on $\mathbb{R}^N$. By (5-9) and (5-10) this implies

$$\int_{\mathbb{R}^N} u_{\varepsilon_n}^{p-2} \tilde w_n^2 \to \int_{\mathbb{R}^N} u_0^{p-2} w^2 \quad \text{as } n \to \infty. \tag{A-16}$$

We also have

$$\int_{\mathbb{R}^N} \bigl(|\nabla(\tilde w_n - w)|^2 + V_{\varepsilon_n}(\tilde w_n - w)^2\bigr) = o(1) + \int_{\mathbb{R}^N} \bigl(|\nabla \tilde w_n|^2 - |\nabla w|^2 + V_{\varepsilon_n}[\tilde w_n^2 - w^2 - 2(\tilde w_n - w)w]\bigr),$$

where, since $|\tilde{w}_n - w| \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} V_{\varepsilon_n}(\tilde{w}_n - w)w\right| \le \|V\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\tilde{w}_n - w||w| \to 0 \quad \text{as } n \to \infty.$$

Moreover,

$$\int_{\mathbb{R}^N} V_{\varepsilon_n} w^2 \to \int_{\mathbb{R}^N} w^2 \quad \text{as } n \to \infty$$

by (5-13) and Lebesgue's theorem. Consequently,

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla\tilde{w}_n|^2+V_{\varepsilon_n}\tilde{w}_n^2) &= \int_{\mathbb{R}^N}(|\nabla w|^2+w^2)+\int_{\mathbb{R}^N}(|\nabla(\tilde{w}_n-w)|^2+V_{\varepsilon_n}(\tilde{w}_n-w)^2)+o(1)\\ &\ge \|w\|_{H^1}^2+\min\bigl\{1,\inf_{\mathbb{R}^N} V\bigr\}\|\tilde{w}_n-w\|_{H^1}^2+o(1) \ge \|w\|_{H^1}^2+2\delta_+\|\tilde{w}_n-w\|_{H^1}^2+o(1),\end{aligned}$$

and together with (A-4), (A-12) and (A-16) this implies

$$\begin{aligned}\langle C_{\varepsilon_n} w_n, w_n\rangle_{H^1} &= \langle A_{\varepsilon_n}\tilde{w}_n, \tilde{w}_n\rangle_{H^1} \ge \langle B^0 w, w\rangle_{H^1} + 2\delta_+\|\tilde{w}_n - w\|_{H^1}^2 + o(1)\\ &\ge 2\delta_+\|w\|_{H^1}^2 + 2\delta_+\|\tilde{w}_n - w\|_{H^1}^2 + o(1) = 2\delta_+\|w_n\|_{H^1}^2 + o(1) = 2\delta_+ + o(1).\end{aligned}$$

This contradicts (A-15), and hence (A-14) follows.

In the following, we let $M \in \mathbb{R}^{N\times N}$ denote the Hessian of the potential V at 0 which is nondegenerate by assumption. Then there exists a basis of eigenvectors $b^1, \dots, b^N \in \mathbb{R}^N$ of M corresponding to the eigenvalues $\mu_1 \le \cdots \le \mu_N$, where

$$\mu_i < 0 \quad \text{for } i \le m_V \qquad \text{and} \qquad \mu_i > 0 \quad \text{for } i > m_V.$$

We then let $w^1, \dots, w^N \in \operatorname{span}(\partial_1 u_0, \dots, \partial_N u_0)$ be defined by

$$w^i := \sum_{j=1}^N b^i_j\, \partial_j u_0 \quad \text{for } i = 1, \dots, N,$$

and we define the subspaces $\widetilde{Y}_\pm \subset Y$ by

$$\widetilde{Y}_- := \operatorname{span}(w^1, \dots, w^m) \quad \text{and} \quad \widetilde{Y}_+ := \operatorname{span}(w^{m+1}, \dots, w^N).$$

By (A-7) and construction, there exists $\tilde{\delta} > 0$ such that for $\varepsilon > 0$ sufficiently small we have

$$\langle C_\varepsilon w, w\rangle_{H^1} \le -\tilde{\delta}\varepsilon^2\|w\|_{H^1}^2 \quad \text{for } w \in \widetilde{Y}_- \qquad \text{and} \qquad \langle C_\varepsilon w, w\rangle_{H^1} \ge \tilde{\delta}\varepsilon^2\|w\|_{H^1}^2 \quad \text{for } w \in \widetilde{Y}_+. \tag{A-17}$$

We now consider the spaces

$$\widetilde{X} := \operatorname{span}(u_0) \oplus \widetilde{Y}_- \quad \text{and} \quad \widetilde{Z} := Z \oplus \widetilde{Y}_+.$$

Then (5-22) follows once we have shown that

$$\sup_{w\in\widetilde{X}, \|w\|_{H^1}=1} \langle C_\varepsilon w, w\rangle_{H^1} < 0, \tag{A-18}$$

$$\inf_{w\in\widetilde{Z}, \|w\|_{H^1}=1} \langle C_\varepsilon w, w\rangle_{H^1} > 0 \tag{A-19}$$

for $\varepsilon > 0$ sufficiently small. We only show (A-19); the proof of (A-18) is very similar but simpler. Suppose by contradiction that (A-19) does not hold true for $\varepsilon > 0$ sufficiently small. Then there exist $\varepsilon_n \in (0, \varepsilon_0)$ and $w_n \in \widetilde{Z}$ with $\|w_n\|_{H^1} = 1$ for $n \in \mathbb{N}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\langle C_{\varepsilon_n} w_n, w_n \rangle_{H^1} \le 0 \quad \text{as } n \to \infty. \tag{A-20}$$

With $w_n^1 := P_Z w_n \in Z$ and $w_n^2 := P_Y w_n \in \widetilde{Y}^+$ we have, by (A-11), (A-14) and (A-17),

$$\begin{aligned}\langle C_{\varepsilon_n} w_n, w_n \rangle_{H^1} &= \langle C_{\varepsilon_n} w_n^1, w_n^1 \rangle_{H^1} + \langle C_{\varepsilon_n} w_n^2, w_n^2 \rangle_{H^1} + 2\langle C_{\varepsilon_n} w_n^2, w_n^1 \rangle_{H^1} \\ &\ge \delta_+ \|w_n^1\|_{H^1}^2 + \tilde{\delta} \|w_n^2\|_{H^1}^2 \varepsilon_n^2 + O(\|w_n^1\|_{H^1} \varepsilon_n^2).\end{aligned}$$

Passing to a subsequence, we may assume that either $\|w_n^1\|_{H^1} \to 0$ and $\|w_n^2\|_{H^1} \to 1$ as $n \to \infty$, or that $\|w_n^1\|_{H^1} \ge c$ for some constant $c > 0$ and all $n \in \mathbb{N}$. In the first case, we deduce that

$$\langle C_{\varepsilon_n} w_n, w_n \rangle_{H^1} \ge \tilde{\delta} \varepsilon_n^2 + o(\varepsilon_n^2)$$

and in the second case we obtain that

$$\langle C_{\varepsilon_n} w_n, w_n \rangle_{H^1} \ge \delta_+ c^2 + o(1)$$

as $n \to \infty$. In both cases we arrive at a contradiction to (A-20), and thus (A-19) is proved. As remarked before, (A-18) is obtained similarly by using (A-13) and the first inequality in (A-17). The proof of (5-22) is thus finished.

Acknowledgement

The authors wish to thank the referee for valuable comments and corrections.

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Received 11 Jul 2017. Revised 6 Apr 2018. Accepted 12 Aug 2018.

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ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1215

SCALE-INVARIANT FOURIER RESTRICTION TO A HYPERBOLIC SURFACE

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This result sharpens the bilinear-to-linear deduction of Lee and Vargas for extension estimates on the hyperbolic paraboloid in $\mathbb{R}^3$ to the sharp line, leading to the first scale-invariant restriction estimates, beyond the Stein–Tomas range, for a hypersurface on which the principal curvatures have different signs.

1. Introduction

We consider the Fourier restriction/extension problem for the hyperbolic paraboloid

$$S := \{(\tau, \xi) \in \mathbb{R}^{1+2} : \tau = \xi_1 \xi_2\}.$$

We denote by $\mathcal{E}$ the extension operator,

$$\mathcal{E} f(t, x) := \int_{\mathbb{R}^2} e^{i(t,x)(\xi_1\xi_2,\xi)} f(\xi)\, d\xi. \tag{1-1}$$

For consistency of exponents, we will consider the problem of establishing $L^r \to L^{2s}$ extension estimates for $\mathcal{E}$, and we are primarily interested in the case when $r = s'$.

Lee [2006] and Vargas [2005] independently established an essentially optimal L^2-based bilinear adjoint restriction estimate for S. This result states that if f and g are supported in 1×1 axis-parallel rectangles that are separated from one another by a distance 1 in the horizontal direction and 1 in the vertical direction, then

$$\|\mathcal{E} f \mathcal{E} g\|_s \lesssim \|f\|_2 \, \|g\|_2, \quad s > \tfrac{5}{3}. \tag{1-2}$$

This two-parameter separation of the tiles is both necessary and troublesome. On the one hand, necessity can be seen by considering the case when each of $f_\pm$ is supported on a $\frac{1}{2}$-neighborhood of $(\pm 1, 0)$. On the other hand, the separation leads to difficulty in deducing linear restriction estimates from the bilinear ones. Indeed, the natural analogue of the Whitney decomposition approach of [Tao, Vargas, and Vega 1998] produces a sum in two scales, length and width, rather than a single distance scale, leading to a loss of the scaling line in the distinct approaches of [Lee 2006] and [Vargas 2005].

The purpose of this note is to overcome this obstacle and recover the sharp line.

Theorem 1.1. *With $\mathcal{E}$ as in* (1-1), *assume that the estimate*

$$\|\mathcal{E} f \mathcal{E} g\|_s \lesssim \|f\|_r \, \|g\|_r \tag{1-3}$$

MSC2010: 42B20.
Keywords: Fourier restriction, bilinear restriction, hyperbolic restriction.

holds for some $\frac{3}{2} < s < 2$ *and* $\frac{r}{2} < s < r'$*, whenever* f *and* g *are supported on* 1×1*, axis-parallel rectangles that are separated from one another by a distance* 1 *in both the horizontal and vertical directions. Then* $\mathcal{E}$ *is of restricted strong type* $(s', 2s)$*, and consequently of strong type* $(\tilde{s}', 2\tilde{s})$ *for all* $\tilde{s} > s$*.*

To put the hypothesis on s in context, we recall that for $s \leq \frac{3}{2}$, linear extension estimates are known to be impossible; that for $s > \frac{3}{2}$, $2s > s'$; and that for $s \geq 2$, linear extension estimates are already known, [Tomas 1975].

As is well known, a (local, linear) $L^{r_0} \to L^{2s_0}$ extension estimate for some $r_0 > s_0'$ allows us, by interpolation with the L^2-based bilinear extension estimate (1-2), to establish the L^r-based bilinear extension estimate (1-3) for some $s > s_0$ and $\frac{r}{2} < s < r'$. Replacing s_0 with s is a loss (whose magnitude depends on the distance from (r_0^{-1}, s_0^{-1}) to the scaling line), but $r < s'$ is a gain in the sense that the corresponding linear extension estimate $\mathcal{E} : L^r \to L^{2s}$ is false.

Lee [2006] and Vargas [2005] independently used the bilinear extension estimate (1-2) to prove that

$$\|\mathcal{E} f\|_{2s} \lesssim \|f\|_{L^r} \tag{1-4}$$

for all $s > \frac{5}{3}$, $r > s'$, and f supported in the unit ball. Cho and Lee [2017] used the polynomial partitioning argument from [Guth 2016] to prove (1-4) for f supported in the unit ball and $2s = r > 3.25$; this was subsequently improved by Kim [2017] to the range $2s > 3.25$ and $r > s'$. Using these results and the discussion in the preceding paragraph, Theorem 1.1 immediately yields the following slight improvement on Kim's result.

Corollary 1.2. *For* $2s > 3.25$*, the extension operator* $\mathcal{E}$ *is bounded from* $L^{s'}$ *to* L^{2s}*.*

To the author's knowledge, this is the first scalable restriction estimate for a negatively curved hypersurface, beyond the Stein–Tomas range ($s = 2$).

Terminology. A constant will be said to be admissible if it depends only on s, r. The inequality $A \lesssim B$ means that $A \leq CB$ for some implicit, admissible constant C, and implicit constants will be allowed to change from line to line. A dyadic interval is an interval of the form $[m2^{-n}, (m+1)2^{-n}]$ for some $m, n \in \mathbb{Z}$, and $\mathcal{I}_n$ denotes the set of all dyadic intervals of length 2^{-n}. A tile is a product of two dyadic intervals, and $\mathcal{D}_{J,K}$ denotes the set of all $2^{-J} \times 2^{-K}$ tiles. We denote by π_1, π_2 the projections $\pi_j : \mathbb{R}^2 \to \mathbb{R}$, $\pi_j(x) = x_j$. We use $\mathcal{H}^1$ for the one-dimensional Hausdorff measure. Finally, we use log to denote the base-2 logarithm.

Outline of proof. To prove our restricted strong-type estimate, it suffices to bound the extension of a characteristic function. Our starting point is the bilinear-to-linear deduction of [Vargas 2005], which shows that, under the hypotheses of Theorem 1.1, the extension of the characteristic function of a set Ω with roughly constant (vertical) fiber length obeys the scalable restriction estimate $\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim |\Omega|^{\frac{1}{s'}}$. In [Vargas 2005], off-scaling estimates are obtained by subdividing a set Ω in the unit cube into subsets having constant fiber length. Off-scaling contributions from those subsets with very short fibers are small (because the sets themselves are small), and adding these amounts to summing a convergent geometric series.

We wish to remain on the sharp line, so we must be more careful. Our first step, taken in Section 2, is to understand when Vargas's constant fiber length estimate can be improved. To this end, we prove a dichotomy result: If Ω has constant fiber length, then either Ω is highly structured (more precisely, Ω is nearly a tile), or we have a better bound on the extension of χ_Ω. Roughly speaking, this reduces matters to controlling the extension of a union of tiles τ_k each having height 2^{-k}, which is the task of Section 3. We can estimate

$$\|\mathcal{E}\chi_{\bigcup \tau_k}\|_{2s} \lesssim \Big(\sum \|\mathcal{E}\chi_{\tau_k}\|_{2s}^{2s}\Big)^{\frac{1}{2s}} + \text{off-diagonal terms},$$

where the off-diagonal terms involve products $\mathcal{E}\chi_{\tau_k}\mathcal{E}\chi_{\tau_{k'}}$, with $|k-k'|$ large. Boundedness of the main term follows from Vargas's estimate and convexity ($2s > s'$). It remains to bound the off-diagonal terms, for which it suffices to prove a bilinear estimate with decay:

$$\|\mathcal{E}\chi_{\tau_k}\mathcal{E}\chi_{\tau_{k'}}\|_s \lesssim 2^{-c_0|k-k'|}\max\{|\tau_k|, |\tau_{k'}|\}^{\frac{1}{s'}},$$

and we prove this by combining the bilinear extension estimate for separated tiles with a further decomposition.

Of course, we have lied. In Section 2, our dichotomy is not that a constant fiber length set Ω is either a tile or has zero extension, and so we still have remainder terms that must be summed. To address this, we argue more quantitatively than has been suggested above: Any constant fiber length set can be approximated by a union of tiles, where the number of tiles and tightness of the approximation depends on the sharpness of our estimate $\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim |\Omega|^{\frac{1}{s'}}$; then we must bound extensions of sets $\bigcup_k \bigcup_{\tau\in\mathcal{T}_k} \tau$, where $\mathcal{T}_k \subseteq \mathcal{D}_{j(k),k}$ may be large (but fortunately, not too large).

2. An inverse problem related to Vargas's linear estimate

To prove Theorem 1.1, it suffices to prove that $\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim |\Omega|^{\frac{1}{s'}}$ for all measurable sets Ω. By scaling, it suffices to consider Ω contained in the unit cube $[-1, 1]^2$. Vargas [2005] proved the following.

Theorem 2.1 [Vargas 2005]. *For each $K \geq 0$, let*

$$\Omega(K) := \{\xi \in \Omega : \mathcal{H}^1(\pi_1^{-1}(\xi_1) \cap \Omega) \sim 2^{-K}\}. \tag{2-1}$$

Then under the hypotheses of Theorem 1.1, for any measurable set $\Omega' \subseteq \Omega(K)$,

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim |\Omega(K)|^{\frac{1}{s'}}. \tag{2-2}$$

This version differs slightly from the one stated in [Vargas 2005], but it follows from the same proof. In proving the next proposition, we will review Vargas's argument, so the reader may verify the above-stated version below.

Our first step is to solve an inverse problem: Characterize those sets $\Omega = \Omega(K)$ for which the inequality in (2-2) can be reversed.

Proposition 2.2. *Assume that the hypotheses of Theorem 1.1 hold. Let $\Omega \subseteq [-1, 1]^2$ be a measurable set, and assume that $\Omega = \Omega(K)$ for some integer $K \geq 0$. Choose a nonnegative integer J such that*

$|\pi_1(\Omega)| \sim 2^{-J}$, *and let* $\varepsilon \lesssim 1$ *denote the smallest dyadic number such that*

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \le \varepsilon |\Omega|^{\frac{1}{s'}}$$

for every measurable $\Omega' \subseteq \Omega$. *Then* $\Omega = \bigcup_{0<\delta\le\varepsilon} \Omega_\delta$, *with the union taken over dyadic* δ. *For each* δ, $\Omega_\delta \subseteq \bigcup_{\tau\in\mathcal{T}_\delta} \tau$, *where* $\mathcal{T}_\delta \subseteq \mathcal{D}_{J,K}$ *has cardinality at most* $O(\delta^{-C})$, *with* C *an admissible constant. For each subset* $\Omega' \subseteq \Omega_\delta$, $\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \delta|\Omega|^{\frac{1}{s'}}$.

Proof of Proposition 2.2. It suffices to produce a union that contains almost every point of Ω, as a set of measure zero makes no contribution to the extension. Our decomposition will be done in three stages. Our first decomposition will be of Ω into sets Ω^1_η, with $\pi_1(\Omega^1_\eta)$ nearly an interval, $I \in \mathcal{I}_J$. Our second decomposition will be of Ω^1_η into sets $\Omega^2_{\eta,\rho}$, $\rho \le \eta$, each of which is nearly a product of I with a set of measure 2^{-K}. Our third decomposition will be of $\Omega^2_{\eta,\rho}$ into sets $\Omega^3_{\eta,\rho,\delta}$, $\delta \le \rho$, each of which is nearly a product of I with an interval in $\mathcal{I}_K$. The product of two dyadic intervals is a tile, so we take $\Omega_\delta := \bigcup_{\rho\ge\delta}\bigcup_{\eta\ge\rho}\Omega^3_{\eta,\rho,\delta}$; the $(\log\delta^{-1})^2$ factor that arises from taking this union is harmless.

Let $S := \pi_1(\Omega)$. We know that $|S| \sim 2^{-J}$ and that $S \subseteq [-1, 1]$. Let $\xi_1 \in S$, and for each $0 < \eta < \varepsilon$, let $I_\eta(\xi_1)$ be the maximal dyadic interval $I \ni \xi_1$ satisfying $|I \cap S| \ge \eta^C |I|$, if such an interval exists. We record that $|I_\eta(\xi_1)| \le \eta^{-C} 2^{-J}$, and if ξ_1 is a Lebesgue point of S, then $|I_\eta(\xi_1)| > 0$. Let

$$T_\eta := \{\xi_1 \in S : |I_\eta(\xi_1)| \ge \eta^C 2^{-J}\},$$

and let $S_\varepsilon := T_\varepsilon$, $S_\eta := T_\eta \setminus T_{2\eta}$ for dyadic $0 < \eta < \varepsilon$. Then a.e. (indeed, every Lebesgue) point of S is contained in a unique S_η. We set $\Omega^1_\eta := \Omega \cap \pi_1^{-1}(S_\eta)$.

Lemma 2.3. *For each* $0 < \eta \le \varepsilon$, S_η *is contained in a union of* $O(\eta^{-3C})$ *dyadic intervals* $I \in \mathcal{I}_J$, *and for each* $\eta < \varepsilon$ *and each subset* $\Omega' \subseteq \Omega^1_\eta$,

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \eta^2 |\Omega|^{\frac{1}{s'}}. \tag{2-3}$$

Proof of Lemma 2.3. By construction, S_η is covered by dyadic intervals I of length $|I| \ge \eta^C|S|$, in which S has density $|I \cap S| \ge \eta^C |I|$. The density of each such interval in S is $|I \cap S| \ge \eta^{2C}|S|$, and so the collection of maximal (hence pairwise disjoint) dyadic intervals with these properties has cardinality at most η^{-2C}. Moreover, from the density estimate, we see that $|I| \le \eta^{-C}2^{-J}$, so these intervals are covered by a total of η^{-3C} intervals in $\mathcal{I}_J$.

To establish (2-3), we will optimize Vargas's proof of Theorem 2.1. Performing a Whitney decomposition in each variable ξ_1, ξ_2 separately and applying the almost orthogonality lemma from [Tao, Vargas, and Vega 1998] (for which it is important that $s \le 2$),

$$\|\mathcal{E}\chi_{\Omega'}\|^2_{2s} \lesssim \sum_{k,j}\Bigg(\sum_{\tau\sim\tau'\in\mathcal{D}_{j,k}} \|\mathcal{E}\chi_{\Omega'\cap\tau}\mathcal{E}\chi_{\Omega'\cap\tau'}\|^s_s\Bigg)^{\frac{1}{s}},$$

where we say that $\tau \sim \tau'$ if τ and τ' are 2^{-j} separated in the horizontal direction and 2^{-k} separated in the vertical direction.

By rescaling our hypothesis, (1-3), for f, g supported on tiles in $\mathcal{D}_{j,k}$ that are separated by a distance 2^{-k} in the vertical direction and 2^{-j} in the horizontal direction,

$$\|\mathcal{E} f \mathcal{E} g\|_s \lesssim 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \|f\|_r \|g\|_r. \tag{2-4}$$

Thus

$$\begin{aligned}\|\mathcal{E}\chi_{\Omega'}\|_{2s}^2 &\lesssim \sum_{k,j} 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \Big(\sum_{\tau\in\mathcal{D}_{j,k}} |\Omega'\cap\tau|^{\frac{2s}{r}}\Big)^{\frac{1}{s}} \\ &\lesssim \sum_{k,j} 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \max_{\tau\in\mathcal{D}_{j,k}} |\Omega'\cap\tau|^{\frac{2}{r}-\frac{1}{s}} |\Omega'|^{\frac{1}{s}}. \end{aligned} \tag{2-5}$$

Our hypotheses on r, s imply that all exponents in the above sum are positive. To bound this double sum, Vargas used the inequality

$$|\Omega'\cap\tau| \lesssim \min\{2^{-j}, 2^{-J}\} \min\{2^{-k}, 2^{-K}\}. \tag{2-6}$$

The definition of Ω^1_η will allow us to improve on this bound.

For $I_j \in \mathcal{I}_j$, we trivially have $|I_j \cap S_\eta| \le \min\{|I_j|, |S_\eta|\} \le \min\{2^{-j}, 2^{-J}\}$, but when $|j-J| < \frac{C}{4}\log\eta^{-1}$, we can do rather better. Suppose that $|j-J| \le \frac{C}{4}\log\eta^{-1}$. Since

$$|I_j| = 2^{-j} \ge \eta^{\frac{C}{4}} 2^{-J} \ge (2\eta)^C 2^{-J}$$

(provided η is sufficiently small), $I_j \cap S_\eta \neq \varnothing$ implies that $I_j \cap S_\eta \not\subseteq T_{2\eta}$, whence

$$|I_j \cap S_\eta| \le |I_j \cap S| \le (2\eta)^C |I_j| = (2\eta)^C 2^{-j} \lesssim \eta^{\frac{3C}{4}} \min\{2^{-j}, 2^{-J}\},$$

where for the last inequality, we used $2^{-j} \le \eta^{-\frac{C}{4}} 2^{-J}$.

Inserting this gain and $|\Omega'\cap(I_j\times I_k)| \le |S_\eta \cap I_j| \min\{2^{-k}, 2^{-K}\}$ into (2-5), and summing the resulting geometric series gives

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \eta^{C'} 2^{-(J+K)(1-\frac{3}{2s})} |\Omega'|^{\frac{1}{2s}} \lesssim \eta^{C'} |\Omega|^{\frac{1}{s'}}$$

for $C' > 0$ some admissible constant dictated by C, r, s; we can reverse engineer C so that $C' = 2$. □

We now turn to our second decomposition. Although $\pi_1(\Omega^1_\eta)$ may be (roughly) thought of as a union of a small number of intervals, an individual horizontal slice $\pi_2^{-1}(\xi_2) \cap \Omega^1_\eta$ might be much smaller. Our next step is to decompose into sets where the size of a nonempty slice is roughly comparable to the size of the projection of the whole. (Sets with this property are nearly products.)

Fix $0 < \eta \le \varepsilon$. For dyadic $0 < \rho \le \eta$, we define

$$V_\rho = \{\xi_2 \in \pi_2(\Omega^1_\eta) : \mathcal{H}^1(\pi_2^{-1}(\xi_2) \cap \Omega^1_\eta) \ge \rho^C 2^{-J}\},$$

and set $U_\eta := V_\eta$, $U_\rho := V_\rho \setminus V_{2\rho}$ for $\rho < \eta$. We define $\Omega^2_{\eta,\rho} := \pi_2^{-1}(U_\rho) \cap \Omega^1_\eta$.

Lemma 2.4. *For each $0 < \rho < \eta \le \varepsilon$, and each subset $\Omega' \subseteq \Omega^2_{\eta,\rho}$, we have $\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \rho^2 |\Omega|^{\frac{1}{s'}}$.*

Proof of Lemma 2.4. Let $\tau \in \mathcal{D}_{j,k}$ and $\Omega' \subseteq \Omega^2_{\eta,\rho}$. Then $\tau \cap \Omega'$ has vertical and horizontal fiber lengths at most

$$\int \chi_{\tau\cap\Omega'}(\xi_1,\xi_2)\,d\xi_2 \lesssim \min\{2^{-K},2^{-k}\}, \quad \int \chi_{\tau\cap\Omega'}(\xi_1,\xi_2)\,d\xi_1 \lesssim \min\{\rho^C 2^{-J},2^{-j}\},$$

respectively, and projections of size at most

$$|\pi_1(\tau\cap\Omega')| \lesssim \min\{2^{-J},2^{-j}\}, \quad |\pi_2(\tau\cap\Omega')| \lesssim 2^{-k}.$$

By Fubini, we can bound $|\tau\cap\Omega'|$ by the measure of the projection times the maximum fiber length, so

$$|\tau\cap\Omega'| \lesssim \min\{2^{-(J+K)},\ 2^{-(j+K)},\ 2^{-(j+k)},\ \rho^C 2^{-(J+k)}\}. \tag{2-7}$$

To utilize (2-7), we let $C' = \frac{C}{2}$ and subdivide $\mathbb{Z}^2 = R_1\cup R_2\cup R_3\cup R_4$, where

$$\begin{aligned}
R_1 &:= \{(j,k) : J - C'\log\rho^{-1} \ge j,\ K \ge k\} \cup \{(j,k) : J \ge j,\ K - C'\log\rho^{-1} \ge k\},\\
R_2 &:= \{(j,k) : j \ge J + C'\log\rho^{-1},\ K \ge k\} \cup \{(j,k) : j \ge J,\ K - C'\log\rho^{-1} \ge k\},\\
R_3 &:= \{(j,k) : j \ge J + C'\log\rho^{-1},\ k \ge K\} \cup \{(j,k) : j \ge J,\ k \ge K + C'\log\rho^{-1}\},\\
R_4 &:= \{(j,k) : J + C'\log\rho^{-1} \ge j,\ k + C'\log\rho^{-1} \ge K\}.
\end{aligned}$$

Now we insert (2-7) into (2-5) to obtain

$$\begin{aligned}
\|\mathcal{E}\chi_{\Omega'}\|_{2s}^2 \lesssim{}& \sum_{R_1} 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} 2^{-\frac{2(J+K)}{r}} + \sum_{R_2} 2^{-j(2-\frac{3}{s})} 2^{k(\frac{2}{s}+\frac{2}{r}-2)} 2^{-\frac{J}{s}} 2^{-\frac{2K}{r}}\\
&+ \sum_{R_3} 2^{-(j+k)(2-\frac{3}{s})} 2^{-\frac{J+K}{s}} + \rho^{C(\frac{2}{r}-\frac{1}{s})} \sum_{R_4} 2^{-k(2-\frac{3}{s})} 2^{j(\frac{2}{s}+\frac{2}{r}-2)} 2^{-\frac{2J}{r}} 2^{-\frac{K}{s}}.
\end{aligned}$$

As $\frac{2}{s}+\frac{2}{r}-2$ and $2-\frac{3}{s}$ are both positive, it is a simple matter to sum each of these terms, obtaining

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s}^2 \lesssim \big(\rho^{(\frac{2}{s}+\frac{2}{r}-2)C'} + \rho^{(2-\frac{3}{s})C'} + \rho^{C(\frac{2}{r}-\frac{1}{s}) - C'(2-\frac{3}{s}) - C'(\frac{2}{s}+\frac{2}{r}-2)}\big) 2^{-\frac{2(J+K)}{s'}}.$$

Since $\frac{2}{r}-\frac{1}{s} > 0$ and $|\Omega| \sim 2^{-(J+K)}$, we obtain the bound claimed in the lemma by choosing C sufficiently large. □

Now we turn to our third decomposition. A single $\Omega^2_{\eta,\rho}$ is "nearly" a product, but $\pi_2(\Omega^2_{\eta,\rho})$ might be far from an interval. However, we may simply perform the first decomposition, with the roles of the coordinate indices reversed. Indeed, our sets satisfy analogous hypotheses to those in Lemma 2.3 (i.e., the hypotheses of Proposition 2.2 with the indices reversed) when $\rho < \eta$, because $|\mathcal{H}^1(\pi_2^{-1}(\xi_2)\cap\Omega^2_{\eta,\rho})| \sim \rho^C 2^{-J}$ for all $\xi_2 \in \pi_2(\Omega^2_{\eta,\rho})$; when $\rho = \eta$, we may abuse notation slightly by decomposing $\Omega^2_{\eta,\eta}$ into $\log\eta^{-1}$ sets $\Omega^2_{\eta,\rho}$ with the same property.

We complete the proof of Proposition 2.2 by taking unions as described at the outset. The factors of η^2 and ρ^2 in Lemmas 2.3 and 2.4 (and the factor of δ^2 in the analogue for $\Omega^3_{\eta,\rho,\delta}$) mean that the resulting factor of $(\log\delta^{-1})^2$ is indeed harmless. □

3. Extensions of characteristic functions of near tiles

We recall the definition (2-1) of $\Omega(K)$. For each K, we define $J(K)$ to be an integer such that $|\Omega(K)| \sim 2^{-J(K)-K}$. Let $\mathcal{K}(\varepsilon)$ denote the collection of all $K \in \mathbb{Z}_{\geq 0}$ for which ε is the smallest dyadic number such that $\|\mathcal{E}\chi_{\Omega'}\|_{2s} \leq \varepsilon |\Omega(K)|^{\frac{1}{s'}}$ holds for all $\Omega' \subseteq \Omega(K)$. Then Proposition 2.2 gives us a decomposition $\Omega(K) = \bigcup_{0<\delta\leq\varepsilon} \Omega_\delta(K)$, where for each δ, we have $\Omega_\delta(K) \subseteq \bigcup_{\tau\in\mathcal{T}_\delta(K)} \tau$ for some $\mathcal{T}_\delta(K) \subseteq \mathcal{D}_{J(K),K}$ of cardinality $\#\mathcal{T}_\delta(K) \lesssim \delta^{-C}$.

Lemma 3.1. *For $0 < \delta \leq \varepsilon$, under the hypotheses of Theorem 1.1,*

$$\Big\| \sum_{K\in\mathcal{K}(\varepsilon)} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} \lesssim (\log \delta^{-1})^{4s} \sum_{K\in\mathcal{K}(\varepsilon)} \|\mathcal{E}\chi_{\Omega_\delta(K)}\|_{2s}^{2s} + \delta |\Omega|^{\frac{2s}{s'}}.$$

Proof of Lemma 3.1. To prove the lemma, it suffices to prove

$$\Big\| \sum_{K\in\mathcal{K}} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} \lesssim \sum_{K\in\mathcal{K}} \|\mathcal{E}\chi_{\Omega_\delta(K)}\|_{2s}^{2s} + \delta^2 |\Omega|^{\frac{2s}{s'}},$$

with $\mathcal{K} \subseteq \mathcal{K}(\varepsilon)$ chosen so that $\mathcal{K}$ and $J(\mathcal{K})$ are both $A\log\delta^{-1}$-separated, with A a sufficiently large admissible constant. (A will be much larger than the constant C in Proposition 2.2.) Since $s < 2$, the triangle inequality gives

$$\Big\| \sum_{K\in\mathcal{K}} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} = \int \Big| \sum_{\boldsymbol{K}\in\mathcal{K}^4} \prod_{i=1}^{4} \mathcal{E}\chi_{\Omega_\delta(K_i)} \Big|^{\frac{s}{2}} \lesssim \sum_{K\in\mathcal{K}} \Big\| \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} + \sum{}' \Big\| \prod_{i=1}^{4} \mathcal{E}\chi_{\Omega_\delta(K_i)} \Big\|_{\frac{s}{2}}^{\frac{s}{2}},$$

where $\sum'$ indicates a sum taken on quadruples $\boldsymbol{K} = (K_1, K_2, K_3, K_4) \in \mathcal{K}^4$, with at least two entries distinct. The following lemma will be useful in controlling this sum.

Lemma 3.2. *If $K, K' \in \mathcal{K}$, and $J := J(K)$, $J' := J(K')$, then*

$$\|\mathcal{E}\chi_{\Omega_\delta(K)}\mathcal{E}\chi_{\Omega_\delta(K')}\|_s \lesssim 2^{-c_0|K-K'|} \max\{|\Omega(K)|, |\Omega(K')|\}^{\frac{2}{s'}} \tag{3-1}$$

for some admissible constant $c_0 > 0$.

Proof of Lemma 3.2. Inequality (3-1) is an immediate consequence of Cauchy–Schwarz and (2-2) whenever

$$|\Omega(K)|^{\frac{1}{s'}} |\Omega(K')|^{\frac{1}{s'}} \lesssim 2^{-\frac{|K-K'|}{s'}} \max\{|\Omega(K)|, |\Omega(K')|\}^{\frac{2}{s'}}.$$

This inequality holds whenever $K = K'$, $J = J'$, $J < J'$ and $K < K'$, or $J > J'$ and $K > K'$.

By symmetry, this leaves us to prove (3-1) when $K < K'$ and $J > J'$. By Proposition 2.2 and the separation condition on $\mathcal{K}$, it suffices to prove that

$$\|\mathcal{E}\chi_{\tau\cap\Omega_\delta(K)}\mathcal{E}\chi_{\tau'\cap\Omega_\delta(K')}\|_s \lesssim \delta^{-C} 2^{-c_0|K-K'|} |\Omega(K)|^{\frac{1}{s'}} |\Omega(K')|^{\frac{1}{s'}} \tag{3-2}$$

for tiles $\tau \in \mathcal{T}_\delta(K)$, $\tau' \in \mathcal{T}_\delta(K')$.

Note that our conditions on J, J', K, K' mean that τ is taller than τ', and τ' is wider than τ. By translating, we may assume that the y-axis forms the center line of τ and that the x-axis forms the center line of τ'. Now our tiles are contained in $[-2,2]^2$, and we decompose:

$$\tau = \bigcup_{k=0}^{K'} \tau_k, \quad \tau' = \bigcup_{j=0}^{J} \tau'_j,$$

where

$$\tau_k = \begin{cases} \tau \cap \{\xi : |\xi_2| \sim 2^{-k}\}, & k < K', \\ \tau \cap \{\xi : |\xi_2| \lesssim 2^{-K'}\}, & k = K' \end{cases} \quad \text{and} \quad \tau'_j = \begin{cases} \tau' \cap \{\xi : |\xi_1| \sim 2^{-j}\}, & j < J, \\ \tau' \cap \{\xi : |\xi_1| \lesssim 2^{-J}\}, & j = J. \end{cases}$$

By the (2-parameter) Littlewood–Paley square function estimate (the two-parameter version can be proved using Khintchine's inequality), the fact that $s < 2$, and the triangle inequality,

$$\begin{aligned} \|\mathcal{E}\chi_{\tau \cap \Omega_\delta(K)} \mathcal{E}\chi_{\tau' \cap \Omega_\delta(K')}\|_s^s &\lesssim \int \Bigl(\sum_{k=0}^{K'} \sum_{j=1}^{J} |\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}|^2 \Bigr)^{\frac{s}{2}} \\ &\lesssim \sum_{k=0}^{K'} \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s^s. \end{aligned} \tag{3-3}$$

We begin with the sum over those terms with $k = K'$. By Cauchy–Schwarz and (2-2),

$$\sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{K'} \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s^s \lesssim \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{K'} \cap \Omega(K)}\|_{2s}^s \, \|\mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_{2s}^s \lesssim \sum_{j=0}^{J} |\tau_{K'}|^{\frac{s}{s'}} \, |\tau'_j|^{\frac{s}{s'}}.$$

Because of the way the τ'_j were defined, we have at most two nonempty τ'_j with $j \leq J'$. This, combined with the bound $|\tau'_j| \leq \min\{2^{-(j-J')}, 1\}|\tau'|$ gives $\sum_j |\tau'_j|^{\frac{s}{s'}} \lesssim |\tau'|^{\frac{s}{s'}}$ (despite the fact that $s < s'$). Since $|\tau_{K'}| \sim 2^{-(K'-K)}|\tau|$, $|\tau| \sim |\Omega(K)|$, and $|\tau'| \sim |\Omega(K')|$,

$$\sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{K'} \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s^s \lesssim 2^{-(K'-K)\frac{s}{s'}} \, |\Omega(K)|^{\frac{s}{s'}} \, |\Omega(K')|^{\frac{s}{s'}}.$$

In the case $j = J$, a similar argument implies that

$$\sum_{k=0}^{K'} \|\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_J \cap \Omega(K')}\|_s^s \lesssim 2^{-(J-J')\frac{s}{s'}} \, |\Omega(K)|^{\frac{s}{s'}} \, |\Omega(K')|^{\frac{s}{s'}} \sim 2^{-(K'-K)\frac{s}{s'}} \, |\Omega(K)|^{\frac{2s}{s'}}.$$

In the cases $k < K'$ and $j < J$, we have a gain, due to our bilinear extension estimate. If $k < K'$ and $j < J$, then τ_k is a (subset of four) tile(s) in $\mathcal{D}_{J,\max\{k,K\}}$, τ_j is a (subset of four) tile(s) in $\mathcal{D}_{\max\{j,J'\},K'}$, and these tiles are separated by a distance 2^{-k} in the vertical direction 2^{-j} in the horizontal direction. These tiles are thus contained in separated tiles in $\mathcal{D}_{j,k}$, so by (2-4),

$$\|\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s \lesssim 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \, |\tau_k \cap \Omega(K)|^{\frac{1}{r}} \, |\tau'_j \cap \Omega(K')|^{\frac{1}{r}}.$$

From our observation above that we have at most two values of j (resp. k) in our sum with $j \le J'$ (resp. $k \le K$), our assumption that $r < s'$ gives

$$
\begin{aligned}
\sum_{j=0}^{J}\sum_{k=0}^{K'} 2^{(j+k)(2+\frac{2s}{r}-2s)}\, |\tau_k \cap \Omega(K)|^{\frac{s}{r}}\, |\tau_j' \cap \Omega(K')|^{\frac{s}{r}} \\
&\le \sum_{j=0}^{J}\sum_{k=0}^{K'} 2^{(j+k)(2+\frac{2s}{r}-2s)}\, |\tau_k|^{\frac{s}{r}}\, |\tau_j'|^{\frac{s}{r}} \\
&\lesssim 2^{(J'+K)(2+\frac{2s}{r}-2s)} |\tau|^{\frac{s}{r}}\, |\tau'|^{\frac{s}{r}} \sim \delta^{-C} 2^{(J'+K)(2+\frac{2s}{r}-2s)}\, |\Omega(K)|^{\frac{s}{r}}\, |\Omega(K')|^{\frac{s}{r}} \\
&\lesssim \delta^{-C} 2^{(J-J'+K'-K)(1+\frac{s}{r}-s)}\, |\Omega(K)|^{\frac{s}{s'}}\, |\Omega(K')|^{\frac{s}{s'}},
\end{aligned}
$$

which, by (3-3) and $\frac{1}{s}+\frac{1}{r}-1>0$, is stronger than (3-2). ☐

We return to the proof of Lemma 3.1.

Let $K_1, K_2, K_3, K_4 \in \mathcal{K}$, not all equal. Rearranging indices if needed, we may assume that $N_1 := K_1 + J(K_1)$ is minimal among all $N_i := K_i + J(K_i)$ and that $|K_1 - K_4| \ge \frac{1}{2}|K_i - K_j|$ for all i, j. Thus $|\Omega(K_1)|$ is maximal. By Hölder's inequality and Lemma 3.2,

$$
\Big\| \prod_{i=1}^{4} \mathcal{E}\chi_{\Omega_\delta(K_i)} \Big\|_{\frac{s}{2}} \lesssim 2^{-c_0|K_1-K_4|} |\Omega(K_1)|^{\frac{4}{s'}}.
$$

Therefore

$$
\sum{}' \Big\| \prod_{i=1}^{4} \mathcal{E}\chi_{\Omega_\delta(K_i)} \Big\|_{\frac{s}{2}}^{\frac{s}{2}} \lesssim \sum_{K_1\in\mathcal{K}} \sum_{K_1\ne K_4\in\mathcal{K}} |K_4-K_1|^2\, 2^{-c_0|K_4-K_1|} |\Omega(K_1)|^{\frac{2s}{s'}}.
$$

Because $2s > s'$ and $\mathcal{K}$ is $A\log\delta^{-1}$-separated for some very large A, this error term is bounded by $\delta^{C}|\Omega|^{\frac{2s}{s'}}$. ☐

Proof of Theorem 1.1. We decompose Ω by fiber length as in (2-1), $\Omega = \bigcup \Omega(K)$, then decompose the fiber lengths according to the exactness of Vargas's estimate as at the beginning of Section 3, $\mathbb{Z}_{\ge 0} = \bigcup_{0\le\varepsilon\lesssim 1} \mathcal{K}(\varepsilon)$, and finally apply the decomposition in Proposition 2.2, $\Omega(K) = \bigcup_{0<\delta\le\varepsilon} \Omega_\delta(K)$. By the triangle inequality,

$$
\|\mathcal{E}\chi_\Omega\|_{2s} \le \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\le\varepsilon} \Big\| \sum_{K\in\mathcal{K}(\varepsilon)} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}.
$$

Thus by Lemma 3.1 and Proposition 2.2,

$$
\begin{aligned}
\|\mathcal{E}\chi_\Omega\|_{2s} &\lesssim \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\le\varepsilon} \Big\{ (\log\delta^{-1})^{4s} \sum_{K\in\mathcal{K}(\varepsilon)} \|\mathcal{E}\chi_{\Omega_\delta(K)}\|_{2s}^{2s} + \delta|\Omega|^{\frac{2s}{s'}} \Big\}^{\frac{1}{2s}} \\
&\lesssim \Big\{ \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\le\varepsilon} (\log\delta^{-1})^2 \Big(\sum_{K\in\mathcal{K}(\varepsilon)} \delta^{2s} |\Omega(K)|^{\frac{2s}{s'}} \Big)^{\frac{1}{2s}} \Big\} + |\Omega|^{\frac{1}{s'}}.
\end{aligned}
$$

Since $2s > s'$ and the $\Omega(K)$ are disjoint, we may use the triangle inequality for $\ell^{\frac{2s}{s'}}$ to sum the volumes of the $\Omega(K)$ in the preceding, and, finally, we sum a geometric series to obtain

$$\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\leq\varepsilon} (\log\delta^{-1})^2 \delta |\Omega|^{\frac{1}{s'}} \lesssim |\Omega|^{\frac{1}{s'}}. \qquad \square$$

Acknowledgements

This work has been supported in part by a grant from the National Science Foundation (DMS-1600458), and was carried out in part while the author was in residence at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, during the Spring of 2017, a visit that was supported in part by a National Science Foundation grant to MSRI (DMS-1440140). The author would like to thank Sanghyuk Lee, Andreas Seeger, and Ana Vargas, from whom she learned of this problem and some of its history. She would also like to thank the anonymous referees and Benjamin Bruce, as the manuscript benefited greatly from their detailed comments and suggestions.

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Received 8 Sep 2017. Revised 6 Aug 2018. Accepted 17 Sep 2018.

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ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1225

STEADY THREE-DIMENSIONAL ROTATIONAL FLOWS: AN APPROACH VIA TWO STREAM FUNCTIONS AND NASH–MOSER ITERATION

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We consider the stationary flow of an inviscid and incompressible fluid of constant density in the region $D = (0, L) \times \mathbb{R}^2$. We are concerned with flows that are periodic in the second and third variables and that have prescribed flux through each point of the boundary ∂D. The Bernoulli equation states that the "Bernoulli function" $H := \frac{1}{2}|v|^2 + p$ (where v is the velocity field and p the pressure) is constant along stream lines, that is, each particle is associated with a particular value of H. We also prescribe the value of H on ∂D. The aim of this work is to develop an existence theory near a given constant solution. It relies on writing the velocity field in the form $v = \nabla f \times \nabla g$ and deriving a degenerate nonlinear elliptic system for f and g. This system is solved using the Nash–Moser method, as developed for the problem of isometric embeddings of Riemannian manifolds; see, e.g., the book by Q. Han and J.-X. Hong (2006). Since we can allow H to be nonconstant on ∂D, our theory includes three-dimensional flows with nonvanishing vorticity.

1. Introduction

The Euler equation for an inviscid and incompressible fluid of constant density is given by

$$(v \cdot \nabla)v = -\nabla p, \quad \operatorname{div} v = 0,$$

if in addition the velocity field v is independent of time. As we are concerned with stationary flows on $D = (0, L) \times \mathbb{R}^2$ that are periodic in the second and third variables, it is useful to introduce the cell of the periodic lattice

$$\mathcal{P} = (0, L) \times (0, P_1) \times (0, P_2),$$

where $L > 0$ and the periods $P_1, P_2 > 0$ are given; in particular integrations will mainly be over $\mathcal{P}$ and maxima of continuous functions considered on $\overline{\mathcal{P}}$. Any constant vector field $\bar{v}$ is a solution on D with constant pressure $\bar{p}$. Such a field can always be written in the form $\bar{v} = \nabla \bar{f} \times \nabla \bar{g}$ for some linear functions $\bar{f}, \bar{g}$. If the real-valued functions

$$(x, y, z) \mapsto f_0(x, y, z), \quad (x, y, z) \mapsto g_0(x, y, z), \quad (x, y, z) \in D,$$

are near 0 and (P_1, P_2)-periodic in (y, z), one may try looking for a velocity field of the form

$$v^* = \nabla(\bar{f} + f_0 + f^*) \times \nabla(\bar{g} + g_0 + g^*)$$

for unknown functions f^* and g^* that vanish at the boundaries $x = 0$ and $x = L$. The functions f_0 and g_0 can be interpreted as encoding a perturbation of the boundary conditions at $x = 0$ and $x = L$ given by $\bar{f}$

MSC2010: 35Q31, 76B03, 76B47, 35G60, 58C15.
Keywords: incompressible flows, vorticity, boundary conditions, Nash–Moser iteration method.

and $\bar{g}$. If f_0 and g_0 vanish at $x = 0$ and $x = L$, then nothing is gained with respect to the case $f_0 = g_0 = 0$ on D.

In the following theorem, the Sobolev spaces $W^{n,p}_{\text{loc}}(D)$ and $H^n_{\text{loc}}(D)$ consist of functions defined on D such that, when restricted to every bounded open subset $D_b \subset D$, they belong to $W^{n,p}(D_b)$ and $H^n(D_b)$. Note that, in contrast with the usual definition, $\overline{D}_b$ is not required to be included in D. Moreover, $\mathcal{Q}$ is the parallelogram in $\mathbb{R}^2$ spanned by RP_1e_1 and RP_2e_2, where

$$R = \begin{pmatrix} \partial_2 \bar{f} & \partial_3 \bar{f} \\ \partial_2 \bar{g} & \partial_3 \bar{g} \end{pmatrix}$$

is the Jacobian matrix of $(\bar{f}, \bar{g})$ with respect to (y, z) and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Theorem 1.1. *Let $j \in \mathbb{N}_0$ and assume that the first component of $\bar{v}$ does not vanish. Then it is possible to choose $\bar{\epsilon} > 0$ such that if*

- *$H_0 \in C^{11+j}(\mathbb{R}^2)$ is periodic with respect to the lattice in $\mathbb{R}^2$ generated by RP_1e_1 and RP_2e_2 (not necessarily the fundamental periods, this remark holding generally throughout),*
- *$c_1, c_2 \in \mathbb{R}$,*
- *$f_0, g_0 \in H^{13+j}_{\text{loc}}(D) = W^{13+j,2}_{\text{loc}}(D)$, P_1-periodic in y and P_2-periodic in z,*
- *$\|(f_0, g_0)\|^2_{H^{13+j}(\mathcal{P})} + \|H_0\|^2_{C^{11+j}(\overline{\mathcal{Q}})} + |c|^2 < \bar{\epsilon}^2$,*

then there exists $(f^*, g^*) \in H^{6+j}_{\text{loc}}(D)$ satisfying

- *f^*, g^* are P_1-periodic in y and P_2-periodic in z,*
- $$f^*, g^* \text{ vanish when } x \in \{0, L\}, \tag{1}$$
- *$v^* := \nabla(\bar{f} + f_0 + f^*) \times \nabla(\bar{g} + g_0 + g^*)$ is a solution to the Euler equation*

$$(v^* \cdot \nabla)v^* = -\nabla p^*, \qquad \operatorname{div} v^* = 0 \quad \text{on } D,$$

with

$$p^* = -\tfrac{1}{2}|v^*|^2 + H(\bar{f} + f_0 + f^*, \bar{g} + g_0 + g^*) \quad \text{and} \quad H(f, g) = c_1 f + c_2 g + H_0(f, g) \quad \text{for all } f, g \in \mathbb{R}. \tag{2}$$

Moreover, there exists a constant $C > 0$ (independent of (f_0, g_0), H_0 and c) such that

$$\|(f^*, g^*)\|_{H^{6+j}(\mathcal{P})} \le C\bar{\epsilon}.$$

The solution is locally unique in the following sense. Let H be as above (but H_0 can be assumed of class C^2 only), $f, g, \tilde{f}, \tilde{g} \in C^3(\overline{D})$ with $(f - \bar{f}, g - \bar{g})$, $(\tilde{f} - \bar{f}, \tilde{g} - \bar{g})$ both (P_1, P_2)-periodic in y and z, and

$$(f(x, y, z), g(x, y, z)) = (\tilde{f}(x, y, z), \tilde{g}(x, y, z)) \quad \text{for all } (x, y, z) \in \{0, L\} \times \mathbb{R}^2.$$

Assume that $v = \nabla f \times \nabla g$ and $\tilde{v} = \nabla \tilde{f} \times \nabla \tilde{g}$ are both solutions to the Euler equation with pressures $-\frac{1}{2}|v|^2 + H(f, g)$ and $-\frac{1}{2}|\tilde{v}|^2 + H(\tilde{f}, \tilde{g})$, respectively. If $(\nabla f, \nabla g)$ and $(\nabla \tilde{f}, \nabla \tilde{g})$ are in a sufficiently small open convex neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ and $\|H_0\|_{C^2(\overline{\mathcal{Q}})}$ is sufficiently small, then $(f, g) = (\tilde{f}, \tilde{g})$ on $[0, L] \times \mathbb{R}^2$.

Remarks. • Observe that $\nabla_{(f,g)} H(\bar{f} + f_0 + f^*, \bar{g} + g_0 + g^*)$ is P_1-periodic in y and P_2-periodic in z. In general the choice $(f^*, g^*) = -(f_0, g_0)$ is not allowed, as (f^*, g^*) is required to vanish at $x = 0$ and $x = L$, but not (f_0, g_0). When H is constant, the choice $(f^*, g^*) = -(f_0, g_0)$ leads to the constant solution $v^* = \bar{v}$, provided that f_0 and g_0 vanish when $x \in \{0, L\}$. However, when H is not constant, (1) and (2) do not allow to choose $(f^*, g^*) = -(f_0, g_0)$. Indeed, if $(f^*, g^*) = -(f_0, g_0)$, then $v^* = \bar{v}$ and p^* should be constant, which is not compatible with (2) when H is not constant.

• If H_0, f_0 and g_0 are C^∞ smooth, we obtain solutions of arbitrarily high regularity. However, we don't necessarily obtain C^∞ smooth solutions since $\bar{\epsilon}$ depends on j. It might be possible to obtain smooth solutions by applying other versions of the Nash–Moser theorem, for example an analytic version, but that's outside the scope of the paper.

• The uniqueness assertion implies that the solution $(\bar{f} + f_0 + f^*, \bar{g} + g_0 + g^*)$ only depends on f_0 and g_0 through their boundary values.

• On the other hand, it is possible for two different sets of data to give rise to the same velocity field v (see the Appendix for more details).

The following example illustrates the relationship with Beltrami flows (flows such that, at each point of D, the vorticity is parallel to the velocity) and the role of the boundary conditions at $x = 0$ and $x = L$.

Example. Let $\bar{f}(x, y, z) = y$, $\bar{g}(x, y, z) = z$, $c_1, c_2 = 0$ and $H_0 = 0$, so that $\bar{v} = (1, 0, 0)$. Let $f_0(x, y, z) = \delta x \sin(2\pi z/P_2)$ and $g_0 = 0$, and let (f^*, g^*) be given by Theorem 1.1 (for $|\delta|$ small enough). Remember that f^* and g^* vanish at $x = 0$ and $x = L$. The pointwise flux of v^* at $x = 0$ and $x = L$ is the constant 1:

$$v_1^* = \partial_y(\bar{f} + f_0)\, \partial_z(\bar{g} + g_0) - \partial_z(\bar{f} + f_0)\, \partial_y(\bar{g} + g_0) = 1.$$

Let us prove that v^* is not irrotational by assuming the opposite. Then v_1^* would be a (P_1, P_2)-periodic function in y and z that is harmonic. By the maximum principle, $v_1^* = 1$ and thus (v_2^*, v_3^*) would be x-independent. The functions v_2^* and v_3^* would also be harmonic and thus they would be constant, and v^* would be a constant vector field. Hence the map that sends a fluid parcel when $x = 0$ to its position when $x = L$ would be a translation. But this is impossible because $\bar{f} + f_0 + f^*$ is preserved along every parcel trajectory and its level sets at $x = 0$ (that is, the level sets of $\bar{f} + f_0$ at $x = 0$) cannot be sent by a translation to its level sets at $x = L$. Although v^* is not an irrotational flow, it is a Beltrami flow because $H = 0$. As the flux through the boundaries $x = 0$ and $x = L$ does not vanish, the proportionality factor between the velocity and the vorticity cannot be constant (using also the periodicity in the y- and z-directions). Beltrami flows have been considered in many papers, for example in [Enciso and Peralta-Salas 2015] (Beltrami flows with constant proportionality factors) and [Kaiser et al. 2000] (with nonconstant proportionality factors).

The representation $v = \nabla f \times \nabla g$ can be seen as a generalization of the stream function representation $v = \nabla^\perp \psi$ for planar divergence-free stationary flows, in which the stream function ψ is replaced by a pair of functions f and g (note that f and g are constant on stream lines). This representation always holds locally near regular points of the velocity field; see, e.g., [Barbarosie 2011]. For the reader's convenience, we give in the Appendix a self-contained proof when v_1 is nonvanishing that the representation holds globally in D with additional (P_1, P_2)-periodicity with respect to y and z for ∇f and ∇g.

In this formulation, the Euler equation has a particularly helpful variational structure [Keller 1996]; see also [Buffoni 2012]. Namely, the pair of functions (f, g) will be called admissible for the present purpose if

- f and g are of class $C^2(\bar{D})$,
- ∇f and ∇g are P_1-periodic in y and P_2-periodic in z,
- $(f(x, y, z), g(x, y, z)) = (\tilde{f}_0(x, y, z), \tilde{g}_0(x, y, z))$ for all $(x, y, z) \in \{0, L\} \times \mathbb{R}^2$,

where $\tilde{f}_0$ and $\tilde{g}_0$ are two fixed functions of class $C^2(\bar{D})$ such that $\nabla \tilde{f}_0$ and $\nabla \tilde{g}_0$ are P_1-periodic in y and P_2-periodic in z. Under these conditions, $v = \nabla f \times \nabla g$ is divergence-free and the first component

$$v_1 = (\nabla f \times \nabla g) \cdot (1, 0, 0) = \partial_y f \, \partial_z g - \partial_y g \, \partial_z f = \partial_y \tilde{f}_0 \, \partial_z \tilde{g}_0 - \partial_y \tilde{g}_0 \, \partial_z \tilde{f}_0$$

of v is prescribed on $\{0, L\} \times \mathbb{R}^2$. In order to get a better insight into the set of admissible (f, g), note that $f(x, y, z) - a_1 y - a_2 z$ and $g(x, y, z) - a_3 y - a_4 z$ are P_1-periodic in y and P_2-periodic in z for some constants $a_1, a_2, a_3, a_4 \in \mathbb{R}$. The boundary condition ensures that $a_1, a_2, a_3, a_4 \in \mathbb{R}$ do not depend on the particular admissible pair of functions (f, g).

We also assume that the function $H : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 and that $\partial_f H$ and $\partial_g H$ composed with every admissible pair (f, g) are (P_1, P_2)-periodic in y and z. The latter is equivalent to requiring that $\nabla_{(f,g)} H$ is periodic with respect to the lattice generated by $P_1(a_1, a_3)$ and $P_2(a_2, a_4)$.

Let $(\tilde{f}, \tilde{g})$ be admissible and assume that $(\tilde{f}, \tilde{g})$ is a critical point of the integral functional

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{2} |\nabla f \times \nabla g|^2 + H(f, g) \right\} dx\, dy\, dz \tag{3}$$

defined on the set of admissible pairs (f, g). Let us check that $\tilde{v} := \nabla \tilde{f} \times \nabla \tilde{g}$ is a solution to the Euler equation with $\tilde{p} = -\frac{1}{2} |\tilde{v}|^2 + H(\tilde{f}, \tilde{g})$. We consider admissible variations (f_s, g_s), that is, maps $(s, x, y, z) \to (f_s(x, y, z), g_s(x, y, z))$ of class $C^2([-1, 1] \times \bar{D})$ such that $(f_0, g_0) = (\tilde{f}, \tilde{g})$, (f_1, g_1) is admissible and

$$(f_s, g_s) = ((1 - s) f_0 + s f_1,\ (1 - s) g_0 + s g_1) \quad \text{for all } s \in (-1, 1).$$

The meaning of critical point is that the integral functional at (f_s, g_s) as a function of s has a vanishing derivative at $s = 0$ for every admissible variation (f_s, g_s). If in addition we assume that $(f_1 - f_0, g_1 - g_0)$ is compactly supported in $\mathcal{P}$, we get the Euler–Lagrange equation

$$\begin{pmatrix} -\operatorname{div}(\nabla \tilde{g} \times (\nabla \tilde{f} \times \nabla \tilde{g})) + \partial_f H(\tilde{f}, \tilde{g}) \\ -\operatorname{div}((\nabla \tilde{f} \times \nabla \tilde{g}) \times \nabla \tilde{f})) + \partial_g H(\tilde{f}, \tilde{g}) \end{pmatrix} = 0. \tag{4}$$

Because of the periodicity assumption on $\nabla \tilde{f}$ and $\nabla \tilde{g}$, more general admissible variations (f_s, g_s) do not provide additional knowledge and, thanks to the periodicity condition on $\partial_f H(\tilde{f}, \tilde{g})$ and $\partial_g H(\tilde{f}, \tilde{g})$, (4) holds true on all of D. Equation (4) can also be written

$$\nabla\tilde{g}\cdot\operatorname{rot}\tilde{v}+\partial_f H(\tilde{f},\tilde{g})=0 \quad\text{and}\quad -\operatorname{rot}\tilde{v}\cdot\nabla\tilde{f}+\partial_g H(\tilde{f},\tilde{g})=0, \quad \text{with } \tilde{v}=\nabla\tilde{f}\times\nabla\tilde{g}. \tag{5}$$

It then follows that

$$\begin{aligned}\tilde{v}\times\operatorname{rot}\tilde{v}&=(\nabla\tilde{f}\times\nabla\tilde{g})\times\operatorname{rot}\tilde{v}=(\nabla\tilde{f}\cdot\operatorname{rot}\tilde{v})\nabla\tilde{g}-(\nabla\tilde{g}\cdot\operatorname{rot}\tilde{v})\nabla\tilde{f}\\&=\partial_f H(\tilde{f},\tilde{g})\nabla\tilde{f}+\partial_g H(\tilde{f},\tilde{g})\nabla\tilde{g}=\nabla_{(x,y,z)}H(\tilde{f},\tilde{g}).\end{aligned} \tag{6}$$

The identity, see, e.g., [Serrin 1959, p. 151],

$$\nabla\big(\tfrac{1}{2}|\tilde{v}|^2\big)=\tilde{v}\times\operatorname{rot}\tilde{v}+(\tilde{v}\cdot\nabla)\tilde{v}$$

gives

$$(\tilde{v}\cdot\nabla)\tilde{v}-\nabla\big(\tfrac{1}{2}|\tilde{v}|^2\big)+\nabla_{(x,y,z)}H(\tilde{f},\tilde{g})=0,$$

which is equivalent to the classical Euler equation for inviscid, incompressible and time-independent flows

$$(\tilde{v}\cdot\nabla)\tilde{v}+\nabla\tilde{p}=0 \quad \text{with } \tilde{p}=-\tfrac{1}{2}|\tilde{v}|^2+H(\tilde{f},\tilde{g}).$$

$H(\tilde{f},\tilde{g})$ can be seen as the Bernoulli function, which is preserved by the flow since $\nabla_{(x,y,z)}(H(\tilde{f},\tilde{g}))\cdot\tilde{v}=0$ by (6).

The aim of the paper is to develop an existence theory in a small neighborhood of $(\bar{f},\bar{g})\in C^\infty(\bar{D})$ when

- $\nabla\bar{f}$ and $\nabla\bar{g}$ are constant, and
- the first component of $\bar{v}=\nabla\bar{f}\times\nabla\bar{g}$ does not vanish.

If we perturb (4) into the equation

$$\begin{pmatrix}-\epsilon(\partial_y^2\tilde{f}+\partial_z^2\tilde{f})-\operatorname{div}(\nabla\tilde{g}\times(\nabla\tilde{f}\times\nabla\tilde{g}))+\partial_f H(\tilde{f},\tilde{g})\\-\epsilon(\partial_y^2\tilde{g}+\partial_z^2\tilde{g})-\operatorname{div}((\nabla\tilde{f}\times\nabla\tilde{g})\times\nabla\tilde{f}))+\partial_g H(\tilde{f},\tilde{g})\end{pmatrix}=0$$

and then linearize this perturbed equation, the obtained linear problem is coercive [Kohn and Nirenberg 1965], provided that $\epsilon>0$. The linearization of (4) can thus be described as "degenerate", the x-direction being however nondegenerate [loc. cit.]. In Section 2, we analyze the linear operator obtained from the linearization of (4) and its invertibility, following the classical work [Kohn and Nirenberg 1965] for noncoercive boundary value problems. The analysis of the linearized problem relies on the particular structure of the integral functional (3). The main point is that its quadratic part is positive definite (see Proposition 2.3 for a precise statement). The local uniqueness result is obtained as a corollary.

The Nash–Moser iteration method [Moser 1961; Zehnder 1975] has been applied to noncoercive problems in previous works, like [Kiremidjian 1978; Han and Hong 2006]. The approach we shall follow is the one described in Section 6 of [Han and Hong 2006] for the embedding problem of Riemannian manifolds with nonnegative Gauss curvature. The details are given in Section 3. For simplicity, we have restricted ourselves as in [loc. cit.] to periodicity conditions with respect to (y,z). A key ingredient are tame estimates for the inverse of the linearization, which are obtained in Section 2 using suitable commutator estimates.

Alber [1992] deals with a closely related setting. The steady Euler equation is considered in a bounded, simply connected, smooth domain $\Omega \subset \mathbb{R}^3$. There are three boundary conditions: (1) the flux through $\partial\Omega$ is given by a function $f : \partial\Omega \to \mathbb{R}$, (2) a condition on the vorticity flux through the entrance set $\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) < 0\} := \partial\Omega_-$ and (3) a condition on the Bernoulli function on $\partial\Omega_-$. Under precise assumptions, existence and uniqueness are obtained near a solution v_0 with small vorticity when the boundary conditions (2) and (3) are slightly modified. In the present paper, boundary condition (2) is, roughly speaking, replaced by a condition on the Bernoulli function on the exit set. These more symmetric boundary conditions might be a first step to considering flows which are periodic in x, which is a natural geometry in the study of water waves. Our approach also has the benefit of using a variational structure.

Note that the stationary Euler equation also appears as a model in ideal magnetohydrodynamics, with v replaced by the magnetic field B, the vorticity $\operatorname{rot} v$ replaced by the current density J (up to a constant multiple) and the Bernoulli function H replaced by the negative of the fluid pressure p. Grad and Rubin [1958] derived a variational principle for this problem which is rather close to the one considered here, see, e.g., [loc. cit., Theorem 1], although they did not use it to construct solutions. Moreover the above example is related to their Theorems 3 and 5 and to a remark that follows their Theorem 5. A recent work that relies on this variational principle for Euler flows is [Slobodeanu 2015]; it is formulated in a more general geometric framework. An iterative method, not of Nash–Moser type, is developed in [Kaiser et al. 2000] to get Beltrami flows with nonconstant proportionality factors. The boundary conditions there have the same flavor as the ones in [Alber 1992]. Writing a divergence-free velocity field v in the form $v = \nabla f \times \nabla g$ may also be useful for irrotational flows, as it could lead to helpful changes of variables; see [Plotnikov 1980].

2. Linearization

The variational structure of (4) allows one to study its linearization with the help of the quadratic part of the integral functional (3) around an admissible pair (f, g). From now on we shall call a pair (f, g) admissible if

(Ad1) f and g are of class $C^3(\bar{D})$,

(Ad2) ∇f and ∇g are (P_1, P_2)-periodic in y and z.

The quadratic part is given by

$$(F, G) \mapsto \int_{\mathcal{P}} \Big\{ \tfrac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \\ + \tfrac{1}{2} \big(\partial_f^2 H(f, g) F^2 + 2 \partial_f \partial_g H(f, g) F G + \partial_g^2 H(f, g) G^2 \big) \Big\} \, dx \, dy \, dz,$$

where (F, G) is assumed admissible in the sense that

(Ad′1) F and G are in the Sobolev space $H^1_{\mathrm{loc}}(D)$,

(Ad′2) F and G are (P_1, P_2)-periodic in y and z, (Ad$'$3) $(F, G) = 0$ on ∂D in the sense of traces.

Condition (Ad$'$3) is introduced because we shall assume later that the restriction of (f, g) to ∂D is a priori given.

Given an admissible pair (f, g), we shall call H admissible if

(Ad$''$) $H \in C^2(\mathbb{R}^2)$ and $H''(f, g)$ is (P_1, P_2)-periodic in y and z.

In this section we will mostly think of $H''(f, g)$ as a given function of (x, y, z) rather than a composition. The quadratic part can be written $\frac{1}{2}B_{(f,g)}((F, G), (F, G))$, where $B_{(f,g)}$ is the symmetric bilinear form

$$\begin{aligned}B_{(f,g)}&((F, G), (\delta F, \delta G))\\ &= \int_{\mathcal{P}} \big\{(\nabla F \times \nabla g + \nabla f \times \nabla G) \cdot (\nabla \delta F \times \nabla g + \nabla f \times \nabla \delta G)\\ &\qquad\qquad + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla \delta G) + (\nabla f \times \nabla g) \cdot (\nabla \delta F \times \nabla G)\\ &\qquad\qquad\quad + \partial_f^2 H(f, g) F \delta F + \partial_f \partial_g H(f, g)(F \delta G + G \delta F) + \partial_g^2 H(f, g) G \delta G\big\}\, dx\, dy\, dz.\end{aligned}$$

This section contains two kinds of results: firstly, we bound from below the quadratic part and, secondly, we study the regularity of solutions to the linearization of problem (4) at (f, g). A preliminary observation is that the quadratic part is not coercive at (f, g) in the sense that there is no $\alpha > 0$ such that, for all admissible (F, G),

$$\tfrac{1}{2}B_{(f,g)}((F, G), (F, G)) \geq \int_{\mathcal{P}} \{\alpha(|\nabla F|^2 + |\nabla G|^2) - \alpha^{-1}(F^2 + G^2)\}\, dx\, dy\, dz.$$

For example, taking $G = 0$, the quadratic part becomes

$$F \mapsto \int_{\mathcal{P}} \big(\tfrac{1}{2}|\nabla F \times \nabla g|^2 + \tfrac{1}{2}\partial_f^2 H(f, g) F^2\big)\, dx\, dy\, dz.$$

In the particular case $f(x, y, z) = y$, $g(x, y, z) = z$, $H = 0$ and $P_1 = P_2 = 1$, the integral reduces to

$$\frac{1}{2}\int_{\mathcal{P}} (F_x^2 + F_y^2)\, dx\, dy\, dz.$$

Choosing F_n of the form

$$F_n(x, y, z) = \phi(x)\cos(2\pi n z),$$

where $\phi \in C^\infty(\mathbb{R}, [0, 1])$ is compactly supported in $(0, 1)$ and takes the value 1 on $\big(\frac{1}{4}, \frac{3}{4}\big)$, we find that the quadratic part and $\|(F_n, G)\|_{L^2(\mathcal{P})}$ have positive constant values along the sequence $\{(F_n, G)\}_{n\geq 1}$. However, $\|(\nabla F_n, \nabla G)\|_{L^2(\mathcal{P})} \to \infty$ and thus α as above cannot exist. For a general pair (f, g), we instead fix $(x_0, y_0, z_0) \in \mathcal{P}$ such that $\nabla g(x_0, y_0, z_0) \neq 0$ and consider F_n which is (P_1, P_2)-periodic in (y, z) and when restricted to $\mathcal{P}$ is given by

$$F_n(x, y, z) = \phi(x, y, z)\cos(n g(x, y, z)),$$

where $\phi \in C^\infty(\overline{\mathcal{P}}, [0, 1])$ is compactly supported in $\mathcal{P}$, with $\phi(x_0, y_0, z_0) = 1$. By choosing n large enough, one again obtains that α cannot exist. In fact, we have made the stronger observation that, for all

$\alpha > 0$, there exists a sequence $\{(F_n, G_n)\}$ of admissible pairs such that

$$\tfrac{1}{2}B_{(f,g)}((F_n, G_n), (F_n, G_n)) + \alpha^{-1}\int_{\mathcal{P}} (F_n^2 + G_n^2)\, dx\, dy\, dz$$

remains bounded, but $\{(F_n, G_n)\}$ does not have any subsequence converging in $L^2(\mathcal{P})$. This has implications for the regularity of the solutions to the linearized problem, as described below.

Nevertheless, in Theorem 2.1, we bound from below the quadratic part in a rougher way. The term $\int_{\mathcal{P}} \frac{1}{2}|\nabla F \times \nabla g + \nabla f \times \nabla G|^2\, dx\, dy\, dz$ turns out to be rather nice, as shown in the first part of the proof, because it is bounded from below by $\int_{\mathcal{P}}\{(v\cdot\nabla F)^2 + (v\cdot\nabla G)^2\}\, dx\, dy\, dz$ (under the simplifying assumption (7), otherwise there is an additional factor). With the help of a Poincaré inequality and thanks to the Dirichlet boundary condition at $x = 0$ and $x = L$, $\int_{\mathcal{P}}\{(v\cdot\nabla F)^2 + (v\cdot\nabla G)^2\}\, dx\, dy\, dz$ can in turn be bounded from below by a positive constant times $\|(F, G)\|^2_{L^2(\mathcal{P})}$. In the second and third parts of the proof of Theorem 2.1, we bound from below the second term of the quadratic part, that is, $\int_{\mathcal{P}}(\nabla f \times \nabla g)\cdot(\nabla F \times \nabla G)\, dx\, dy\, dz$: it cannot become too negative with respect to $\int_{\mathcal{P}} \frac{1}{2}|\nabla F \times \nabla g + \nabla f \times \nabla G|^2\, dx\, dy\, dz$. In these estimates, it is assumed that $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$. To get a better feeling for the term $\int_{\mathcal{P}}(\nabla f \times \nabla g)\cdot(\nabla F \times \nabla G)\, dx\, dy\, dz$, observe that it vanishes when v is irrotational because (see the beginning of the second step)

$$\int_{\mathcal{P}} (\nabla f \times \nabla g)\cdot(\nabla F \times \nabla G)\, dx\, dy\, dz = \frac{1}{2}\int_{\mathcal{P}} \operatorname{rot} v \cdot (F\nabla G - G\nabla F)\, dx\, dy\, dz.$$

As we allow v to be slightly rotational, this term needs careful estimates.

As a consequence of Theorem 2.1, the integral functional is strictly convex in a neighborhood of $(\bar{f}, \bar{g})$, which implies local uniqueness of a solution to (4) (but not existence at this stage); see Theorem 2.2.

With the aim to apply the technique of elliptic regularization [Kohn and Nirenberg 1965], we consider for $\epsilon \in [0, 1]$ the regularized quadratic part

$$\begin{aligned}(F, G) \mapsto \int_{\mathcal{P}} \Big\{ &\tfrac{1}{2}|\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g)\cdot(\nabla F \times \nabla G) + \tfrac{1}{2}\epsilon(|\nabla F|^2 + |\nabla G|^2) \\ &+ \tfrac{1}{2}\big(\partial_f^2 H(f, g)F^2 + 2\partial_f\partial_g H(f, g)FG + \partial_g^2 H(f, g)G^2\big)\Big\}\, dx\, dy\, dz \\ := {}& \tfrac{1}{2}B^{\epsilon}_{(f,g)}((F, G), (F, G)).\end{aligned}$$

All the obtained estimates are uniform in $\epsilon \in [0, 1]$, but, in addition, the problem becomes elliptic for $\epsilon \in (0, 1]$.

For every admissible $(f, g) \in C^3(\overline{D})$, we introduce the following system for $(\mu, \nu) \in L^2_{\text{loc}}(D)$ that is (P_1, P_2)-periodic in y and z, and for $(F, G) \in H^2_{\text{loc}}(D)$ admissible in the sense of (Ad′1)–(Ad′3):

$$\begin{aligned}\mu &= -\operatorname{div}\big(\nabla g \times (\nabla F \times \nabla g + \nabla f \times \nabla G) + \nabla G \times (\nabla f \times \nabla g)\big) - \epsilon\Delta F + \partial_f^2 H(f, g)F + \partial_f\partial_g H(f, g)G, \\ \nu &= -\operatorname{div}\big((\nabla F \times \nabla g + \nabla f \times \nabla G) \times \nabla f + (\nabla f \times \nabla g) \times \nabla F\big) - \epsilon\Delta G + \partial_f\partial_g H(f, g)F + \partial_g^2 H(f, g)G.\end{aligned}$$

The right-hand side is the linear operator related to the regularized quadratic part. This system also makes sense in a weak form if, instead of $(F, G) \in H^2_{\text{loc}}(D)$, we ask that $(F, G) \in H^1_{\text{loc}}(D)$. Given (μ, ν) in any higher-order Sobolev space, the main issue of Section 2 is to study the regularity of a solution (F, G), aiming at estimates of the Sobolev norms, uniformly in $\epsilon \in [0, 1]$. Such a pair (F, G) is easily proved to be unique and its existence for $\epsilon \in (0, 1]$ follows from the fact that the system is elliptic. The same particular case as above gives more insight into this system. Setting $\mu = \nu = 0$, $\epsilon = 0$, $G = 0$, $f(x, y, z) = y$, $g(x, y, z) = z$ and $P_1 = P_2 = 1$, we get

$$-\operatorname{div}(\partial_1 F, \partial_2 F, 0) + \partial_f^2 H(f, g) F = 0,$$
$$-\operatorname{div}(0, -\partial_3 F, 2\partial_2 F) + \partial_f \partial_g H(f, g) F = 0.$$

Keeping only the second-order terms and forgetting the boundary and periodicity conditions, we see that $F(x, y, z) = \cos(z)$ is a solution to both equations. Hence the regularity theory in [Agmon et al. 1964] cannot be used when $\epsilon = 0$, $f(x, y, z) = y$, $g(x, y, z) = z$ and $P_1 = P_2 = 1$.

In Proposition 2.4, we explain how the general system allows one to express $\partial_{11}^2 F$ and $\partial_{11}^2 G$ with respect to the other second-order partial derivatives of F and G, and lower-order terms, involving μ and ν too. After iterative differentiations, this also yields expressions for higher-order derivatives that contain at least two partial derivatives with respect to x. In a more general setting, this is developed in [Kohn and Nirenberg 1965].

For $i \in \{2, 3\}$, multiplying both sides of each equation of the system by $(-1)^r \partial_i^{2r} F$ and $(-1)^r \partial_i^{2r} G$, respectively, summing the two equations and then integrating by parts many times, $B_{(f,g)}(\partial_i^r F, \partial_i^r F)$ arises, with additional bilinear terms in (F, G) that turn out to involve at most r partial derivatives of F and G for each of the two components of each bilinear term. We can make some of these additional terms small if v is near $\bar{v}$ (here, the hypothesis that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant is used; see the remarks following Theorem 2.7). This crucial observation is developed in [Kohn and Nirenberg 1965] in a more general framework, and is presented here in our specific setting in Theorem 2.5. The quadratic part gives then control on the $L^2(\mathcal{P})$-norms of $\partial_i^r F$ and $\partial_i^r G$, but also on the $L^2(\mathcal{P})$-norms of $\partial_1 \partial_i^r F$ and $\partial_1 \partial_i^r G$. Hence the $L^2(\mathcal{P})$-norms of $\partial_i^r F$, $\partial_i^r G$, $\partial_1 \partial_i^r F$ and $\partial_1 \partial_i^r G$ are controlled by the $L^2(\mathcal{P})$-norms of $\partial_i^r \mu$ and $\partial_i^r \nu$ and by a small factor times the $H^r(\mathcal{P})$-norms of F and G. With all these tools, we get the estimate of Theorem 2.8 at the end of Section 2, in which the norm of (f, g) in some Sobolev space also appears, the order of which is under sufficient control. Although we follow ideas from [loc. cit.] (see in particular Theorem 2′), explicit estimates allow one to get explicit regularity results for the solutions obtained by the Nash–Moser procedure. It may be expected that these estimates could be improved and thus also the statements on regularity, but we do not strive in the present work to be optimal. The lack of compactness mentioned above prevents us from proving C^∞ smoothness of the solution using the method behind Theorem 2 in [loc. cit.].

Our first aim is to find conditions that ensure that $B_{(f,g)}$ is positive definite. In [Buffoni 2012], a minimizer of a more general integral functional could be found in some space of general flows, in a very similar spirit as in [Brenier 1999]. Hence it could be expected that, under appropriate conditions, the quadratic part is nonnegative at a solution of (4). In the proof of the following theorem, we also rely on Poincaré's inequality to get the stronger result that the quadratic part is positive definite for (f, g) (not necessarily a solution to (4)) sufficiently close to $(\bar{f}, \bar{g})$ and H'' sufficiently small (see Theorem 2.1). For simplicity, we shall assume in the following statement that

$$|\nabla \bar{f}|^2 + |\nabla \bar{g}|^2 + \sqrt{(|\nabla \bar{f}|^2 + |\nabla \bar{g}|^2)^2 - 4|\bar{v}|^2} \leq 2, \quad \bar{v} := \nabla \bar{f} \times \nabla \bar{g}. \tag{7}$$

As for (small) $\lambda > 0$, equation (4) remains invariant under the transformation

$$(\tilde{f}, \tilde{g}) \to (\lambda \tilde{f}, \lambda \tilde{g}), \quad H \to \lambda^4 H(\lambda^{-1}\cdot, \lambda^{-1}\cdot),$$

there is no loss of generality.

Theorem 2.1. *Assume that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant, that the first component of $\bar{v}$ does not vanish and that* (7) *holds true. For admissible (f, g) and (F, G),*

$$\begin{aligned} B_{(f,g)}&((F, G), (F, G)) \\ &\geq \int_{\mathcal{P}} \Big\{ \tfrac{1}{16}(v\cdot\nabla F)^2 + \tfrac{1}{16}(v\cdot\nabla G)^2 + (1 - O(\|v'\|_{C(\overline{\mathcal{P}})}))\frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{16L^2}(F^2+G^2) \\ &\qquad + \partial_f^2 H(f,g)F^2 + 2\partial_f\partial_g H(f,g)FG + \partial_g^2 H(f,g)G^2 \Big\}\, dx\, dy\, dz \qquad (8) \end{aligned}$$

holds if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ (independent of H admissible).

Notation. The notation $u = O(v)$ means that the norm (or absolute value) of u is less than a constant times v in the relevant domain. We also use the notation $u \lesssim v$ to indicate that there exists a constant $C > 0$ (independent of u and v) such that $u \leq Cv$.

Remark. It is not essential that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant for this result to hold. The result would still remain true if we instead were to require that $\operatorname{rot} \bar{v} = 0$ (the other hypotheses remaining the same) and replace the coefficient $1 - O(\|v'\|_{C(\overline{\mathcal{P}})})$ in (8) by $\exp(-4L\|(v/v_1)'\|_{C(\overline{\mathcal{P}})})$. This might be useful for considering perturbations of other irrotational flows. See, however, the remarks following Theorem 2.7.

Proof. Under the hypotheses of the theorem, we can assume that the first component of the velocity field $v = \nabla f \times \nabla g$ never vanishes (like the one of $\bar{v}$). We study the various terms separately.

First step: Let us first show that

$$\begin{aligned} \int_{\mathcal{P}} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2\, dx\, dy\, dz &\geq \int_{\mathcal{P}} \{(v\cdot\nabla F)^2 + (v\cdot\nabla G)^2\}\, dx\, dy\, dz \\ &\geq (1 - O(\|v'\|_{C(\overline{\mathcal{P}})}))\frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{L^2} \int_{\mathcal{P}} (F^2+G^2)\, dx\, dy\, dz \end{aligned}$$

if $(\nabla f, \nabla g)$ is near enough to $(\nabla \bar{f}, \nabla \bar{g})$ in $C^1(\overline{\mathcal{P}})$.

To this end, write

$$\nabla F \times \nabla g + \nabla f \times \nabla G = a\nabla f + b\nabla g + c\nabla f \times \nabla g.$$

By taking the scalar product of both sides with ∇f, ∇g and $\nabla f \times \nabla g$ successively, we get

$$\begin{cases} (\nabla g \times \nabla f)\cdot\nabla F = a|\nabla f|^2 + b\nabla f\cdot\nabla g, \\ (\nabla g \times \nabla f)\cdot\nabla G = a\nabla f\cdot\nabla g + b|\nabla g|^2, \\ (\nabla g \times (\nabla f \times \nabla g))\cdot\nabla F + ((\nabla f \times \nabla g)\times\nabla f)\cdot\nabla G = c|\nabla f \times \nabla g|^2 \end{cases}$$

and

$$\begin{aligned}
a &= \frac{-|\nabla g|^2(v\cdot\nabla F)+(\nabla f\cdot\nabla g)(v\cdot\nabla G)}{|\nabla f|^2|\nabla g|^2-(\nabla f\cdot\nabla g)^2} = \frac{-|\nabla g|^2(v\cdot\nabla F)+(\nabla f\cdot\nabla g)(v\cdot\nabla G)}{|v|^2},\\
b &= \frac{-|\nabla f|^2(v\cdot\nabla G)+(\nabla f\cdot\nabla g)(v\cdot\nabla F)}{|v|^2},\\
c &= \frac{(v\times\nabla f)\cdot\nabla G+(\nabla g\times v)\cdot\nabla F}{|v|^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathcal{P}} |\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz\\
&\quad\ge \int_{\mathcal{P}} |a\nabla f+b\nabla g|^2\,dx\,dy\,dz\\
&\quad= \int_{\mathcal{P}} (a\ \ b)\begin{pmatrix}|\nabla f|^2 & \nabla f\cdot\nabla g\\ (\nabla f\cdot\nabla g) & |\nabla g|^2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}dx\,dy\,dz\\
&\quad= \int_{\mathcal{P}} \frac{1}{|v|^4}\big(v\cdot\nabla F\ \ v\cdot\nabla G\big)\\
&\qquad\qquad\times\begin{pmatrix}-|\nabla g|^2 & \nabla f\cdot\nabla g\\ \nabla f\cdot\nabla g & -|\nabla f|^2\end{pmatrix}\begin{pmatrix}|\nabla f|^2 & \nabla f\cdot\nabla g\\ \nabla f\cdot\nabla g & |\nabla g|^2\end{pmatrix}\begin{pmatrix}-|\nabla g|^2 & \nabla f\cdot\nabla g\\ \nabla f\cdot\nabla g & -|\nabla f|^2\end{pmatrix}\begin{pmatrix}v\cdot\nabla F\\ v\cdot\nabla G\end{pmatrix}dx\,dy\,dz\\
&\quad= \int_{\mathcal{P}} \frac{1}{|v|^4}\big(v\cdot\nabla F\ \ v\cdot\nabla G\big)\begin{pmatrix}-|\nabla g|^2 & \nabla f\cdot\nabla g\\ \nabla f\cdot\nabla g & -|\nabla f|^2\end{pmatrix}\begin{pmatrix}-|v|^2 & 0\\ 0 & -|v|^2\end{pmatrix}\begin{pmatrix}v\cdot\nabla F\\ v\cdot\nabla G\end{pmatrix}dx\,dy\,dz\\
&\quad= \int_{\mathcal{P}} \frac{1}{|v|^2}\big(v\cdot\nabla F\ \ v\cdot\nabla G\big)\begin{pmatrix}|\nabla g|^2 & -\nabla f\cdot\nabla g\\ -\nabla f\cdot\nabla g & |\nabla f|^2\end{pmatrix}\begin{pmatrix}v\cdot\nabla F\\ v\cdot\nabla G\end{pmatrix}dx\,dy\,dz\\
&\quad\ge \int_{\mathcal{P}} \frac{|\nabla f|^2+|\nabla g|^2-\sqrt{(|\nabla f|^2+|\nabla g|^2)^2-4|v|^2}}{2|v|^2}\{(v\cdot\nabla F)^2+(v\cdot\nabla G)^2\}\,dx\,dy\,dz
\end{aligned}$$

because the eigenvalues of

$$\begin{pmatrix}|\nabla g|^2 & -\nabla f\cdot\nabla g\\ -\nabla f\cdot\nabla g & |\nabla f|^2\end{pmatrix}$$

are $\frac{1}{2}\big(|\nabla f|^2+|\nabla g|^2\pm\sqrt{(|\nabla f|^2+|\nabla g|^2)^2-4|v|^2}\big)$. By the simplifying assumption (7),

$$\int_{\mathcal{P}} |\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz \ge \int_{\mathcal{P}} \{(v\cdot\nabla F)^2+(v\cdot\nabla G)^2\}\,dx\,dy\,dz$$

if $(\nabla f, \nabla g)$ is near enough to $(\nabla\bar{f}, \nabla\bar{g})$ in $C(\overline{\mathcal{P}})$.

To obtain the second inequality of the first step, we now use Poincaré's inequality in one dimension by relying on the fact that F and G vanish on $\{0, L\}\times(0, P_1)\times(0, P_2)$, and then integrate with respect to the two remaining variables. We use again that the first component of $\bar{v}$ does not vanish and that v is in some small neighborhood of $\bar{v}$, so that the first component of v does not vanish either. Given $(\tilde{y}, \tilde{z})\in\mathbb{R}^2$, let $\Gamma_{(\tilde{y},\tilde{z})}\colon [0, L] \to \mathbb{R}^2$ be the function of the variable $\tilde{x} \in [0, L]$ satisfying

$$\Gamma'_{(\tilde{y},\tilde{z})}(\tilde{x}) = \frac{1}{v_1(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x}))}(v_2(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})), v_3(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})))$$

with the initial condition $\Gamma_{(\tilde{y},\tilde{z})}(0) = (\tilde{y}, \tilde{z})$. By Theorem 7.2 of Chapter 1 in [Coddington and Levinson 1955] on the regularity of solutions of ODEs, the map $(\tilde{x}, \tilde{y}, \tilde{z}) \to \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})$ is of class $C^2(\bar{P})$.

Moreover the Jacobian determinant of the map $(\tilde{y}, \tilde{z}) \to \Gamma_{(\tilde{y},\tilde{z})}(s)$ is given by

$$\exp \int_0^s \operatorname{div}_{(y,z)}(v_2/v_1, v_3/v_1)|_{(\tilde{x},\Gamma_{(\tilde{y},\tilde{z})}(\tilde{x}))}\, d\tilde{x}.$$

Given $\tilde{x} \in (0, L)$, we associate to $(\tilde{x}, \tilde{y}, \tilde{z})$ the point

$$(x, y, z) = (\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})).$$

Observe that $x = \tilde{x}$. We denote by $J(\tilde{x}, \tilde{y}, \tilde{z})$ the Jacobian determinant and obtain

$$J(s, \tilde{y}, \tilde{z}) = \exp \int_0^s \operatorname{div}_{(y,z)}(v_2/v_1, v_3/v_1)|_{(\tilde{x},\Gamma_{(\tilde{y},\tilde{z})}(\tilde{x}))}\, d\tilde{x} = 1 + O(\|v'\|_{C(\overline{\mathcal{P}})})$$

uniformly in $(s, \tilde{y}, \tilde{z}) \in \overline{\mathcal{P}}$ if v is near enough to $\bar{v}$ in $C^1(\overline{\mathcal{P}})$.

Setting

$$\widetilde{F}(\tilde{x}, \tilde{y}, \tilde{z}) = F(x, y, z), \quad \widetilde{G}(\tilde{x}, \tilde{y}, \tilde{z}) = G(x, y, z), \quad \tilde{v}_1(\tilde{x}, \tilde{y}, \tilde{z}) = v_1(x, y, z),$$

we get

$$\partial_1 \widetilde{F}(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{d}{d\tilde{x}} F(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})) = \nabla F \cdot \begin{pmatrix} 1 \\ v_2/v_1 \\ v_3/v_1 \end{pmatrix} \quad \text{at } (\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})),$$

$$\tilde{v}_1 \partial_1 \widetilde{F} = v \cdot \nabla F, \quad \tilde{v}_1 \partial_1 \widetilde{G} = v \cdot \nabla G$$

and

$$\begin{aligned}
\int_{\mathcal{P}} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\}\, dx\, dy\, dz &= \int_{\mathcal{P}} \{(\tilde{v}_1 \partial_1 \widetilde{F})^2 + (\tilde{v}_1 \partial_1 \widetilde{G})^2\} J(\tilde{x}, \tilde{y}, \tilde{z})\, d\tilde{x}\, d\tilde{y}\, d\tilde{z} \\
&\geq \min_{\overline{\mathcal{P}}}(\tilde{v}_1^2 J) \int_{(0,P_1)\times(0,P_2)} \left\{ \int_0^L \{(\partial_1 \widetilde{F})^2 + (\partial_1 \widetilde{G})^2\}\, d\tilde{x} \right\} d\tilde{y}\, d\tilde{z} \\
&\geq \frac{\pi^2 \min_{\overline{\mathcal{P}}} \tilde{v}_1^2 J}{L^2} \int_{(0,P_1)\times(0,P_2)} \left\{ \int_0^L (\widetilde{F}^2 + \widetilde{G}^2) d\tilde{x} \right\} d\tilde{y}\, d\tilde{z} \\
&\geq \frac{\pi^2 \min_{\overline{\mathcal{P}}} \tilde{v}_1^2 J}{L^2 \max_{\overline{\mathcal{P}}} J} \int_{\mathcal{P}} (F^2 + G^2)\, dx\, dy\, dz \\
&\geq (1 - O(\|v'\|_{C(\overline{\mathcal{P}})})) \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{L^2} \int_{\mathcal{P}} (F^2 + G^2)\, dx\, dy\, dz
\end{aligned}$$

if v is in some small neighborhood of $\bar{v}$ in $C^1(\overline{\mathcal{P}})$.

Second step: We now deal with the term $\int_{\mathcal{P}}(\nabla f\times\nabla g)\cdot(\nabla F\times\nabla G)\,dx\,dy\,dz$. Write

$$\operatorname{rot} v=\alpha v+\beta v\times\nabla f+\gamma\nabla g\times v,$$

with

$$\alpha=\frac{\operatorname{rot} v\cdot v}{|v|^2},\quad \beta=\frac{\operatorname{rot} v\cdot\nabla g}{|v|^2},\quad \gamma=\frac{\operatorname{rot} v\cdot\nabla f}{|v|^2}.$$

We get

$$\begin{aligned}\int_{\mathcal{P}}(\nabla f\times\nabla g)\cdot(\nabla F\times\nabla G)\,dx\,dy\,dz&=\frac{1}{2}\int_{\mathcal{P}}v\cdot\operatorname{rot}(F\nabla G-G\nabla F)\,dx\,dy\,dz\\&=\frac{1}{2}\int_{\mathcal{P}}\operatorname{rot} v\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz\end{aligned}$$

because

$$0=\int_{\mathcal{P}}\operatorname{div}(v\times(F\nabla G-G\nabla F))\,dx\,dy\,dz=\int_{\mathcal{P}}(\operatorname{rot} v\cdot(F\nabla G-G\nabla F)-v\cdot\operatorname{rot}(F\nabla G-G\nabla F))\,dx\,dy\,dz.$$

Hence

$$\begin{aligned}&\int_{\mathcal{P}}(\nabla f\times\nabla g)\cdot(\nabla F\times\nabla G)\,dx\,dy\,dz\\&\quad=\frac{1}{2}\int_{\mathcal{P}}(\alpha v+\beta v\times\nabla f+\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz\\&\quad=\frac{1}{2}\int_{\mathcal{P}}\big\{\alpha(\nabla F\times\nabla g+\nabla f\times\nabla G)\cdot(G\nabla f-F\nabla g)\\&\qquad\qquad+(\beta v\times\nabla f+\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\big\}\,dx\,dy\,dz\\&\quad\geq\int_{\mathcal{P}}\big\{-\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2-\alpha^2(G^2|\nabla f|^2+F^2|\nabla g|^2)\\&\qquad\qquad+\tfrac{1}{2}(\beta v\times\nabla f+\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\big\}\,dx\,dy\,dz.\end{aligned}$$

The (absolute value of the) first term in this expression does not create problems because it can be controlled by one eighth of the term studied in the first step. Neither does the second term because it can also be controlled by any fraction of the term studied in the first step (as the second term is quadratic in (F,G) and $|\alpha|$ is as small as needed if $\operatorname{rot} v$ is near enough to $\operatorname{rot}\bar{v}=0$). The aim of the next step is to deal with the last term.

Third step: The aim of this step it to get control of the term

$$\frac{1}{2}\int_{\mathcal{P}}(\beta v\times\nabla f+\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz.$$

First, using $\nabla(FG)=G\nabla F+F\nabla G$, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{P}}(\beta v\times\nabla f)\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz\\&\qquad=\frac{1}{2}\int_{\mathcal{P}}(\beta v\times\nabla f)\cdot\nabla(FG)\,dx\,dy\,dz-\int_{\mathcal{P}}(\beta v\times\nabla f)\cdot(G\nabla F)\,dx\,dy\,dz\\&\qquad=-\frac{1}{2}\int_{\mathcal{P}}FG(\beta\operatorname{rot} v+\nabla\beta\times v)\cdot\nabla f\,dx\,dy\,dz-\int_{\mathcal{P}}(\beta v\times\nabla f)\cdot(G\nabla F)\,dx\,dy\,dz.\end{aligned}$$

Similarly, we can rewrite

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz\\ &=-\frac{1}{2}\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot\nabla(FG)\,dx\,dy\,dz+\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G)\,dx\,dy\,dz\\ &=-\frac{1}{2}\int_{\mathcal{P}}FG(\gamma\operatorname{rot}v+\nabla\gamma\times v)\cdot\nabla g\,dx\,dy\,dz+\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G)\,dx\,dy\,dz.\end{aligned}$$

As

$$|-\beta Fv\cdot(\nabla F\times\nabla g+\nabla f\times\nabla G)|\le 2\beta^2F^2|v|^2+\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2$$

and

$$\begin{aligned}0&=\int_{\mathcal{P}}\operatorname{div}\big(v\times\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)\big)\,dx\,dy\,dz\\ &=\int_{\mathcal{P}}\operatorname{rot}v\cdot\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)\,dx\,dy\,dz-\int_{\mathcal{P}}v\cdot\operatorname{rot}\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)\,dx\,dy\,dz,\end{aligned}\tag{9}$$

we have

$$\begin{aligned}\int_{\mathcal{P}}\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz\\ &\ge\int_{\mathcal{P}}\{-\beta Fv\cdot(\nabla F\times\nabla g+\nabla f\times\nabla G)-2\beta^2F^2|v|^2\}\,dx\,dy\,dz\\ &=\int_{\mathcal{P}}\big\{v\cdot\big(\operatorname{rot}\big(-\beta\tfrac{1}{2}F^2\nabla g+\beta FG\nabla f\big)+\tfrac{1}{2}F^2\nabla\beta\times\nabla g\\ &\qquad-FG\nabla\beta\times\nabla f-\beta G\nabla F\times\nabla f\big)-2\beta^2F^2|v|^2\big\}\,dx\,dy\,dz\\ &\overset{(9)}{=}\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\beta\tfrac{1}{2}F^2\nabla g+\beta FG\nabla f\big)+\tfrac{1}{2}F^2v\cdot(\nabla\beta\times\nabla g)\\ &\qquad-FGv\cdot(\nabla\beta\times\nabla f)-\beta Gv\cdot(\nabla F\times\nabla f)-2\beta^2F^2|v|^2\big\}\,dx\,dy\,dz\end{aligned}$$

and therefore

$$\begin{aligned}-\int_{\mathcal{P}}(\beta v\times\nabla f)\cdot(G\nabla F)\,dx\,dy\,dz\\ &=\int_{\mathcal{P}}\beta Gv\cdot(\nabla F\times\nabla f)\,dx\,dy\,dz\\ &\ge-\int_{\mathcal{P}}\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz\\ &\quad+\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)+\tfrac{1}{2}F^2v\cdot(\nabla\beta\times\nabla g)\\ &\qquad-FGv\cdot(\nabla\beta\times\nabla f)-2\beta^2F^2|v|^2\big\}\,dx\,dy\,dz.\end{aligned}$$

In the previous computations, substitute f and F by $-g$ and $-G$, g and G by f and F, and β by γ, yielding

$$\begin{aligned}&\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G)\,dx\,dy\,dz\\&\quad\geq-\int_{\mathcal{P}}\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz\\&\qquad+\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\tfrac{1}{2}\gamma G^2\nabla f+\gamma FG\nabla g\big)+\tfrac{1}{2}G^2v\cdot(\nabla\gamma\times\nabla f)-FGv\cdot(\nabla\gamma\times\nabla g)-2\gamma^2G^2|v|^2\big\}\,dx\,dy\,dz.\end{aligned}$$

Adding the different contributions, we find that

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{P}}(\beta v\times\nabla f+\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz\\&\quad\geq-\int_{\mathcal{P}}\tfrac{1}{4}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz\\&\qquad+\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\tfrac{1}{2}\beta F^2\nabla g+\tfrac{1}{2}FG\beta\nabla f\big)+\tfrac{1}{2}F^2v\cdot(\nabla\beta\times\nabla g)-\tfrac{1}{2}FGv\cdot(\nabla\beta\times\nabla f)-2\beta^2F^2|v|^2\big\}\,dx\,dy\,dz\\&\qquad+\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\tfrac{1}{2}\gamma G^2\nabla f+\tfrac{1}{2}FG\gamma\nabla g\big)+\tfrac{1}{2}G^2v\cdot(\nabla\gamma\times\nabla f)-\tfrac{1}{2}FGv\cdot(\nabla\gamma\times\nabla g)-2\gamma^2G^2|v|^2\big\}\,dx\,dy\,dz.\end{aligned}$$

All the absolute values of these terms are controlled by multiples of the term studied in the first step. Moreover $|\nabla\beta|$ and $|\nabla\gamma|$ become small if $(\nabla f,\nabla g)$ is near enough to $(\nabla\bar f,\nabla\bar g)$ in $C^2(\overline{\mathcal{P}})$.

<u>Last step</u>:

$$\begin{aligned}&\int_{\mathcal{P}}\big\{\tfrac{1}{2}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2+(\nabla f\times\nabla g)\cdot(\nabla F\times\nabla G)\big\}\,dx\,dy\,dz\\&\overset{\text{step 2}}{\geq}\int_{\mathcal{P}}\big\{\tfrac{3}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2-\alpha(G^2|\nabla f|^2+F^2|\nabla g|^2)\\&\qquad\qquad+\tfrac{1}{2}(\beta v\times\nabla f+\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\big\}\,dx\,dy\,dz\\&\overset{\text{step 3}}{\geq}\int_{\mathcal{P}}\big\{\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2-\alpha(G^2|\nabla f|^2+F^2|\nabla g|^2)\big\}\,dx\,dy\,dz\\&\qquad+\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\beta\tfrac{1}{2}F^2\nabla g+\tfrac{1}{2}FG\beta\nabla f\big)+\tfrac{1}{2}F^2v\cdot(\nabla\beta\times\nabla g)-\tfrac{1}{2}FGv\cdot(\nabla\beta\times\nabla f)-2\beta^2F^2|v|^2\big\}\,dx\,dy\,dz\\&\qquad+\int_{\mathcal{P}}\big\{\operatorname{rot}v\cdot\big(-\gamma\tfrac{1}{2}G^2\nabla f+\tfrac{1}{2}FG\gamma\nabla g\big)+\tfrac{1}{2}G^2v\cdot(\nabla\gamma\times\nabla f)-\tfrac{1}{2}FGv\cdot(\nabla\gamma\times\nabla g)-2\gamma^2G^2|v|^2\big\}\,dx\,dy\,dz\\&\overset{\text{step 1}}{\geq}\int_{\mathcal{P}}\tfrac{1}{16}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz\\&\overset{\text{step 1}}{\geq}\int_{\mathcal{P}}\left\{\tfrac{1}{32}(v\cdot\nabla F)^2+\tfrac{1}{32}(v\cdot\nabla G)^2+(1-O(\|v'\|_{C(\overline{\mathcal{P}})}))\frac{\pi^2\min_{\overline{\mathcal{P}}}v_1^2}{32L^2}(F^2+G^2)\right\}dx\,dy\,dz\end{aligned}$$

if $(\nabla f,\nabla g)$ is in some small neighborhood of $(\nabla\bar f,\nabla\bar g)$ in $C^2(\overline{\mathcal{P}})$ (independent of H). □

Theorem 2.1 implies local uniqueness of solutions (existence will be discussed later).

Theorem 2.2. *Assume that* (f, g) *and* $(\tilde{f}, \tilde{g})$ *and are admissible* (*see* (Ad1)–(Ad2) *above*) *such that*

$$(f(x, y, z), g(x, y, z)) = (\tilde{f}(x, y, z), \tilde{g}(x, y, z)) \quad \text{for all } (x, y, z) \in \{0, L\} \times \mathbb{R}^2,$$

and both (f, g) *and* $(\tilde{f}, \tilde{g})$ *are solutions to* (4). *In addition let* $(\bar{f}, \bar{g})$ *be as in Theorem 2.1 and* H *be as in Theorem 1.1* (*but* H_0 *can be assumed of class* C^2 *only*). *If* $(\nabla f, \nabla g)$ *and* $(\nabla \tilde{f}, \nabla \tilde{g})$ *are in a sufficiently small open convex neighborhood of* $(\nabla \bar{f}, \nabla \bar{g})$ *in* $C^2(\overline{\mathcal{P}})$ *and* $\|H_0''\|_{C(\overline{\mathcal{Q}})}$ *is sufficiently small, then* $(f, g) = (\tilde{f}, \tilde{g})$ *on* $[0, L] \times \mathbb{R}^2$.

Proof. If they were not equal, we could consider

$$(f_\theta, g_\theta) = \theta(\tilde{f}, \tilde{g}) + (1 - \theta)(f, g)$$

for θ in some slightly larger interval than [0, 1]. The map

$$\theta \to \int_{\mathcal{P}} \left\{ \tfrac{1}{2} |\nabla f_\theta \times \nabla g_\theta|^2 + H(f_\theta, g_\theta) \right\} dx\, dy\, dz$$

would be of class C^2, its derivative would vanish at $\theta = 0$ and $\theta = 1$, and its second derivative would be strictly positive on [0, 1] (by Theorem 2.1), which is a contradiction. □

Remark. The proof of Theorem 2.2 relies on the local convexity of the functional (3). It is natural to wonder if local convexity may lead to existence too. Theorem 2.1 shows that the quadratic form $B_{(f,g)}((F, G), (F, G))$ is positive definite if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ (independent of H as long as $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ is sufficiently small). However, as mentioned above, the quadratic form is not coercive at $(f, g) = (\bar{f}, \bar{g})$. This feature creates difficulties in getting good a priori bounds on minimizing sequences. One can hope that they may converge in some weak sense to some kind of weak solution and indeed such kind of results, in a more general setting, are obtained in [Buffoni 2012]. One can also wonder if some kind of regularization of the integral functional followed by a limiting process could lead to regular solutions. If this were feasible, it seems likely that it would rely on a regularity analysis similar to the one that follows. We leave these considerations for further works.

To implement a Nash–Moser iteration, we introduce for $\epsilon \in [0, 1]$ the *regularized quadratic form*

$$\begin{aligned}(F, G) \mapsto \int_{\mathcal{P}} \Big\{ &\tfrac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) + \tfrac{1}{2}\epsilon(|\nabla F|^2 + |\nabla G|^2) \\ &+ \tfrac{1}{2}\big(\partial_f^2 H(f, g) F^2 + 2\partial_f \partial_g H(f, g) F G + \partial_g^2 H(f, g) G^2\big) \Big\}\, dx\, dy\, dz,\end{aligned}$$

which is clearly also positive definite if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ and $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ is small enough, uniformly in $\epsilon \in [0, 1]$, and coercive for a fixed $\epsilon \in (0, 1]$. Again, the regularized quadratic form can be written $\frac{1}{2} B^{\epsilon}_{(f,g)}((F, G), (F, G))$, where $B^{\epsilon}_{(f,g)}$ is the corresponding symmetric bilinear form.

For an admissible $(f, g) \in C^3(\overline{D})$ (see (Ad1)–(Ad2) above), we are interested in the map $(\mu, \nu) \mapsto (F, G)$ defined as follows:

- $(F, G) \in H^1_{\text{loc}}(D)$ is admissible in the sense of (Ad′1)–(Ad′3).
- $(\mu, \nu) \in L^2_{\text{loc}}(D)$ is (P_1, P_2)-periodic in y and z.

- For all $\delta F, \delta G \in H^1_{\mathrm{loc}}(D)$ that are admissible in the sense of (Ad′1)–(Ad′3)

$$B^{\epsilon}_{(f,g)}((F,G),(\delta F,\delta G)) = \int_{\mathcal{P}} (\mu\delta F + \nu\delta G)\,dx\,dy\,dz. \tag{10}$$

If (f, g) is admissible and (F, G) is admissible in $H^2_{\mathrm{loc}}(D)$, (10) is equivalent to the system

$$\begin{aligned}\mu &= -\operatorname{div}\big(\nabla g\times(\nabla F\times\nabla g+\nabla f\times\nabla G)+\nabla G\times(\nabla f\times\nabla g)\big)-\epsilon\Delta F+\partial_f^2 H(f,g)F+\partial_f\partial_g H(f,g)G,\\ \nu &= -\operatorname{div}\big((\nabla F\times\nabla g+\nabla f\times\nabla G)\times\nabla f+(\nabla f\times\nabla g)\times\nabla F\big)-\epsilon\Delta G+\partial_f\partial_g H(f,g)F+\partial_g^2 H(f,g)G.\end{aligned} \tag{11}$$

In particular, if $\epsilon = 0$, then the linear operator related to $B^{\epsilon}_{(f,g)}$ is the linearization of (4) around (f, g).

Thanks to the fact that the regularized quadratic form is positive definite, (F, G) is uniquely defined by (μ, ν). We leave for later the issue of the existence of (F, G) and its regularity, as dealt with in [Kohn and Nirenberg 1965].

Proposition 2.3. *Assume that $\nabla\bar f$ and $\nabla\bar g$ are constant, that the first component of $\bar v$ does not vanish and that* (7) *holds true. If f, g (admissible) are of class $C^3(\bar D)$ and H (admissible) is of class $C^2(\mathbb{R}^2)$, $(\nabla f, \nabla g)$ is in some small enough neighborhood of $(\nabla\bar f, \nabla\bar g)$ in $C^2(\bar{\mathcal{P}})$ and $\|H''(f,g)\|_{C(\bar{\mathcal{P}})}$ is small enough, then*

$$B^{\epsilon}_{(f,g)}((F,G),(F,G)) \ge \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v\cdot\nabla F)^2 + \tfrac{1}{16}(v\cdot\nabla G)^2 + \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{32L^2}(F^2+G^2) \right\} dx\,dy\,dz. \tag{12}$$

Moreover

$$\|(F,G)\|_{L^2(\mathcal{P})} \le \frac{32L^2}{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2} \|(\mu,\nu)\|_{L^2(\mathcal{P})} \tag{13}$$

and

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v\cdot\nabla F)^2 + \tfrac{1}{16}(v\cdot\nabla G)^2 \right\} dx\,dy\,dz \le \frac{32L^2}{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2} \|(\mu,\nu)\|^2_{L^2(\mathcal{P})}$$

for all periodic $(\mu,\nu) \in L^2_{\mathrm{loc}}(D)$ and all admissible $(F, G) \in H^1_{\mathrm{loc}}(D)$ satisfying (10). *These estimates are uniform in $\epsilon \in [0, 1]$.*

Proof. Assuming $|v'|$ and $|H''(f,g)|$ small enough (as we can), we get in (8)

$$(1 - O(\|v'\|_{C(\bar{\mathcal{P}})}))\frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{32L^2}(F^2+G^2) + \tfrac{1}{2}\big(\partial_f^2 H(f,g)F^2 + 2\partial_f\partial_g H(f,g)FG + \partial_g^2 H(f,g)G^2\big) \ge \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{64L^2}(F^2+G^2)$$

and inequality (12) follows from (8). Applying (10) to $(\delta F, \delta G) = (F, G)$,

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v\cdot\nabla F)^2 + \tfrac{1}{16}(v\cdot\nabla G)^2 + \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{32L^2}(F^2+G^2) \right\} dx\,dy\,dz \le B^{\epsilon}_{(f,g)}((F,G),(F,G)) \le \|(\mu,\nu)\|_{L^2(\mathcal{P})}\|(F,G)\|_{L^2(\mathcal{P})},$$

$$\|(F,G)\|_{L^2(\mathcal{P})} \le \frac{32L^2}{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2} \|(\mu,\nu)\|_{L^2(\mathcal{P})}$$

and

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla F)^2 + \tfrac{1}{16}(v \cdot \nabla G)^2 \right\} dx\, dy\, dz \le \frac{32L^2}{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2} \|(\mu, \nu)\|^2_{L^2(\mathcal{P})}. \qquad \square$$

Proposition 2.4. *Assume that the first component of $\bar{v}$ does not vanish and that $(\nabla f, \nabla g)$ is near enough to $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$. Then system* (11) *allows one to express the partial derivatives $\partial^2_{11} F$ and $\partial^2_{11} G$ linearly with respect to μ, ν, the other second-order partial derivatives of F and G, the first-order partial derivatives of F and G, and F and G. The coefficients of these two linear expressions are rational functions of f', g', f'', g'', $H''(f, g)$, ϵ (without singularities on $\overline{D}$). More precisely,*

$$\begin{aligned}\partial^2_{11} F &= a_1\mu + a_2\nu + a_3\partial^2_{12}F + a_4\partial^2_{13}F + a_5\partial^2_{22}F + a_6\partial^2_{23}F + a_7\partial^2_{33}F + a_8\partial^2_{12}G + a_9\partial^2_{13}G + a_{10}\partial^2_{22}G \\ &\quad + a_{11}\partial^2_{23}G + a_{12}\partial^2_{33}G + a_{13}\partial_1 F + a_{14}\partial_2 F + a_{15}\partial_3 F + a_{16}\partial_1 G + a_{17}\partial_2 G + a_{18}\partial_3 G + a_{19}F + a_{20}G,\end{aligned}$$

where each a_i, $1 \le i \le 20$, is of the form

$$a_i = \frac{Q_i}{v_1^2 + \epsilon|(\partial_2 f, \partial_3 f, \partial_2 g, \partial_3 g)|^2 + \epsilon^2}$$

for some polynomial

$$Q_i = \begin{cases} Q_i(f', g', \epsilon), & 1 \le i \le 12, \\ Q_i(f'', g''), & 13 \le i \le 18, \\ Q_i(H''), & 19 \le i \le 20. \end{cases}$$

The denominator does not vanish on $\overline{D}$ because $(\nabla f, \nabla g)$ is supposed near enough to $(\nabla \bar{f}, \nabla \bar{g})$ and $\epsilon \in [0, 1]$. Moreover, for all integers $1 \le i \le 20$ and $\ell \ge 0$,

$$\|a_i\|_{C^\ell(\overline{\mathcal{P}})} = \begin{cases} O(\|(f, g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 1 \le i \le 12, \\ O(\|(f, g)\|_{C^{\ell+2}(\overline{\mathcal{P}})} + 1), & 13 \le i \le 18, \\ O(\|H''(f, g)\|_{C^\ell(\overline{\mathcal{P}})} + \|(f, g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 19 \le i \le 20 \end{cases}$$

if all norms are well-defined. Analogous results hold for $\partial^2_{11} G$ and all the estimates are uniform in $\epsilon \in [0, 1]$.

Proof. If we keep only the second-order terms in (F, G), we get

$$\begin{aligned}\mu &= \nabla g \cdot \operatorname{rot}(\nabla F \times \nabla g + \nabla f \times \nabla G) - \epsilon \Delta F + \cdots, \\ \nu &= -\operatorname{rot}(\nabla F \times \nabla g + \nabla f \times \nabla G) \cdot \nabla f - \epsilon \Delta G + \cdots.\end{aligned}$$

Observe that

$$\operatorname{rot}(\nabla F \times \nabla g) = \Delta g \nabla F - \Delta F \nabla g + F'' \nabla g - g'' \nabla F$$

and thus

$$\begin{aligned}\mu &= \nabla g \cdot ((F'' - \Delta F I)\nabla g) - \nabla g \cdot ((G'' - \Delta G I)\nabla f) - \epsilon \Delta F + \cdots, \\ \nu &= -\nabla f \cdot ((F'' - \Delta F I)\nabla g) + \nabla f \cdot ((G'' - \Delta G I)\nabla f) - \epsilon \Delta G + \cdots,\end{aligned}$$

where I is the identity matrix. To see that this allows one to express $\partial^2_{11} F$ and $\partial^2_{11} G$ with respect to μ, ν, the other second-order partial derivatives of F and G, and the first-order partial derivatives of F and G, and F and G, it is sufficient to study

$$\mu = -\partial_{11}^2 F \nabla g \cdot (J\nabla g) + \partial_{11}^2 G \nabla f \cdot (J\nabla g) - \epsilon \partial_{11}^2 F + \cdots,$$
$$\nu = \partial_{11}^2 F \nabla f \cdot (J\nabla g) - \partial_{11}^2 G \nabla f \cdot (J\nabla f) - \epsilon \partial_{11}^2 G + \cdots,$$

where J is the diagonal matrix with entries (0, 1, 1) on the diagonal and the remainders now also contain the other second-order partial derivatives of F and G. The discriminant of this system for $(\partial_{11}^2 F, \partial_{11}^2 G)$ is

$$\begin{aligned}(|J\nabla g|^2+\epsilon)(|J\nabla f|^2+\epsilon) - ((J\nabla f)\cdot(J\nabla g))^2 &= |(J\nabla f)\times(J\nabla g)|^2 + \epsilon|J\nabla f|^2 + \epsilon|J\nabla g|^2 + \epsilon^2 \\ &= v_1^2 + \epsilon|J\nabla f|^2 + \epsilon|J\nabla g|^2 + \epsilon^2.\end{aligned}$$

We estimate $\|a_i\|_{C^\ell(\overline{\mathcal{P}})}$, $1 \le i \le 20$, using the inequality

$$\|\xi(u_1,\dots,u_N)\|_{C^k(\overline{\mathcal{P}})} \le C\|\xi\|_{C^k}(1 + \|u_1\|_{C^k(\overline{\mathcal{P}})} + \cdots + \|u_N\|_{C^k(\overline{\mathcal{P}})}) \tag{14}$$

for $\xi \in C^k([-M, M]^N)$ and $u_j \in C^k(\overline{\mathcal{P}})$ with $\|u_j\|_{C(\overline{\mathcal{P}})} \le M$ for $1 \le j \le N$, which, e.g., follows by interpolation in C^k spaces, see, e.g., of [Hamilton 1982, Theorem 2.2.1, p. 143], and the Faà di Bruno formula. Hence

$$O(\|a_i\|_{C^\ell(\overline{\mathcal{P}})}) = \begin{cases} O(\|(f,g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 1 \le i \le 12, \\ O(\|(f,g)\|_{C^{\ell+2}(\overline{\mathcal{P}})} + 1), & 13 \le i \le 18, \\ O(\|H''(f,g)\|_{C^\ell(\overline{\mathcal{P}})} + \|(f,g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 19 \le i \le 20. \end{cases}$$ □

We now study to which extent $B^\epsilon_{(f,g)}$ commutes with differentiations in y and z, following the general approach of [Kohn and Nirenberg 1965].

Theorem 2.5. *Let $(\nabla f, \nabla g)$ be in any bounded subset of $C^1(\overline{\mathcal{P}})$, $r \in \{1, 2, 3, \dots\}$, $(f, g) \in C^{r+2}(\overline{D})$, $H \in C^{r+2}(\mathbb{R}^2)$ and $(F, G) \in H^{2r+1}_{\mathrm{loc}}(D)$ (all admissible). Then, for $j \in \{2, 3\}$,*

$$\begin{aligned}&B^\epsilon_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^\epsilon_{(f,g)}((F,G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G)) \\ &\quad = \sum_{p\in\mathcal{S}} \int_{\mathcal{P}} \partial_j^{2r-s_p-t_p} L_p\, \partial_j^{s_p} u_p\, \partial_j^{t_p} v_p \,dx\,dy\,dz + \sum_{p\in\widetilde{\mathcal{S}}} \int_{\mathcal{P}} \partial_j^{r-\tilde{s}_p} \tilde{L}_p\, \partial_j^{\tilde{s}_p} \tilde{u}_p\, \partial_j^r \tilde{v}_p \,dx\,dy\,dz,\end{aligned}$$

where, for each p in some finite sets $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ of indices,

$$0 \le s_p \le t_p \le r-1, \quad 2 \le 2r - s_p - t_p \le r+1, \quad 0 \le \tilde{s}_p \le r-1$$

and

$$\{u_p, v_p\} \subset \{\partial_1 F, \partial_2 F, \partial_3 F, \partial_1 G, \partial_2 G, \partial_3 G\}, \quad \{\tilde{u}_p, \tilde{v}_p\} \subset \{F, G\}.$$

For each p, the coefficient $L_p(x, y, z)$ is a polynomial of all partial derivatives of f and g of order 1, *while $\tilde{L}_p$ is a second-order partial derivative of H (with respect to f and g). Moreover we have the following estimate, where the dependence on r is more explicitly stated*:

$$\Bigg\| \sum_{\substack{p\in\mathcal{S}: \\ s_p=t_p=r-1}} \partial_j^{2r-s_p-t_p} L_p \Bigg\|_{C(\overline{\mathcal{P}})} = \Bigg\| \sum_{\substack{p\in\mathcal{S}: \\ s_p=t_p=r-1}} \partial_j^2 L_p \Bigg\|_{C(\overline{\mathcal{P}})} = O(r^2)\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})} \tag{15}$$

(*the function* $O(r^2)$ *being independent of* $f, g, F, G, H''(f,g)$ *and* ϵ). *Finally, for the other indices* p,

$$\begin{aligned} \|\partial_j^{2r-s_p-t_p} L_p\|_{C(\overline{\mathcal{P}})} &= O(\|(\nabla f, \nabla g)\|_{C^{2r-s_p-t_p}(\overline{\mathcal{P}})} + 1), \quad p \in \mathcal{S}, \\ \|\partial_j^{r-\tilde{s}_p} \tilde{L}_p\|_{C(\overline{\mathcal{P}})} &= O(\|H''(f,g)\|_{C^{r-\tilde{s}_p}(\overline{\mathcal{P}})}), \qquad p \in \widetilde{\mathcal{S}}, \end{aligned} \tag{16}$$

where the constants in the estimates may depend on r.

Remarks. The expression

$$B^{\epsilon}_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^{\epsilon}_{(f,g)}((F,G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G))$$

would vanish if $(\nabla f, \nabla g)$ and $H''(f,g)$ were independent of y and z, and the statement allows one to estimate its size otherwise. In the statement, we add the property $s_p \le t_p$. In fact we shall omit this property in the proof, as it is easy to get it by renaming s_p and t_p. The statement would be much easier if we would aim at the weaker inequality $0 \le s_p \le t_p \le r$ (the proof would then rely on straightforward integrations by parts). The crucial regularity gain $s_p, t_p \le r-1$ has been explored in a general setting in [Kohn and Nirenberg 1965].

Proof. The typical term of $B^{\epsilon}_{(f,g)}((F,G),(F,G))$ is of either of the form

$$\int_{\mathcal{P}} 2L(x,y,z)u(x,y,z)v(x,y,z)\,dx\,dy\,dz,$$

where

$$\{u, v\} \subset \{\partial_1 F, \partial_2 F, \partial_3 F, \partial_1 G, \partial_2 G, \partial_3 G\}$$

and the coefficient $L(x,y,z)$ can be expressed as a polynomial of the partial derivatives of f and g of order 1, or of the form

$$\int_{\mathcal{P}} 2\tilde{L}(x,y,z)\tilde{u}(x,y,z)\tilde{v}(x,y,z)\,dx\,dy\,dz,$$

where

$$\{\tilde{u}, \tilde{v}\} \subset \{F, G\}$$

and $\tilde{L}$ is equal to $\partial_f^2 H(f,g)$, $2\partial_f \partial_g H(f,g)$ or $\partial_g^2 H(f,g)$. The typical term of

$$B^{\epsilon}_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^{\epsilon}_{(f,g)}((F,G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G))$$

is therefore either of the form

$$\int_{\mathcal{P}} \big(2L\partial_j^r u \partial_j^r v - (-1)^r L v \partial_j^{2r} u - (-1)^r L u \partial_j^{2r} v\big)\,dx\,dy\,dz$$

or

$$\int_{\mathcal{P}} \big(2\tilde{L}\partial_j^r \tilde{u} \partial_j^r \tilde{v} - (-1)^r \tilde{L} \tilde{v} \partial_j^{2r} \tilde{u} - (-1)^r \tilde{L} \tilde{u} \partial_j^{2r} \tilde{v}\big)\,dx\,dy\,dz.$$

We only give the details for the first type of term since the argument for the second is similar but simpler (move r derivatives using integration by parts).

We get as in [Kohn and Nirenberg 1965] (but in a simpler setting)

$$
\begin{aligned}
\int_{\mathcal{P}} -(-1)^r L v \partial_j^{2r} u \, dx\, dy\, dz &= \int_{\mathcal{P}} \partial_j^{r+1}(Lv)\partial_j^{r-1} u \, dx\, dy\, dz \\
&= \int_{\mathcal{P}} \sum_{k=0}^{r+1} \binom{r+1}{k} \partial_j^{r+1-k} L \partial_j^k v \partial_j^{r-1} u \, dx\, dy\, dz \\
&= \int_{\mathcal{P}} L \partial_j^{r+1} v \partial_j^{r-1} u \, dx\, dy\, dz + \int_{\mathcal{P}} (r+1)\partial_j L \partial_j^r v \partial_j^{r-1} u \, dx\, dy\, dz \\
&\quad + \int_{\mathcal{P}} \tfrac{1}{2} r(r+1) \partial_j^2 L \partial_j^{r-1} v \partial_j^{r-1} u \, dx\, dy\, dz \\
&\quad\quad + \int_{\mathcal{P}} \sum_{k=0}^{r-2} \binom{r+1}{k} \partial_j^{r+1-k} L \partial_j^k v \partial_j^{r-1} u \, dx\, dy\, dz
\end{aligned}
$$

and thus, together with the equality one gets by permuting u and v,

$$
\begin{aligned}
&\int_{\mathcal{P}} \big(2L\partial_j^r u \partial_j^r v - (-1)^r L v \partial_j^{2r} u - (-1)^r L u \partial_j^{2r} v\big) \, dx\, dy\, dz \\
&\quad = \int_{\mathcal{P}} L \partial_j^2 (\partial_j^{r-1} u \partial_j^{r-1} v) \, dx\, dy\, dz \\
&\quad\quad + \int_{\mathcal{P}} (r+1)\partial_j L \partial_j (\partial_j^{r-1} u \partial_j^{r-1} v) \, dx\, dy\, dz + \int_{\mathcal{P}} r(r+1)\partial_j^2 L \partial_j^{r-1} u \partial_j^{r-1} v \, dx\, dy\, dz \\
&\quad\quad + \int_{\mathcal{P}} \sum_{k=0}^{r-2} \binom{r+1}{k} \partial_j^{r+1-k} L (\partial_j^k v \partial_j^{r-1} u + \partial_j^k u \partial_j^{r-1} v) \, dx\, dy\, dz \\
&\quad = r^2 \int_{\mathcal{P}} \partial_j^2 L \partial_j^{r-1} u \partial_j^{r-1} v \, dx\, dy\, dz + \int_{\mathcal{P}} \sum_{k=0}^{r-2} \binom{r+1}{k} \partial_j^{r+1-k} L (\partial_j^k v \partial_j^{r-1} u + \partial_j^k u \partial_j^{r-1} v) \, dx\, dy\, dz.
\end{aligned}
$$

With respect to the j-th variable, L is differentiated at most $r+1$ times, and u and v at most $r-1$ times. Moreover the term containing $\partial_j^{r-1} u \partial_j^{r-1} v$ is given by

$$
r^2 \int_{\mathcal{P}} \partial_j^2 L \partial_j^{r-1} u \partial_j^{r-1} v \, dx\, dy\, dz,
$$

where

$$
\|\partial_j^2 L\|_{C(\overline{\mathcal{P}})} = O(\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})})
$$

(using the fact that $(\nabla f, \nabla g)$ is supposed to be in some bounded subset of the algebra $C^1(\overline{\mathcal{P}})$). To get (16), we use (14) with $k = 2r - s_p - t_p$ and $\xi = L$. □

In the two following results, everything is uniform in $\epsilon \in [0, 1]$ and we do not state explicitly the dependence on ϵ.

Proposition 2.6. *If* $(f, g, H) \in C^3(\overline{D}) \times C^3(\overline{D}) \times C^3(\mathbb{R}^2)$ *is admissible*, $(\nabla f, \nabla g)$ *is in some small enough neighborhood of* $(\nabla \bar{f}, \nabla \bar{g})$ *in* $C^2(\overline{\mathcal{P}})$ *and* $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ *is small enough, then*

$$
\|(F, G)\|_{H^1(\mathcal{P})} = O(\|H''(f, g)\|_{C^1(\overline{\mathcal{P}})} + 1)\|(\mu, \nu)\|_{H^1(\mathcal{P})} \tag{17}
$$

and

$$\sum_{j\in\{2,3\}}\int_{\mathcal{P}}\left\{\tfrac{1}{16}(v\cdot\nabla\partial_j F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j G)^2\right\}dx\,dy\,dz=O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}$$

for all periodic $(\mu,\nu)\in H^1_{\mathrm{loc}}(D)$ *and all admissible* $(F,G)\in H^3_{\mathrm{loc}}(D)$ *satisfying* (10).

Proof. In Theorem 2.5, we consider $r=1$. Applying (10) to $(\delta F,\delta G)=-(\partial_j^2F,\partial_j^2G)$ with $j\in\{2,3\}$ and using Proposition 2.3, we get

$$\begin{aligned}
&\int_{\mathcal{P}}\left\{\tfrac{1}{16}(v\cdot\nabla\partial_j F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j G)^2+\frac{\pi^2\min_{\overline{\mathcal{P}}}v_1^2}{32L^2}((\partial_jF)^2+(\partial_jG)^2)\right\}dx\,dy\,dz\\
&\quad\le B^{\epsilon}_{(f,g)}((\partial_jF,\partial_jG),(\partial_jF,\partial_jG))\\
&\quad=B^{\epsilon}_{(f,g)}((F,G),-(\partial_j^2F,\partial_j^2G))+\{B^{\epsilon}_{(f,g)}((\partial_jF,\partial_jG),(\partial_jF,\partial_jG))-B^{\epsilon}_{(f,g)}((F,G),-(\partial_j^2F,\partial_j^2G))\}\\
&\quad\overset{(10)}{=}\int_{\mathcal{P}}(\partial_j\mu\partial_jF+\partial_j\nu\partial_jG)\,dx\,dy\,dz+\{B^{\epsilon}_{(f,g)}((\partial_jF,\partial_jG),(\partial_jF,\partial_jG))-B^{\epsilon}_{(f,g)}((F,G),-(\partial_j^2F,\partial_j^2G))\}\\
&\quad\overset{(15),(16)}{\le}\|(\partial_j\mu,\partial_j\nu)\|_{L^2(\mathcal{P})}\|(\partial_jF,\partial_jG)\|_{L^2(\mathcal{P})}+O(\|(\partial_j\nabla f,\partial_j\nabla g)\|_{C^1(\overline{\mathcal{P}})})\|(F,G)\|^2_{H^1(\mathcal{P})}\\
&\qquad\qquad+O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})})\|(F,G)\|_{L^2(\mathcal{P})}\|(\partial_jF,\partial_jG)\|_{L^2(\mathcal{P})}\\
&\quad\le\|(\partial_j\mu,\partial_j\nu)\|_{L^2(\mathcal{P})}\|(\partial_jF,\partial_jG)\|_{L^2(\mathcal{P})}+O(\|(\partial_j\nabla f,\partial_j\nabla g)\|_{C^1(\overline{\mathcal{P}})})\|(F,G)\|^2_{H^1(\mathcal{P})}\\
&\qquad\qquad+\delta^{-1}O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})})^2\|(F,G)\|^2_{L^2(\mathcal{P})}+\delta\|(\partial_jF,\partial_jG)\|^2_{L^2(\mathcal{P})}.
\end{aligned}$$

If, in addition,

$$\|(\partial_2\nabla f,\partial_3\nabla f,\partial_2\nabla g,\partial_3\nabla g)\|_{C^1(\overline{\mathcal{P}})}<\delta$$

and $\delta>0$ is small enough, we get (note that the coefficient 32 is replaced by 64, and later by 128)

$$\begin{aligned}
&\sum_{j\in\{2,3\}}\int_{\mathcal{P}}\left\{\tfrac{1}{16}(v\cdot\nabla\partial_j F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j G)^2+\frac{\pi^2\min_{\overline{\mathcal{P}}}v_1^2}{64L^2}((\partial_jF)^2+(\partial_jG)^2)\right\}dx\,dy\,dz\\
&\qquad\lesssim\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}+\delta^{-1}(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})}+1)^2\|(F,G)\|^2_{L^2(\mathcal{P})}+\delta\|(\partial_1F,\partial_1G)\|^2_{L^2(\mathcal{P})}\\
&\qquad\overset{(13)}{\lesssim}\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}+\delta^{-1}(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{L^2(\mathcal{P})}+\delta\|(\partial_1F,\partial_1G)\|^2_{L^2(\mathcal{P})}.
\end{aligned}$$

Using the last inequality in Proposition 2.3 to estimate $\|\partial_1F\|^2_{L^2(\mathcal{P})}$ and $\|\partial_1G\|^2_{L^2(\mathcal{P})}$ (using also the fact that the first component of v never vanishes), we obtain

$$\|(\partial_1F,\partial_1G)\|^2_{L^2(\mathcal{P})}=O(\|(\mu,\nu,\partial_2F,\partial_2G,\partial_3F,\partial_3G)\|^2_{L^2(\mathcal{P})})$$

and

$$\begin{aligned}
&\sum_{j\in\{2,3\}}\int_{\mathcal{P}}\left\{\tfrac{1}{16}(v\cdot\nabla\partial_j F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j G)^2\right\}dx\,dy\,dz+\frac{\pi^2\min_{\overline{\mathcal{P}}}v_1^2}{128L^2}\|(\nabla F,\nabla G)\|^2_{L^2(\mathcal{P})}\\
&\qquad\qquad=O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}.
\end{aligned}$$

We get (17) by combining this with (13). □

By induction, we get the following theorem.

Theorem 2.7. *Let $r \geq 1$ be an integer, $(f, g) \in H^{r+4}_{\mathrm{loc}}(D)$ (admissible) be in some small enough neighborhood of $(\bar f, \bar g)$ in $H^5(\mathcal{P})$, $H \in C^2(\mathbb{R}^2)$ be admissible, $H''(f, g) \in C^r(\overline{\mathcal{P}})$ and $H''(f, g)$ be small enough in $C(\overline{\mathcal{P}})$. There exists a constant $C_r > 0$ such that, if*

$$\|(\partial_2\nabla f, \partial_3\nabla f, \partial_2\nabla g, \partial_3\nabla g)\|_{C^1(\overline{\mathcal{P}})} < C_r^{-1}, \tag{18}$$

then

$$\sum_{j\in\{2,3\}} \int_{\mathcal{P}} \big\{ \tfrac{1}{16}(v\cdot\nabla\partial_j^r F)^2 + \tfrac{1}{16}(v\cdot\nabla\partial_j^r G)^2 \big\}\, dx\, dy\, dz + \|(F, G)\|^2_{H^r(\mathcal{P})}$$
$$\leq C_r\|(\mu, \nu)\|^2_{H^r(\mathcal{P})} + C_r\|(\mu, \nu)\|^2_{H^1(\mathcal{P})}(\|(f, g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f, g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2 \tag{19}$$

for all periodic $(\mu, \nu) \in H^r(\mathcal{P})$ and all admissible $(F, G) \in H^{2r+1}_{\mathrm{loc}}(D)$ satisfying (10).

Remarks. • In (18), all terms in the norm are differentiated at least once with respect to y or z. In the first sentence of the statement, the small neighborhood and the small bound on the size of $H''(f, g)$ in $C(\overline{\mathcal{P}})$ are independent of $r \geq 1$. The constant C_r can depend on them, on r, $\bar f$ and $\bar g$, but not on H, f and g.

• The r-dependence in (18) is due to the appearance of r in the estimate (15) in Theorem 2.5; see also (23) below.

• Unlike Theorem 2.1 where the constancy of $\bar v$ was not essential it really does matter here; see (18).

Proof. As the result is already known for $r = 1$ (see Proposition 2.6) let us assume that $r \geq 2$.

First step: We first bound from above

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v\cdot\nabla\partial_j^r F)^2 + \tfrac{1}{16}(v\cdot\nabla\partial_j^r G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32L^2}((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx\, dy\, dz$$

for $j \in \{2, 3\}$. We shall deal with $\partial_1^r F$ and $\partial_1^r G$ in the third and fourth steps. Applying (10) to $(\delta F, \delta G) = (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G)$ with $j \in \{2, 3\}$, and using Proposition 2.3 we get

$$\begin{aligned}
&\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v\cdot\nabla\partial_j^r F)^2 + \tfrac{1}{16}(v\cdot\nabla\partial_j^r G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32L^2}((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx\, dy\, dz \\
&\quad\leq B^{\epsilon}_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) \\
&\quad= B^{\epsilon}_{(f,g)}((F, G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G)) \\
&\qquad\qquad + \{B^{\epsilon}_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^{\epsilon}_{(f,g)}((F, G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G))\} \\
&\quad\overset{(10)}{=} \int_{\mathcal{P}} (\partial_j^r\mu\,\partial_j^r F + \partial_j^r\nu\,\partial_j^r G)\, dx\, dy\, dz \\
&\qquad\qquad + \{B^{\epsilon}_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^{\epsilon}_{(f,g)}((F, G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G))\}.
\end{aligned}$$

By Theorem 2.5,

$$\int_{\mathcal{P}}\left\{\tfrac{1}{16}(v\cdot\nabla\partial_j^r F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j^r G)^2+\frac{\pi^2\min_{\overline{\mathcal{P}}} v_1^2}{32L^2}((\partial_j^r F)^2+(\partial_j^r G)^2)\right\}dx\,dy\,dz$$
$$\le \|(\partial_j^r\mu,\partial_j^r\nu)\|_{L^2(\mathcal{P})}\|(\partial_j^r F,\partial_j^r G)\|_{L^2(\mathcal{P})}+O(r^2)\|(\partial_j\nabla f,\partial_j\nabla g)\|_{C^1(\overline{\mathcal{P}})}\|(F,G)\|^2_{H^r(\mathcal{P})}$$
$$+\sum O(\|(f,g)\|_{H^{k_1+3}(\mathcal{P})}+1)\|(F,G)\|_{H^{k_2+1}(\mathcal{P})}\|(F,G)\|_{H^{k_3+1}(\mathcal{P})}$$
$$+\sum O(\|H''(f,g)\|_{C^{r-k_4}(\overline{\mathcal{P}})})\|(F,G)\|_{H^{k_4}(\mathcal{P})}\|(F,G)\|_{H^r(\mathcal{P})},\quad (20)$$

where the sums are over all integers $k_1,k_2,k_3\ge 0$ such that

$$k_1+k_2+k_3=2r,\quad k_1\le r+1,\quad k_2\le k_3\le r-1,\quad k_2+k_3<2r-2$$

(this implies $k_1>2$ and, as $r\ge 2$, $k_2+k_3>0$) and $0\le k_4\le r-1$. Here and in the following estimates, we only indicate the r-dependence in the coefficients of $\|(F,G)\|_{H^r(\mathcal{P})}$. We don't keep track of the r-dependence of the lower-order terms.

By standard interpolation in Sobolev spaces based on the equality

$$k_j+1=\frac{r-1-k_j}{r-1}\cdot 1+\frac{k_j}{r-1}\cdot r,\quad j=2,3,$$

see, e.g., [Han and Hong 2006, Section 4.3], the first sum can be estimated by

$$\sum O(\|(f,g)\|_{H^{k_1+3}(\mathcal{P})}+1)\|(F,G)\|_{H^1(\mathcal{P})}^{(k_1-2)/(r-1)}\|(F,G)\|_{H^r(\mathcal{P})}^{(2r-k_1)/(r-1)}$$
$$=\sum\{O(\|(f,g)\|_{H^{k_1+3}(\mathcal{P})}+1)^{(2(r-1))/(k_1-2)}\delta^{-(2r-k_1)/(k_1-2)}\|(F,G)\|^2_{H^1(\mathcal{P})}\}^{(k_1-2)/(2(r-1))}$$
$$\times\{\delta\|(F,G)\|^2_{H^r(\mathcal{P})}\}^{(2r-k_1)/(2(r-1))},$$

where $\delta>0$ will be chosen as small as needed. The choice of $\delta>0$ can depend on r, $\bar f$ and $\bar g$, but not on (F,G), (μ,ν), H, f and g. In what follows, we write explicitly some negative powers of δ, even when they can be merged with other positive factors, for example those referred to in the notation $\lesssim$ (possibly depending on r, $\bar f$ and $\bar g$). By Young's inequality for products, $xy\le p^{-1}x^p+q^{-1}y^q$ with $p=2(r-1)/(k_1-2)$, $q=2(r-1)/(2r-k_1)$, and interpolation based on the equality

$$k_1+3=\frac{r+1-k_1}{r-1}\cdot 5+\frac{k_1-2}{r-1}\cdot(r+4),$$

this can in turn be estimated by

$$\delta\|(F,G)\|^2_{H^r(\mathcal{P})}+\sum\delta^{-(2r-k_1)/(k_1-2)}O(\|(f,g)\|_{H^{k_1+3}(\mathcal{P})}+1)^{2(r-1)/(k_1-2)}\|(F,G)\|^2_{H^1(\mathcal{P})}$$
$$\lesssim\delta\|(F,G)\|^2_{H^r(\mathcal{P})}+\sum\delta^{-(2r-k_1)/(k_1-2)}$$
$$\times(\|(f,g)\|_{H^5(\mathcal{P})}+1)^{2(r+1-k_1)/(k_1-2)}(\|(f,g)\|_{H^{r+4}(\mathcal{P})}+1)^2\|(F,G)\|^2_{H^1(\mathcal{P})}.$$

By Proposition 2.6, the sum is thus estimated above:

$$\sum(\|(f,g)\|_{H^{k_1+3}(\mathcal{P})}+1)\|(F,G)\|_{H^{k_2+1}(\mathcal{P})}\|(F,G)\|_{H^{k_3+1}(\mathcal{P})}$$
$$\lesssim\delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}+\delta\|(F,G)\|^2_{H^r(\mathcal{P})}.\quad (21)$$

We have also used that, by assumption, (f,g) is in some small enough neighborhood of $(\bar f,\bar g)$ in $H^5(\mathcal{P})$.

The second sum can similarly be estimated as follows:

$$\sum \|H''(f,g)\|_{C^{r-k_4}(\overline{\mathcal{P}})}\|(F,G)\|_{H^{k_4}(\mathcal{P})}\|(F,G)\|_{H^r(\mathcal{P})}$$
$$\lesssim \sum \|H''(f,g)\|_{C(\overline{\mathcal{P}})}^{k_4/r}\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}^{(r-k_4)/r}\|(F,G)\|_{L^2(\mathcal{P})}^{(r-k_4)/r}\|(F,G)\|_{H^r(\mathcal{P})}^{(r+k_4)/r}$$
$$\overset{(13)}{\lesssim} \delta^{-2r}\|H''(f,g)\|^2_{C^r(\overline{\mathcal{P}})}\|(\mu,\nu)\|^2_{L^2(\mathcal{P})}+\delta\|(F,G)\|^2_{H^r(\mathcal{P})}. \tag{22}$$

Let us now choose

$$\|(\partial_2\nabla f,\partial_3\nabla f,\partial_2\nabla g,\partial_3\nabla g)\|_{C^1(\overline{\mathcal{P}})}<r^{-2}\delta. \tag{23}$$

If δ is small enough (this is allowed by assumption (18)), then, by (20)–(22) (note that the coefficient 32 is replaced by 64),

$$\sum_{j\in\{2,3\}}\int_{\mathcal{P}}\Big\{\tfrac{1}{16}(v\cdot\nabla\partial_j^r F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j^r G)^2+\frac{\pi^2\min_{\overline{\mathcal{P}}} v_1^2}{64L^2}((\partial_j^r F)^2+(\partial_j^r G)^2)\Big\}\,dx\,dy\,dz+\|(F,G)\|^2_{L^2(\mathcal{P})}$$
$$\lesssim \|(F,G)\|^2_{L^2(\mathcal{P})}+\delta^{-1}\|(\mu,\nu)\|^2_{H^r(\mathcal{P})}+\delta\|(F,G,\partial_1 F,\partial_1 G)\|^2_{H^{r-1}(\mathcal{P})}$$
$$+\delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})}+\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}$$

because, for $\hat r=r$,

$$\sum_{|\alpha_2|+|\alpha_3|\le\hat r}\|(\partial^\alpha F,\partial^\alpha G)\|^2_{L^2(\mathcal{P})}\lesssim\|(F,G)\|^2_{L^2(\mathcal{P})}+\sum_{j\in\{2,3\}}\|(\partial_j^{\hat r}F,\partial_j^{\hat r}G))\|^2_{L^2(\mathcal{P})}, \tag{24}$$

where the sum is over all multi-indices $\alpha=(\alpha_2,\alpha_3)\in\mathbb{N}_0^2$ such that $|\alpha_2|+|\alpha_3|\le\hat r$ and ∂^α is the corresponding partial derivative with respect to the variables (y,z). Thanks to the induction hypothesis,

$$\sum_{j\in\{2,3\}}\int_{\mathcal{P}}\Big\{\tfrac{1}{16}(v\cdot\nabla\partial_j^r F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j^r G)^2+\frac{\pi^2\min_{\overline{\mathcal{P}}} v_1^2}{64L^2}((\partial_j^r F)^2+(\partial_j^r G)^2)\Big\}\,dx\,dy\,dz+\|(F,G)\|^2_{L^2(\mathcal{P})}$$
$$\lesssim \delta^{-1}\|(\mu,\nu)\|^2_{H^r(\mathcal{P})}+\delta\|(\partial_1 F,\partial_1 G)\|^2_{H^{r-1}(\mathcal{P})}$$
$$+\delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})}+\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}. \tag{25}$$

<u>Second step</u>: Let us now deal with the terms containing only one partial derivative with respect to x and $r-1$ partial derivatives with respect to y or z. By induction, we know that

$$\sum_{j\in\{2,3\}}\int_{\mathcal{P}}\{\tfrac{1}{16}(v\cdot\nabla\partial_j^{r-1}F)^2+\tfrac{1}{16}(v\cdot\nabla\partial_j^{r-1}G)^2\}\,dx\,dy\,dz+\|(F,G)\|^2_{H^{r-1}(\mathcal{P})}$$
$$\le C_{r-1}\|(\mu,\nu)\|^2_{H^{r-1}(\mathcal{P})}+C_{r-1}\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}(\|(f,g)\|_{H^{r+3}(\mathcal{P})}+\|H''(f,g)\|_{C^{r-1}(\overline{\mathcal{P}})}+1)^2$$

and thus

$$\sum_{j\in\{2,3\}}\|(\partial_1\partial_j^{r-1}F,\partial_1\partial_j^{r-1}G)\|^2_{L^2(\mathcal{P})}$$
$$\lesssim\sum_{j\in\{2,3\}}\|(\partial_2\partial_j^{r-1}F,\partial_2\partial_j^{r-1}G,\partial_3\partial_j^{r-1}F,\partial_3\partial_j^{r-1}G)\|^2_{L^2(\mathcal{P})}$$
$$+\|(\mu,\nu)\|^2_{H^{r-1}(\mathcal{P})}+\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}(\|(f,g)\|_{H^{r+3}(\mathcal{P})}+\|H''(f,g)\|_{C^{r-1}(\overline{\mathcal{P}})}+1)^2$$

because the first component of v never vanishes. Together with the first step and thanks to (24) with $\hat{r}=r$, this gives

$$\begin{aligned}
&\|(F,G)\|^2_{L^2(\mathcal{P})}+\sum_{j\in\{2,3\}}\|(\partial_1\partial_j^{r-1}F,\partial_1\partial_j^{r-1}G)\|^2_{L^2(\mathcal{P})}\\
&\qquad\lesssim \delta^{-1}\|(\mu,\nu)\|^2_{H^r(\mathcal{P})}+\delta\|(\partial_1F,\partial_1G)\|^2_{H^{r-1}(\mathcal{P})}\\
&\qquad\qquad+\delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})}+\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}.
\end{aligned}$$

Applying (24) to $\hat{r}=r-1$ and to $(\partial_1F,\partial_1G)$, we obtain for small enough δ

$$\begin{aligned}
&\|(F,G)\|^2_{L^2(\mathcal{P})}+\|(\partial_1F,\partial_1G)\|^2_{L^2(\mathcal{P})}+\sum_{j\in\{2,3\}}\|(\partial_j^{r-1}\partial_1F,\partial_j^{r-1}\partial_1G)\|^2_{L^2(\mathcal{P})}\\
&\qquad\lesssim \delta^{-1}\|(\mu,\nu)\|^2_{H^r(\mathcal{P})}+\delta\|(\partial_1^2F,\partial_1^2G)\|^2_{H^{r-2}(\mathcal{P})}\\
&\qquad\qquad+\delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})}+\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}+1)^2\|(\mu,\nu)\|^2_{H^1(\mathcal{P})}. \qquad (26)
\end{aligned}$$

Third step: We now deal with partial derivatives in which F and G are differentiated at least twice with respect to x. We estimate these using induction on the number of partial derivatives with respect to x for a fixed r. In the special case $r=2$ there is only one second-order partial derivative to estimate, and we simply note directly using Proposition 2.4 that

$$\begin{aligned}
\|(\partial_1^2F,\partial_1^2G)\|_{L^2(\mathcal{P})}&\lesssim\|(\mu,\nu)\|_{L^2(\mathcal{P})}+\|(\partial_2\nabla F,\partial_2\nabla G,\partial_3\nabla F,\partial_3\nabla G)\|_{L^2(\mathcal{P})}+\|(F,G)\|_{H^1(\mathcal{P})}\\
&\overset{(17)}{\lesssim}(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})}+1)\|(\mu,\nu)\|_{H^1(\mathcal{P})}+\|(\partial_2\nabla F,\partial_2\nabla G,\partial_3\nabla F,\partial_3\nabla G)\|_{L^2(\mathcal{P})}.
\end{aligned}$$

Next, let $r>2$ and B_s be a differential operator of order $r-2$ in (x,y,z) that consists of an iteration of $r-2$ partial derivatives, exactly s of which are with respect to x $(0\le s\le r-2)$. Differentiating $r-2$ times the expressions for ∂_1^2F and ∂_1^2G in Proposition 2.4, we get

$$\begin{aligned}
\|(B_s\partial_1^2F,B_s\partial_1^2G)\|_{L^2(\mathcal{P})}&\lesssim\sum_{k=0}^{r-2}(\|(f,g)\|_{H^{r+1-k}(\mathcal{P})}+1)\|(\mu,\nu)\|_{H^k(\mathcal{P})}\\
&\quad+\sum_{k=0}^{r-2}(\|(f,g)\|_{H^{r+2-k}(\mathcal{P})}+1)\|(F,G)\|_{H^{k+1}(\mathcal{P})}\\
&\quad+\sum_{k=0}^{r-2}\|H''\|_{C^{r-2-k}(\overline{\mathcal{P}})}\|(F,G)\|_{H^k(\mathcal{P})}\\
&\quad+(\|(f,g)\|_{H^3(\mathcal{P})}+1)(\|D_s(\partial_2F,\partial_2G)\|_{L^2(\mathcal{P})}+\|E_s(\partial_3F,\partial_3G)\|_{L^2(\mathcal{P})}),
\end{aligned}$$

where D_s and E_s are matricial differential operators of order $r-1$ in (x,y,z), but at most of order $s+1$ when seen as differential operators in x (their coefficients being constants). The terms involving E_s and D_s come from applying B_s to the terms in Proposition 2.4 involving $\partial^2_{\alpha\beta}F$ or $\partial^2_{\alpha\beta}G$ with $(\alpha,\beta)\neq(1,1)$. The last inequality allows one to estimate differential expressions of order $s+2$ with respect to x by differential expressions of orders at most $s+1$ with respect to x.

We get again by interpolation and Young's inequality

$$
\begin{aligned}
&\|(B_s\partial_1^2 F, B_s\partial_1^2 G)\|_{L^2(\mathcal{P})} \\
&\quad\lesssim (\|(f,g)\|_{H^{r+1}(\mathcal{P})}+1)\|(\mu,\nu)\|_{L^2(\mathcal{P})} + (\|(f,g)\|_{H^3(\mathcal{P})}+1)\|(\mu,\nu)\|_{H^{r-2}(\mathcal{P})} \\
&\qquad + (\|(f,g)\|_{H^{r+3}(\mathcal{P})}+1)\|(F,G)\|_{L^2(\mathcal{P})} + (\|(f,g)\|_{H^4(\mathcal{P})}+1)\|(F,G)\|_{H^{r-1}(\mathcal{P})} \\
&\qquad + \|H''(f,g)\|_{C^{r-2}(\bar{\mathcal{P}})}\|(F,G)\|_{L^2(\mathcal{P})} + \|H''(f,g)\|_{C(\bar{\mathcal{P}})}\|(F,G)\|_{H^{r-2}(\mathcal{P})} \\
&\qquad + (\|(f,g)\|_{H^3(\mathcal{P})}+1)(\|D_s(\partial_2 F,\partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F,\partial_3 G)\|_{L^2(\mathcal{P})}) \\
&\quad\lesssim (\|(f,g)\|_{H^{r+3}(\mathcal{P})} + \|H''(f,g)\|_{C^{r-2}(\bar{\mathcal{P}})}+1)\|(\mu,\nu)\|_{L^2(\mathcal{P})} \\
&\qquad + \|(\mu,\nu)\|_{H^{r-2}(\mathcal{P})} + \|(F,G)\|_{H^{r-1}(\mathcal{P})} + \|D_s(\partial_2 F,\partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F,\partial_3 G)\|_{L^2(\mathcal{P})} \\
&\quad\lesssim (\|(f,g)\|_{H^{r+3}(\mathcal{P})} + \|H''(f,g)\|_{C^{r-1}(\bar{\mathcal{P}})}+1)\|(\mu,\nu)\|_{H^1(\mathcal{P})} \\
&\qquad + \|(\mu,\nu)\|_{H^{r-1}(\mathcal{P})} + \|D_s(\partial_2 F,\partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F,\partial_3 G)\|_{L^2(\mathcal{P})},
\end{aligned}
$$

where we've used the induction hypothesis (19) with r replaced by $r-1$ in the last step. By induction on s, we get the estimate

$$
\begin{aligned}
\|(B_s\partial_1^2 F, B_s\partial_1^2 G)\|_{L^2(\mathcal{P})} &\lesssim (\|(f,g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f,g)\|_{C^r(\bar{\mathcal{P}})}+1)\|(\mu,\nu)\|_{H^1(\mathcal{P})} \\
&\quad + \|(\mu,\nu)\|_{H^r(\mathcal{P})} + \sum_{j\in\{2,3\}} \|(\partial_j^{r-1}\partial_1 F, \partial_j^{r-1}\partial_1 G)\|_{L^2(\mathcal{P})} + \delta\|(\partial_1^2 F,\partial_1^2 G)\|_{H^{r-2}(\mathcal{P})},
\end{aligned}
$$

thanks to (24) applied to (F, G) and $(\partial_1 F, \partial_1 G)$, and to (25). Hence, choosing δ sufficiently small

$$
\begin{aligned}
\|(\partial_1^2 F, \partial_1^2 G)\|_{H^{r-2}(\mathcal{P})} &\lesssim (\|(f,g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f,g)\|_{C^r(\bar{\mathcal{P}})}+1)\|(\mu,\nu)\|_{H^1(\mathcal{P})} \\
&\quad + \|(\mu,\nu)\|_{H^r(\mathcal{P})} + \sum_{j\in\{2,3\}} \|(\partial_j^{r-1}\partial_1 F, \partial_j^{r-1}\partial_1 G)\|_{L^2(\mathcal{P})}. \qquad (27)
\end{aligned}
$$

Combining (27) with (26) and again choosing δ sufficiently small allows us to estimate all partial derivatives of order r with precisely one derivative with respect to x. Substitution of the resulting estimate into (27) gives us control of all derivatives with at least two derivatives with respect to x.

Conclusion: The estimate of the statement follows from the three steps. □

Let us deal with the case $\epsilon = 0$ with the help of the technique of elliptic regularization introduced and well explained in [Kohn and Nirenberg 1965]; see, e.g, p. 449, the beginning of the proof of Theorem 2 and the proof of Theorem 2′ in that work. Firstly, when $\epsilon > 0$, one deduces from this a priori estimate the existence of an admissible solution $(F, G) \in H^r(\mathcal{P})$ given any $(\mu, \nu) \in H^r(\mathcal{P})$, by approximating (f, g), $H''(f, g)$ itself and (μ, ν) by smooth functions. The existence of (F, G) is ensured because the problem is elliptic in this case. Secondly, as the above estimate holds uniformly in $\epsilon \in (0, 1]$, the existence persists when taking the limit $\epsilon \to 0$. Thus we get the following theorem.

Theorem 2.8. *Let $\epsilon = 0$, $r \geq 1$ be an integer, $(f, g) \in H^{r+4}_{\mathrm{loc}}(D)$ (admissible) be in some small enough neighborhood of $(\bar f, \bar g)$ in $H^5(\mathcal{P})$, $H \in C^2(\mathbb{R}^2)$ be admissible, $H''(f, g) \in C^r(\bar D)$ and $H''(f, g)$ be small* *enough in $C(\overline{\mathcal{P}})$. There exists a constant $C_r > 0$ such that if*

$$\|(\partial_2\nabla f, \partial_3\nabla f, \partial_2\nabla g, \partial_3\nabla g)\|_{C^1(\overline{\mathcal{P}})} < C_r^{-1},$$

then for any periodic $(\mu, \nu) \in H^r_{\mathrm{loc}}(D)$ there exists an admissible $(F, G) \in H^r_{\mathrm{loc}}(D)$ satisfying (10) (*with $\epsilon = 0$*) *and*

$$\|(F, G)\|^2_{H^r(\mathcal{P})} \le C_r \|(\mu, \nu)\|^2_{H^r(\mathcal{P})} + C_r \|(\mu, \nu)\|^2_{H^1(\mathcal{P})}(\|(f, g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f, g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2.$$

This result remains true without the simplifying hypothesis (7).

3. A solution by the Nash–Moser method

In this section we shall take $\bar{f}$ and $\bar{g}$ to be some fixed linear functions and let R be the corresponding Jacobian matrix with respect to (y, z) as in the Introduction.

Let us define three decreasing sequences of Banach spaces.

Definition of the Banach spaces $\mathcal{U}_k$. For each integer $k \ge 2$, let $\mathcal{U}_k$ be the real linear space of all (F, G) in $H^k_{\mathrm{loc}}(D)$ satisfying (Ad′2) and (Ad′3). We define the norm $\|\cdot\|_k$ on $\mathcal{U}_k$ as

$$\|(F, G)\|^2_k = \|F\|^2_{H^k(\mathcal{P})} + \|G\|^2_{H^k(\mathcal{P})}.$$

Definition of the Banach spaces $\mathcal{V}_k$. For each integer $k \ge 0$, let $\mathcal{V}_k$ be the real linear space of all (μ, ν) in $H^k_{\mathrm{loc}}(D)$ that satisfy the periodicity condition (Ad′2) almost everywhere. We define the norm $\|\cdot\|_k$ on $\mathcal{V}_k$ by

$$\|(\mu, \nu)\|^2_k = \|\mu\|^2_{H^k(\mathcal{P})} + \|\nu\|^2_{H^k(\mathcal{P})}.$$

Definition of the Banach spaces $\mathcal{W}_k$. For each integer $k \ge 4$, let $\mathcal{W}_k$ be the real linear space of (f_0, g_0, H_0, c) such that

(i) $f_0, g_0 \in H^k_{\mathrm{loc}}(D)$ satisfy the periodicity condition (Ad′2),

(ii) $H_0 \in C^{k-2}(\mathbb{R}^2)$ is periodic with respect to the lattice generated by RP_1e_1 and RP_2e_2, and $c \in \mathbb{R}^2$.

Note that (ii) ensures that $H_0(\bar{f} + f_0 + f_1, \bar{g} + g_0 + g_1)$ satisfies (Ad′2) for all $(f_1, g_1) \in \mathcal{U}_k$.

We define the norm $\|\cdot\|_k$ on $\mathcal{W}_k$ by

$$\|(f_0, g_0, H_0, c)\|^2_k = \|f_0\|^2_{H^k(\mathcal{P})} + \|g_0\|^2_{H^k(\mathcal{P})} + \|H_0\|^2_{C^{k-2}(\overline{\mathcal{Q}})} + |c|^2.$$

Given $(f_0, g_0, H_0, c) \in \mathcal{W}_4$, with $H_0 \in C^3(\mathbb{R}^2)$, we define the map $\mathcal{F}\colon \mathcal{U}_4 \to \mathcal{V}_2$ by

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \to \mathcal{F}\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} -\operatorname{div}(\nabla g \times (\nabla f \times \nabla g)) + \partial_f H(f, g) \\ -\operatorname{div}((\nabla f \times \nabla g) \times \nabla f)) + \partial_g H(f, g) \end{pmatrix}$$

with $f = \bar{f} + f_0 + f_1,\ g = \bar{g} + g_0 + g_1$ and $H(f, g) = c_1 f + c_2 g + H_0(f, g)$.

The following theorem results directly from Theorem 2.8 and (14) (with $\xi = H$).

Theorem 3.1. *Let $k \geq 1$ be an integer and suppose that $(f_0, g_0, H_0, c) \in \mathcal{W}_{k+4}$, $(f_1, g_1) \in \mathcal{U}_{k+4}$, $\|H_0''\|_{C(\bar{\mathcal{Q}})}$ is small enough, and (f, g) is in some small enough neighborhood of $(\bar{f}, \bar{g})$ in $H^5(\mathcal{P})$, with*

$$f = \bar{f} + f_0 + f_1, \quad g = \bar{g} + g_0 + g_1 \quad \text{and} \quad H(f, g) = c_1 f + c_2 g + H_0(f, g).$$

There exists a constant $M_k > 0$ such that if

$$\|(\partial_2 \nabla f, \partial_3 \nabla f, \partial_2 \nabla g, \partial_3 \nabla g)\|_{C^1(\bar{\mathcal{P}})} < M_k^{-1},$$

we get the following. Given any $(\mu, \nu) \in \mathcal{V}_k$, there exists a unique $(F, G) \in \mathcal{U}_k$ satisfying (10) with $\epsilon = 0$. It also satisfies

$$\|(F, G)\|_k \leq M_k \|(\mu, \nu)\|_k + M_k \|(\mu, \nu)\|_1 (\|(f_1, g_1)\|_{H^{k+4}(\mathcal{P})} + 1)$$

and

$$\|(F, G)\|_0 \leq M_0 \|(\mu, \nu)\|_0$$

for some constant $M_0 > 0$ independent of k.

Remark. The constants M_k in Theorem 3.1 can also depend on (f_0, g_0, H_0, c) and $(\bar{f}, \bar{g})$.

Let us state Theorem 6.3.1 in [Han and Hong 2006]. There Ω is a smooth domain in $\mathbb{R}^n$ or a rectangle with the sides parallel to the coordinate axes and with periodic boundary conditions with respect to $n - 1$ coordinates. The corresponding Sobolev spaces are simply denoted by H^k.

Theorem 3.2. *Suppose $\mathcal{F}(w)$ is a nonlinear differential operator of order m in a domain $\Omega \subset \mathbb{R}^n$, given by*

$$\mathcal{F}(w) = \Gamma(x, w, \partial w, \ldots, \partial^m w),$$

where Γ is smooth (see, however, the remark below).

Suppose that d_0, d_1, d_2, d_3, s_0 and $\tilde{s}$ are nonnegative integers with

$$d_0 \geq m + [n/2] + 1$$

and

$$\tilde{s} \geq \max\{3m + 2d_* + [n/2] + 2, m + d_* + d_0 + 1, m + d_2 + d_3 + 1\},$$

where $d_* = \max\{d_1, d_3 - s_0 - 1\}$. Assume that, for any $h \in H^{\tilde{s}+d_1} = H^{\tilde{s}+d_1}(\Omega)$ and $w \in H^{\tilde{s}+d_2}$ with

$$\|w\|_{H^{d_0}} \leq r_0 := 1,$$

the linear equation

$$\mathcal{F}'(w)\rho = h \tag{28}$$

admits a solution $\rho \in H^{\tilde{s}}$ satisfying for any $s = 0, 1, \ldots, \tilde{s}$

$$\|\rho\|_{H^s} \leq c_s (\|h\|_{H^{s+d_1}} + (s - s_0)^+ (\|w\|_{H^{s+d_2}} + 1)\|h\|_{H^{d_3}}),$$

where c_s is a positive constant independent of h, w and ρ. Then there exists a positive constant μ_*, depending only on $\Omega, c_s, m, d_0, d_1, d_2, d_3, s_0$ and $\tilde{s}$, such that if

$$\|\mathcal{F}(0)\|_{H^{\tilde{s}-m}} \leq \mu_*^2, \tag{29}$$

the equation $\mathcal{F}(w) = 0$ admits an $H^{\tilde{s}-m-d_*-1}$ solution w in Ω.

Remarks. • By inspecting the proof in [Han and Hong 2006], we see that it holds as well for systems of $N \geq 1$ differential equations. Moreover the constant $r_0 = 1$ can be replaced by any fixed value $r_0 > 0$ by multiplying appropriately functions by constant factors.

• Also the solution w is the limit in $H^{\tilde{s}-m-d_*-1}$ of sums of solutions in $H^{\tilde{s}}$ to linear equations of type (28). See in [loc. cit.] equations (6.3.14) and (6.3.15), and the proof of Theorem 6.3.1 on p. 103.

• We can relax the condition that Γ is smooth. Let $\hat{c} > 0$ be such that, for all $w \in H^{d_0}$ with $\|w\|_{H^{d_0}} \leq r_0$, we have

$$\|w\|_{C^m(\overline{\Omega})} \leq \hat{c},$$

and define $\Sigma \subset \mathbb{R}^{N+Nn+Nn^2+\cdots+Nn^m}$ as the ball of radius $\hat{c}$ centered at the origin. In the proof, the map Γ appears in the various estimates via $\|\mathcal{F}(0)\|_{H^{\tilde{s}-m}}$ and via "constants" depending on

$$\|\partial_\alpha \partial_\beta \Gamma\|_{C^{\tilde{s}-m}(\overline{\Omega}\times\overline{\Sigma})},$$

where ∂_α and ∂_β are all possible partial derivatives with respect to $w, \ldots, \partial^m w$. See (14) and, in [loc. cit.], the proof of $(P_3)_{\ell+1}$ on p. 101. It therefore suffices to assume that Γ is of class $C^{\tilde{s}-m+2}$.

• From [loc. cit.] it follows that there exists a constant $C > 0$ such that $\|w\|_{H^{\tilde{s}-m-d_*-1}} \leq C\mu_*^2$. More precisely, see in [loc. cit.] the last estimate in the proof of $(P_1)_{l+1}$ on p. 100, (6.3.31) and the proof of Theorem 6.3.1 on p. 103.

To apply this theorem, we need to check (29). For this reason, we shall stay near a solution (namely $(f_1, g_1) = 0$) to an unperturbed problem (namely $(f_0, g_0) = 0$ and $H = 0$), so that (29) is satisfied, and rely on the fact that all relevant "constants" (in particular μ^*) for the perturbed problem can be chosen equal to those of the unperturbed problem.

Theorem 3.3. *Let $j \geq 0$ be an integer, $R > 0$ arbitrary and $\delta > 0$ sufficiently small and assume that $(f_0, g_0, H_0, c) \in \mathcal{W}_{13+j}$ with $\|(f_0, g_0, H_0, c)\|_{13+j} < R$ and $\|(f_0, g_0, H_0, 0)\|_5 < \delta$. It is possible to choose $\epsilon > 0$ (independent of (f_0, g_0, H_0, c), but depending on $(\bar{f}, \bar{g})$, j, R and δ) such that if $\|\mathcal{F}(0, 0)\|_{7+j} < \epsilon$ then there exists $(f^*, g^*) \in \mathcal{U}_{6+j}$ satisfying $\mathcal{F}(f^*, g^*) = 0$.*

Proof. We choose $r_0 > 0$ small enough so that Theorem 3.1 with $k = 9 + j$ can be applied for all $(f_1, g_1) \in \mathcal{U}_5$ in the closed ball of radius r_0 centered at the origin. Let $\hat{c} > 0$ be such that

$$\|(f_1, g_1)\|_{C^2(\overline{\mathcal{P}})} \leq \hat{c}$$

for all $(f_1, g_1) \in \mathcal{U}_5$ in this ball, and define $\Sigma \subset \mathbb{R}^{2+6+18}$ as the ball of radius $\hat{c}$ centered at the origin.

We apply Theorem 3.2 with $m = 2$, $\Omega = \mathcal{P} \subset \mathbb{R}^n$, $n = 3$, $d_0 = 5$, $d_1 = 0$, $d_2 = 4$, $d_3 = 1$, $s_0 = 1$, $d_* = 0$ and $\tilde{s} = 9 + j$. We get

$$\tilde{s} + d_1 = 9 + j, \quad \tilde{s} + d_2 = 13 + j, \quad \tilde{s} - m = 7 + j, \quad \tilde{s} - m - d_* - 1 = 6 + j$$

and a solution $(f^*, g^*) \in H^{6+j}(\mathcal{P})$. Let the map $\Gamma \colon \mathcal{P} \times \mathbb{R}^{1+1+3+3+9+9} \to \mathbb{R}^2$ be such that

$$\mathcal{F}(f_1, g_1) = \Gamma(x, y, z, f_1, g_1, f_1', g_1', f_1'', g_1'').$$

It appears in the various estimates also via "constants" depending on $\|\partial_\alpha\partial_\beta\Gamma\|_{C^{\tilde{s}-m}(\overline{\mathcal{P}}\times\overline{\Sigma})}$, where ∂_α and ∂_β are all possible partial derivatives with respect to f_1, g_1, f_1', g_1', f_1'' or g_1''. Observe that $(f_0, g_0, H_0, c) \in \mathcal{W}_{13+j}$ implies $(f_0, g_0, H_0, c) \in C^{\tilde{s}+2}(\overline{\mathcal{P}}) \times C^{\tilde{s}+2}(\overline{\mathcal{P}}) \times C^{\tilde{s}+2}(\overline{\mathcal{Q}}) \times \mathbb{R}^2$ and $\partial_\alpha\partial_\beta\Gamma \in C^{\tilde{s}-m}(\overline{\mathcal{P}}\times\overline{\Sigma})$. As (f^*, g^*) is the limit in $H^{6+j}(\mathcal{P})$ of sums of solutions in $\mathcal{U}_{9+j}$ to equations of type (10) (with $\epsilon = 0$), it satisfies (Ad′3) and thus belongs to $\mathcal{U}_{6+j}$. □

As a corollary, we get the following simplified statement.

Theorem 3.4. *Assume that $H_0 \in C^{11+j}$ and $f_0, g_0 \in H^{13+j}$. It is possible to choose $\bar{\epsilon} > 0$ such that if $\|(f_0, g_0, H_0, c)\|_{13+j} < \bar{\epsilon}$, then there exists $(f^*, g^*) \in \mathcal{U}_{6+j}$ satisfying $\mathcal{F}(f^*, g^*) = 0$.*

Theorem 1.1 is a reformulation of this last result and Theorem 2.2.

Appendix: Representation of divergence-free vector fields

The fact that the vector field $\nabla f \times \nabla g$ is divergence-free if f and g are C^2 is easily checked using the formula $\operatorname{div}(u \times v) = v \cdot \operatorname{rot} u - u \cdot \operatorname{rot} v$. A local converse near points where v is nonzero has been known for a long time; see, e.g., [Barbarosie 2011; Cartan 1967, Chapter 3, Exercise 14]. A local converse that can be seen as a global converse under additional conditions can be found in Appendix I in [Grad and Rubin 1958]. In the present appendix, we give for the reader's convenience a self-contained proof that a divergence-free vector field $v \in C^2(\overline{D})$ can be represented globally in this form if v is periodic in y and z and $v_1 \neq 0$ in $\overline{D}$, and that f and g can be chosen to be of the form "linear plus periodic". Our argument is essentially a simple version of an elementary proof of global equivalence of volume forms on compact connected manifolds due to [Moser 1965].

For a given point $(x, y, z) \in \overline{D}$ we solve the system of ODEs $\phi' = v(\phi)$, with $\phi(0) = (x, y, z)$, and let $T = T(x, y, z)$ be the unique time such that $\phi_1(-T; x, y, z) = 0$ (here we use that $\inf_{\overline{D}} |v_1| > 0$ and $\sup_{\overline{D}} |v| < \infty$). We define the C^2 functions $Y, Z\colon \overline{D} \to \mathbb{R}^2$ by

$$Y\colon (x, y, z) \mapsto \phi_2(-T; x, y, z) \quad \text{and} \quad Z\colon (x, y, z) \mapsto \phi_3(-T; x, y, z).$$

The functions Y and Z are invariants of the vector field v and therefore $\nabla Y \times \nabla Z = \lambda v$ for some function λ. Using the fact that v is divergence-free, it is easily established that λ is another invariant and therefore

$$\nabla Y \times \nabla Z = \frac{1}{v_1(0, Y, Z)} v$$

in view of the relations $Y(0, y, z) = y$ and $Z(0, y, z) = z$. If $F, G\colon \mathbb{R}^2 \to \mathbb{R}^2$ and

$$f(x, y, z) = F(Y(x, y, z), Z(x, y, z)), \quad g(x, y, z) = G(Y(x, y, z), Z(x, y, z)),$$

then

$$\nabla f \times \nabla g = (\partial_1 F\, \partial_2 G - \partial_2 F\, \partial_1 G) \nabla Y \times \nabla Z.$$

Thus in order to have $\nabla f \times \nabla g = v$ we must find F and G with

$$\partial_1 F(Y, Z)\, \partial_2 G(Y, Z) - \partial_2 F(Y, Z)\, \partial_1 G(Y, Z) = v_1(0, Y, Z).$$

If it weren't for the periodicity conditions, this would be trivial. We describe next how to make a choice which respects these conditions (the choice is not unique).

Note that $v_1(0, Y, Z)$ is P_1-periodic in Y and P_2-periodic in Z. Let

$$\alpha = \frac{1}{P_1 P_2} \int_0^{P_1} \int_0^{P_2} v_1(0, Y, Z)\, dY\, dZ$$

and write $v_1(0, Y, Z) = a(Y)\, b(Y, Z)$, where

$$a(Y) = \frac{1}{P_2} \int_0^{P_2} v_1(0, Y, Z)\, dZ \quad \text{and} \quad b(Y, Z) = \frac{v_1(0, Y, Z)}{a(Y)},$$

so that

$$\frac{1}{P_1} \int_0^{P_1} a(Y)\, dY = \alpha \quad \text{and} \quad \frac{1}{P_2} \int_0^{P_2} b(Y, Z)\, dZ = 1.$$

We choose

$$F(Y) = \int_0^Y a(s)\, ds \quad \text{and} \quad G(Y, Z) = \int_0^Z b(Y, s)\, ds.$$

Note that F and G (and hence f and g) are C^2 and that the map

$$\Psi : (Y, Z) \mapsto (F(Y), G(Y, Z))$$

from $\mathbb{R}^2$ to itself is bijective. It is easily verified that

$$\partial_1 F(Y)\, \partial_2 G(Y, Z) = a(Y)\, b(Y, Z) = v_1(0, Y, Z),$$

that $F(Y) - \alpha Y$ is P_1-periodic and that $G(Y, Z) - Z$ is (P_1, P_2)-periodic. Finally, by the periodicity of v and standard ODE theory, it follows that $(Y(x, y, z), Z(x, y, z)) - (y, z)$ is P_1 periodic in y and P_2-periodic in z, and therefore so is $(f(x, y, z), g(x, y, z)) - (\alpha y, z)$. This concludes the proof.

As mentioned above, the representation $v = \nabla f \times \nabla g$ is not unique. Indeed, if $\Phi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ satisfies

$$\det \Phi' = \partial_1 \Phi_1 \partial_2 \Phi_2 - \partial_2 \Phi_1 \partial_1 \Phi_2 = 1,$$

then $(\tilde{f}, \tilde{g}) = \Phi(f, g)$ also satisfies $\nabla \tilde{f} \times \nabla \tilde{g} = v$. Moreover, $(\tilde{f}, \tilde{g})$ is also linear plus (P_1, P_2)-periodic in (y, z) if $\Phi(f, g) = T(f, g) + \Phi_0(f, g)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and Φ_0 is $(\alpha P_1, P_2)$-periodic.

Note that T is bijective, since otherwise one could find a nonzero linear functional ℓ annihilating its range. This would cause $\ell \circ \Phi$ to be periodic, and thus $\ell \circ \Phi$ would have a critical point at which $\det \Phi'$ would vanish. As T is bijective, Φ is proper and hence bijective by the global inversion theorem (using again $\det \Phi' = 1$).

Conversely, if $v = \nabla \tilde{f} \times \nabla \tilde{g}$ for some C^2 functions $\tilde{f}$ and $\tilde{g}$, then $\tilde{f}$ and $\tilde{g}$ are constant along the streamlines of v. Hence

$$(\tilde{f}(x, y, z), \tilde{g}(x, y, z)) = (\tilde{f}(0, Y, Z), \tilde{g}(0, Y, Z))$$

with $(Y, Z) = (Y(x, y, z), Z(x, y, z))$ as above, and we obtain $(\tilde{f}, \tilde{g}) = \Phi(f, g)$, where $\Phi = (\tilde{f}, \tilde{g})|_{x=0} \circ \Psi^{-1}$ is C^2. Moreover, Φ is linear plus $(\alpha P_1, P_2)$-periodic and $\det \Phi' = 1$.

Let us finally note that the Bernoulli function $H = \frac{1}{2}|v|^2 + P$ can clearly be written as a function of (f, g) since it is constant on streamlines. Denoting this function also by $H(f, g)$, we find that if (f, g) is transformed to $(\tilde{f}, \tilde{g}) = \Phi(f, g)$ with Φ as above, then H is transformed to $H \circ \Phi^{-1}$.

Acknowledgments

The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Nonlinear water waves" where work on this paper was undertaken. This work was supported by EPSRC grant no. EP/K032208/1. Wahlén has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 678698). Finally, the authors are grateful to the referees for helpful comments.

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Received 18 Sep 2017. Revised 26 Jul 2018. Accepted 18 Oct 2018.

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ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1259

SPARSE BOUNDS FOR THE DISCRETE CUBIC HILBERT TRANSFORM

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Consider the discrete cubic Hilbert transform defined on finitely supported functions f on $\mathbb{Z}$ by

$$H_3 f(n) = \sum_{m \neq 0} \frac{f(n-m^3)}{m}.$$

We prove that there exists $r < 2$ and universal constant C such that for all finitely supported f, g on $\mathbb{Z}$ there exists an (r, r)-sparse form $\Lambda_{r,r}$ for which

$$|\langle H_3 f, g\rangle| \le C \Lambda_{r,r}(f, g).$$

This is the first result of this type concerning discrete harmonic analytic operators. It immediately implies some weighted inequalities, which are also new in this setting.

1. Introduction

The purpose of this paper is to initiate a theory of sparse domination for discrete operators in harmonic analysis. We do so in the simplest nontrivial case; it will be clear that there is a much richer theory to be uncovered.

Our main result concerns the discrete cubic Hilbert transform, defined for finitely supported functions f on $\mathbb{Z}$ by

$$H_3 f(x) = \sum_{n \neq 0} \frac{f(x-n^3)}{n}.$$

It is known [Stein and Wainger 1990; Ionescu and Wainger 2006] that this operator extends to a bounded linear operator on $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$ for all $1 < p < \infty$. We prove a sparse bound, which in turn proves certain weighted inequalities. Both results are entirely new.

By an *interval* we mean a set $I = \mathbb{Z} \cap [a, b]$ for $a < b \in \mathbb{R}$. For $1 \le r < \infty$, set

$$\langle f \rangle_{I,r} := \left[\frac{1}{|I|} \sum_{x \in I} |f(x)|^r \right]^{1/r}.$$

We say a collection of intervals $\mathcal{S}$ is *sparse* if there are subsets $E_S \subset S \subset \mathbb{Z}$ with (a) $|E_S| > \frac{1}{4}|S|$, uniformly in $S \in \mathcal{S}$, and (b) the sets $\{E_S : S \in \mathcal{S}\}$ are pairwise disjoint. For sparse collections $\mathcal{S}$, consider sparse

Lacey supported in part by grant NSF-DMS-1600693.
MSC2010: 11L03, 42A05.
Keywords: Hilbert transform, cubic integers, sparse bound, exponential sum, circle method,

bisublinear forms

$$\Lambda_{\mathcal{S},r,s}(f,g) := \sum_{S\in\mathcal{S}} |S| \langle f\rangle_{S,r} \langle g\rangle_{S,s}.$$

Frequently we will suppress the collection $\mathcal{S}$, and if $r = s = 1$, we will suppress this dependence as well.

The main result of this paper is the following theorem.

Theorem 1.1. *There is a choice of $1 < r < 2$ and constant $C > 0$ so that for all f, g that are finitely supported on $\mathbb{Z}$ there is a sparse collection of intervals $\mathcal{S}$ so that*

$$|\langle H_3 f, g\rangle| \le C \Lambda_{\mathcal{S},r,r}(f,g).$$

The beauty of sparse operators is that they are both positive and highly localized operators. In particular, many of their mapping properties can be precisely analyzed. As an immediate corollary [Bernicot et al. 2016, §6] we obtain weighted inequalities, holding in an appropriate intersection of Muckenhoupt and reverse Hölder weight classes.

Corollary 1.2. *There exists $1 < r < 2$ so that for all weights $w^{-1}, w \in A_2 \cap RH_r$ we have*

$$\|H_3 : \ell^2(w) \mapsto \ell^2(w)\| \lesssim 1.$$

For instance, one can take

$$w(x) = [1+|x|]^a \quad \text{for } -\tfrac{1}{2} < a < \tfrac{1}{2}.$$

The concept of a sparse bound originated in [Lerner 2013; Conde-Alonso and Rey 2016; Lacey 2017], so it is new, in absolute terms, as well as this area. On the other hand, the study of norm inequalities for discrete arithmetic operators has been under active investigation for over 30 years. However, *no weighted inequalities have ever been proved in this setting.*

The subject of discrete norm inequalities of this type began with the breakthrough work of Bourgain [1988a; 1988b] on arithmetic ergodic theorems. He proved, for instance, the following theorem.

Theorem 1A. *Let P be a polynomial on $\mathbb{Z}$ which takes integer values. Then the maximal function M_P below maps $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$ for all $1 < p < \infty$:*

$$M_P f(x) = \sup_N \frac{1}{N} \sum_{n=1}^{N} |f(x - p(n))|.$$

Subsequently, attention turned to a broader understanding of Bourgain's work, including its implications for singular integrals and Radon transforms [Ionescu et al. 2007; Stein and Wainger 1990]. The fine analysis needed to obtain results in all ℓ^p spaces was developed by Ionescu and Wainger [2006]. This theme is ongoing, with recent contributions in [Mirek et al. 2015; 2017; 2018], while other variants of these questions can be found in [Krause and Lacey 2017; Pierce 2010].

Initiated by Lerner [2013] as a remarkably simple proof of the so-called A_2 theorem, the study of sparse bounds for operators has recently been an active topic. The norm control provided in [Lerner 2013] was improved to a pointwise control for Calderón–Zygmund operators in [Lacey 2017; Conde-Alonso and Rey 2016]. The paper [Culiuc et al. 2016] proved sparse bounds for the bilinear Hilbert transform, in the language of sparse forms, pointing to the applicability of sparse bounds outside the classical Calderón–Zygmund setting. That point of view is crucial for this paper.

Two papers [Lacey and Spencer 2017; Krause and Lacey 2018] have proved sparse bounds for *random* discrete operators, a much easier setting than the current one. A core technique of these papers reappears in Section 4. Sparse bounds continue to be explored in a variety of settings [Benea et al. 2017; Bernicot et al. 2016; Karagulyan 2016; Lacey and Mena Arias 2017; Hytönen et al. 2017].

We recall some aspects of known techniques in sparse bounds in Section 2. These arguments and results are formalized in a new notation, which makes the remaining quantitative proof more understandable. In particular, we define a "sparse norm" and formalize some of its properties. Our main theorem above is a sparse bound for a Fourier multiplier. In Section 3, we describe a decomposition of this Fourier multiplier, which has a familiar form within the discrete harmonic analysis literature. The multiplier is decomposed into "minor" and "major" arc components, which require dramatically different methods to control. Concerning the minor arcs, there is one novel aspect of the decomposition, a derivative condition which has a precursor in [Krause and Lacey 2017]. Using this additional feature, the minor arcs are controlled in Section 4 through a variant of an argument in [Lacey and Spencer 2017]. The major arcs are the heart of the matter, and are addressed in Section 5.

An expert in the subject of discrete harmonic analysis will recognize that there are many possible extensions of the main result of this paper. We have chosen to present the main techniques in the simplest nontrivial example. Many variants and extensions to our main theorem hold, but all the ones we are aware of are more complicated than this one.

2. Generalities

We collect some additional notation, beginning with the one term that is not standard, namely the sparse operators. Given an operator T acting on finitely supported functions on $\mathbb{Z}$, and index $1 \le r, s < \infty$, we set

$$\|T : \operatorname{Sparse}(r, s)\| \tag{2.1}$$

to be the infimum over constants $C > 0$ so that for all finitely supported functions f, g on $\mathbb{Z}$,

$$|\langle Tf, g\rangle| \le C \sup \Lambda_{r,s}(f, g),$$

where the supremum is over all sparse forms. In particular, the "sparse norm" in (2.1) satisfies a triangle inequality:

$$\Big\| \sum_j T_j : \operatorname{Sparse}(r, s) \Big\| \le \sum_j \|T_j : \operatorname{Sparse}(r, s)\|.$$

We collect some quantitative estimates for different operators; hence the notation. As the notation indicates, it suffices to exhibit a single sparse bound for $\langle Tf, g\rangle$.

It is known that the Hardy–Littlewood maximal function

$$M_{\mathrm{HL}} f = \sup_N \frac{1}{2N+1} \sum_{j=-N}^{N} |f(x-j)|$$

satisfies a sparse bound. This is even a classical result.

Theorem 2B. *We have*

$$\|M_{\mathrm{HL}} : \mathrm{Sparse}(1,1)\| \lesssim 1.$$

The following is a deep fact about sparse bounds that is at the core of our main theorem.

Theorem 2C [Conde-Alonso and Rey 2016; Lacey 2017]. *Let T_K be the convolution with any Calderón–Zygmund kernel. For a Hilbert space $\mathcal{H}$, and viewing T_K as acting on $\mathcal{H}$-valued functions, we have the sparse bound*

$$\|T_K : \mathrm{Sparse}(1,1)\| < \infty.$$

We make the natural extension of the definition of the sparse form to vector-valued functions, namely $\langle f\rangle_I = |I|^{-1} \sum_{x\in I} \|f\|_{\mathcal{H}}$.

Recall that K is a *Calderón–Zygmund kernel on* $\mathbb{R}$ if $K : \mathbb{R}\setminus\{0\} \to \mathbb{C}$ satisfies

$$\sup_{x\in\mathbb{R}\setminus\{0\}} |xK(x)| + \left|x^2 \frac{d}{dx} K(x)\right| < \infty, \tag{2.2}$$

and T_K acts boundedly from L^2 to L^2. The kernels that we will encounter are small perturbations of $1/x$. Restricting a Calderón–Zygmund kernel to the integers, we have a kernel which satisfies Theorem 2C.

In a different direction, we will accumulate a range of sparse operator bounds at different points of our argument. Yet there is, in a sense, a unique maximal sparse operator, once a pair of functions f, g are specified. Thus we need not specify the exact sparse form which proves our main theorem.

Lemma 2.3 [Lacey and Mena Arias 2017, Lemma 4.7]. *Given finitely supported functions f, g and choices of $1 \le r, s < \infty$, there is a sparse form $\Lambda^*_{r,s}$, and constant $C > 0$ so that for any other sparse form $\Lambda_{r,s}$ we have*

$$\Lambda_{r,s}(f,g) \le C\Lambda^*_{r,s}(f,g).$$

A couple of elementary estimates, which we will appeal to, are in this next proposition. The use of these inequalities in the sparse-bound setting appeared in [Lacey and Spencer 2017].

Proposition 2.4. *Let $T_K f(x) = \sum_n K(n) f(x-n)$ be convolution with kernel K. Assuming that K is finitely supported on interval $[-N, N]$ we have the inequalities*

$$\|T_K : \mathrm{Sparse}(r,s)\| \lesssim N^{1/r+1/s-1} \|T_K : \ell^r \mapsto \ell^{s'}\|, \quad 1 \le r, s < \infty. \tag{2.5}$$

The two instances of the above inequality we will use are $(r,s) = (1,1), (2,2)$. In the latter case, one should observe that the power of N above is zero.

Proof. Let $\mathcal{I}$ be a partition of $\mathbb{Z}$ into intervals of length $2N$. Assume that if $I, I' \in \mathcal{I}$ with $\operatorname{dist}(I, I') \leq 1$, then either $f\mathbf{1}_I$ or $f\mathbf{1}_{I'}$ are identically zero. Then,

$$\begin{aligned}|\langle T_K f, g\rangle| &\leq \sum_{I\in\mathcal{I}} \langle f\mathbf{1}_I, T_K^*(g\mathbf{1}_{3I})\rangle \\ &\leq \|T_K : \ell^r \mapsto \ell^{s'}\| \sum_{I\in\mathcal{I}} \|f\mathbf{1}_{3I}\|_r \, \|g\mathbf{1}_{3I}\|_s \\ &\lesssim N^{1/r+1/s-1} \|T_K : \ell^r \mapsto \ell^{s'}\| \sum_{I\in\mathcal{I}} |3I| \cdot \langle f\rangle_{3I,r} \langle g\rangle_{3I,s}. \end{aligned}$$ □

The definition of sparse collections has a useful variant. Let $0 < \eta \leq \frac{1}{4}$. We say a collection of intervals $\mathcal{S}$ is *η-sparse* if there are subsets $E_S \subset S \subset \mathbb{Z}$ with (a) $|E_S| > \eta|S|$, uniformly in $S \in \mathcal{S}$, and (b) the sets $\{E_S : S \in \mathcal{S}\}$ are pairwise disjoint.

Lemma 2.6. *For each f, g there is a $\frac{1}{2}$-sparse form Λ so that for all η-sparse forms Λ^η, we have*

$$\Lambda^\eta(f, g) \lesssim \eta^{-1}\Lambda(f, g), \quad 0 < \eta < \tfrac{1}{4}.$$

Proof. Let $\mathcal{S}^\eta$ be the sparse collection of intervals associated to Λ^η. Using shifted dyadic grids [Hytönen et al. 2013, Lemma 2.5], we can, without loss of generality, assume that $\mathcal{S}^\eta$ consists of dyadic intervals. It follows that we have the uniform Carleson measure estimate

$$\sum_{J\in\mathcal{S}:J\subset I} |J| \lesssim \eta^{-1}|I|, \quad I \in \mathcal{S}^\eta.$$

Then, for an integer $J \lesssim \eta^{-1}$, we can decompose $\mathcal{S}^\eta$ into subcollections $\mathcal{S}_j$, for $1 \leq j \leq J$, so that each collection $\mathcal{S}_j$ is $\frac{1}{2}$-sparse.

Now, with f, g fixed, by Lemma 2.3, there is a single sparse operator Λ so that uniformly in $1 \leq j \leq J$ we have

$$\Lambda_{\mathcal{S}_j}(f, g) \lesssim \Lambda(f, g).$$ □

A variant of the sparse operator will appear, one with a "long tails" average. Define

$$\{f\}_S = \frac{1}{|S|} \sum_x \frac{|f(x)|}{(1+\operatorname{dist}(x, S)/|S|)^3}. \tag{2.7}$$

Lemma 2.8. *For all finitely supported f, g, there is a sparse operator Λ so that for any sparse collection $\mathcal{S}_0$ there holds*

$$\sum_{S\in\mathcal{S}_0} |S|\{f\}_S\{g\}_S \lesssim \Lambda(f, g).$$

Proof. For integers $t > 0$ let $\mathcal{S}_t = \{2^t S : S \in \mathcal{S}\}$. Assuming that $\mathcal{S}_0$ is $\frac{1}{2}$-sparse, it follows that $\mathcal{S}_t$ is 2^{-t-1}-sparse, for $t > 0$. Appealing to the power decay in (2.7),

$$\sum_{S\in\mathcal{S}_0} |S|\{f\}_S\{g\}_S \lesssim \sum_{t=0}^{\infty} 2^{-2t} \Lambda_{\mathcal{S}_t}(f, g).$$

But by Lemma 2.6, there is a fixed $\frac{1}{2}$-sparse form $\Lambda(f, g)$ so that

$$\Lambda_{\mathcal{S}_t}(f, g) \lesssim 2^t \Lambda(f, g), \quad t > 0. \qquad \square$$

Throughout, $e(x) := e^{2\pi i x}$, and $\varepsilon > 0$ is a fixed small absolute constant. For a function $f \in \ell^2(\mathbb{Z})$, the (inverse) Fourier transform of f is defined as

$$\mathcal{F} f(\beta) := \sum_{n \in \mathbb{Z}} f(n) e(-\beta n),$$
$$\mathcal{F}^{-1} g(n) = \int_{\mathbb{T}} g(\beta) e(\beta n)\, d\beta.$$

We will define operators as Fourier multipliers. Namely, given a function $M : \mathbb{T} \mapsto \mathbb{C}$, we define the associated linear operator by

$$\mathcal{F}[\Phi_M f](\beta) = M(\beta) \mathcal{F} f(\beta). \tag{2.9}$$

The notation $\mathcal{F}^{-1} M = \check{M}$ will be convenient. As above, for kernel K, the operator T_K will denote convolution with respect to K. Thus, $\Phi_M = T_{\check{M}}$.

3. The main decomposition

We prove the main result by decomposition of the Fourier multiplier

$$M(\beta) := \sum_{m \neq 0} \frac{e(-\beta m^3)}{m}. \tag{3.1}$$

In this section, we detail the decomposition, which is done in the standard way, with one new point needed.

The kernel. Let $\{\psi_j\}_{j \geq 0}$ be a dyadic resolution of $1/t$, where $\psi_j(x) = 2^{-j} \psi(2^{-j} x)$ is a smooth odd function satisfying $|\psi(x)| \leq 1_{[1/4,1]}(|x|)$. In particular

$$\sum_{k \geq 0} \psi_k(t) = \frac{1}{t}, \quad |t| \geq 1. \tag{3.2}$$

The major arcs. The rationals in the torus are the union over $s \in \mathbb{N}$ of the collections $\mathcal{R}_s$ given by

$$\mathcal{R}_s := \{B/Q \in \mathbb{T} : (B, Q) = 1,\ 2^{s-1} \leq Q < 2^s\}. \tag{3.3}$$

Namely the denominator of the rationals is held approximately fixed. For all rationals $B/Q \in \mathcal{R}_s$, define the j-th major box at B/Q to be

$$\mathfrak{M}_j(B/Q) := \{\beta \in \mathbb{T} : |\beta - B/Q| \leq 2^{(\varepsilon-3)j}\}, \quad s \leq \varepsilon j.$$

Collect the major arcs, defining

$$\mathfrak{M}_j := \bigcup_{(B,Q)=1 : Q \leq 2^{6j\varepsilon}} \mathfrak{M}_j(B/Q). \tag{3.4}$$

Note in particular that for a sufficiently small ε, in the union above no two distinct major arcs $\mathfrak{M}_j(B/Q)$ intersect. That is, if $B_1/Q_1 \neq B_2/Q_2$, suppose that $\beta \in \mathfrak{M}_j(B_1/Q_1) \cup \mathfrak{M}_j(B_2/Q_2)$. Then

$$2^{-6j\varepsilon} \leq |B_1/Q_1 - B_2/Q_2| \leq |B_1/Q_1 - \beta| + |B_2/Q_2 - \beta| \leq 2^{(\varepsilon-3)j+1},$$

which is a contradiction for $\varepsilon < \frac{2}{7}$.

Multipliers. We use the notation below for the decomposition of the multiplier:

$$M_j(\beta) := \sum_{m\in\mathbb{Z}} e(-\beta m^3)\psi_j(m), \tag{3.5}$$

$$H_j(y) := \int_{\mathbb{R}} e(-yt^3)\psi_j(t)\,dt \quad \text{(continuous analog of } M_j\text{)}, \tag{3.6}$$

$$S(B/Q) := \frac{1}{Q}\sum_{r=0}^{Q-1} e(-B/Q\cdot r^3) \quad \text{(Gauss sum)},$$

$$L_{j,s}(\beta) := \sum_{B/Q\in\mathcal{R}_s} S(B/Q)H_j(\beta - B/Q)\chi_s(\beta - B/Q), \tag{3.7}$$

where χ is a smooth even bump function with $\mathbb{1}_{[-1/10,1/10]} \leq \chi \leq \mathbb{1}_{[-1/5,1/5]}$ and $\chi_s(t) = \chi(10^s t)$,

$$L_j(\beta) := \sum_{s\leq j\varepsilon} L_{j,s}(\beta), \quad j \geq 1, \tag{3.8}$$

$$L^s(\beta) := \sum_{j\geq s/\varepsilon} L_{j,s}(\beta), \quad s \geq 1, \tag{3.9}$$

$$L(\beta) := \sum_{s=1}^{\infty} L^s(\beta) = \sum_{j=1}^{\infty} L_j(\beta),$$

$$E_j(\beta) := M_j(\beta) - L_j(\beta), \quad j \geq 1, \tag{3.10}$$

$$E(\beta) := \sum_{j=1}^{\infty} E_j(\beta). \tag{3.11}$$

Therefore, by construction, $M(\beta) = L(\beta) + E(\beta)$ for all $\beta \in \mathbb{T}$. Our motivation for introducing the above decomposition is that the discrete multiplier M_j is well-approximated by its continuous analog L_j on the major arcs in $\mathfrak{M}_j$. And off of the major arcs, the multiplier is otherwise small.

Theorem 1.1 is proved by showing that there exist $1 < r < 2$ and $\kappa > 0$ such that

$$\|\Phi_{E_j} : \mathrm{Sparse}(r, r)\| \lesssim 2^{-\kappa j}, \quad j \geq 1, \tag{3.12}$$

$$\|\Phi_{L^s} : \mathrm{Sparse}(r, r)\| \lesssim 2^{-\kappa s}, \quad s \geq 1. \tag{3.13}$$

Indeed, from the above inequalities, it follows that

$$\|\Phi_L : \mathrm{Sparse}(r, r)\| \leq \sum_{s=1}^{\infty} \|\Phi_{L^s} : \mathrm{Sparse}(r, r)\| \lesssim \sum_{s=1}^{\infty} 2^{-\kappa s} \lesssim 1,$$

$$\|\Phi_E : \mathrm{Sparse}(r, r)\| \le \sum_{j=1}^{\infty} \|\Phi_{E_j} : \mathrm{Sparse}(r, r)\| \lesssim \sum_{j=1}^{\infty} 2^{-\kappa j} \lesssim 1.$$

Therefore, our main theorem follows from

$$\|\Phi_M : \mathrm{Sparse}(r, r)\| \le \|\Phi_L : \mathrm{Sparse}(r, r)\| + \|\Phi_E : \mathrm{Sparse}(r, r)\| \lesssim 1.$$

We prove the "minor arcs" estimate (3.12) in Section 4 and the "major arcs" estimate (3.13) in Section 5.

The next theorem gives quantitative estimates for the Gauss sums (3.15) and the multipliers E_j defined in (3.10) that are essential to our proof of Theorem 1.1.

Theorem 3.14. *For absolute choices of* $\varepsilon > 0$,

$$|S(B/Q)| \lesssim 2^{-\varepsilon s}, \quad B/Q \in \mathcal{R}_s,\ s \ge 1, \tag{3.15}$$

$$\|E_j(\beta)\|_\infty \lesssim 2^{-\varepsilon j}, \quad j \ge 1, \tag{3.16}$$

$$\left\| \frac{d^2}{d\beta^2} E_j(\beta) \right\|_\infty \lesssim 2^{7j}, \quad j \ge 1. \tag{3.17}$$

The first two are well-known estimates. The estimate (3.15) is the Gauss sum bound, see [Hua 1982], while the estimate (3.16) is gotten by combining Lemmas 3.21 and 3.18. The only unfamiliar estimate is the derivative bound (3.17), but our claim is very weak and follows from elementary considerations.

The details of a proof of Theorem 3.14 are represented in the literature [Stein and Wainger 1990; Krause and Lacey 2017]. We indicate the details. A central lemma is this approximation of M_j defined in (3.5), in terms of L_j defined in (3.8).

Lemma 3.18. *For* $1 \le s \le \varepsilon j$, $B/Q \in \mathcal{R}_s$, *we have the approximation*

$$M_j(\beta) = L_j(\beta) + O(2^{(2\varepsilon-1)j}), \quad \beta \in \mathfrak{M}_j(B/Q).$$

Proof. We closely follow the argument in [Krause and Lacey 2017]. There are two estimates to prove:

$$|M_j(\beta) - S(B/Q)H_j(\beta - B/Q)| \lesssim 2^{(2\varepsilon-1)j}, \tag{3.19}$$

$$|L_j(\beta) - S(B/Q)H_j(\beta - B/Q)| \lesssim 2^{(2\varepsilon-1)j}, \tag{3.20}$$

both estimates holding uniformly over $\beta \in \mathfrak{M}_j(B/Q)$, and $B/Q \in \mathcal{R}_s$.

For the second estimate (3.20), it follows from the definitions of L_j and $L_{j,s}$ in (3.7), as well as the disjointess of the major arcs, that

$$\begin{aligned} |L_j(\beta) - S(B/Q)H_j(\beta - B/Q)| &= |L_{j,s}(\beta) - S(B/Q)H_j(\beta - B/Q)| \\ &\le |S(B/Q)H_j(\beta - B/Q)|(\mathbf{1}_{\mathfrak{M}_j(B/Q)} - \chi(10^s(\beta - B/Q))) \\ &\lesssim \sup_{|\beta| > (1/2)10^{s-1}} |H_j(\beta)| \lesssim 10^{-s}. \end{aligned}$$

The last bound is a standard van der Corput estimate.

We turn to (3.19). Write $\beta = B/Q + \eta$, where $|\eta| \le 2^{(\varepsilon-3)j}$. For all positive m in the support of ψ_j, decompose these integers into their residue classes mod Q, i.e., $m = pQ + r$, where $0 \le r < Q \le 2^{j\varepsilon}$ and the p-values are integers in $[c, d]$, with $c = d/8 \simeq 2^j/Q$ to cover the support of ψ_j. The argument of the exponential in (3.1) is, modulo 1, given by

$$\beta(pQ+r)^3 = (B/Q+\eta)(pQ+r)^3 \equiv r^3 B/Q + (pQ)^3\eta + O(2^{j(2\varepsilon-1)}).$$

Then the sum over all positive integers m in the support of ψ_j can be written as

$$\begin{aligned}\sum_{p\in[c,d]} \sum_{r=0}^{Q-1} &[e(-r^3 B/Q - (pQ)^3\eta) + O(2^{(2\varepsilon-1)j})]\psi_j(pQ+r) \\ &= \sum_{r=0}^{Q-1} e(-r^3 \cdot B/Q) \times \sum_{p\in[c,d]} e(-\eta(pQ)^3)\psi_j(pQ) + O(2^{(2\varepsilon-1)j}) \\ &= S(B, Q) \times Q \sum_{p\in[c,d]} e(-\eta(pQ)^3)\psi_j(pQ) + O(2^{(2\varepsilon-1)j}).\end{aligned}$$

For fixed $p \in [c, d]$ and $0 \le t \le Q$, we have

$$\begin{aligned}|e(-\eta(pQ)^3)\psi_j(pQ) - e(-\eta(pQ+t)^3)\psi_j(pQ+t)| \\ &\lesssim |e(-\eta(pQ)^3) - e(-\eta(pQ+t)^3)|2^{-j} + |\psi_j(pQ) - \psi_j(pQ+t)| \\ &\lesssim 2^{(2\varepsilon-2)j}.\end{aligned}$$

Therefore,

$$Q \sum_{p\in[c,d]} e(-\eta(pQ)^3)\psi_j(pQ) = \int_0^\infty e(-\eta t^3)\psi_j(t)\,dt + O(2^{(2\varepsilon-1)j}).$$

The analogous computation for negative values of m yields

$$\sum_{m<0} e(-\beta m^3)\psi_j(m) = S(B, Q) \times \int_{-\infty}^0 e(-\eta t^3)\psi_j(t)\,dt + O(2^{(2\varepsilon-1)j}),$$

and combining the two estimates with the notation in (3.11) leads to the desired conclusion. □

We also need control of M_j and L_j, defined in (3.8) on the minor arcs, which are the open components of the complement of $\mathfrak{M}_j$ defined in (3.4).

Lemma 3.21. *There is a $\delta = \delta(\varepsilon)$ so that uniformly in $j \ge 1$,*

$$|M_j(\beta)| + |L_j(\beta)| \lesssim 2^{-\delta j}, \quad \beta \notin \mathfrak{M}_j.$$

This estimate is essentially present in [Krause and Lacey 2017]. The bound $|M_j(\beta)| \lesssim 2^{-\delta j}$ for $\beta \notin \mathfrak{M}_j$ can be seen from [Bourgain 1989, Lemma 5.4], and is a consequence of a fundamental estimate of Weyl [Iwaniec and Kowalski 2004, Theorem 8.1]. The corresponding bound on L_j is an easy consequence of the van der Corput estimate $|H_j(y)| \lesssim 2^{-j}|y|^{-1/3}$.

4. Minor arcs

Recalling the sparse-form notation (2.1) and the Fourier multiplier notation (2.9), we now proceed to the proof of the bound in (3.12).

Lemma 4.1. *There exists $\kappa > 0$ and $1 < r < 2$ such that*

$$\|\Phi_{E_j} : \mathrm{Sparse}(r, r)\| \lesssim 2^{-\kappa j}, \quad j \geq 1.$$

Proof. We only need the L^∞ bound on E_j given in (3.16), and the derivative condition (3.17). In particular, these two conditions imply

$$|\mathcal{F}^{-1} E_j(m)| \lesssim \min\left\{2^{-\varepsilon j}, \frac{2^{7j}}{1+m^2}\right\}. \tag{4.2}$$

Write $\mathcal{F}^{-1}E_j = \check{E}_{j,0} + \check{E}_{j,1}$, where $\check{E}_{j,0}(m) = [\mathcal{F}^{-1}E_j(m)]\mathbf{1}_{[-2^{10j},2^{10j}]}(m)$. It follows immediately from (4.2) that

$$\|T_{\check{E}_{j,1}} : \ell^2 \mapsto \ell^2\| \lesssim \|\check{E}_{j,1}\|_1 \lesssim 2^{-3j}.$$

(Recall that T_K denotes convolution with respect to kernel K.) But, it follows that $T_K f \lesssim M_{\mathrm{HL}} f$, where the latter is the maximal function. And so by Theorem 2B, we have

$$\|T_{\check{E}_{j,1}} : \mathrm{Sparse}(1, 1)\| \lesssim 2^{-3j}.$$

It remains to provide a sparse bound for $T_{\check{E}_{j,0}}$ (which is the interesting case). We are in a position to use (2.5), with $N \simeq 2^{10j}$. We have for $1 < r < 2$

$$\|T_{\check{E}_{j,0}} : \mathrm{Sparse}(r, r)\| \lesssim 2^{10j(2/r-1)}\|T_{\check{E}_{j,0}} : \ell^r \mapsto \ell^{r'}\|. \tag{4.3}$$

Notice that $2/r - 1$ can be made arbitrarily small. We need to estimate the operator norm above. But, we have the two estimates

$$\|T_{\check{E}_{j,0}} : \ell^s \mapsto \ell^{s'}\| \lesssim 2^{-\varepsilon j}, \quad s = 1, 2.$$

The case of $s = 1$ follows from (4.2), and the case of $s = 2$ from Plancherel and (3.16). We therefore see that we have a uniformly small estimate on the norm of $T_{\check{E}_{j,0}}$ from $\ell^r \mapsto \ell^{r'}$ for $1 < r < 2$. For $0 < 2 - r \ll \varepsilon$, we have the desired bound in (4.3). □

5. Major arcs

The following estimate is the core of the main result, Theorem 1.1. Recalling the definition of L^s in (3.9), the notation for Fourier multipliers (2.9) and the sparse norm notation (2.1), we have this, which verifies the bound in (3.13):

Lemma 5.1. *There exist $\kappa > 0$ and $1 < r < 2$ such that*

$$\|\Phi_{L^s} : \mathrm{Sparse}(r, r)\| \lesssim 2^{-\kappa s}, \quad s \geq 1.$$

Combining the "major arcs" estimate in Lemma 5.1 with the "minor arcs" estimate in Lemma 4.1, the proof of Theorem 1.1 is complete.

The remainder of this section is taken up with the proof of the lemma. The central facts are (1) the Gauss sum bound (3.15); (2) the sparse bound for Hilbert-space-valued singular integrals Theorem 2C, which is applied to Fourier projections of f and g onto the major arcs; (3) an argument to pass from a sparse operator applied to the aforementioned Fourier projections to a sparse bound in terms of just f and g.

Step 1. We define our Hilbert-space-valued functions, where the Hilbert space will be the finite-dimensional space $\ell^2(\mathcal{R}_s)$. Recall that the rationals $\mathcal{R}_s$ are defined in (3.3), and the functions χ_s are defined in (3.7). Given $f \in \ell^2$, set

$$f_s = \{f_{s,B/Q} : B/Q \in \mathcal{R}_s\} := \{\chi_{s-1} * (\mathrm{Mod}_{-B/Q} f) : B/Q \in \mathcal{R}_s\}. \tag{5.2}$$

Above, $\mathrm{Mod}_\lambda f(x) = e(\lambda x) f(x)$ is modulation by λ. The intervals

$$\{[B/Q - 10^{-s}, B/Q + 10^{-s}] : B/Q \in \mathcal{R}_s\} \tag{5.3}$$

are pairwise disjoint, so that by Bessel's theorem, we have

$$\|f_s\|_{\ell^2(\ell^2(\mathcal{R}_s))} = \|\{f_{s,B/Q} : B/Q \in \mathcal{R}_s\}\|_{\ell^2(\ell^2(\mathcal{R}_s))} \le \|f\|_2.$$

Step 2. The inner product we are interested in can be viewed as one acting on $\ell^2(\mathcal{R}_s)$ functions. Observe that the Fourier multiplier associated to L^s enjoys the equalities below. Beginning from (3.9) and (3.7),

$$\begin{aligned}
\langle \Phi_{L^s} f, g\rangle &= \sum_{B/Q\in\mathcal{R}_s} \sum_{j\ge s/\varepsilon} S(B,Q)\cdot \langle H_j(\beta - B/Q)\chi_s(\beta - B/Q)\mathcal{F}f(\beta), \mathcal{F}g(\beta)\rangle \\
&= \sum_{B/Q\in\mathcal{R}_s} \sum_{j\ge s/\varepsilon} S(B,Q)\cdot \langle H_j(\beta)\chi_s(\beta) f(\beta + B/Q), \mathcal{F}g(\beta + B/Q)\rangle \\
&= \sum_{B/Q\in\mathcal{R}_s} \sum_{j\ge s/\varepsilon} S(B,Q)\cdot \langle H_j(\beta)\chi_s(\beta)\mathcal{F}f_{s,B/Q}(\beta), \mathcal{F}g_{s,B/Q}(\beta)\rangle.
\end{aligned}$$

Crucially, above we have removed some modulation factors to get a fixed multiplier acting on a Hilbert-space-valued function. Continuing the equalities, we have

$$= \sum_{B/Q\in\mathcal{R}_s} S(B,Q)\langle \Phi_{H^s} f_{s,B/Q}, g_{s,B/Q}\rangle, \quad \text{where } H^s = \sum_{j\ge s/\varepsilon} H_j. \tag{5.4}$$

We address the Gauss sums $S(B,Q)$ above. Recalling (3.15) and setting $f_s' = \{\lambda_{B/Q} f_{s,B/Q}\}$ for appropriate choice of $|\lambda_{B/Q}| = 1$, we have

$$|\langle \Phi_{L^s} f_s, g_s\rangle| \lesssim 2^{-\varepsilon s}\langle \Phi_{H^s} f_s', g_s\rangle. \tag{5.5}$$

Above we have gained a geometric decay in s.

On the right-hand side of (5.5), we have an operator acting on Hilbert-space-valued functions. Noting that $\|f_s'\|_{\ell^2(\mathcal{R}_s)} = \|f_s\|_{\ell^2(\mathcal{R}_s)}$ pointwise, we are free to replace f_s' in (5.5) by simply f_s, as defined in (5.2). The remaining estimate to prove is that there is a choice of $1 < r < 2$ and sparse operator $\Lambda_{r,r}$ so that

$$|\langle \Phi_{H^s} f_s, g_s\rangle| \lesssim 2^{(\varepsilon/4)s}\Lambda_{r,r}(f, g). \tag{5.6}$$

Note in particular that we will allow small geometric growth in this estimate, which will be absorbed into the geometric decay in (5.5).

Step 3. The principal step is the application of the sparse bound in Theorem 2C. From the definitions in (3.6) and (5.4), we have

$$H^s(\beta) = \sum_{j \geq s/\varepsilon} H_j(\beta) = \sum_{j \geq s/\varepsilon} \int e(-\beta t^3)\psi_j(t)\,dt.$$

By the choice of ψ in (3.2), it follows that the integrand on the right equals $e(-\beta t^3)dt/t$ for $t > 2^{s/\varepsilon+1}$. And, in particular,

$$H^s(\beta) = \frac{1}{3} \sum_{j \geq s/\varepsilon} \int e(-\beta s) \frac{\psi_j(s^{1/3})}{s^{2/3}}\,ds.$$

But ψ is odd; hence so is $\psi_j(s^{1/3})/s^{2/3}$. It follows that $\check{H}^s$ is a Calderón–Zygmund kernel; that is, it meets the conditions in (2.2). Thus, the operator we are considering is convolution with respect to $\check{H}^s$, namely $\Phi_{H^s} = T_{\check{H}^s}$.

Therefore, from Theorem 2C, we have the following inequality for the expression in (5.4):

$$|\langle T_{\check{H}^s} f_s, g_s\rangle| \lesssim \Lambda_{1,1}(f_s, g_s). \tag{5.7}$$

There is one additional fact: all the intervals used in the definition of the sparse form in (5.7) above have length at least $2^{3(s/\varepsilon-2)}$. This is a simple consequence of $\check{H}^s(x)\mathbf{1}_{[-2^{3(s/\varepsilon-2)},2^{3(s/\varepsilon-2)}]} \equiv 0$.

Step 4. We should emphasize that (5.7) has a small abuse of notation: The sparse form is computed on the vector-valued functions f_s and g_s. That is, the implied averages have to be made relative to the $\ell^2(\mathcal{R}_s)$-norm. The last step is to remove the norm. Namely, we show that there is a choice of $1 < r < 2$ and sparse form $\Lambda_{r,r}$ so that

$$\Lambda_{1,1}(f_s, g_s) \lesssim 2^{(\varepsilon/4)s} \Lambda_{r,r}(f, g). \tag{5.8}$$

Combining this estimate with (5.7) proves (5.6), completing the proof.

The proof of (5.8) is reasonably routine. It will be crucial that we have the estimate $\sharp\mathcal{R}_s \lesssim 2^{2s}$. Let $\mathcal{S}$ be the sparse collection of intervals associated with the sparse form $\Lambda_{1,1}(f_s, g_s)$. As noted, we are free to assume that for all $S \in \mathcal{S}$, we have $|S| \geq 10^{s/4\varepsilon}$. Recall the definition of f_s in (5.2). Write $f_s = f_s^{S,0} + f_s^{S,1}$, where

$$f_s^{S,0} := \{\chi_{s-1} * (\mathrm{Mod}_{-B/Q}(f\mathbf{1}_{2S})) : B/Q \in \mathcal{R}_s\}.$$

Above, we have localized the support of f to the interval $2S$. The same decomposition is used on the functions g and g_s. By subadditivity, we have

$$\Lambda_{1,1}(f_s, g_s) \leq \Lambda_{1,1}(f_s^{S,0}, g_s^{S,0}) \tag{5.9}$$

$$+ \Lambda_{1,1}(f_s^{S,1}, g_s^{S,0}) + \Lambda_{1,1}(f_s^{S,0}, g_s^{S,1}) \tag{5.10}$$

$$+ \Lambda_{1,1}(f_s^{S,1}, g_s^{S,1}). \tag{5.11}$$

The crux of the matter is this estimate: For each interval $S \in \mathcal{S}$, we have

$$\langle f_s^{S,0} \rangle_S \lesssim 2^{s(2-r)/r} \langle f \rangle_{2S,r}, \quad 1 < r < 2. \tag{5.12}$$

And, the fraction $(2-r)/r$ in the exponent can be made arbitrarily small by taking $0 < 2-r$ very small. Indeed, using the disjointness of the intervals in (5.3), and Plancherel, we have

$$\langle f_s^{S,0} \rangle_{S,2} \lesssim \langle f \rangle_{2S,2}. \tag{5.13}$$

Second, it is trivial that

$$\langle \chi_{s-1} * (\mathrm{Mod}_{-B/Q}\, f \mathbf{1}_{2S}) \rangle_S \lesssim \langle f \rangle_{2S}$$

and by simply summing over the bounded number of choices of $B/Q \in \mathcal{R}_s$, we have

$$\langle f_s^{S,0} \rangle_S \lesssim 2^{2s} \langle f \rangle_{2S}.$$

Interpolating between this and (5.13) proves (5.12). With that inequality in hand, we have, for $0 < 2-r$ sufficiently small,

$$\sum_{S \in \mathcal{S}} |S| \langle f_s^{S,0} \rangle_S \langle g_s^{S,0} \rangle_S \lesssim 2^{s(\varepsilon/4)} \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{2S,r} \langle g \rangle_{2S,r}.$$

If the family $\mathcal{S}$ is $\frac{1}{2}$-sparse, then the family $\{2S : S \in \mathcal{S}\}$ is $\frac{1}{4}$-sparse, so we have our desired bound for the term on the right in (5.9).

There are three more terms, in (5.10) and (5.11), which are all much smaller. Recall the notation $\{f\}$ of (2.7). Since χ, as chosen in (3.7), is smooth, and the length of $S \in \mathcal{S}$ is much larger than 10^s, we have

$$\langle \chi_{s-1} * (\mathrm{Mod}_{-B/Q}\, f \mathbf{1}_{\mathbb{R}\setminus 2S}) \rangle_S \lesssim 2^{-100s} \{f\}_S, \quad B/Q \in \mathcal{R}_s.$$

Summing this estimate over all 2^{2s} choices $B/Q \in \mathcal{R}_s$, we see that each of the three terms in (5.10) and (5.11) are at most

$$2^{-s} \sum_{S \in \mathcal{S}} |S| \{f\}_S \{g\}_S.$$

It remains to bound this last bilinear form, which is the task taken up in Lemma 2.8. This completes the argument for (5.8).

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Received 9 Nov 2017. Accepted 5 Jul 2018.

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ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1273

ON THE DIMENSION AND SMOOTHNESS OF RADIAL PROJECTIONS

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This paper contains two results on the dimension and smoothness of radial projections of sets and measures in Euclidean spaces.

To introduce the first one, assume that $E, K \subset \mathbb{R}^2$ are nonempty Borel sets with $\dim_{\mathrm{H}} K > 0$. Does the radial projection of K to some point in E have positive dimension? Not necessarily: E can be zero-dimensional, or E and K can lie on a common line. I prove that these are the only obstructions: if $\dim_{\mathrm{H}} E > 0$, and E does not lie on a line, then there exists a point in $x \in E$ such that the radial projection $\pi_x(K)$ has Hausdorff dimension at least $(\dim_{\mathrm{H}} K)/2$. Applying the result with $E = K$ gives the following corollary: if $K \subset \mathbb{R}^2$ is a Borel set which does not lie on a line, then the set of directions spanned by K has Hausdorff dimension at least $(\dim_{\mathrm{H}} K)/2$.

For the second result, let $d \geq 2$ and $d-1 < s < d$. Let μ be a compactly supported Radon measure in $\mathbb{R}^d$ with finite s-energy. I prove that the radial projections of μ are absolutely continuous with respect to $\mathcal{H}^{d-1}$ for every centre in $\mathbb{R}^d \setminus \operatorname{spt}\mu$, outside an exceptional set of dimension at most $2(d-1)-s$. In fact, for x outside an exceptional set as above, the proof shows that $\pi_{x\sharp}\mu \in L^p(S^{d-1})$ for some $p > 1$. The dimension bound on the exceptional set is sharp.

1. Introduction

This paper studies visibility and radial projections. Given $x \in \mathbb{R}^d$, define the radial projection $\pi_x : \mathbb{R}^d \setminus \{x\} \to S^{d-1}$ by

$$\pi_x(y) = \frac{y-x}{|y-x|}.$$

A Borel set $K \subset \mathbb{R}^2$ will be called

- *invisible from* x if $\mathcal{H}^{d-1}(\pi_x(K \setminus \{x\})) = 0$, and
- *totally invisible from* x if $\dim_{\mathrm{H}} \pi_x(K \setminus \{x\}) = 0$.

Above, $\dim_{\mathrm{H}}$ stands for Hausdorff dimension and $\mathcal{H}^s$ stands for s-dimensional Hausdorff measure. I will only consider Hausdorff dimension in this paper, as many of the results below would be much easier for box dimension. The study of (in-)visibility has a long tradition in geometric measure theory. For many

Orponen is supported by the Academy of Finland via the project *Quantitative rectifiability in Euclidean and non-Euclidean spaces*, grant number 309365. The research was also partially supported by travel grants from the *Väisälä fund* and *Mathematics fund* of the *Finnish Academy of Science and Letters*.
MSC2010: primary 28A80; secondary 28A78.
Keywords: Hausdorff dimension, fractals, radial projections, visibility.

more results and questions than I can introduce here, see Section 6 of [Mattila 2004]. The basic question is the following: given a Borel set $K \subset \mathbb{R}^d$, how large can the sets

$$\mathrm{Inv}(K) = \{x \in \mathbb{R}^d : K \text{ is invisible from } x\},$$
$$\mathrm{Inv}_T(K) := \{x \in \mathbb{R}^d : K \text{ is totally invisible from } x\}$$

be? Clearly $\mathrm{Inv}_T(K) \subset \mathrm{Inv}(K)$, and one generally expects $\mathrm{Inv}_T(K)$ to be significantly smaller than $\mathrm{Inv}(K)$. The existing results fall roughly into the following three categories:

(1) What happens if $\dim_{\mathrm{H}} K > d-1$?

(2) What happens if $\dim_{\mathrm{H}} K \le d-1$?

(3) What happens if $0 < \mathcal{H}^{d-1}(K) < \infty$?

Cases (1) and (3) are the most classical, having already been studied (for $d = 2$) in [Marstrand 1954]. Given $s > 1$, Marstrand proved that any Borel set $K \subset \mathbb{R}^2$ with $0 < \mathcal{H}^s(K) < 1$ is visible (that is, not invisible) from Lebesgue almost every point $x \in \mathbb{R}^2$, and also from $\mathcal{H}^s$-almost every point $x \in K$. Unifying Marstrand's results, and their generalisations to $\mathbb{R}^d$, the following sharp bound was recently established by Mattila and the author in [Mattila and Orponen 2016; Orponen 2018]:

$$\dim_{\mathrm{H}} \mathrm{Inv}(K) \le 2(d-1) - \dim_{\mathrm{H}} K \tag{1.1}$$

for all Borel sets $K \subset \mathbb{R}^d$ with $d-1 < \dim_{\mathrm{H}} K \le d$. This paper contains a variant of the bound (1.1) for measures; see Section 1B.

The visibility of sets K in Case (3) depends on their rectifiability. I will restrict the discussion to the case $d = 2$ for now. It is easy to show that 1-rectifiable sets which are not $\mathcal{H}^1$-almost surely covered by a single line are visible from all points in $\mathbb{R}^2$, with possibly one exception; see [Orponen and Sahlsten 2011]. On the other hand, if $K \subset \mathbb{R}^2$ is purely 1-unrectifiable, then the sharp bound

$$\dim_{\mathrm{H}}[\mathbb{R}^2 \setminus \mathrm{Inv}(K)] = \dim_{\mathrm{H}}\{x \in \mathbb{R}^2 : K \text{ is visible from } x\} \le 1$$

was obtained by Marstrand, building on Besicovitch's projection theorem. For generalisations, improvements and constructions related to the bound above, see [Mattila 1981, Theorem 5.1; Csörnyei 2000; 2001]. Marstrand raised the question — which remains open to the best of my knowledge — whether it is possible that $\mathcal{H}^1(\mathbb{R}^2 \setminus \mathrm{Inv}(K)) > 0$: in particular, can a purely 1-unrectifiable set be visible from a positive fraction of its own points? For purely 1-unrectifiable self-similar sets $K \subset \mathbb{R}^2$ one has $\mathrm{Inv}(K) = \mathbb{R}^2$, as shown by Simon and Solomyak [2006/07].

1A. *The first main result.* Case (2) has received less attention. To simplify the discussion, assume that $\dim_{\mathrm{H}} K = 1$ and $\mathcal{H}^1(K) = 0$, so that $\mathrm{Inv}(K) = \mathbb{R}^2$, and the relevant question becomes the size of $\mathrm{Inv}_T(K)$. The radial projections π_p fit the influential *generalised projections* framework of [Peres and Schlag 2000]. If $K \subset \mathbb{R}^2$ is a Borel set with arbitrary dimension $s \in [0, 2]$, then it follows from Theorem 7.3 of that paper that

$$\dim_{\mathrm{H}} \mathrm{Inv}_T(K) \le 2 - s. \tag{1.2}$$

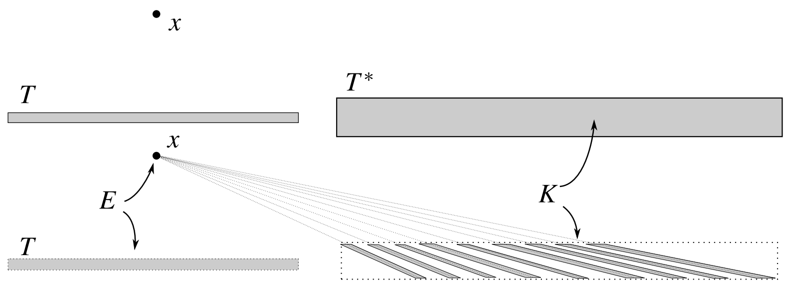

Figure 1. What is the next step in the construction of E?

When $s > 1$, the bound (1.2) is a weaker version of (1.1), but the benefit of (1.2) is that it holds without any restrictions on s. In particular, if $s = 1$, one obtains

$$\dim_{\mathrm{H}} \mathrm{Inv}_T(K) \le 1. \tag{1.3}$$

This bound is sharp for a trivial reason: consider the case, where K lies on a single line $\ell \subset \mathbb{R}^2$. Then, $\mathrm{Inv}_T(K) = \ell$. The starting point for this paper was the question: are there essentially different examples manifesting the sharpness of (1.3)? The answer turns out to be negative in a very strong sense. Here are the first main results of the paper:

Theorem 1.4 (weak version). *Assume that $K \subset \mathbb{R}^2$ is a Borel set with* $\dim_{\mathrm{H}} K > 0$. *Then, at least one of the following holds*:

- $\dim_{\mathrm{H}} \mathrm{Inv}_T(K) = 0$.
- $\mathrm{Inv}_T(K)$ *is contained on a line.*

In fact, more is true. For $K \subset \mathbb{R}^2$, define

$$\mathrm{Inv}_{1/2}(K) := \left\{x \in \mathbb{R}^2 : \dim_{\mathrm{H}} \pi_x(K \setminus \{x\}) < \tfrac{1}{2} \dim_{\mathrm{H}} K\right\}.$$

Then, if $\dim_{\mathrm{H}} K > 0$, one evidently has $\mathrm{Inv}_T(K) \subset \mathrm{Inv}_{1/2}(K) \subset \mathrm{Inv}(K)$.

Theorem 1.5 (strong version). *Theorem 1.4 holds with* $\mathrm{Inv}_T(K)$ *replaced by* $\mathrm{Inv}_{1/2}(K)$. *That is, if $E \subset \mathbb{R}^2$ is a Borel set with* $\dim_{\mathrm{H}} E > 0$, *not contained on a line, then there exists $x \in E$ such that* $\dim_{\mathrm{H}} \pi_x(K \setminus \{x\}) \ge (\dim_{\mathrm{H}} K)/2$.

Remark 1.6. A closely related result is Theorem 1.6 in [Bond, Łaba and Zahl 2016]; with some imagination, part (a) of that theorem can be viewed as a "single scale" variant of Theorem 1.5, although at this scale, their Theorem 1.6(a) contains more information than Theorem 1.5. As far as I can tell, proving the Hausdorff dimension statement in this context presents a substantial extra challenge, so Theorem 1.5 is not easily implied by the results in [Bond, Łaba and Zahl 2016].

Example 1.7. Figure 1 depicts the main challenge in the proofs of Theorems 1.4 and 1.5. The set E has $\dim_{\mathrm{H}} E > 0$, and consists of something inside a narrow tube T, plus a point $x \notin T$. Then, Theorem 1.4 states that $E \not\subset \mathrm{Inv}_T(K)$ for any compact set $K \subset \mathbb{R}^2$ with $\dim_{\mathrm{H}} K > 0$. So, in order to find a counterexample to Theorem 1.5, all one needs to do is find K by a standard "Venetian blind" construction in such a way that $\dim_H K > 0$ and $\dim_H \pi_y(K) = 0$ for all $y \in E$. The first steps are obvious: to begin with, require that $K \subset T^*$ for another narrow tube parallel to T; see Figure 1. Then $\pi_y(K)$ is small for all $y \in T$. To handle the special point $x \in E$, split the contents of T^* into a finite collection of new narrow tubes in such a way that $\pi_x(K)$ is small. In this manner, $\pi_y(K)$ can be made arbitrarily small for all $y \in E$ (in the sense of ϵ-dimensional Hausdorff content, for instance, for any prescribed $\epsilon > 0$). It is quite instructive to think why the construction cannot be completed: why cannot the Venetian blinds be iterated further (for both E and K) so that, at the limit, $\dim_H \pi_y(K) = 0$ for all $x \in E$?

Theorem 1.5 has the following immediate consequence:

Corollary 1.8 (corollary to Theorem 1.5). *Assume that $K \subset \mathbb{R}^2$ is a Borel set not contained on a line. Then the set of unit vectors spanned by K, namely*

$$S(K) := \left\{ \frac{x-y}{|x-y|} \in S^1 : x, y \in K \text{ and } x \neq y \right\},$$

satisfies $\dim_H S(K) \geq (\dim_H K)/2$.

Proof. If $\dim_H K = 0$, there is nothing to prove. Otherwise, Theorem 1.5 implies that $K \not\subset \mathrm{Inv}_{1/2}(K)$, whence $\dim_H S(K) \geq \dim_H \pi_x(K \setminus \{x\}) \geq (\dim_H K)/2$ for some $x \in K$. □

Corollary 1.8 is probably not sharp, and the following conjecture seems plausible:

Conjecture 1.9. *Assume that $K \subset \mathbb{R}^2$ is a Borel set not contained on a line. Then* $\dim_H S(K) = \min\{\dim_H K, 1\}$.

This follows from Marstrand's result, discussed in Case (1) above, when $\dim_H K > 1$. For $\dim_H K \leq 1$, Conjecture 1.9 is closely connected with continuous sum-product problems, which means that significant improvements over Corollary 1.8 will, most likely, require new technology. It would, however, be interesting to know if an ϵ-improvement over Corollary 1.8 is possible, combining the proof below with ideas from [Katz and Tao 2001], and using the discretised sum-product theorem of [Bourgain 2003].

I have the referee to thank for pointing out that a natural discrete variant of Conjecture 1.9 has been solved by P. Ungar [1982]: a set of $n \geq 3$ points in the plane, not all on a single line, determine at least $n-1$ distinct directions.

1B. *The second main result.* The second main result is a version of the estimate (1.1) for measures. Fix $d \geq 2$, and denote the space of compactly supported Radon measures on $\mathbb{R}^d$ by $\mathcal{M}(\mathbb{R}^d)$. For $\mu \in \mathcal{M}(\mathbb{R}^d)$, write

$$\mathcal{S}(\mu) := \{x \in \mathbb{R}^d \setminus \operatorname{spt}\mu : \pi_{x\sharp}\mu \text{ is not absolutely continuous with respect to } \mathcal{H}^{d-1}|_{S^{d-1}}\}.$$

Note that whenever $x \in \mathbb{R}^d \setminus \operatorname{spt}\mu$, the projection π_x is continuous on $\operatorname{spt}\mu$, and $\pi_{x\sharp}\mu$ is well-defined. One can check that the family of projections $\{\pi_x\}_{x \in \mathbb{R}^d \setminus \operatorname{spt}\mu}$ fits in the *generalised projections* framework of [Peres and Schlag 2000], and indeed Theorem 7.3 in that paper yields

$$\dim_H \mathcal{S}(\mu) \leq 2d - 1 - s, \tag{1.10}$$

whenever $d-1 < s < d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ has finite s-energy (see (1.12) for a definition). Combining this bound with standard arguments shows that if $K \subset \mathbb{R}^d$ is a Borel set with $d-1 < \dim_{\mathrm{H}} K \leq d$, then

$$\dim_{\mathrm{H}} \mathrm{Inv}(K) = \dim_{\mathrm{H}}\{x \in \mathbb{R}^d : \mathcal{H}^{d-1}(\pi_x(K)) = 0\} \leq 2d - 1 - \dim_{\mathrm{H}} K.$$

This is weaker than the sharp bound (1.1), so it is natural to ask whether the bound (1.10) for measures could be lowered to match (1.1). The answer is affirmative:

Theorem 1.11. *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ and*

$$I_s(\mu) := \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^s} < \infty \tag{1.12}$$

for some $s > d-1$, then $\dim_{\mathrm{H}} \mathcal{S}(\mu) \leq 2(d-1) - s$.

The bound is sharp, essentially because (1.1) is, and Theorem 1.11 implies (1.1). More precisely, following [Orponen 2018, Section 2.2], there exist compact sets $K \subset \mathbb{R}^d$ of any dimension $\dim_{\mathrm{H}} K \in (d-1, d)$ such that

$$\dim_{\mathrm{H}}[\mathrm{Inv}(K) \setminus K] = 2(d-1) - \dim K.$$

Then, the sharpness of Theorem 1.11 follows by considering Frostman measures supported on K, and noting that $\mathcal{S}(\mu) \supset \mathrm{Inv}(K) \setminus K$ whenever $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathrm{spt}\,\mu \subset K$.

An open question is the validity of Theorem 1.11 for $s = d-1$. If $I_{d-1}(\mu) < \infty$, Theorem 7.3 in [Peres and Schlag 2000] implies that $\mathcal{L}^d(\mathcal{S}(\mu)) = 0$, but I do not even know if $\dim_{\mathrm{H}} \mathcal{S}(\mu) < d$.

Theorem 1.11 does not immediately follow from the proof of (1.1) in [Mattila and Orponen 2016; Orponen 2018], as the argument in those papers was somewhat indirect. Having said that, many observations from the previous papers still play a role in the new proof. Theorem 1.11 will be deduced from the next statement concerning L^p-densities:

Theorem 1.13. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be as in Theorem 1.5. For $p \in (1,2)$, write*

$$\mathcal{S}_p(\mu) := \{x \in \mathbb{R}^d \setminus \mathrm{spt}\,\mu : \pi_{x\sharp}\mu \notin L^p(S^{d-1})\}.$$

Then $\dim_{\mathrm{H}} \mathcal{S}_p(\mu) \leq 2(d-1) - s + \delta(p)$, where $\delta(p) > 0$, and $\delta(p) \to 0$ as $p \searrow 1$.

Note that the claim is vacuous for "large" values of p. The dependence of $\delta(p) > 0$ on p is effective and not very hard to track; see (3.5).

Remark 1.14. Theorem 1.13 can be viewed as an extension of Falconer's exceptional set estimate [1982]. I only discuss the planar case. Falconer proved that if $I_s(\mu) < \infty$ for some $1 < s < 2$, then the orthogonal projections of μ to all 1-dimensional subspaces are in L^2, outside an exceptional set of dimension at most $2-s$. Now, orthogonal projections can be viewed as radial projections from points on the line at infinity. Alternatively, if the reader prefers a more rigorous statement, Falconer's proof shows that if $\ell \subset \mathbb{R}^2$ is any fixed line outside the support of μ, then all the radial projections of μ to points on ℓ are in L^2, outside an exceptional set of dimension at most $2-s$. In comparison, Theorem 1.13 states that the radial projections of μ to points in $\mathbb{R}^2 \setminus \mathrm{spt}\,\mu$ are in L^p for some $p > 1$, outside an exceptional set of dimension at most $2-s$. So, the size of the exceptional set remains the same even if the "fixed line ℓ" is removed from the statement. The price to pay is that the projections only belong to some L^p with $p > 1$ (possibly) smaller than 2. I do not know if the reduction in p is necessary, or an artefact of the proof.

2. Proof of Theorem 1.5

If $\ell \subset \mathbb{R}^2$ is a line, I denote by $T(\ell, \delta)$ the open (infinite) tube of width 2δ, with ℓ "running through the middle", that is, $\mathrm{dist}(\ell, \mathbb{R}^2 \setminus T(\ell, \delta)) = \delta$. The notation $B(x, r)$ stands for a closed ball with centre $x \in \mathbb{R}^2$ and radius $r > 0$. The notation $A \lesssim B$ means that there is an absolute constant $C \geq 1$ such that $A \leq CB$.

Lemma 2.1. *Assume that μ is a Borel probability measure on $B(0, 1) \subset \mathbb{R}^2$, and $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(T(\ell, \delta)) \leq \epsilon$ for all lines $\ell \subset \mathbb{R}^2$.*

Proof. Assume not, so there exists $\epsilon > 0$, a sequence of positive numbers $\delta_1 > \delta_2 > \cdots > 0$ with $\delta_i \searrow 0$ and a sequence of lines $\{\ell_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$ with $\mu(T(\ell_i, \delta_i)) \geq \epsilon$. Since $\operatorname{spt} \mu \subset B(0, 1)$, one has $\ell_i \cap B(0, 1) \neq \varnothing$ for all $i \in \mathbb{N}$. Consequently, there exists a subsequence $(i_j)_{j \in \mathbb{N}}$ and a line $\ell \subset \mathbb{R}^2$ such that $\ell_j \to \ell$ in the Hausdorff metric. Then, for any given $\delta > 0$, there exists $j \in \mathbb{N}$ such that

$$B(0, 1) \cap T(\ell_{i_j}, \delta_{i_j}) \subset T(\ell, \delta),$$

so that $\mu(T(\ell, \delta)) \geq \epsilon$. It follows that $\mu(\ell) \geq \epsilon$, a contradiction. □

The next lemma contains all the information needed to prove Theorem 1.5. I state two versions: the first one is slightly easier to read and apply, while the second one is slightly more detailed.

Lemma 2.2. *Assume that μ, ν are Borel probability measures with compact supports $K, E \subset B(0, 1)$, respectively. Assume that both measures μ and ν satisfy a Frostman condition with exponents $\kappa_\mu, \kappa_\nu \in (0, 2]$, respectively:*

$$\mu(B(x, r)) \leq C_\mu r^{\kappa_\mu} \quad \text{and} \quad \nu(B(x, r)) \leq C_\nu r^{\kappa_\nu} \tag{2.3}$$

for all balls $B(x, r) \subset \mathbb{R}^2$ and for some constants $C_\mu, C_\nu \geq 1$. Assume further that $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Fix also

$$0 < \tau < \tfrac{1}{2}\kappa_\mu \quad \text{and} \quad \epsilon > 0,$$

and write $\delta_k := 2^{-(1+\epsilon)^k}$.

Then, there exists a compact subset $K' \subset K$ with

$$\mu(K') \geq \tfrac{1}{2},$$

a number $\eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0$, an index $k_0 = k_0(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N}$, and a point $x \in E$ with the following property. If $k > k_0$, and $T(\ell_1, \delta_k), \ldots, T(\ell_N, \delta_k)$ is a family of δ_k-tubes of cardinality $N \leq \delta_k^{-\tau}$, each containing x, then

$$\mu\biggl(K' \cap \bigcup_{j=1}^{N} T(\ell_j, \delta_k)\biggr) \leq \delta_k^{\eta}. \tag{2.4}$$

Roughly speaking, the conclusion (2.4) means that K' has a radial projection of dimension $\geq \tau$ relative to the viewpoint $x \in E$, since only a tiny fraction of K' can be covered by $\leq \delta_k^{-\tau}$ tubes of width $2\delta_k$ containing x.

The set $K' \subset K$ and the point $x \in E$ will be found by induction on the scales δ_k. To set the scene for the induction, it is convenient to state a more detailed version of the lemma:

Lemma 2.5. *Assume that μ, ν are Borel probability measures with compact supports $K, E \subset B(0,1)$, respectively. Assume that both measures μ and ν satisfy a Frostman condition with exponents $\kappa_\mu, \kappa_\nu \in (0,2]$, respectively*:

$$\mu(B(x,r)) \leq C_\mu r^{\kappa_\mu} \quad \textit{and} \quad \nu(B(x,r)) \leq C_\nu r^{\kappa_\nu}$$

for all balls $B(x,r) \subset \mathbb{R}^2$ and for some constants $C_\mu, C_\nu \geq 1$. Assume further that $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Fix also

$$0 < \tau < \tfrac{1}{2}\kappa_\mu \quad \textit{and} \quad \epsilon > 0,$$

and write $\delta_k := 2^{-(1+\epsilon)^k}$.

Then, there exist numbers $\beta = \beta(\kappa_\mu, \kappa_\nu, \tau) > 0$, $\eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0$, and an index $k_0 = k_0(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N}$ with the following properties. For all $k \geq k_0$, there exist

(a) *compact sets $K \supset K_{k_0} \supset K_{k_0+1} \cdots$ with*

$$\mu(K_k) \geq 1 - \sum_{k_0 \leq j < k} \left(\tfrac{1}{4}\right)^{j-k_0+1} \geq \tfrac{1}{2}, \tag{2.6}$$

(b) *compact sets $E \supset E_{k_0} \supset E_{k_0+1} \cdots$ with $\nu(E_k) \geq \delta_k^\beta$*

with the following property: *if $k > k_0$, $x \in E_k$, and $T(\ell_1, \delta_k), \ldots, T(\ell_N, \delta_k)$ is a family of tubes of cardinality $N \leq \delta_k^{-\tau}$, each containing x, then*

$$\mu\left(K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta_k)\right) \leq \delta_k^\eta. \tag{2.7}$$

Remark 2.8. The index k_0 can be chosen as large as desired; this will be clear from the proof below. It will also be used on many occasions, without separate remark, that δ_k can be assumed very small for all $k \geq k_0$. I also record that Lemma 2.2 follows from Lemma 2.5: simply take K' to be the intersection of all the sets K_j, $j \geq k_0$, and let $x \in E$ be any point in the intersection of all the sets E_j, $j \geq k_0$.

Proof. As stated above, the proof is by induction, starting at the largest scale k_0, which will be presently defined. Fix $\eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0$ and

$$\Gamma = \Gamma(\epsilon, \kappa_\mu, \kappa_\nu, \tau) \in \mathbb{N}. \tag{2.9}$$

The number Γ will be specified at the very end of the proof, right before (2.34), and there will be several requirements for the number η; see (2.24), (2.30), and (2.33). Applying Lemma 2.1, first pick an index $k_1 = k_1(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N}$ such that $\mu(T(\ell, \delta_{k_1})) \leq \left(\tfrac{1}{4}\right)^{\Gamma+1}$ for all tubes $T(\ell, \delta_{k_1}) \subset \mathbb{R}^2$, and

$$\delta_{k-\Gamma}^\eta \leq \left(\tfrac{1}{4}\right)^{k-\Gamma+1}, \quad k \geq k_1. \tag{2.10}$$

Set $k_0 := k_1 + \Gamma$. Then, the following holds for all $k \in \{k_0, \dots, k_0 + \Gamma\}$. For any subset $K' \subset K$, and any tube $T(\ell, \delta_{k-\Gamma}) \subset \mathbb{R}^2$, one has

$$\mu(K' \cap T(\ell, \delta_{k-\Gamma})) \leq \mu(T(\ell, \delta_{k_1})) \leq \left(\tfrac{1}{4}\right)^{\Gamma+1} \leq \left(\tfrac{1}{4}\right)^{k-k_0+1}. \tag{2.11}$$

Define

$$K_k := K \quad \text{and} \quad E_k := E, \quad k_1 \leq k \leq k_0.$$

(The definitions of E_k, K_k for $k_1 \leq k < k_0$ are only given for notational convenience.)

I start by giving an outline of how the induction will proceed. Assume that, for a certain $k \geq k_0$, the sets K_k and E_k have been constructed such that:

(i) The condition (2.11) is satisfied with $K' = K_k$, and for all tubes $T(\ell, \delta_{k-\Gamma})$ with $T(\ell, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \varnothing$.

(ii) K_k and E_k satisfy the measure lower bounds (a) and (b) from the statement of the lemma.

Under the conditions (i)–(ii), I claim that it is possible to find subsets $K_{k+1} \subset K_k$ and $E_{k+1} \subset E_k$ satisfying (ii) at level $k+1$, and also the nonconcentration condition (2.7) at level $k+1$. This is why (2.7) is only claimed to hold for $k > k_0$, and no one is indeed claiming that it holds for the sets K_{k_0} and E_{k_0}. These sets satisfy (i), however, which should be viewed as a weaker substitute for (2.7) at level k, which is just strong enough to guarantee (2.7) at level $k+1$. There is one obvious question at this point: if (i) at level k gives (2.7) at level $k+1$, then where does one get (i) back at level $k+1$?

If $k+1 \in \{k_0, \dots, k_0+\Gamma\}$, the condition (i) is simply guaranteed by the choice of k_0 (one does not even need to assume that $T(\ell, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \varnothing$). For $k+1 > k_0 + \Gamma$, this is no longer true. However, for $k+1 > \Gamma + k_0$, one has $k+1-\Gamma > k_0$, and thus $K_{k+1-\Gamma}$ and $E_{k+1-\Gamma}$ have already been constructed to satisfy (2.7). In particular, if $E_{k+1-\Gamma} \cap T(\ell, \delta_{k+1-\Gamma}) \neq \varnothing$, then

$$\mu(K_{k+1} \cap T(\ell, \delta_{k+1-\Gamma})) \leq \mu(K_{k+1-\Gamma} \cap T(\ell, \delta_{k+1-\Gamma})) \leq \delta_{k+1-\Gamma}^{\eta} \leq \left(\tfrac{1}{4}\right)^{(k+1)-k_0+1} \tag{2.12}$$

by (2.7) and (2.10). This means that (i) is satisfied at level $k+1$, and the induction may proceed.

So, it remains to prove that (i)–(ii) at level k imply (ii) and (2.7) at level $k+1$. To avoid clutter, I write

$$\delta := \delta_{k+1}.$$

Assume that the sets K_k, E_k have been constructed for some $k \geq k_0$ satisfying (i)–(ii). The main task is to understand the structure of the set of points $x \in E_k$ for which (2.7) fails. To this end, we define the set $\mathrm{Bad}_k \subset E_k$ as follows: $x \in \mathrm{Bad}_k$ if and only if $x \in E_k$, and there exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \dots, T(\ell_N, \delta)$, each containing x, such that

$$\mu\left(K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta)\right) > \delta^{\eta}. \tag{2.13}$$

Note that if $\mathrm{Bad}_k = \varnothing$, then one can simply define $E_{k+1} := E_k$ and $K_{k+1} := K_k$, and (ii) and (2.7) (at level $k+1$) are clearly satisfied.

Instead of analysing Bad_k directly, it is useful to split it up into "directed" pieces, and digest the pieces individually. To make this precise, let S be the "space of directions"; for concreteness, I identify S with

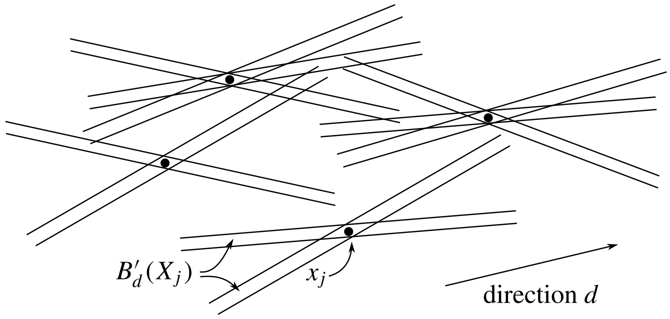

Figure 2. The set Bad_k^d.

the upper half of the unit circle. Then, if $T = T(\ell, \delta) \subset \mathbb{R}^2$ is a tube, I denote by $\mathrm{dir}(T)$ the unique vector $e \in S$ such that $\ell \| e$.

Recall the small parameter $\eta > 0$, and partition S into $D = \delta^{-\eta}$ arcs $J_1, \dots, J_D$ of length $\sim \delta^{\eta}$.[1] For $d \in \{1, \dots, D\}$ fixed ("d" for "direction"), consider the set Bad_k^d: it consists of those points $x \in E_k$ such that there exist $N \le \delta^{-\tau}$ tubes $T(\ell_1, \delta), \dots, T(\ell_N, \delta)$, each containing x, with $\mathrm{dir}(T(\ell_i, \delta)) \in J_d$, and satisfying

$$\mu\left(K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta)\right) > \delta^{2\eta}.$$

Since the direction of every possible tube in $\mathbb{R}^2$ belongs to one of the arcs J_i, and there are only $D = \delta^{-\eta}$ arcs in total, one has

$$\mathrm{Bad}_k \subset \bigcup_{d=1}^{D} \mathrm{Bad}_k^d. \tag{2.14}$$

The next task is to understand the structure of Bad_k^d for a fixed direction $d \in \{1, \dots, D\}$. I claim that Bad_k^d looks like a garden of flowers, with all the petals pointing in direction J_d; see Figure 2 for a rough idea. To make the statement more precise, I introduce an additional piece of notation. Fix $X \subset K_k$, and let $B_d(X)$ consist of those points $x \in E_k$ such that X can be covered by $N \le \delta^{-\tau}$ tubes $T(\ell_1, \delta), \dots, T(\ell_N, \delta)$, with directions $\mathrm{dir}(T(\ell_i, \delta)) \in J_d$, and each containing x. Then, note that

$$\mathrm{Bad}_k^d = \{x \in E_k : \text{there exists } X \subset K_k \text{ with } \mu(X) > \delta^{2\eta} \text{ and } x \in B_d(X)\}. \tag{2.15}$$

The sets $B_d(X)$ also have the trivial but useful property that

$$X \subset X' \subset K_k \quad \Longrightarrow \quad B_d(X') \subset B_d(X).$$

There are two steps in establishing the "garden" structure of Bad_k^d: first, one needs to find the "flowers", and second, one needs to check that the sets obtained actually look like flowers in a nontrivial sense. I

[1] Here, it might be better style to pick another letter, say $\alpha > 0$, in place of η, since the two parameters play slightly different roles in the proof. Eventually, however, one would end up considering $\min\{\eta, \alpha\}$, and it seems a bit cleaner to let $\eta > 0$ be a "jack of all trades" from the start.

start with the former task. Assuming that $\mathrm{Bad}_k^d \neq \varnothing$, pick any point $x_1 \in \mathrm{Bad}_k^d$ and an associated subset $X_1 \subset K_k$ with

$$\mu(X_1) > \delta^{2\eta} \quad \text{and} \quad x_1 \in B_d(X_1).$$

Then, assume that $x_1, \dots, x_m \in \mathrm{Bad}_k^d$ and $X_1, \dots, X_m$ have already been chosen with the properties above, and further satisfying

$$\mu(X_i \cap X_j) \leq \tfrac{1}{2}\delta^{4\eta}, \quad 1 \leq i < j \leq m. \tag{2.16}$$

Then, see if there still exists a subset $X_{m+1} \subset K_k$ with the following three properties: $\mu(X_{m+1}) > \delta^{2\eta}$, $B_d(X_{m+1}) \neq \varnothing$, and $\mu(X_{m+1} \cap X_i) \leq \delta^{4\eta}/2$ for all $1 \leq i \leq m$. If such a set no longer exists, stop; if it does, pick $x_{m+1} \in B_d(X_{m+1})$, and add X_{m+1} to the list.

It follows from the "competing" conditions $\mu(X_i) > \delta^{2\eta}$, and (2.16), that the algorithm needs to terminate in at most

$$M \leq 2\delta^{-4\eta} \tag{2.17}$$

steps. Indeed, assume that the sets $X_1, \dots, X_M$ have already been constructed, and consider the following chain of inequalities:

$$\begin{aligned}
\frac{1}{M} + \frac{1}{M(M-1)} \sum_{i_1 \neq i_2} \mu(X_{i_1} \cap X_{i_2}) &\geq \frac{1}{M^2} \sum_{i_1, i_2 = 1}^{M} \mu(X_{i_1} \cap X_{i_2}) \\
&= \frac{1}{M^2} \int \sum_{i_1, i_2 = 1}^{M} \mathbf{1}_{X_{i_1} \cap X_{i_2}}(x)\, d\mu(x) \\
&= \frac{1}{M^2} \int [\mathrm{card}\{1 \leq i \leq M : x \in X_i\}]^2 \, d\mu(x) \\
&\geq \frac{1}{M^2} \left(\int \mathrm{card}\{1 \leq i \leq M : x \in X_i\}\, d\mu(x) \right)^2 \\
&= \frac{1}{M^2} \left(\sum_{i=1}^{M} \mu(X_i) \right)^2 > \delta^{4\eta}.
\end{aligned}$$

Thus, if $M > 2\delta^{-4\eta}$, there exists a pair X_{i_1}, X_{i_2} with $i_1 \neq i_2$ such that $\mu(X_{i_1} \cap X_{i_2}) > \delta^{4\eta}/2$, and the algorithm has already terminated earlier. This proves (2.17).

With the sets $X_1, \dots, X_M$ now defined, write

$$B_d'(X_j) := \left\{ x \in E_k : \text{there exists } X' \subset X_j \text{ with } \mu(X') > \tfrac{1}{2}\delta^{4\eta} \text{ and } p \in B_d(X') \right\}.$$

I claim that

$$\mathrm{Bad}_k^d \subset \bigcup_{j=1}^{M} B_d'(X_j). \tag{2.18}$$

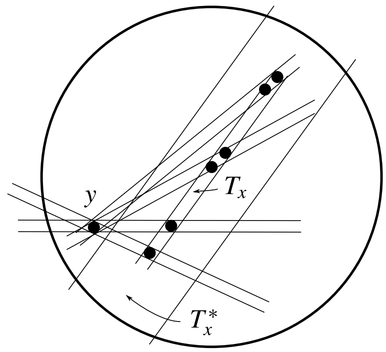

Figure 3. Covering $X_j \cap T_x$ by tubes centred at points outside T_x^*.

Indeed, if $x \in \mathrm{Bad}_k^d$, then $x \in B_d(X)$ for some $X \subset K_k$ with $\mu(X) > \delta^{2\eta}$ by (2.15). It follows that

$$\mu(X \cap X_j) > \tfrac{1}{2}\delta^{4\eta} \tag{2.19}$$

for one of the sets X_j, $1 \le j \le M$, because either $X \in \{X_1, \dots, X_M\}$ and (2.19) is clear (all the sets X_j even satisfy $\mu(X_j) > \delta^{2\eta}$), or else (2.19) must hold by virtue of X *not* having been added to the list $X_1, \dots, X_M$ in the algorithm. But (2.19) implies that $x \in B'_d(X_j)$, since $X' = X \cap X_j \subset X_j$ satisfies $\mu(X') > \delta^{4\eta}/2$ and $x \in B_d(X) \subset B_d(X')$.

According to (2.17) and (2.18) the set Bad_k^d can be covered by $M \le 2\delta^{-4\eta}$ sets of the form $B'_d(X_j)$; see Figure 2. These sets are the "flowers", and their structure is explored in the next lemma:

Lemma 2.20. *The following holds if* $\delta = \delta_{k+1}$ *and* $\eta > 0$ *are small enough* (*the latter depending on* κ_μ, τ *here*). *For* $1 \le d \le D$ *and* $1 \le j \le M$ *fixed, the set* $B'_d(X_j)$ *can be covered by* $\le 4\delta^{-8\eta}$ *tubes of the form* $T = T(\ell, \delta^\rho)$, *where* $\mathrm{dir}(T) \in J_d$ *and* $\rho = \rho(\kappa_\mu, \tau) > 0$. *The tubes can be chosen to contain the point* $x_j \in B_d(X_j)$.

Proof. Fix $1 \le j \le M$ and $x \in B'_d(X_j)$. Recall the point $x_j \in B_d(X_j)$ from the definition of X_j. By definition of $x \in B'_d(X_j)$, there exists a set $X' \subset X_j$ with $\mu(X') > \delta^{4\eta}/2$ and $x \in B_d(X')$. Unwrapping the definitions further, there exist $N \le \delta^{-\tau}$ tubes $T(\ell_1, \delta), \dots, T(\ell_N, \delta)$, the union of which covers X', and each satisfies $\mathrm{dir}(T(\ell_i, \delta)) \in J_d$ and $x \in T(\ell_i, \delta)$. In particular, one of these tubes, say $T_x = T(\ell_i, \delta)$, has

$$\mu(X_j \cap T_x) \ge \mu(X' \cap T_x) \ge \mu(X') \cdot \delta^\tau \ge \tfrac{1}{2}\delta^{4\eta+\tau} \ge \tfrac{1}{4}\delta^{8\eta+\tau}. \tag{2.21}$$

(The final inequality is just a triviality at this point, but is useful for technical purposes later.) Here comes perhaps the most basic geometric observation in the proof: if the measure lower bound (2.21) holds for some δ-tube T — this time T_x — and a sufficiently small $\eta > 0$ (crucially so small that $8\eta + \tau < \kappa_\mu/2$), then the whole set $B_d(X_j)$ is actually contained in a neighbourhood of T, called T^*, because $X_j \cap T$ is so difficult to cover by δ-tubes centred at points outside T^*; see Figure 3. In particular, in the present case,

$$x_j \in B_d(X_j) \subset T(\ell_i, \delta^{4\rho}) =: T_x^* \tag{2.22}$$

for a suitable constant $\rho = \rho(\kappa_\mu, \tau) > 0$, specified in (2.24). To see this formally, pick $y \in B(0,1) \setminus T_x^*$, and argue as follows to show that $y \notin B_d(X_j)$. First, any δ-tube T containing y and intersecting $T_x \cap B(0,1)$ makes an angle $\gtrsim \delta^{4\rho}$ with T_x. It follows that

$$\operatorname{diam}(T \cap T_x \cap B(0,1)) \lesssim \delta^{1-4\rho},$$

and consequently $\mu(T \cap T_x \cap B(0,1)) \lesssim C_\mu \delta^{\kappa_\mu(1-4\rho)}$. So, in order to cover $X_j \cap T_x$ (let alone the whole set X_j) it takes by (2.21)

$$\gtrsim \frac{\mu(X_j \cap T_x)}{C_\mu \delta^{\kappa_\mu(1-4\rho)}} \geq \frac{\delta^{8\eta+\tau-\kappa_\mu(1-4\rho)}}{4C_\mu} \geq \frac{\delta^{8\eta-\kappa_\mu/2+8\rho}}{4C_\mu} \tag{2.23}$$

tubes T containing y. But if

$$0 < 8\eta < \frac{\kappa_\mu/2 - \tau}{2} \quad \text{and} \quad 8\rho = \frac{\kappa_\mu/2 - \tau}{2}, \tag{2.24}$$

then the number on the right-hand side of (2.23) is far larger than $\delta^{-\tau}$, which means that $y \notin B_d(X_j)$, and proves (2.22).

Recall the statement of Lemma 2.20, and compare it with the previous accomplishment: (2.22) states that if $x \in B'_d(X_j)$, then x lies in a certain tube of width $\delta^{4\rho}$ (namely T_x), which has direction in J_d, and also contains x_j. This sounds a bit like the statement of the lemma, but there is a problem: in principle, every point $x \in B'(X_j)$ could give rise to a different tube T_x. So, it essentially remains to show that all these $\delta^{4\rho}$-tubes T_x can be covered by a small number of tubes of width δ^ρ. To begin with, note that the ball $B_j := B(x_j, \delta^{2\rho})$ can be covered by a single tube of width δ^ρ, in any direction desired. So, to prove the lemma, it remains to cover $B'_d(X_j) \setminus B_j$.

Note that if x, y satisfy $|x - y| \geq \delta^{2\rho}$, then the direction of any $\delta^{4\rho}$-tube containing both x, y lies in a fixed arc $J(x, y) \subset S$ of length $|J(x, y)| \lesssim \delta^{4\rho}/\delta^{2\rho} = \delta^{2\rho}$. As a corollary, the union of all $\delta^{4\rho}$-tubes containing x, y, intersected with $B(0,1)$, is contained in a single tube of width $\sim \delta^{2\rho}$. In particular, this union (still intersected with $B(0,1)$) is contained in a single δ^ρ-tube, assuming that $\delta > 0$ is small; this tube can be chosen to be a δ^ρ-tube around an arbitrary $\delta^{4\rho}$-tube containing both x and y.

The tube-cover of $B'_d(X_j) \setminus B_j$ can now be constructed by adding one tube at a time. First, assume that there is a point $y_1 \in B'_d(X_j) \setminus B_j$ left to be covered, and find a tube $T(\ell_1, \delta^{4\rho})$ containing both y_1 and x_j, with direction in J_d; existence follows from (2.22). Add the tube $T(\ell_1, \delta^\rho)$ to the tube-cover of $B'_d(X_j) \setminus B_j$, and recall from the previous paragraph that $T(\ell_1, \delta^\rho)$ now contains $T \cap B(0,1)$ for *any* $\delta^{4\rho}$-tube $T \supset \{y_1, x_j\}$ (of which $T = T(\ell_1, \delta^{4\rho})$ is just one example). Finally, by the definition of $y_1 \in B'_d(X_j)$, associate to y_1 a subset $X'_1 \subset X_j$ with

$$\mu(X'_1) > \tfrac{1}{2}\delta^{4\eta} \quad \text{and} \quad y_1 \in B_d(X'_1). \tag{2.25}$$

Assume that the points $y_1, \ldots, y_H \in B'_d(X_j) \setminus B_j$, along with the associated tubes $\{y_i, x_j\} \subset T(\ell_i, \delta^{4\rho}) \subset T(\ell_i, \delta^\rho)$, and subsets $X'_i \subset X_j$, as in (2.25), have already been constructed. Assume inductively that

$$\mu(X'_{i_1} \cap X'_{i_2}) \leq \tfrac{1}{4}\delta^{8\eta}, \quad 1 \leq i_1 < i_2 \leq H. \tag{2.26}$$

To proceed, pick any point $y_{H+1} \in B'_d(X_j) \setminus B_j$, and associate to y_{H+1} a subset $X'_{H+1} \subset X_j$ with $\mu(X'_{H+1}) > \delta^{4\rho}/2$ and $y_{H+1} \in B_d(X'_{H+1})$. Then, test whether (2.26) still holds, that is, whether $\mu(X'_{H+1} \cap X'_i) \leq \delta_{k+1}^{8\eta}/4$ for all $1 \leq i \leq H$. If such a point y_{H+1} can be chosen, run the argument from the previous paragraph, first locating a tube $T(\ell_{H+1}, \delta^{4\rho})$ containing both y_{H+1} and p_j, with direction in J_d, and finally adding $T(\ell_{H+1}, \delta^{\rho})$ to the tube-cover under construction.

The "competing" conditions $\mu(X'_i) > \delta^{4\eta}/2$ and (2.26) guarantee that the algorithm terminates in

$$H \leq 4\delta^{-8\eta}$$

steps. The argument is precisely the same as that used to prove (2.17), so I omit it. Once the algorithm has terminated, I claim that all points of $B'_d(X_j) \setminus B_j$ are covered by the tubes $T(\ell_i, \delta^{\rho})$, with $1 \leq i \leq H$. To see this, pick $y \in B'_d(X_j) \setminus B_j$, and a subset $X' \subset X_j$ with $\mu(X') > \delta^{4\eta}/2$, and $y \in B_d(X')$. Since the algorithm has already terminated, it must be the case that

$$\mu(X' \cap X'_i) > \tfrac{1}{4}\delta^{8\eta}$$

for some index $1 \leq i \leq H$. Since $X'' := X' \cap X'_i \subset X'$ and consequently $y \in B_d(X'')$, one can find a tube $T_y = T(\ell_y, \delta) \ni y$, with $\operatorname{dir}(T_y) \in J_d$, satisfying

$$\mu(X'_i \cap T_y) \geq \mu(X'' \cap T_y) \geq \mu(X'') \cdot \delta^{\tau} > \tfrac{1}{4}\delta^{8\eta+\tau}.$$

This lower bound is precisely the same as in (2.21). Hence, it follows from the same argument which gave (2.22) that

$$y_i \in B_d(X'_i) \subset T(\ell_y, \delta^{4\rho}).$$

Since $X'_i \subset X_j$, we also have $x_j \in B_d(X_j) \subset B_d(X'_i) \subset T(\ell_q, \delta^{4\rho})$. So,

$$\{y, y_i, x_j\} \subset B(0,1) \cap T(\ell_y, \delta^{4\rho}). \tag{2.27}$$

In particular, $T(\ell_y, \delta^{4\rho})$ is a $\delta^{4\rho}$-tube containing both y_i, x_j, and hence

$$B(0,1) \cap T(\ell_y, \delta^{4\rho}) \subset T(\ell_i, \delta^{\rho}).$$

Combined with (2.27), this yields $y \in T(\ell_i, \delta^{\rho})$, as claimed. This concludes the proof of Lemma 2.20. □

Combining (2.17)–(2.18) with Lemma 2.20, the structural description of Bad_k^d is now complete: Bad_d^k is covered by

$$\leq M \cdot 4\delta^{-8\eta} \leq 8\delta^{-12\eta} \tag{2.28}$$

tubes of width δ^{ρ}, with directions in J_d. For nonadjacent $d_1, d_2 \in \{1, \ldots, D\}$ (the ordering of indices corresponds to the ordering of the arcs $J_d \subset S$), the covering tubes are then fairly transversal. This is can be used to infer that most points in E_k do not lie in many different sets Bad_k^d. Indeed, consider the set BadBad_k of those points in $\mathbb{R}^2$ which lie in (at least) two sets $\mathrm{Bad}_k^{d_1}$ and $\mathrm{Bad}_k^{d_2}$ with $|d_2 - d_1| > 1$. By Lemma 2.20, such points lie in the intersection of some pair of tubes $T_1 = T(\ell_1, \delta^{\rho})$ and $T_2 = T(\ell_2, \delta^{\rho})$ with $\operatorname{dir}(T_i) \in J_{d_i}$. The angle between these tubes is $\gtrsim \delta^{\eta}$, whence

$$\operatorname{diam}(T_1 \cap T_2) \lesssim \delta^{\rho-\eta},$$

and consequently

$$\nu(T_1 \cap T_2) \lesssim C_\nu \delta^{\kappa_\nu(\rho-\eta)} \le C_\nu \delta^{\kappa_\nu \rho - 2\eta}. \tag{2.29}$$

For $d \in \{1, \ldots, D\}$ fixed, there correspond $\lesssim \delta^{-12\eta}$ tubes in total, as pointed out in (2.28). So, the number of pairs T_1, T_2, as above, is bounded by

$$\lesssim D^2 \cdot \delta^{-24\eta} \le \delta^{-26\eta}.$$

Consequently, by (2.29),

$$\nu(\mathrm{BadBad}_k) \lesssim C_\nu \delta^{-28\eta + \kappa_\nu \rho}.$$

This upper bound is far smaller than $\delta_k^\beta/2 \le \nu(E_k)/2$, taking $0 < \max\{\beta, 28\eta\} < \kappa_\nu\rho/2$, so that

$$0 < \beta < \kappa_\nu \rho - 28\eta. \tag{2.30}$$

For such choices of β, η, the next task is then to choose $E_{k+1} \subset E_k$ such that $\nu(E_{k+1}) \ge \delta_{k+1}^\beta$. Start by writing $G_k := E_k \setminus \mathrm{BadBad}_k$, so that

$$\nu(G_k) \ge \tfrac{1}{2}\nu(E_k) \ge \tfrac{1}{2}\delta_k^\beta$$

by the choice of β. Now, either

$$\nu(G_k \cap \mathrm{Bad}_k) \ge \tfrac{1}{2}\nu(G_k) \quad \text{or} \quad \nu(G_k \cap \mathrm{Bad}_k) < \tfrac{1}{2}\nu(G_k). \tag{2.31}$$

The latter case is quick and easy: set $E_{k+1} := G_k \setminus \mathrm{Bad}_k$ and $K_{k+1} := K_k$. Then $\nu(E_{k+1}) \ge \nu(E_k)/4 \ge \delta_{k+1}^\beta$ (assuming that $k \ge k_0$ is large enough). Moreover, the set E_{k+1} no longer contains any points in Bad_k, so (2.7) is satisfied at level $k+1$ by the very definition of Bad_k; see (2.13).

So, it remains to treat the first case in (2.31). Start by recalling from (2.14) that Bad_k is covered by the sets Bad_k^d, $1 \le d \le D$, so

$$\nu(G_k \cap \mathrm{Bad}_k^d) \ge \frac{\nu(G_k)}{2D} \ge \tfrac{1}{4}\delta^\eta \delta_k^\beta = \tfrac{1}{4}\delta^{\eta + \beta/(1+\epsilon)}$$

for some fixed $d \in \{1, \ldots, D\}$. Then, recall from (2.28) that Bad_k^d can be covered by $\le 8\delta^{-12\eta}$ tubes of the form $T(\ell, \delta^\rho)$ with directions in J_d. It follows that there exists a fixed tube $T_0 = T(\ell_0, \delta^\rho)$ such that

$$\mathrm{dir}(T_0) \in J_d \quad \text{and} \quad \nu(G_k \cap T_0 \cap \mathrm{Bad}_k^d) \ge \tfrac{1}{32}\delta^{13\eta + \beta/(1+\epsilon)}. \tag{2.32}$$

So, to ensure $\nu(G_k \cap T_0 \cap \mathrm{Bad}_k^d) \ge \delta^\beta$, choose $\eta > 0$ so small that

$$13\eta + \frac{\beta}{1+\epsilon} < \beta. \tag{2.33}$$

To convince the reader that there is no circular reasoning at play, I gather here all the requirements for β and η (harvested from (2.24), (2.30), and (2.33)):

$$0 < \beta < \frac{\kappa_\nu \rho}{2} \quad \text{and} \quad 0 < \eta < \min\left\{\frac{\kappa_\mu/2 - \tau}{2}, \frac{\kappa_\nu \rho}{56}, \frac{\epsilon\beta}{13(1+\epsilon)}\right\}.$$

With such choices of β, η, recalling (2.32), and assuming that δ is small enough, the set

$$E_{k+1} := G_k \cap T_0 \cap \mathrm{Bad}_k^d$$

satisfies $\nu(E_{k+1}) \geq \delta^{\beta}$, which is statement (b) from the lemma. It remains to define K_{k+1}. To this end, recall that T_0 is a tube around the line $\ell_0 \subset \mathbb{R}^2$. Define

$$K_{k+1} := K_k \setminus T(\ell_0, \delta^{\eta/2}).$$

Then, assuming that $\eta/2$ has the form $\eta/2 = (1+\epsilon)^{-\Gamma-1}$ for an integer $\Gamma = \Gamma(\epsilon, \kappa_\mu, \kappa_\nu, \tau) \in \mathbb{N}$ (this is finally the integer from (2.9)), one has

$$\delta^{\eta/2} = \delta_{k-\Gamma}. \tag{2.34}$$

Since $T(\ell_0, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \varnothing$, it follows from the induction hypothesis (i) that

$$\mu(K_k \cap T(\ell_0, \delta_{k-\Gamma})) \leq \left(\tfrac{1}{4}\right)^{k-k_0+1}.$$

Consequently,

$$\mu(K_{k+1}) \geq \mu(K_k) - \left(\tfrac{1}{4}\right)^{k-k_0+1} \geq 1 - \sum_{k_0 \leq j < k+1} \left(\tfrac{1}{4}\right)^{j-k_0+1},$$

which is the desired lower bound from (a) of the statement of the lemma. So, it remains to verify the nonconcentration condition (2.7) for E_{k+1} and K_{k+1}. To this end, pick $x \in E_{k+1}$. First, observe that every tube $T = T(\ell, \delta)$ which contains x and has nonempty intersection with $K_{k+1} \subset B(0,1) \setminus T(\ell, \delta^{\eta/2})$ forms an angle $\gtrsim \delta^{\eta/2}$ with T_0. In particular, this angle is far larger than δ^{η}. Since $\mathrm{dir}(T_0) \in J_d$ by (2.32), this implies that $\mathrm{dir}(T) \in J_{d'}$ for some $|d' - d| > 1$.

Now, if the nonconcentration condition (2.7) still fails for $x \in E_{k+1}$, there would exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, each containing x, and with

$$\mu\left(K_{k+1} \cap \bigcup_{i=1}^{N} T(\ell_i, \delta)\right) > \delta^{\eta}.$$

By the pigeonhole principle, it follows that the tubes $T(\ell_i, \delta)$ with $\mathrm{dir}(T_i) \in J_{d'}$ for some fixed arc $J_{d'}$ cover a set $X \subset K_{k+1} \subset K_k$ of measure $\mu(X) > \delta^{2\eta}$. This means precisely that $x \in \mathrm{Bad}_k^{d'}$, and by the observation in the previous paragraph, $|d - d'| > 1$. But $x \in E_{k+1} \subset \mathrm{Bad}_k^d$ by definition, so this would imply that $x \in \mathrm{BadBad}_k$, contradicting the fact that $x \in E_{k+1} \subset G_k$. This completes the proof of (2.7), and the lemma. □

The proof of Theorem 1.5 is now quite standard:

Proof of Theorem 1.5. Write $s := \dim_{\mathrm{H}} K$, and assume that $s > 0$ and $\dim_{\mathrm{H}} E > 0$. Make a counter-assumption: E is not contained on a line, but $\dim_{\mathrm{H}} \pi_x(K) < s/2$ for all $x \in E$. Then, find $t < s/2$, and a positive-dimensional subset $\widetilde{E} \subset E$ not contained on any single line, with $\dim_{\mathrm{H}} \pi_x(K) \leq t$ for all $x \in \widetilde{E}$ (if your first attempt at $\widetilde{E}$ lies on some line ℓ, simply add a point $x_0 \in E \setminus \ell$ to $\widetilde{E}$, and replace t by

$\max\{t, \dim_{\mathrm{H}} \pi_{x_0}(K)\} < s/2$). So, now $\widetilde{E}$ satisfies the same hypotheses as E, but with "$< s/2$" replaced by "$\le t < s/2$". Thus, without loss of generality, one may assume that

$$\dim_{\mathrm{H}} \pi_x(K) \le t < \tfrac{1}{2}s, \quad x \in E. \tag{2.35}$$

Using Frostman's lemma, pick probability measures μ, ν, with $\operatorname{spt}\mu \subset K$ and $\operatorname{spt}\nu \subset E$, satisfying the growth bounds (2.3) with exponents $0 < \kappa_\mu < s$ and $\kappa_\nu > 0$. Pick, moreover, κ_μ so close to s that

$$\tfrac{1}{2}\kappa_\mu > t. \tag{2.36}$$

Observe that $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Indeed, if $\mu(\ell) > 0$ for some line $\ell \subset \mathbb{R}^2$, then there exists $x \in E \setminus \ell$ by assumption, and

$$\dim_{\mathrm{H}} \pi_x(K) \ge \dim_{\mathrm{H}} \pi_x(\operatorname{spt}\mu \cap \ell) \ge \kappa_\mu > t,$$

violating (2.35) at once. Finally, by restricting the measures μ and ν slightly, one may assume that they have disjoint supports.

In preparation for using Lemma 2.2, fix $\epsilon > 0$, $0 < \tau < \kappa_\mu/2$ in such a way that

$$\frac{\tau}{(1+\epsilon)^2} > t. \tag{2.37}$$

This is possible by (2.36). Then, apply Lemma 2.2 to find the set $K' \subset \operatorname{spt}\mu \subset K$ with

$$\mu(K') \ge \tfrac{1}{2},$$

the parameters $\eta > 0$ and $k_0 \in \mathbb{N}$, and the point $x \in E$ satisfying (2.4). I claim that

$$\dim_{\mathrm{H}} \pi_x(K') \ge \frac{\tau}{(1+\epsilon)^2}, \tag{2.38}$$

which violates (2.35) by (2.37). If not, cover $\pi_x(K)$ efficiently by arcs $J_1, J_2, \ldots$ of lengths restricted to the values $\delta_k = 2^{-(1+\epsilon)^k}$, with $k \ge k_0$. More precisely: assuming that (2.38) fails, start with an arbitrary efficient cover $\widetilde{J}_1, \widetilde{J}_2, \ldots$ by arcs of length $|\widetilde{J}_i| \le \delta_{k_0}$, satisfying

$$\sum_{j \ge 1} |\widetilde{J}_j|^{\tau/(1+\epsilon)^2} \le 1.$$

Then, replace each $\widetilde{J}_j$ by the shortest concentric arc $J_j \supset \widetilde{J}_j$, whose length is of the form δ_k. Note that $\ell(J_j) \le \ell(\widetilde{J}_j)^{1/(1+\epsilon)}$, so that

$$\sum_{j \ge 1} |J_j|^{\tau/(1+\epsilon)} \le \sum_{j \ge 1} |\widetilde{J}_j|^{\tau/(1+\epsilon)^2} \le 1.$$

The arcs $J_1, J_2, \ldots$ now cover $\pi_x(K')$, and there are $\le \delta_k^{-\tau/(1+\epsilon)}$ arcs of any fixed length δ_k. Since $x \notin K'$, for every $k \ge k_0$ there exists a collection of tubes $\mathcal{T}_k$ of the form $T(\ell, \delta_k) \ni x$, such that $|\mathcal{T}_k| \lesssim \delta_k^{-\tau/(1+\epsilon)}$ (the implicit constant depends on $\operatorname{dist}(x, K')$), and

$$K' \subset \bigcup_{k \ge k_0} \bigcup_{T \in \mathcal{T}_k} T.$$

In particular $|\mathcal{T}_k| \le \delta_k^{-\tau}$, assuming that δ_k is small enough for all $k \ge k_0$. Recall that $\mu(K') \ge \frac{1}{2}$. Hence, by the pigeonhole principle, one can find $k \in \mathbb{N}$ such that the following holds: there is a subset $K'_k \subset K'$ with $\mu(K'_k) \ge 1/(100k^2)$ such that K'_k is covered by the tubes in $\mathcal{T}_k$. But $1/(100k^2)$ is far larger than δ_k^{η}, so this is explicitly ruled out by nonconcentration estimate (2.4). This contradiction completes the proof. □

3. Proof of Theorem 1.11

This section contains the proof of Theorem 1.13, which evidently implies Theorem 1.11. Fix $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \setminus \operatorname{spt} \mu$. For a suitable constant $c_d > 0$ to be determined shortly, consider the weighted measure

$$\mu_x := c_d k_x \, d\mu,$$

where $k_x := |x-y|^{1-d}$ is the $(d-1)$-dimensional Riesz kernel, translated by x. A main ingredient in the proof of Theorem 1.13 is the following identity:

Lemma 3.1. *Let $\mu \in C_0(\mathbb{R}^d)$ (that is, μ is a continuous function with compact support) and $\nu \in \mathcal{M}(\mathbb{R}^d)$. Assume that* $\operatorname{spt} \mu \cap \operatorname{spt} \nu = \varnothing$. *Then, for $p \in (0, \infty)$,*

$$\int \|\pi_{x\sharp}\mu_x\|^p_{L^p(S^{d-1})} \, d\nu(x) = \int_{S^{d-1}} \|\pi_{e\sharp}\mu\|^p_{L^p(\pi_{e\sharp}\nu)} \, d\mathcal{H}^{d-1}(e).$$

Here, and for the rest of the paper, π_e stands for the orthogonal projection onto $e^\perp \in G(d, d-1)$.

Proof. Start by assuming that also $\nu \in C_0(\mathbb{R}^d)$. Fix $x \in \mathbb{R}^d$. The first aim is to find an explicit expression for the density $\pi_x\mu_x$ on S^{d-1}, so fix $f \in C(S^{d-1})$ and compute as follows, using the definition of the measure μ_x, integration in polar coordinates, and choosing the constant $c_d > 0$ appropriately:

$$\begin{aligned}\int f(e) \, d[\pi_{x\sharp}\mu_x](e) &= \int f(\pi_x(y)) \, d\mu_x(y) = c_d \int \frac{f(\pi_x(y))}{|x-y|^{d-1}} \, d\mu(y) \\ &= \int_{S^{d-1}} f(e) \int_{\mathbb{R}} \mu(x+re) \, dr \, d\mathcal{H}^{d-1}(e) \\ &= \int_{S^{d-1}} f(e) \cdot \pi_{e\sharp}\mu(\pi_e(x)) \, d\mathcal{H}^{d-1}(e).\end{aligned}$$

Since the equation above holds for all $f \in C(S^{d-1})$, one infers that

$$\pi_{x\sharp}\mu_x = [e \mapsto \pi_{e\sharp}\mu(\pi_e(x))] \, d\mathcal{H}^{d-1}|_{S^{d-1}}. \tag{3.2}$$

Now, one may prove the lemma by a straightforward computation, starting with

$$\begin{aligned}\int \|\pi_{x\sharp}\mu_x\|^p_{L^p(S^{d-1})} \, d\nu(x) &= \iint_{S^{d-1}} [\pi_{x\sharp}\mu_x(e)]^p \, d\mathcal{H}^{d-1}(e) \, d\nu(x) \\ &= \int_{S^{d-1}} \int_{e^\perp} \int_{\pi_e^{-1}\{w\}} [\pi_{e\sharp}\mu(\pi_e(x))]^p \nu(x) \, d\mathcal{H}^1(x) \, d\mathcal{H}^{d-1}(w) \, d\mathcal{H}^{d-1}(e).\end{aligned}$$

Note that if $x \in \pi_e^{-1}\{w\}$, then $\pi_e(x) = w$, so the expression $[\cdots]^p$ above is independent of x. Hence,

$$\begin{aligned}\int \|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) &= \int_{S^{d-1}} \int_{e^\perp} [\pi_{e\sharp}\mu(w)]^p \left(\int_{\pi_e^{-1}\{w\}} \nu(x)\, d\mathcal{H}^1(x)\right) d\mathcal{H}^{d-1}(w)\, d\mathcal{H}^1(e) \\ &= \int_{S^{d-1}} \int_{e^\perp} [\pi_{e\sharp}\mu(w)]^p \pi_{e\sharp}\nu(w)\, d\mathcal{H}^{d-1}(w)\, d\mathcal{H}^{d-1}(e) \\ &= \int_{S^{d-1}} \|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu)}^p \, d\mathcal{H}^{d-1}(e),\end{aligned}$$

as claimed.

Finally, if $\nu \in \mathcal{M}(\mathbb{R}^d)$ is arbitrary, not necessarily smooth, note that

$$x \mapsto \|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p$$

is continuous, assuming that $\mu \in C_0(\mathbb{R}^d)$, as we do (to check the details, it is helpful to infer from (3.2) that $\pi_x\mu_x \in L^\infty(S^{d-1})$ uniformly in x, since the projections $\pi_{e\sharp}\mu$ clearly have bounded density, uniformly in $e \in S^{d-1}$). Thus, if $(\psi_n)_{n\in\mathbb{N}}$ is a standard approximate identity on $\mathbb{R}^d$, one has

$$\int \|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) = \lim_{n\to\infty} \int_{S^{d-1}} \|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu_n)}^p \, d\mathcal{H}^{d-1}(e), \tag{3.3}$$

with $\nu_n = \nu * \psi_n$. Since $\pi_{e\sharp}\nu_n$ converges weakly to $\pi_{e\sharp}\nu$ for any fixed $e \in S^{d-1}$, and $\pi_{e\sharp}\mu \in C_0(e^\perp)$, it is easy to see that the right-hand side of (3.3) equals

$$\int_{S^{d-1}} \|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu)}^p \, d\mathcal{H}^{d-1}(e). \qquad \square$$

Here is one more (classical) tool required in the proof of Theorem 1.13:

Lemma 3.4. *Let $0 < \sigma < d/2$, and let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a measure with* $\operatorname{spt}\mu \subset B(0,1)$ *and $I_{d-2\sigma}(\mu) < \infty$. Then*

$$\|f\|_{L^1(\mu)} \lesssim_{d,\sigma} \sqrt{I_{d-2\sigma}(\mu)}\|f\|_{H^\sigma(\mathbb{R}^d)}$$

for all continuous functions $f \in H^\sigma(\mathbb{R}^d)$, where

$$\|f\|_{H^\sigma(\mathbb{R}^d)} := \left(\int |\hat{f}(\xi)|^2 |\xi|^{2\sigma}\, d\xi\right)^{1/2}.$$

Proof. See Theorem 17.3 in [Mattila 2015]. Since f is assumed continuous here, $|f|$ is pointwise bounded by the maximal function $\widetilde{M}f$ appearing in [Mattila 2015, Theorem 17.3]. $\square$

Proof of Theorem 1.13. Fix $2(d-1) - s < t < d-1$. It suffices to prove that if $\nu \in \mathcal{M}(\mathbb{R}^d)$ is a fixed measure with $I_t(\nu) < \infty$, and $\operatorname{spt}\mu \cap \operatorname{spt}\nu = \varnothing$, then

$$\pi_{x\sharp}\mu_x \in L^p(S^{d-1}) \quad \text{for } \nu \text{ a.e. } x \in \mathbb{R}^d,$$

whenever

$$1 < p \le \min\left\{2 - \frac{t}{(d-1)}, \frac{t}{2(d-1)-s}\right\}. \tag{3.5}$$

I will treat the numbers d, p, s, t as "fixed" from now on, and in particular the implicit constants in the $\lesssim$ notation may depend on d, p, s, t. Note that the right-hand side of (3.5) lies in $(1, 2)$, so this is a nontrivial range of p's. Fix p as in (3.5). The plan is to show that

$$\int \|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) \lesssim I_t(\nu)^{1/2p} I_s(\mu)^{1/2} < \infty. \tag{3.6}$$

This will be done via Lemma 3.1, but one first needs to reduce to the case $\mu \in C_0(\mathbb{R}^d)$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a standard approximate identity on $\mathbb{R}^d$, and write $\mu_n = \mu * \psi_n$. Then $\pi_{x\sharp}(\mu_n)_x$ converges weakly to $\pi_{x\sharp}\mu_x$ for any fixed $x \in \operatorname{spt}\nu \subset \mathbb{R}^d \setminus \operatorname{spt}\mu$:

$$\int f(e)\, d[\pi_{x\sharp}\mu_x(e)] = \lim_{n\to\infty} \int f(e)\, d\pi_{x\sharp}(\mu_n)_x(e), \quad f \in C(S^{d-1}).$$

It follows that

$$\|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p \le \liminf_{n\to\infty} \|\pi_{x\sharp}(\mu_n)_x\|_{L^p(S^{d-1})}^p, \quad x \in \operatorname{spt}\nu,$$

and consequently

$$\int \|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) \le \liminf_{n\to\infty} \int \|\pi_{x\sharp}(\mu_n)_x\|_{L^p(S^{d-1})}^p \, d\nu(x)$$

by Fatou's lemma. Now, it remains to find a uniform upper bound for the terms on the right-hand side; the only information about μ_n, which we will use, is that $I_s(\mu_n) \lesssim I_s(\mu)$. With this in mind, I simplify notation by defining $\mu_n := \mu$. For the remainder of the proof, one should keep in mind that $\pi_{e\sharp}\mu \in C_0^\infty(e^\perp)$ for $e \in S^{d-1}$, so the integral of $\pi_{e\sharp}\mu$ with respect to various Radon measures on $e^\perp$ is well-defined, and the Fourier transform of $\pi_{e\sharp}\mu$ on $e^\perp$ (identified with $\mathbb{R}^{d-1}$) is a rapidly decreasing function.

We start by appealing to Lemma 3.1:

$$\int \|\pi_{x\sharp}\mu_x\|_{L^p(S^{d-1})}^p \, d\nu(x) = \int_{S^{d-1}} \|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu)}^p \, d\mathcal{H}^{d-1}(e). \tag{3.7}$$

The next task is to estimate the $L^p(\pi_{e\sharp}\nu)$-norms of $\pi_{e\sharp}\mu$ individually, for $e \in S^{d-1}$ fixed. I start by recording the standard fact, see for example the proof of Theorem 9.3 in [Mattila 1995], that $I_t(\pi_{e\sharp}\nu) < \infty$ for $\mathcal{H}^{d-1}$-almost every $e \in S^{d-1}$; I will only consider those $e \in S^{d-1}$ satisfying this condition. Recall that $1 < p \le t/[2(d-1) - s]$. Fix $f \in L^q(\pi_{e\sharp}\nu)$, with $q = p'$ and $\|f\|_{L^q(\pi_{e\sharp}\nu)} = 1$, and note that

$$I_{2(d-1)-s}(f\, d\pi_{e\sharp}\nu) = \iint \frac{f(x) f(y)\, d\pi_{e\sharp}\nu(x)\, d\pi_{e\sharp}\nu(y)}{|x-y|^{2(d-1)-s}} \lesssim I_t(\pi_{e\sharp}\nu)^{1/p}$$

by Hölder's inequality. It now follows from Lemma 3.4 (applied in $e^\perp \cong \mathbb{R}^{d-1}$ with $\sigma = [s-(d-1)]/2$) that

$$\begin{aligned}\int \pi_{e\sharp}\mu \cdot f\, d\pi_{e\sharp}\nu &\lesssim \sqrt{I_{2(d-1)-s}(f\, d\pi_{e\sharp}\nu)}\, \|\pi_{e\sharp}\mu\|_{H^{[s-(d-1)]/2}} \\ &\lesssim (I_t(\pi_{e\sharp}\nu))^{1/2p} \left(\int_{e^\perp} |\widehat{\pi_{e\sharp}\mu}(\xi)|^2 |\xi|^{s-(d-1)}\, d\xi \right)^{1/2}.\end{aligned}$$

Since the function $f \in L^q(\pi_{e\sharp}\nu)$ with $\|f\|_{L^q(\pi_{e\sharp}\nu)} = 1$ was arbitrary, one may infer by duality that

$$\|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu)} \lesssim (I_t(\pi_{e\sharp}\nu))^{1/2p}\left(\int_{e^\perp} |\widehat{\pi_{e\sharp}\mu}(\xi)|^2 |\xi|^{s-(d-1)}\, d\xi\right)^{1/2}.$$

Now it is time to estimate (3.7). This uses duality once more, so fix $f \in L^q(S^{d-1})$ with $\|f\|_{L^q(S^{d-1})} = 1$. Then, write

$$\begin{aligned}\int_{S^{d-1}} &\|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu)} \cdot f(e)\, d\mathcal{H}^{d-1}(e) \\ &\lesssim \int_{S^{d-1}} (I_t(\pi_{e\sharp}\nu))^{1/2p}\left(\int_{e^\perp} |\widehat{\pi_{e\sharp}\mu}(\xi)|^2 |\xi|^{s-(d-1)}\, d\xi\right)^{1/2} \cdot f(e)\, d\mathcal{H}^{d-1}(e) \\ &\lesssim \left(\int_{S^{d-1}} I_t(\pi_{e\sharp}\nu)^{1/p} \cdot f(e)^2\, d\mathcal{H}^{d-1}(e)\right)^{1/2}\left(\int_{S^{d-1}}\int_{e^\perp} |\widehat{\pi_{e\sharp}\mu}(\xi)|^2 |\xi|^{s-(d-1)}\, d\xi\, d\mathcal{H}^{d-1}(e)\right)^{1/2}.\end{aligned}$$

The second factor is bounded by $\lesssim I_s(\mu)^{1/2} < \infty$, using (generalised) integration in polar coordinates; see for instance (2.6) in [Mattila and Orponen 2016]. To tackle the first factor, say "I", write $f^2 = f \cdot f$ and use Hölder's inequality again:

$$I \lesssim \left(\int_{S^{d-1}} I_t(\pi_{e\sharp}\nu) \cdot f(e)^p\, d\mathcal{H}^{d-1}(e)\right)^{1/2p} \cdot \|f\|_{L^q(S^{d-1})}^{1/2}.$$

The second factor equals 1. To see that the first factor is also bounded, note that if $B(e, r) \subset S^{d-1}$ is a ball, then

$$\int_{B(e,r)} f^p\, d\mathcal{H}^{d-1} \leq (\mathcal{H}^{d-1}(B(e,r)))^{2-p} \cdot \left(\int_{S^{d-1}} f^q\, d\mathcal{H}^{d-1}\right)^{p-1} \lesssim r^{(d-1)(2-p)}.$$

Thus, $\sigma = f^p\, d\mathcal{H}^{d-1}$ is a Frostman measure on S^{d-1} with exponent $(d-1)(2-p)$. Now, it is well known (and first observed by Kaufman [1968]) that

$$\int_{S^{d-1}} I_t(\pi_{e\sharp}\nu)\, d\sigma(e) = \iiint_{S^{d-1}} \frac{d\sigma(e)}{|\pi_e(x) - \pi_e(y)|^t}\, d\nu(x)\, d\nu(y) \lesssim I_t(\nu),$$

as long as $t < (d-1)(2-p)$, which is implied by (3.5). Hence $I \lesssim I_t(\nu)^{1/2p}$, and finally

$$\int_{S^{d-1}} \|\pi_{e\sharp}\mu\|_{L^p(\pi_{e\sharp}\nu)} \cdot f(e)\, d\mathcal{H}^{d-1}(e) \lesssim I_t(\nu)^{1/2p} I_s(\mu)^{1/2}$$

for all $f \in L^q(S^{d-1})$ with $\|f\|_{L^q(S^{d-1})} = 1$. By duality, it follows that

$$(3.7) \lesssim I_t(\nu)^{1/2p} I_s(\mu)^{1/2} < \infty.$$

This proves (3.6), using (3.7). The proof of Theorem 1.13 is complete. □

Acknowledgements

I started working on the questions while taking part in the research programme *Fractal Geometry and Dynamics* at Institut Mittag-Leffler. I am grateful to the organisers for letting me participate, and to the staff of the institute for making my stay very pleasant. I would also like to thank Tamás Keleti and Pablo Shmerkin for stimulating conversations, both on this project and several related topics. I thank Pablo for explicitly asking whether Theorem 1.11 is true. The visit at the institute was enabled financially by travel grants from the *Väisälä fund* and the *Mathematics fund* of the *Finnish Academy of Science and Letters*.

I am grateful to the referee for a careful reading of the manuscript, and for many helpful suggestions.

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Received 23 Nov 2017. Revised 31 Jul 2018. Accepted 16 Sep 2018.

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ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1295

CARTAN SUBALGEBRAS OF TENSOR PRODUCTS OF FREE QUANTUM GROUP FACTORS WITH ARBITRARY FACTORS

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Let $\mathbb{G}$ be a free (unitary or orthogonal) quantum group. We prove that for any nonamenable subfactor $N \subset L^\infty(\mathbb{G})$ which is an image of a faithful normal conditional expectation, and for any σ-finite factor B, the tensor product $N \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras. This generalizes our previous work that provides the same result when B is finite. In the proof, we establish Ozawa–Popa and Popa–Vaes's weakly compact action on the continuous core of $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} B$ as the one *relative to* B, by using an operator-valued weight to B and the central weak amenability of $\widehat{\mathbb{G}}$.

1. Introduction

Let M be a von Neumann algebra. A *Cartan subalgebra* $A \subset M$ is an abelian von Neumann subalgebra which is an image of a faithful normal conditional expectation such that (i) A is maximal abelian and (ii) the normalizer $\mathcal{N}_M(A)$ generates M as a von Neumann algebra [Feldman and Moore 1977]. Here $\mathcal{N}_M(A)$ is given by $\{u \in \mathcal{U}(M) \mid uAu^* = A\}$.

The group measure space construction of Murray and von Neumann gives a typical example of a Cartan subalgebra. Indeed, the canonical subalgebra $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma$ is Cartan whenever the given action $\Gamma \curvearrowright (X,\mu)$ is free. More generally, one can associate any (not necessarily free) group action with a Cartan subalgebra by its orbit equivalence relation. Conversely when M has separable predual, any Cartan subalgebra $A \subset M$ is realized by an orbit equivalence relation (with a cocycle), and hence by a group action. Thus the notion of Cartan subalgebras is closely related to group actions. In particular if M has no Cartan subalgebras, then it cannot be constructed by any group actions. It was an open problem to find such a von Neumann algebra.

The first result in this direction was given by Connes [1975]. He constructed a II_1 factor which is not isomorphic to its opposite algebra, so it is particularly not isomorphic to any group action (without cocycle) von Neumann algebra. Voiculescu [1996] then provided a complete solution to this problem, by proving free group factors $L\mathbb{F}_n$ $(n \geq 2)$ have no Cartan subalgebras. He used his celebrated *free entropy* technique, and it was later developed to give other examples [Shlyakhtenko 2000; Jung 2007].

After these pioneering works, Ozawa and Popa [2010] introduced a completely new framework to study this subject. Among other things, they proved that free group factors are *strongly solid*, that is, for any diffuse amenable subalgebra $A \subset L\mathbb{F}_n$, the von Neumann algebra generated by the normalizer $\mathcal{N}_{L\mathbb{F}_n}(A)$

MSC2010: primary 46L10, 46L36; secondary 58B32.
Keywords: von Neumann algebra, type III factor, Cartan subalgebra.

remains amenable. Since $L\mathbb{F}_n$ itself is nonamenable, this immediately yields that $L\mathbb{F}_n$ has no Cartan subalgebras. Note that strong solidity is stable under taking subalgebras and hence any nonamenable subfactor of $L\mathbb{F}_n$ also has no Cartan subalgebras.

The proof of Ozawa and Popa consist of two independent steps. First, by using weak amenability of $\mathbb{F}_n$, they observed that the normalizer group acts *weakly compactly* on a given amenable subalgebra. Second, combining this weakly compact action with *Popa's deformation and intertwining techniques* [Popa 2006a; 2006b], they constructed a state which is central with respect to the normalizer group. Thus they obtained that the normalizer group generates an amenable von Neumann algebra. Since these techniques are applied to any finite crossed product $B \rtimes \mathbb{F}_n$ with the W*CMAP (weak* completely metric approximation property, see Section 2D), they also proved that for any finite factor B with the W*CMAP, the tensor product $L\mathbb{F}_n \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras.

To remove the W*CMAP assumption on $B \rtimes \mathbb{F}_n$, Popa and Vaes [2014a] introduced a notion of *relative weakly compact action*. This is an appropriate "relativization" of the first step above in the view of the relative tensor product $L^2(B \rtimes \mathbb{F}_n) \otimes_B L^2(B \rtimes \mathbb{F}_n)$. In particular this *only* requires the weak amenability of $\mathbb{F}_n$. Thus by modifying the proof in the second step above, they obtained, among other things, the tensor product $L\mathbb{F}_n \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras for any finite factor B.

The aim of the present paper is to develop these techniques to study type III von Neumann algebras. More specifically we replace the free group factor $L\mathbb{F}_n$ with the *free quantum group factor*, which is a type III factor in most cases. We have already studied this [Isono 2015a; 2015b] when B is finite. In the general case however, namely, when B is a type III factor, we could not provide a satisfactory answer to this problem, and this will be discussed in this article.

We note that the first solution to the Cartan subalgebra problem for type III factors in our framework was obtained by Houdayer and Ricard [2011]. They followed the proof of [Ozawa and Popa 2010] by exploiting techniques in [Chifan and Houdayer 2010], that is, the use of Popa's deformation and intertwining techniques together with the *continuous core decomposition*. While Houdayer and Ricard followed the idea of [Ozawa and Popa 2010], our approach in [Isono 2015a; 2015b] was based on [Popa and Vaes 2014b]. In particular, in the second step above, we made use of *Ozawa's condition* (*AO*) [2004] (or *biexactness*, see Section 2C) at the level of the continuous core. In this article, we stand again on the use of biexactness, and we will further develop techniques of [Isono 2015b]. See [Boutonnet et al. 2014] for other examples of type III factors with no Cartan subalgebras, and [Chifan and Sinclair 2013; Chifan et al. 2013] for other works on Cartan subalgebras of biexact group von Neumann algebras.

The following theorem is the main observation of this article. This should be regarded as a generalization of [Isono 2015b, Theorem B], and this allows us to obtain a satisfactory answer to the Cartan problem in the type III setting. See Section 2 for items in this theorem.

Theorem A. *Let $\mathbb{G}$ be a compact quantum group with the Haar state h, and B a type* III_1 *factor with a faithful normal state φ_B. Put $M := L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} B$ and $\varphi := h \otimes \varphi_B$. Let $C_{\varphi_B}(B)$ and $C_\varphi(M)$ be continuous cores of B and M with respect to φ_B and φ, and regard $C_{\varphi_B}(B)$ as a subset of $C_\varphi(M)$. Let* Tr *be a semifinite trace on $C_\varphi(M)$ with $\mathrm{Tr}|_{C_{\varphi_B}(B)}$ semifinite, and $p \in C_\varphi(M)$ a projection with $\mathrm{Tr}(p) < \infty$.*

Assume that $\widehat{\mathbb{G}}$ is biexact and centrally weakly amenable with Cowling–Haagerup constant 1. *Then for any amenable von Neumann subalgebra $A \subset pC_\varphi(M)p$, we have either one of the following conditions*:

(i) *We have $A \preceq_{C_\varphi(M)} C_{\varphi_B}(B)$.*

(ii) *The von Neumann algebra $\mathcal{N}_{pC_\varphi(M)p}(A)''$ is amenable relative to $C_{\varphi_B}(B)$.*

As a consequence of the main theorem, we obtain the following corollary. This is the desired one since our main example, free quantum groups, satisfies the assumptions in this corollary. See [Isono 2015b, Theorem C] for other examples of quantum groups satisfying these assumptions. Below we say that an inclusion of von Neumann algebras $A \subset M$ is *with expectation* if there is a faithful normal conditional expectation.

Corollary B. *Let $\mathbb{G}$ be a compact quantum group as in Theorem A. Then for any nonamenable subfactor $N \subset L^\infty(\mathbb{G})$ with expectation and any σ-finite factor B, the tensor product $N \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras.*

For the proof of Theorem A, we will establish a weakly compact action on the continuous core of $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} B$ as the one *relative to B*. The central weak amenability of $\widehat{\mathbb{G}}$ is used to find approximation maps on the continuous core which are relative to $B \rtimes \mathbb{R}$. Then combined with the amenability of $\mathbb{R}$, we construct appropriate approximation maps on the core relative to B. In this process, since B is not with expectation in the core, we use operator-valued weights instead. This is our strategy for the first step.

For the second step, although we go along a very similar line to [Isono 2015b], we need a rather different (and general) approach to the proof. We note that this is why we assume only biexactness of $\widehat{\mathbb{G}}$, and do not need the notion of *condition* $(AOC)^+$ as in [Isono 2015a; 2015b].

This paper is organized as follows. In Section 2, we recall fundamental facts for our paper, such as Tomita–Takesaki theory, free quantum groups, biexactness, weak amenability, and Popa's intertwining techniques.

In Section 3, we study a generalization of the relative weakly compact action on the continuous core by constructing appropriate approximation maps on the core. The main tools for this construction are: operator-valued weights; central weak amenability; and weak containment, together with the amenability of $\mathbb{R}$. This is the most technical part of this paper.

In Section 4, we prove the main theorem. We follow the proof of [Popa and Vaes 2014b; Isono 2015b], using the weakly compact action given in Section 3.

2. Preliminaries

2A. *Tomita–Takesaki theory and operator-valued weights.* We first recall some notions in Tomita–Takesaki theory. We refer the reader to [Takesaki 1979] for this theory, and to [Haagerup 1979a; 1979b] and [Takesaki 1979, Chapter IX, §4] for operator-valued weights.

Let M be a von Neumann algebra and φ a faithful normal semifinite weight on M. Put $\mathfrak{n}_\varphi := \{x \in M \mid \varphi(x^*x) < \infty\}$ and denote by $\Lambda_\varphi : \mathfrak{n}_\varphi \to L^2(M, \varphi)$ the canonical embedding. We denote the *modular operator, modular conjugation*, and *modular action* for $M \subset \mathbb{B}(L^2(M, \varphi))$ by Δ_φ, J_φ and σ^φ respectively. The Hilbert space $L^2(M, \varphi)$ with J_φ and with its *positive cone* $\mathcal{P}_\varphi$ is called the *standard representation* *for M* [Takesaki 1979, Chapter IX, §1] and does not depend on the choice of φ. Any state on M is represented by a vector state, from which the vector is uniquely chosen from $\mathcal{P}_\varphi$. Any element $\alpha \in \mathrm{Aut}(M)$ is written as $\alpha = \mathrm{Ad}\, u$ by a unique $u \in \mathbb{B}(L^2(M, \varphi))$ which preserves the standard representation structure. The crossed product $M \rtimes_{\sigma^\varphi} \mathbb{R}$ by the modular action is called the *continuous core* [loc. cit., Chapter XII, §1] and is written as $C_\varphi(M)$, which is equipped with the dual weight $\hat{\varphi}$ and the canonical trace $\mathrm{Tr}_\varphi := \hat{\varphi}(h_\varphi^{-1}\,\cdot\,)$, where h_φ is a self-adjoint positive closed operator affiliated with $L\mathbb{R}$. For any other faithful normal semifinite weight ψ, there is a family of unitaries $([D\varphi, D\psi]_t)_{t\in\mathbb{R}}$ in M called the *Connes cocycle* [loc. cit., Definition VIII.3.4]. This gives a cocycle conjugate for modular actions of φ and ψ, and hence there is a $*$-isomorphism

$$\Pi_{\psi,\varphi} : C_\varphi(M) \to C_\psi(M), \qquad \Pi_{\psi,\varphi}(x) = x \quad (x \in M), \qquad \Pi_{\psi,\varphi}(\lambda_t^\varphi) = [D\psi, D\varphi]_t^* \lambda_t^\psi \quad (t \in \mathbb{R}).$$

It holds that $\Pi_{\psi,\varphi} \circ \Pi_{\varphi,\omega} = \Pi_{\psi,\omega}$ for any other ω on M, and $\Pi_{\psi\circ E_M,\varphi\circ E_M}|_{C_\varphi(M)} = \Pi_{\psi,\varphi}$ for any $M \subset N$ with expectation E_M. It preserves traces $\mathrm{Tr}_\psi \circ \Pi_{\psi,\varphi} = \mathrm{Tr}_\varphi$ [loc. cit., Theorem XII.6.10(iv)]. So the pair $(C_\varphi(M), \mathrm{Tr}_\varphi)$ does not depend on the choice of φ, and we call Tr_φ the canonical trace. A von Neumann algebra is said to be a *type III_1 factor* if its continuous core is a II_∞ factor.

Let $B \subset M$ be any inclusion of von Neumann algebras. We denote by $\widehat{M}^+$ the *extended positive cone* of M. For any *operator-valued weight* $T : \widehat{M}^+ \to \widehat{B}^+$, we use the notation

$$\mathfrak{n}_T := \{x \in M \mid \|T(x^*x)\|_\infty < +\infty\},$$
$$\mathfrak{m}_T := (\mathfrak{n}_T)^*\mathfrak{n}_T = \left\{\textstyle\sum_{i=1}^n x_i^* y_i \;\middle|\; n \geq 1,\ x_i, y_i \in \mathfrak{n}_T \text{ for all } 1 \leq i \leq n\right\}.$$

Then T has a unique extension $T : \mathfrak{m}_T \to B$ as a B-bimodule linear map. In this paper, all the operator-valued weights that we consider are assumed to be *faithful, normal* and *semifinite*. Note that since the operator-valued weight is nothing but a weight when $B = \mathbb{C}$, we may also extend a faithful normal semifinite weight φ on $\mathfrak{m}_\varphi$.

For any inclusion $B \subset M$ of von Neumann algebras with faithful normal weights φ_B and φ_M on B and M respectively, the modular actions of them satisfy $\sigma^{\varphi_M}|_B = \sigma^{\varphi_B}$ if and only if there is an operator-valued weight E_B from M to B which satisfies $\varphi_B \circ E_B = \varphi_M$, and E_B is determined uniquely by this equality [loc. cit., Theorem IX.4.18]. We call E_B the *operator-valued weight from (M, φ_M) to (B, φ_B)*. In this case, the cores satisfy the inclusion $C_{\varphi_B}(B) \subset C_{\varphi_M}(M)$ since $\sigma^{\varphi_M}|_B = \sigma^{\varphi_B}$. When $\varphi_M|_B = \varphi_B$, E_B is a faithful normal conditional expectation [loc. cit., Theorem IX.4.2].

Let M be a von Neumann algebra and φ a faithful normal semifinite weight on M. Put $L^2(M) := L^2(M, \varphi)$ and let α be an action of $\mathbb{R}$ on M. In this article, as a representation of $M \rtimes_\alpha \mathbb{R}$, we use that for any $\xi \in L^2(\mathbb{R}) \otimes L^2(M) \simeq L^2(\mathbb{R}, M)$ and $s, t \in \mathbb{R}$,

$$M \ni x \mapsto \pi_\alpha(x), \qquad (\pi_\alpha(x)\xi)(s) := \alpha_{-s}(x)\xi(s),$$
$$L\mathbb{R} \ni \lambda_t \mapsto 1_M \otimes \lambda_t, \quad ((1 \otimes \lambda_t)\xi)(s) := \xi(s - t).$$

Let $C_c(\mathbb{R}, M)$ be the set of all $*$-strongly continuous functions from $\mathbb{R}$ to M with compact supports. Then there is an embedding

$$\hat{\pi}_\alpha : C_c(\mathbb{R}, M) \ni f \mapsto \int_{\mathbb{R}} (1 \otimes \lambda_t)\pi_\alpha(f(t))\, dt \in M \rtimes_\alpha \mathbb{R},$$

where the integral here should be understood as the map $T \in \mathbb{B}(L^2(\mathbb{R}) \otimes L^2(M))$ given by

$$\langle T\xi, \eta\rangle = \int_{\mathbb{R}} \langle (1 \otimes \lambda_t)\pi_\alpha(f(t))\xi, \eta\rangle \, dt$$

for all $\xi, \eta \in L^2(\mathbb{R}) \otimes L^2(M)$. We note that by

$$(f * g)(t) := \int_{\mathbb{R}} \alpha_s(f(t+s))g(-s)\, ds \quad \text{and} \quad f^\sharp(t) := \alpha_t^{-1}(f(-t)^*) \quad \text{for } f, g \in C_c(\mathbb{R}, M) \text{ and } t \in \mathbb{R},$$

$C_c(\mathbb{R}, M)$ is a $*$-algebra, so that $\hat{\pi}_\alpha$ is a $*$-homomorphism. For $f \in C_c(\mathbb{R}, M)$ and $x \in M$, we define $(f \cdot x)(t) := f(t)x$ for $t \in G$. Let $C_c(\mathbb{R}, M)\mathfrak{n}_\varphi \subset C_c(\mathbb{R}, M)$ be the set of linear spans of $f \cdot x$ for $f \in C_c(\mathbb{R}, M)$ and $x \in \mathfrak{n}_\varphi$. With this notation, the dual weight satisfies

$$\hat{\varphi}(\hat{\pi}_\alpha(g)^*\hat{\pi}_\alpha(f)) = \varphi((g^\sharp * f)(0)) = \int_{\mathbb{R}} \varphi(g(t)^* f(t))\, dt \quad \text{for any } f, g \in C_c(\mathbb{R}, M)\mathfrak{n}_\varphi$$

[Takesaki 1979, Theorem X.1.17]. The modular objects of $\hat{\varphi}$ are given by

$$\sigma_t^{\hat{\varphi}}|_M = \sigma_t^{\varphi} \quad \text{and} \quad \sigma_t^{\hat{\varphi}}(\lambda_s) = \lambda_s[D(\varphi \circ \alpha_s), D\varphi]_t \quad \text{for } s, t \in \mathbb{R},$$
$$(J_{\hat{\varphi}}\xi)(t) = u^*(t) J_\varphi \xi(-t) \quad \text{for } t \in \mathbb{R} \text{ and } \xi \in L^2(\mathbb{R}, L^2(M)),$$

where $u(t)$ is the unitary such that $\alpha_t = \operatorname{Ad} u(t)$, which preserves the standard structure of $L^2(M, \varphi)$. In particular $\sigma^{\hat{\varphi}}$ globally preserves M and so there is a canonical operator-valued weight E_M from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to (M, φ). By the equality $\varphi \circ E_M = \hat{\varphi}$, it holds that for any $f, g \in C_c(\mathbb{R}, M)$,

$$E_M(\hat{\pi}_\alpha(g)^*\hat{\pi}_\alpha(f)) = (g^\sharp * f)(0) = \int_{\mathbb{R}} g(t)^* f(t)\, dt.$$

Here we prove a few lemmas.

Lemma 2.1. *Let (N, φ_N) and (B, φ_B) be von Neumann algebras with faithful normal semifinite weights with $\varphi_N(1) = 1$. Let α^B be an action of $\mathbb{R}$ on B, and put $M := N \mathbin{\bar{\otimes}} B$, $\varphi := \varphi_N \otimes \varphi_B$, $\alpha := \sigma^{\varphi_N} \otimes \alpha^B$. Let E_M, E_B, $E_{B\rtimes\mathbb{R}}$ be the canonical operator-valued weights from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to (M, φ), from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to (B, φ_B), and from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to $(B \rtimes_{\alpha^B} \mathbb{R}, \hat{\varphi}_B)$ respectively. Then we have $E_{B\rtimes\mathbb{R}} \circ E_M = E_B$.*

Proof. Let P_N be the one-dimensional projection from $L^2(N, \varphi_N)$ onto $\mathbb{C}\Lambda_{\varphi_N}(1_N)$ and observe that the compression map by $P_N \otimes 1_B \otimes 1_{L^2(\mathbb{R})}$ on $N \mathbin{\bar{\otimes}} B \mathbin{\bar{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$ gives a normal conditional expectation $E : M \rtimes_\alpha \mathbb{R} \to B \rtimes_{\alpha^B} \mathbb{R}$ satisfying $E((x \otimes b)\lambda_t) = \varphi_N(x) b\lambda_t$ for $x \in N$, $b \in B$, and $t \in \mathbb{R}$. It is faithful on $M \rtimes_\alpha \mathbb{R}$ since it is faithful on $N \mathbin{\bar{\otimes}} B \mathbin{\bar{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$. A simple computation shows that $E = E_{B\rtimes\mathbb{R}}$ and $E_{B\rtimes\mathbb{R}}((x \otimes b)\lambda_t) = \varphi_N(x) b\lambda_t$ for $x \in N$, $b \in B$, and $t \in \mathbb{R}$. In particular $E_{B\rtimes\mathbb{R}}|_M$ is the canonical conditional expectation E_B^M from (M, φ) to (B, φ_B). Then by definition, $\varphi_B \circ E_B^M \circ E_M = \varphi \circ E_M = \hat{\varphi}$, and hence $E_B^M \circ E_M = E_B$. Since $E_B^M \circ E_M = E_{B\rtimes\mathbb{R}} \circ E_M$, we obtain the conclusion. □

We next recall the following well-known fact. We include a proof for the reader's convenience.

Lemma 2.2. *Let M be a type III_1 factor and N a von Neumann algebra. Then the center of the continuous core of $M \mathbin{\bar{\otimes}} N$ coincides with the center of N.*

Proof. Since M is a type III_1 factor, there is a faithful normal semifinite weight φ_M on M such that $(M_{\varphi_M})' \cap M = \mathbb{C}$ [Takesaki 1979, Theorem XII.1.7], where M_{φ_M} is the fixed point algebra of the modular action of φ_M. Let φ_N be a faithful normal semifinite weight on N and put $\varphi := \varphi_M \otimes \varphi_N$. Observe that the center of $C_\varphi(M \mathbin{\overline{\otimes}} N)$ is contained in

$$(M_{\varphi_M} \otimes \mathbb{C}1_{L^2(N)\otimes L^2(\mathbb{R})})' \cap M \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R})) = \mathbb{C}1_{L^2(M,\varphi_M)} \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R})).$$

On the other hand, since $\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N))$ commutes with $L\mathbb{R}$, it is contained in $(M \mathbin{\overline{\otimes}} N)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$; see, e.g., [Houdayer and Ricard 2011, Proposition 2.4]. Hence

$$\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N)) \subset \mathbb{C} \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R})) \cap (M \mathbin{\overline{\otimes}} N)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R} = \mathbb{C} \mathbin{\overline{\otimes}} N_{\varphi_N} \mathbin{\overline{\otimes}} L\mathbb{R}.$$

Finally since $\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N))$ commutes with M, and N_{φ_N} commutes with M and $L\mathbb{R}$, (up to exchanging positions of M and N) we have

$$\mathcal{Z}(C_\varphi(N \mathbin{\overline{\otimes}} M)) \subset M' \cap N_{\varphi_N} \mathbin{\overline{\otimes}} \mathbb{C} \mathbin{\overline{\otimes}} L\mathbb{R} = N_{\varphi_N} \mathbin{\overline{\otimes}} (M' \cap \mathbb{C} \mathbin{\overline{\otimes}} L\mathbb{R}) = N_{\varphi_N} \mathbin{\overline{\otimes}} \mathbb{C}1,$$

where we used $M' \cap \mathbb{C} \mathbin{\overline{\otimes}} L\mathbb{R} \subset \mathcal{Z}(C_{\varphi_M}(M)) = \mathbb{C}$. Since $N' \cap N_{\varphi_N} = \mathcal{Z}(N)$, we conclude that $\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N)) = \mathcal{Z}(N)$. Since all continuous cores are isomorphic with each other, preserving the position of $M \mathbin{\overline{\otimes}} N$, for any other faithful normal semifinite weight ψ, we obtain $\mathcal{Z}(C_\psi(M \mathbin{\overline{\otimes}} N)) = \mathcal{Z}(N)$. □

2B. ***Relative tensor products, basic constructions and weak containments.*** Let M and N be von Neumann algebras and H a Hilbert space. Throughout this paper, we denote *opposite* objects with a circle superscript (e.g., $N^\circ := N^{\mathrm{op}}$, $x^\circ := x^{\mathrm{op}} \in N^\circ$, $(xy)^\circ = y^\circ x^\circ$ for $x, y \in N$). We say that H is a *left M-module* (resp. a *right N-module*) if there is a normal unital injective $*$-homomorphism $\pi_H : M \to \mathbb{B}(H)$ (resp. $\theta_H : N^\circ \to \mathbb{B}(H)$). We say H is an *M-N-bimodule* if H is a left M-module and a right N-module with commuting ranges. The *standard bimodule* of M is a standard representation $L^2(M)$ as an M-bimodule, where the right action is given by $M^\circ \ni x^\circ \mapsto Jx^*J \in M' \subset \mathbb{B}(L^2(M))$.

Let N be a von Neumann algebra, φ a faithful normal semifinite weight, and $H = H_N$ a right N-module with the right action θ. A vector $\xi \in H$ is said to be *left φ-bounded* if there is a constant $C > 0$ such that $\|\theta(x^\circ)\xi\| \le C\|J_\varphi \Lambda_\varphi(x^*)\|$ for all $x \in \mathfrak{n}_\varphi^*$. We denote by $D(H, \varphi)$ all left φ-bounded vectors in H. It is known that the subspace $D(H, \varphi) \subset H$ is always dense [Takesaki 1979, Lemma IX.3.3(iii)]. For $\xi \in D(H, \varphi)$, define a bounded operator

$$L_\xi : L^2(N, \varphi) \to H;\; L_\xi J_\varphi \Lambda_\varphi(a^*) = \theta(a^\circ)\xi.$$

It is easy to verify that

$$\theta(x^\circ)L_\xi = L_\xi J_\varphi x^* J_\varphi \quad (x \in N),$$
$$L_\xi L_\eta^* \in \theta(N^\circ)' \quad \text{and} \quad L_\eta^* L_\xi \in (J_\varphi N J_\varphi)' = N \quad (\xi, \eta \in D(H, \varphi)),$$
$$x L_\xi y = L_{x\theta(\sigma_{i/2}^\varphi(y)^\circ)\xi} \quad (x \in \theta(N^\circ)',\, y \in N_a),$$

where $N_a \subset N$ is the subalgebra consisting of all *analytic* elements with respect to (σ_t^φ) (see [Takesaki 1979, Lemma IX.3.3(v)] for the third statement). For a left N-module $K = {}_N K$, the *relative tensor product* $H \otimes_N K$ is defined as the Hilbert space obtained by separation and compression of $D(H, \varphi) \otimes_{\mathrm{alg}} K$ with an inner product $\langle \xi_1 \otimes_N \eta_1, \xi_2 \otimes_N \eta_2 \rangle := \langle L_{\xi_2}^* L_{\xi_1} \eta_1, \eta_2 \rangle_K$. When $H = {}_M H_N$ is an M-N-bimodule and $K = {}_N K_A$ is an N-A-bimodule for von Neumann algebras M and A, the Hilbert space $H \otimes_N K$ is an M-A-bimodule given by $\pi(x)\theta(a^\circ)(\xi \otimes_N \eta) := (\pi_H(x)\xi) \otimes_B (\theta_K(a^\circ)\eta)$ for $x \in M,\ a \in A,\ \xi \in D(H, \varphi)$ and $\eta \in K$.

Since all standard representations $L^2(M)$ of M are isomorphic as M-bimodules, when we consider $H = K = L^2(M)$ and $N \subset M$, the Hilbert space $L^2(M) \otimes_N L^2(M)$ is determined canonically, and does not depend on the choice of a faithful normal semifinite weight φ on M with $L^2(M) = L^2(M, \varphi)$.

Let $B \subset M$ be an inclusion of von Neumann algebras and φ a faithful normal semifinite weight on M. The *basic construction* of the inclusion $B \subset M$ is defined by

$$\langle M, B \rangle := (J_\varphi B J_\varphi)' \cap \mathbb{B}(L^2(M, \varphi)).$$

Since all standard representations are canonically isomorphic, the basic construction does not depend on the choice of φ. Assume that the inclusion $B \subset M$ is with an operator-valued weight E_B. Fix a faithful normal semifinite weight φ_B on B and put $\varphi := \varphi_B \circ E_B$. Here we observe that any $x \in \mathfrak{n}_{E_B} \cap \mathfrak{n}_\varphi$ is left φ-bounded and $L_{\Lambda_\varphi(x)} \Lambda_{\varphi_B}(a) = \Lambda_\varphi(xa)$ for $a \in \mathfrak{n}_{\varphi_B}$. Indeed, for any analytic $a \in \mathfrak{n}_{\varphi_B} \cap \mathfrak{n}_{\varphi_B}^*$, we have $J_{\varphi_B} \Lambda_{\varphi_B}(a^*) = \Delta_{\varphi_B}^{1/2} \Lambda_{\varphi_B}(a) = \Lambda_{\varphi_B}(\sigma_{-i/2}^{\varphi_B}(a))$, see, e.g., the equation just before [Takesaki 1979, Lemma VIII.2.4], and hence by Lemma V.III.3.18(ii) of the same work,

$$L_{\Lambda_\varphi(x)} \Lambda_{\varphi_B}(\sigma_{-i/2}^{\varphi_B}(a)) = L_{\Lambda_\varphi(x)} J_{\varphi_B} \Lambda_{\varphi_B}(a^*) = J_\varphi a^* J_\varphi \Lambda_\varphi(x) = \Lambda_\varphi(x \sigma_{-i/2}^{\varphi}(a)).$$

Since $\sigma_{-i/2}^{\varphi_B}(a) = \sigma_{-i/2}^{\varphi}(a)$ (because $\sigma_t^\varphi|_B = \sigma_t^{\varphi_B}$ for $t \in \mathbb{R}$, and the analytic extension is unique if exists), this means that $L_{\Lambda_\varphi(x)} \Lambda_{\varphi_B}(b) = \Lambda_\varphi(xb)$ for any analytic $b \in \mathfrak{n}_{\varphi_B} \cap \mathfrak{n}_{\varphi_B}^*$. At the same time, we can define a bounded operator $L_x : \Lambda_{\varphi_B}(a) \mapsto \Lambda_\varphi(xa)$ for $a \in \mathfrak{n}_{\varphi_B}$ (use $x \in \mathfrak{n}_{E_B}$). So the map $L_{\Lambda_\varphi(x)}$ has a bounded extension on $L^2(B, \varphi_B)$ and coincides with L_x, as desired. Now it is easy to verify that

$$L_{\Lambda_\varphi(y)}^* L_{\Lambda_\varphi(x)} = E_B(y^* x) \in (J_\varphi B J_\varphi)' = B \subset \mathbb{B}(L^2(B, \varphi_B)) \quad (x, y \in \mathfrak{n}_{E_B} \cap \mathfrak{n}_\varphi).$$

We will use this formula for calculations in the proposition below and in Section 3.

Here we observe that a relative tensor product has a useful identification. We will use this proposition in Sections 3 and 4.

Proposition 2.3. *Let N and B be von Neumann algebras, and α^N and α^B actions of $\mathbb{R}$ on N and B respectively. Put $M := N \mathbin{\overline{\otimes}} B$ and $\alpha := \alpha^N \otimes \alpha^B$, and define $H := L^2(M \rtimes_\alpha \mathbb{R}) \otimes_B L^2(M \rtimes_\alpha \mathbb{R})$ as an $M \rtimes_\alpha \mathbb{R}$-bimodule with left and right actions π_H and θ_H.*

Then there is a

$U : H \to L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})$ such that, putting $\tilde{\pi}_H := \operatorname{Ad} U \circ \pi_H$ and $\tilde{\theta}_H := \operatorname{Ad} U \circ \theta_H$,

- $\tilde{\pi}_H(M \rtimes_\alpha \mathbb{R}) \subset \mathbb{B}(L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B)) \otimes \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}$,

$$\tilde{\pi}_H(\lambda_t) = \lambda_t \otimes 1_N \otimes 1_B \quad \textit{and} \quad \tilde{\pi}_H(x) = \pi_\alpha(x) \quad (t \in \mathbb{R},\ x \in N \mathbin{\overline{\otimes}} B = M);$$

- $\tilde{\theta}_H((M \rtimes_\alpha \mathbb{R})^\circ) \subset \mathbb{C}1_{L^2(\mathbb{R})} \otimes \mathbb{C}1_N \otimes \mathbb{B}(L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R}))$

$$\tilde{\theta}_H(\lambda_t^\circ) = 1_B \otimes 1_N \otimes \rho_t \quad \textit{and} \quad \tilde{\theta}_H(y^\circ) = \theta_\alpha(y^\circ) \quad (t \in \mathbb{R},\ y \in B \mathbin{\overline{\otimes}} N \simeq M),$$

$$\textit{where } (\theta_\alpha(y^\circ)\xi)(s) := \alpha_s(y)^\circ \xi(s) \quad \textit{for } \xi \in L^2(\mathbb{R}, L^2(B) \otimes L^2(N)) \textit{ and } s \in \mathbb{R}.$$

Proof. We fix a faithful normal semifinite weight φ_B on B and put $\varphi := \varphi_N \otimes \varphi_B$. Denote by $\hat{\varphi}$ the dual weight of φ and then the standard representation of $M \rtimes_\alpha \mathbb{R}$ is given by

$$L^2(M \rtimes_\alpha \mathbb{R}, \hat{\varphi}) = L^2(N, \varphi_N) \otimes L^2(B, \varphi_B) \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}, L^2(N, \varphi_N) \otimes L^2(B, \varphi_B)).$$

For simplicity we put $L^2(N) := L^2(N, \varphi_N)$ and $L^2(B) := L^2(B, \varphi_B)$. Let E_B be the canonical operator-valued weight from $\widetilde{M}$ to B given by $\hat{\varphi} = \varphi_B \circ E_B$. Then for $E_B^M := \varphi_N \otimes \mathrm{id}_B$ on M and for the canonical operator-valued weight E_M from $(M \rtimes \mathbb{R}, \hat{\varphi})$ to (M, φ), we have $\hat{\varphi} = \varphi \circ E_M = \varphi_B \circ E_B^M \circ E_M$, and hence $E_B = E_B^M \circ E_M$ by the uniqueness condition. Observe then for any $f, g \in C_c(\mathbb{R}, M)$,

$$E_B(\hat{\pi}_\alpha(g)^* \hat{\pi}_\alpha(f)) = \int_{\mathbb{R}} E_B^M(g(t)^* f(t))\, dt.$$

Define a well-defined linear map

$$V : \Lambda_\varphi(\mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B}) \otimes_{\mathrm{alg}} J_\varphi \Lambda_\varphi(\mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N}) \to L^2(N) \otimes L^2(B) \otimes L^2(N)$$

by $V(\Lambda_\varphi(x \otimes a) \otimes J_\varphi \Lambda_\varphi(b \otimes y)) := \Lambda_{\varphi_N}(x) \otimes a J_{\varphi_B} \Lambda_{\varphi_B}(b) \otimes J_{\varphi_N} \Lambda_{\varphi_N}(y)$. We then define a linear map

$$U : L^2(\mathbb{R}, L^2(N) \otimes L^2(B)) \otimes_B L^2(\mathbb{R}, L^2(B) \otimes L^2(N)) \to L^2(\mathbb{R} \times \mathbb{R}, L^2(N) \otimes L^2(B) \otimes L^2(N))$$

by $(U(f \otimes_B J_{\hat{\varphi}} g))(t, s) := V(\Lambda_\varphi(f(t)) \otimes J_\varphi \Lambda_\varphi(g(-s)))$ for $f \in C_c(\mathbb{R}, N \otimes_{\mathrm{alg}} B)(\mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B})$ and $g \in C_c(\mathbb{R}, B \otimes_{\mathrm{alg}} N)(\mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N})$. (Note that we are identifying $\Lambda_{\hat{\varphi}}(\hat{\pi}_\alpha(f))$ and $\Lambda_{\hat{\varphi}}(\hat{\pi}_\alpha(g))$ as f and g.) We have to show that it is a well-defined unitary map. For $f_i \in C_c(\mathbb{R}, N \otimes_{\mathrm{alg}} B)(\mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B})$ and $g_i \in C_c(\mathbb{R}, B \otimes_{\mathrm{alg}} N)(\mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N})$, straightforward but rather complicated computations yield, on the one hand,

$$\Big\| \sum_i f_i \otimes_B J_{\hat{\varphi}} g_i \Big\|_2^2 = \sum_{i,j} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle F_{j,i} J_\varphi \Lambda_\varphi(g_i(-s)), J_\varphi \Lambda_\varphi(g_j(-s)) \rangle\, ds\, dt,$$

where $F_{j,i} := E_B^M(f_j(t)^* f_i(t))$, and on the other hand,

$$\Big\| U \sum_i (f_i \otimes_B J_{\hat{\varphi}} g_i) \Big\|_2^2 = \sum_{i,j} \int_{\mathbb{R}\times\mathbb{R}} \big\langle V\big(\Lambda_\varphi(f_i(t)) \otimes J_\varphi \Lambda_\varphi(g_i(-s))\big), V\big(\Lambda_\varphi(f_j(t)) \otimes J_\varphi \Lambda_\varphi(g_i(-s))\big) \big\rangle\, dt\, ds.$$

Hence if we show

$$\langle V(\Lambda_\varphi(x) \otimes J_\varphi \Lambda_\varphi(a)), V(\Lambda_\varphi(y) \otimes J_\varphi \Lambda_\varphi(b)) \rangle = \langle E_B^M(y^* x) J_\varphi \Lambda_\varphi(a), J_\varphi \Lambda_\varphi(b) \rangle$$

for any $x, y \in \mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B}$ and $a, b \in \mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N}$, then U is a well-defined unitary map. However this equation follows easily if we use elementary elements.

Finally $L^2(\mathbb{R} \times \mathbb{R}, L^2(N) \otimes L^2(B) \otimes L^2(N))$ is canonically isomorphic to $L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})$, where the first (resp. the second) variable in $\mathbb{R} \times \mathbb{R}$ corresponds to $L\mathbb{R}$ of the left one (resp. the right one) in the Hilbert space. It is then easy to see that $\tilde{\pi}_H$ and $\tilde{\theta}_H$ satisfy the desired condition. □

Let M and N be von Neumann algebras, and let H and K be M-N-bimodules. We denote by π_H and θ_H (resp. π_K and θ_K) left and right actions on H (resp. K). We say that K *is weakly contained in* H, denoted by $K \prec H$, if for any $\varepsilon > 0$, finite subsets $\mathcal{E} \subset M$ and $\mathcal{F} \subset N$, and any vector $\xi \in K$, there are vectors $(\eta_i)_{i=1}^n \subset H$ such that

$$\left| \sum_{i=1}^{n} \langle \pi_H(x)\theta_H(y^\circ)\eta_i, \eta_i \rangle_H - \langle \pi_K(x)\theta_K(y^\circ)\xi, \xi \rangle_K \right| < \varepsilon \quad (x \in \mathcal{E},\ y \in \mathcal{F}).$$

This is equivalent to saying that the algebraic $*$-homomorphism given by $\pi_H(x)\theta_H(y^\circ) \mapsto \pi_K(x)\theta_K(y^\circ)$ for $x \in M$ and $y \in N$ is bounded on $*$-alg$\{\pi_H(M), \theta_H(N^\circ)\}$. We denote by $\nu_{K,H}$ the associated $*$-homomorphism for $K \prec H$.

Let M and N be *σ-finite* von Neumann algebras and let X be a self-dual M-N-correspondence (i.e., a Hilbert N-module with a normal left M-action, see [Paschke 1973, Section 3] for self-duality and normality). Then the *interior tensor product*, see, e.g., [Lance 1995, Section 4], $H(X) := X \otimes_N L^2(N)$ is an M-N-bimodule. Conversely if H is an M-N-bimodule, then one can define a self-dual M-N-correspondence (i.e., a W*-Hilbert N-module with a left M-action)

$$X(H) := \{T : L^2(N) \to H \mid \text{bounded, } N^\circ\text{-module linear map}\}.$$

They in fact give a one-to-one correspondence between M-N-bimodules and self-dual M-N-correspondences, up to unitary equivalence; see [Baillet et al. 1988, Theorem 2.2] and [Rieffel 1974, Proposition 6.10]. By [Anantharaman-Delaroche 1990, §1.12, Proposition], $K \prec H$ if and only if $X(K) \prec X(H)$ in the following sense: for any σ-weak neighborhood $\mathcal{V}$ of $0 \in N$, finite subsets $\mathcal{E} \subset M$ and $\mathcal{F} \subset N$, and any $\xi \in X(K)$, there are vectors $(\eta_i)_{i=1}^n \subset X(H)$ such that

$$\sum_{i=1}^{n} \langle \eta_i, x\eta_i y \rangle_{X(H)} - \langle \xi, x\xi y \rangle_{X(K)} \in \mathcal{V} \quad (x \in \mathcal{E},\ y \in \mathcal{F}).$$

Suppose that $M = N$, $L^2(M) = K$, and $M = X(K)$. Then if $L^2(M) \prec H$, putting $\xi := 1_M$, for any finite subset $\mathcal{E} \subset M$ and for any σ-weak neighborhood $\mathcal{V}$ of $0 \in N$, there are vectors $(\eta_i)_{i=1}^n \subset X(H)$ such that

$$\sum_{i=1}^{n} \langle \eta_i, x\eta_i \rangle_{X(H)} - x \in \mathcal{V} \quad (x \in \mathcal{E}).$$

So putting $\psi_{(\mathcal{E},\mathcal{V})}(x) := \sum_{i=1}^n \langle \eta_i, x\eta_i \rangle_{X(H)}$ for $x \in M$, we find a net $(\psi_i)_i$ such that each ψ_i is given by a sum of compression maps by vectors in $X(H)$ and such that it converges to id_M in the point σ-weak topology. In this case, up to replacing η_i, we may assume that each ψ_i is a contraction [Anantharaman-Delaroche and Havet 1990, Lemma 2.2]. Then it is known that the existence of such a net is equivalent to $L^2(M) \prec H$ as follows, although we do not need this equivalence. See Proposition 2.4 of the same work for a more general statement.

Proposition 2.4. *Let M be a σ-finite von Neumann algebra and H an M-bimodule. Then $L^2(M) \prec H$ as M-bimodules if and only if there is a net $(\psi_i)_i$ of normal contractive completely positive (c.c.p.) maps on M, which converges to* id_M *point σ-weakly, such that each ψ_i is a finite sum of $\langle \eta, \cdot\, \eta \rangle_{X(H)}$ for some $\eta \in X(H)$.*

We recall the following well-known fact. This will be used in Section 3.

Lemma 2.5. *Let $B \subset M$ be an inclusion of σ-finite von Neumann algebras with an operator-valued weight E_B. Then the vector space $\mathfrak{n}_{E_B}$ is a pre-Hilbert B-module with the inner product $\langle x, y\rangle := E_B(x^*y)$ for $x, y \in \mathfrak{n}_{E_B}$, and its self-dual completion $\bar{\mathfrak{n}}_{E_B}$ is an M-B-correspondence.*

Let X be the self-dual completion of the interior tensor product $\bar{\mathfrak{n}}_{E_B} \otimes_B M$. Then as an M-M-correspondence, X is the unique one corresponding to the M-bimodule $L^2(M) \otimes_B L^2(M)$, using the one-to-one correspondence above.

Proof. It is easy to see that the B-valued inner product on $\mathfrak{n}_{E_B}$ in the statement is well-defined, so that $\mathfrak{n}_{E_B}$ is a pre-Hilbert B-module with a left M-action. Since the left M-action is faithful on $\mathfrak{n}_{E_B}$, so is on the self-dual completion; see, e.g., [Paschke 1973, Corollary 3.7]. This left M-action is normal, since the functional $M \ni x \mapsto \omega(\langle \xi, x\eta\rangle)$ is normal for all $\omega \in M_*$ and $\xi, \eta \in \mathfrak{n}_{E_B}$, and hence for all $\xi, \eta \in \bar{\mathfrak{n}}_{E_B}$ by [Paschke 1976, Lemma 2.3]. Thus $\bar{\mathfrak{n}}_{E_B}$ is an M-B-correspondence.

Let X be as in the statement. Then as in the first paragraph, it is easy to see that it is really an M-M-correspondence (i.e., the left M-action is well-defined, injective, and normal). Let us fix faithful normal states φ_B and φ on B and M respectively. Then the interior tensor product $X \otimes_M L^2(M, \varphi)$ is canonically identified as $L^2(M, \varphi_B \circ E_M) \otimes_B L^2(M, \varphi)$, so that X is identified as $X(L^2(M) \otimes_B L^2(M))$. □

2C. *Free quantum groups and biexactness.* For compact quantum groups, we refer the reader to [Woronowicz 1998; Maes and Van Daele 1998].

Let $\mathbb{G}$ be a compact quantum group. In this paper, we use the following notation, which will only be used in Section 4. We denote the Haar state by h, the set of equivalence classes of all irreducible unitary corepresentations by $\mathrm{Irred}(\mathbb{G})$, and right and left regular representations by ρ and λ respectively. We regard $C_{\mathrm{red}}(\mathbb{G}) := \rho(C(\mathbb{G}))$ as our main object and we frequently omit ρ when we see the dense Hopf $*$-algebra. The GNS representation of h is written as $L^2(\mathbb{G})$ and it has a decomposition $L^2(\mathbb{G}) = \sum_{x\in\mathrm{Irred}(\mathbb{G})} \oplus (H_x \otimes H_{\bar{x}})$. Along the decomposition, the modular operator of h is of the form $\Delta_h^{it} = \sum_{x\in\mathrm{Irred}(\mathbb{G})} \oplus (Q_x^{it} \otimes Q_{\bar{x}}^{-it})$ for some positive matrices Q_x.

Let F be a matrix in $\mathrm{GL}(n, \mathbb{C})$. The *free unitary quantum group* (resp. *free orthogonal quantum group*) for F [Wang 1995; Van Daele and Wang 1996] is the C*-algebra $C(A_u(F))$ (resp. $C(A_o(F))$) defined as the universal unital C*-algebra generated by all the entries of a unitary n by n matrix $u = (u_{i,j})_{i,j}$ satisfying that $F(u_{i,j}^*)_{i,j}F^{-1}$ is a unitary (resp. $F(u_{i,j}^*)_{i,j}F^{-1} = u$). We simply say that $\mathbb{G}$ is a *free quantum group* if $\mathbb{G}$ is a free unitary or orthogonal quantum group.

Here we recall the notion of biexactness introduced in [Isono 2015b, Definition 3.1], based on the group case [Brown and Ozawa 2008, Lemma 15.1.2].

Definition 2.6. Let $\mathbb{G}$ be a compact quantum group. We say that the dual $\widehat{\mathbb{G}}$ is *biexact* if it satisfies following conditions:

(i) $\widehat{\mathbb{G}}$ is exact (i.e., $C_{\mathrm{red}}(\mathbb{G})$ is exact).

(ii) There exists a unital completely positive (u.c.p.) map $\Theta : C_{\mathrm{red}}(\mathbb{G}) \otimes_{\min} C_{\mathrm{red}}(\mathbb{G})^\circ \to \mathbb{B}(L^2(\mathbb{G}))$ such that

$$\Theta(a \otimes b^\circ) - ab^\circ \in \mathbb{K}(L^2(\mathbb{G})) \quad \text{for any } a, b \in C_{\mathrm{red}}(\mathbb{G}).$$

Biexactness of free quantum groups was proved in [Vergnioux 2005; Vaes and Vergnioux 2007; Vaes and Vander Vennet 2010]. See [Isono 2015b, Theorem C] for other examples of biexact quantum groups.

Theorem 2.7. *Let $\mathbb{G}$ be a free quantum group* (more generally, a compact quantum group in [Isono 2015b, Theorem C]). *Then the dual $\widehat{\mathbb{G}}$ is biexact.*

2D. ***Central weak amenability and the W*CMAP.*** Let $\mathbb{G}$ be a compact quantum group. Denote the dense Hopf $*$-algebra by $\mathscr{C}(\mathbb{G})$. To any element $a \in \ell^\infty(\widehat{\mathbb{G}})$ we can associate a linear map m_a on $\mathscr{C}(\mathbb{G})$, given by $(m_a \otimes \iota)(u^x) = (1 \otimes a p_x)u^x$ for any $x \in \mathrm{Irred}(\mathbb{G})$, where $p_x \in c_0(\widehat{\mathbb{G}})$ is the canonical projection onto the x-component. We say $\widehat{\mathbb{G}}$ is *weakly amenable* (*with Cowling–Haagerup constant 1*) if there exists a net $(a_i)_i$ of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that:

- Each a_i has finite support; namely, $a_i p_x = 0$ except for finitely many $x \in \mathrm{Irred}(\mathbb{G})$.
- $(a_i)_i$ converges to 1 pointwise; namely, $a_i p_x$ converges to p_x in $\mathbb{B}(H_x)$ for any $x \in \mathrm{Irred}(\mathbb{G})$.
- Each m_{a_i} is extended on $L^\infty(\mathbb{G})$ as a completely contractive (say c.c.) map.

Note that, since a_i is finitely supported, each m_{a_i} is actually a map from $L^\infty(\mathbb{G})$ to $\mathscr{C}(\mathbb{G})$. We say $\widehat{\mathbb{G}}$ is *centrally weakly amenable* if each $a_i p_x$ above is taken as a scalar matrix for all i and $x \in \mathrm{Irred}(\mathbb{G})$. In this case, the associated multiplier m_{a_i} commutes with the modular action of the Haar state. This commutativity is important to us since such multipliers can be extended naturally on the continuous core with respect to the Haar state. Indeed, the maps $m_{a_i} \otimes \mathrm{id}_{L^2(\mathbb{R})}$ on $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$ restrict to approximation maps on the core. With this phenomenon in mind, we introduce the following terminology.

Definition 2.8. Let M be a von Neumann algebra and φ a fixed faithful normal state on M. We say that M has the *weak* completely metric approximation property with respect to* φ (or *φ-W*CMAP*, in short) if there exists a net $(\psi_i)_i$ of normal c.c. maps on M such that:

- Each ψ_i commutes with σ^φ; that is, $\psi_i \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \psi_i$ for all i and $t \in \mathbb{R}$.
- Each ψ_i is a finite sum of $\varphi(b^* \cdot a)z$ for some $a, b, z \in M$.
- ψ_i converges to id_M in the point σ-weak topology.

It is easy to see that the central weak amenability of $\widehat{\mathbb{G}}$ implies the W*CMAP with respect to the Haar state.

Weak amenability of the free quantum group was first obtained in [Freslon 2013], using the Haagerup property [Brannan 2012]. This is for the Kac type and hence is equivalent to the central weak amenability. The general case was solved later in [De Commer et al. 2014] and its proof in fact shows the central weak amenability as follows.

Theorem 2.9. *Let $\mathbb{G}$ be a free quantum group* (more generally a quantum group in [Isono 2015b, Theorem C]). *Then the dual $\widehat{\mathbb{G}}$ is centrally weakly amenable.*

*In particular there is a net $(\psi_i)_i$ of normal c.c. maps on $L^\infty(\mathbb{G})$, possessing the W*CMAP with respect to the Haar state, such that $\psi_i(L^\infty(\mathbb{G})) \subset \mathscr{C}(\mathbb{G})$ for all i.*

2E. *Popa's intertwining techniques.* Popa [2006a; 2006b] introduced a powerful tool called *intertwining techniques*. This is one of the main ingredients in the recent development of the von Neumann algebra theory. Here we introduce the one defined and studied in [Houdayer and Isono 2017, Definition 4.1 and Theorem 4.3] which treats general von Neumann algebras.

Definition 2.10. Let M be any σ-finite von Neumann algebra, 1_A and 1_B any nonzero projections in M, $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras with expectation. We say that A *embeds with expectation into B inside M* and write $A \preceq_M B$ if there exist projections $e \in A$ and $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that the inclusion $\theta(eAe) \subset fBf$ is with expectation and $av = v\theta(a)$ for all $a \in eAe$.

Theorem 2.11. *Keep the same notation as in Definition 2.10 and assume that A is finite. Then the following conditions are equivalent*:

(1) *We have* $A \preceq_M B$.

(2) *There exists no net* $(w_i)_{i\in I}$ *of unitaries in* $\mathcal{U}(A)$ *such that* $E_B(b^* w_i a) \to 0$ *in the* σ-$*$-*strong topology for all* $a, b \in 1_A M 1_B$, *where* E_B *is a fixed faithful normal conditional expectation from* $1_B M 1_B$ *onto* B.

For the proof of Corollary B, we prove a lemma. In the proof below, we make use of the *ultraproduct* von Neumann algebras [Ocneanu 1985]. We will actually use a more general one used in [Houdayer and Isono 2017], which treats a general directed set instead of $\mathbb{N}$. Recall from Section 2 of that paper that for any σ-finite von Neumann algebra M and any free ultrafilter $\mathcal{U}$ on a directed set I, we may define the *ultraproduct von Neumann algebra* $M^{\mathcal{U}}$, using $\ell^\infty(I) \mathbin{\overline{\otimes}} M$. In the proof below, we only need the following elementary properties: with the standard notation $(x_i)_{\mathcal{U}} \in M^{\mathcal{U}}$ for $(x_i)_{i\in I}$:

- $M \subset M^{\mathcal{U}}$ is with expectation by $E_{\mathcal{U}}((x_i)_{\mathcal{U}}) := \lim_{i\to\mathcal{U}} x_i$.
- For any σ-finite von Neumann algebras $A \subset M$ with expectation E_A, $A^{\mathcal{U}} \subset M^{\mathcal{U}}$ is with expectation defined by $E_{A^{\mathcal{U}}}((x_i)_{\mathcal{U}}) := (E_A(x_i))_{\mathcal{U}}$.
- If the subalgebra A is finite, then any norm bounded net $(a_i)_{i\in I}$ determines an element $(a_i)_{\mathcal{U}}$ in $M^{\mathcal{U}}$.

Lemma 2.12. *Let* (B, φ_B) *and* (N, φ_N) *be von Neumann algebras with faithful normal states. Put* $M := B \mathbin{\overline{\otimes}} N$, $\varphi := \varphi_B \otimes \varphi_N$, $E_B = \mathrm{id}_B \otimes \varphi_N$ *and* $E_N = \varphi_B \otimes \mathrm{id}_N$. *Let* $p \in M$ *be a projection and* $A \subset pMp$ *a von Neumann subalgebra with expectation. Fix* $a := (a_i)_{i\in I} \in \ell^\infty(I) \mathbin{\overline{\otimes}} A$ *and a free ultrafilter* $\mathcal{U}$ *on I such that* $(a_i)_{\mathcal{U}} \in A^{\mathcal{U}}$. *Then* $E_{B^{\mathcal{U}}}(y^* a x) = 0$ *for all* $x, y \in M$ *if and only if* $E_N \circ E_{\mathcal{U}}(c^* a b)$ *for all* $b, c \in B^{\mathcal{U}}$.

In particular, if A is finite, then $A \preceq_M B$ *if and only if* $A \preceq_{B \mathbin{\overline{\otimes}} N_0} B$ *for any* $N_0 \subset N$ *with expectation* E_{N_0} *such that* $\varphi_N \circ E_{N_0} = \varphi_N$, $p \in B \mathbin{\overline{\otimes}} N_0$ *and* $A \subset p(B \mathbin{\overline{\otimes}} N_0)p$.

Proof. Observe first that $E_{B^{\mathcal{U}}}(y^* a x) = 0$ for all $x, y \in M$ if and only if $E_{B^{\mathcal{U}}}((1 \otimes y^*) a (1 \otimes x)) = 0$ for all $x, y \in N$, which is equivalent to

$$\left\langle E_{B^{\mathcal{U}}}((1 \otimes y^*) a (1 \otimes x)) \Lambda_{\varphi_B^{\mathcal{U}}}(b),\, \Lambda_{\varphi_B^{\mathcal{U}}}(c) \right\rangle_{\varphi_B^{\mathcal{U}}} = 0$$

for all $x, y \in N$ and $b, c \in B^{\mathcal{U}}$. Writing $b = (b_i)_{\mathcal{U}}$ and $c = (c_i)_{\mathcal{U}}$, we calculate that

$$\begin{aligned}\big\langle E_{B^{\mathcal{U}}}((1\otimes y^*)a(1\otimes x))\Lambda_{\varphi_B^{\mathcal{U}}}(b), \Lambda_{\varphi_B^{\mathcal{U}}}(c)\big\rangle_{\varphi_B^{\mathcal{U}}} &= \lim_{i\to\mathcal{U}}\big\langle E_B((1\otimes y^*)a_i(1\otimes x))\Lambda_{\varphi_B}(b_i), \Lambda_{\varphi_B}(c_i)\big\rangle_{\varphi_B} \\ &= \lim_{i\to\mathcal{U}} \varphi_B\big(c_i^* E_B((1\otimes y^*)a_i(1\otimes x))b_i\big) \\ &= \lim_{i\to\mathcal{U}} \varphi_B\circ E_B((c_i^*\otimes y^*)a_i(b_i\otimes x)) \\ &= \lim_{i\to\mathcal{U}} \varphi_N\circ E_N((c_i^*\otimes y^*)a_i(b_i\otimes x)) \\ &= \lim_{i\to\mathcal{U}} \varphi_N\big(y^* E_N((c_i^*\otimes 1)a_i(b_i\otimes 1))x\big) \\ &= \varphi_N\big(y^* E_N(\lim_{i\to\mathcal{U}}((c_i^*\otimes 1)a_i(b_i\otimes 1)))x\big) \\ &= \varphi_N\big(y^* E_N\circ E_{\mathcal{U}}((c^*\otimes 1)a(b\otimes 1))x\big).\end{aligned}$$

Then since functionals of the form $\varphi_N(y^*\cdot x)$ for $x, y \in N$ are norm dense in N_*, the final term above is zero for all $x, y \in N$ if and only if $E_N\circ E_{\mathcal{U}}((c^*\otimes 1)a(b\otimes 1)) = 0$. Thus we proved that $E_{B^{\mathcal{U}}}(y^*ax) = 0$ for all $x, y \in M$ if and only if $E_N\circ E_{\mathcal{U}}((c^*\otimes 1)a(b\otimes 1)) = 0$ for all $b, c \in B^{\mathcal{U}}$.

For the second half of the statement, suppose that A is finite and $A \not\preceq_{B\overline{\otimes}N_0} B$. We will show $A \not\preceq_M B$. Since A is finite, there is a net $(u_i)_{i\in I} \subset \mathcal{U}(A)$ for a directed set I such that $E_B(y^*u_ix) \to 0$ strongly as $i\to\infty$ for all $x, y \in B\overline{\otimes}N_0$. Fix any cofinal ultrafilter $\mathcal{U}$ on I. Since A is finite, $u := (u_i)_{\mathcal{U}} \in A^{\mathcal{U}}$ and hence $E_{B^{\mathcal{U}}}(y^*ux) = 0$ for all $x, y \in B\overline{\otimes}N_0$. By the first half of the statement, this is equivalent to $E_{N_0}\circ E_{\mathcal{U}}(c^*ub) = 0$ for all $b, c \in B^{\mathcal{U}}$. Then since $E_{\mathcal{U}}(c^*ub)$ is contained in $B\overline{\otimes}N_0$ and since $E_N|_{B\overline{\otimes}N_0} = (\varphi_B\otimes \mathrm{id}_N)|_{B\overline{\otimes}N_0} = E_{N_0}$, we have $E_N\circ E_{\mathcal{U}}(c^*ub) = 0$ for all $b, c \in B^{\mathcal{U}}$, which is in turn equivalent to $E_{B^{\mathcal{U}}}(y^*ux) = 0$ for $x, y \in M$ by the first half of the statement. Since this holds for arbitrary $\mathcal{U}$ on I, we conclude that $E_B(y^*u_ix) \to 0$ $*$-strongly as $i\to\infty$ for all $x, y \in M$. Thus we proved that $A \not\preceq_{B\overline{\otimes}N_0} B$ implies $A \not\preceq_M B$. ☐

3. Weakly compact actions

In this section, we define and study weakly compact actions on continuous cores. The main observation is Theorem 3.10, and the key item for the proof is Lemma 3.3.

3A. ***Relative amenability and approximation maps.*** In this subsection, we recall relative amenability for general von Neumann algebras introduced in [Isono 2017], which generalizes [Ozawa and Popa 2010; Popa and Vaes 2014a].

Definition 3.1. Let $B \subset M$ be von Neumann algebras, $p \in M$ a projection and $A \subset pMp$ a von Neumann subalgebra with expectation E_A. We say that the pair (A, E_A) *is injective relative to B in M*, and write $(A, E_A) \lessdot_M B$, if there exists a conditional expectation from $p\langle M, B\rangle p$ onto A which restricts to E_A on pMp.

Using amenability of $\mathbb{R}$ and the notion of relative amenability, we prove a lemma for approximation maps on the continuous core. For this we fix the following notation.

Let (M, φ) be a von Neumann algebra with a faithful normal semifinite weight, and $\widetilde{M} := M \rtimes \mathbb{R}$ the continuous core of M with the modular action σ^{φ}. We denote by $\hat{\varphi}$ the dual weight of φ, and by E_M the canonical operator-valued weight from $\widetilde{M}$ to M given by $\hat{\varphi} = \varphi \circ E_M$. We denote by $M \rtimes_{\mathrm{alg}} G$ all the linear spans of $x\lambda_t$ for $x \in M$ and $t \in G$, which is a $*$-strongly dense subalgebra in $\widetilde{M}$.

Lemma 3.2. *In this setting, we have*

$$_{\widetilde{M}}L^2(\widetilde{M})_{\widetilde{M}} \prec {}_{\widetilde{M}}L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M})_{\widetilde{M}}.$$

Proof. Recall first that

$$M \rtimes \mathbb{R} = (M^\circ \otimes 1)' \cap \{\Delta_\varphi^{it} \otimes \rho_t \mid t \in \mathbb{R}\}', \quad \langle M \rtimes \mathbb{R}, M\rangle = (M^\circ \otimes 1)',$$

where ρ is the right regular representation. Since $\mathbb{R}$ is amenable, there are positive functionals $(f_n)_n \subset L^1(\mathbb{R})$ with $\|f_n\|_1 = 1$ satisfying $\lambda_g f_n - f_n \to 0$ weakly for all $g \in \mathbb{R}$. For each n, define a positive map

$$F_n : \mathbb{B}(L^2(M) \otimes L^2(\mathbb{R})) \to \mathbb{B}(L^2(M) \otimes L^2(\mathbb{R}))$$

by

$$F_n(T) := \int_{\mathbb{R}} (\Delta_\varphi^{it} \otimes \rho_t)\, T\, (\Delta_\varphi^{it} \otimes \rho_t)^* f_n(t) \cdot dt.$$

Since $\|F_n\| = 1$, we can take a cluster point of $(F_n)_n$, which we write as F. Then it satisfies

$$(\Delta_\varphi^{it} \otimes \rho_t) F(T) (\Delta_\varphi^{it} \otimes \rho_t)^* = F(T)$$

for all $t \in \mathbb{R}$ and hence F is a conditional expectation onto $\{\Delta_\varphi^{it} \otimes \rho_t \mid t \in \mathbb{R}\}'$. It is easy to see that $F(T) \in (M^\circ \otimes 1)'$ for any $T \in (M^\circ \otimes 1)'$. Hence F restricts to a conditional expectation from $\langle M \rtimes \mathbb{R}, M\rangle$ onto $M \rtimes \mathbb{R}$. We obtain $(M \rtimes \mathbb{R}, \mathrm{id}) \lessdot_{M \rtimes \mathbb{R}} M$. Finally since $M \rtimes \mathbb{R}$ is semifinite, using [Isono 2017, Theorem A.5], we get the conclusion. □

Lemma 3.3. *In this setting, there is a net $(\omega_j)_j$ of c.c.p. maps on $\widetilde{M}$ such that $\omega_j \to \mathrm{id}_{\widetilde{M}}$ point σ-weakly and each ω_j is a finite sum of $\lambda_q^* E_M(z^* \cdot y)\lambda_p$ for some $y, z \in \mathfrak{n}_{E_M}$ and $p, q \in \mathbb{R}$.*

Proof. By Lemma 3.2 and Proposition 2.4, there is a net $(\omega_j)_j$ of c.c.p. maps on $\widetilde{M}$ such that $\omega_j \to \mathrm{id}_{\widetilde{M}}$ point σ-weakly and each ω_j is a finite sum of $\langle \eta, \cdot\, \eta\rangle_{X(L^2(\widetilde{M})\otimes_M L^2(\widetilde{M}))}$ for some $\eta \in X(L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M}))$. We first replace each η in ω_j with some "algebraic" element in $X(L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M}))$.

By Lemma 2.5, the self dual completion X of $\bar{\mathfrak{n}}_{E_M} \otimes_{\mathrm{alg}} \widetilde{M}$ is identified as the one corresponding to $L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M})$. We denote by X_0 the image of $\bar{\mathfrak{n}}_{E_M} \otimes_{\mathrm{alg}} \widetilde{M}$ in X. By [Paschke 1976, Lemma 2.3], $X_0 \subset X$ is dense in the *s-topology*; that is, for any $\eta \in X$ there is a net $(\eta_i)_i \subset X_0$ such that $\langle \eta - \eta_i, \eta - \eta_i\rangle_X \to 0$ in the σ-weak topology in $\widetilde{M}$. In our case, since $\mathfrak{n}_{E_B} \subset \bar{\mathfrak{n}}_{E_B}$ is dense in the s-topology and since $M \rtimes_{\mathrm{alg}} G \subset \widetilde{M}$ is $*$-strongly dense, the image of $\mathfrak{n}_{E_M} \otimes_{\mathrm{alg}} (M \rtimes_{\mathrm{alg}} G)$ in X is dense in the s-topology. Hence we may replace each vector $\eta \in X$, appearing in ω_j above, with the one represented by elements in $\mathfrak{n}_{E_M} \otimes_{\mathrm{alg}} (M \rtimes_{\mathrm{alg}} G)$.

Thus, we may assume that each ω_j is a finite sum of $\lambda_q^* E_M(z^* \cdot y)\lambda_p$ for some $y, z \in \mathfrak{n}_{E_M}$ and $p, q \in \mathbb{R}$. However the completely bounded (c.b.) norms of the resulting net $(\omega_j)_j$ are no longer uniformly bounded.

So we have to again replace $(\omega_j)_j$ with c.c.p. maps. For this, we assume that, up to convex combinations, the convergence $\omega_j \to \mathrm{id}_{\widetilde{M}}$ is in the point strong topology.

Recall from (the first half of) the proof of [Anantharaman-Delaroche 1990, Lemma 2.2] that if we put $\varphi_i(x) := c_j\omega_j(x)c_j$ for $x \in \widetilde{M}$, where $c_j := 2(1+\omega_j(1))^{-1}$, then the net $(\varphi_i)_i$ satisfies that each φ_i is c.c.p. and that $\varphi_i \to \mathrm{id}_{\widetilde{M}}$ in the point strong topology. We will replace c_j with elements in $M \rtimes_{\mathrm{alg}} G$. For this, fix j and observe that, since $1+\omega_j(1)$ is in $M \rtimes_{\mathrm{alg}} G$, each c_j is actually contained in $\mathrm{C}^*\{M \rtimes_{\mathrm{alg}} G\}$, which is the norm closure of $M \rtimes_{\mathrm{alg}} G$. So there is a sequence $(a_n)_n$ in $M \rtimes_{\mathrm{alg}} G$ such that $\|a_n\|_\infty \le \|c_j^{1/2}\|_\infty$ and $\|a_n - c_j^{1/2}\|_\infty \to 0$. Put $b_n := a_n^* a_n \in M \rtimes_{\mathrm{alg}} G$ and observe that it satisfies $\|b_n\|_\infty \le \|c_j\|_\infty$ and $\|b_n - c_j\|_\infty \to 0$. It then holds that for any $x \in \widetilde{M}$,

$$\|c_j\omega_j(x)c_j - b_n\omega_j(x)b_n\|_\infty \le 2\|c_j\|_\infty \,\|\omega_j\|_{\mathrm{cb}}\, \|x\|_\infty \,\|c_j - b_n\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Now fix any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset (\widetilde{M})_1$ such that $1 \in \mathcal{F}$, and choose b_n such that

$$\|c_j\omega_j(x)c_j - b_n\omega_j(x)b_n\|_\infty < \varepsilon$$

for all $x \in \mathcal{F}$. Then since $1 \in \mathcal{F}$, we have

$$\|b_n\omega_j(\,\cdot\,)b_n\|_{\mathrm{cb}} = \|b_n\omega_j(1)b_n\|_\infty < \|c_j\omega_j(1)c_j\|_\infty + \varepsilon \le 1 + \varepsilon.$$

So $(1+\varepsilon)^{-1}b_n\omega_j(\,\cdot\,)b_n$ is a c.c.p. map which is still close to $c_j\omega_j(\,\cdot\,)c_j$ on $\mathcal{F}$. Thus we proved that for any j there is a net of c.c.p. maps converging to $c_j\omega_j(\,\cdot\,)c_j$ in the point *norm* topology such that each map is a finite sum of $\lambda_q^* E_M(z^* \cdot y)\lambda_p$ for some $y, z \in \mathfrak{n}_{E_M}$ and $p, q \in G$. Using this observation, since $c_j\omega_j(\,\cdot\,)c_j \to \mathrm{id}_{\widetilde{M}}$ as $j \to \infty$ in the point strong topology, it is easy to construct a desired net. □

3B. ***Definition of weakly compact actions.*** We introduce the following notion, which is an appropriate generalization of [Ozawa and Popa 2010, Definition 3.1] in our setting; see also [Popa and Vaes 2014a, Theorem 5.1]. Indeed, in the definition below, if we take $\mathcal{M} = M \mathbin{\overline{\otimes}} M^\circ$, this coincides with the original definition of weakly compact actions.

Definition 3.4. Let M be a semifinite von Neumann algebra with trace Tr, and let $\mathcal{M}$ be a von Neumann algebra which contains M and M° as von Neumann subalgebras, which we denote by $\pi(M)$ and $\theta(M^\circ)$, such that $[\pi(M), \theta(M^\circ)] = 0$.

Let $p \in M$ be a projection with $\mathrm{Tr}(p) = 1$, $A \subset pMp$ be a von Neumann subalgebra, and $\mathcal{G} \le \mathcal{N}_{pMp}(A)$ a subgroup. We say that the adjoint action of $\mathcal{G}$ on A is *weakly compact for* $(M, \mathrm{Tr}, \pi, \theta, \mathcal{M})$ if there is a net $(\xi_i)_i$ of unit vectors in the positive cone of $L^2(\mathcal{M})$ such that

(i) $\langle \pi(x)\xi_i, \xi_i\rangle_{L^2(\mathcal{M})} \to \mathrm{Tr}(pxp)$ for any $x \in M$;

(ii) $\|\pi(a)\theta(\bar{a})\xi_i - \xi_i\|_{L^2(\mathcal{M})} \to 0$ for any $a \in \mathcal{U}(A)$;

(iii) $\|\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\xi_i - \xi_i\|_{L^2(\mathcal{M})} \to 0$ for any $u \in \mathcal{G}$.

Here $\bar{a}$ means $(a^\circ)^*$ and $\mathcal{J}_{\mathcal{M}}$ is the modular conjugation for $L^2(\mathcal{M})$.

Remark 3.5. In this definition, since $\mathcal{J}_{\mathcal{M}}\xi_i = \xi_i$ for all i, condition (ii) for $a \in \mathcal{U}(A)$ implies condition (iii) for $a \in \mathcal{U}(A)$. Hence up to replacing $\mathcal{G}$ with the group generated by $\mathcal{U}(A)$ and $\mathcal{G}$, we may always assume that $\mathcal{G}$ contains $\mathcal{U}(A)$.

Below we record a characterization for weakly compact actions.

Proposition 3.6. *Keep the notation in Definition 3.4. The following conditions are equivalent*:

(1) *The group $\mathcal{G}$ acts on A as a weakly compact action for* $(M, \mathrm{Tr}, \pi, \theta, \mathcal{M})$.

(2) *There exists a net $(\omega_i)_i$ of normal states on $\mathcal{M}$ such that*

  (i) $\omega_i(\pi(x)) \to \mathrm{Tr}(pxp)$ *for any $x \in pMp$;*

  (ii) $\omega_i(\pi(a)\theta(\bar{a})) \to 1$ *for any $a \in \mathcal{U}(A)$;*

  (iii) $\|\omega_i \circ \mathrm{Ad}(\pi(u)\theta(\bar{u})) - \omega_i\| \to 0$ *for any $u \in \mathcal{G}$.*

(3) *There is a $\mathcal{G}$-central state ω on $\mathcal{M}$ such that for any $x \in M$ and $a \in \mathcal{U}(A)$*

$$\omega(x) = \mathrm{Tr}(pxp) \quad \textit{and} \quad \omega(\pi(a)\theta(\bar{a})) = 1.$$

(4) *There is a state Ω on $\mathbb{B}(L^2(\mathcal{M}))$ such that for any $x \in M$, $a \in \mathcal{U}(A)$ and $u \in \mathcal{G}$,*

$$\Omega(x) = \mathrm{Tr}(pxp), \quad \Omega(\pi(a)\theta(\bar{a})) = 1, \quad \textit{and} \quad \Omega((\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}) = 1.$$

Proof. This theorem follows from well-known arguments; see, e.g., the proof of [Ozawa and Popa 2010, Theorem 2.1]. So we give a sketch of proofs.

If (1) holds, then put $\Omega := \mathrm{Lim}_i \langle \cdot\, \xi_i, \xi_i \rangle_{L^2(\mathcal{M})}$ and obtain (4). If (4) holds, then the restriction of Ω on $\mathcal{M}$ gives (3). If (3) holds, then we can approximate ω by a net of normal states $(\omega_i)_i \subset \mathcal{M}_*$ weakly. Then by the Hahn–Banach separation theorem, up to convex combinations, we may assume that the convergence is in the norm and obtain (2). Finally if (2) holds, then for each i one can find a unique $\xi_i \in L^2(\mathcal{M})$ which is in the positive cone such that $\omega_i = \langle \cdot\, \xi_i, \xi_i \rangle_{L^2(\mathcal{M})}$. By the Powers–Størmer inequality [Takesaki 1979, Theorem IX.1.2(iv)], we obtain

$$\|\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\xi_i - \xi_i\|^2 \le \|\omega_i \circ \mathrm{Ad}(\pi(u^*)\theta(u^\circ)) - \omega_i\| \to 0$$

for any $u \in \mathcal{G}$ and hence (1) holds. □

3C. ***W*CMAP with respect to a state produces approximation maps on continuous cores.*** We construct a family of approximation maps on continuous cores by assuming the W*CMAP with respect to a state.

For this, we fix the following setting. Let N and B be von Neumann algebras and φ_N and φ_B faithful normal states on N and B respectively. Put

$$M := N \mathbin{\overline{\otimes}} B, \quad \varphi := \varphi_N \otimes \varphi_B, \quad E_N := \mathrm{id}_N \otimes \varphi_B, \quad E_B := \varphi_N \otimes \mathrm{id}_B,$$

and we regard $\widetilde{B} := B \rtimes_{\sigma^{\varphi_B}} \mathbb{R}$ and $\widetilde{N} := N \rtimes_{\sigma^{\varphi_N}} \mathbb{R}$ as subalgebras of $\widetilde{M} := M \rtimes_{\sigma^{\varphi}} \mathbb{R}$. We denote by E_M the canonical operator-valued weight from $\widetilde{M}$ to M given by $\hat{\varphi} = \varphi \circ E_M$, where $\hat{\varphi}$ is the dual weight on $\widetilde{M}$. We also denote by E_B the canonical operator-valued weight from $\widetilde{M}$ to B given by $\hat{\varphi} = \varphi_B \circ E_B$.

Lemma 3.7. *Let $\omega : \widetilde{M} \to \widetilde{M}$ and $\psi : N \to N$ be c.b. maps given by*

$$\omega := \lambda_q^* E_M(z^* \cdot y)\lambda_p \quad \text{and} \quad \psi := \sum_{i=1}^{n} \varphi_N(z_i^* \cdot y_i)c_i$$

for some $p, q \in \mathbb{R}$, $y, z \in \mathfrak{n}_{E_M}$ and $c_i, y_i, z_i \in N$. Suppose $\psi \circ \sigma_t^{\varphi_N} = \sigma_t^{\varphi_N} \circ \psi$ for all $t \in \mathbb{R}$, so that the map $\tilde{\psi} := \psi \otimes \mathrm{id}_B \otimes \mathrm{id}_{L^2(\mathbb{R})}$ on $M \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$ induces the map $\widetilde{M} \to \widetilde{M}$ given by $\tilde{\psi}(x\lambda_t) = (\psi \otimes \mathrm{id}_B)(x)\lambda_t$ for $x \in M$ and $t \in \mathbb{R}$. Then the composition $\tilde{\psi} \circ \omega$ is given by

$$\tilde{\psi} \circ \omega(x) = \sum_{i=1}^{n} \lambda_q^* E_B\big(\sigma_q^{\varphi_N}(z_i^*) z^* x y \sigma_p^{\varphi_N}(y_i)\big)\lambda_p c_i, \quad x \in \widetilde{M}.$$

Proof. Recall from the proof of Lemma 2.1 that the canonical conditional expectation from $(\widetilde{M}, \hat{\varphi})$ to $(\widetilde{B}, \hat{\varphi}_B)$ is given by $E_{B \rtimes \mathbb{R}}((x \otimes b)\lambda_t) = \varphi_N(x) b \lambda_t$ for $x \in N$, $b \in B$ and $t \in \mathbb{R}$. For $x \in \widetilde{M}$, we calculate that

$$\begin{aligned}
\tilde{\psi} \circ \omega(x) &= \tilde{\psi}(\lambda_q^* E_M(z^* x y)\lambda_p) \\
&= \sum_{i=1}^{n} (\varphi_N(z_i^* \cdot y_i) \otimes \mathrm{id}_B \otimes \mathrm{id}_{L^2(\mathbb{R})})(\lambda_q^* E_M(z^* x y)\lambda_p) c_i \\
&= \sum_{i=1}^{n} E_{B \rtimes \mathbb{R}}(z_i^* \lambda_q^* E_M(z^* x y)\lambda_p y_i) c_i \\
&= \sum_{i=1}^{n} \lambda_q^* E_{B \rtimes \mathbb{R}} \circ E_M(\sigma_q^{\varphi_N}(z_i^*) z^* x y \sigma_p^{\varphi_N}(y_i))\lambda_p c_i.
\end{aligned}$$

Since $E_{B \rtimes \mathbb{R}} \circ E_M = E_B$ by Lemma 2.1, we obtain the conclusion. □

Lemma 3.8. *Suppose that N has the φ_N-W*CMAP. Then there exists a net $(\varphi_\lambda)_\lambda$ of c.c. maps on $\widetilde{M}$ such that $\varphi_\lambda \to \mathrm{id}_{\widetilde{M}}$ point σ-weakly and such that each φ_λ is a finite sum of $d^* E_B(z^* \cdot y)c$ for some $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$.*

Proof. Fix a net $(\psi_i)_i$ of normal c.c. maps on N as in Definition 2.8 and put $(\tilde{\psi}_i)_i$ as in the statement of the previous lemma. Let $(\omega_j)_j$ be a net of c.c.p. maps on $\widetilde{M}$ given by Lemma 3.3. Then by Lemma 3.7 the composition $\tilde{\psi}_i \circ \omega_j$ is a finite sum of $d^* E_B(z^* \cdot y)c$ for some $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$. Since $\lim_i(\lim_j \tilde{\psi}_i \circ \omega_j) = \mathrm{id}_{\widetilde{M}}$ in the point σ-weak topology, it is easy to show that for any finite subset $\mathcal{F} \subset \widetilde{M}$ and any σ-weak neighborhood $\mathcal{V}$ of 0, there are i and j such that $\tilde{\psi}_i \circ \omega_j(x) - x \in \mathcal{V}$ for all $x \in \mathcal{F}$. So putting this $\tilde{\psi}_i \circ \omega_j$ as $\varphi_{(\mathcal{F},\mathcal{V})}$, one can construct a desired net $(\varphi_\lambda)_\lambda := (\varphi_{(\mathcal{F},\mathcal{V})})_{(\mathcal{F},\mathcal{V})}$. □

3D. ***Relative weakly compact actions on continuous cores.*** We keep the notation from the previous subsection, such as $M = N \mathbin{\overline{\otimes}} B$ and $\varphi = \varphi_N \otimes \varphi_B$. Let Tr be an arbitrary semifinite trace on $\widetilde{M}$, $p \in \widetilde{M}$ a projection with $\mathrm{Tr}(p) = 1$, and $A \subset p\widetilde{M}p$ a von Neumann subalgebra with expectation E_A. In this subsection, we prove that under some assumptions on A and M, the normalizer of A in pMp acts on A as a weakly compact action with an appropriate representation.

Since our proof is a generalization of the one of [Popa and Vaes 2014a, Theorem 5.1], we make use of the following notation, which is similar to notation used in that theorem:

$$\begin{aligned}
H &:= L^2(\widetilde{M}, \hat{\varphi}) \otimes_B L^2(\widetilde{M}, \mathrm{Tr}), \text{ with left, right actions } \pi_H, \theta_H, \\
\mathcal{M}_H &:= \mathrm{W}^*\{\pi_H(\widetilde{M}), \theta_H(\widetilde{M}^\circ)\} \subset \mathbb{B}(H), \\
\mathcal{H} &:= (\theta_H(p)H) \otimes_A pL^2(\widetilde{M}, \mathrm{Tr}), \\
\pi_{\mathcal{H}} &: \widetilde{M} \ni x \mapsto (x \otimes_B p^\circ) \otimes_A p \in \mathbb{B}(\mathcal{H}), \\
\theta_{\mathcal{H}} &: \widetilde{M}^\circ \ni y^\circ \mapsto (1 \otimes_B p^\circ) \otimes_A y^\circ \in \mathbb{B}(\mathcal{H}), \\
\mathcal{M} &:= \mathrm{W}^*\{\pi_{\mathcal{H}}(\widetilde{M}), \theta_{\mathcal{H}}(\widetilde{M}^\circ)\} \subset \mathbb{B}(\mathcal{H}).
\end{aligned}$$

As we observed in Proposition 3.6, we actually use the weakly compact action with the standard representation of $\mathcal{M}$. So we first observe that $\mathcal{M}$ admits a useful identification as a crossed product, and so its standard representation is taken as a simple form.

Lemma 3.9. *Let $X \subset \mathcal{M}$ be the von Neumann subalgebra generated by $\pi_{\mathcal{H}}(B)$ and $\theta_{\mathcal{H}}(\widetilde{M}^\circ)$, and let $X \subset \mathbb{B}(L^2(X))$ be a standard representation, so that B and $\widetilde{M}^\circ$ acts on $L^2(X)$. Then $\mathcal{M}$ is isomorphic to the crossed product von Neumann algebra $\mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ by the diagonal action $\sigma^{\varphi_N} \otimes \alpha^X$, where α^X is given by $\alpha_t^X(\pi_{\mathcal{H}}(b)\theta_{\mathcal{H}}(y^\circ)) = \pi_{\mathcal{H}}(\sigma_t^{\varphi_B}(b))\theta_{\mathcal{H}}(y^\circ)$ for $t \in \mathbb{R}$, $b \in B$, and $y \in \widetilde{M}$.*

In particular the standard representation of $\mathcal{M}$ is given by $L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(X)$ with the following representation: for any $\xi \in L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(X) = L^2(\mathbb{R}, L^2(N) \otimes L^2(X))$ and $s \in \mathbb{R}$,

$$\begin{aligned}
&L\mathbb{R} \ni \lambda_t \mapsto \lambda_t \otimes 1_N \otimes 1_X, && ((\lambda_t \otimes 1_N \otimes 1_X)\xi)(s) := \xi(s-t), \\
&N \ni x \mapsto \pi_{\sigma^{\varphi_N}}(x) \otimes 1_X, && ((\pi_{\sigma^{\varphi_N}}(x) \otimes 1_X)\xi)(s) := (\sigma_{-s}^{\varphi_N}(x) \otimes 1_X)\xi(s), \\
&B \ni b \mapsto \pi_{\sigma^{\varphi_B}}(b)_{13}, && ((\pi_{\sigma^{\varphi_B}}(b)_{13})\xi)(s) := (1_N \otimes \sigma_{-s}^{\varphi_B}(b))\xi(s), \\
&\widetilde{M}^\circ \ni y^\circ \mapsto 1_{L^2(\mathbb{R})} \otimes 1_N \otimes y^\circ, && ((1_{\mathbb{R}} \otimes 1_N \otimes y^\circ)\xi)(s) := (1_N \otimes y^\circ)\xi(s).
\end{aligned}$$

Proof. By Proposition 2.3, H is isomorphic to $L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})$. Since the right $\widetilde{M}$-action acts only on the right three Hilbert spaces, the Hilbert space $\mathcal{H} = H \otimes_A pL^2(\widetilde{M}, \mathrm{Tr})$ is identified as $L^2(\mathbb{R}) \otimes L^2(N) \otimes K$, where

$$K := \theta_H(p^\circ)(L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})) \otimes_A pL^2(\widetilde{M}, \mathrm{Tr}).$$

Note that $\widetilde{M}^\circ$ acts on K by $\theta_{\mathcal{H}}$, and B acts on $L^2(\mathbb{R}) \otimes K$ by $\pi_{\mathcal{H}}$, so that X acts on $L^2(\mathbb{R}) \otimes K$. More precisely we have $X \subset L^\infty(\mathbb{R}) \mathbin{\overline{\otimes}} \mathbb{C}1_N \mathbin{\overline{\otimes}} \mathbb{B}(K)$.

Let W be a unitary on $L^2(\mathbb{R}) \otimes L^2(N)$ given by $(W\xi)(t) := \Delta_{\varphi_N}^{it}\xi(t)$ for $t \in \mathbb{R}$ and $\xi \in L^2(\mathbb{R}) \otimes L^2(N) = L^2(\mathbb{R}, L^2(N))$. It satisfies that for any $f \in L^\infty(\mathbb{R})$, $t \in \mathbb{R}$, and $x \in N$,

$$W\pi_{\sigma^{\varphi_N}}(x)W^* = 1_{L^2(\mathbb{R})} \otimes x, \quad W(\lambda_t \otimes 1_N)W^* = \lambda_t \otimes \Delta_{\varphi_N}^{it}, \quad \text{and} \quad W(f \otimes 1_N)W^* = f \otimes 1_N.$$

Let next V be a unitary on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ defined similarly to W exchanging $\Delta_{\varphi_N}^{it}$ with λ_t, so that it satisfies for $t \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R})$,

$$V(1 \otimes \lambda_t)V^* = \lambda_t \otimes \lambda_t \quad \text{and} \quad V(1 \otimes f)V^* = 1 \otimes f.$$

Define then a unitary on $L^2(\mathbb{R}) \otimes \mathcal{H}$ by $U := (V \otimes 1_N \otimes 1_K)(1_{L^2(\mathbb{R})} \otimes W \otimes 1_K)$. One can show that $\operatorname{Ad} U = \mathrm{id}$ on $\mathbb{C}1_{L^2(\mathbb{R})} \otimes X \subset \mathbb{C}1_{L^2(\mathbb{R})} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}) \mathbin{\overline{\otimes}} \mathbb{C}1_N \mathbin{\overline{\otimes}} \mathbb{B}(K)$, and

$$\begin{aligned} \operatorname{Ad} U(1_{L^2(\mathbb{R})} \otimes \lambda_t \otimes 1_N \otimes 1_K) &= (\lambda_t \otimes \lambda_t \otimes \Delta_{\varphi_N}^{it} \otimes 1_K) && \text{for } t \in \mathbb{R}, \\ \operatorname{Ad} U(1_{L^2(\mathbb{R})} \otimes \pi_{\sigma^{\varphi_N}}(x) \otimes 1_K) &= (1_{L^2(\mathbb{R})} \otimes 1_{L^2(\mathbb{R})} \otimes x \otimes 1_K) && \text{for } x \in N. \end{aligned}$$

Then $\operatorname{Ad} U(\mathcal{M})$ is identified as the crossed product von Neumann algebra $\mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ given by the $\mathbb{R}$-action $\sigma^{\varphi_N} \otimes \alpha^X$, where α^X is given by $\operatorname{Ad}(\lambda_t \otimes 1_N \otimes 1_K)$ using $X \subset L^\infty(\mathbb{R}) \otimes \mathbb{C}1_N \otimes \mathbb{B}(K)$, which is exactly the action given in the statement. Finally one can choose the standard representation of $\mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ as in the statement and we can end the proof. □

Now we prove the main observation of this section. This is a generalization of [Ozawa and Popa 2010, Theorem 3.5] and [Popa and Vaes 2014a, Theorem 5.1]. Since we already obtained approximation maps for $\widetilde{M}$ in Lemma 3.8, which are "relative to B", almost the same arguments as the above-cited theorems work. However, since our approximation maps are not defined directly on $\mathcal{M}_H$, we need a stronger assumption on the subalgebra A; namely, we need *amenability*, instead of relative amenability. See Step 1 in the proof below and observe that we really need amenability for a subalgebra $Q \subset pMp$.

Theorem 3.10. *Keep the setting above and suppose the following conditions*:

- *The algebra B is a type III_1 factor.*
- *The algebra A is amenable.*
- *The algebra N has the φ_N-W*CMAP.*

Then $\mathcal{N}_{p\widetilde{M}p}(A)$ acts on A as a weakly compact action for $(\widetilde{M}, \mathrm{Tr}, \pi_{\mathcal{H}}, \theta_{\mathcal{H}}, \mathcal{M})$.

Proof. The proof consists of several steps. For any von Neumann subalgebra $Q \subset p\widetilde{M}p$, we denote by $\mathcal{C}_{H,Q}$ (resp. $\mathcal{M}_{H,Q}$) the C*-algebra (resp. the von Neumann algebra) generated by $\pi_H(p\widetilde{M}p)\theta_H(Q^\circ)$.

Step 1. Using the φ_N-W*CMAP of N, we construct a net of normal functionals on $\mathcal{M}_H$ which are contractive on $\mathcal{M}_{H,Q}$ for any amenable Q.

In this step, we show that there is a net $(\mu_i)_i$ of normal functional on $\mathcal{M}_H$ such that

- $\mu_i(\pi_H(a)\theta_H(b^\circ)) = \mathrm{Tr}(p\varphi_i(a)pbp)$ for all $a, b \in \widetilde{M}$,
- we have $\|\mu_i|_{\mathcal{M}_{H,Q}}\| \le 1$ for any amenable von Neumann subalgebra $Q \subset p\widetilde{M}p$.

By Lemma 3.8, there exists a net $(\varphi_i)_i$ of c.c. maps on $\widetilde{M}$ such that $\varphi_i \to \mathrm{id}_{\widetilde{M}}$ point σ-weakly and that each φ_i is a finite sum of $d^* E_B(z^* \cdot y)c$ for $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$. Observe that for any functional $d^* E_B(z^* \cdot y)c$ for some $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$, one can define an associated normal functional on $\mathcal{M}_H$ by

$$\mathcal{M}_H \ni T \mapsto \left\langle T(\Lambda_{\hat{\varphi}}(y) \otimes_B \Lambda_{\mathrm{Tr}}(cp)), \Lambda_{\hat{\varphi}}(z) \otimes_B \Lambda_{\mathrm{Tr}}(dp)\right\rangle_H.$$

In this way, since φ_i is a finite sum of such maps, one can associate each φ_i with a normal functional on $\mathcal{M}_H$, which we denote by μ_i. Then by the formula $L^*_{\Lambda_{\hat{\varphi}}(z)} a L_{\Lambda_{\hat{\varphi}}(y)} = E_B(z^* a y)$ for $x, y \in \mathfrak{n}_{E_B} \cap \mathfrak{n}_\varphi$ and $a \in \widetilde{M}$, it is easy to verify that $\mu_i(\pi_H(a)\theta_H(b^\circ)) = \mathrm{Tr}(p\varphi_i(a)pbp)$ for $a, b \in \widetilde{M}$. We need to show that $\|\mu_i|_{\mathcal{M}_{H,Q}}\| \le 1$ for any amenable $Q \subset p\widetilde{M}p$. For this, since μ_i is normal, we have only to show that $\|\mu_i|_{\mathcal{C}_{H,Q}}\| \le 1$.

By Lemma 3.11 below, since B is a type III_1 factor, the $*$-algebra generated by $\pi_H(\widetilde{M})$ and $\theta_H(\widetilde{M}^\circ)$ is isomorphic to $\widetilde{M}\otimes_{\mathrm{alg}}\widetilde{M}^\circ$. So for any amenable $Q\subset p\widetilde{M}p$, the C*-algebra generated by $\pi_H(\widetilde{M})\theta_H(Q^\circ)$ is isomorphic to $\widetilde{M}\otimes_{\min}Q^\circ$. Hence one can define c.c. maps $\varphi_i\otimes\mathrm{id}_{Q^\circ}$ on $\mathcal{C}_{H,Q}$. Since Q is amenable, one has

$$ {}_{\widetilde{M}}L^2(\widetilde{M}p)_Q \prec {}_{\widetilde{M}}(\theta_H(p^\circ)H)_Q. $$

Finally if we denote by ν the associated $*$-homomorphism with this weak containment, then the functional $T\mapsto\langle\nu\circ(\varphi_i\otimes\mathrm{id}_{Q^\circ})(T)\Lambda_{\mathrm{Tr}}(p),\,\Lambda_{\mathrm{Tr}}(p)\rangle_{\mathrm{Tr}}$ coincides with μ_i on $\mathcal{C}_{H,Q}$, and hence we obtain $\|\mu_i|_{\mathcal{C}_{H,Q}}\|\le 1$. Thus we obtained a desired net $(\mu_i)_i$.

Step 2. Using the amenability of A, the absolute values of normal functionals $(\mu_i)_i$ constructed in Step 1 satisfy desired properties on $\mathcal{M}_{H,A}$.

Before this step, recall from the first part of the proof of [Ozawa and Popa 2010, Theorem 3.5] that for any C*-algebra C, any state ω on C and any partial isometry $u\in C$ with $p:=uu^*$ and $q:=u^*u$, one has

$$ \max\bigl\{\|\omega(\,\cdot\,u^*)-\omega(\,\cdot\,q)\|^2,\,\|\omega(u\,\cdot\,u^*)-\omega(q\,\cdot\,q)\|^2\bigr\}\le 4(\omega(p)+\omega(q)-\omega(u)-\omega(u^*)). $$

Let $(\mu_i)_i$ be a net constructed in Step 1. For notational simplicity, for any amenable von Neumann subalgebra $Q\subset p\widetilde{M}p$ we denote by μ_i^Q the restriction of μ_i on $\mathcal{M}_{H,Q}$.

Claim. *For any amenable Q, one has*

$$ \|\mu_i^Q\|\to 1 \quad\textit{and}\quad \|\mu_i^Q-|\mu_i^Q|\|\to 0, $$

where $|\mu_i^Q|$ is the absolute value of μ_i^Q.

Proof of Claim. By Step 1, we know $\|\mu_i^Q\|\le 1$ and hence $\|\mu_i^Q\|\to 1$, since $\mu_i(\pi_H(p)\theta_H(p^\circ))\to 1$. Let $\mu_i^Q=|\mu_i^Q|(\,\cdot\,u_i)$ be the polar decomposition with a partial isometry $u_i\in\mathcal{M}_{H,Q}$. For $p_i:=u_iu_i^*$ and $q_i:=u_i^*u_i$, it holds that

$$ |\mu_i^Q|=\mu_i^Q(\,\cdot\,u_i^*),\qquad |\mu_i^Q|=|\mu_i^Q|(q_i\,\cdot\,q_i),\qquad\text{and}\qquad \mu_i^Q=\mu_i^Q(\,\cdot\,p_i)=\mu_i^Q(q_i\,\cdot\,). $$

The final equation says that $\mu_i^Q(p_i)=\mu_i^Q(1_Q)\to 1$. Then by the inequality at the beginning of this step, we have

$$ \begin{aligned} \|\mu_i^Q-|\mu_i^Q|\|^2 &= \||\mu_i^Q|(\,\cdot\,u_i^*)-|\mu_i^Q|(\,\cdot\,q_i)\|^2\\ &\le 4\bigl(|\mu_i^Q|(p_i)+|\mu_i^Q|(q_i)-|\mu_i^Q|(u_i)-|\mu_i^Q|(u_i^*)\bigr)\\ &\le 4\bigl(\|\mu_i^Q\|+\|\mu_i^Q\|-2\,\mathrm{Re}(\mu_i^Q(p_i))\bigr)\to 0. \end{aligned} $$ □

Put $\omega_i:=|\mu_i^A|/\|\mu_i^A\|$. In this step, we show that $(\omega_i)_i$ satisfies the following conditions:

(1) $\omega_i(\pi_H(x)\theta_H(p^\circ))\to\mathrm{Tr}(pxp)$ for all $x\in p\widetilde{M}p$.

(2) $\omega_i(\pi_H(a)\theta_H(\bar{a}))\to 1$ for all $a\in\mathcal{U}(A)$.

(3) $\|\omega_i\circ\mathrm{Ad}(\pi_H(u)\theta_H(\bar{u}))-\omega_i\|_{\mathcal{M}_{H,A}^*}\to 0$ for all $u\in\mathcal{N}_{p\widetilde{M}p}(A)$.

Since $\|\mu_i^A\| \to 1$ and $\|\mu_i^A - |\mu_i^A|\| \to 0$, to verify these three conditions, we have only to show that $(\mu_i)_i$ satisfies the same conditions. Then by construction, it is easy to verify (i) and (ii). So we will check only the final condition.

Fix $u \in \mathcal{N}_{p\widetilde{M}p}(A)$ and recall that the von Neumann algebra A^u generated by A and u is amenable [Ozawa and Popa 2010, Lemma 3.4]. Hence by Step 1, $\||\mu_i^{A^u}| - \mu_i^{A^u}\|_{\mathcal{M}^*_{H,A^u}} \to 0$. Combined with the inequality at the beginning of this step, putting $U := \pi_H(u)\theta_H(\bar{u})$, we have

$$\begin{aligned}\lim_i \|\mu_i^A \circ \operatorname{Ad} U - \mu_i^A\|^2_{\mathcal{M}^*_{H,A}} &\le \lim_i \|\mu_i^{A^u} \circ \operatorname{Ad} U - \mu_i^{A^u}\|^2_{\mathcal{M}^*_{H,A^u}} \\ &= \lim_i \||\mu_i^{A^u}| \circ \operatorname{Ad} U - |\mu_i^{A^u}|\|^2_{\mathcal{M}^*_{H,A^u}} \\ &\le \lim_i 4(2 - 2\operatorname{Re}(|\mu_i^{A^u}|(U))) \\ &= \lim_i 4(2 - 2\operatorname{Re}(\mu_i^{A^u}(U))) = 0.\end{aligned}$$

Thus we proved that the net $(\omega_i)_i$ of normal states on $\mathcal{M}_H$ satisfies conditions (i), (ii) and (iii) above.

Step 3. Using a normal u.c.p. map from $\mathcal{M}$ to $\mathcal{M}_{H,A}$, we obtain desired functionals on $\mathcal{M}$.

In this step, we first construct a normal u.c.p. map $\mathcal{E} : \mathcal{M} \to \mathcal{M}_{H,A}$ satisfying

$$\mathcal{E}(\pi_{\mathcal{H}}(a)\theta_{\mathcal{H}}(b^\circ)) = \pi_H(pap)\theta_H(E_A(pbp)^\circ) \quad \text{for any } a, b \in \widetilde{M},$$

where E_A is the unique Tr-preserving conditional expectation from $p\widetilde{M}p$ onto A.

For this, observe first that for any right A-module K with the right action θ_K, there is an isometry $V_K : K \to K \otimes_A pL^2(\widetilde{M}, \mathrm{Tr})$ given by $V\xi = \xi \otimes_A \Lambda_{\mathrm{Tr}}(p)$ for any left Tr-bounded vector $\xi \in K$. Indeed, using the fact $\Lambda_{\mathrm{Tr}}(p) = J_{\mathrm{Tr}}\Lambda_{\mathrm{Tr}}(p)$, one has

$$\|V\xi\| = \|\xi \otimes_A \Lambda_{\mathrm{Tr}}(p)\| = \|L_\xi \Lambda_{\mathrm{Tr}}(p)\|_{2,\mathrm{Tr}} = \|L_\xi \Lambda_{\mathrm{Tr}}(p)\|_{2,\mathrm{Tr}} = \|\theta_K(p^\circ)\xi\|_K = \|\xi\|_K.$$

Hence, since $\pi_H(p)\theta_H(p^\circ)H$ is a right A-module, one can define an isometry

$$V : \pi_H(p)\theta_H(p^\circ)H \to \pi_{\mathcal{H}}(p)\theta_{\mathcal{H}}(p^\circ)\mathcal{H} \subset \mathcal{H}, \quad V\xi := \xi \otimes_A \Lambda_{\mathrm{Tr}}(p).$$

It is then easy to verify that

$$V^*\pi_{\mathcal{H}}(a)\theta_{\mathcal{H}}(b^\circ)V = \pi_H(pap)\theta_H(E_A(pbp)^\circ) \quad \text{for any } a, b \in \widetilde{M}.$$

Thus we obtain a normal u.c.p. map $\mathcal{E} : \mathcal{M} \to \mathcal{M}_{H,A}$ by $\mathcal{E}(T) := V^*TV$.

Let now $(\omega_i)_i$ be the net of normal states on $\mathcal{M}_{H,A}$ constructed in Step 2. By conditions (i) and (ii) on $(\omega_i)_i$, it is easy to see that normal states $\gamma_i := \omega_i \circ \mathcal{E}$ on $\mathcal{M}$ satisfy

(i)′ $\gamma_i(\pi_{\mathcal{H}}(x)) \to \tau(pxp)$ for all $x \in \widetilde{M}$;

(ii)′ $\gamma_i(\pi_{\mathcal{H}}(a)\theta_{\mathcal{H}}(\bar{a})) \to 1$ for all $a \in \mathcal{U}(A)$.

Finally since E_A satisfies $E_A \circ \operatorname{Ad} u = \operatorname{Ad} u \circ E_A$ for any $u \in \mathcal{N}_{p\widetilde{M}p}(A)$, one has

$$\gamma_i \circ \operatorname{Ad}(\pi_{\mathcal{H}}(u)\theta_{\mathcal{H}}(\bar{u})) = \omega_i \circ \operatorname{Ad}(\pi_{\mathcal{H}}(u)\theta_{\mathcal{H}}(\bar{u})) \circ \mathcal{E}$$

on $\pi_{\mathcal{H}}(\widetilde{M})\theta_{\mathcal{H}}(\widetilde{M})$, and hence on $\mathcal{M}$ by normality. So condition (iii) on $(\omega_i)_i$ shows

(iii)′ $\|\gamma_i \circ \mathrm{Ad}(\pi_{\mathcal{H}}(u)\theta_{\mathcal{H}}(\bar{u})) - \gamma_i\| \to 0$ for all $u \in \mathcal{N}_{p\widetilde{M}p}(A)$.

Thus the net $(\gamma_i)_i$ on $\mathcal{M}$ satisfies conditions (i)′, (ii)′ and (iii)′. By Proposition 3.6(2), we conclude that $\mathcal{N}_{p\widetilde{M}p}(A)$ acts on A weakly compactly for $(\widetilde{M}, \mathrm{Tr}, \pi_{\mathcal{H}}, \theta_{\mathcal{H}}, \mathcal{M})$. □

We prove a lemma used in the proof above.

Lemma 3.11. *Assume that B is a type* III_1 *factor. Then the $*$-algebra generated by $\pi_H(\widetilde{M})$ and $\theta_H(\widetilde{M}^\circ)$ is isomorphic to $\widetilde{M} \otimes_{\mathrm{alg}} \widetilde{M}^\circ$.*

Proof. Let $\nu : \widetilde{M} \otimes_{\mathrm{alg}} \widetilde{M}^\circ \to *\text{-alg}\{\pi_H(\widetilde{M}), \theta_H(\widetilde{M}^\circ)\}$ be a $*$-homomorphism given by $\nu(x \otimes y^\circ) = \pi_H(x)\theta_H(y^\circ)$ for $x, y \in \widetilde{M}$. We will show that ν is injective.

Assume that $\nu\big(\sum_{i=1}^n x_i \otimes y_i^\circ\big) = \sum_{i=1}^n \pi_H(x_i)\theta_H(y_i^\circ) = 0$ for some $x_i, y_i \in \widetilde{M}$. We may assume $y_i \neq 0$ for all i. Put

$$X := \begin{bmatrix} \pi_H(x_1) & \pi_H(x_2) & \cdots & \pi_H(x_n) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad Y := \begin{bmatrix} \theta_H(y_1^\circ) & 0 & \cdots & 0 \\ \theta_H(y_2^\circ) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_H(y_n^\circ) & 0 & \cdots & 0 \end{bmatrix}$$

and observe $XY = 0$. We regard them as elements in $\mathbb{B}(H) \otimes \mathbb{M}_n$. Let p be the left support projection of Y which is contained in $\theta_H(\widetilde{M}^\circ) \otimes \mathbb{M}_n$ and satisfies $Xp = 0$. Since $Xupu^* = 0$ for any unitary $u \in \mathbb{B}(H) \otimes \mathbb{M}_n$ which commutes with X, and since $\theta_H(\widetilde{M}^\circ) \otimes \mathbb{C}^n$ commutes with X (where $\mathbb{C}^n \subset \mathbb{M}_n$ is the diagonal embedding), we have $Xz = 0$ for $z := \sup\{upu^* \mid u \in \mathcal{U}(\theta_H(\widetilde{M}^\circ) \otimes \mathbb{C}^n)\}$. Observe that z is contained in

$$(\theta_H(\widetilde{M}^\circ) \otimes \mathbb{M}_n) \cap (\theta_H(\widetilde{M}^\circ) \otimes \mathbb{C}^n)' = \theta_H(\mathcal{Z}(\widetilde{M})^\circ) \otimes \mathbb{C}^n$$

and hence we can write $z = (z_i)_{i=1}^n$ for some $z_i \in \theta_H(\mathcal{Z}(\widetilde{M})^\circ)$. Then the condition $Xz = 0$ is equivalent to $\pi_H(x_i)z_i = 0$ for all i. Observe also that $z_i \neq 0$ for all i. Indeed, since $z \geq p$ and $pY = Y$, we have $zY = Y$ and hence $z_i\theta_H(y_i^\circ) = \theta_H(y_i^\circ)$. This implies $z_i \neq 0$ since we assume $y_i \neq 0$ for all i.

Now we claim that $\pi_H(x_i)z_i = 0$ is equivalent to $x_i = 0$ or $z_i = 0$. Once we prove the claim, since $z_i \neq 0$, we have $x_i = 0$ and so $\sum_{i=1}^n x_i \otimes y_i^\circ = 0$, which gives the injectivity of ν.

By Lemma 2.2, the center of $\widetilde{M}$ coincides with $\mathcal{Z}(N)$. Then by Proposition 2.3, we identify $H = L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B, \psi_B) \otimes L^2(N) \otimes L^2(\mathbb{R})$ on which we have

$$\pi_H(\widetilde{M}) \subset \mathbb{B}(L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B, \psi_B)) \otimes \mathbb{C}1_{L^2(N)\otimes L^2(\mathbb{R})},$$
$$\theta_H(\widetilde{M}^\circ) \subset \mathbb{C}1_{L^2(\mathbb{R})\otimes L^2(N)} \otimes \mathbb{B}(L^2(B, \psi_B) \otimes L^2(N) \otimes L^2(\mathbb{R})).$$

In particular $\theta_H(\mathcal{Z}(\widetilde{M})^\circ) = \theta_H(\mathcal{Z}(N)) \subset \mathbb{C}1_{L^2(\mathbb{R})\otimes L^2(N)\otimes L^2(B,\psi_B)} \otimes \mathbb{B}(L^2(N) \otimes L^2(\mathbb{R}))$, and hence the C*-algebra generated by $\pi_H(\widetilde{M})$ and $\theta_H(\mathcal{Z}(\widetilde{M})^\circ)$ is isomorphic to $\widetilde{M} \otimes_{\min} \mathcal{Z}(\widetilde{M})^\circ$. Thus since $z_i \in \theta_H(\mathcal{Z}(\widetilde{M})^\circ)$, the condition $\pi_H(x_i)z_i = 0$ is equivalent to $x_i = 0$ or $z_i = 0$. □

4. Proof of Theorem A

To prove Theorem A we follow the proof of [Isono 2015b, Theorem B], which originally comes from the one of [Popa and Vaes 2014b, Theorem 1.4].

4A. ***Some general lemmas.*** Let $\mathbb{G}$ be a compact quantum group with the Haar state h and put $N_0 := C_{\mathrm{red}}(\mathbb{G}) \subset L^\infty(G) =: N$ and $\varphi_N := h$. Let (X, φ_X) be a von Neumann algebra with a faithful normal semifinite weight. Let α^X be an action of $\mathbb{R}$ on X and put $\alpha := \sigma^{\varphi_N} \otimes \alpha^X$ and $\mathcal{M} := (N \mathbin{\overline{\otimes}} X) \rtimes_\alpha \mathbb{R}$.

In this setting, we prove two general lemmas. We use the following general fact for quantum groups.

- For any $x \in \mathrm{Irred}(\mathbb{G})$, there is an orthonormal basis $\{u^x_{i,j}\}_{i,j} \subset C_{\mathrm{red}}(\mathbb{G})$ of H_x with $\lambda^x_{i,j} > 0$ such that $\sigma^h_t(u^x_{i,j}) = \lambda^x_{i,j} u^x_{i,j}$ for all $t \in \mathbb{R}$.

Recall that all the linear spans of such a basis, which is usually called a dense Hopf $*$-algebra, make a norm-dense $*$-subalgebra of $C_{\mathrm{red}}(\mathbb{G})$. We note that each matrix $(u^x_{i,j})_{i,j}$ may not be a unitary, since we assume $\{u^x_{i,j}\}_{i,j}$ is orthonormal (i.e., they are normalized).

Convention. Throughout this section, we fix such a basis $\{u^x_{i,j}\}^x_{i,j}$. For notation simplicity, we identify any subset $\mathcal{E} \subset \mathrm{Irred}(\mathbb{G})$ (possibly $\mathcal{E} = \mathrm{Irred}(\mathbb{G})$) with the set $\{u^x_{i,j} \mid x \in \mathcal{E}, i, j\}$.

Note that this identification will not cause any confusion, since in proofs of this section we only use the property that $\mathcal{E} \subset \mathrm{Irred}(\mathbb{G})$ is a finite set.

Here we record an elementary lemma.

Lemma 4.1. *For any $a \in N_0$, the element $\pi_{\sigma^{\varphi_N}}(a) \in N \rtimes_{\sigma^{\varphi_N}} \mathbb{R} \subset \mathbb{B}(L^2(N) \otimes L^2(\mathbb{R}))$ is contained in $N_0 \otimes_{\min} C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the set of all norm continuous bounded functions on $\mathbb{R}$.*

Proof. We may assume that a is an eigenvector; namely, $\sigma^{\varphi_N}_t(a) = \lambda^{it} a$ for some $\lambda > 0$. Then since $(\pi_{\sigma^{\varphi_N}}(a)\xi)(t) = \sigma^{\varphi_N}_{-t}(a)\xi(t) = \lambda^{-it} a\xi(t)$ for $t \in \mathbb{R}$, one has $\pi_{\sigma^{\varphi_N}}(a) = a \otimes f$, where $f \in C_b(\mathbb{R})$ is given by $f(t) := \lambda^{-it}$. Hence we get the conclusion. □

We fix a faithful normal semifinite weight φ_X on X and put $\psi := \varphi_N \otimes \varphi_X$ with its dual weight $\hat{\psi}$. Recall that the compression map $P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})}$, where P_N is the one-dimensional projection from $L^2(N)$ onto $\mathbb{C}\Lambda_{\varphi_N}(1_N)$, is a conditional expectation $E_{X \rtimes \mathbb{R}} : \mathcal{M} \to X \rtimes \mathbb{R}$, which satisfies $\hat{\psi} = \hat{\varphi}_X \circ E_{X \rtimes \mathbb{R}}$ (this was shown in the first half of the proof of Lemma 2.1). For any $a \in \mathcal{M}$ and $f \in C_c(\mathbb{R}, \mathcal{M})\mathfrak{n}_\psi$, we denote by af an element in $C_c(\mathbb{R}, \mathcal{M})\mathfrak{n}_\psi$ given by $t \mapsto \alpha_{-t}(a) f(t)$. Observe that $\Lambda_{\hat{\psi}}(\hat{\pi}_\alpha(af)) = \pi_\alpha(a)\Lambda_{\hat{\psi}}(\hat{\pi}_\alpha(f))$. A simple computation shows that for any $a, b \in N$ and $f, g \in C_c(\mathbb{R}, X)\mathfrak{n}_{\varphi_X}$,

$$\langle af, bg\rangle_{\hat{\psi}} = \langle a, b\rangle_{\varphi_N} \langle f, g\rangle_{\hat{\varphi}_X}.$$

Observe that all the linear spans of uf for $u \in \mathrm{Irred}(\mathbb{G})$ and $f \in C_c(\mathbb{R}, X)\mathfrak{n}_{\varphi_X}$ are dense in $L^2(N) \otimes L^2(X) \otimes L^2(\mathbb{R})$. So if $\{f_\lambda\}_\lambda \subset C_c(\mathbb{R}, X)\mathfrak{n}_{\varphi_X}$ is an orthonormal basis in $L^2(X) \otimes L^2(\mathbb{R})$, then the set $\{uf_\lambda\}_{u,\lambda}$ is an orthonormal basis of $L^2(N) \otimes L^2(X) \otimes L^2(\mathbb{R})$. Along this basis, any $a \in \mathfrak{n}_{\hat{\psi}}$ can be decomposed in $L^2(N) \otimes L^2(X) \otimes L^2(\mathbb{R})$ as, for some $\alpha_{u,\lambda} \in \mathbb{C}$,

$$\Lambda_{\hat{\psi}}(a) = \sum_{u,\lambda} \alpha_{u,\lambda} u f_\lambda = \sum_{u,\lambda} \alpha_{u,\lambda} \pi_{\varphi_N}(u) \Lambda_{\hat{\psi}}(\hat{\pi}_\alpha(f_\lambda)) = \sum_u \pi_{\sigma^{\varphi_N}}(u) a_u,$$

where $a_u = \sum_\lambda \alpha_{u,\lambda} f_\lambda \in L^2(\mathbb{R}, X)$. If we apply $(P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\pi_{\sigma^{\varphi_N}}(v^*)$ for some $v \in \mathrm{Irred}(\mathbb{G})$ to this decomposition, then on the one hand

$$(P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\pi_{\sigma^{\varphi_N}}(v^*)\, \Lambda_{\hat{\psi}}(a) = (P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\, \Lambda_{\hat{\psi}}(v^* a) = \Lambda_{\hat{\psi}}(E_{X \rtimes \mathbb{R}}(v^* a))$$

and on the other hand

$$(P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\pi_{\sigma^{\varphi_N}}(v^*) \sum_u \pi_{\sigma^{\varphi_N}}(u) a_u = \sum_u \varphi_N(v^* u) a_u = \varphi_N(v^* v) a_v = a_v.$$

Hence we have $a_v = \Lambda_{\hat{\psi}}(E_{X \rtimes \mathbb{R}}(v^* a))$ for all $v \in \mathrm{Irred}(\mathbb{G})$. Thus we observe that any element $a \in \mathfrak{n}_{\hat{\psi}}$ has the *Fourier expansion* in the sense that

$$\Lambda_{\hat{\psi}}(a) = \sum_u \pi_{\sigma^{\varphi_N}}(u) a_u = \sum_u \Lambda_{\hat{\psi}}(u E_{X \rtimes \mathbb{R}}(u^* a)), \quad \text{where } a_u = \Lambda_{\hat{\psi}}(E_{X \rtimes \mathbb{R}}(u^* a)).$$

Using this property, we can prove the following lemma. We omit the proof, since it is straightforward.

Lemma 4.2. *Let $\mathcal{M}_0 \subset \mathcal{M}$ be the C*-subalgebra generated by N_0 and $X \rtimes \mathbb{R}$. Then one has*

$$\begin{aligned}\mathcal{M}_0 &= \overline{\mathrm{span}}^{\mathrm{norm}}\{ax \mid a \in N_0,\ x \in X \rtimes \mathbb{R}\}\\ &= \overline{\mathrm{span}}^{\mathrm{norm}}\{xa \mid a \in N_0,\ x \in X \rtimes \mathbb{R}\}.\end{aligned}$$

4B. *Proof of Theorem A.* Let $\mathbb{G}$ be a compact quantum group with the Haar state h and put $N_0 := C_{\mathrm{red}}(\mathbb{G}) \subset L^\infty(\mathbb{G}) =: N$ and $\varphi_N := h$. Let (B, φ_B) be a von Neumann algebra with a faithful normal state. We keep the notation from Sections 3C and 3D, such as M, φ, $\widetilde{B}$, $\widetilde{M}$, Tr, p, A, $\mathcal{H}$, $\pi_{\mathcal{H}}$, $\theta_{\mathcal{H}}$, $\mathcal{M}$, except for the Hilbert space H (which is used just below in a different manner). Assume that $\mathrm{Tr}\,|_{\widetilde{B}}$ is semifinite. Recall that by Lemma 3.9, $\mathcal{M} = \mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ with the standard representation $L^2(\mathcal{M}) = L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(X)$. Set $\pi := \pi_{\mathcal{H}}$ and $\theta := \theta_{\mathcal{H}}$ for simplicity, and we sometimes omit π and θ by regarding $\widetilde{M}$, $\widetilde{M}^\circ$ as subsets of $\mathcal{M}$. Using Proposition 2.3, we put

$$\begin{aligned}H &:= L^2(\mathcal{M}) \otimes_X L^2(\mathcal{M}) = L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N) \otimes L^2(X) \otimes L^2_r(N) \otimes L^2_r(\mathbb{R}),\\ K &:= L^2(\mathcal{M}) \otimes_{(N \overline{\otimes} X)} L^2(\mathcal{M}) = L^2_\ell(\mathbb{R}) \otimes L^2(N) \otimes L^2(X) \otimes L^2_r(\mathbb{R}),\end{aligned}$$

and we denote by π_H, ρ_H, π_K and ρ_K corresponding left and right actions of $\mathcal{M}$. Here we are using symbols ℓ and r for $L^2(\mathbb{R})$ and $L^2(N)$, so that π_H and π_K act on $L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N) \otimes L^2(X)$ and $L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N) \otimes L^2(X)$ respectively, and θ_H and θ_K act on $L^2(X) \otimes L^2_r(N) \otimes L^2_r(\mathbb{R})$ and $L^2(N) \otimes L^2(X) \otimes L^2_r(\mathbb{R})$ respectively. We denote by $\nu_{K,H}$ the corresponding $*$-homomorphism as $\mathcal{M}$-bimodules, which is *not* bounded in general.

In this setting, we prove two lemmas. The first one uses biexactness of quantum groups, which corresponds to [Isono 2015a, Lemma 4.1.3], while the second one uses Popa's intertwining techniques, which corresponds to [Isono 2015a, Lemma 4.1.2; 2015b, Lemma 4.4]. See also [Popa and Vaes 2014b, Sections 3.2 and 3.5] for the origins of them.

Lemma 4.3. *Assume that $\widehat{\mathbb{G}}$ is biexact with a u.c.p. map Θ as in the definition of biexactness. Let $\mathcal{M}_0$ be the C*-algebra generated by N_0 and $\mathbb{R} \ltimes X$. Then Θ can be extended to a u.c.p. map*

$$\widetilde{\Theta} : \mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0)\} \to \mathbb{B}(K)$$

which satisfies, using the flip $\Sigma_{12} : K \simeq L^2(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})$,

$$\Sigma_{12}\big(\widetilde{\Theta}(\pi_H(xa)\theta_H(b^\circ y^\circ)) - \pi_K(xa)\theta_K(b^\circ y^\circ)\big)\Sigma_{12} \in \mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))$$

for any $a, b \in N_0$ *and* $x, y \in \mathbb{R} \ltimes X$.

Proof. By applying flip maps, we identify

$$H = L^2_\ell(N) \otimes L^2_r(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}),$$
$$K = L^2(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}).$$

We define a u.c.p. map $\widetilde{\Theta}$ by

$$\widetilde{\Theta} := \Theta \otimes \mathrm{id}_{L^2_\ell(\mathbb{R})} \otimes \mathrm{id}_{L^2(X)} \otimes \mathrm{id}_{L^2_r(\mathbb{R})} : N_0 \otimes_{\min} N_0^\circ \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})) \to \mathbb{B}(K).$$

Observe that by Lemma 4.1, $\pi_H(\mathcal{M}_0)$ and $\rho_H(\mathcal{M}_0)$ are contained in

$$N_0 \otimes_{\min} N_0^\circ \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})).$$

Recall that for $a, b \in N$, $\pi_H(a)$ and $\theta_H(b^\circ)$ are given by $\pi_{\sigma^{\varphi_N}}(a)$ on $L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N)$ and $\theta_{\sigma^{\varphi_N}}(b^\circ)$ on $L^2_r(N) \otimes L^2_r(\mathbb{R})$. So if a and b are eigenvectors, they are of the form $\pi_H(a) = f \otimes a$ and $\theta_H(b^\circ) = b^\circ \otimes g$ for some $f, g \in C_b(\mathbb{R})$ by Lemma 4.1. It then holds that for any $x, y \in \mathbb{R} \ltimes X$,

$$\begin{aligned}
&\widetilde{\Theta}(\pi_H(xa)\theta_H(b^\circ y^\circ)) - \pi_K(xa)\theta_K(b^\circ y^\circ) \\
&\quad = \widetilde{\Theta}(\pi_H(x)\pi_H(a)\theta_H(b^\circ)\theta_H(y^\circ)) - \pi_K(x)\pi_K(a)\theta_K(b^\circ)\theta_K(y^\circ) \\
&\quad = \widetilde{\Theta}\big(\pi_H(x)(a \otimes b^\circ \otimes f \otimes 1_{L^2(X)} \otimes g)\theta_H(y^\circ)\big) - \pi_K(x)(ab^\circ \otimes f \otimes 1_{L^2(X)} \otimes g)\theta_K(y^\circ) \\
&\quad = \pi_K(x)\big((\Theta(a \otimes b^\circ) - ab^\circ) \otimes f \otimes 1_{L^2(X)} \otimes g\big)\theta_K(y^\circ).
\end{aligned}$$

Since $\Theta(a \otimes b^\circ) - ab^\circ \in \mathbb{K}(L^2(N))$ and $\pi_K(x), \theta_K(y^\circ) \in \mathbb{C}1_N \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))$, the last term above is contained in $\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})))$. Then by Lemma 4.2, we obtain the conclusion. □

Lemma 4.4. *Let* Ω *be a state on* $\mathbb{B}(K)$ *satisfying for any* $x \in \widetilde{M}$ *and* $a \in \mathcal{U}(A)$,

$$\Omega(\pi_K(\pi(x))) = \mathrm{Tr}(pxp) \quad \textit{and} \quad \Omega(\pi_K(\pi(a)\theta(\bar{a}))) = 1.$$

If $A \not\preceq_{\widetilde{M}} \widetilde{B}$, *then using the flip* $\Sigma_{12} : K \simeq L^2(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})$, *it holds that*

$$\Omega \circ \mathrm{Ad}(\Sigma_{12})\big(\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))\big) = 0.$$

Proof. Since Ω is a state, by the Cauchy–Schwarz inequality, we have only to show that

$$\Omega \circ \mathrm{Ad}(\Sigma_{12})(\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{C}1_{L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})}) = 0.$$

In this setting we can follow the proof of [Isono 2015b, Lemma 4.4]. Indeed suppose by contradiction that there exist $\delta > 0$ and a finite subset $\mathcal{F} \subset \mathrm{Irred}(\mathbb{G})$ such that

$$\Omega(1_{L^2_\ell(\mathbb{R})} \otimes P_{\mathcal{F}} \otimes 1_{L^2(X) \otimes L^2_r(\mathbb{R})}) > \delta,$$

where $P_{\mathcal{F}}$ is the orthogonal projection onto $\sum_{x\in\mathcal{F}} H_x \otimes H_{\bar{x}}$. Then the argument in [loc. cit., Lemma 4.4] works by replacing $\|\cdot\|$ with Ω. Hence we omit the proof. □

Now we are in position to prove the main theorem. We actually prove the following more general theorem. Theorem A then follows immediately with Theorem 3.10.

Theorem 4.5. *Let $A \subset p\widetilde{M}p$ be a von Neumann subalgebra and $\mathcal{G} \leq \mathcal{N}_{p\widetilde{M}p}(A)$ a subgroup. Assume the following three conditions*:

(A) *The group $\mathcal{G}$ acts on A by conjugation as a weakly compact action for $(\widetilde{M}, \pi, \theta, \mathcal{M})$.*

(B) *The quantum group $\widehat{\mathbb{G}}$ is biexact and centrally weakly amenable.*

(C) *We have $A \not\preceq_{\widetilde{M}} \widetilde{B}$.*

Then there is a $(\mathcal{U}(A) \cup \mathcal{G})$-central state on $p\langle \widetilde{M}, \widetilde{B}\rangle p$ which coincides with Tr *on $p\widetilde{M}p$. In particular the von Neumann algebra generated by A and $\mathcal{G}$ is amenable relative to $\widetilde{B}$.*

Proof. By Remark 3.5, we may assume $\mathcal{U}(A) \subset \mathcal{G}$. Recall from Lemma 3.2 that as $\mathcal{M}$-bimodules,

$$L^2(\mathcal{M}) \prec L^2(\mathcal{M}) \otimes_{(N\otimes X)} L^2(\mathcal{M}) = K,$$

and we denote by ν the associated $*$-homomorphism. Let $(\xi_i)_i \subset L^2(\mathcal{M})$ be a net for the given weakly compact action of $\mathcal{G}$ and put a state $\Omega(X) := \mathrm{Lim}_i \langle \nu(X)\xi_i, \xi_i\rangle_{L^2(\mathcal{M})}$ on $\mathrm{C}^*\{\pi_K(\mathcal{M}), \theta_K(\mathcal{M}^\circ)\}$. Observe that it satisfies

(i)′ $\Omega(\pi_K(\pi(x))) = \mathrm{Tr}(pxp)$ for any $x \in \widetilde{M}$;

(ii)′ $\Omega(\pi_K(\pi(a)\theta(\bar{a}))) = 1$ for any $a \in \mathcal{U}(A)$;

(iii)′ $\Omega(\pi_K(\pi(u)\theta(\bar{u}))\theta_K(\pi(u^*)^\circ\theta(u^\circ)^\circ)) = 1$ for any $u \in \mathcal{G}$.

Note that since $\mathcal{J}_{\mathcal{M}}\xi_i = \xi_i$, we also have $\Omega(\theta_K(\pi(x)^\circ)) = \mathrm{Tr}(pxp)$ for any $x \in \widetilde{M}$. Denote by $\nu_{K,H}$ the (not necessarily bounded) $*$-homomorphism for $\mathcal{M}$-bimodules H and K. Here we claim that, using the biexactness of $\widehat{\mathbb{G}}$, the functional $\widetilde{\Omega} := \Omega \circ \nu_{K,H}$ satisfies the following boundedness condition.

Claim. *The functional $\widetilde{\Omega}$ is bounded on* $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$.

Proof of Claim. We first extend Ω on $\mathbb{B}(K)$ by the Hahn–Banach theorem. Then by Lemma 4.4, using assumption (C) and conditions (i)′ and (ii)′, one has

$$\Omega \circ \mathrm{Ad}(\Sigma_{12})\big(\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))\big) = 0.$$

Let Θ be a u.c.p. map for biexactness of $\widehat{\mathbb{G}}$ and denote by $\widetilde{\Theta}$ the extension given in Lemma 4.3. Define a state on $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$ by $\widehat{\Omega} := \Omega \circ \widetilde{\Theta}$. Then conclusions of Lemmas 4.3 and 4.4 show that for any $a, b \in N_0$ and $x, y \in \mathbb{R} \ltimes X$,

$$\widehat{\Omega}(\pi_H(xa)\theta_H(b^\circ y^\circ)) = \Omega \circ \widetilde{\Theta}(\pi_H(xa)\theta_H(b^\circ y^\circ)) = \Omega(\pi_K(xa)\theta_K(b^\circ y^\circ)).$$

This means that the functional $\widetilde{\Omega}$ coincides with $\widehat{\Omega}$ on $*$-alg$\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$, and hence it is a state on $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$ since so is $\widehat{\Omega}$. □

We next show that the above boundedness extends partially, using the central weak amenability and a normality of $\widetilde{\Omega}$. This is the second use of the weak amenability. Recall that $\mathcal{M}$ is generated by a copy of $\widetilde{M}$ and $\widetilde{M}^\circ$. We put $\widetilde{M}_0 \subset \mathcal{M}_0$ as the C*-subalgebra generated by $\widetilde{B}$ and N_0, and note that Lemma 4.2 is applied to $\widetilde{M}_0$.

Claim. *The functional $\widetilde{\Omega}$ is bounded on*

$$\mathrm{C}^*\{\pi_H(\widetilde{M}), \pi_H(\widetilde{M}^\circ), \theta_H(\widetilde{M}^\circ), \theta_H(\widetilde{M})\} =: \mathfrak{A},$$

where $\theta_H(\widetilde{M})$ should be understood as $\theta_H((\widetilde{M}^\circ)^\circ)$.

Proof of Claim. Let $(\psi_i)_i$ be a net of finite-rank normal c.c. maps on N as in Theorem 2.9. Up to convex combinations, we may assume $\psi_i \to \mathrm{id}_N$ in the point $*$-strong topology. For each i we put $\psi_i^\circ := J_N \psi_i(J_N \cdot J_N) J_N$ as a normal c.c. map on N°. For each i, since ψ_i commutes with the modular action, one can define a normal c.c. map on $\mathfrak{A}$ by

$$\Psi_i := \mathrm{id}_{L^2_\ell(\mathbb{R})} \otimes \psi_i \otimes \mathrm{id}_{L^2(X)} \otimes \psi_i^\circ \otimes \mathrm{id}_{L^2_r(\mathbb{R})}.$$

Observe that the restriction of Ψ_i on $\pi_H(\widetilde{M})$ defines a normal c.c. map $\tilde{\psi}_i : \widetilde{M} \to \widetilde{M}_0$ (use Lemma 4.2). The same holds for $\theta_H(\widetilde{M}^\circ)$ and define $\tilde{\psi}_i^\circ$ similarly. Then with the formula $\|\pi_H(z)\|_{2,\widetilde{\Omega}} = \|zp\|_{2,\mathrm{Tr}} = \|\theta_H(\bar{z})\|_{2,\widetilde{\Omega}}$ for $z \in \widetilde{M}$ and by the Cauchy–Schwarz inequality, it holds that for any $a, b, x, y \in \widetilde{M}$

$$\begin{aligned}
&\left|\widetilde{\Omega} \circ \Psi_i(\pi_H(ax^\circ)\theta_H(b^\circ y)) - \widetilde{\Omega}(\pi_H(ax^\circ)\theta_H(b^\circ y))\right| \\
&\qquad = \left|\widetilde{\Omega}(\pi_H(\tilde{\psi}_i(a)x^\circ)\theta_H(\tilde{\psi}_i^\circ(b^\circ)y)) - \widetilde{\Omega}(\pi_H(ax^\circ)\theta_H(b^\circ y))\right| \\
&\qquad \le \|\tilde{\psi}_i(a)^* - a^*\|_{2,\mathrm{Tr}}\|x\|_\infty \|b\|_\infty \|y\|_\infty + \|\tilde{\psi}_i(b)^* - b^*\|_{2,\mathrm{Tr}} \|a\|_\infty \|x\|_\infty \|y\|_\infty \\
&\qquad \to 0 \quad \text{as } i \to \infty.
\end{aligned}$$

Hence $\widetilde{\Omega} \circ \Psi_i$ converges pointwisely to $\widetilde{\Omega}$ on the norm-dense $*$-subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ generated by $\pi_H(\widetilde{M}), \pi_H(\widetilde{M}^\circ), \theta_H(\widetilde{M}^\circ)$, and $\theta_H(\widetilde{M})$. Observe that $\|\widetilde{\Omega} \circ \Psi_i|_{\mathfrak{A}}\| \le 1$ for all i, since the range of Ψ_i is contained in $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$ and $\widetilde{\Omega}$ is bounded by 1 on this C*-algebra by the previous claim. So we conclude $\|\widetilde{\Omega}|_{\mathfrak{A}}\| \le 1$, as desired. □

Observe that $\widetilde{\Omega}$ is a state, since it is positive on $\mathfrak{A}_0$ by construction, and $\widetilde{\Omega}(1) = 1$. By the Hahn–Banach theorem, we extend $\widetilde{\Omega}$ from $\mathfrak{A}$ to $\mathbb{B}(H)$ and we still denote it by $\widetilde{\Omega}$. By construction, it satisfies that for all $x \in \widetilde{M}$ and $u \in \mathcal{G}$,

$$\widetilde{\Omega}(\pi_H(x)) = \mathrm{Tr}(pxp) \quad \text{and} \quad \widetilde{\Omega}\big(\pi_H(\pi(u)\theta(\bar{u}))\theta_H(\pi(u^*)^\circ\theta(u^\circ)^\circ)\big) = 1.$$

Putting $U(u) := \pi_H(\pi(u)\theta(\bar{u}))\theta_H(\pi(u^*)^\circ\theta(u^\circ)^\circ)$, the second condition implies $\widetilde{\Omega}(Y) = \widetilde{\Omega}(U(u)YU(u)^*)$ for any $u \in \mathcal{G}$ and $Y \in \mathbb{B}(H)$. Recall that since $H = L^2(\mathcal{M}) \otimes_X L^2(\mathcal{M})$, regarding $L^2(\mathcal{M})$ as an $\langle \mathcal{M}, \mathbb{R} \ltimes X\rangle$-$X$-bimodule, the basic construction $\langle \mathcal{M}, \mathbb{R} \ltimes X\rangle$ acts on H on the left, which we again denote by π_H, and its image commutes with $\theta_H(\mathcal{M}^\circ)$. So if $Y \in \langle \mathcal{M}, \mathbb{R} \ltimes X\rangle \cap \theta(\widetilde{M}^\circ)'$, then

$$\widetilde{\Omega}(\pi_H(Y)) = \widetilde{\Omega}(U(u)\pi_H(Y)U(u)^*) = \widetilde{\Omega}(\pi_H(\pi(u))\pi_H(Y)\pi_H(\pi(u))^*)$$

for any $u \in \mathcal{G}$. So the state $\widetilde{\Omega} \circ \pi_H$ is a $\mathcal{G}$-central state on $\langle \mathcal{M}, \mathbb{R} \ltimes X \rangle \cap \theta(\widetilde{M}^\circ)'$. Finally since $\widetilde{M} L^2(\mathbb{R} \ltimes X) \subset L^2(\mathcal{M})$ is dense, the von Neumann subalgebra in $\langle \mathcal{M}, \mathbb{R} \ltimes X \rangle \cap \theta(\widetilde{M}^\circ)'$ generated by $\widetilde{M}$ and $e_{\mathbb{R} \ltimes X} := 1_{L^2(\mathbb{R})} \otimes P_N \otimes 1_X$, where P_N is the 1-dimensional projection onto $\mathbb{C}\Lambda_{\varphi_N}(1_N)$, is canonically identified as $\langle \widetilde{M}, \widetilde{B} \rangle$ (by the fact that $e_{\mathbb{R} \ltimes X}\, a\, e_{\mathbb{R} \ltimes X} = E_{\widetilde{B}}(a) e_{\mathbb{R} \ltimes X}$ for $a \in \widetilde{M}$). Thus the restriction of $\widetilde{\Omega} \circ \pi_H$ on $\langle \widetilde{M}, \widetilde{B} \rangle$ is a $\mathcal{G}$-central state which coincides with Tr on $p\widetilde{M}p$. Using the normality on $p\widetilde{M}p$ and by the Cauchy–Schwarz inequality, we obtain that $\mathcal{G}''$ is amenable relative to $\widetilde{B}$ in $\widetilde{M}$. □

4C. *Proof of Corollary B.*

Proof of Corollary B. Put $M := N \mathbin{\overline{\otimes}} B \supset N_0 \mathbin{\overline{\otimes}} B =: M_0$ and suppose that $A \subset M_0$ is a Cartan subalgebra. We will deduce a contradiction. For this, let R_∞ be the AFD III_1 factor and $A_0 \subset R_\infty$ a Cartan subalgebra. Up to exchanging B and A with $B \mathbin{\overline{\otimes}} R_\infty$ and $A \mathbin{\overline{\otimes}} A_0$ respectively, we assume that B is a type III_1 factor (see, e.g., Lemma 2.2).

Let ψ_{N_0} and τ_A be faithful normal states on N_0 and A respectively, and E_{N_0} and E_A faithful normal conditional expectations from N to N_0 and from M_0 to A respectively. Put

$$\psi_A := \tau_A \circ E_A, \quad \psi_N := \psi_{N_0} \circ E_{N_0}, \quad \psi := \psi_N \otimes \varphi_B, \quad \varphi := h \otimes \varphi_B$$

and $E_{M_0} := E_{N_0} \otimes \mathrm{id}_B$. Then since all continuous cores are isomorphic, we have $\Pi_{\psi_A \circ E_{M_0}, \psi} : C_\psi(M) \to C_{\psi_A \circ E_{M_0}}(M)$, which restricts to $\Pi_{\psi_A, \psi_{N_0} \otimes \varphi_B} : C_{\psi_{N_0} \otimes \varphi_B}(M_0) \to C_{\psi_A}(M_0)$. Recall that $A \mathbin{\overline{\otimes}} L\mathbb{R} \subset C_{\psi_A}(M_0)$ is a Cartan subalgebra, see, e.g., [Houdayer and Ricard 2011, Proposition 2.6], and hence so is the image

$$\widetilde{A} := \Pi_{\varphi, \psi_A \circ E_{N_0}}(A \mathbin{\overline{\otimes}} L\mathbb{R}) \subset \Pi_{\varphi, \psi_A \circ E_{N_0}}(C_{\psi_A}(M_0)) =: \mathcal{N}.$$

Claim. *There is a conditional expectation $E : \langle C_\varphi(M), C_{\varphi_B}(B) \rangle \to \mathcal{N}$ which is faithful and normal on $C_\varphi(M)$.*

Proof. We first show $A \not\preceq_M B$. Indeed, if $A \preceq_M B$, then we have $A \preceq_{M_0} B$ by Lemma 2.12. So by [Houdayer and Isono 2017, Lemma 4.9], one has $N_0 = B' \cap M_0 \preceq_{M_0} A' \cap M_0 = A$, which is a contradiction. Hence we have $A \not\preceq_M B$.

We apply [Boutonnet et al. 2014, Proposition 2.10] (this holds if A is finite by exactly the same proof) and get $\widetilde{A} \not\preceq_{C_\varphi(M)} C_{\varphi_B}(B)$. Fix any projection $p \in \widetilde{A}$ with $\mathrm{Tr}(p) < \infty$, where Tr is the canonical trace on the core, and observe $p\widetilde{A}p \not\preceq_{C_\varphi(M)} C_{\varphi_B}(B)$ by definition. We apply Theorem A to $p\widetilde{A}p$ and get that $\mathcal{N}_{pC_\varphi(M)p}(p\widetilde{A}p)''$ is amenable relative to $C_{\varphi_B}(B)$. Observe that $\mathcal{N}_{pC_\varphi(M)p}(p\widetilde{A}p)'' = p(\mathcal{N}_{C_\varphi(M)}(\widetilde{A})'')p$; see, e.g., [Houdayer and Ricard 2011, Proposition 2.7]. Combined with [Isono 2017, Remark 3.3], there is a conditional expectation $E_p : p\langle C_\varphi(M), C_{\varphi_B}(B)\rangle p \to p\mathcal{N}p$ which restricts to the Tr-preserving expectation on $pC_\varphi(M)p$. Taking a net $(p_i)_i$ of Tr-finite projections converging to 1 weakly, one can construct a desired conditional expectation by $E(x) := \sigma\text{-weak}\,\mathrm{Lim}_i\, E_{p_i}(p_i x p_i)$ for $x \in \langle C_\varphi(M), C_{\varphi_B}(B) \rangle$. □

We apply [Isono 2017, Theorem 3.2] to the conclusion of the claim and get that M_0 is amenable relative to B in M. Hence there is a conditional expectation $F : \langle M, B \rangle \to M_0$ which is faithful and normal on M. Using the identification $\langle M, B \rangle = \mathbb{B}(L^2(M)) \mathbin{\overline{\otimes}} B$, we can construct a conditional expectation from $\mathbb{B}(L^2(M))$ onto N_0, which means N_0 is injective. This is a contradiction. □

Acknowledgements

The author would like to thank Yuki Arano, Kei Hasegawa, and Reiji Tomatsu for useful comments on the relative amenability, and Narutaka Ozawa and Stefaan Vaes for fruitful conversations. This research was carried out while he was visiting the University of California, Los Angeles. He gratefully acknowledges their kind hospitality. He was supported by JSPS, Research Fellow of the Japan Society for the Promotion of Science.

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Received 11 Jan 2018. Accepted 16 Sep 2018.

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mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1325

COMMUTATORS OF MULTIPARAMETER FLAG SINGULAR INTEGRALS AND APPLICATIONS

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We introduce the iterated commutator for the Riesz transforms in the multiparameter flag setting, and prove the upper bound of this commutator with respect to the symbol b in the flag BMO space. Our methods require the techniques of semigroups, harmonic functions and multiparameter flag Littlewood–Paley analysis. We also introduce the *big* commutator in this multiparameter flag setting and prove the upper bound with symbol b in the flag little bmo space by establishing the "exponential-logarithmic" bridge between this flag little bmo space and the Muckenhoupt A_p weights with flag structure. As an application, we establish the div-curl lemmas with respect to the appropriate Hardy spaces in the multiparameter flag setting.

1. Introduction and statement of main results

The Calderón–Zygmund theory of singular integrals has been central to the success and applicability of modern harmonic analysis in the last fifty years. This theory has had extensive applications to other fields of mathematics such as complex analysis, geometric measure theory and partial differential equations. In the setting of Euclidean spaces $\mathbb{R}^n$, a notable property of standard Calderón–Zygmund singular integrals, shared with the Hardy–Littlewood maximal operator, is that these operators commute with the classical one-parameter family of dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta x_1, \ldots, \delta x_n)$ for $\delta > 0$. See for example [Stein 1993].

The product Calderón–Zygmund theory in harmonic analysis was introduced in the 1970s and has been studied extensively since then. The model case is a tensor product of classical singular integral operators; such operators arise in the context of questions about summation of multiple variable Fourier series. Early key work in this field includes that of Chang and R. Fefferman [1980; 1982; 1985], R. Fefferman [1986; 1987; 1999], R. Fefferman and Stein [1982], C. Fefferman and Stein [1972], Gundy and Stein [1979], Journé [1985], and Pipher [1986]. Included in these works are the identification of appropriate notions of product BMO space and product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.

More recently, the theory of (iterated) commutators has been developed in connection with the Chang–Fefferman BMO space, including paraproducts and multiparameter div-curl lemmas; see, for example, [Dalenc and Ou 2016; Ferguson and Lacey 2002; Ferguson and Sadosky 2000; Lacey et al. 2009; 2010; 2012; Lacey and Terwilleger 2009]. In contrast with the classical Euclidean setting, the product Calderón–Zygmund singular integrals and the *strong* maximal function operator commute with the multiparameter dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta_1 x_1, \ldots, \delta_n x_n)$ for $\delta = (\delta_1, \ldots, \delta_n) \in (0, \infty)^n$.

MSC2010: 42B30, 42B20, 42B35.
Keywords: multiparameter flag setting, flag commutator, Hardy space, BMO space, div-curl lemma .

A new type of multiparameter structure, which lies in between one-parameter and tensor product, was introduced by Müller, Ricci and Stein in [Müller et al. 1995; 1996], where they studied the L^p boundedness of Marcinkiewicz multipliers $m(\mathcal{L}, iT)$ on the Heisenberg group, where $\mathcal{L}$ is the sub-Laplacian and T is the central invariant vector field, with m being a multiplier of Marcinkiewicz-type. They showed that such Marcinkiewicz multipliers can be characterized by a convolution operator $f * K$, where K is a so-called *flag* convolution kernel. This multiparameter flag structure is not explicit, but only *implicit* in the sense that one cannot formulate it in terms of an explicit dilation δ acting on x. Later, the notion of flag kernels (having singularities on appropriate flag varieties) and the properties of the corresponding singular integrals were then extended to the higher-step case by Nagel, Ricci and Stein [Nagel et al. 2001] on Euclidean space and their applications on certain quadratic CR submanifolds of $\mathbb{C}^n$. Recently, Nagel, Ricci, Stein and Wainger [Nagel et al. 2012; 2018] established the theory of singular integrals with flag kernels in a more general setting of homogeneous groups. They proved that, on a homogeneous group, singular integral operators with flag kernels are bounded on L^p, $1 < p < \infty$, and form an algebra. (See also [Głowacki 2010] for related work.) Associated to this implicit multiparameter flag structure, the Hardy space $H^1_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ and BMO space $\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ were introduced by Han, Lu and Sawyer [Han and Lu 2008; Han et al. 2014] through their creation of a flag-type Littlewood–Paley theory. More recently, Han, Lee, and the second and fifth authors [Han et al. 2016a] established a full characterization of $H^1_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ via appropriate flag-type nontangential, radial maximal functions, Littlewood–Paley theory via Poisson integrals, the flag-type Riesz transforms, as well as flag atomic decompositions.

In the multiparameter setting, the dilation structure $\delta \cdot x = (\delta_1 x_1, \dots, \delta_n x_n)$ for $\delta := (\delta_1, \dots, \delta_n) \in (0, \infty)^n$ determines a geometry that is reflected by axes-parallel rectangles of arbitrary side-lengths. Indeed, the strong maximal function is defined as the supremum of averages over such rectangles, and the Chang–Fefferman product BMO space can also be characterized using such rectangles. When it comes to the flag setting, the lack of an explicit dilation structure makes its geometry much more obscure. However, from the study of properties of the flag singular integrals, such as the flag Riesz transforms that will be introduced below, one realizes that the flag geometry can be reflected by axes-parallel rectangles with certain restriction on the side-lengths. For example, the flag rectangles in $\mathbb{R}^n \times \mathbb{R}^m$ are the ones of the form $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$ with $\ell(I) \le \ell(J)$. Compared to the multiparameter setting, the restriction $\ell(I) \le \ell(J)$ gives rise to new difficulties. For instance, a very useful trick in the study of problems in the multiparameter setting is to take a sequence of rectangles $\{I \times J_i\}$ and let J_i shrink to a point y_0 as $i \to \infty$. This can usually effectively reduce the problem to one-parameter. However, in the flag setting, such an operation is not allowed anymore. Other intrinsic difficulties of the flag setting can be better described from the analytic perspective, which will be discussed below.

A commutator of a classical Calderón–Zygmund singular integral with a BMO function is a bounded operator on L^p with norm equivalent to the BMO norm of the symbol [Coifman et al. 1976]. Modern methods of proving the upper bound of these commutators in the multiparameter product setting rely upon the existence of a wavelet basis for $L^2(\mathbb{R}^n)$, such as the Meyer wavelets or Haar wavelets; see for example [Lacey et al. 2009; Dalenc and Ou 2016]. It turns out that the behavior of the commutator is straightforward to analyze in terms of the wavelet basis. One method of proof shows that the commutator can be written as a linear combination of paraproducts and simple wavelet analogs of the Calderón–Zygmund operator in question. The other approach uses the wavelet basis to dominate the commutator by a composition of sparse operators. In the flag setting, we lack a suitable wavelet basis and this approach is not available. Essentially, the wavelet basis requires the construction of a suitable multiresolution analysis, which we do not have in this flag setting. Hence, instead of the wavelet basis, we resort to using a method based on heat semigroups and flag-type Littlewood–Paley theory, exploiting the connection between the Reisz transforms and the Laplacian.

We now recall the flag Riesz transforms as studied in [Han et al. 2016a]. We use $R_j^{(1)}$ to denote the j-th Riesz transform on $\mathbb{R}^{n+m}$, $j = 1, 2, \dots, n+m$, and we use $R_k^{(2)}$ to denote the k-th Riesz transform on $\mathbb{R}^m$, $k = 1, 2, \dots, m$. Namely, we have that for $g^{(1)} \in L^2(\mathbb{R}^{n+m})$,

$$R_j^{(1)} g^{(1)}(x) = \text{p.v. } c_{n+m} \int_{\mathbb{R}^{n+m}} \frac{x_j - y_j}{|x-y|^{n+m+1}} g^{(1)}(y)\, dy, \quad x \in \mathbb{R}^{n+m},$$

and for $g^{(2)} \in L^2(\mathbb{R}^m)$,

$$R_k^{(2)} g^{(2)}(z) = \text{p.v. } c_m \int_{\mathbb{R}^m} \frac{w_j - z_j}{|w-z|^{m+1}} g^{(2)}(w)\, dw, \quad z \in \mathbb{R}^m.$$

For $f \in L^2(\mathbb{R}^{n+m})$, we set

$$R_{j,k}(f) = R_j^{(1)} * R_k^{(2)} *_2 f; \tag{1-1}$$

that is, $R_{j,k}$ is the composition of $R_j^{(1)}$ and $R_k^{(2)}$. Note that the flag structure appears in $R_{j,k}$.

Given two functions $b, f \in L^2(\mathbb{R}^{n+m})$, we first recall the usual definition of commutator

$$[b, R_j^{(1)}](f)(x_1, x_2) := b(x_1, x_2) R_j^{(1)} * f(x_1, x_2) - R_j^{(1)} * (bf)(x_1, x_2). \tag{1-2}$$

The commutator can also act only on the second variable:

$$[b, R_k^{(2)}]_2(f)(x_1, x_2) := b(x_1, x_2) R_k^{(2)} *_2 f(x_1, x_2) - R_k^{(2)} *_2 (bf)(x_1, x_2). \tag{1-3}$$

Iterated commutators arise in the study of commutators of multiparameter singular integral operators which are tensor products. In the flag setting, our iterated commutator takes the following form:

Definition 1.1. Given two functions $b, f \in L^2(\mathbb{R}^{n+m})$, the iterated commutator in the flag setting of $\mathbb{R}^n \times \mathbb{R}^m$ is

$$\begin{aligned}[][b, R_j^{(1)}], R_k^{(2)}]_2(f) := b(x_1, x_2) R_j^{(1)} * R_k^{(2)} *_2 f(x_1, x_2) - R_j^{(1)} * (b \cdot R_k^{(2)} *_2 f)(x_1, x_2) \\ - R_k^{(2)} *_2 (b \cdot R_j^{(1)} * f)(x_1, x_2) + R_k^{(2)} *_2 R_j^{(1)} * (b \cdot f)(x_1, x_2).\end{aligned}$$

We point out that another possible definition via $[[b, R_k^{(2)}]_2, R_j^{(1)}](f)$ turns out to be equivalent; see Proposition 2.5 in Section 2.

We also introduce the *big* commutator in the flag setting as follows.

Definition 1.2. Given two functions $b, f \in L^2(\mathbb{R}^{n+m})$, the big commutator in the flag setting of $\mathbb{R}^n \times \mathbb{R}^m$ is

$$[b, R_{j,k}](f)(x) := b(x) R_{j,k}(f)(x) - R_{j,k}(bf)(x). \tag{1-4}$$

The main results, below, of this paper relate iterated and big commutator bounds to flag BMO spaces. As the definition of the space $\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ is very technical, we refer the reader to Section 2, Definition 2.4 for details.

Theorem 1.3. *Suppose $b \in \mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ and $1 < p < \infty$. Then for every $j = 1, \ldots, n+m$, $k = 1, \ldots, m$, $f \in L^p(\mathbb{R}^{n+m})$,*

$$\|[[b, R_j^{(1)}], R_k^{(2)}]_2(f)\|_{L^p(\mathbb{R}^{n+m})} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)} \|f\|_{L^p(\mathbb{R}^{n+m})}. \tag{1-5}$$

Lacking methods related to analyticity ([Ferguson and Sadosky 2000] for the Hilbert transform) or wavelets [Lacey et al. 2009; 2010; Dalenc and Ou 2016], we instead obtain this upper bound using the duality argument and the tools of semigroups, harmonic function extensions and techniques from multiparameter analysis.

Next, we introduce the little flag BMO space. The flag structure has a geometry which is reflected by the axes-parallel rectangles $R = I \times J \subset \mathbb{R}^{n+m}$ satisfying $\ell(I) \leq \ell(J)$, the collection of which is referred to as *flag rectangles*, denoted by $\mathcal{R}_{\mathcal{F}}$. One can then define the little flag BMO space and the flag-type Muckenhoupt weights $A_{\mathcal{F},p}$ with respect to $\mathcal{R}_{\mathcal{F}}$.

Definition 1.4. A locally integrable function b is in *little flag BMO space*, denoted by $\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$, if

$$\|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)} := \sup_{R\in\mathcal{R}_{\mathcal{F}}} \frac{1}{|R|} \int_R |b(x, y) - \langle b\rangle_R| \, dx \, dy < \infty, \tag{1-6}$$

where $\langle b\rangle_R = (1/|R|) \int_R b(x_1, x_2) \, dx_1 \, dx_2$.

Theorem 1.5. *Suppose $T_{\mathcal{F}}$ is a flag singular integral operator on $\mathbb{R}^n \times \mathbb{R}^m$, $b \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ and $1 < p < \infty$. Then for $f \in L^p(\mathbb{R}^{n+m})$,*

$$\|[b, T_{\mathcal{F}}](f)\|_{L^p(\mathbb{R}^{n+m})} \lesssim \|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)} \|f\|_{L^p(\mathbb{R}^{n+m})}. \tag{1-7}$$

In the above, the flag singular integral $T_{\mathcal{F}}$ can be taken as the Riesz transform $R_{j,k}$. The class of flag singular integral operators $T_{\mathcal{F}}$ naturally generalize the Riesz transforms $R_{j,k}$ and are assumed to be associated to kernels having a standard flag structure. We refer the reader to Definition 4.4 in Section 4 for its precise definition. To obtain this upper bound, we study the little flag BMO space $\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ and find the connection with the John–Nirenberg BMO space on $\mathbb{R}^{n+m}$ and on $\mathbb{R}^m$. We also establish the bridge between functions in $\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ and weights in $A_{\mathcal{F},p}$. These structures lead to the upper bound for $[b, R_{j,k}](f)$.

As application, the commutator estimates obtained above imply certain versions of div-curl lemmas, which seem to be first of their kind in the flag setting. Roughly speaking, a div-curl lemma says that if vector fields E and B initially in L^2 have some cancellation (e.g., divergence or curl zero) then one can expect their dot product $E \cdot B$ to belong to a better space of functions instead of just L^1 (as provided for by Cauchy–Schwarz). The cancellation conditions allow one to deduce some type of cancellation, e.g., $\int E \cdot B = 0$, suggesting that the function should belong to a suitable Hardy space since it is integrable and has mean zero. The algebraic structure of $E \cdot B$ coupled with the duality between Hardy spaces and BMO spaces then points to the use of the commutator theorem to arrive at the membership of $E \cdot B$ in the Hardy space; different commutator results suggest different div-curl lemmas that can be explored. In the classical one-parameter setting, the div-curl lemma says that given two vector fields, one with divergence zero and the other with curl zero, their dot product belongs to a Hardy space [Coifman et al. 1993]. Later on, Lacey, Petermichl, and the fourth and the fifth authors proved multiple versions of div-curl lemmas in the multiparameter setting [Lacey et al. 2012], which are expected since the multiparameter setting offers several different interpretations of the Hardy and BMO spaces. Thus, it is natural that our Theorems 1.3 and 1.5 lead to two versions of flag-type div-curl lemmas.

First, consider vector fields on $\mathbb{R}^n \times \mathbb{R}^m$ that take values in $\mathcal{M}_{n+m,m}$ and are associated with the flag structure (see Section 5 for the precise definitions and details). We establish the div-curl lemma in the flag setting with respect to the flag Hardy space below, which is a consequence of Theorem 1.3.

Theorem 1.6. *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Suppose that E, B are vector fields on $\mathbb{R}^n \times \mathbb{R}^m$ taking the values in $\mathcal{M}_{n+m,m}$, associated with the flag structure. Moreover, suppose $E = E^{(1)} *_2 E^{(2)} \in L^p_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m; \mathcal{M}_{n+m,m})$ and $B = B^{(1)} *_2 B^{(2)} \in L^q_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m; \mathcal{M}_{n+m,m})$ satisfy*

$$\operatorname{div}_{(x,y)} E_j^{(1)}(x, y) = 0 \quad \textit{and} \quad \operatorname{curl}_{(x,y)} B_j^{(1)}(x, y) = 0 \quad \textit{for all } k$$

and

$$\operatorname{div}_y E_k^{(2)}(x, y) = 0 \quad \textit{and} \quad \operatorname{curl}_y B_k^{(2)}(x, y) = 0 \quad \textit{for all } x \in \mathbb{R}^n, \textit{ for all } j.$$

Then $E \cdot B$ belongs to the flag Hardy space $H^1_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ with

$$\|E \cdot B\|_{H^1_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \|E\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m; \mathcal{M}_{n+m,m})} \|B\|_{L^q(\mathbb{R}^n \times \mathbb{R}^m; \mathcal{M}_{n+m,m})}. \tag{1-8}$$

We also prove another version of the div-curl lemma in the flag setting, which is with respect to the Hardy spaces on $\mathbb{R}^{n+m}$ and on $\mathbb{R}^m$, respectively. This version relies on the intermediate result in the proof of Theorem 1.5, namely, the structure of the flag little bmo space.

Theorem 1.7. *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Suppose that E, B are vector fields on $\mathbb{R}^n \times \mathbb{R}^m$ taking the values in $\mathbb{R}^{n+m}$. Moreover, suppose $E \in L^p(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^{n+m})$ and $B \in L^q(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^{n+m})$ satisfy*

$$\operatorname{div}_{(x,y)} E(x, y) = 0 \quad \textit{and} \quad \operatorname{curl}_{(x,y)} B(x, y) = 0$$

and

$$\operatorname{div}_y E(x, y) = 0 \quad \textit{and} \quad \operatorname{curl}_y B(x, y) = 0 \quad \textit{for all } x \in \mathbb{R}^n.$$

Then we have

$$\|E \cdot B\|_{H^1(\mathbb{R}^{n+m})} \lesssim \|E\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^{n+m})} \|B\|_{L^q(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^{n+m})}, \tag{1-9}$$

and

$$\int_{\mathbb{R}^m} \|E(\,\cdot\,, y) \cdot_2 B(\,\cdot\,, y)\|_{H^1(\mathbb{R}^m)}\, dy \lesssim \|E\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^{n+m})} \|B\|_{L^q(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^{n+m})}, \tag{1-10}$$

where

$$E(x, y) \cdot_2 B(x, y) := \sum_{k=1}^{m} E_{n+k}(x, y) B_k(x, y).$$

It is known that the div-curl lemma in the classical setting has many applications in PDE and compensated compactness [Coifman et al. 1993]. Similarly, we expect that the flag-type div-curl lemmas described above would have interesting implications in these directions as well. For instance, following the ideas in [Coifman et al. 1993], one can study weak convergence problems in the flag Hardy space. And it would be interesting to know whether one can use the flag-type regularity (implied by our div-curl lemmas) of certain nonlinear quantities to obtain improved regularity results for certain nonlinear PDE.

This paper is organized as follows. In Section 2 we provide necessary preliminaries with respect to the flag structures. In Section 3 we study the flag iterated commutators as in Definition 1.1 and prove Theorem 1.3. In Section 4 we give a complete treatment of the flag little bmo spaces and flag-type Muckenhoupt A_p weights, toward the proof of Theorem 1.5. In the last section, we apply the boundedness of flag commutators from Theorems 1.3 and 1.5 to establish the flag div-curl results, Theorems 1.6 and 1.7.

2. Preliminaries in the flag setting

Recall the classical Poisson kernel on $\mathbb{R}^n$:

$$P(x) := \frac{c_n}{(1+|x|^2)^{(n+1)/2}}.$$

And we define

$$P_t(x) := \frac{1}{t^n} P\left(\frac{x}{t}\right).$$

For $f \in L^1(\mathbb{R}^n)$, let $F(x,t) := P_t * f(x)$. Then we have the following standard pointwise estimates for the Poisson integral; see in particular [Stein 1993].

Proposition 2.1. *Suppose $f \in L^1(\mathbb{R}^n)$. Then*

$$\sup_{(x,t)\in\mathbb{R}^{n+1}_+} t^{n+k} |\nabla^k_{x,t} F(x,t)| \le C\|f\|_{L^1(\mathbb{R}^n)}. \tag{2-1}$$

We now recall the flag Poisson kernel given by

$$P(x,y) = P^{(1)} *_{\mathbb{R}^m} P^{(2)}(x,y) = \int_{\mathbb{R}^m} P^{(1)}(x, y-z) P^{(2)}(z)\, dz,$$

where

$$P^{(1)}(x,y) = \frac{c_{n+m}}{(1+|x|^2+|y|^2)^{(n+m+1)/2}} \quad \text{and} \quad P^{(2)}(z) = \frac{c_m}{(1+|z|^2)^{(m+1)/2}}$$

are the classical Poisson kernels on $\mathbb{R}^{n+m}$ and $\mathbb{R}^m$, respectively. Then we have

$$P_{t_1,t_2}(x,y) = P^{(1)}_{t_1} *_{\mathbb{R}^m} P^{(2)}_{t_2}(x,y).$$

We define the Lusin area function with respect to $u = P_{t_1,t_2} * f$ as follows.

Definition 2.2. For $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$ and $u(x_1,x_2,t_1,t_2) = P_{t_1,t_2} * f(x_1,x_2)$, the Lusin area integral of $u(x_1,x_2,t_1,t_2)$, denoted by $S_{\mathcal{F}}(u)$, is defined by

$$S_{\mathcal{F}}(u)(x_1,x_2) = \left\{ \int_{\mathbb{R}^{n+1}_+} \int_{\mathbb{R}^{m+1}_+} \chi_{t,s}(x_1-w_1, x_2-w_2) |t_1 \nabla^{(1)} t_2 \nabla^{(2)} u(w_1,w_2,t_1,t_2)|^2 \frac{dw_1\, dt_1}{t_1^{n+m+1}} \frac{dw_2\, dt_2}{t_2^{m+1}} \right\}^{\frac{1}{2}},$$

where $\nabla^{(1)} = (\partial_{t_1}, \partial_{w_{1,1}} \cdots \partial_{w_{1,n}}, \partial_{w_{2,1}} \cdots \partial_{w_{2,m}})$ is the standard gradient on $\mathbb{R}^{n+m+1}$, and $\nabla^{(2)} = (\partial_{t_2}, \partial_{w_{2,1}} \cdots \partial_{w_{2,m}})$ is the standard gradient on $\mathbb{R}^{m+1}$, and

$$\chi_{t_1,t_2}(x_1, x_2) := \chi^{(1)}_{t_1} *_{\mathbb{R}^m} \chi^{(2)}_{t_2}(x_1, x_2), \tag{2-2}$$

$\chi^{(1)}_{t_1}(x_1, x_2) := {t_1}^{-(n+m)}\chi^{(1)}(x_1/t_1, x_2/t_1)$, $\chi^{(2)}_{t_2}(z) := {t_2}^{-m}\chi^{(2)}(z/t_2)$, and $\chi^{(1)}(x, y)$ and $\chi^{(2)}(z)$ are the indicator functions of the unit balls of $\mathbb{R}^{n+m}$ and $\mathbb{R}^m$, respectively.

Definition 2.3. The flag Hardy space $H^1_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the collection of $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$ such that $S_{\mathcal{F}}(u) \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$. The norm of $H^1_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined by

$$\|f\|_{H^1_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)} = \|S_{\mathcal{F}}(u)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{2-3}$$

We now recall the definition of the flag BMO space.

Definition 2.4. The flag BMO space $\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the collection of $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)} := \sup_{\Omega}\biggl(\frac{1}{|\Omega|}\int_{T(\Omega)} |t_1\nabla^{(1)} t_2\nabla^{(2)} u(w_1, w_2, t_1, t_2)|^2 \,\frac{dw_1\, dt_1 dw_2\, dt_2}{t_1 t_2}\biggr)^{\frac{1}{2}} < \infty, \tag{2-4}$$

where the supremum is taken over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with finite measures, and $T(\Omega) = \bigcup_{R\subset\Omega} T(R)$ with $R = I \times J$, $\ell(I) \le \ell(I)$ and $T(R) = I \times \bigl(\frac{1}{2}\ell(I), \ell(I)\bigr] \times J \times \bigl(\frac{1}{2}\ell(J), \ell(J)\bigr]$.

Proposition 2.5. *Given two functions $b, f \in L^2(\mathbb{R}^{n+m})$, we have*

$$[[b, R^{(1)}_j], R^{(2)}_k]_2(f) = [[b, R^{(2)}_k]_2, R^{(1)}_j](f). \tag{2-5}$$

Proof. By definition, we see that

$$\begin{aligned}
[[b, R^{(1)}_j], R^{(2)}_k]_2(f)(x_1, x_2) &= [b, R^{(1)}_j]R^{(2)}_k *_2 f(x_1, x_2) - R^{(2)}_k *_2 ([b, R^{(1)}_j](f))(x_1, x_2)\\
&= b(x_1, x_2)R^{(1)}_j * R^{(2)}_k *_2 f(x_1, x_2) - R^{(1)}_j * (b \cdot R^{(2)}_k *_2 f)(x_1, x_2)\\
&\qquad - R^{(2)}_k *_2 (b \cdot R^{(1)}_j * f - R^{(1)}_j * (b \cdot f))(x_1, x_2)\\
&= b(x_1, x_2)R^{(1)}_j * R^{(2)}_k *_2 f(x_1, x_2) - R^{(1)}_j * (b \cdot R^{(2)}_k *_2 f)(x_1, x_2)\\
&\qquad - R^{(2)}_k *_2 (b \cdot R^{(1)}_j * f)(x_1, x_2) + R^{(2)}_k *_2 R^{(1)}_j * (b \cdot f)(x_1, x_2).
\end{aligned}$$

And we also have

$$\begin{aligned}
[[b, R^{(2)}_k]_2, R^{(1)}_j](f)(x_1, x_2) &= [b, R^{(2)}_k]_2 R^{(1)}_j * f(x_1, x_2) - R^{(1)}_j * ([b, R^{(2)}_k]_2(f))(x_1, x_2)\\
&= b(x_1, x_2)R^{(2)}_k *_2 R_j * f(x_1, x_2) - R^{(2)}_k *_2 (b \cdot R^{(1)}_j * f)(x_1, x_2)\\
&\qquad - R^{(1)}_j * (b \cdot R^{(2)}_k *_2 f - R^{(2)}_k *_2 (b \cdot f))(x_1, x_2)\\
&= b(x_1, x_2)R^{(2)}_k *_2 R^{(1)}_j * f(x_1, x_2) - R^{(2)}_k *_2 (b \cdot R^{(1)}_j * f)(x_1, x_2)\\
&\qquad - R^{(1)}_j * (b \cdot R^{(2)}_k *_2 f)(x_1, x_2) + R^{(1)}_j * R^{(2)}_k *_2 (b \cdot f)(x_1, x_2).
\end{aligned}$$

It is direct to see that, by changing of variables,

$$\begin{aligned} R_k^{(2)} *_2 R_j^{(1)} * f(x_1,x_2) &= \int R_k^{(2)}(x_2-z) R_j^{(1)}(x_1-y_1, z-y_2) f(y_1,y_2)\,dz\,dy_1\,dy_2 \\ &= \int R_k^{(2)}(\tilde{z}-y_2) R_j^{(1)}(x_1-y_1, x_2-\tilde{z}) f(y_1,y_2)\,d\tilde{z}\,dy_1\,dy_2 \\ &= \int R_j^{(1)}(x_1-y_1, x_2-\tilde{z}) R_k^{(2)}(\tilde{z}-y_2) f(y_1,y_2)\,d\tilde{z}\,dy_1\,dy_2 \\ &= R_j^{(1)} * R_k^{(2)} *_2 f(x_1,x_2), \end{aligned}$$

which implies that (2-5) holds. □

3. Upper bound of the iterated commutator $[[b, R_i^{(1)}], R_j^{(2)}]_2$

In this section, we prove Theorem 1.3, i.e., the upper bound of the iterated commutator $[[b, R_i^{(1)}], R_j^{(2)}]_2$. As we pointed out earlier, in the flag setting, there is lack of a suitable wavelet basis or Haar basis and hence the approaches in [Lacey et al. 2009; Dalenc and Ou 2016] are not available. We establish a fundamental duality argument (Lemma 3.3) with respect to general flag-type area integrals and flag Carleson measures, and then apply the technique of harmonic expansion to obtain the full versions of flag-type Carleson measure inequalities (Proposition 3.5), which plays the role of "paraproducts". Then, by considering the bilinear form associated with the iterated commutator $[[b, R_i^{(1)}], R_j^{(2)}]_2$ and by integration by parts, we can decompose the bilinear form into a summation of different versions of "paraproducts". Then the upper bound of the iterated commutator $[[b, R_i^{(1)}], R_j^{(2)}]_2$ follows from applying Proposition 3.5 to each "paraproducts".

Extension via flag Poisson operator. For any $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$, we define the flag Poisson integral of f by

$$F(x_1,x_2,t_1,t_2) := P_{t_1,t_2} * f(x_1,y_2), \tag{3-1}$$

where

$$P_{t_1,t_2}(x_1,x_2) = P_{t_1}^{(1)} *_{\mathbb{R}^m} P_{t_2}^{(2)}(x_1,x_2). \tag{3-2}$$

Since $P(x_1,x_2) \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$, it easy to see that $F(x_1,x_2,t_1,t_2)$ is well-defined. Moreover, for any fixed t_1 and t_2, we know $P_{t_1,t_2} * f(x_1,x_2)$ is a bounded C^∞ function and the function $F(x_1,x_2,t_1,t_2)$ is harmonic in (x_1,x_2,t_1) and (x_2,t_2), respectively. $F(x_1,x_2,t_1,t_2)$ is the flag harmonic extension of f to $\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{m+1}$. More precisely,

$$\begin{aligned} \Delta_{\mathbb{R}^{n+m+1}} F(x_1,x_2,t_1,t_2) &= (\partial_{t_1}^2 + \Delta_{x_1,x_2}) F(x_1,x_2,t_1,t_2) = 0 \quad \text{in } \mathbb{R}_+^{n+m+1}, \\ \Delta_{\mathbb{R}^{m+1}} F(x_1,x_2,t_1,t_2) &= (\partial_{t_2}^2 + \Delta_{x_2}) F(x_1,x_2,t_1,t_2) = 0 \quad \text{in } \mathbb{R}_+^{m+1}, \end{aligned} \tag{3-3}$$

and

$$\begin{aligned} \lim_{t_1\to 0} \partial_{t_1} F(x_1,x_2,t_1,t_2) &= -(\Delta_{x_1,x_2})^{\frac12} P^{(2)} *_{\mathbb{R}^m} f(x_1,x_2) \quad \text{on } \mathbb{R}^{n+m}, \\ \lim_{t_2\to 0} \partial_{t_2} F(x_1,x_2,t_1,t_2) &= -(\Delta_{x_2})^{\frac12} P^{(1)} * f(x_1,x_2) \quad \text{on } \mathbb{R}^{n+m}, \end{aligned}$$

$$\begin{aligned}
&\lim_{t_1\to 0} F(x_1,x_2,t_1,t_2) = P^{(2)} *_{\mathbb{R}^m} f(x_1,x_2) && \text{on } \mathbb{R}^{n+m},\\
&\lim_{t_2\to 0} F(x_1,x_2,t_1,t_2) = P^{(1)} * f(x_1,x_2) && \text{on } \mathbb{R}^{n+m},\\
&\lim_{t_1\to 0,\ t_2\to 0} F(x_1,x_2,t_1,t_2) = f(x_1,x_2) && \text{on } \mathbb{R}^{n+m},\\
&\lim_{|(x_1,x_2,t_1)|\to\infty} F(x_1,x_2,t_1,t_2) = 0,\\
&\lim_{|(x_2,t_2)|\to\infty} F(x_1,x_2,t_1,t_2) = 0.
\end{aligned}$$

We then have the following lemma providing a connection between the boundary values f and the flag harmonic extension F. This follows from the decay of the flag harmonic extensions of f and repeated applications of integration by parts in the variables t_1 and t_2.

Lemma 3.1. *For $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$, let F be the same as in* (3-1). *Then we have*

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x_1,x_2)\,dx_1\,dx_2 = \int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\partial^2_{t_1}\, t_2\partial^2_{t_2} F(x_1,x_2,t_1,t_2)\,dx_1\,dx_2\,dt_1\,dt_2. \tag{3-4}$$

Proof. We start from the right-hand side of (3-4). We write

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} & t_1\partial^2_{t_1}\, t_2\partial^2_{t_2} F(x_1,x_2,t_1,t_2)\,dx_1\,dx_2\,dt_1\,dt_2\\
&= \int_{\mathbb{R}^{m+1}_+} t_2\,\partial^2_{t_2} P^{(2)}_{t_2} *_{\mathbb{R}^m} \biggl(\int_{\mathbb{R}^{n+1}_+} t_1\,\partial^2_{t_1} P^{(1)}_{t_1} * f(x_1,x_2)\,dx_1\,dt_1\biggr)\,dx_2\,dt_2\\
&= \int_{\mathbb{R}^m} \biggl(\int_{\mathbb{R}^{n+1}_+} t_1\,\partial^2_{t_1} P^{(1)}_{t_1} * f(x_1,x_2)\,dx_1\,dt_1\biggr)\,dx_2,
\end{aligned}$$

where the last equality follows from decay of the flag harmonic extensions of f and using integration by parts in the variable t_2. To continue, we write the right-hand side of the last equality above as

$$\int_{\mathbb{R}^{n+m+1}_+} t_1\,\partial^2_{t_1} P^{(1)}_{t_1} * f(x_1,x_2)\,dx_1\,dx_2\,dt_1 = \int_{\mathbb{R}^{n+m}} f(x_1,x_2)\,dx_1\,dx_2,$$

which yields (3-4). Again, the last equality follows from decay of the flag harmonic extensions of f and using integration by parts in the variable t_1. □

Flag area functions and estimates. We also have a more general version of the area function.

Definition 3.2. For a function $G(x_1,x_2,t_1,t_2)$ defined on $\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+$, the general flag-type Lusin area integral of G is defined by

$$S_{\mathcal{F},L}(G)(x_1,x_2) := \biggl\{\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+} \chi_{t,s}(x_1-w_1,x_2-w_2)|G(w_1,w_2,t_1,t_2)|^2\,\frac{dw_1\,dt_1}{t_1^{n+m+1}}\frac{dw_2\,dt_2}{t_2^{m+1}}\biggr\}^{\frac12}. \tag{3-5}$$

Lemma 3.3. *Suppose $F(x_1,x_2,t_1,t_2)$ and $G(x_1,x_2,t_1,t_2)$ are defined on $\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+$. Then the following estimate holds*:

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+} F(x_1,x_2,t_1,t_2)G(x_1,x_2,t_1,t_2)\,dx_1\,dx_2\,dt_1\,dt_2\\ &\quad\le C\sup_{\Omega\subset\mathbb{R}^n\times\mathbb{R}^m}\biggl(\frac{1}{|\Omega|}\int_{T(\Omega)} t_1\,t_2\,|F(y_1,y_2,t_1,t_2)|^2\,dy_1\,dy_2\,dt_1\,dt_2\biggr)^{1/2}\\ &\quad\quad\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\biggl(\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+}\chi_{t_1,t_2}(x_1-y_1,x_2-y_2)|G(y_1,y_2,t_1,t_2)|^2\,\frac{dy_1\,dy_2\,dt_1\,dt_2}{t_1^{n+m+1}t_2^{m+1}}\biggr)^{1/2}dx_1\,dx_2.\end{aligned}\tag{3-6}$$

Proof. Suppose both factors on the right-hand side above are finite, since otherwise there is nothing to prove. We also note that the second factor is actually $\|S_{\mathcal{F}}(G)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}$.

We now let

$$\Omega_k:=\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m:\ S_{\mathcal{F},L}(G)(x_1,x_2)>2^k\}$$

and define

$$B_k:=\bigl\{R=I_1\times I_2:\ |(I_1\times I_2)\cap\Omega_k|>\tfrac12|I_1\times I_2|,\ |(I_1\times I_2)\cap\Omega_{k+1}|\le\tfrac12|I_1\times I_2|\bigr\},$$

where I_1 and I_2 are dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ with side-lengths $\ell(I)$ and $\ell(J)$ satisfying $\ell(I)\le\ell(J)$. Moreover, we define

$$\Omega_k=\bigcup_{R\in B_k}R\quad\text{and}\quad\widetilde{\Omega}_k=\bigl\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m:\ M_{\text{flag}}(\chi_{\Omega_k})(x_1,x_2)>\tfrac12\bigr\}.$$

Next, we have

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+} F(x_1,x_2,t_1,t_2)G(x_1,x_2,t_1,t_2)\,dx_1\,dx_2\,dt_1\,dt_2\\ &\qquad=\sum_k\sum_{R\in B_k}\int_{T(R)}\sqrt{t_1t_2}F(x_1,x_2,t_1,t_2)\,\frac{G(x_1,x_2,t_1,t_2)}{\sqrt{t_1t_2}}\,dx_1\,dx_2\,dt_1\,dt_2\\ &\qquad\le\sum_k\biggl(\sum_{R\in B_k}\int_{T(R)}t_1t_2|F(x_1,x_2,t_1,t_2)|^2\,dx_1\,dx_2\,dt_1\,dt_2\biggr)^{1/2}\\ &\qquad\qquad\qquad\times\biggl(\sum_{R\in B_k}\int_{T(R)}|G(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}\biggr)^{1/2}\\ &\qquad=\sum_k\biggl(\frac{1}{|\Omega_k|}\sum_{R\in B_k}\int_{T(R)}t_1t_2|F(x_1,x_2,t_1,t_2)|^2\,dx_1\,dx_2\,dt_1\,dt_2\biggr)^{1/2}\\ &\qquad\qquad\qquad\times\biggl(|\Omega_k|\sum_{R\in B_k}\int_{T(R)}|G(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}\biggr)^{1/2}\\ &\qquad\le\sum_k\biggl(\frac{1}{|\Omega_k|}\int_{T(\Omega_k)}t_1t_2|F(x_1,x_2,t_1,t_2)|^2\,dx_1\,dx_2\,dt_1\,dt_2\biggr)^{1/2}\\ &\qquad\qquad\qquad\times\biggl(|\widetilde{\Omega}_k|\sum_{R\in B_k}\int_{T(R)}|G(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}\biggr)^{1/2}\end{aligned}$$

$$\le \sup_{\Omega\subset\mathbb{R}^n\times\mathbb{R}^m}\left(\frac{1}{|\Omega|}\int_{T(\Omega)} t_1t_2|F(x_1,x_2,t_1,t_2)|^2\,dx_1\,dx_2\,dt_1\,dt_2\right)^{1/2} \times\sum_k\left(|\widetilde{\Omega}_k|\sum_{R\in B_k}\int_{T(R)}|G(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}\right)^{1/2}.$$

As for the second factor in the last inequality above, note that

$$\begin{aligned}
2^{2k}|\widetilde{\Omega}_k\backslash\Omega_k| &\ge \int_{\widetilde{\Omega}_k\backslash\Omega_k} S_{\mathcal{F},L}(G)(x_1,x_2)^2\,dx_1\,dx_2\\
&= \int_{\widetilde{\Omega}_k\backslash\Omega_k}\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+}\chi_{t_1,t_2}(x_1-y_1,x_2-y_2)|G(y_1,y_2,t_1,t_2)|^2\,\frac{dy_1\,dy_2\,dt_1\,dt_2}{t_1^{n+m+1}t_2^{m+1}}\,dx_1\,dx_2\\
&= \int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+}\int_{\widetilde{\Omega}_k\backslash\Omega_k}\chi_{t_1,t_2}(x_1-y_1,x_2-y_2)\,dx_1\,dx_2\,|G(y_1,y_2,t_1,t_2)|^2\,\frac{dy_1\,dy_2\,dt_1\,dt_2}{t_1^{n+m+1}t_2^{m+1}}\\
&\approx \int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+}|G(y_1,y_2,t_1,t_2)|^2\,\frac{dy_1\,dy_2\,dt_1\,dt_2}{t_1t_2}\\
&\ge \sum_{R\in B_k}\int_{T(R)}|G(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+} F(x_1,x_2,t_1,t_2)G(x_1,x_2,t_1,t_2)\,dx_1\,dx_2\,dt_1\,dt_2\\
&\qquad\le \sup_{\Omega\subset\mathbb{R}^n\times\mathbb{R}^m}\left(\frac{1}{|\Omega|}\int_{T(\Omega)} t_1t_2|F(x_1,x_2,t_1,t_2)|^2\,dx_1\,dx_2\,dt_1\,dt_2\right)^{1/2}\sum_k(|\widetilde{\Omega}_k|2^{2k}|\widetilde{\Omega}_k\backslash\Omega_k|)^{1/2}\\
&\qquad\le \sup_{\Omega\subset\mathbb{R}^n\times\mathbb{R}^m}\left(\frac{1}{|\Omega|}\int_{T(\Omega)} |t_1t_2F(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}\right)^{1/2}\sum_k|\Omega_k|2^k\\
&\qquad\le \sup_{\Omega\subset\mathbb{R}^n\times\mathbb{R}^m}\left(\frac{1}{|\Omega|}\int_{T(\Omega)} |t_1t_2F(x_1,x_2,t_1,t_2)|^2\,\frac{dx_1\,dx_2\,dt_1\,dt_2}{t_1t_2}\right)^{1/2}\|S_{\mathcal{F},L}(G)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)},
\end{aligned}$$

which gives (3-6). □

From Lemma 3.3 above and the definition of $\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$, we can obtain the following corollary immediately.

Corollary 3.4. *Suppose $G(x_1,x_2,t_1,t_2)$ is defined on $\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+$, and $F(x_1,x_2,t_1,t_2) := P_{t_1,t_2} * f(x_1,x_2)$, where $f\in\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$. Then we have*

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{m+1}_+}|\nabla^{(1)}\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,|G(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|f\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|S_{\mathcal{F},L}(G)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{3-7}$$

Moreover, based on Lemma 3.3, we can also establish the following estimates.

Proposition 3.5. *Suppose* $F(x_1,x_2,t_1,t_2)=P_{t_1,t_2}*f(x_1,x_2)$, $G(x_1,x_2,t_1,t_2)=P_{t_1,t_2}*g(x_1,x_2)$, *and* $B(x_1,x_2,t_1,t_2)=P_{t_1,t_2}*b(x_1,x_2)$. *Then we have*

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|\nabla^{(1)}\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-8}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|\nabla^{(1)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-9}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-10}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-11}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|\nabla^{(1)}\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-12}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|\nabla^{(1)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-13}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)|&\\ \times|\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2&\\ \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},&\end{aligned}\tag{3-14}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_1,x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac{1}{2}}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-15}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(1)}\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac{1}{2}}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac{1}{2}}(-\Delta_{x_2})^{\frac{1}{2}}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-16}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(1)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac{1}{2}}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac{1}{2}}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-17}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac{1}{2}}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac{1}{2}}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-18}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac{1}{2}}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-19}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(1)}\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac{1}{2}}(-\Delta_{x_2})^{\frac{1}{2}}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-20}$$

$$\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|\,|\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(1)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2 \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac{1}{2}}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-21}$$

$$\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)||\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2$$
$$\le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-22}$$

$$\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)||\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2$$
$$\le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{3-23}$$

Proof. We first point out that for $f\in C_0^\infty(\mathbb{R}^{n+m})$, $F(x_1,x_2,t_1,t_2)=P_{t_1,t_2}*f(x_1,x_2)$,

$$\sup_{\substack{(y_1,y_2,t_1,t_2)\\ \chi_{t_1,t_2}(x_1-y_1,x_2-y_2)\neq0}}|F(y_1,y_2,t_1,t_2)|\le\sup_{\substack{(y_1,y_2,t_1,t_2)\\ |x_1-y_1|<t_1+t_2,|x_2-y_2|<t_2}}|P_{t_1,t_2}*f(y_1,y_2)|$$
$$\le M_1(M_2(f(\cdot_1,\cdot))(\cdot_2))(x_1,x_2),$$

where M_1 and M_2 are the Hardy–Littlewood maximal functions on $\mathbb{R}^{n+m}$ and $\mathbb{R}^m$, respectively.

Next, based on the estimate above and from the property of the Poisson semigroup, we have

$$\sup_{\substack{(y_1,y_2,t_1,t_2)\\ \chi_{t_1,t_2}(x_1-y_1,x_2-y_2)\neq0}}|\partial_{t_1}\partial_{t_2}F(y_1,y_2,t_1,t_2)|\le\sup_{\substack{(y_1,y_2,t_1,t_2)\\ |x_1-y_1|<t_1+t_2,|x_2-y_2|<t_2}}|P_{t_1,t_2}*((-\Delta_{(1)})^{\frac12}(-\Delta_{(2)})^{\frac12}f)(y_1,y_2)|$$
$$\le M_1\big(M_2(((-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}f)(\cdot_1,\cdot))(\cdot_2)\big)(x_1,x_2).$$

Also, we have

$$\sup_{\substack{(y_1,y_2,t_1,t_2)\\ \chi_{t_1,t_2}(x_1-y_1,x_2-y_2)\neq0}}|\nabla_{y_1,y_2}\nabla_{y_2}F(y_1,y_2,t_1,t_2)|\le\sup_{\substack{(y_1,y_2,t_1,t_2)\\ |x_1-y_1|<t_1+t_2,|x_2-y_2|<t_2}}|P_{t_1,t_2}*(\nabla_{\cdot_1,\cdot_2}\nabla_{\cdot_2}f)(y_1,y_2)|$$
$$\le M_1\big(M_2((\nabla_{\cdot_1,\cdot_2}\nabla_{\cdot_2}f)(\cdot_1,\cdot))(\cdot_2)\big)(x_1,x_2).$$

Then, we first consider (3-8). Based on the estimates above and Corollary 3.4, we have

$$\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)||\nabla_{x_1,x_2}\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G(x_1,x_2,t_1,t_2)| \times|\nabla^{(1)}\nabla^{(2)}F(x_1,x_2,t_1,t_2)|\,dx_1\,dx_2\,dt_1\,dt_2$$
$$\le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\int_{\mathbb{R}^n\times\mathbb{R}^m}S_{\mathcal{F},L}(t_1t_2\nabla_{x_1,x_2}\nabla_{x_2}\nabla^{(1)}\nabla^{(2)}G)(x_1,x_2) \times\Big(M_1\big(M_2(((-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}f)(\cdot_1,\cdot))(\cdot_2)\big)(x_1,x_2) +M_1\big(M_2((\nabla_{\cdot_1,\cdot_2}\nabla_{\cdot_2}f)(\cdot_1,\cdot))(\cdot_2)\big)(x_1,x_2)\Big)\,dx_1\,dx_2$$
$$\le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\int_{\mathbb{R}^n\times\mathbb{R}^m}S_{\mathcal{F}}\big(\nabla_{x_1,x_2}\nabla_{x_2}(-\Delta_{x_1,x_2})^{-\frac12}(-\Delta_{x_2})^{-\frac12}(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}G\big)(x_1,x_2) \times\Big(M_1\big(M_2(((-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}f)(\cdot_1,\cdot))(\cdot_2)\big)(x_1,x_2) +M_1\big(M_2((\nabla_{\cdot_1,\cdot_2}\nabla_{\cdot_2}(-\Delta_{\cdot_1,\cdot_2})^{-\frac12}(-\Delta_{\cdot_2})^{-\frac12}(-\Delta_{\cdot_1,\cdot_2})^{\frac12}(-\Delta_{\cdot_2})^{\frac12}f)(\cdot_1,\cdot))(\cdot_2)\big)(x_1,x_2)\Big)\,dx_1\,dx_2$$
$$\le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|(-\Delta_{x_1,x_2})^{\frac12}(-\Delta_{x_2})^{\frac12}f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{3-24}$$

where in the second inequality the area function $S_{\mathcal{F}}$ is defined as in Definition 2.2, and the last inequality follows from Hölder's inequality and boundedness of the maximal functions as well as the boundedness of the flag Riesz transforms. Hence we see that (3-8) holds.

By using an estimate similar to that above, we can obtain the estimates in (3-9)–(3-23). We omit the details here since they are straightforward. □

Upper bound for iterated commutators.

Theorem 3.6. *For every $b \in \mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$, $g \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and for any $i = 1, 2, \ldots, m+n$, $j = 1, \ldots, n$, there exits a positive constant C depending only on p, n and m such that*

$$\|[[b, R_i^{(1)}], R_j^{(2)}]_2(g)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{3-25}$$

Proof. Recall that

$$\begin{aligned}[][[b, R_i^{(1)}], R_j^{(2)}]_2(g)(x_1,x_2) = b(x_1,x_2)R_i^{(1)} * R_j^{(2)} *_2 g(x_1,x_2) - R_i^{(1)} * (b\cdot R_j^{(2)} *_2 g)(x_1,x_2)\\ - R_j^{(2)} *_2 (b\cdot R_i^{(1)} * g)(x_1,x_2) + R_j^{(2)} *_2 R_i^{(1)} * (b\cdot g)(x_1,x_2).\end{aligned}$$

Hence, for every $f \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, we have

$$\begin{aligned}\langle f, [[b, R_i^{(1)}], R_j^{(2)}]_2(g)\rangle = \langle f\cdot b, R_i^{(1)} * R_j^{(2)} *_2 g\rangle + \langle R_i^{(1)} * f, b\cdot R_j^{(2)} *_2 g\rangle\\ + \langle R_j^{(2)} *_2 f, b\cdot R_i^{(1)} * g\rangle + \langle R_j^{(2)} *_2 R_i^{(1)} * f, b\cdot g\rangle.\end{aligned}$$

Denote by B, F, G the flag harmonic extensions of the functions b, f, g, respectively, as defined in (3-1). And for each fixed i, j, denote by $(R_i^{(1)} * f)^\sim$, $(R_j^{(2)} *_2 f)^\sim$ and $(R_i^{(1)} * R_j^{(2)} *_2 f)^\sim$ the flag harmonic extensions of $R_i^{(1)} * f$, $R_j^{(2)} *_2 f$ and $R_i^{(1)} * R_j^{(2)} *_2 f$.

Then we write

$$\begin{aligned}&\langle f, [[b, R_i^{(1)}], R_j^{(2)}]_2(g)\rangle\\ &= \int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1\partial_{t_1}^2\, t_2\partial_{t_2}^2\big(F\cdot B\cdot(R_i^{(1)} * R_j^{(2)} *_2 g)^\sim + (R_i^{(1)} * f)^\sim\cdot B\cdot(R_j^{(2)} *_2 g)^\sim\\ &\qquad + (R_j^{(2)} *_2 f)^\sim\cdot B\cdot(R_i^{(1)} * g)^\sim + (R_j^{(2)} *_2 R_i^{(1)} * f)^\sim\cdot B\cdot G\big)\,dx_1\,dx_2\,dt_1\,dt_2.\end{aligned} \tag{3-26}$$

We now claim that the right-hand side of (3-26) is bounded by

$$C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{3-27}$$

To see this, we compute the derivatives $t_1\partial_{t_1}^2\, t_2\partial_{t_2}^2$ for the integrand in the right-hand side of (3-26). Then we have the following terms:

$$\begin{aligned}\mathcal{C}_1 = \int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} &\Big(t_1\partial_{t_1}^2\, t_2\partial_{t_2}^2 B\cdot F\cdot(R_i^{(1)} * R_j^{(2)} *_2 g)^\sim + t_1\partial_{t_1}^2\, t_2\partial_{t_2}^2 B\cdot(R_i^{(1)} * f)^\sim\cdot(R_j^{(2)} *_2 g)^\sim\\ &+ t_1\partial_{t_1}^2\, t_2\partial_{t_2}^2 B\cdot(R_j^{(2)} *_2 f)^\sim\cdot(R_i^{(1)} * g)^\sim\\ &+ t_1\partial_{t_1}^2\, t_2\partial_{t_2}^2 B\cdot(R_j^{(2)} *_2 R_i^{(1)} * f)^\sim\cdot G\Big)\,dx_1\,dx_2\,dt_1\,dt_2;\end{aligned} \tag{3-28}$$

$$\begin{aligned}\mathcal{C}_2=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1\partial_{t_1}^2\, t_2\partial_{t_2}B\cdot\partial_{t_2}(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1\partial_{t_1}^2\, t_2\partial_{t_2}B\cdot\partial_{t_2}((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1\partial_{t_1}^2\, t_2\partial_{t_2}B\cdot\partial_{t_2}((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1\partial_{t_1}^2\, t_2\partial_{t_2}B\cdot\partial_{t_2}((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-29}$$

$$\begin{aligned}\mathcal{C}_3=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1\,\partial_{t_1}t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1\,\partial_{t_1}t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1\,\partial_{t_1}t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1\,\partial_{t_1}t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-30}$$

$$\begin{aligned}\mathcal{C}_4=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1\,\partial_{t_1}t_2\,\partial_{t_2}B\cdot\partial_{t_1}\partial_{t_2}(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1\,\partial_{t_1}t_2\,\partial_{t_2}B\cdot\partial_{t_1}\partial_{t_2}((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1\,\partial_{t_1}t_2\,\partial_{t_2}B\cdot\partial_{t_1}\partial_{t_2}((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1\,\partial_{t_1}t_2\,\partial_{t_2}B\cdot\partial_{t_1}\partial_{t_2}((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-31}$$

$$\begin{aligned}\mathcal{C}_5=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1\,\partial_{t_1}t_2B\cdot\partial_{t_1}\partial_{t_2}^2(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1\,\partial_{t_1}t_2B\cdot\partial_{t_1}\partial_{t_2}^2((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1\,\partial_{t_1}t_2B\cdot\partial_{t_1}\partial_{t_2}^2((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1\,\partial_{t_1}t_2B\cdot\partial_{t_1}\partial_{t_2}^2((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-32}$$

$$\begin{aligned}\mathcal{C}_6=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1t_2\,\partial_{t_2}B\cdot\partial_{t_1}^2\partial_{t_2}(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1t_2\partial_{t_2}B\cdot\partial_{t_1}^2\partial_{t_2}((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1t_2\,\partial_{t_2}B\cdot\partial_{t_1}^2\partial_{t_2}((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1t_2\,\partial_{t_2}B\cdot\partial_{t_1}^2\partial_{t_2}((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-33}$$

$$\begin{aligned}\mathcal{C}_7=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}^2(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}^2((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}^2((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1t_2\,\partial_{t_2}^2B\cdot\partial_{t_1}^2((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-34}$$

$$\begin{aligned}\mathcal{C}_8=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1t_2\,\partial_{t_1}^2B\cdot\partial_{t_2}^2(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1t_2\,\partial_{t_1}^2B\cdot\partial_{t_2}^2((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1t_2\,\partial_{t_1}^2B\cdot\partial_{t_2}^2((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1t_2\,\partial_{t_1}^2B\cdot\partial_{t_2}^2((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2; \end{aligned}\tag{3-35}$$

$$\begin{aligned}\mathcal{C}_9=\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} & t_1t_2B\cdot\partial_{t_1}^2\partial_{t_2}^2(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^{\sim})+t_1t_2B\cdot\partial_{t_1}^2\partial_{t_2}^2((R_i^{(1)}*f)^{\sim}\cdot(R_j^{(2)}*_2 g)^{\sim})\\ &+t_1t_2B\cdot\partial_{t_1}^2\partial_{t_2}^2((R_j^{(2)}*_2 f)^{\sim}\cdot(R_i^{(1)}*g)^{\sim})\\ &+t_1t_2B\cdot\partial_{t_1}^2\partial_{t_2}^2((R_j^{(2)}*_2 R_i^{(1)}*f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2. \end{aligned}\tag{3-36}$$

We first consider $\mathcal{C}_1$. Note that $\partial_{t_2}^2 B = -\Delta_{x_2} B = -\nabla_{x_2} \cdot \nabla_{x_2} B$ and that $\partial_{t_1}^2 B = -\Delta_{x_1,x_2} B = -\nabla_{x_1,x_2} \cdot \nabla_{x_1,x_2} B$. So, integration by parts gives

$$\begin{aligned}\mathcal{C}_1 = \int_{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+} & t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_1,x_2} \nabla_{x_2} (F \cdot (R_i^{(1)} * R_j^{(2)} *_2 g)^{\sim}) \\ & + t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_1,x_2} \nabla_{x_2} ((R_i^{(1)} * f)^{\sim} \cdot (R_j^{(2)} *_2 g)^{\sim}) \\ & + t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_1,x_2} \nabla_{x_2} ((R_j^{(2)} *_2 f)^{\sim} \cdot (R_i^{(1)} * g)^{\sim}) \\ & + t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_1,x_2} \nabla_{x_2} ((R_j^{(2)} *_2 R_i^{(1)} * f)^{\sim} \cdot G) \, dx_1 \, dx_2 \, dt_1 \, dt_2 \\ =: {} & \mathcal{C}_{1,1} + \mathcal{C}_{1,2} + \mathcal{C}_{1,3} + \mathcal{C}_{1,4}.\end{aligned}$$

For the first term, it is clear that

$$\begin{aligned}\mathcal{C}_{1,1} &= \int_{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+} t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_1,x_2} \nabla_{x_2} F \cdot (R_i^{(1)} * R_j^{(2)} *_2 g)^{\sim} \, dx_1 \, dx_2 \, dt_1 \, dt_2 \\ &= \int_{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+} t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_1,x_2} F \cdot \nabla_{x_2} (R_i^{(1)} * R_j^{(2)} *_2 g)^{\sim} \, dx_1 \, dx_2 \, dt_1 \, dt_2 \\ &= \int_{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+} t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot \nabla_{x_2} F \cdot \nabla_{x_1,x_2} (R_i^{(1)} * R_j^{(2)} *_2 g)^{\sim} \, dx_1 \, dx_2 \, dt_1 \, dt_2 \\ &= \int_{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+} t_1 t_2 \nabla_{x_1,x_2} \nabla_{x_2} B \cdot F \cdot \nabla_{x_1,x_2} \nabla_{x_2} (R_i^{(1)} * R_j^{(2)} *_2 g)^{\sim} \, dx_1 \, dx_2 \, dt_1 \, dt_2 \\ &=: \mathcal{C}_{1,1,1} + \mathcal{C}_{1,1,2} + \mathcal{C}_{1,1,3} + \mathcal{C}_{1,1,4}.\end{aligned}$$

It is direct that $\mathcal{C}_{1,1,1}$ and $\mathcal{C}_{1,1,4}$ can be handled by using (3-9), and $\mathcal{C}_{1,1,2}$ and $\mathcal{C}_{1,1,3}$ can be handled by using (3-10), which gives

$$\mathcal{C}_{1,1} \le C \|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)} \|g\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \|f\|_{L^{p'}(\mathbb{R}^n \times \mathbb{R}^m)}.$$

Symmetrically we obtain the estimate for $\mathcal{C}_{1,4}$, and using similar estimates we can handle $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$. All these three terms are have the same upper as $\mathcal{C}_{1,1}$ above.

Next, for $\mathcal{C}_2$, note that $\partial_{t_1}^2 B = -\Delta_{x_1,x_2} B = -\nabla_{x_1,x_2} \cdot \nabla_{x_1,x_2} B$. Thus, similar to the term $\mathcal{C}_1$, by integration by parts, we have

$$\begin{aligned}\mathcal{C}_2 = -\int_{\mathbb{R}^{n+1}_+ \times \mathbb{R}^{m+1}_+} & t_1 t_2 \nabla_{x_1,x_2} \partial_{t_2} B \cdot \nabla_{x_1,x_2} \partial_{t_2} (F \cdot (R_i^{(1)} * R_j^{(2)} *_2 g)^{\sim}) \\ & + t_1 t_2 \nabla_{x_1,x_2} \partial_{t_2} B \cdot \nabla_{x_1,x_2} \partial_{t_2} ((R_i^{(1)} * f)^{\sim} \cdot (R_j^{(2)} *_2 g)^{\sim}) \\ & + t_1 t_2 \nabla_{x_1,x_2} \partial_{t_2} B \cdot \nabla_{x_1,x_2} \partial_{t_2} ((R_j^{(2)} *_2 f)^{\sim} \cdot (R_i^{(1)} * g)^{\sim}) \\ & + t_1 t_2 \nabla_{x_1,x_2} \partial_{t_2} B \cdot \nabla_{x_1,x_2} \partial_{t_2} ((R_j^{(2)} *_2 R_i^{(1)} * f)^{\sim} \cdot G) \, dx_1 \, dx_2 \, dt_1 \, dt_2 \\ =: {} & \mathcal{C}_{2,1} + \mathcal{C}_{2,2} + \mathcal{C}_{2,3} + \mathcal{C}_{2,4}.\end{aligned}$$

Again, the upper bounds from the four terms above can be obtained by applying Proposition 3.5, and they are all controlled by

$$C \|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)} \|g\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \|f\|_{L^{p'}(\mathbb{R}^n \times \mathbb{R}^m)}.$$

The term $\mathcal{C}_3$ can be handled symmetrically to $\mathcal{C}_2$ and we obtain the same upper bounds.

For the term $\mathcal{C}_4$, by noting that $|\partial_{t_1}\partial_{t_2}B(x_1,x_2,t_1,t_2)|$ is bounded by $|\nabla^{(1)}\nabla^{(2)}B(x_1,x_2,t_1,t_2)|$, we obtain that $\mathcal{C}_4$ is bounded by

$$C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},$$

where we apply again the upper bounds in Proposition 3.5.

We now turn to the term $\mathcal{C}_9$. We first point out the following equalities:

$$\begin{aligned}
\partial_{t_1}(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim(x_1,x_2) &= -c\,\partial_{(x_1,x_2),i}\ (R_j^{(2)}*_2 g)^\sim(x_1,x_2),\\
\partial_{t_1}^2(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim(x_1,x_2) &= -c\,\partial_{t_1}\partial_{(x_1,x_2),i}\ (R_j^{(2)}*_2 g)^\sim(x_1,x_2),\\
\partial_{t_2}(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim(x_1,x_2) &= -c\,\partial_{x_{2,j}}\ (R_i^{(1)}*g)^\sim(x_1,x_2),\\
\partial_{t_2}^2(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim(x_1,x_2) &= -c\,\partial_{t_2}\partial_{x_{2,j}}\ (R_i^{(1)}*g)^\sim(x_1,x_2),\\
\partial_{t_1}(R_i^{(1)}*f)^\sim &= -c\,\partial_{(x_1,x_2),i}(f)^\sim,\\
\partial_{t_1}^2(R_i^{(1)}*f)^\sim &= -c\,\partial_{t_1}\partial_{(x_1,x_2),i}(f)^\sim,\\
\partial_{t_2}(R_j^{(2)}*_2 g)^\sim &= -c\,\partial_{x_{2,j}}(g)^\sim,\\
\partial_{t_2}^2(R_j^{(2)}*_2 g)^\sim &= -c\,\partial_{t_2}\partial_{x_{2,j}}(g)^\sim.
\end{aligned}$$

Then for the term $\mathcal{C}_9$, we get

$$\begin{aligned}
&\partial_{t_1}^2\partial_{t_2}^2\big(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim+(R_i^{(1)}*f)^\sim\cdot(R_j^{(2)}*_2 g)^\sim+(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim+(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot G\big)\\
&=4\,\partial_{(x_1,x_2),i}\partial_{t_1}\partial_{x_{2,j}}\partial_{t_2}(FG)\\
&\qquad-2\nabla_{x_1,x_2}\,\partial_{x_{2,j}}\partial_{t_2}(\nabla_{x_1,x_2}(R_i^{(1)}*f)^\sim\cdot G)-2\nabla_{x_1,x_2}\,\partial_{x_{2,j}}\partial_{t_2}(F\cdot\nabla_{x_1,x_2}(R_i^{(1)}*g)^\sim)\\
&\qquad+2\nabla_{x_1,x_2}\,\partial_{x_{2,j}}\partial_{t_2}(\nabla_{x_1,x_2}F\cdot(R_i^{(1)}*g)^\sim)+2\nabla_{x_1,x_2}\,\partial_{x_{2,j}}\partial_{t_2}((R_i^{(1)}*f)^\sim\cdot\nabla_{x_1,x_2}G)\\
&\quad-2\,\partial_{(x_1,x_2),i}\partial_{t_1}\nabla_{x_2}(\nabla_{x_2}(R_j^{(2)}*f)^\sim\cdot G)\\
&\qquad+\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}\nabla_{x_2}(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot G)+\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_2}(R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_1,x_2}(R_i^{(1)}*g)^\sim)\\
&\qquad-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}\nabla_{x_2}(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim)-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_2}(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot\nabla_{x_1,x_2}G)\\
&\quad-2\,\partial_{(x_1,x_2),i}\partial_{t_1}\nabla_{x_2}(F\cdot\nabla_{x_2}(R_j^{(2)}*g)^\sim)\\
&\qquad+\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}(R_i^{(1)}*f)^\sim\cdot\nabla_{x_2}(R_j^{(2)}*_2 g)^\sim)+\nabla_{x_1,x_2}\nabla_{x_2}(F\cdot\nabla_{x_1,x_2}\nabla_{x_2}(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim)\\
&\qquad-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}\nabla_{x_2}(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim)-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_2}(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot\nabla_{x_1,x_2}G)\\
&\quad+2\,\partial_{(x_1,x_2),i}\partial_{t_1}\nabla_{x_2}(\nabla_{x_2}F\cdot(R_j^{(2)}*g)^\sim)\\
&\qquad-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}\nabla_{x_2}(R_i^{(1)}*f)^\sim\cdot(R_j^{(2)}*_2 g)^\sim)-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_2}F\cdot\nabla_{x_1,x_2}(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim)\\
&\qquad+\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}\nabla_{x_2}F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim)+\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_2}(R_i^{(1)}*f)^\sim\cdot\nabla_{x_1,x_2}(R_j^{(2)}*_2 g)^\sim)\\
&\quad+2\,\partial_{(x_1,x_2),i}\partial_{t_1}\nabla_{x_2}((R_j^{(2)}*f)^\sim\cdot\nabla_{x_2}G)\\
&\qquad-\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}(R_i^{(1)}*R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_2}G)-\nabla_{x_1,x_2}\nabla_{x_2}((R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_1,x_2}\nabla_{x_2}(R_i^{(1)}*g)^\sim)\\
&\qquad+\nabla_{x_1,x_2}\nabla_{x_2}(\nabla_{x_1,x_2}(R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_2}(R_i^{(1)}*g)^\sim)+\nabla_{x_1,x_2}\nabla_{x_2}((R_i^{(1)}*R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_1,x_2}\nabla_{x_2}G).
\end{aligned}$$

Thus, we input the above 25 terms back into the right-hand side of $\mathcal{C}_9$ and obtain the terms as follows:

$$\begin{aligned}\mathcal{C}_9 &= \int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2B\cdot\partial_{t_1}^2\partial_{t_2}^2\big(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim+(R_i^{(1)}*f)^\sim\cdot(R_j^{(2)}*_2 g)^\sim \\ &\qquad +(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim+(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot G\big)\,dx_1\,dx_2\,dt_1\,dt_2 \\ &= 4\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2\,\partial_{(x_1,x_2),i}\,\partial_{x_{2,j}}B\cdot\partial_{t_1}\partial_{t_2}(FG)\,dx_1\,dx_2\,dt_1\,dt_2 \\ &\quad -2\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2\nabla_{x_1,x_2}\,\partial_{x_{2,j}}B\cdot\partial_{t_2}(\nabla_{x_1,x_2}(R_i^{(1)}*f)^\sim\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2 \\ &\quad \cdots+\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1t_2\nabla_{x_1,x_2}\nabla_{x_2}B\cdot((R_i^{(1)}*R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_1,x_2}\nabla_{x_2}G)\,dx_1\,dx_2\,dt_1\,dt_2 \\ &= \mathcal{C}_{9,1}+\mathcal{C}_{9,2}+\cdots+\mathcal{C}_{9,25},\end{aligned}$$

where we get all these terms from the equality $\partial_{t_1}^2\partial_{t_2}^2(\cdots)$ by integration by parts and taking all the gradients or partial derivatives with respect to x_1, x_2 to the function B. By applying Proposition 3.5 to all these terms, we obtain that they are all controlled by

$$C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

Next we consider the term $\mathcal{C}_5$, which can be considered as a cross term in between $\mathcal{C}_1$ and $\mathcal{C}_9$. To continue, we write

$$\begin{aligned}&\partial_{t_2}^2(F\cdot(R_i^{(1)}*R_j^{(2)}*_2 g)^\sim+(R_i^{(1)}*f)^\sim\cdot(R_j^{(2)}*_2 g)^\sim+(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim+(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot G) \\ &\quad=\partial_{t_2}^2(F\cdot(R_j^{(2)}*_2(R_i^{(1)}*g))^\sim+(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim) \\ &\qquad\qquad+\partial_{t_2}^2((R_i^{(1)}*f)^\sim\cdot(R_j^{(2)}*_2 g)^\sim+(R_j^{(2)}*_2(R_i^{(1)}*f))^\sim\cdot G) \\ &\quad=E_1+E_2.\end{aligned}$$

For the term E_1, we write

$$\begin{aligned}E_1&=-2\,\partial_{x_{2,j}}\partial_{t_2}(F\cdot(R_i^{(1)}*g)^\sim)+\nabla_{x_2}\big(\nabla_{x_2}(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim+F\cdot\nabla_{x_2}(R_j^{(2)}*_2 R_i^{(1)}*g)^\sim \\ &\qquad -\nabla_{x_2}F\cdot(R_j^{(2)}*_2 R_i^{(1)}*g)^\sim-(R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_2}(R_i^{(1)}*g)^\sim\big).\end{aligned}$$

For the term E_2, we write

$$\begin{aligned}E_2&=-2\,\partial_{x_{2,j}}\partial_{t_2}((R_i^{(1)}*f)^\sim\cdot G)+\nabla_{x_2}\big(\nabla_{x_2}(R_j^{(2)}*_2 R_i^{(1)}*f)^\sim\cdot G+(R_i^{(1)}*f)^\sim\cdot\nabla_{x_2}(R_j^{(2)}*_2 g)^\sim \\ &\qquad -\nabla_{x_2}(R_i^{(1)}*f)^\sim\cdot(R_j^{(2)}*_2 g)^\sim-(R_j^{(2)}*_2 f)^\sim\cdot\nabla_{x_2}G\big).\end{aligned}$$

As a consequence, by substituting the above 10 terms in the right-hand side of the equalities E_1 and E_2 back into the term $\mathcal{C}_5$, we have

$$\begin{aligned}\mathcal{C}_5&=2\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1\,\partial_{t_1}t_2\,\partial_{x_{2,j}}B\cdot\partial_{t_1}\partial_{t_2}(F\cdot(R_i^{(1)}*g)^\sim)\,dx_1\,dx_2\,dt_1\,dt_2 \\ &\quad-\int_{\mathbb{R}_+^{n+1}\times\mathbb{R}_+^{m+1}} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}(\nabla_{x_2}(R_j^{(2)}*_2 f)^\sim\cdot(R_i^{(1)}*g)^\sim)\,dx_1\,dx_2\,dt_1\,dt_2\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}(F\cdot\nabla_{x_2}(R^{(2)}_j *_2 R^{(1)}_i * g)^{\sim})\,dx_1\,dx_2\,dt_1\,dt_2\\
&+\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}(\nabla_{x_2}F\cdot(R^{(2)}_j *_2 R^{(1)}_i * g)^{\sim})\,dx_1\,dx_2\,dt_1\,dt_2\\
&+\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}((R^{(2)}_j *_2 f)^{\sim}\cdot\nabla_{x_2}(R^{(1)}_i * g)^{\sim})\,dx_1\,dx_2\,dt_1\,dt_2\\
&+2\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\,\partial_{x_{2,j}}B\cdot\partial_{t_1}\partial_{t_2}((R^{(1)}_i * f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2\\
&-\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}(\nabla_{x_2}(R^{(2)}_j *_2 R^{(1)}_i * f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2\\
&-\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}((R^{(1)}_i * f)^{\sim}\cdot\nabla_{x_2}(R^{(2)}_j *_2 g)^{\sim})\,dx_1\,dx_2\,dt_1\,dt_2\\
&+\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}(\nabla_{x_2}(R^{(1)}_i * f)^{\sim}\cdot(R^{(2)}_j *_2 g)^{\sim})\,dx_1\,dx_2\,dt_1\,dt_2\\
&+\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1\,\partial_{t_1}t_2\nabla_{x_2}B\cdot\partial_{t_1}((R^{(2)}_j *_2 f)^{\sim}\cdot\nabla_{x_2}G)\,dx_1\,dx_2\,dt_1\,dt_2\\
=:&\ \mathcal{C}_{5,1}+\cdots+\mathcal{C}_{5,10}.
\end{aligned}
$$

By applying Proposition 3.5 to these terms, we obtain that they are all controlled by

$$C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

The estimates for the term $\mathcal{C}_6$ can be handled symmetrically, and we get the same upper bound for $\mathcal{C}_6$ as that for $\mathcal{C}_5$ above.

For the term $\mathcal{C}_7$, first note that $\partial^2_{t_2}B=-\Delta_{x_2}B=-\nabla_{x_2}\cdot\nabla_{x_2}B$. Hence we can write

$$
\begin{aligned}
\mathcal{C}_7=-\int_{\mathbb{R}^{n+1}_+\times\mathbb{R}^{m+1}_+} t_1t_2\nabla_{x_2}B\cdot\nabla_{x_2}\,\partial^2_{t_1}(F\cdot(R^{(1)}_i * R^{(2)}_j *_2 g)^{\sim}+(R^{(1)}_i * f)^{\sim}\cdot(R^{(2)}_j *_2 g)^{\sim}\\
+(R^{(2)}_j *_2 f)^{\sim}\cdot(R^{(1)}_i * g)^{\sim}+(R^{(2)}_j *_2 R^{(1)}_i * f)^{\sim}\cdot G)\,dx_1\,dx_2\,dt_1\,dt_2.
\end{aligned}
$$

Similar to the calculation in the terms E_1 and E_2 in the estimate of $\mathcal{C}_5$, we can now decompose

$$\partial^2_{t_1}(F\cdot(R^{(1)}_i * R^{(2)}_j *_2 g)^{\sim}+(R^{(1)}_i * f)^{\sim}\cdot(R^{(2)}_j *_2 g)^{\sim}+(R^{(2)}_j *_2 f)^{\sim}\cdot(R^{(1)}_i * g)^{\sim}+(R^{(2)}_j *_2 R^{(1)}_i * f)^{\sim}\cdot G)$$

into 10 terms, which further gives

$$\mathcal{C}_7=\mathcal{C}_{7,1}+\cdots+\mathcal{C}_{7,10}.$$

Then by applying Proposition 3.5 to these terms, we obtain that they are all controlled by

$$C\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|g\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

The estimates for the term $\mathcal{C}_8$ can be handled symmetrically, and we get the same upper bound for $\mathcal{C}_7$ above. □

4. Upper bound of the big commutator $[b, R_{j,k}]$

We derive a general upper bound result for commutators of any flag singular integral. The proof is based on the $A_{\mathcal{F},p}$ weighted estimate of flag singular integral operators and a Cauchy integral trick that goes back to the work of Coifman, Rochberg, and Weiss [Coifman et al. 1976]. Roughly speaking, this technique allows one to bootstrap the weighted estimate for an arbitrary linear operator to that of its commutators of any order. This is the first time this idea is explored in the multiparameter flag setting. In fact, although not needed for our upper bound proof, we demonstrate the bootstrapping result in the general higher-order, two-weight setting.

A_p weight and little bmo in the flag setting. To begin with, we define the Muckenhoupt A_p weights in the flag setting, which consists of positive, locally integrable functions w satisfying

$$[w]_{A_{\mathcal{F},p}} := \sup_{R\in\mathcal{R}_{\mathcal{F}}} \left(\frac{1}{|R|}\int_R w(x,y)\,dx\,dy\right)\left(\frac{1}{|R|}\int_R w(x,y)^{1-p'}\,dx\,dy\right)^{p-1} < \infty, \quad 1<p<\infty, \tag{4-1}$$

where p' denotes the Hölder conjugate of p. The following result of [Wu 2014] provides a way of approaching the $A_{\mathcal{F},p}$ weights via the classical weights:

$$A_{\mathcal{F},p} = A_p \cap A_p^{(2)} \quad \text{for all } 1<p<\infty, \tag{4-2}$$

where A_p is the classical Muckenhoupt A_p class of weights on $\mathbb{R}^{n+m}$, and $A_p^{(2)}$ consists of weights $w(x,y)$ such that $w(x,\cdot)\in A_p$ with uniformly bounded characteristics for a.e. fixed $x\in\mathbb{R}^n$.

We first show that a similar relation holds true for $\mathrm{bmo}_{\mathcal{F}}$, which will be a useful tool for us in the study of this space.

Lemma 4.1. *Let* $\mathrm{BMO}(\mathbb{R}^{n+m})$ *denote the classical John–Nirenberg BMO space on* $\mathbb{R}^{n+m}$, *and* $\mathrm{BMO}^{(2)}(\mathbb{R}^m)$ *be the space consisting of functions* $f(x,y)$ *such that* $f(x,\cdot)\in\mathrm{BMO}(\mathbb{R}^m)$ *for a.e. fixed* $x\in\mathbb{R}^n$ *with uniformly bounded norm. There holds*

$$\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m}) = \mathrm{BMO}(\mathbb{R}^{n+m})\cap\mathrm{BMO}^{(2)}(\mathbb{R}^m)$$

with comparable norms.

Proof. The inclusion

$$\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m}) \subset \mathrm{BMO}(\mathbb{R}^{n+m})\cap\mathrm{BMO}^{(2)}(\mathbb{R}^m)$$

can be easily verified. Indeed, the inclusion $\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})\subset\mathrm{BMO}(\mathbb{R}^{n+m})$ is obvious from the definition. Now fix $x\in\mathbb{R}^n$. For any cube $J\subset\mathbb{R}^m$, one can find a sequence of cubes $I_k\subset\mathbb{R}^n$ such that $\ell(I_k)\le\ell(J)$ and I_k shrinks to the point $\{x\}$ as $k\to\infty$. The containment thus follows from the Lebesgue differentiation theorem.

The other inclusion ("$\supset$") of the lemma follows from Proposition 4.2 below, which establishes the exp-log connection between $A_{\mathcal{F},p}$ weights and $\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})$, much as in the one-parameter and the product settings. □

Proposition 4.2. *Suppose w is a weight and $1 < p < \infty$. We have*

(i) *if $w \in A_{\mathcal{F},p}$, then* $\log w \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})$;

(ii) *if* $\log w \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})$, *then $w^\eta \in A_{\mathcal{F},p}$ for sufficiently small $\eta > 0$.*

Proof. One observes directly from the definition that

$$A_{\mathcal{F},p} \subset A_{\mathcal{F},q} \quad \text{for all } 1 < p \le q < \infty,$$

and

$$w \in A_{\mathcal{F},p} \Longleftrightarrow w^{1-p'} \in A_{\mathcal{F},p'} \quad \text{for all } 1 < p < \infty.$$

Therefore, it suffices to prove the case $p = 2$.

We first prove (i). Suppose $w \in A_{\mathcal{F},2}$ and let $\gamma = \log w$. Then, for any $R \in \mathcal{R}_{\mathcal{F}}$ the $A_{\mathcal{F},2}$ condition implies

$$\left(\frac{1}{|R|}\int_R e^{\gamma(x,y)-\langle\gamma\rangle_R}\,dx\,dy\right)\left(\frac{1}{|R|}\int_R e^{\langle\gamma\rangle_R-\gamma(x,y)}\,dx\,dy\right) \le [w]_{A_{\mathcal{F},2}} < \infty.$$

By Jensen's inequality we have each of the factors above is at least 1 and at most $[w]_{A_{\mathcal{F},2}}$. Therefore, the inequality

$$\frac{1}{|R|}\int_R e^{|\gamma(x,y)-\langle\gamma\rangle_R|}\,dx\,dy \le 2[w]_{A_{\mathcal{F},2}}$$

holds, which, using the trivial estimate $t \le e^t$, implies

$$\frac{1}{|R|}\int_R |\gamma(x,y)-\langle\gamma\rangle_R|\,dx\,dy \le 2[w]_{A_{\mathcal{F},2}}.$$

Hence, $\gamma \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})$.

We now prove (ii). Let $\gamma = \log w \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})$; it follows from Lemma 4.1 that $\gamma \in \mathrm{BMO}(\mathbb{R}^{n+m})$ and $\gamma \in \mathrm{BMO}^{(2)}(\mathbb{R}^m)$. According to the classical exp-log connection between BMO and A_2, there hold for sufficiently small $\eta > 0$

$$\begin{aligned} &e^{\eta\gamma(\cdot,\cdot)} \in A_2(\mathbb{R}^{n+m}),\\ &e^{\eta\gamma(x,\cdot)} \in A_2(\mathbb{R}^m) \quad \text{uniformly in } x \in \mathbb{R}^n. \end{aligned}$$

Hence, (4-2) implies $e^{\eta\gamma} \in A_{\mathcal{F},2}$ for sufficiently small $\eta > 0$, which completes the proof. □

Upper bound of the commutator. Given an operator T, define its k-th order commutator as

$$C^k_{\vec{b}}(T) := [b_k, [b_{k-1}, \dots, [b_1, T]\cdots]],$$

where each b_j is a function on $\mathbb{R}^n \times \mathbb{R}^m$ for all $1 \le j \le k$.

Theorem 4.3. *Let ν be a fixed weight on $\mathbb{R}^n \times \mathbb{R}^m$, $1 < p < \infty$, and T be a linear operator satisfying*

$$\|T\|_{L^p(\mu)\to L^p(\lambda)} \le C_{n,m,p,T}([\mu]_{A_{\mathcal{F},p}}, [\lambda]_{A_{\mathcal{F},p}}),$$

where $C_{n,m,p,T}(\cdot,\cdot)$ *is an increasing function of both components, with* $\mu, \lambda \in A_{\mathcal{F},p}$ *and* $\mu/\lambda = \nu^p$. *For* $k \geq 1$, *let* $b_j \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$, $1 \leq j \leq k$; *then there holds*

$$\|C^k_{\vec{b}}(T)\|_{L^p(\mu)\to L^p(\lambda)} \leq C_{n,m,p,k,T}([\mu]_{A_{\mathcal{F},p}}, [\lambda]_{A_{\mathcal{F},p}}) \prod_{j=1}^{k} \|b_j\|_{\mathrm{bmo}_{\mathcal{F}}}.$$

Assuming Theorem 4.3, in order to derive an (even unweighted) upper estimate for commutator of operator T, it suffices to know the corresponding weighted estimate for T itself. When T is a flag singular integral operator (which includes the flag Riesz transform $R_{j,k}$), such a result was obtained by Han, Lin and Wu [Han et al. 2016b].

Definition 4.4. A flag singular integral $T_{\mathcal{F}} : f \mapsto \mathcal{K} * f$ is defined via a flag kernel $\mathcal{K}$ on $\mathbb{R}^n \times \mathbb{R}^m$, which is a distribution on $\mathbb{R}^{n+m}$ that coincides with a C^∞ function away from the coordinate subspace $\{(0, y)\} \subset \mathbb{R}^{n+m}$ and satisfies:

(i) (differential inequalities) For each $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$

$$|\partial_x^\alpha \partial_y^\beta \mathcal{K}(x, y)| \lesssim |x|^{-n-|\alpha|}(|x| + |y|)^{-m-|\beta|}$$

for all $(x, y) \in \mathbb{R}^{n+m}$ with $|x| \neq 0$.

(ii) (cancellation conditions)

$$\left|\int_{\mathbb{R}^m} \partial_x^\alpha \mathcal{K}(x, y)\psi_1(\delta y)\, dy\right| \leq C_\alpha |x|^{-n-|\alpha|}$$

for every multi-index α and for every normalized bump function ψ_1 on $\mathbb{R}^m$ and every $\delta > 0$;

$$\left|\int_{\mathbb{R}^n} \partial_y^\beta \mathcal{K}(x, y)\psi_2(\delta y)\, dy\right| \leq C_\beta |y|^{-m-|\beta|}$$

for every multi-index β and for every normalized bump function ψ_2 on $\mathbb{R}^n$ and every $\delta > 0$;

$$\left|\int_{\mathbb{R}^{n+m}} \mathcal{K}(x, y)\psi_3(\delta_1 x, \delta_2 y)\, dx\, dy\right| \leq C$$

for every normalized bump function ψ_3 on $\mathbb{R}^{n+m}$ and every $\delta_1, \delta_2 > 0$.

Theorem 4.5 [Han et al. 2016b, Remark 1.4]. *Let* $1 < p < \infty$ *and* $w \in A_{\mathcal{F},p}(\mathbb{R}^{n+m})$; *there holds*

$$\|T_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^{n+m})} \leq C_p \|f\|_{L^p_w(\mathbb{R}^{n+m})} \quad \textit{for all } f \in L^p_w(\mathbb{R}^{n+m}).$$

Applying Theorem 4.3 (with the choice $\mu = \lambda = w$) together with Theorem 4.5, one obtains immediately the following.

Corollary 4.6. *Let* $w \in A_{\mathcal{F},p}$, $1 < p < \infty$, *and* T *be a flag singular integral operator as defined above. For any* $k \geq 1$, $\vec{b} = (b_1, \ldots, b_k)$ *where* $b_j \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$, $j = 1, \ldots, k$, *there holds*

$$\|C^k_{\vec{b}}(T)\|_{L^p(w)\to L^p(w)} \leq C_{n,m,p,k,w,T} \prod_{j=1}^{k} \|b_j\|_{\mathrm{bmo}_{\mathcal{F}}}.$$

Obviously, the result above in the first-order unweighted case is precisely the desired upper bound estimate in Theorem 1.5.

The core of the proof of Theorem 4.3 lies in a complex function representation of the commutators and the Cauchy integral formula. This method has been widely used to obtain upper estimates for linear and multilinear commutators in various settings; see [Chung et al. 2012; Coifman et al. 1976; Hytönen 2016; Bényi et al. 2017; Kunwar and Ou 2017] for examples. The main new challenge in our problem is the unique structure of the little flag BMO space and flag weights, which for instance doesn't seem to fall into the category of spaces recently studied in [Bényi et al. 2017].

Proof of Theorem 4.3. Observe that

$$C^k_{\vec{b}}(T) = \partial_{z_1}\cdots\partial_{z_k}F(\vec{0}), \quad F(\vec{z}) := e^{\sum_{j=1}^k b_1 z_1} T e^{-\sum_{j=1}^k b_j z_j},$$

which generalizes a classical formula representing higher-order commutators. We remark that when all the symbol functions b_j are the same, one can work instead with a simpler formula using single variable complex functions and their k-th order derivatives. According to the Cauchy integral formula on polydiscs,

$$C^k_{\vec{b}}(T) = \frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{F(\vec{z})\,dz_1\cdots dz_k}{z_1^2\cdots z_k^2},$$

where each integral is over any closed path around the origin in the corresponding variable. For fixed $(\delta_1,\dots,\delta_k)$ which will be determined later, there holds by the Minkowski inequality

$$\begin{aligned}
&\|C^k_{\vec{b}}(T)\|_{L^p(\mu)\to L^p(\lambda)}\\
&\quad\le \frac{1}{(2\pi)^k}\oint_{|z_1|=\delta_1}\cdots\oint_{|z_k|=\delta_k}\|T\|_{L^p(e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\mu)\to L^p(e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\lambda)}\frac{|dz_1|\cdots|dz_k|}{\delta_1^2\cdots\delta_k^2}\\
&\quad\le \frac{1}{(2\pi)^k}\oint_{|z_1|=\delta_1}\cdots\oint_{|z_k|=\delta_k} C_{n,m,p,T}([e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\mu]_{A_{\mathcal{F},p}}, [e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\lambda]_{A_{\mathcal{F},p}})\frac{|dz_1|\cdots|dz_k|}{\delta_1^2\cdots\delta_k^2},
\end{aligned}$$

where we have used the fact that $(e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\mu, e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\lambda)$ is a pair of weights satisfying

$$\frac{e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\mu}{e^{p\operatorname{Re}(\sum_{j=1}^k b_j z_j)}\lambda} = \frac{\mu}{\lambda} = \nu^p.$$

Now we choose $\{\delta_j\}$ according to Lemma 4.7 below, which is the key ingredient of the proof concerning the relation between $A_{\mathcal{F},p}$ weights and little flag BMO functions. Let

$$\delta_1 := \frac{\epsilon_{n,m,p}}{\max((\mu)_{A_{\mathcal{F},p}}, (\lambda)_{A_{\mathcal{F},p}})\|b_1\|_{\mathrm{bmo}_{\mathcal{F}}}},$$

where for any $w \in A_{\mathcal{F},p}$

$$(w)_{A_{\mathcal{F},p}} := \max([w]_{A_{\mathcal{F},p}}, [\sigma]_{A_{\mathcal{F},p'}}). \tag{4-3}$$

Here we have used the notation $\sigma := w^{1-p'}$ to denote the dual weight of w, and the relevant property of $(w)_{A_{\mathcal{F},p}}$ to us is that

$$(w)_{A_{\mathcal{F},p}} = \max([w]_{A_{\mathcal{F},p}}, [w]^{p'-1}_{A_{\mathcal{F},p}}) = [w]^{\max(1,p'-1)}_{A_{\mathcal{F},p}}.$$

Recursively, for any $j \geq 2$, choose

$$\delta_j := \frac{\epsilon_{n,m,p}}{\sup_{\{z_t\}:\, |z_1|=\delta_1,\ldots,|z_{j-1}|=\delta_{j-1}} \max((e^{p\operatorname{Re}(\sum_{t=1}^{j-1} b_t z_t)}\mu)_{A_{\mathcal{F},p}}, (e^{p\operatorname{Re}(\sum_{t=1}^{j-1} b_t z_t)}\lambda)_{A_{\mathcal{F},p}})\|b_j\|_{\mathrm{bmo}_{\mathcal{F}}}}.$$

Then applying Lemma 4.7 iteratively shows that

$$[e^{p\operatorname{Re}(\sum_{j=1}^{k} b_j z_j)}\mu]_{A_{\mathcal{F},p}} \leq C_{n,m,p}[e^{p\operatorname{Re}(\sum_{j=1}^{k-1} b_j z_j)}\mu]_{A_{\mathcal{F},p}} \leq \cdots \leq C^k_{n,m,p}[\mu]_{A_{\mathcal{F},p}},$$

and similarly

$$[e^{p\operatorname{Re}(\sum_{j=1}^{k} b_j z_j)}\lambda]_{A_{\mathcal{F},p}} \leq C^k_{n,m,p}[\lambda]_{A_{\mathcal{F},p}},$$

which in turn via the monotonicity of $C_{n,m,p,T}(\,\cdot\,,\,\cdot\,)$ leads to

$$C_{n,m,p,T}([e^{p\operatorname{Re}(\sum_{j=1}^{k} b_j z_j)}\mu]_{A_{\mathcal{F},p}}, [e^{p\operatorname{Re}(\sum_{j=1}^{k} b_j z_j)}\lambda]_{A_{\mathcal{F},p}}) \leq C'_{n,m,p,k,T}([\mu]_{A_{\mathcal{F},p}}, [\lambda]_{A_{\mathcal{F},p}}).$$

Therefore,

$$\begin{aligned}\|C^k_{\vec{b}}(T)\|_{L^p(\mu)\to L^p(\lambda)} &\leq \frac{1}{\delta_1\cdots\delta_k} C'_{n,m,p,k,T}([\mu]_{A_{\mathcal{F},p}}, [\lambda]_{A_{\mathcal{F},p}}) \\ &\leq C_{n,m,p,k,T}([\mu]_{A_{\mathcal{F},p}}, [\lambda]_{A_{\mathcal{F},p}}) \prod_{j=1}^{k} \|b_j\|_{\mathrm{bmo}_{\mathcal{F}}}.\end{aligned}$$ □

Lemma 4.7. *Let $w \in A_{\mathcal{F},p}$, $1 < p < \infty$, and $b \in \mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$. There are constants $\epsilon_{n,m,p}, C_{n,m,p} > 0$ such that*

$$[e^{\operatorname{Re}(bz)}w]_{A_{\mathcal{F},p}} \leq C_{n,m,p}[w]_{A_{\mathcal{F},p}}$$

whenever $z \in \mathbb{C}$ satisfies

$$|z| \leq \frac{\epsilon_{n,m,p}}{\|b\|_{\mathrm{bmo}_{\mathcal{F}}}(w)_{A_{\mathcal{F},p}}},$$

where $(w)_{A_{\mathcal{F},p}}$ is defined as in (4-3).

Proof. This estimate is a consequence of (4-2), Lemma 4.1 and a one-parameter version proven by Hytönen [2016], which states that for any $w \in A_p$, the classical Muckenhoupt A_p class on $\mathbb{R}^d$, $1 < p < \infty$, there exist $\epsilon_{d,p}, C_{d,p} > 0$ such that

$$[e^{\operatorname{Re}(bz)}w]_{A_p} \leq C_{d,p}[w]_{A_p}$$

for all $z \in \mathbb{C}$ with

$$|z| \leq \frac{\epsilon_{n,p}}{\|b\|_{\mathrm{BMO}}(w)_{A_p}}.$$

To see this, by (4-2) and Lemma 4.1, given $w \in A_{\mathcal{F},p}$ and $b \in \mathrm{bmo}_{\mathcal{F}}$, there hold $w \in A_p \cap A_p^{(2)}$ and $b \in \mathrm{BMO}(\mathbb{R}^{n+m}) \cap \mathrm{BMO}^{(2)}(\mathbb{R}^m)$. Hence, taking $\epsilon_{n,m,p} > 0$ sufficiently small, for all $z \in \mathbb{C}$ satisfying

$$|z| \leq \frac{\epsilon_{n,m,p}}{\|b\|_{\mathrm{bmo}_{\mathcal{F}}}(w)_{A_{\mathcal{F},p}}},$$

one has

$$[e^{\operatorname{Re}(bz)}w]_{A_p} \leq C_{n+m,p}[w]_{A_p} \leq C_{n,m,p}[w]_{A_{\mathcal{F},p}}$$

and

$$[e^{\mathrm{Re}(b(x,\cdot)z)}w(x,\cdot)]_{A_p} \le C_{m,p}[w(x,\cdot)]_{A_p} \le C_{n,m,p}[w]_{A_{\mathcal{F},p}} \quad \text{a.e. } x \in \mathbb{R}^n,$$

by observing that

$$\|b\|_{\mathrm{bmo}_{\mathcal{F}}} \gtrsim \max\Big(\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+m})}, \sup_{x\in\mathbb{R}^n} \|b(x,\cdot)\|_{\mathrm{BMO}^{(2)}(\mathbb{R}^m)}\Big)$$

and that

$$(w)_{A_{\mathcal{F},p}} \gtrsim \max([w]_{A_p}, \sup_{x\in\mathbb{R}^n}[w(x,\cdot)]_{A_p}). \qquad \square$$

5. Applications: div-curl lemmas in the flag setting

Let $E^{(1)}$ be a vector field on $\mathbb{R}^{n+m}$ taking the values in $\mathbb{R}^{n+m}$, and let $E^{(2)}$ be a vector field on $\mathbb{R}^m$ taking the values in $\mathbb{R}^m$. Now let $\mathcal{M}_{n+m,m}$ denote the set of all $(n+m)\times m$ matrices. We now consider the following version of vector fields on $\mathbb{R}^n \times \mathbb{R}^m$ taking the values in $\mathcal{M}_{n+m,m}$, associated with the flag structure:

$$E = E^{(1)} *_2 E^{(2)} := \begin{bmatrix} E_1^{(1)} *_2 E_1^{(2)} & \dots & E_1^{(1)} *_2 E_m^{(2)} \\ \vdots & \cdots & \vdots \\ E_{n+m}^{(1)} *_2 E_1^{(2)} & \dots & E_{n+m}^{(1)} *_2 E_m^{(2)} \end{bmatrix}, \tag{5-1}$$

where

$$E_j^{(1)} *_2 E_k^{(2)}(x,y) = \int_{\mathbb{R}^m} E_j^{(1)}(x, y-z) E_k^{(2)}(z)\,dz.$$

Next we consider the following L^p space via projections. Suppose $1 < p < \infty$. We define $L^p_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m; \mathcal{M}_{n+m,m})$ to be the set of vector fields E in $L^p(\mathbb{R}^n \times \mathbb{R}^m; \mathcal{M}_{n+m,m})$ such that there exist $r_1, r_2 \in (1,\infty)$ with $1/r_1 + 1/r_2 = 1/p + 1$, $E^{(1)} \in L^{r_1}(\mathbb{R}^{n+m}; \mathbb{R}^{n+m})$, $E^{(2)} \in L^{r_2}(\mathbb{R}^m; \mathbb{R}^m)$ and that $E = E^{(1)} *_2 E^{(2)}$; moreover,

$$\|E\|_{L^p_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})} := \inf \|E^{(1)}\|_{L^{r_1}(\mathbb{R}^{n+m};\mathbb{R}^{n+m})} \|E^{(2)}\|_{L^{r_2}(\mathbb{R}^m;\mathbb{R}^m)},$$

where the infimum is taken over all possible $r_1, r_2 \in (1,\infty)$, $E^{(1)} \in L^{r_1}(\mathbb{R}^{n+m}; \mathbb{R}^{n+m})$, $E^{(2)} \in L^{r_2}(\mathbb{R}^m; \mathbb{R}^m)$.

Given two matrices $A, B \in \mathcal{M}_{n+m,m}$, we define the "dot product" between A and B by

$$A \cdot B = \sum_{j=1}^{n+m} \sum_{k=1}^{m} A_{j,k} B_{j,k}.$$

We point out that this is the Hilbert–Schmidt inner product for two matrices and more generally this is referred to as the Schur product of two matrices.

Proof of Theorem 1.6. Note that B is a vector field on $\mathbb{R}^n \times \mathbb{R}^m$ taking the values in $\mathcal{M}_{n+m,m}$, associated with the flag structure (5-1). Then there exist certain vector fields $B^{(1)}$ on $\mathbb{R}^{n+m}$ taking the values in $\mathbb{R}^{n+m}$ and $B^{(2)}$ on $\mathbb{R}^m$ taking the values in $\mathbb{R}^m$ such that $B = B^{(1)} *_2 B^{(2)}$ and that

$$\|B\|_{L^q_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})} \approx \inf \|B^{(1)}\|_{L^{q_1}(\mathbb{R}^{n+m};\mathbb{R}^{n+m})} \|B^{(2)}\|_{L^{q_2}(\mathbb{R}^m;\mathbb{R}^m)}$$

with $1/q_1 + 1/q_2 = 1/q + 1$.

Thus, $\mathrm{curl}_{(x,y)} B^{(1)} = 0$ implies that there exists $\phi^{(1)} \in L^q(\mathbb{R}^{n+m})$ such that

$$B^{(1)} = (R_1^{(1)}\phi^{(1)}, \ldots, R_{n+m}^{(1)}\phi^{(1)})$$

with $\|B^{(1)}\|_{L^{q_1}(\mathbb{R}^{n+m};\mathbb{R}^{n+m})} \approx \|\phi^{(1)}\|_{L^{q_1}(\mathbb{R}^{n+m})}$. Again, $\mathrm{curl}_y B^{(2)} = 0$ implies that there exists $\phi^{(2)} \in L^{q_2}(\mathbb{R}^{n+m})$ such that

$$B^{(2)} = (R_1^{(2)}\phi^{(2)}, \ldots, R_m^{(2)}\phi^{(2)})$$

with $\|B^{(2)}\|_{L^{q_2}(\mathbb{R}^m;\mathbb{R}^m)} \approx \|\phi^{(2)}\|_{L^{q_2}(\mathbb{R}^m)}$. As a consequence we get that the matrix B has elements

$$B_{j,k} = R_{j,k} * \phi, \quad j = 1, \ldots, n+m, \; k = 1, \ldots, m,$$

where $\phi = \phi^{(1)} *_2 \phi^{(2)}$ and $\|B\|_{L^q_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})} \approx \|\phi\|_{L^q(\mathbb{R}^{n+m})}$.

Similarly, note that E is a vector field on $\mathbb{R}^n \times \mathbb{R}^m$ taking the values in $\mathcal{M}_{n+m,m}$, associated with the flag structure (5-1). Then there exist certain vector fields $E^{(1)}$ on $\mathbb{R}^{n+m}$ taking the values in $\mathbb{R}^{n+m}$ and $E^{(2)}$ on $\mathbb{R}^m$ taking the values in $\mathbb{R}^m$ such that $E = E^{(1)} *_2 E^{(2)}$ and that

$$\|E\|_{L^p_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})} \approx \inf \|E^{(1)}\|_{L^{p_1}(\mathbb{R}^{n+m};\mathbb{R}^{n+m})} \|E^{(2)}\|_{L^{p_2}(\mathbb{R}^m;\mathbb{R}^m)}$$

with $1/p_1 + 1/p_2 = 1/p + 1$.

Thus, the conditions $\mathrm{div}_{(x,y)} E^{(1)} = 0$ and $\mathrm{div}_y E^{(2)} = 0$ imply

$$\sum_{j=1}^{n+m} R_j^{(1)} * E_j^{(1)}(x, y) = 0 \quad \text{and} \quad \sum_{k=1}^{m} R_k^{(2)} *_2 E_k^{(2)}(y) = 0.$$

Hence, we get

$$\sum_{j=1}^{n+m} R_j^{(1)} * E_{j,k}(x, y) = 0 \quad \text{and} \quad \sum_{k=1}^{m} R_k^{(2)} *_2 E_{j,k}(x, y) = 0.$$

With these facts, we have

$$\begin{aligned} E(x, y) \cdot B(x, y) &= \sum_{j=1}^{n+m}\sum_{k=1}^{m} E_{j,k}(x, y) B_{j,k}(x, y) = \sum_{j=1}^{n+m}\sum_{k=1}^{m} E_{j,k}(x, y) R_{j,k} * \phi(x, y) \\ &= \sum_{j=1}^{n+m}\sum_{k=1}^{m} \big\{ E_{j,k}(x, y) R_{j,k} * \phi(x, y) + R_j^{(1)} * E_{j,k}(x, y) R_k^{(2)} *_2 \phi(x, y) \\ &\qquad + R_k^{(2)} *_2 E_{j,k}(x, y) R_j^{(1)} * \phi(x, y) + R_{j,k} * E_{j,k}(x, y)\phi(x, y) \big\}. \end{aligned}$$

Now testing this equality over all functions in the flag BMO space, i.e., for every $b \in \mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$, and then unravelling the expression with the Riesz transforms we see that

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} E(x, y) \cdot B(x, y)\, b(x, y)\, dx\, dy = \sum_{j=1}^{n+m}\sum_{k=1}^{m} \int_{\mathbb{R}^n\times\mathbb{R}^m} [[b, R_j^{(1)}], R_k^{(2)}]_2(E_{j,k})(x, y)\phi(x, y)\, dx\, dy.$$

Then based on Theorem 1.3, since $b \in \mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$ we have that each of the above commutators is a bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ with norm controlled by the norm of b, i.e., $\|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}$.

As a consequence, we get

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^m} E(x,y)\cdot B(x,y)\,b(x,y)\,dx\,dy\right| \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|E\|_{L^p_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})}\|\phi\|_{L^q(\mathbb{R}^{n+m})}$$
$$\lesssim \|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|E\|_{L^p_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})}\|B\|_{L^q_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})}.$$

Then from the duality of $H^1_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$ with $\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$, we obtain

$$\|E\cdot B\|_{H^1_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|E\|_{L^p_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})}\|B\|_{L^q_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m;\mathcal{M}_{n+m,m})}. \qquad \square$$

Proof of Theorem 1.7. Suppose that E, B are vector fields on $\mathbb{R}^n\times\mathbb{R}^m$ taking values in $\mathbb{R}^{n+m}$. Moreover, suppose $E\in L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})$ and $B\in L^q(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})$ satisfy

$$\mathrm{div}_{(x,y)}\, E(x,y)=0 \quad \text{and} \quad \mathrm{curl}_{(x,y)}\, B(x,y)=0$$

and

$$\mathrm{div}_{y}\, E(x,y)=0 \quad \text{and} \quad \mathrm{curl}_{y}\, B(x,y)=0 \quad \text{for all } x\in\mathbb{R}^n.$$

We now define the projection operator $\mathcal{P}$ as

$$\mathcal{P}E=\left(E_1+R^{(1)}_1\left(\sum_{k=1}^{n+m}R^{(1)}_k E_k\right),\dots,E_{n+m}+R^{(1)}_{n+m}\left(\sum_{k=1}^{n+m}R^{(1)}_k E_k\right)\right).$$

Then by definition, it is direct that

$$\mathrm{div}_{(x,y)}\,\mathcal{P}E=0$$

since

$$\sum_{j=1}^{n+m}R^{(1)}_j\left(E_j+R^{(1)}_j\left(\sum_{k=1}^{n+m}R^{(1)}_k E_k\right)\right)=0. \tag{5-2}$$

Moreover, we also have $\mathcal{P}\circ\mathcal{P}E=\mathcal{P}E$. Next, we point out that applying $[b,\mathcal{P}]$ to the vector field E, we can get that the j-th component is given by

$$\sum_{k=1}^{n+m}[b,R^{(1)}_j R^{(1)}_k](E_k).$$

Suppose now $b\in\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$. Then from Lemma 4.1 we know

$$\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^{n+m})=\mathrm{BMO}(\mathbb{R}^{n+m})\cap\mathrm{BMO}^{(2)}(\mathbb{R}^m)$$

with comparable norms. Hence, we have $b\in\mathrm{BMO}(\mathbb{R}^{n+m})$ with

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+m})}\lesssim\|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

With all these observations, an application of the Coifman, Rochberg and Weiss theorem demonstrates that $[b,\mathcal{P}](E)$ is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})$ with

$$\|[b,\mathcal{P}](E)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+m})}\|E\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}$$
$$\lesssim\|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|E\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}.$$

As a consequence, from the definition of $[b, P]$ and (5-2) we get

$$\begin{aligned}\left|\int_{\mathbb{R}^{n+m}} E(x,y)\cdot B(x,y)\,b(x,y)\,dx\,dy\right| &= \left|\int_{\mathbb{R}^{n+m}} [b,P]E(x,y)\cdot B(x,y)\,dx\,dy\right| \\ &\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^{n+m})}\|E\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}\|B\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})} \\ &\lesssim \|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|E\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}\|B\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}.\end{aligned}$$

Thus we get that $E\cdot B$ is in $H^1(\mathbb{R}^{n+m})$ with

$$\|E\cdot B\|_{H^1(\mathbb{R}^{n+m})}\lesssim \|E\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}\|B\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}.$$

To show the second result, we now define the projection operator $\mathcal{P}^{(2)}$ as

$$\mathcal{P}^{(2)}E=\left(E_{n+1}+R_1^{(2)}\left(\sum_{k=1}^m R_k^{(2)}E_{n+k}\right),\ldots,E_{n+m}+R_{n+m}^{(1)}\left(\sum_{k=1}^m R_k^{(1)}E_{n+k}\right)\right).$$

Then, again, by definition, we have

$$\mathrm{div}_y\,\mathcal{P}^{(2)}E=0$$

since

$$\sum_{j=1}^m R_j^{(2)}\left(E_{n+j}+R_j^{(2)}\left(\sum_{k=1}^m R_k^{(2)}E_{n+k}\right)\right)=0. \tag{5-3}$$

Now fix $x\in\mathbb{R}^n$; by using the definition of $\mathcal{P}^{(2)}$ and the fact (5-3), we get that for $b\in\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$,

$$\int_{\mathbb{R}^m} E(x,y)\cdot_2 B(x,y)b(x,y)\,dy=\int_{\mathbb{R}^m}[b(x,\cdot),\mathcal{P}^{(2)}]E(x,y)\psi(x,y)\,dy.$$

Integrating the above equality over $\mathbb{R}^n$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} E(x,y)\cdot_2 B(x,y)b(x,y)\,dy\,dx\right| &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}[b(x,\cdot),\mathcal{P}^{(2)}]E(x,y)\cdot_2 B(x,y)\,dy\,dx\right| \\ &\lesssim \int_{\mathbb{R}^n}\|b(x,\cdot)\|_{\mathrm{BMO}(\mathbb{R}^n)}\|E(x,\cdot)\|_{L^p(\mathbb{R}^m)}\|B(x,\cdot)\|_{L^q(\mathbb{R}^m)}\,dx \\ &\lesssim \|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\int_{\mathbb{R}^n}\|E(x,\cdot)\|_{L^p(\mathbb{R}^m)}\|B(x,\cdot)\|_{L^q(\mathbb{R}^m)}\,dx \\ &\lesssim \|b\|_{\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)}\|E\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n;\mathbb{R}^{n+m})}\|B\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n;\mathbb{R}^{n+m})}.\end{aligned}$$

Here we use again Lemma 4.1 and Hölder's inequality. Taking the supremum over all $b\in\mathrm{bmo}_{\mathcal{F}}(\mathbb{R}^n\times\mathbb{R}^m)$ we obtain that

$$\int_{\mathbb{R}^m}\|E(\cdot,y)\cdot_2 B(\cdot,y)\|_{H^1(\mathbb{R}^m)}\,dy\lesssim \|E\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}\|B\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R}^{n+m})}. \qquad \square$$

Acknowledgments

Duong, Li and Pipher are supported by ARC DP 160100153. Duong and Li are also supported by Macquarie University Research Seeding Grant. Wick's research is supported in part by National Science

Foundation grant no. DMS-1560955. This paper started in July 2016 during Li's visit with Pipher at Brown University. Li would like to thank the Department of Mathematics of Brown University for its hospitality. Part of this material is based upon work supported by the National Science Foundation under grant no. DMS-1440140 while Ou and Pipher were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2017 semester.

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Received 19 Feb 2018. Revised 24 Jul 2018. Accepted 16 Sep 2018.

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mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 12, No. 5, 2019
dx.doi.org/10.2140/apde.2019.12.1357

ROKHLIN DIMENSION: ABSORPTION OF MODEL ACTIONS

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We establish a connection between Rokhlin dimension and the absorption of certain model actions on strongly self-absorbing C^*-algebras. Namely, as to be made precise in the paper, let G be a well-behaved locally compact group. If $\mathcal{D}$ is a strongly self-absorbing C^*-algebra and $\alpha : G \curvearrowright A$ is an action on a separable, $\mathcal{D}$-absorbing C^*-algebra that has finite Rokhlin dimension with commuting towers, then α tensorially absorbs every semi-strongly self-absorbing G-action on $\mathcal{D}$. In particular, this is the case when α satisfies any version of what is called the Rokhlin property, such as for $G = \mathbb{R}$ or $G = \mathbb{Z}^k$. This contains several existing results of similar nature as special cases. We will in fact prove a more general version of this theorem, which is intended for use in subsequent work. We will then discuss some nontrivial applications. Most notably it is shown that for any $k \geq 1$ and on any strongly self-absorbing Kirchberg algebra, there exists a unique $\mathbb{R}^k$-action having finite Rokhlin dimension with commuting towers up to (very strong) cocycle conjugacy.

Introduction

The present work is a continuation of the author's quest to study fine structure and classification of certain C^*-dynamics by employing ideas related to tensorial absorption. In previous work, the theory of (semi-)strongly self-absorbing actions on C^*-algebras [Szabó 2017b; 2018b; 2018c] was developed, closely following the important results established in the classical theory of strongly self-absorbing C^*-algebras by Toms and Winter [2007] and others [Kirchberg 2006; Dadarlat and Winter 2009]. Strongly self-absorbing C^*-algebras have historically emerged by example [Jiang and Su 1999], and now play a central role in the structure theory of simple nuclear C^*-algebras; see for example [Kirchberg and

This work was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92), and the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement 746272.
MSC2010: 46L55.
Keywords: Rokhlin dimension, C*-dynamical system, strongly self-absorbing C*-algebra.

Phillips 2000; Rørdam 2004; Elliott and Toms 2008; Winter and Zacharias 2010; Winter 2010; 2012; 2014; Matui and Sato 2012b; 2014a; Bosa et al. 2015; Castillejos et al. 2018]. Roughly speaking, a tensorial factorization of the form $A \cong A \otimes \mathcal{D}$ — for a given C*-algebra A and a strongly self-absorbing C*-algebra $\mathcal{D}$ — provides sufficient space to perform nontrivial manipulations on elements inside A, which often gives rise to structural properties of particular interest for classification. The underlying motivation behind [Szabó 2017b; 2018b; 2018c] is the idea that this kind of phenomenon should persist at the level of C*-dynamics if one is interested in classification of group actions up to cocycle conjugacy; in fact some much earlier work [Kishimoto 2001; 2002; Izumi and Matui 2010; 2012; Goldstein and Izumi 2011; Matui and Sato 2012a; 2014b] has (sometimes implicitly) used this idea to reasonable success. It was further demonstrated in [Szabó 2017b; 2018a] how this approach can indeed give rise to new insights about classification or rigidity of group actions on certain C*-algebras, in particular strongly self-absorbing ones.

Starting from Connes' groundbreaking work [1975; 1976; 1977] on injective factors, which involved classification of single automorphisms, the Rokhlin property in its various forms became a key tool to classify actions of amenable groups on von Neumann algebras [Jones 1980; Ocneanu 1985; Sutherland and Takesaki 1989; Kawahigashi et al. 1992; Katayama et al. 1998; Masuda 2007]. It did not take long for these ideas to reach the realm of C*-algebras. Initially appearing in [Herman and Jones 1982] and [Herman and Ocneanu 1984], the Rokhlin property for single automorphisms and its applications for classification were perfected in works of Kishimoto and various collaborators [Kishimoto 1995; 1996b; 1998a; 1998b; Bratteli et al. 1993; 1995; Evans and Kishimoto 1997; Elliott et al. 1998; Bratteli and Kishimoto 2000; Nakamura 2000]. Further work pushed these techniques to actions of infinite higher-rank groups as well [Nakamura 1999; Katsura and Matui 2008; Matui 2008; 2010; 2011; Izumi and Matui 2010; 2012; 2018]. The case of finite groups was treated in [Izumi 2004a; 2004b], where it was shown that such actions with the Rokhlin property have a particularly rigid theory; see also [Santiago 2015; Gardella and Santiago 2016; Gardella 2014a; 2014b; 2017; Barlak and Szabó 2016; Barlak et al. 2017]. In contrast to von Neumann algebras, however, the Rokhlin property for actions on C*-algebras has too many obstructions in general, ranging from obvious ones like lack of projections to more subtle ones of K-theoretic nature.

Rokhlin dimension is a notion of dimension for actions of certain groups on C*-algebras and was first introduced by Hirshberg, Winter and Zacharias [Hirshberg et al. 2015]. Several natural variants of Rokhlin dimension have been introduced, and all of them have in common that they generalize (to some degree) the Rokhlin property for actions of either finite groups or the integers. The theory has been extended and applied in many following works, such as [Szabó 2015; Hirshberg and Phillips 2015; Szabó et al. 2017; Gardella 2017; Hirshberg et al. 2017; Liao 2016; 2017; Brown et al. 2018; Gardella et al. 2017]. In short, the advantage of working with Rokhlin dimension is that it is both more prevalent and more flexible than the Rokhlin property, but is yet often strong enough to deduce interesting structural properties of the crossed product, such as finite nuclear dimension [Winter and Zacharias 2010].

A somewhat stronger version of Rokhlin dimension, namely with commuting towers, has been considered from the very beginning as a variant that was also compatible with respect to the absorption of strongly self-absorbing C*-algebras. Although the assumption of commuting towers initially only looked like a minor technical assumption, it was eventually discovered that it can make a major difference in some cases, such as for actions of finite groups [Hirshberg and Phillips 2015].

The purpose of this paper is to showcase a decisive connection between finite Rokhlin dimension with commuting towers and the absorption of semi-strongly self-absorbing model actions. The following describes a variant of the main result; see Theorem 4.4:

Theorem A. *Let G be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Suppose that the quotient G/N contains a discrete, normal, cocompact subgroup that is residually finite and has a box space with finite asymptotic dimension. Let A be a separable* C*-*algebra with an action $\alpha : G \curvearrowright A$. Let $\gamma : G \curvearrowright \mathcal{D}$ be a semi-strongly self-absorbing action that is unitarily regular. Suppose that $\alpha|_N$ is $\gamma|_N$-absorbing. If the Rokhlin dimension of α with commuting towers relative to N is finite, then it follows that α is γ-absorbing.*

Since many assumptions in this theorem are fairly technical at first glance, it may be helpful for the reader to keep in mind some special cases. For example, the above assumptions on the pair $N \subset G$ are satisfied when the quotient G/N above is isomorphic to either $\mathbb{R}$ or $\mathbb{Z}$. In this case, the theorem states that as long as the action α satisfies a suitable Rokhlin-type criterion relative to N, tensorial absorption of the G-action γ can be detected by restricting to the N-actions, even though this restriction procedure (a priori) comes with great loss of dynamical information. This is most apparent when the normal subgroup N is trivial, which is yet another important special case; see Corollary 5.1:

Corollary B. *Let G be a second-countable, locally compact group. Suppose that G contains a discrete, normal, cocompact subgroup that is residually finite and has a box space with finite asymptotic dimension. Let A be a separable* C*-*algebra with an action $\alpha : G \curvearrowright A$. Suppose that $\mathcal{D}$ is a strongly self-absorbing* C*-*algebra with $A \cong A \otimes \mathcal{D}$. If the Rokhlin dimension of α with commuting towers is finite, then it follows that α is γ-absorbing for every semi-strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{D}$.*

Here it may be useful to keep in mind that any version of what is called the Rokhlin property for $G = \mathbb{R}$ or $G = \mathbb{Z}^k$ will automatically imply finite Rokhlin dimension with commuting towers, and is therefore covered by Corollary B. This is in turn a generalization of [Hirshberg and Winter 2007, Theorem 1.1; Hirshberg et al. 2015, Theorems 5.8, 5.9; 2017, Theorem 5.3; Szabó et al. 2017, Theorem 9.6; Gardella and Lupini 2018, Theorem 4.50(2)]. We will in fact only apply the corollary within this paper, with a particular focus on the special case where the action is assumed to have the Rokhlin property. Some immediate applications of Corollary B will be discussed in Section 5. The main nontrivial application is pursued in Section 6, which is as follows; see Theorem 6.7 and Corollary 6.11:

Theorem C. *Let $\mathcal{D}$ be a strongly self-absorbing Kirchberg algebra. Then up to* (*very strong*) *cocycle conjugacy, there is a unique action $\gamma : \mathbb{R}^k \curvearrowright \mathcal{D}$ that has finite Rokhlin dimension with commuting towers.*

We note that a strongly self-absorbing C*-algebra is a Kirchberg algebra precisely when it is traceless. Kirchberg algebras are (by convention) the separable, simple, nuclear, purely infinite C*-algebras, whose celebrated classification is due to [Kirchberg and Phillips 2000; Phillips 2000; Kirchberg 2003] and which constitutes a prominent special case of the Elliott classification program. We note that all other strongly self-absorbing C*-algebras are conjectured to be quasidiagonal — see [Tikuisis et al. 2017, Corollary 6.7] — and so any Rokhlin flows on them would induce Rokhlin flows on the universal UHF algebra, which do not exist; see [Kishimoto 1996a, page 600; 1998a, page 289; Hirshberg et al. 2017, Section 2]. In particular, the underlying problem above is only interesting to consider in the purely infinite case.

Although the theorem above is not too far off from being a very special case of [Szabó 2017a] for ordinary flows, this result is entirely new for $k \geq 2$, and is in fact the first classification result for $\mathbb{R}^k$-actions on C*-algebras up to cocycle conjugacy.

The proof goes via induction in the number k of flows generating the action. In order to achieve a major part of the induction step, the corollary above is used in order to see that any two $\mathbb{R}^k$-actions as in the statement absorb each other tensorially. However, in order for this to make sense, it has to be at least established beforehand (as part of the induction step) that any such action has equivariantly approximately inner flip. This is achieved via a relative Kishimoto-type approximate cohomology-vanishing argument inspired by [Kishimoto 2002, Section 3], which combines arguments related to the Rokhlin property for $\mathbb{R}^k$-actions with arguments related to the structure theory of semi-strongly self-absorbing actions.

At this moment it seems unclear whether or not to expect a similarly rigid situation for Rokhlin $\mathbb{R}^k$-actions on general Kirchberg algebras, as is the case for $k = 1$ [Szabó 2017a]. In general, in order to implement a more general classification of this sort, it would require a technique for both constructing and manipulating cocycles for $\mathbb{R}^k$-actions (where $k \geq 2$) with the help of the Rokhlin property, which may potentially be much more complicated than for $k = 1$. In essence, our approach based on ideas related to strong self-absorption works because the main result allows one to bypass the need to bother with general cocycles for all of $\mathbb{R}^k$, but instead requires one only to consider individual copies of $\mathbb{R}$ inside $\mathbb{R}^k$ at a time (represented by the flows generating the $\mathbb{R}^k$-action), enabling an induction process.

In forthcoming work, the full force of the aforementioned main result of this paper (Theorem 4.4) will form the basis of further uniqueness results regarding actions of certain discrete amenable groups on strongly self-absorbing C*-algebras.

1. Preliminaries

Notation 1.1. Unless specified otherwise, we will stick to the following notational conventions:

- G denotes a locally compact Hausdorff group.
- A and B denote C*-algebras.
- The symbols α, β, γ are used to denote point-norm continuous actions on C*-algebras. Since continuity is always assumed in this context, we will simply refer to them as actions.
- If $\alpha : G \curvearrowright A$ is an action, then A^α denotes the fixed-point algebra of A.
- If F is a finite subset inside some set M, we often write $F \Subset M$.
- If (X, d) is some metric space with elements $a, b \in X$, then we write $a =_\varepsilon b$ as a shorthand for $d(a, b) \leq \varepsilon$.

We first recall some needed definitions and notation.

Definition 1.2 (cf. [Packer and Raeburn 1989, Definition 3.2] and [Szabó 2018b; 2017b, Section 1]). Let $\alpha : G \curvearrowright A$ be an action. Consider a strictly continuous map $w : G \to \mathcal{U}(\mathcal{M}(A))$:

(i) w is called an α-1-cocycle if one has $w_g\alpha_g(w_h) = w_{gh}$ for all $g, h \in G$. In this case, the map $\alpha^w : G \to \mathrm{Aut}(A)$ given by $\alpha^w_g = \mathrm{Ad}(w_g) \circ \alpha_g$ is again an action, and is called a cocycle perturbation of α. Two G-actions on A are called exterior equivalent if one of them is a cocycle perturbation of the other.

(ii) Assume that w is an α-1-cocycle. It is called an approximate coboundary if there exists a sequence of unitaries $x_n \in \mathcal{U}(\mathcal{M}(A))$ such that $x_n\alpha_g(x_n^*) \xrightarrow{\mathrm{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two G-actions on A are called strongly exterior equivalent if one of them is a cocycle perturbation of the other via an approximate coboundary.

(iii) Assume w is an α-1-cocycle. It is called an asymptotic coboundary if there exists a strictly continuous map $x : [0, \infty) \to \mathcal{U}(\mathcal{M}(A))$ with $x_0 = \mathbf{1}$ and such that $x_t\alpha_g(x_t^*) \xrightarrow{\mathrm{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two G-actions on A are called very strongly exterior equivalent if one of them is a cocycle perturbation of the other via an asymptotic coboundary.

(iv) Let $\beta : G \curvearrowright B$ be another action. The actions α and β are called cocycle conjugate, written $\alpha \simeq_{\mathrm{cc}} \beta$ if there exists an isomorphism $\psi : A \to B$ such that $\psi^{-1} \circ \beta \circ \psi$ and α are exterior equivalent. If ψ can be chosen such that $\psi^{-1} \circ \beta \circ \psi$ and α are strongly exterior equivalent, then α and β are called strongly cocycle conjugate, written $\alpha \simeq_{\mathrm{scc}} \beta$. If ψ can be chosen such that $\psi^{-1} \circ \beta \circ \psi$ and α are very strongly exterior equivalent, then α and β are called very strongly cocycle conjugate, written $\alpha \simeq_{\mathrm{vscc}} \beta$.

Note that for a cocycle w, the cocycle identity applied to $g = h = e$ yields $w_e = w_e^2$, and hence $w_e = \mathbf{1}$. This is implicitly used in many calculations without further mention.

Definition 1.3 (cf. [Kirchberg 2006, Definition 1.1] and [Szabó 2018b, Section 1]). Let A be a C*-algebra and let $\alpha : G \curvearrowright A$ be an action of a locally compact group:

(i) The sequence algebra of A is given as

$$A_\infty = \ell^\infty(\mathbb{N}, A)/\big\{(x_n)_n \;\big|\; \lim_{n\to\infty} \|x_n\| = 0\big\}.$$

There is a standard embedding of A into A_∞ by sending an element to its constant sequence. We shall always identify $A \subset A_\infty$ this way, unless specified otherwise.

(ii) Pointwise application of α on representing sequences defines a (not necessarily continuous) G-action α_∞ on A_∞. Let

$$A_{\infty,\alpha} = \{x \in A_\infty \mid [g \mapsto \alpha_{\infty,g}(x)] \text{ is continuous}\}$$

be the continuous part of A_∞ with respect to α.

(iii) For some C*-subalgebra $B \subset A_\infty$, the (corrected) relative central sequence algebra is defined as

$$F(B, A_\infty) = (A_\infty \cap B')/\,\mathrm{Ann}(B, A_\infty).$$

(iv) If $B \subset A_\infty$ is α_∞-invariant, then the G-action α_∞ on A_∞ induces a (not necessarily continuous) G-action $\tilde{\alpha}_\infty$ on $F(B, A_\infty)$. Let

$$F_\alpha(B, A_\infty) = \{y \in F_\alpha(B, A_\infty) \mid [g \mapsto \tilde{\alpha}_{\infty,g}(y)] \text{ is continuous}\}$$

be the continuous part of $F(B, A_\infty)$ with respect to α.

(v) When $B = A$, we write $F(A, A_\infty) = F_\infty(A)$ and $F_\alpha(A, A_\infty) = F_{\infty,\alpha}(A)$.

Definition 1.4 [Barlak and Szabó 2016, Definition 3.3]. Let G be a second-countable, locally compact group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable C*-algebras. An equivariant $*$-homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ is called (equivariantly) sequentially split if there exists a $*$-homomorphism $\psi : (B, \beta) \to (A_{\infty,\alpha}, \alpha_\infty)$ such that $\psi(\varphi(a)) = a$ for all $a \in A$.

Definition 1.5. Let G be a second-countable, locally compact group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on unital C*-algebras. Let $\varphi_1, \varphi_2 : (A, \alpha) \to (B, \beta)$ be two unital and equivariant $*$-homomorphisms. We say that φ_1 and φ_2 are approximately G-unitarily equivalent if the following holds. For every $\mathcal{F} \Subset A$, $\varepsilon > 0$, and compact set $K \subseteq G$, there exists a unitary $v \in \mathcal{U}(B)$ such that

$$\max_{a \in \mathcal{F}} \|\varphi_2(a) - v\varphi_1(a)v^*\| \leq \varepsilon, \quad \max_{g \in K} \|v - \beta_g(v)\| \leq \varepsilon.$$

Definition 1.6 [Szabó 2018b, Definitions 3.1, 4.1]. Let $\mathcal{D}$ be a separable, unital C*-algebra and G a second-countable, locally compact group. Let $\gamma : G \curvearrowright \mathcal{D}$ be an action. We say that:

(i) γ is a strongly self-absorbing action if the equivariant first-factor embedding

$$\mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G-unitarily equivalent to an isomorphism.

(ii) γ is semi-strongly self-absorbing if it is strongly cocycle conjugate to a strongly self-absorbing action.

Definition 1.7 [Szabó 2018c, Definition 2.17]. Let G be a second-countable, locally compact group. An action $\alpha : G \curvearrowright A$ on a unital C*-algebra is called unitarily regular if for every $\varepsilon > 0$ and compact set $K \subseteq G$, there exists $\delta > 0$ such that for every pair of unitaries

$$u, v \in \mathcal{U}(A) \quad \text{with} \max_{g \in K} \max\{\|\alpha_g(u) - u\|, \|\alpha_g(v) - v\|\} \leq \delta,$$

there exists a continuous path of unitaries $w : [0, 1] \to \mathcal{U}(A)$ satisfying

$$w(0) = \mathbf{1}, \quad w(1) = uvu^*v^*, \quad \max_{0 \leq t \leq 1} \max_{g \in K} \|\alpha_g(w(t)) - w(t)\| \leq \varepsilon.$$

Let us recall some of the main results from [Szabó 2017b; 2018b; 2018c], which we will use throughout. We will also use the perspective given in [Barlak and Szabó 2016, Section 4].

Theorem 1.8 (cf. [Szabó 2018b, Theorems 3.7, 4.7]). *Let G be a second-countable, locally compact group. Let A be a separable* C**-algebra and $\alpha : G \curvearrowright A$ an action. Let $\mathcal{D}$ be a separable, unital* C**-algebra and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing action. The following are equivalent*:

(i) *α and $\alpha \otimes \gamma$ are strongly cocycle conjugate.*

(ii) *α and $\alpha \otimes \gamma$ are cocycle conjugate.*

(iii) *There exists a unital, equivariant $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$.*

(iv) *The equivariant first-factor embedding $\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \to (A \otimes \mathcal{D}, \alpha \otimes \gamma)$ is sequentially split.*

If γ is moreover unitarily regular, then these statements are equivalent to

(v) *α and $\alpha \otimes \gamma$ are very strongly cocycle conjugate.*

Remark. For the rest of this paper, an action α satisfying condition (i) from above is called γ-absorbing or γ-stable. In the particular case that γ is the trivial G-action on a strongly self-absorbing C*-algebra $\mathcal{D}$, we will say that α is equivariantly $\mathcal{D}$-stable.

Remark 1.9. Unitary regularity for an action is a fairly mild technical assumption. It can be seen as the equivariant analog of the C*-algebraic property that the commutator subgroup inside the unitary group lies in the connected component of the unit. Unitary regularity holds automatically under equivariant $\mathcal{Z}$-stability, but also in other cases; see [Szabó 2018c, Proposition 2.19 and Example 6.4].

Theorem 1.10 [Szabó 2018c, Theorem 5.9]. *A semi-strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{D}$ is unitarily regular if and only if the class of all separable γ-absorbing G-C*-dynamical systems is closed under equivariant extensions.*

We will extensively use the following without much mention:

Proposition 1.11 [Brown 2000]. *Let G be a second-countable, locally compact group. Let A be a C*-algebra and $\alpha : G \curvearrowright A$ an action. Let $x \in A_{\infty,\alpha}$ and $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ be a bounded sequence representing x. Then $(x_n)_n$ is a continuous element with respect to the componentwise action of α on $\ell^\infty(\mathbb{N}, A)$.*

2. Box spaces and partitions of unity over groups

Definition 2.1. Let G be a second-countable, locally compact group. A residually compact approximation of G is a decreasing sequence $H_{n+1} \subseteq H_n \subseteq G$ of normal, discrete, cocompact subgroups in G with $\bigcap_{n\in\mathbb{N}} H_n = \{1\}$. If G is a discrete group, then the subgroups H_n will have finite index, in which case we call the sequence $(H_n)_n$ a residually finite approximation.

Remark 2.2. In the above setting, the sequence $(H_n)_n$ is automatically a residually finite approximation of the discrete group H_1.

Recall the definition of a box space; see [Roe 2003, Definition 10.24; Khukhro 2012].

Definition 2.3. Let Γ be a countable discrete group and $\mathcal{S} = (H_n)_n$ a residually finite approximation of Γ. Let d be a proper, right-invariant metric on Γ. For every $n \in \mathbb{N}$, denote by $\pi_n : \Gamma \to \Gamma/H_n$ the quotient map and by $\pi_{n*}(d)$ the push-forward metric on Γ/H_n that is induced by d. The box space of Γ along $\mathcal{S}$, denoted by $\square_{\mathcal{S}}\Gamma$, is the coarse disjoint union of the sequence of finite metric spaces $(\Gamma/H_n, \pi_{n*}(d))$.

The main purpose of this section will be to prove the following technical lemma:

Lemma 2.4. *Let G be a second-countable, locally compact group and $\mathcal{S} = (H_n)_n$ a residually compact approximation of G. Assume that the box space $\square_{\mathcal{S}} H_1$ has finite asymptotic dimension d. Then for every $\varepsilon > 0$ and compact set $K \subset G$, there exists $n \in \mathbb{N}$ and continuous, compactly supported functions $\mu^{(0)}, \dots, \mu^{(d)} : G \to [0, 1]$ satisfying*:

(a) *For every $l = 0, \dots, d$ and $h \in H_n \setminus \{1\}$, we have*
$$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)}) \cdot h = \varnothing.$$

(b) *For every $g \in G$, we have*
$$\sum_{l=0}^{d} \sum_{h \in H_n} \mu^{(l)}(gh) = 1.$$

(c) *For every $l = 0, \dots, d$ and $g \in K$, we have*
$$\|\mu^{(l)}(g \cdot \_) - \mu^{(l)}\|_\infty \leq \varepsilon.$$

Remark 2.5. In the case that $G = \Gamma$ is a discrete group and $\mathcal{S}$ is a residually finite approximation, this is precisely [Szabó et al. 2017, Lemma 2.13]. In order to prove Lemma 2.4, we shall convince ourselves that the desired functions can be constructed from finitely supported functions with similar properties on the cocompact subgroup H_1. For this, we first have to observe a slightly improved version of [Szabó et al. 2017, Lemma 2.13] in the discrete case.

Lemma 2.6. *Let Γ be a countable discrete group and $\mathcal{S} = (H_n)_n$ a residually finite approximation of Γ. Assume that the box space $\square_{\mathcal{S}} \Gamma$ has finite asymptotic dimension d. Then for every $\varepsilon > 0$ and finite set $F \Subset \Gamma$, there exists $n \in \mathbb{N}$ and finitely supported functions $\nu^{(0)}, \dots, \nu^{(d)} : \Gamma \to [0, 1]$ satisfying*:

(a) *For every $l = 0, \dots, d$ and $h \in H_n \setminus \{1\}$, we have*
$$g_1 h g_2^{-1} \notin F \quad \textit{for all } g_1, g_2 \in \operatorname{supp}(\nu^{(l)}).$$

(b) *For every $g \in \Gamma$, we have*
$$\sum_{l=0}^{d} \sum_{h \in H_n} \nu^{(l)}(gh) = 1.$$

(c) *For every $l = 0, \dots, d$ and $g \in F$, we have*
$$\|\nu^{(l)}(g \cdot \_) - \nu^{(l)}\|_\infty \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $F \subset G$ be given. We apply [Szabó et al. 2017, Lemma 2.13] and choose some n and finitely supported functions $\theta^{(0)}, \dots, \theta^{(d)} : \Gamma \to [0, 1]$ satisfying
$$\operatorname{supp}(\theta^{(l)}) \cap \operatorname{supp}(\theta^{(l)}) \cdot h_n = \varnothing \quad \text{for all } h_n \in H_n \setminus \{1\}, \tag{2-1}$$
as well as properties (b) and (c). Combining property (2-1) and (c), we see that if $g_1, g_2 \in \operatorname{supp}(\theta^{(l)})$ and $h \in H_n \setminus \{1\}$ are such that $g_1 h g_2^{-1} = g_1(g_2 h^{-1}) \in F$, then we get
$$|\theta^{(l)}(g_1)| = |\theta^{(l)}(g_1 h g_2^{-1} \cdot g_2 h^{-1})| \overset{(c)}{\leq} \varepsilon + |\theta^{(l)}(g_2 h^{-1})| \overset{(2\text{-}1)}{=} \varepsilon. \tag{2-2}$$

Let us define new functions $\kappa^{(l)} : \Gamma \to [0, 1]$ via

$$\kappa^{(l)}(g) = (\theta^{(l)}(g) - \varepsilon)_+. \tag{2-3}$$

These new functions clearly still satisfy property (c). For any $g_1, g_2 \in \operatorname{supp}(\kappa^{(l)})$, we evidently have $g_1, g_2 \in \operatorname{supp}(\theta^{(l)})$, so assuming $g_1 h g_2^{-1} \in F$ for some $h \in H_n \setminus \{1\}$ would imply $\kappa^{(l)}(g_1) = 0$ by (2-2) and (2-3), a contradiction. In particular we obtain property (a) for these functions.

Lastly, note that property (a) implies that any sum as in (b) can have at most $d+1$ nonvanishing summands, and thus we may estimate for all $g \in \Gamma$ that

$$1 = \sum_{l=0}^{d} \sum_{h \in H_n} \theta^{(l)}(gh) \geq \sum_{l=0}^{d} \sum_{h \in H_n} \kappa^{(l)}(gh) \geq \left(\sum_{l=0}^{d} \sum_{h \in H_n} \theta^{(l)}(gh) \right) - (d+1)\varepsilon = 1 - (d+1)\varepsilon.$$

So let us yet again define new functions $\nu^{(l)} : \Gamma \to [0, 1]$ via

$$\nu^{(l)}(g) = \left(\sum_{l=0}^{d} \sum_{h \in H_n} \kappa^{(l)}(gh) \right)^{-1} \kappa^{(l)}(g).$$

By our previous calculation, we have

$$\kappa^{(l)} \leq \nu^{(l)} \leq \frac{1}{1-(d+1)\varepsilon} \kappa^{(l)}.$$

For these functions, property (a) will still hold, while property (b) holds by construction. Moreover property (c) holds with regard to the tolerance

$$\eta_\varepsilon := \varepsilon + \frac{2(d+1)\varepsilon}{1-(d+1)\varepsilon}$$

in place of ε. Since $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$, this means that the functions $\nu^{(l)}$ will have the desired property after rescaling ε. This shows our claim. □

Lemma 2.7. *Let G be a locally compact group and $H \subset G$ a closed and cocompact subgroup. Let μ be a left-invariant Haar measure on H:*

(i) *There exists a compactly supported continuous function $C : G \to [0, \infty)$ satisfying the equation*

$$\int_H C(gh)\, d\mu(h) = 1 \quad \text{for all } g \in G.$$

(ii) *Assume furthermore that G is amenable. Let $\varepsilon > 0$ and let $K \subset G$ be a compact subset. Then there exists a function C as above with the additional property that*

$$\|C(g \cdot \_) - C\|_\infty \leq \varepsilon$$

for all $g \in K$.

Proof. (i): As H is a cocompact subgroup, there exists some compact set $K_H \subset G$ such that $G = K_H \cdot H$. By Urysohn–Tietze, we may choose a compactly supported continuous function $c : G \to [0, 1]$ with $c|_{K_H} = 1$. Define the compact set $K_c \subset H$ via

$$K_c = (K_H^{-1} \cdot \operatorname{supp}(c)) \cap H.$$

Then for every $g \in G$, there is some $h_0 \in H$ with $gh_0 \in K_H$. We have

$$\operatorname{supp}(c(gh_0 \cdot \_)) \cap H = ((gh_0)^{-1} \cdot \operatorname{supp}(c)) \cap H \subset K_c.$$

Thus, we get that

$$0 < \int_H c(gh)\, d\mu(h) = \int_H c(gh_0 h)\, d\mu(h) \leq \mu(K_c) < \infty.$$

Note that by the properties of the Haar measure, the assignment

$$\mathcal{I} : G \to (0, \infty), \quad g \mapsto \int_H c(gh)\, d\mu(h),$$

is H-periodic. Then the above computation shows that this assignment yields a well-defined, continuous function on G, which by H-periodicity and cocompactness of H can be viewed as a continuous function on the compact space G/H. Thus the image of this function is compact. In particular, its (pointwise) multiplicative inverse is also bounded and continuous. Let us define

$$C : G \to [0, \infty), \quad g \mapsto \mathcal{I}(g)^{-1} c(g).$$

Then this again yields a continuous function on G with compact support, but with the property that

$$\int_H C(gh)\, d\mu(h) = 1 \quad \text{for all } g \in G. \tag{2-4}$$

(ii): Let us now additionally assume that G is amenable. Let $\varepsilon > 0$ and $K \subset G$ be given as in the statement. Let ρ^G denote a right-invariant Haar measure on G. It follows from [Emerson and Greenleaf 1967] that we may find some compact set $J \subset G$ with $\rho^G(J) > 0$ such that $\rho^G(J \Delta (J \cdot K)) \leq \varepsilon / \|C\|_\infty \cdot \rho^G(J)$. Define $C' : G \to [0, \infty)$ via

$$C'(g) = \frac{1}{\rho^G(J)} \cdot \int_J C(xg)\, d\rho^G(x).$$

Clearly C' is yet another continuous function with compact support contained in $J^{-1} \cdot \operatorname{supp}(C)$. Given any element $g \in G$, we compute

$$\begin{aligned}
\int_H C'(gh)\, d\mu(h) &= \int_H \frac{1}{\rho^G(J)} \Big(\int_J C(xgh)\, d\rho^G(x) \Big)\, d\mu(h) \\
&= \frac{1}{\rho^G(J)} \int_J \Big(\int_H C(xgh)\, d\mu(h) \Big)\, d\rho^G(x) \\
&\overset{\text{(2-4)}}{=} \frac{1}{\rho^G(J)} \int_J 1\, d\rho^G(x) = 1.
\end{aligned}$$

Furthermore, we have for any $g_K \in K$ and $g \in G$ that

$$\begin{aligned}
|C'(g_K g) - C'(g)| &= \frac{1}{\rho^G(J)} \cdot \Big| \int_J C(x g_K g)\, d\rho^G(x) - \int_J C(xg)\, d\rho^G(x) \Big| \\
&\leq \frac{1}{\rho^G(J)} \cdot \|C\|_\infty \cdot \rho^G(J \Delta J g_K) \\
&\leq \varepsilon.
\end{aligned}$$

This shows the last part of the claim. □

Proof of Lemma 2.4. We first remark that since the box space $\square_S H_1$ has finite asymptotic dimension, it also has property A, and therefore H_1 is amenable; see [Nowak and Yu 2012, Theorems 4.3.6 and 4.4.6; Roe 2003, Proposition 11.39]. As H_1 is a discrete cocompact normal subgroup in G, we also see that G is amenable.

Let $\varepsilon > 0$ and $K \subset G$ be given. Then there exists a function $C : G \to [0, \infty)$ as in Lemma 2.7 for H_1 in place of H, with the property that

$$\|C(g \cdot \_) - C\|_\infty \leq \varepsilon \quad \text{for all } g \in K. \tag{2-5}$$

Let us denote the support of C by $S = \mathrm{supp}(C)$. As H_1 is discrete in G and S is compact, there exists a finite set $F \subset H_1$ with

$$h_1 \in F \quad \text{whenever } h_1 \in H \text{ and } S \cap Sh_1 \neq \varnothing. \tag{2-6}$$

Applying Lemma 2.6, there exists some n and finitely supported functions $\nu^{(0)}, \dots, \nu^{(d)} : H_1 \to [0, 1]$ satisfying the following properties:[1]

$$h_1 h_n h_2^{-1} \notin F \quad \text{for all } h_1, h_2 \in \mathrm{supp}(\nu^{(l)}) \text{ and } h_n \in H_n \setminus \{1\}, \tag{2-7}$$

$$1 = \sum_{l=0}^{d} \sum_{h_n \in H_n} \nu^{(l)}(h_1 h_n) \quad \text{for all } h_1 \in H_1. \tag{2-8}$$

We define $\mu^{(l)} : G \to [0, \infty)$ for $l = 0, \dots, d$ via

$$\mu^{(l)}(g) = \sum_{h_1 \in H_1} C(gh_1^{-1}) \nu^{(l)}(h_1).$$

Since $\nu^{(l)}$ is finitely supported on H_1, we see that $\mu^{(l)}$ is a finite sum of continuous functions with compact support, and hence $\mu^{(l)} \in \mathcal{C}_c(G)$.

We claim that these functions have the desired properties. Let us verify (a), which is equivalent to the statement that

$$\mu^{(l)}(g) \cdot \mu^{(l)}(gh_n^{-1}) = 0 \quad \text{for all } g \in G \text{ and } h_n \in H_n \setminus \{1\}.$$

Fix an element $h_n \in H_n \setminus \{1\}$ for the moment. We compute

$$\begin{aligned} \mu^{(l)}(g) \cdot \mu^{(l)}(gh_n^{-1}) &= \sum_{h_1, h_2 \in H_1} C(gh_1^{-1}) C(gh_n^{-1} h_2^{-1}) \nu^{(l)}(h_1) \nu^{(l)}(h_2) \\ &= \sum_{h_1, h_2 \in H_1} C(gh_1^{-1}) C(gh_2^{-1}) \nu^{(l)}(h_1) \nu^{(l)}(h_2 h_n^{-1}). \end{aligned}$$

We claim that each individual summand is zero. Indeed, suppose $h_1, h_2 \in H_1$ are such that

$$\nu^{(l)}(h_1) \nu^{(l)}(h_2 h_n^{-1}) > 0.$$

[1]Note that we will reserve the notation h_1, h_2 for elements in H_1, whereas h_n will denote an element in the smaller subgroup H_n for $n > 2$.

Then $h_1 \in \operatorname{supp}(\nu^{(l)})$ and $h_2 \in \operatorname{supp}(\nu^{(l)}) \cdot h_n$, which implies $h_1 h_2^{-1} \notin F$ by (2-7). By our choice of F, we obtain

$$\operatorname{supp}(C(\_ \cdot h_1^{-1})) \cap \operatorname{supp}(C(\_ \cdot h_2^{-1})) \subseteq Sh_1 \cap Sh_2 = (Sh_1 h_2^{-1} \cap S) \cdot h_2 \overset{(2\text{-}6)}{=} \varnothing,$$

and in particular $C(gh_1^{-1})C(gh_2^{-1}) = 0$. This finishes the proof that each summand of the above sum is zero and shows property (a).

Let us now show property (b). We calculate for every $g \in G$ that

$$\begin{aligned}\sum_{l=0}^{d} \sum_{h_n \in H_n} \mu^{(l)}(gh_n) &= \sum_{l=0}^{d} \sum_{h_n \in H_n} \sum_{h_1 \in H_1} C(gh_n h_1^{-1}) \nu^{(l)}(h_1) \\ &= \sum_{l=0}^{d} \sum_{h_n \in H_n} \sum_{h_1 \in H_1} C(gh_1^{-1}) \nu^{(l)}(h_1 h_n) \\ &= \sum_{h_1 \in H_1} C(gh_1^{-1}) \Big(\sum_{l=0}^{d} \sum_{h_n \in H_n} \nu^{(l)}(h_1 h_n) \Big) \\ &\overset{(2\text{-}8)}{=} \sum_{h_1 \in H_1} C(gh_1) \\ &\overset{\text{Lem. 2.7}}{=} 1.\end{aligned}$$

Let us now turn to (c). Given any $g \in G$ and $g_K \in K$, we compute

$$\begin{aligned}|\mu^{(l)}(g_K g) - \mu^{(l)}(g)| &= \Big| \sum_{h_1 \in H_1} (C(g_K g h_1^{-1}) - C(gh_1^{-1})) \nu^{(l)}(h_1) \Big| \\ &\overset{(2\text{-}8)}{\leq} \sup_{h_1 \in H_1} |C(g_K g h_1^{-1}) - C(gh_1^{-1})| \\ &\leq \|C(g_K \cdot \_) - C\|_\infty \overset{(2\text{-}6)}{\leq} \varepsilon.\end{aligned}$$

As $g \in G$ was arbitrary, this finishes the proof. □

Remark. Let G be a locally compact group and $H \subset G$ a closed, cocompact subgroup. For any C*-algebra A, we may naturally view $\mathcal{C}(G/H, A)$ as a C*-subalgebra of (right-)H-periodic functions inside $\mathcal{C}_b(G, A)$ by assigning a function f to the function f' given by $f'(g) = f(gH)$.

In what follows, we will briefly establish a technical result that allows one to perturb *approximately* H-periodic functions in $\mathcal{C}_b(G, A)$ to *exactly* H-periodic functions in a systematic way.

Lemma 2.8. *Let G be a locally compact group and $H \subset G$ a closed, cocompact subgroup. Let A be a* C*-*algebra. Then there exists a conditional expectation $E : \mathcal{C}_b(G, A) \to \mathcal{C}(G/H, A)$ with the following property.*

For every $\varepsilon > 0$ and compact set $K \subset G$, there exists $\delta > 0$ and a compact set $J \subset H$ such that the following holds:

If $f \in \mathcal{C}_b(G, A)$ *satisfies*

$$\max_{g\in K}\max_{h\in J}\|f(g)-f(gh)\|\le\delta,$$

then

$$\|f-E(f)\|_{\infty,K}\le\varepsilon.$$

Proof. Let μ be a left-invariant Haar measure on H. Let $C \in \mathcal{C}_c(G)$ be a function as in Lemma 2.7. Then we define

$$E:\mathcal{C}_b(G,A)\to\mathcal{C}(G/H,A),\quad E(f)(gH)=\int_H C(gh)f(gh)\,d\mu(h).$$

Since C is compactly supported and the Haar measure μ is left-invariant, it is clear that E is well-defined and indeed a conditional expectation. Let $\varepsilon > 0$ and $K \subset G$ be given. Let S be the compact support of C. Then the set $J := (K^{-1}S) \cap H$ is compact in H with the property that

$$g\in K \text{ and } gh\in S \quad\Longrightarrow\quad h\in J \tag{2-9}$$

for all $h \in H$. Set

$$\delta=\frac{\varepsilon}{1+\mu(J)\cdot\|C\|_\infty}.$$

For every $f \in \mathcal{C}_b(G, A)$ with

$$\max_{g\in K}\max_{h\in J}\|f(g)-f(gh)\|\le\delta,$$

it follows for every $g \in K$ that

$$\begin{aligned}\|f(g)-E(f)(gH)\| &= \left\|\left(\int_H C(gh)\,d\mu(h)\right)f(g)-\int_H C(gh)f(gh)\,d\mu(h)\right\|\\ &\overset{(2\text{-}9)}{=}\left\|\int_J C(gh)(f(g)-f(gh))\,d\mu(h)\right\|\\ &\le \mu(J)\cdot\|C\|_\infty\cdot\delta\le\varepsilon.\end{aligned}$$

This shows our claim. ☐

Corollary 2.9. *Let G be a locally compact group and* $H \subset G$ *a closed, cocompact subgroup. Let A and B be two* C**-algebras. Then for every* $\varepsilon > 0$, $F \Subset B$ *and compact set* $K \subset G$, *there exists* $\delta > 0$ *and a compact set* $J \subset H$ *such that the following holds*:

If $\Theta : B \to \mathcal{C}_b(G, A)$ *is a c.p.c. map with*

$$\max_{g\in K}\max_{h\in J}\|\Theta(b)(g)-\Theta(b)(gh)\|\le\delta\quad\textit{for all } b\in F,$$

then there exists a c.p.c. map $\Psi : B \to \mathcal{C}(G/H, A)$ *with*

$$\max_{g\in K}\|\Psi(b)(gH)-\Theta(b)(g)\|\le\varepsilon\quad\textit{for all } b\in F.$$

Proof. Let $E : \mathcal{C}_b(G, A) \to \mathcal{C}(G/H, A)$ be a conditional expectation as in Lemma 2.8. Given a triple (ε, F, K), choose $\delta > 0$ and $J \subset H$ so that the property in Lemma 2.8 holds for all $f \in \mathcal{C}_b(G, A)$ with respect to the pair (ε, K). Then we can directly conclude that if Θ is a map as in the statement, then $\Psi = E \circ \Theta$ has the desired property. ☐

3. Systems generated by order-zero maps with commuting ranges

The following notation and observations are [Hirshberg et al. 2017, Lemma 6.6] and originate in [Hirshberg et al. 2015, Section 5].

Notation 3.1. Let $D_1, \dots, D_n$ be finitely many unital C*-algebras. For $t \in [0, 1]$ and $j = 1, \dots, n$, we define

$$D_j^{(t)} := \begin{cases} D_j, & t > 0, \\ \mathbb{C} \cdot \mathbf{1}_{D_j}, & t = 0. \end{cases}$$

Given moreover a tuple $\vec{t} = (t_1, \dots, t_n) \in [0, 1]^n$, let us define

$$D^{(\vec{t})} := D_1^{(t_1)} \otimes_{\max} D_2^{(t_2)} \otimes_{\max} \cdots \otimes_{\max} D_n^{(t_n)}.$$

Consider the simplex

$$\Delta^{(n)} := \{\vec{t} \in [0, 1]^n \mid t_1 + \cdots + t_n = 1\}$$

and set

$$\mathcal{E}(D_1, \dots, D_n) := \{f \in \mathcal{C}(\Delta^{(n)}, D_1 \otimes_{\max} \cdots \otimes_{\max} D_n) \mid f(\vec{t}) \in D^{(\vec{t})}\}.$$

In the case that $D_j = D$ are all the same C*-algebra, we will write

$$\mathcal{E}(D_1, \dots, D_n) =: \mathcal{E}(D, n)$$

instead. For every $j = 1, \dots, n$, we will consider the canonical c.p.c. order-zero map

$$\eta_j : D_j \to \mathcal{E}(D_1, \dots, D_n)$$

given by

$$\eta_j(d_j)(\vec{t}) = t_j \cdot (\mathbf{1}_{D_1} \otimes \cdots \otimes \mathbf{1}_{D_{j-1}} \otimes d_j \otimes \mathbf{1}_{D_{j+1}} \otimes \cdots \otimes \mathbf{1}_{D_n}).$$

One easily checks that the ranges of the maps η_j generate $\mathcal{E}(D_1, \dots, D_n)$ as a C*-algebra.

Proposition 3.2. *Let $D_1, \dots, D_n$ be unital C*-algebras. Then the C*-algebra $\mathcal{E}(D_1, \dots, D_n)$, together with the c.p.c. order-zero maps $\eta_j : D_j \to \mathcal{E}(D_1, \dots, D_n)$, satisfies the following universal property:*

If B is any unital C-algebra and $\psi_j : D_j \to B$ for $j = 1, \dots, n$ are c.p.c. order-zero maps with pairwise commuting ranges and*

$$\psi_1(\mathbf{1}_{D_1}) + \cdots + \psi_n(\mathbf{1}_{D_n}) = \mathbf{1}_B,$$

then there exists a unique unital $*$-homomorphism $\Psi : \mathcal{E}(D_1, \dots, D_n) \to B$ such that $\Psi \circ \eta_j = \psi_j$ for all $j = 1, \dots, n$.

Notation 3.3. Let G be a second-countable, locally compact group. Let $D_1, \dots, D_n$ be unital C*-algebras with continuous actions $\alpha^{(j)} : G \curvearrowright D_j$ for $j = 1, \dots, n$. Then the G-action on $\mathcal{C}(\Delta^{(n)}, D_1 \otimes_{\max} \cdots \otimes_{\max} D_n)$ defined fibrewise by $\alpha^{(1)} \otimes_{\max} \cdots \otimes_{\max} \alpha^{(n)}$ restricts to a well-defined action

$$\mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)}) : G \curvearrowright \mathcal{E}(D_1, \dots, D_n).$$

We will again write $\mathcal{E}(\alpha, n) := \mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)})$ in the special case that all $(D_j, \alpha^{(j)}) = (D, \alpha)$ are the same C*-dynamical system.

Remark 3.4. By the universal property in Proposition 3.2, the G-action $\mathcal{E}(\alpha^{(1)},\dots,\alpha^{(n)})$ defined in Notation 3.3 is uniquely determined by the identity $\mathcal{E}(\alpha^{(1)},\dots,\alpha^{(n)})_g\circ\eta_j=\eta_j\circ\alpha^{(j)}_g$ for all $j=1,\dots,n$ and $g\in G$.

This immediately allows us obtain the following equivariant version of Proposition 3.2 as a consequence: Let B be any unital C*-algebra with an action $\beta: G\curvearrowright B$. If $\psi_j:(D_j,\alpha^{(j)})\to(B,\beta)$ are equivariant c.p.c. order-zero maps with pairwise commuting ranges and $\psi_1(\mathbf{1}_{D_1})+\dots+\psi_n(\mathbf{1}_{D_n})=\mathbf{1}_B$, then there exists a unique unital equivariant $*$-homomorphism

$$\Psi:(\mathcal{E}(D_1,\dots,D_n),\mathcal{E}(\alpha^{(1)},\dots,\alpha^{(n)}))\to(B,\beta)$$

satisfying $\Psi\circ\eta_j=\psi_j$ for all $j=1,\dots,n$.

Remark 3.5. Let us now also convince ourselves of a different natural way to view the C*-algebras from Notation 3.1.

For this, let us first consider the case $n=2$, so we have two unital C*-algebras D_1 and D_2. Notice that $[0,1]$ is naturally homeomorphic to the simplex $\Delta^{(2)}=\{(t_1,t_2)\in[0,1]^2\mid t_1+t_2=1\}$ via the assignment $t\mapsto(t,t-1)$. In this way we may see that there is a natural isomorphism

$$\begin{aligned}\mathcal{E}(D_1,D_2)&\stackrel{\text{def}}{=}\{f\in\mathcal{C}(\Delta^{(2)},D_1\otimes_{\max}D_2)\mid f(0,1)\in D_1\otimes\mathbf{1},\ f(1,0)\in\mathbf{1}\otimes D_2\}\\&\cong\{f\in\mathcal{C}([0,1],D_1\otimes_{\max}D_2)\mid f(0)\in D_1\otimes\mathbf{1},\ f(1)\in\mathbf{1}\otimes D_2\}\\&=:D_1\star D_2.\end{aligned}$$

In particular, we see that the notation $\mathcal{E}(D_1,D_2)$ is consistent with [Szabó 2018c, Definition 5.1]. As pointed out in Remark 5.2 of the same paper, the assignment $(D_1,D_2)\mapsto\mathcal{E}(D_1,D_2)$ on pairs of unital C*-algebras therefore generalizes the join construction for pairs of compact spaces, which gives rise to the notation $D_1\star D_2$.

Let now $n\geq 2$ and let $D_1,\dots,D_{n+1}$ be unital C*-algebras. The simplex $\Delta^{(n+1)}$ is homeomorphic to $[0,1]\times\Delta^{(n)}$ via the assignment

$$(t_1,\vec{t})\mapsto\begin{cases}(1,\vec{t}), & t_1=0,\\(1-t_1,\vec{t}/(1-t_1)), & t_1\neq 0\end{cases}$$

for $(\vec{t},t_{n+1})\in\Delta^{(n+1)}$. Keeping this in mind, we see that there is a natural map

$$\Phi: D_1\star\mathcal{E}(D_2,\dots,D_{n+1})\to\mathcal{E}(D_1,\dots,D_{n+1})$$

given by[2]

$$\Phi(f)(t_1,\vec{t})=\begin{cases}f(1)(\vec{t}), & t_1=0,\\f(1-t_1)(\vec{t}/(1-t_1)), & t_1\neq 0\end{cases}$$

for $(t_1,\vec{t})\in\Delta^{(n+1)}$. It is a simple exercise to see that this is a well-defined isomorphism. This shows that it makes sense to view the C*-algebra $\mathcal{E}(D_1,\dots,D_n)$ as the n-fold join $D_1\star\dots\star D_n$. We can also

[2]The reader should keep in mind that an element f in the domain is a continuous function on $[0,1]$ whose values are in turn (certain) continuous functions from $\Delta^{(n)}$ to the tensor product $D_1\otimes_{\max}\dots\otimes_{\max}D_{n+1}$.

observe that this isomorphism is natural in each C*-algebra, and therefore becomes equivariant as soon as we equip each C*-algebra D_j with an action $\alpha^{(j)}$ of some group G.

Henceforth, we will in particular write

$$D^{\star n} := \mathcal{E}(D, n) \quad \text{and} \quad \alpha^{\star n} := \mathcal{E}(\alpha, n)$$

for a unital C*-algebra D and some group action $\alpha : G \curvearrowright D$.

Remark 3.6. By the definition of the join of two C*-algebras D_1 and D_2, there is a natural short exact sequence

$$0 \longrightarrow \mathcal{C}_0(0,1) \otimes D_1 \otimes_{\max} D_2 \longrightarrow D_1 \star D_2 \longrightarrow D_1 \oplus D_2 \longrightarrow 0.$$

Given some $n \geq 1$ and a unital C*-algebra D, we have $D^{\star n+1} \cong D \star (D^{\star n})$, and therefore a special case of the above yields the short exact sequence

$$0 \longrightarrow \mathcal{C}_0(0,1) \otimes D \otimes_{\max} D^{\star n} \longrightarrow D^{\star n+1} \longrightarrow D \oplus D^{\star n} \longrightarrow 0.$$

Again by naturality, we note that this short exact sequence is automatically equivariant if we additionally equip D with a group action.

We now come to the main observation about C*-dynamical systems arising in this fashion, which will be crucial in proving our main result:

Lemma 3.7. *Let G be a second-countable, locally compact group. Let A be a separable, unital C*-algebra with an action $\alpha : G \curvearrowright A$. Suppose that $\gamma : G \curvearrowright \mathcal{D}$ is a semi-strongly self-absorbing and unitarily regular action. If α is γ-absorbing, then so is the action $\alpha^{\star n} : G \curvearrowright A^{\star n}$ for all $n \geq 2$.*

Proof. This follows directly from Remark 3.6 and Theorem 1.10 by induction. □

Remark 3.8. It ought to be mentioned that Lemma 3.7 does not depend in any way on the fact that one considers the n-fold join over the same C*-algebra and the same action. The analogous statement is valid for more general joins of the form

$$\alpha^{(1)} \star \cdots \star \alpha^{(n)} : G \curvearrowright A_1 \star \cdots \star A_n$$

by virtually the same argument.

In fact, by putting in a bit more work, one could likely prove an equivariant version of [Hirshberg et al. 2007, Theorem 4.6] for $\mathcal{C}_0(X)$-G-C*-algebras with $\dim(X) < \infty$ whose fibres absorb a given semi-strongly self-absorbing and unitarily regular action. This would contain Lemma 3.7 as a special case since the G-C*-algebra $A_1 \star \cdots \star A_n$ is in fact a $\mathcal{C}(\Delta^{(n)})$-$G$-C*-algebra with each fibre being isomorphic to some finite tensor product of the A_j. We will never need this level of generality within this paper, however.

4. Rokhlin dimension with commuting towers

The following notion generalizes analogous definitions made in [Hirshberg et al. 2015; 2017; Szabó et al. 2017; Gardella 2017].

Definition 4.1 (cf. [Hirshberg et al. 2017, Definition 4.1]). Let G be a second-countable, locally compact group. Let $\alpha : G \curvearrowright A$ be an action on a separable C*-algebra:

(i) Let $H \subset G$ be a closed, cocompact subgroup. The Rokhlin dimension of α with commuting towers relative to H, denoted by $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, H)$, is the smallest natural number d such that there exist equivariant c.p.c. order-zero maps

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(G/H), G\text{-shift}) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$$

with pairwise commuting ranges such that $\mathbf{1} = \varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1})$.

(ii) If $\mathcal{S} = (G_k)_k$ denotes a decreasing sequence of closed, cocompact subgroups, then we define the Rokhlin dimension of α with commuting towers relative to $\mathcal{S}$ via

$$\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \mathcal{S}) = \sup_{k\in\mathbb{N}} \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, G_k).$$

(iii) Let $N \subset G$ be any closed, normal subgroup. The Rokhlin dimension of α with commuting towers relative to N is defined as

$$\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, N) := \sup\{\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, H) \mid H \subseteq G \text{ closed, cocompact}, N \subseteq H\}.$$

(iv) Lastly, the Rokhlin dimension of α with commuting towers is defined as

$$\begin{aligned}\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) &:= \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \{1\}) \\ &= \sup\{\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, H) \mid H \subseteq G \text{ closed, cocompact}\}.\end{aligned}$$

We note that, even though the second half of Definition 4.1 always makes sense, these concepts are not expected to be of any practical use when G (or G/N) is not assumed to have enough closed cocompact subgroups, or to admit at least some residually compact approximation.

Notation 4.2. Let G be a second-countable, locally compact group. Given a decreasing sequence $\mathcal{S} = (G_k)_k$ of closed, cocompact subgroups, we will define

$$G/\mathcal{S} = \varprojlim G/G_k.$$

This is a metrizable, compact space,[3] which carries a natural continuous G-action induced by the left G-shift on each building block G/G_k; in particular we will call the resulting action also just the G-shift and denote it by

$$\sigma^{\mathcal{S}} : G \curvearrowright G/\mathcal{S}.$$

In the sequel, we will adopt the perspective of the associated G-C*-dynamical system, which is given as the equivariant inductive limit

$$\mathcal{C}(G/\mathcal{S}) = \varinjlim \mathcal{C}(G/G_k).$$

[3] This construction generalizes the profinite completion of a discrete residually finite group along a chosen separating sequence of normal subgroups of finite index.

We will moreover consider $\mathcal{C}(G/\mathcal{S})^{\star n}$ for $n \geq 2$. With some abuse of terminology, we will use the term "G-shift" also to refer to the canonical action on this C*-algebra (or the underlying space) that is induced by the n-fold tensor products of the G-shift on each fibre.

Lemma 4.3. *Let G be a second-countable, locally compact group. Let $\alpha : G \curvearrowright A$ be an action on a separable* C**-algebra. Let $\mathcal{S} = (G_k)_k$ be a decreasing sequence of closed, cocompact subgroups. Let $d \geq 0$ be some natural number. Then the following are equivalent*:

(i) $\dim^{c}_{\mathrm{Rok}}(\alpha, \mathcal{S}) \leq d$.

(ii) *There exist equivariant c.p.c. order-zero maps*

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(G/\mathcal{S}), G\text{-shift}) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$$

with pairwise commuting ranges such that $\mathbf{1} = \varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1})$.

(iii) *There exists a unital G-equivariant $*$-homomorphism*

$$(\mathcal{C}(G/\mathcal{S})^{\star(d+1)}, G\text{-shift}) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty).$$

(iv) *The first-factor embedding*

$$\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \to (A \otimes \mathcal{C}(G/\mathcal{S})^{\star(d+1)}, \alpha \otimes (G\text{-shift}))$$

is G-equivariantly sequentially split.

Proof. The equivalence (i)⇔(ii) follows from a standard reindexing trick such as Kirchberg's ε-test [2006, Lemma A.1], using the equivariant inductive limit structure of $\mathcal{C}(G/\mathcal{S})$ as pointed out in Notation 4.2. We will leave the details to the reader.

The equivalence (ii)⇔(iii) is a direct consequence of Proposition 3.2 and Remark 3.5, and the equivalence (iii)⇔(iv) is a direct consequence of [Barlak and Szabó 2016, Lemma 4.2]. □

The purpose of this section is to prove the following theorem, which can be regarded as the main result of the paper. Some of its nontrivial applications will be discussed in the subsequent sections. See in particular Corollary 5.1 for a possibly more accessible special case of this theorem.

Theorem 4.4. *Let G be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Denote by $\pi_N : G \to G/N$ the quotient map. Let $\mathcal{S}_1 = (H_k)_k$ be a residually compact approximation of G/N, and set $G_k = \pi_N^{-1}(H_k)$ for all $k \in \mathbb{N}$ and $\mathcal{S}_0 = (G_k)_k$. Let A be a separable* C**-algebra and $\mathcal{D}$ a strongly self-absorbing* C**-algebra. Let $\alpha : G \curvearrowright A$ be an action and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing, unitarily regular action. Suppose that for the restrictions to the N-actions, we have $\alpha|_N \simeq_{\mathrm{cc}} (\alpha \otimes \gamma)|_N$. If*

$$\mathrm{asdim}(\Box_{\mathcal{S}_1} H_1) < \infty \quad \text{and} \quad \dim^{c}_{\mathrm{Rok}}(\alpha, \mathcal{S}_0) < \infty,$$

then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.

We note that Theorem A is a direct consequence of this result. The hypothesis that G/N has some discrete, normal, residually finite, cocompact subgroup admitting a box space with finite asymptotic dimension means that there is choice for $\mathcal{S}_1$ as required by the above statement. The hypothesis that α has finite Rokhlin dimension with commuting towers means that the value $\dim^c_{\mathrm{Rok}}(\alpha, \mathcal{S}_0)$ has a finite uniform upper bound, for any possible choice of $\mathcal{S}_1$. Hence the statement of Theorem A follows.

The proof of Theorem 4.4 will occupy the rest of this section. The first and most difficult step is to convince ourselves of a very special case of Theorem 4.4, which involves the technical preparation below and from Section 2.

For convenience, we isolate the following lemma, which is a consequence of Proposition 1.11, the Winter–Zacharias structure theorem for order-zero maps, along with the Choi–Effros lifting theorem [1976]; see also [Winter and Zacharias 2009, Section 3].

Lemma 4.5. *Let G be a second-countable, locally compact group. Let A be a separable* C^**-algebra and B a separable, unital and nuclear* C^**-algebra. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions. Let $\kappa : (B, \beta) \to (A_{\infty,\alpha}, \alpha_\infty)$ be an equivariant c.p.c. order-zero map. Then κ can be represented by a sequence of c.p.c. maps $\kappa_n : B \to A$ satisfying*

(a) $\|\kappa_n(xy)\kappa(\mathbf{1}) - \kappa_n(x)\kappa_n(y)\| \to 0$,

(b) $\max_{g\in K} \|(\kappa_n \circ \gamma_g)(x) - (\alpha_g \circ \kappa_n)(x)\| \to 0$,

for all $x, y \in B$ and compact subsets $K \subset G$.

The proof of the following is based on a standard reindexing trick. In the short proof below, precise references are provided for completeness, although we note that this might not be the most elegant or direct way to show these statements.

Lemma 4.6. *Let G be a second-countable, locally compact group. Suppose that $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$, and $\gamma : G \curvearrowright \mathcal{D}$ are actions on separable* C^**-algebras. Assume furthermore that $\mathcal{D}$ is unital, that γ is semi-strongly self-absorbing, and that $\beta \simeq_{\mathrm{cc}} \beta \otimes \gamma$:*

(i) *Suppose that there exists an equivariant $*$-homomorphism $(A, \alpha) \to (B, \beta)$ that is G-equivariantly sequentially split. Then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.*

(ii) *Suppose that B is unital and that there exists an equivariant and unital $*$-homomorphism from (B, β) to $(F_{\infty,\alpha}, \tilde{\alpha}_\infty)$. Then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.*

Proof. (i): By Theorem 1.8, the statement $\beta \simeq_{\mathrm{cc}} \beta \otimes \gamma$ is equivalent to the equivariant first-factor embedding

$$\mathrm{id}_B \otimes \mathbf{1} : (B, \beta) \to (B \otimes \mathcal{D}, \beta \otimes \gamma)$$

being sequentially split. Let $\varphi : (A, \alpha) \to (B, \beta)$ be sequentially split. By [Barlak and Szabó 2016, Proposition 3.7], the composition $\varphi \otimes \mathbf{1}_{\mathcal{D}} = (\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{D}}) \circ \varphi$ is also sequentially split. However, we also have

$$\varphi \otimes \mathbf{1}_{\mathcal{D}} = (\varphi \otimes \mathrm{id}_{\mathcal{D}}) \circ (\mathrm{id}_A \otimes \mathbf{1}_{\mathcal{D}}),$$

which implies that $\mathrm{id}_A \otimes \mathbf{1}_{\mathcal{D}}$ is also sequentially split. This implies the claim that $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.

(ii): By [Barlak and Szabó 2016, Lemma 4.2], it follows that the embedding

$$\mathrm{id}_A \otimes \mathbf{1}_B : (A, \alpha) \to (A \otimes_{\max} B, \alpha \otimes \beta)$$

is sequentially split. Since we assumed that β is γ-absorbing, so is $\alpha \otimes \beta$, and so the claim arises as a special case of (i). ☐

The following is a special case of Theorem 4.4, as the process of tensorially stabilizing any action $\alpha : G \curvearrowright A$ with $(\mathcal{C}(G/\mathcal{S}), \sigma^{\mathcal{S}})$ causes the Rokhlin dimension relative to $\mathcal{S}$ to collapse to zero by definition. This explains why the statement below makes no explicit reference to Rokhlin dimension. Its proof is by far the most technical part of this paper:

Lemma 4.7. *Let G be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Denote by $\pi_N : G \to G/N$ the quotient map. Let $\mathcal{S}_1 = (H_k)_k$ be a residually compact approximation of G/N, and set $G_k = \pi_N^{-1}(H_k)$ for all $k \in \mathbb{N}$ and $\mathcal{S}_0 = (G_k)_k$. Let A be a separable* C*-*algebra and $\mathcal{D}$ a strongly self-absorbing* C*-*algebra. Let $\alpha : G \curvearrowright A$ be an action and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing, unitarily regular action. Suppose that for the restrictions to the N-actions, we have $\alpha|_N \simeq_{\mathrm{cc}} (\alpha \otimes \gamma)|_N$. If $\mathrm{asdim}(\Box_{\mathcal{S}_1} H_1) < \infty$, then the G-action*

$$\sigma^{\mathcal{S}_0} \otimes \alpha : G \curvearrowright \mathcal{C}(G/\mathcal{S}_0) \otimes A$$

is γ-absorbing.

Proof. Set $d := \mathrm{asdim}(\Box_{\mathcal{S}_1} H_1)$. Let

$$\tilde{\kappa} : (\mathcal{D}, \gamma|_N) \to (F_{\infty,\alpha|_N}(A), \tilde{\alpha}_\infty|_N)$$

be an N-equivariant, unital $*$-homomorphism. Using [Szabó 2018c, Example 4.4 and Proposition 4.5], we may choose an equivariant c.p.c. order-zero map

$$\kappa : (\mathcal{D}, \gamma|_N) \to (A_{\infty,\alpha|_N} \cap A', \alpha_\infty|_N)$$

that lifts $\tilde{\kappa}$.

Consider a sequence of c.p.c. maps $\kappa_n : B \to A$ lifting κ as in Lemma 4.5. Let us choose finitely many subsequences $\kappa_n^{(l)} : B \to A$ of the maps κ_n for $l = 0, \dots, d$ so that, using Lemma 4.5, each sequence $\kappa_n^{(l)}$ has the following properties for all $a \in A$, $b, b_1, b_2 \in \mathcal{D}$ and compact sets $L \subseteq N$:

$$\|[\kappa_n^{(l)}(b), a]\| \to 0, \tag{4-1}$$

$$\|\kappa_n^{(l)}(b_1 b_2)\kappa_n^{(l)}(\mathbf{1}) - \kappa_n^{(l)}(b_1)\kappa_n^{(l)}(b_2)\| \to 0, \tag{4-2}$$

$$\|(\kappa_n^{(l)}(\mathbf{1}) - \mathbf{1}) \cdot a\| \to 0, \tag{4-3}$$

$$\max_{r \in L} \|(\kappa_n^{(l)} \circ \gamma_r)(b) - (\alpha_r \circ \kappa_n^{(l)})(b)\| \to 0, \tag{4-4}$$

and additionally one has for every compact set $K \subseteq G$ that

$$\max_{g \in K} \|[\kappa_n^{(l_1)}(b_1), (\alpha_g \circ \kappa_n^{(l_2)})(b_2)]\| \to 0 \quad \text{for all } l_1 \neq l_2. \tag{4-5}$$

Let now $\varepsilon > 0$ be a fixed parameter and $1_G \in K \subseteq G$ a fixed compact set. Apply Lemma 2.4 and find k and compactly supported functions $\mu^{(0)}, \dots, \mu^{(d)} \in \mathcal{C}_c(G/N)$, so that for every $l = 0, \dots, d$ we have

$$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)}) \cdot h = \varnothing \quad \text{for all } h \in H_k \setminus \{1\}, \tag{4-6}$$

$$\sum_{l=0}^{d} \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) = 1 \quad \text{for all } g \in G, \tag{4-7}$$

$$\|\mu^{(l)}(\pi_N(g) \cdot \_) - \mu^{(l)}\|_\infty \le \varepsilon \quad \text{for all } g \in K \cup K^{-1}. \tag{4-8}$$

The group H_k is discrete, so we may choose a cross-section $\sigma : H_k \to G_k = \pi_N^{-1}(H_k) \subseteq G$. For each $l = 0, \dots, d$, consider the sequence of c.p.c. maps

$$\Theta_n^{(l)} : \mathcal{D} \to \mathcal{C}_b(G, A)$$

given by

$$\Theta_n^{(l)}(b)(g) = \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) \cdot (\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{g\sigma(h)}^{-1})(b). \tag{4-9}$$

This sum is well-defined because the compact support of the function $\mu^{(l)}$ on G/N meets a set of the form $\pi_N(g) \cdot H_k$ at most once according to (4-6).

We wish to show that given an element $b \in \mathcal{D}$, the functions $\Theta_n^{(l)}(b)$ are approximately G_k-periodic on large compact sets. This is so that we may apply Corollary 2.9 in order to approximate the maps $\Theta_n^{(l)}$ by other maps going into $\mathcal{C}(G/G_k, A)$.

Let $K_{H_k} \subseteq G_k$ and $K_G \subseteq G$ be two compact sets. As H_k is discrete, we observe two facts. First, there exists a compact set $K_N \subseteq N$ and a finite set $1 \in F_k \Subset H_k$ with

$$K_{H_k} \subset \sigma(F_k) \cdot K_N. \tag{4-10}$$

Second, by possibly enlarging F_k if necessary, we may assume by (4-6) that also

$$\mu^{(l)}(\pi_N(g)h) > 0 \quad \text{implies} \quad h \in F_k \qquad \text{for all } g \in K_G. \tag{4-11}$$

Define also

$$K_N' = \bigcup_{h_0, h \in F_k} \sigma(h_0) \cdot K_N \cdot \sigma(h_0^{-1}h)\sigma(h)^{-1} \subseteq N, \tag{4-12}$$

$$K_N'' = \bigcup_{h \in F_k} \sigma(h)^{-1} \cdot K_N' \cdot \sigma(h) \subseteq N. \tag{4-13}$$

As N is a normal subgroup and σ is a cross-section for the quotient map π_N, it follows that these are compact subsets in N.

We compute for all $l = 0, \dots, d,\ b \in \mathcal{D},\ g \in K_G,\ h_0 \in F_k$ and $r \in K_N'$ that

$$\begin{aligned}
&\|(\alpha_{g\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{g\sigma(h_0)}^{-1})(b) - (\alpha_{gr\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{gr\sigma(h_0)}^{-1})(b)\| \\
&\quad = \|(\alpha_{\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) - (\alpha_{r\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{r\sigma(h_0)}^{-1})(\gamma_g^{-1}(b))\| \\
&\quad = \|(\alpha_{\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) - (\alpha_{\sigma(h_0)} \circ \alpha_{\sigma(h_0)^{-1}r\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{\sigma(h_0)^{-1}r\sigma(h_0)}^{-1} \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b))\| \\
&\quad \overset{(4\text{-}13)}{\le} \max_{g \in K_G} \max_{s \in K_N''} \|(\alpha_s \circ \kappa_n^{(l)} \circ \gamma_s^{-1})(\gamma_{g\sigma(h_0)}^{-1}(b)) - \kappa_n^{(l)}(\gamma_{g\sigma(h_0)}^{-1}(b))\| \\
&\quad \xrightarrow{(4\text{-}4)} 0 \quad (\text{uniformly on } K_G, K_N').
\end{aligned}$$

It thus follows for all $l=0,\dots,d,\ b\in\mathcal{D},\ g\in K_G,\ h_0\in F_k$ and $r\in K_N$ that

$$\begin{aligned}
&\|\Theta_n^{(l)}(b)(g)-\Theta_n^{(l)}(b)(g\sigma(h_0)r)\|\\
&\overset{\substack{(4\text{-}9),\\(4\text{-}11)}}{=} \Big\| \sum_{h_1\in F_k} \mu^{(l)}(\pi_N(g)h_1)\cdot(\alpha_{g\sigma(h_1)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_1)})(b)\\
&\qquad\qquad - \sum_{h_2\in h_0^{-1}F_k} \mu^{(l)}(\pi_N(g)h_0h_2)\cdot(\alpha_{g\sigma(h_0)r\sigma(h_2)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_0)r\sigma(h_2)})(b)\Big\|\\
&= \Big\| \sum_{h_1\in F_k} \mu^{(l)}(\pi_N(g)h_1)\cdot(\alpha_{g\sigma(h_1)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_1)})(b)\\
&\qquad\qquad - \sum_{h_2\in F_k} \mu^{(l)}(\pi_N(g)h_2)\cdot(\alpha_{g\sigma(h_0)r\sigma(h_0^{-1}h_2)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_0)r\sigma(h_0^{-1}h_2)})(b)\Big\|\\
&\overset{(4\text{-}6)}{=} \max_{h\in F_k} \|(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h)})(b)-(\alpha_{g\sigma(h_0)r\sigma(h_0^{-1}h)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)r\sigma(h_0^{-1}h)})(b)\|\\
&= \max_{h\in F_k} \|(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h)})(b)-(\alpha_{g\sigma(h_0)r\sigma(h_0^{-1}h)\sigma(h)^{-1}\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_0)r\sigma(h_0^{-1}h)\sigma(h)^{-1}\sigma(h)})(b)\|\\
&\overset{(4\text{-}12)}{=} \max_{h\in F_k}\max_{s\in K_N'} \|(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h)})(b)-(\alpha_{gs\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{gs\sigma(h)})(b)\|\\
&\longrightarrow 0 \quad (\text{uniformly on } K_G,\ K_N).
\end{aligned}$$

Here we have used (4-6) in the third equality in the sense that $\mu^{(l)}(\pi_N(g)h)$ is nonzero for a unique element $h\in F_k$. By (4-10) we get for all $b\in\mathcal{D}$ that

$$\max_{g\in K_G}\max_{t\in K_{H_k}}\|\Theta_n^{(l)}(b)(g)-\Theta_n^{(l)}(b)(gt)\| \overset{(4\text{-}10)}{\leq} \max_{g\in K_G}\max_{h_0\in F_k}\max_{r\in K_N}\|\Theta_n^{(l)}(b)(g)-\Theta_n^{(l)}(b)(g\sigma(h_0)r)\|\longrightarrow 0.$$

Since $K_G\subseteq G$ and $K_{H_k}\subseteq G_k$ were arbitrary compact sets, we are in the position to apply Corollary 2.9. As $\mathcal{D}$ is separable, it follows for every $l=0,\dots,d$ that there exists a sequence of c.p.c. maps

$$\Psi_n^{(l)}: B\to\mathcal{C}(G/G_k,A)$$

so that for every compact set $K_G\subseteq G$ and $b\in\mathcal{D}$, we have

$$\max_{g\in K_G}\|\Psi_n^{(l)}(b)(gG_k)-\Theta_n^{(l)}(b)(g)\|\to 0. \tag{4-14}$$

We now wish to show that these c.p.c. maps are approximately equivariant with regard to γ and $\sigma^{G_k}\otimes\alpha$, where σ^{G_k} is the G-action on $\mathcal{C}(G/G_k)$ induced by the left-translation of G on G/G_k.

Let us fix a compact set $K_G\subseteq G$ as above. Without loss of generality, let us assume that it is large enough so that the quotient map $G\to G/G_k$ is still surjective when restricted to K_G. Given $b\in\mathcal{D}$, set

$$\rho_n(b)=\max_{l=0,\dots,d}\ \max_{g\in K^{-1}K_G}\|\Psi_n^{(l)}(b)(gG_k)-\Theta_n^{(l)}(b)(g)\|. \tag{4-15}$$

Note that by an elementary compactness argument, it follows from (4-14) that for every compact set $J\subset\mathcal{D}$, we have

$$\max_{b\in J}\rho_n(b)\to 0. \tag{4-16}$$

Let $t \in K$, $g \in K_G$ and $b \in \mathcal{D}$ with $\|b\| \le 1$. Then

$$\begin{aligned}(\sigma_t^{G_k} \otimes \alpha_t)((\Psi_n^{(l)})(b))(gG_k) &= \alpha_t(\Psi_n^{(l)}(b)(t^{-1}gG_k)) \\ &\overset{(4\text{-}15)}{=}_{\rho_n(b)} \alpha_t(\Theta_n^{(l)}(b)(t^{-1}gG_k)) \\ &\overset{(4\text{-}9)}{=} \sum_{h \in H_k} \mu^{(l)}(\pi_N(t^{-1}g)h) \cdot (\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{t^{-1}g\sigma(h)}^{-1})(b) \\ &\overset{(4\text{-}6),(4\text{-}8)}{=}_{\varepsilon} \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) \cdot (\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{t^{-1}g\sigma(h)}^{-1})(b) \\ &\overset{(4\text{-}15),(4\text{-}9)}{=}_{\rho_n(\gamma_t(b))} \Psi_n^{(l)}(\gamma_t(b))(gG_k).\end{aligned}$$

Note that as K_G contains a representative for every G_k-orbit in G, these approximations carry over to the $\|\cdot\|_\infty$-norm of the involved functions. Using (4-16), we obtain for all $b \in \mathcal{D}$ with $\|b\| \le 1$ that

$$\limsup_{n\to\infty} \max_{t \in K} \|(\sigma_t^{G_k} \otimes \alpha_t)(\Psi_n^{(l)})(b) - (\Psi_n^{(l)} \circ \gamma_t)(b)\| \le \varepsilon. \tag{4-17}$$

Next, we wish to show that for $l_1 \neq l_2$, the c.p.c. maps $\Psi_n^{(l_1)}$ and $\Psi_n^{(l_2)}$ have approximately commuting ranges as $n \to \infty$. Let $g_1, g_2 \in K_G$ and $b \in \mathcal{D}$ with $\|b\| \le 1$ be given. Then we compute

$$\begin{aligned}&\|[\Psi_n^{(l_1)}(b)(g_1G_k), \Psi_n^{(l_2)}(b)(g_2G_k)]\| \\ &\quad\overset{(4\text{-}15)}{=}_{4\rho_n(b)} \|[\Theta_n^{(l_1)}(b)(g_1), \Theta_n^{(l_2)}(b)(g_2)]\| \\ &\quad\overset{(4\text{-}6),(4\text{-}9),(4\text{-}11)}{\le} \max_{h_1,h_2 \in F_k} \|[(\alpha_{g_1\sigma(h_1)} \circ \kappa_n^{(l_1)} \circ \gamma_{g_1\sigma(h_1)}^{-1})(b), (\alpha_{g_2\sigma(h_2)} \circ \kappa_n^{(l_2)} \circ \gamma_{g_2\sigma(h_2)}^{-1})(b)]\| \\ &\quad= \max_{h_1,h_2 \in F_k} \|[(\kappa_n^{(l_1)} \circ \gamma_{g_1\sigma(h_1)}^{-1})(b), (\alpha_{\sigma(h_1)^{-1}g_1^{-1}g_2\sigma(h_2)} \circ \kappa_n^{(l_2)} \circ \gamma_{g_2\sigma(h_2)}^{-1})(b)]\|\end{aligned}$$

In particular, we obtain for every contraction $b \in \mathcal{D}$ that

$$\begin{aligned}&\max_{g_1,g_2 \in K_G} \|[\Psi_n^{(l_1)}(b)(g_1G_k), \Psi_n^{(l_2)}(b)(g_2G_k)]\| \\ &\quad\le \max_{g_1,g_2 \in K_G} \max_{h_1,h_2 \in F_k} \|[(\kappa_n^{(l_1)} \circ \gamma_{g_1\sigma(h_1)}^{-1})(b), (\alpha_{\sigma(h_1)^{-1}g_1^{-1}g_2\sigma(h_2)} \circ \kappa_n^{(l_2)} \circ \gamma_{g_2\sigma(h_2)}^{-1})(b)]\| + 4\rho_n(b) \\ &\quad\overset{(4\text{-}16),(4\text{-}5)}{\longrightarrow} 0.\end{aligned} \tag{4-18}$$

Here we have used that the convergence in (4-5) automatically holds uniformly when quantifying over b_1, b_2 belonging to some compact subset in $\mathcal{D}$, in this case

$$b_1, b_2 \in \{\gamma_g^{-1}(b) \mid g \in K_G \cdot \sigma(F_k)\}.$$

In exactly the same fashion, one also computes

$$\|[\Psi_n^{(l)}(b), a]\| \longrightarrow 0 \tag{4-19}$$

for all $l = 0, \dots, d$, $b \in \mathcal{D}$, and $a \in A$, by using (4-1) in place of (4-5).

Next, we wish to show that for each $l = 0, \dots, d$, the c.p.c. maps $\Psi_n^{(l)}$ behave approximately like order-zero maps. Let $g \in K_G$. Choose the unique element $h_0 \in F_k$ with $\mu^{(l)}(\pi_N(g)h_0) > 0$. Then it follows for every $b_1, b_2 \in \mathcal{D}$ that

$$\begin{aligned}\Theta_n^{(l)}(b_1)(g)\cdot\Theta_n^{(l)}(b_2)(g) &= \mu^{(l)}(\pi_N(g)h_0)^2\cdot(\alpha_{g\sigma(h_0)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_1)\cdot(\alpha_{g\sigma(h_0)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_2)\\ &= \mu^{(l)}(\pi_N(g)h_0)^2\cdot\alpha_{g\sigma(h_0)}\big((\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_1)\cdot(\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_1)\big).\end{aligned}$$

It follows from this calculation that

$$\begin{aligned}&\|\Theta_n^{(l)}(b_1)\cdot\Theta_n^{(l)}(b_2)-\Theta_n^{(l)}(b_1b_2)\cdot\Theta_n^{(l)}(\mathbf{1})\|_{\infty,K_G}\\ &\qquad\le \max_{s\in K_{G_k}\cdot\sigma(F_k)}\|(\kappa_n^{(l)}\circ\gamma_s^{-1})(b_1)\cdot(\kappa_n^{(l)}\circ\gamma_s^{-1})(b_1)-(\kappa_n^{(l)}\circ\gamma_s^{-1})(b_1b_2)\cdot(\kappa_n^{(l)}\circ\gamma_s^{-1})(\mathbf{1})\|\\ &\qquad\xrightarrow{(4\text{-}7),(4\text{-}8)} 0.\end{aligned}$$

As K_G contains a representative of every G_k-orbit in G, it follows from (4-14) that

$$\|\Psi_n^{(l)}(b_1)\cdot\Psi_n^{(l)}(b_2)-\Psi_n^{(l)}(b_1b_2)\cdot\Psi_n^{(l)}(\mathbf{1})\|\longrightarrow 0 \tag{4-20}$$

for every $b_1, b_2 \in \mathcal{D}$.

Next, we wish to show that the completely positive sum $\sum_{l=0}^d \Psi_n^{(l)}$ behaves approximately like a u.c.p. map upon multiplication with an element of $\mathbf{1}\otimes A$ as $n\to\infty$. Let $g\in K_G$. We have

$$\begin{aligned}\Theta_n^{(l)}(\mathbf{1})(g) &\overset{(4\text{-}9),(4\text{-}10)}{=} \sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h)}^{-1})(\mathbf{1})\\ &= \sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1}).\end{aligned}$$

It follows for all $a\in A$ that

$$\begin{aligned}&\max_{g\in K_G}\Big\|\Big(\mathbf{1}-\sum_{l=0}^d\Theta_n^{(l)}(\mathbf{1})(g)\Big)\cdot a\Big\|\\ &\quad\le \max_{g\in K_G}(d+1)\cdot\max_l\max_{h\in F_k}\|(\alpha_{\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1})-\kappa_n^{(l)}(\mathbf{1})\|+\Big\|\Big(\mathbf{1}-\sum_{l=0}^d\sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\alpha_g\circ\kappa_n^{(l)})(\mathbf{1})\Big)\cdot a\Big\|\\ &\quad\overset{(4\text{-}7)}{=} \max_{g\in K_G}(d+1)\cdot\max_l\max_{h\in F_k}\|(\alpha_{\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1})-\kappa_n^{(l)}(\mathbf{1})\|+\Big\|\Big(\sum_{l=0}^d\sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\mathbf{1}-(\alpha_g\circ\kappa_n^{(l)})(\mathbf{1}))\Big)\cdot a\Big\|\\ &\quad\overset{(4\text{-}6)}{\le} \max_{g\in K_G}\big((d+1)\cdot\max_l\max_{h\in F_k}\|(\alpha_{\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1})-\kappa_n^{(l)}(\mathbf{1})\|+(d+1)\cdot\max_l\|(\mathbf{1}-\kappa_n^{(l)}(\mathbf{1}))\cdot\alpha_g^{-1}(a)\|\big)\\ &\quad\xrightarrow{(4\text{-}3),(4\text{-}4)} 0.\end{aligned}$$

Since K_G contains every G_k-orbit in G, it follows from (4-14) that

$$\Big\|\Big(\mathbf{1}-\sum_{l=0}^d\Psi_n^{(l)}(\mathbf{1})\Big)\cdot(\mathbf{1}\otimes a)\Big\|\to 0 \quad \text{for all } a\in A. \tag{4-21}$$

Let us now summarize everything we have obtained so far. The c.p.c. maps $\Psi_n^{(l)} : \mathcal{D} \to \mathcal{C}(G/G_k, A)$, for $l = 0, \dots, d$ and $n \in \mathbb{N}$ satisfy the following properties for all $b, b_1, b_2 \in \mathcal{D}$ and $a \in A$:

$$\|[\Psi_n^{(l)}(b), \mathbf{1} \otimes a]\| \longrightarrow 0, \tag{4-22}$$

$$\limsup_{n\to\infty} \max_{t\in K} \|((\sigma^{G_k} \otimes \alpha)_t \circ \Psi_n^{(l)})(b) - (\Psi_n^{(l)} \circ \gamma_t)(b)\| \le \varepsilon, \tag{4-23}$$

$$\|[\Psi_n^{(l_1)}(b), \Psi_n^{(l_2)}(b)]\| \longrightarrow 0 \quad \text{for all } l_1 \neq l_2, \tag{4-24}$$

$$\|\Psi_n^{(l)}(b_1) \cdot \Psi_n^{(l)}(b_2) - \Psi_n^{(l)}(b_1 b_2) \cdot \Psi_n^{(l)}(\mathbf{1})\| \longrightarrow 0, \tag{4-25}$$

$$\left\| \left(\mathbf{1} - \sum_{l=0}^{d} \Psi_n^{(l)}(\mathbf{1}) \right) \cdot \mathbf{1} \otimes a \right\| \longrightarrow 0. \tag{4-26}$$

Note that k, and thus the codomain of $\Psi_n^{(l)}$, had to be chosen depending on ε and $K \subseteq G$. However, we have canonical (equivariant) inclusions $\mathcal{C}(G/G_k, A) \subseteq \mathcal{C}(G/\mathcal{S}_0, A)$, which we may compose our maps with. It is then clear that the same properties as in (4-22) up to (4-26) hold, where we replace the action $\sigma^{G_k} : G \curvearrowright \mathcal{C}(G/G_k)$ by $\sigma^{\mathcal{S}_0} : G \curvearrowright \mathcal{C}(G/\mathcal{S}_0)$.

Since A and $\mathcal{D}$ are separable and G is second-countable, we can let the tolerance ε go to zero, let the set $K \subseteq G$ get larger and apply a diagonal sequence argument. Putting the appropriate choices of c.p.c. maps into a single sequence, we can thus obtain c.p.c. maps

$$\psi^{(l)} : B \to (\mathcal{C}(G/\mathcal{S}_0) \otimes A)_\infty, \quad l = 0, \dots, d,$$

that satisfy the following properties for all $g \in G,\ a \in A$, and $b, b_1, b_2 \in \mathcal{D}$:

$$[\psi^{(l)}(b), \mathbf{1} \otimes a] = 0, \tag{4-27}$$

$$(\sigma^{\mathcal{S}_0} \otimes \alpha)_g \circ \psi^{(l)} = \psi^{(l)} \circ \gamma_g, \tag{4-28}$$

$$[\psi^{(l_1)}(b), \psi^{(l_2)}(b)] = 0 \quad \text{for all } l_1 \neq l_2, \tag{4-29}$$

$$\psi^{(l)}(b_1) \cdot \psi^{(l)}(b_2) = \psi^{(l)}(b_1 b_2) \cdot \psi^{(l)}(\mathbf{1}), \tag{4-30}$$

$$\left(\mathbf{1} - \sum_{l=0}^{d} \psi^{(l)}(\mathbf{1}) \right) \cdot \mathbf{1} \otimes a = 0. \tag{4-31}$$

Since $\gamma : G \curvearrowright \mathcal{D}$ is point-norm continuous, (4-28) implies that the image of each map $\psi^{(l)}$ is in the continuous part $(\mathcal{C}(G/\mathcal{S}_0) \otimes A)_{\infty, \sigma^{\mathcal{S}_0} \otimes \alpha}$. In fact it is in the relative commutant of $\mathbf{1} \otimes A$ by (4-27), but then also automatically in the relative commutant of all of $\mathcal{C}(G/\mathcal{S}_0) \otimes A$. This allows us to define equivariant maps

$$\zeta^{(l)} : \mathcal{D} \to F_{\infty, \sigma^{\mathcal{S}_0} \otimes \alpha}(\mathcal{C}(G/\mathcal{S}_0) \otimes A), \quad \zeta^{(l)}(b) = \psi^{(l)}(b) + \mathrm{Ann}(\mathcal{C}(G/\mathcal{S}_0) \otimes A)$$

for all $l = 0, \dots, d$. Then (4-29) implies that these maps have commuting ranges, (4-30) implies that they are c.p.c. order-zero, and finally (4-31) implies the equation $\sum_{l=0}^{d} \zeta^{(l)}(\mathbf{1}) = \mathbf{1}$.

By virtue of Proposition 3.2 and Remark 3.5, this gives rise to a unital equivariant $*$-homomorphism

$$(\mathcal{D}^{\star(d+1)}, \gamma^{\star(d+1)}) \to \big(F_{\infty, \sigma^{\mathcal{S}_0} \otimes \alpha}(\mathcal{C}(G/\mathcal{S}_0) \otimes A), \widetilde{(\sigma^{\mathcal{S}_0} \otimes \alpha)_\infty}\big).$$

As γ is unitarily regular, it follows from Lemma 3.7 that $\gamma^{\star(d+1)}$ is a γ-absorbing action. Applying Lemma 4.6 yields that $\sigma^{\mathcal{S}_0} \otimes \alpha$ is γ-absorbing, which finishes the proof. $\square$

Now we are in a position to prove Theorem 4.4:

Proof of Theorem 4.4. Let $\alpha : G \curvearrowright A$ and $\gamma : G \curvearrowright \mathcal{D}$ be the two actions as in the assumption. Let also $N \subset G,\ H_k \subset G/N$, and $G_k \subset G$ be subgroups as specified in the statement, and denote by $\mathcal{S}_1 = (H_k)_k$ a sequence of subgroups in G/N, and by $\mathcal{S}_0 = (G_k)_k$ a sequence of subgroups in G.

Suppose $\operatorname{asdim}(\square_{\mathcal{S}_1} H_1) < \infty$ and $s := \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \mathcal{S}_0) < \infty$. Using the latter, Lemma 4.3(iv) implies that the equivariant embedding

$$\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \to (A \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star(s+1)}, \alpha \otimes (G\text{-shift}))$$

is G-equivariantly sequentially split. By Lemma 4.6, in order to show that α is γ-absorbing, it suffices to show that the G-C*-algebra $A \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star(s+1)}$ is γ-absorbing. We will show this via induction on s.

For $s = 0$, the claim is precisely Lemma 4.7, and in particular it holds because we assumed that $\operatorname{asdim}(\square_{\mathcal{S}_1} H_1) < \infty$.

Given $s \geq 1$, assume that the claim holds for $s - 1$. It follows by Remark 3.6 that there is an extension of G-C*-algebras of the form

$$0 \longrightarrow J^{(s)} \longrightarrow A \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star(s+1)} \longrightarrow Q^{(s)} \longrightarrow 0,$$

where

$$J^{(s)} = A \otimes \mathcal{C}_0(0,1) \otimes \mathcal{C}(G/\mathcal{S}_0) \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star s},$$
$$Q^{(s)} = A \otimes (\mathcal{C}(G/\mathcal{S}_0) \oplus \mathcal{C}(G/\mathcal{S}_0)^{\star s}).$$

By the induction hypothesis, both the kernel and the quotient of this extension are γ-absorbing G-C*-algebras, and therefore so is the middle by Theorem 1.10. This finishes the induction step and the proof. $\square$

Remark 4.8. We remark that the statement of the main result holds verbatim for cocycle actions instead of genuine actions. Note that the concept of Rokhlin dimension makes sense for cocycle actions with the same definition, since there is still a natural genuine action induced on the central sequence algebra. If $(\alpha, w) : G \curvearrowright A$ is a cocycle action on a separable C*-algebra, then $(\alpha \otimes \mathrm{id}_{\mathcal{K}}, w \otimes \mathbf{1}) : G \curvearrowright A \otimes \mathcal{K}$ is cocycle conjugate to a genuine action by the Packer–Raeburn stabilization trick [1989]. Since both Rokhlin dimension and absorption of a semi-strongly self-absorbing action are invariants under stable (cocycle) conjugacy, the statement of Theorem 4.4 follows for cocycle actions.

5. Some applications

Let us now discuss some immediate applications of the main result. First we wish to point out that the following result arises as a special case.

Corollary 5.1. *Let G be a second-countable, locally compact group. Let $\mathcal{S} = (H_n)_n$ be a residually compact approximation consisting of normal subgroups of G with*

$$\operatorname{asdim}(\square_{\mathcal{S}} H_1) < \infty.$$

Let A be a separable C*-*algebra and $\mathcal{D}$ a strongly self-absorbing* C*-*algebra with $A \cong A \otimes \mathcal{D}$. Let $\alpha : G \curvearrowright A$ be an action with*

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, \mathcal{S}) < \infty.$$

Then $\alpha \simeq_{\mathrm{vscc}} \alpha \otimes \gamma$ for all semi-strongly self-absorbing actions $\gamma : G \curvearrowright \mathcal{D}$.

Proof. Let $\gamma : G \curvearrowright \mathcal{D}$ be a semi-strongly self-absorbing action. Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$ by [Winter 2011], we may replace γ with $\gamma \otimes \mathrm{id}_{\mathcal{Z}}$ for the purpose of showing the claim, as $\gamma \otimes \mathrm{id}_{\mathcal{Z}}$ is again semi-strongly self-absorbing and every $(\gamma \otimes \mathrm{id}_{\mathcal{Z}})$-absorbing action is γ-absorbing. So let us simply assume $\gamma \simeq_{\mathrm{cc}} \gamma \otimes \mathrm{id}_{\mathcal{Z}}$. By Remark 1.9, we may thus assume that γ is unitarily regular. The claim then follows directly from Theorem 4.4 applied to the case $N = \{1\}$. Note that one automatically has absorption with respect to very strong cocycle conjugacy by virtue of Theorem 1.8(v). □

Note that the results below in part refer to Rokhlin dimension without commuting towers, as defined in [Szabó et al. 2017, Section 5]. For the Rokhlin dimension-zero case, the commuting tower assumption is vacuous.

Example 5.2. Let $\mathcal{Q}$ denote the universal UHF algebra. Let Γ be a countable, discrete group and $H \subset \Gamma$ a normal subgroup with finite index. There exists a strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{Q}$ with $\dim_{\mathrm{Rok}}(\gamma, H) = 0$.

Proof. Such an action is constructed as part of [Szabó et al. 2017, Remark 10.8]. Namely, consider the left-regular representation $\lambda^{G/H} : G/H \to \mathcal{U}(M_{|G:H|})$, consider the quotient map $\pi_H : G \to G/H$, and define

$$\gamma_g = \mathrm{id}_{\mathcal{Q}} \otimes \bigotimes_{\mathbb{N}} \mathrm{Ad}(\lambda^{G/H}(\pi_H(g)))$$

as an action on $\mathcal{Q} \cong \mathcal{Q} \otimes M_{[G:H]}^{\otimes\infty}$. As the diagonal embedding $\mathcal{C}(G/H) \subset M_{[G:H]}$ is equivariant, it follows that $\dim_{\mathrm{Rok}}(\gamma, H) = 0$. By [Szabó 2018c, Proposition 6.3], such an action is strongly self-absorbing. □

This in turn has the following consequence regarding the dimension-reducing effect of strongly self-absorbing C*-algebras.

Corollary 5.3. *Let Γ be a countable, discrete, residually finite group that has some box space with finite asymptotic dimension. Let $\alpha : \Gamma \curvearrowright A$ be an action on a separable* C*-*algebra with* $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$:

(1) *If $A \cong A \otimes \mathcal{Q}$, then* $\dim_{\mathrm{Rok}}(\alpha) = 0$.

(2) *If $A \cong A \otimes \mathcal{Z}$, then* $\dim_{\mathrm{Rok}}(\alpha) \leq 1$.

Proof. (1): This follows directly from Example 5.2 and Corollary 5.1.

(2): We have $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{Z}}$, and there exist two c.p.c. order-zero maps $\psi_0, \psi_1 : \mathcal{Q} \to \mathcal{Z}_\infty \cap \mathcal{Z}'$ with $\psi_0(\mathbf{1}) + \psi_1(\mathbf{1}) = \mathbf{1}$; see [Matui and Sato 2014a, Section 5; Sato et al. 2015, Section 6]. Consider two sequences $\psi_{0,n}, \psi_{1,n} : \mathcal{Q} \to \mathcal{Z}$ of c.p.c. maps lifting ψ_0 and ψ_1.

By (1), $\alpha \otimes \mathrm{id}_{\mathcal{Q}}$ has Rokhlin dimension zero. Given any subgroup $H \subset \Gamma$ with finite index, we can find c.p.c. order-zero maps $\mathcal{C}(\Gamma/H) \to A \otimes \mathcal{Q}$ which are approximately equivariant, have approximately central image, and are such that the image of the unit acts approximately like a unit on finite sets. Once we compose such maps with $\mathrm{id}_A \otimes \psi_{i,n}$ for $i = 0, 1$ and large enough n, we may obtain two c.p.c. maps $\kappa_0, \kappa_1 : C(\Gamma/H) \to A \otimes \mathcal{Z}$, which are approximately equivariant, have approximately central image, and so that the element $\kappa_0(\mathbf{1}) + \kappa_1(\mathbf{1})$ approximately acts like a unit on a given finite set in $A \otimes \mathcal{Z}$. But this is what is required by $\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}}) \leq 1$; we leave the finer details to the reader as the proof is quite standard. □

Remark 5.4. The reason why the proof of Corollary 5.3(2) does not yield $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) \leq 1$ is due to the fact that the two order-zero maps $\psi_0, \psi_1 : \mathcal{Q} \to \mathcal{Z}_\infty$ can never have commuting ranges. Indeed, this would imply the existence of a unital $*$-homomorphism $\mathcal{Q} \to \mathcal{Z}_\infty$ via Lemma 3.7, so it is impossible. More concretely, [Hirshberg and Phillips 2017, Example 3.32] exhibits an example of a $\mathbb{Z}_2$-action α on a Kirchberg algebra with $\dim_{\mathrm{Rok}}(\alpha) = 1$ and $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = 2$.

Corollary 5.5. *Let Γ be a discrete, finitely generated, virtually nilpotent group. Let X be a compact metrizable space with finite covering dimension, and $\alpha : \Gamma \curvearrowright X$ a free action by homeomorphisms. Then one has*

$$\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Q}} : \Gamma \curvearrowright \mathcal{C}(X) \otimes \mathcal{Q}) = 0$$

and

$$\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}} : \Gamma \curvearrowright \mathcal{C}(X) \otimes \mathcal{Z}) \leq 1.$$

Proof. By [Szabó et al. 2017, Corollary 7.5], the action $\alpha : \Gamma \curvearrowright \mathcal{C}(X)$ has finite Rokhlin dimension.[4] Since the underlying C*-algebra is abelian, the claim follows from Corollary 5.3. □

6. Multiflows on strongly self-absorbing Kirchberg algebras

In this section, we shall study actions of $\mathbb{R}^k$ on certain C*-algebras satisfying an obvious notion of the Rokhlin property.

Notation 6.1. For $k \geq 2$, we will refer to a continuous action of $\mathbb{R}^k$ on a C*-algebra as a *multiflow*. Let $(e_j)_{1 \leq j \leq k}$ be the standard basis of $\mathbb{R}^k$. Given $\alpha : \mathbb{R}^k \curvearrowright A$, we will define the *generating flows* $\alpha^{(j)} : \mathbb{R} \curvearrowright A$, given by $\alpha^{(j)}_t = \alpha_{te_j}$, for $j = 1, \ldots, k$. We then have

$$\alpha^{(j)}_{t_j} \circ \alpha^{(i)}_{t_i} = \alpha^{(i)}_{t_i} \circ \alpha^{(j)}_{t_j} \quad \text{for all } i, j = 1, \ldots, k \text{ and all } t_i, t_j \in \mathbb{R}.$$

We will also denote by $\alpha^{(\neq j)} : \mathbb{R}^{k-1} \curvearrowright A$ the action generated by the flows $(\alpha^{(i)})_{i \neq j}$. We remark that $\alpha^{(j)}$ reduces naturally to a flow on the fixed point algebra $A^{\alpha^{(\neq j)}}$.

Definition 6.2. Let A be a separable C*-algebra and $\alpha : \mathbb{R}^k \curvearrowright A$ an action. We say that α has the Rokhlin property if $\dim_{\mathrm{Rok}}(\alpha, p\mathbb{Z}^k) = 0$ for all $p > 0$.

Remark 6.3. An obvious question regarding Definition 6.2 is whether this is the same as $\dim_{\mathrm{Rok}}(\alpha) = 0$ when $k \geq 2$, especially because this appears to be (a priori) much more difficult to check. Nevertheless, this turns out to be case. Instead of giving a detailed proof here, let us just roughly sketch the basic idea.

[4] Strictly speaking, only the nilpotent case is proved there. The virtually nilpotent case follows from independent work of Bartels [2017, Section 1].

The condition $\dim_{\mathrm{Rok}}(\alpha) = 0$ in the sense of Definition 4.1 amounts to checking $\dim_{\mathrm{Rok}}(\alpha, H) = 0$ for every closed cocompact subgroup $H \subset \mathbb{R}^k$, or in other words finding approximately equivariant unital embeddings from $\mathcal{C}(\mathbb{R}^k/H)$ into the central sequence algebra of A. This only gets easier when we make H smaller, so we may assume without loss of generality that H is discrete. Since H is a free abelian group and is cocompact in $\mathbb{R}^k$, it has a $\mathbb{Z}$-basis $e_1, \dots, e_k \in H$. We may approximate these elements by $f_1, \dots, f_k \in \mathbb{Q}^k$, which are linearly independent over $\mathbb{Q}$ and span another subgroup H'. By using for example Lemma 2.8 we can then obtain approximately multiplicative and equivariant u.c.p. maps $\mathcal{C}(\mathbb{R}^k/H) \to \mathcal{C}(\mathbb{R}^k/H')$. By the properties of central sequence algebras, we may thus assume without loss of generality that in fact $H \subseteq \mathbb{Q}^k$. Now the same argument as in [Szabó et al. 2017, Example 3.19] allows one to see that H contains a finite-index subgroup of the form $n\mathbb{Z}^k$ for some $n \in \mathbb{N}$. In summary, we obtain $\dim_{\mathrm{Rok}}(\alpha, H) = 0$ for arbitrary H when we assume the Rokhlin property in the sense of Definition 6.2.

Remark 6.4. In the case of flows, i.e., the case $k = 1$ above, Definition 6.2 coincides with the notion of the Rokhlin property from [Kishimoto 1996a]. Let us for now denote by $\sigma^T : \mathbb{R} \curvearrowright \mathcal{C}(\mathbb{R}/T\mathbb{Z})$ the action induced by the $\mathbb{R}$-shift.

Proposition 6.5. *Let A be a separable* C*-*algebra and $\alpha : \mathbb{R}^k \curvearrowright A$ an action. The following are equivalent*:

(i) *α has the Rokhlin property.*

(ii) *For every $j = 1, \dots, k$ and every $p > 0$, there exists a unitary*

$$u \in F_{\infty,\alpha}(A)^{\tilde{\alpha}_\infty^{(\not j)}} \quad \textit{such that} \quad \tilde{\alpha}^{(j)}_{\infty,t}(u) = e^{ipt}u, \quad t \in \mathbb{R}.$$

(iii) *For every $j = 1, \dots, k$ and every $T > 0$, there exists an equivariant and unital $*$-homomorphism*

$$(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^T) \longrightarrow (F_{\infty,\alpha}(A)^{\tilde{\alpha}_\infty^{(\not j)}}, \tilde{\alpha}^{(j)}_\infty).$$

Proof. (i)⇔(iii): Let $T > 0$. One has a canonical equivariant isomorphism

$$(\mathcal{C}(\mathbb{R}^k/T\mathbb{Z}^k), \mathbb{R}^k\text{-shift}) \cong (\mathcal{C}(\mathbb{R}/T\mathbb{Z})^{\otimes k}, \sigma^{T,1} \otimes \cdots \otimes \sigma^{T,k}),$$

where $\sigma^{T,j}$ is the $\mathbb{R}^k$-action on $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ where only the j-th component acts by the $\mathbb{R}$-shift. By definition, α having the Rokhlin property means that for every $T > 0$ the dynamical system on the left embeds into $(F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$. So in particular, when (i) holds, one also obtains an embedding of $(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^{T,j})$ for every $j = 1, \dots, k$, which implies (iii). Conversely, when (iii) holds, for all $T > 0$ one has an embedding of $(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^{T,j})$ into $(F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$ for all $j = 1, \dots, k$. By applying a standard reindexing argument in the central sequence algebra, one may assume that these embeddings have pairwise commuting ranges for all $j = 1, \dots, k$. Therefore one obtains an embedding of the C*-dynamical system given by the tensor product of all $(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^{T,j})$, which we have seen to be the same as the dynamical system $(\mathcal{C}(\mathbb{R}^k/T\mathbb{Z}^k), \mathbb{R}^k\text{-shift})$. In particular this implies (i).

(ii)⇔(iii): This follows directly from functional calculus. A unitary u as in (ii) gives rise to a unital equivariant $*$-homomorphism

$$\varphi_u : \big(\mathcal{C}\big(\mathbb{R}/\tfrac{2\pi}{p}\mathbb{Z}\big), \sigma^{\frac{2\pi}{p}}\big) \longrightarrow (F_{\infty,\alpha}(A)^{\tilde{\alpha}_\infty^{(\not j)}}, \tilde{\alpha}^{(j)}_\infty), \quad \varphi_u(f) = f(u).$$

Conversely, whenever φ is an arbitrary homomorphism between these two dynamical systems, $u = \varphi\big(\big[t + \frac{2\pi}{p}\mathbb{Z} \mapsto e^{ipt}\big]\big)$ yields a unitary as required by (ii). □

Remark 6.6. We note that for $G = \mathbb{R}^k$, the sequence $H_n = (n!) \cdot \mathbb{Z}^k$ yields a residually compact approximation in the sense of Definition 2.1. Now it is well known that $\square_{(H_n)_n}\mathbb{Z}^k$ has finite asymptotic dimension k; see either [Szabó et al. 2017, Sections 2–3] or better yet [Delabie and Tointon 2018]. In particular, Corollary 5.1 is applicable to $\mathbb{R}^k$-actions that have finite Rokhlin dimension with commuting towers, and more specifically it is applicable to $\mathbb{R}^k$-actions with the Rokhlin property.

The following is the main result of this section.

Theorem 6.7. *Let $\mathcal{D}$ be a strongly self-absorbing Kirchberg algebra. Let $k \geq 1$ be a given natural number. Then all continuous $\mathbb{R}^k$-actions on $\mathcal{D}$ with the Rokhlin property are semi-strongly self-absorbing and are mutually (very strongly) cocycle conjugate.*

The approach for proving this result, at least in the way presented here, uses the theory of semi-strongly self-absorbing actions in a crucial way. In such dynamical systems, one has a very strong control over certain (approximately central) unitary paths, which, together with the Rokhlin property, allows one to obtain a relative cohomology-vanishing-type statement. This will be used to deduce inductively that the actions in the statement of Theorem 6.7 have approximately $\mathbb{R}^k$-inner flip. The desired uniqueness for such actions is then achieved by combining this fact with Corollary 5.1, which is a special case of our main result, in a suitable way.

Example 6.8 (see [Bratteli et al. 2007]). Denote by $s_1, s_2, \ldots$ the generators of the Cuntz algebra $\mathcal{O}_\infty$. Define a quasifree flow $\gamma^0 : \mathbb{R} \curvearrowright \mathcal{O}_\infty$ via

$$\gamma^0_t(s_1) = e^{2\pi i t}s_1, \quad \gamma^0_t(s_2) = e^{-2\pi i\sqrt{2}t}s_2, \quad \text{and} \quad \gamma^0_t(s_j) = s_j \quad \text{for } j \geq 3.$$

Then γ^0 has the Rokhlin property by [Bratteli et al. 2007, Theorem 1.1].

In particular, given $k \geq 1$ and any strongly self-absorbing Kirchberg algebra $\mathcal{D}$, the action

$$\mathrm{id}_{\mathcal{D}} \otimes (\underbrace{\gamma^0 \times \cdots \times \gamma^0}_{k \text{ times}}) : \mathbb{R}^k \curvearrowright \mathcal{D} \otimes \mathcal{O}_\infty^{\otimes k} \cong \mathcal{D}$$

is a (k-)multiflow with the Rokhlin property on $\mathcal{D}$, and is in fact (very strongly) cocycle conjugate to every other one by Theorem 6.7.

Let us now implement the strategy outlined above step by step. We begin with the aforementioned cohomology-vanishing-type statement, which involves minimal assumptions about the underlying C*-algebras but otherwise very strong assumptions about the existence of certain unitary paths, which will naturally appear in our intended setup later.

Lemma 6.9. *Let A be a separable unital* C**-algebra. Let $k \geq 1$ and let $\alpha : \mathbb{R}^k \curvearrowright A$ be a continuous action with the Rokhlin property, and fix some $j \in \{1, \ldots, k\}$.*

For every $\varepsilon > 0$, $L > 0$ and $\mathcal{F} \Subset A$, there exists a $T > 0$ and $\mathcal{G} \Subset A$ with the following property:

If $\{w_t\}_{t\in\mathbb{R}} \subset \mathcal{U}(A)$ *is any* $\alpha^{(j)}$-1*-cocycle satisfying*

$$\max_{a\in\mathcal{F}} \max_{0\le t\le T} \|[w_t, a]\| \le \varepsilon,$$

$$\max_{0\le t\le T} \max_{\vec{r}\in[0,1]^{k-1}} \|w_t - \alpha^{(\bar{j})}_{\vec{r}}(w_t)\| \le \varepsilon,$$

and moreover there exists some continuous path of unitaries $u : [0, 1] \to \mathcal{U}(A)$ *with*

$$u(0) = \mathbf{1}, \quad u(1) = w_{-T}, \quad \ell(u) \le L,$$

$$\max_{0\le t\le 1} \max_{\vec{r}\in[0,1]^{k-1}} \|u(t) - \alpha^{(\bar{j})}_{\vec{r}}(u(t))\| \le \varepsilon,$$

$$\max_{0\le t\le 1} \max_{a\in\mathcal{G}} \|[u(t), a]\| \le \varepsilon,$$

then there exists a unitary $v \in \mathcal{U}(A)$ *satisfying*

$$\max_{0\le t\le 1} \|w_t - v\alpha^{(j)}_t(v^*)\| \le 3\varepsilon,$$

$$\max_{a\in\mathcal{F}} \|[v, a]\| \le 3\varepsilon,$$

$$\max_{\vec{r}\in[0,1]^{k-1}} \|v - \alpha^{(\bar{j})}_{\vec{r}}(v)\| \le 3\varepsilon.$$

Proof. Let $T > 0$ and note that we have fixed $j \in \{1, \dots, k\}$ by assumption. By some abuse of notation, let us view σ^T as the $\mathbb{R}^k$-action on $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ such that the j-th coordinate acts as the $\mathbb{R}$-shift and all the other components act trivially. In this way, any $*$-homomorphism as in Proposition 6.5(iii) can be viewed as an $\mathbb{R}^k$-equivariant $*$-homomorphism from $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ to $F_{\infty,\alpha}(A)$. In particular, denote such a homomorphism by θ. We can then obtain a commutative diagram of $\mathbb{R}^k$-equivariant $*$-homomorphisms via

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\hspace{6em}} & (A_{\infty,\alpha}, \alpha_\infty) \\ & {\scriptstyle d\mapsto \mathbf{1}\otimes d} \searrow \quad \nearrow {\scriptstyle f\otimes d\mapsto \theta(f)\cdot d} & \\ & (\mathcal{C}(\mathbb{R}/T\mathbb{Z})\otimes A, \sigma^T\otimes\alpha) & \end{array} \tag{6-1}$$

We will keep this in mind for later.

Now let $\varepsilon > 0$, $L > 0$ and $\mathcal{F} \Subset A$ be as in the statement. Without loss of generality, we assume that $\mathcal{F}$ consists of contractions. We choose $T > L/\varepsilon$ and $\mathcal{G} \Subset A$ to be any finite set of contractions containing $\mathcal{F}$ that is $\varepsilon/2$-dense in the compact subset

$$\{\alpha^{(j)}_{-s}(a) \mid a \in \mathcal{F},\ 0 \le s \le T\}. \tag{6-2}$$

We claim that these do the trick. We note that the rest of the proof below is almost identical to the proofs of [Kishimoto 1996a, Theorem 2.1; Szabó 2017a, Lemma 3.4], respectively, except for some obvious modifications.

Assume that $\{w_t\}_{t\in\mathbb{R}} \subset \mathcal{U}(A)$ is an $\alpha^{(j)}$-1-cocycle satisfying

$$\max_{a\in\mathcal{F}} \max_{0\le t\le T} \|[w_t, a]\| \le \varepsilon, \tag{6-3}$$

$$\max_{0\le t\le T} \max_{\vec{r}\in[0,1]^{k-1}} \|w_t - \alpha^{(\bar{j})}_{\vec{r}}(w_t)\| \le \varepsilon, \tag{6-4}$$

and moreover that there exists some continuous path of unitaries $u : [0, 1] \to \mathcal{U}(A)$ with

$$u(0) = \mathbf{1}, \quad u(1) = w_{-T}, \quad \ell(u) \le L, \tag{6-5}$$

$$\max_{0\le t\le 1} \max_{\vec{r}\in[0,1]^{k-1}} \|u(t) - \alpha^{(\not{j})}_{\vec{r}}(u(t))\| \le \varepsilon, \tag{6-6}$$

$$\max_{0\le t\le 1} \max_{a\in\mathcal{G}} \|[u(t), a]\| \le \varepsilon. \tag{6-7}$$

As $\ell(u) \le L$, we may assume that u is L-Lipschitz by passing to the arc-length parametrization if necessary. We denote by $\kappa : [0, T] \to \mathcal{U}(A)$ the path given by $\kappa_s = u(s/T)$, which is then Lipschitz with respect to the constant $L/T \le \varepsilon$. Let us define a continuous path of unitaries $v : [0, T] \to \mathcal{U}(A)$ via $v_s = w_s\alpha^{(j)}_s(\kappa_s)$. Then by (6-5) it follows that $v(0) = v(T) = \mathbf{1}$. In particular, we may view v as a unitary in $\mathcal{C}(\mathbb{R}/T\mathbb{Z}) \otimes A$.

We have

$$\begin{aligned}
\max_{a\in\mathcal{F}} \|[v, \mathbf{1}\otimes a]\| &= \max_{a\in\mathcal{F}} \max_{0\le s\le T} \|[w_s\alpha^{(j)}_s(\kappa_s), a]\| \\
&\le \max_{a\in\mathcal{F}} \max_{0\le s\le T} \|[w_s, a]\| + \|[\kappa_s, \alpha^{(j)}_{-s}(a)]\| \\
&\overset{(6\text{-}3)}{\le} \varepsilon + \max_{a\in\mathcal{F}} \max_{0\le s\le T} \|[\kappa_s, \alpha^{(j)}_{-s}(a)]\| \\
&\overset{(6\text{-}2)}{\le} 3\varepsilon/2 + \max_{b\in\mathcal{G}} \|[\kappa_s, b]\| \\
&\overset{(6\text{-}7)}{\le} 5\varepsilon/2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\max_{\vec{r}\in[0,1]^{k-1}} \|v - (\sigma^T\otimes\alpha)^{(\not{j})}_{\vec{r}}(v)\| &= \max_{\vec{r}\in[0,1]^{k-1}} \|v - (\mathrm{id}\otimes\alpha)^{(\not{j})}_{\vec{r}}(v)\| \\
&= \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\le s\le T} \|v_s - \alpha^{(\not{j})}_{\vec{r}}(v_s)\| \\
&= \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\le s\le T} \|w_s\alpha^{(j)}_s(\kappa_s) - \alpha^{(\not{j})}_{\vec{r}}(w_s\alpha^{(j)}_s(\kappa_s))\| \\
&= \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\le s\le T} \|w_s\alpha^{(j)}_s(\kappa_s) - \alpha^{(\not{j})}_{\vec{r}}(w_s)\cdot\alpha^{(j)}_s(\alpha^{(\not{j})}_{\vec{r}}(\kappa_s))\| \\
&\le \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\le s\le T} \|w_s - \alpha^{(\not{j})}_{\vec{r}}(w_s)\| + \|\kappa_s - \alpha^{(\not{j})}_{\vec{r}}(\kappa_s)\| \\
&\overset{(6\text{-}4),(6\text{-}6)}{\le} 2\varepsilon.
\end{aligned}$$

Lastly, let us fix $t \in [0, 1]$ and $s \in [0, T]$. If $s \ge t$, then we compute

$$\begin{aligned}
(v(\sigma^T\otimes\alpha)^{(j)}_t(v^*))(s) &= w_s\alpha^{(j)}_s(\kappa_s)\cdot\alpha^{(j)}_t(\alpha^{(j)}_{s-t}(\kappa^*_{s-t})w^*_{s-t}) \\
&= w_s\cdot\alpha^{(j)}_s(\kappa_s\kappa^*_{s-t})\alpha^{(j)}_t(w^*_{s-t}) \\
&\overset{(6\text{-}5)}{=_\varepsilon} w_s\alpha^{(j)}_t(w^*_{s-t}) = w_t.
\end{aligned}$$

On the other hand, if $s \le t$, then in particular $s \le 1$ and $T-1 \le T+s-t \le T$, and we compute

$$
\begin{aligned}
(v(\sigma^T \otimes \alpha)_t^{(j)}(v^*))(s) &= w_s \alpha_s^{(j)}(\kappa_s) \cdot \alpha_t^{(j)}(\alpha_{T+s-t}^{(j)}(\kappa_{T+s-t}^*) w_{T+s-t}^*) \\
&= w_s \alpha_s^{(j)}(\kappa_s) \cdot \alpha_{T+s}^{(j)}(\kappa_{T+s-t}^*) \alpha_t^{(j)}(w_{T+s-t}^*) \\
&\overset{(6\text{-}5)}{=}_{2\varepsilon} w_s \cdot \mathbf{1} \cdot \alpha_{T+s}^{(j)}(w_{-T}^*) \alpha_t^{(j)}(w_{T+s-t}^*) \\
&= w_s \alpha_s^{(j)}(w_T) \alpha_t^{(j)}(w_{T+s-t}^*) \\
&= w_{T+s} \alpha_t^{(j)}(w_{T+s-t}^*) w_t^* \cdot w_t = w_t.
\end{aligned}
$$

Let us summarize what we have accomplished so far. Starting from the existence of the $\alpha^{(j)}$-1-cocycle $\{w_t\}_{t\in\mathbb{R}}$ and the unitary path u with the prescribed properties, we have found a unitary $v \in \mathcal{U}(\mathcal{C}(\mathbb{R}/T\mathbb{Z}) \otimes A)$ satisfying

$$\max_{a\in\mathcal{F}} \|[v, \mathbf{1}\otimes a]\| \le 5\varepsilon/2, \tag{6-8}$$

$$\max_{\vec{r}\in[0,1]^{k-1}} \|v - (\sigma^T \otimes \alpha)_{\vec{r}}^{(\not{j})}(v)\| \le 2\varepsilon, \tag{6-9}$$

$$\max_{0\le t\le 1} \|w_t - v(\sigma^T \otimes \alpha)_t^{(j)}(v^*)\| \le 2\varepsilon. \tag{6-10}$$

By using the commutative diagram (6-1), we may send v into the sequence algebra of A, represent the resulting unitary by a sequence of unitaries in A, and then select a member of this sequence so that it will satisfy the properties in the claim with respect to the parameter 3ε. □

Now record the following useful technical result about semi-strongly self-absorbing actions, which arises as a special case of [Szabó 2018c, Lemma 3.12]:

Lemma 6.10. *Let G be a second-countable, locally compact group. Let $\mathcal{D}$ be a separable, unital* C*-*algebra and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing action. For every $\varepsilon > 0$, $\mathcal{F} \Subset \mathcal{D}$ and compact set $K \subset G$, there exist $\delta > 0$ and $\mathcal{G} \Subset \mathcal{D}$ with the following property*:

Suppose that $u : [0, 1] \to \mathcal{U}(\mathcal{D})$ is a unitary path satisfying

$$u(0) = \mathbf{1}, \quad \max_{0\le t\le 1} \max_{g\in K} \|u(t) - \gamma_g(u(t))\| \le \delta,$$

and

$$\max_{a\in\mathcal{G}} \|[u(1), a]\| \le \delta.$$

Then there exists a unitary path $w : [0, 1] \to \mathcal{U}(\mathcal{D})$ satisfying

$$w(0) = \mathbf{1}, \quad w(1) = u(1),$$

$$\max_{g\in K} \|w(t) - \gamma_g(w(t))\| \le \varepsilon,$$

$$\max_{0\le t\le 1} \max_{a\in\mathcal{F}} \|[w(t), a]\| \le \varepsilon.$$

Moreover, we may choose w in such a way that

$$\|w(t_1) - w(t_2)\| \le \|u(t_1) - u(t_2)\| \quad \textit{for all } 0 \le t_1, t_2 \le 1.$$

We are now ready to prove the main result of this section:

Proof of Theorem 6.7. We will prove this via induction in k. For this purpose, we will include the case $k = 0$, where the claim is true for trivial reasons.

Now let $k \geq 1$ and assume that the claim is true for actions of $\mathbb{R}^{k-1}$. We will then show that the claim is also true for actions of $\mathbb{R}^k$.

Step 1: Let $\alpha : \mathbb{R}^k \curvearrowright \mathcal{D}$ be an action with the Rokhlin property. In a similar fashion as in [Kishimoto 2002, Proposition 3.5], we shall show that α has approximately $\mathbb{R}^k$-inner flip.

Set $B = \mathcal{D} \otimes \mathcal{D}$ and $\beta = \alpha \otimes \alpha$. Denote by Σ the flip automorphism on B, which is equivariant with regard to β. Note that β is still a $\mathbb{R}^k$-action on a strongly self-absorbing Kirchberg algebra with the Rokhlin property. The $\mathbb{R}^{k-1}$-action $\alpha^{(\not k)}$ is semi-strongly self-absorbing by the induction hypothesis. Applying [Szabó 2018c, Proposition 3.6], we find a sequence of unitaries $y_n, z_n \in \mathcal{U}(B)$ satisfying

$$\max_{\vec{r}\in[0,1]^{k-1}} \|y_n - \beta^{(\not k)}_{\vec{r}}(y_n)\| + \|z_n - \beta^{(\not k)}_{\vec{r}}(z_n)\| \xrightarrow{n\to\infty} 0 \tag{6-11}$$

and

$$\Sigma(b) = \lim_{n\to\infty} \mathrm{Ad}(y_n z_n y_n^* z_n^*)(b), \quad b \in B. \tag{6-12}$$

Let us set $Y = [(y_n)_n]$ and $Z = [(z_n)_n]$ with $Y, Z \in B_{\infty,\beta^{(\not k)}}^{\beta_\infty^{(k)}}$. Moreover set $X = YZY^*Z^*$. Note that since $\mathcal{D}$ is a Kirchberg algebra, Corollary 5.1 implies that β is equivariantly $\mathcal{O}_\infty$-absorbing. By [Szabó 2018c, Proposition 2.19(iii)], the unitary X is thus homotopic to the unit inside $B_{\infty,\beta^{(\not k)}}^{\beta_\infty^{(k)}}$. Write $X = \exp(iH_1)\cdots\exp(iH_r)$ for certain self-adjoint elements $H_1, \dots, H_r \in B_{\infty,\beta^{(\not k)}}^{\beta_\infty^{(k)}}$. Set $L' = \|H_1\| + \cdots + \|H_r\|$. For $l = 1, \dots, r$, represent H_l via a sequence of self-adjoint elements $h_{l,n} \in B$ with $\|h_{l,n}\| \leq \|H_l\|$. We define a sequence of continuous paths $x_n : [0, 1] \to \mathcal{U}(B)$ via

$$x_n(t) = \exp(ith_{1,n}) \cdots \exp(ith_{r,n}).$$

Then each of these paths is L'-Lipschitz. By slight abuse of notation we write $X : [0, 1] \to \mathcal{U}(B_{\infty,\beta^{(\not k)}}^{\beta_\infty^{(k)}})$ for $X(t) = [(x_n(t))_n]$, which is then continuous and satisfies $X(0) = \mathbf{1}$ and $X(1) = X$. Also define $x_n = x_n(1)$ for all n.

Since we have $\Sigma(b) = XbX^*$ for all $b \in B$ and β and $\Sigma \circ \beta_t^{(k)} = \beta_t^{(k)} \circ \Sigma$, one has $\Sigma(b) = \beta_{\infty,t}^{(k)}(X) b \beta_{\infty,t}^{(k)}(X^*)$ for all $t \in \mathbb{R}$. It follows that for all $t \in \mathbb{R}$, one has that the element $X\beta_{\infty,t}^{(k)}(X^*)$ commutes with all elements in $B \subset B_\infty$.

Let us for the moment fix some number $T > 0$. Define $u_n^T : [0, 1] \to \mathcal{U}(B)$ via $u_n^T(t) = x_n(t)\beta_{-T}^{(k)}(x_n(t)^*)$. Then u_n^T is a unitary path starting at the unit and with Lipschitz constant $L \leq 2L'$. We have

$$\max_{0\leq t\leq 1} \max_{\vec{r}\in[0,1]^{k-1}} \|u_n^T(t) - \beta^{(\not k)}_{\vec{r}}(u_n^T(t))\| \xrightarrow{n\to\infty} 0$$

as $\beta_{-T}^{(k)} \circ \beta_{\vec{r}}^{(\not k)} = \beta_{\vec{r}}^{(\not k)} \circ \beta_{-T}^{(k)}$ and the elements $x_n(t)$ are approximately $\beta^{(\not k)}$-invariant by construction, and

$$\|[u_n^T(1), b]\| = \|[x_n \beta_{-T}^{(k)}(x_n^*), b]\| \xrightarrow{n\to\infty} 0 \quad \text{for all } b \in B.$$

Due to Lemma 6.10, we may replace the unitary paths u_n^T by ones which become approximately central along the entire path and retain all the other properties. In other words, by changing the path u_n on $(0, 1)$, we may in fact assume

$$\max_{0\le t\le 1} \|[u_n^T(t), b]\| \xrightarrow{n\to\infty} 0 \quad \text{for all } b\in B.$$

Let us consider the sequence of $\beta^{(k)}$-1-cocycles $\{w_t^{(n)}\}_{t\in\mathbb{R}}$ given by $w_t^{(n)} = x_n\beta_t^{(k)}(x_n^*)$. Then by what we have observed before, we have

$$\max_{0\le t\le T} \|[w_t^{(n)}, b]\| \xrightarrow{n\to\infty} 0, \quad b\in B,$$

as well as

$$\max_{0\le t\le T}\ \max_{\vec{r}\in[0,1]^{k-1}} \|w_t^{(n)} - \beta_{\vec{r}}^{(\not k)}(w_t^{(n)})\| \le 2\cdot \max_{\vec{r}\in[0,1]^{k-1}} \|x_n - \beta_{\vec{r}}^{(\not k)}(x_n)\| \xrightarrow{n\to\infty} 0.$$

This puts us into the position to apply Lemma 6.9. Given some small tolerance $\varepsilon>0$ and $\mathcal{F}\Subset\mathcal{D}$, we can choose $T>0$ and $\mathcal{G}\Subset\mathcal{D}$ with respect to the constant $L=2L'$ and with (B,β) in place of (A,α). Without loss of generality, we choose $\mathcal{F}$ in such a way that

$$\Sigma(\mathcal{F}) = \mathcal{F}. \tag{6-13}$$

Then the cocycles $\{w_t^{(n)}\}_{t\in\mathbb{R}}$ and the unitary paths u_n^T (in place of $\{w_t\}_{t\in\mathbb{R}}$ and u in Lemma 6.9) will eventually satisfy the assumptions in Lemma 6.9 for large enough n. By the conclusion of the statement, one finds a unitary $v_n\in\mathcal{U}(B)$ such that

$$\max_{0\le t\le 1} \|w_t^{(n)} - v_n\beta_t^{(k)}(v_n^*)\| = \max_{0\le t\le 1} \|x_n\beta_t^{(k)}(x_n)^* - v_n\beta_t^{(k)}(v_n^*)\| \le 3\varepsilon, \tag{6-14}$$

$$\max_{b\in\mathcal{F}} \|[v_n, b]\| \le 3\varepsilon, \tag{6-15}$$

$$\max_{\vec{r}\in[0,1]^{k-1}} \|v_n - \alpha_{\vec{r}}^{(\not k)}(v_n)\| \le 3\varepsilon. \tag{6-16}$$

We set $\mathbb{U}_n = v_n^* x_n$, which is yet another sequence of unitaries in B. Note that (6-14) translates to

$$\max_{0\le t\le 1} \|\mathbb{U}_n - \beta_t^{(k)}(\mathbb{U}_n)\| \le 3\varepsilon.$$

Together with (6-16) and $X\in B_{\infty,\beta^{(\not k)}}^{\beta_\infty^{(\not k)}}$ this yields

$$\max_{\vec{r}\in[0,1]^k} \|\mathbb{U}_n - \beta_{\vec{r}}(\mathbb{U}_n)\| \le 7\varepsilon$$

for large enough n. Finally, if we combine (6-12), (6-13) and (6-15), we obtain

$$\max_{b\in\mathcal{F}} \|\Sigma(b) - \mathbb{U}_n b\mathbb{U}_n^*\| \le 4\varepsilon$$

for sufficiently large n. Since $\varepsilon>0$ was an arbitrary parameter and $\mathcal{F}\Subset B$ was arbitrary as well, we see that the flip automorphism Σ on B is indeed approximately $\mathbb{R}^k$-inner.

Step 2: Let $\alpha:\mathbb{R}^k\curvearrowright\mathcal{D}$ be an action with the Rokhlin property. Due to the first step, α has approximately $\mathbb{R}^k$-inner flip. By [Szabó 2018b, Proposition 3.3], it follows that the infinite tensor power action

$\alpha^{\otimes\infty} : \mathbb{R}^k \curvearrowright \mathcal{D}^{\otimes\infty}$ is strongly self-absorbing. In view of Remark 6.6, we may apply Corollary 5.1 to α and $\alpha^{\otimes\infty}$ in place of γ, and see that

$$\alpha \simeq_{\mathrm{scc}} \alpha \otimes \alpha^{\otimes\infty} \cong \alpha^{\otimes\infty},$$

which implies that α is semi-strongly self-absorbing.

Step 3: For $i = 0, 1$, let $\alpha^{(i)} : \mathbb{R}^k \curvearrowright \mathcal{D}$ be two actions with the Rokhlin property. By the previous step, they are semi-strongly self-absorbing. If we apply Corollary 5.1 to $\alpha^{(0)}$ in place of α and $\alpha^{(1)}$ in place of γ, then it follows that $\alpha^{(0)} \simeq_{\mathrm{vscc}} \alpha^{(0)} \otimes \alpha^{(1)}$. If we exchange the roles of $\alpha^{(0)}$ and $\alpha^{(1)}$ and repeat this argument, we conclude $\alpha^{(0)} \simeq_{\mathrm{vscc}} \alpha^{(1)}$.

This finishes the induction step and the proof. □

We observe the following consequence as a combination of all of our main results for $\mathbb{R}^k$-actions; this is new even for ordinary flows.

Corollary 6.11. *Let A be a separable* C*-*algebra with $A \cong A \otimes \mathcal{O}_\infty$. Suppose that $\alpha : \mathbb{R}^k \curvearrowright A$ is a multiflow. The following are equivalent*:

(i) *α has the Rokhlin property.*

(ii) *α has finite Rokhlin dimension with commuting towers.*

(iii) *$\alpha \simeq_{\mathrm{vscc}} \alpha \otimes \gamma$ for any multiflow $\gamma : \mathbb{R}^k \curvearrowright \mathcal{O}_\infty$ with the Rokhlin property.*

(iv) *$\alpha \simeq_{\mathrm{vscc}} \alpha \otimes \gamma$ for every multiflow $\gamma : \mathbb{R}^k \curvearrowright \mathcal{O}_\infty$ with the Rokhlin property.*

Proof. This follows directly from Theorem 6.7 and Corollary 5.1. □

Once we combine Corollary 6.11 and Theorem 6.7, we obtain Theorem C as a direct consequence.

The following remains open:

Question 6.12. Let $\alpha : \mathbb{R}^k \curvearrowright A$ be a multiflow on a Kirchberg algebra. Suppose that for every $\vec{r} \in \mathbb{R}^k$ the flow on A given by $t \mapsto \alpha_{t\vec{r}}$ has the Rokhlin property. Does it follow that α has the Rokhlin property?

Acknowledgements

The author would like to express his gratitude to the referee for very useful and detailed comments.

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Received 16 Apr 2018. Revised 12 Sep 2018. Accepted 18 Oct 2018.

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ANALYSIS & PDE

Volume 12 No. 5 2019

2157-5045(2019)12:5;1-L