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BETSY STOVALL

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SCALE-INVARIANT FOURIER RESTRICTION TO A HYPERBOLIC SURFACE

BETSY STOVALL

This result sharpens the bilinear-to-linear deduction of Lee and Vargas for extension estimates on the hyperbolic paraboloid in $\mathbb{R}^3$ to the sharp line, leading to the first scale-invariant restriction estimates, beyond the Stein–Tomas range, for a hypersurface on which the principal curvatures have different signs.

1. Introduction

We consider the Fourier restriction/extension problem for the hyperbolic paraboloid

$$S := \{(\tau, \xi) \in \mathbb{R}^{1+2} : \tau = \xi_1 \xi_2\}.$$

We denote by $\mathcal{E}$ the extension operator,

$$\mathcal{E} f(t, x) := \int_{\mathbb{R}^2} e^{i(t,x)(\xi_1\xi_2,\xi)} f(\xi)\, d\xi. \tag{1-1}$$

For consistency of exponents, we will consider the problem of establishing $L^r \to L^{2s}$ extension estimates for $\mathcal{E}$, and we are primarily interested in the case when $r = s'$.

Lee [2006] and Vargas [2005] independently established an essentially optimal L^2-based bilinear adjoint restriction estimate for S. This result states that if f and g are supported in 1×1 axis-parallel rectangles that are separated from one another by a distance 1 in the horizontal direction and 1 in the vertical direction, then

$$\|\mathcal{E} f\, \mathcal{E} g\|_s \lesssim \|f\|_2\, \|g\|_2, \quad s > \tfrac{5}{3}. \tag{1-2}$$

This two-parameter separation of the tiles is both necessary and troublesome. On the one hand, necessity can be seen by considering the case when each of $f_\pm$ is supported on a $\frac{1}{2}$-neighborhood of $(\pm 1, 0)$. On the other hand, the separation leads to difficulty in deducing linear restriction estimates from the bilinear ones. Indeed, the natural analogue of the Whitney decomposition approach of [Tao, Vargas, and Vega 1998] produces a sum in two scales, length and width, rather than a single distance scale, leading to a loss of the scaling line in the distinct approaches of [Lee 2006] and [Vargas 2005].

The purpose of this note is to overcome this obstacle and recover the sharp line.

Theorem 1.1. *With $\mathcal{E}$ as in* (1-1), *assume that the estimate*

$$\|\mathcal{E} f\, \mathcal{E} g\|_s \lesssim \|f\|_r\, \|g\|_r \tag{1-3}$$

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holds for some $\frac{3}{2} < s < 2$ *and* $\frac{r}{2} < s < r'$, *whenever* f *and* g *are supported on* 1×1, *axis-parallel rectangles that are separated from one another by a distance* 1 *in both the horizontal and vertical directions. Then* $\mathcal{E}$ *is of restricted strong type* $(s', 2s)$, *and consequently of strong type* $(\tilde{s}', 2\tilde{s})$ *for all* $\tilde{s} > s$.

To put the hypothesis on s in context, we recall that for $s \leq \frac{3}{2}$, linear extension estimates are known to be impossible; that for $s > \frac{3}{2}$, $2s > s'$; and that for $s \geq 2$, linear extension estimates are already known, [Tomas 1975].

As is well known, a (local, linear) $L^{r_0} \to L^{2s_0}$ extension estimate for some $r_0 > s_0'$ allows us, by interpolation with the L^2-based bilinear extension estimate (1-2), to establish the L^r-based bilinear extension estimate (1-3) for some $s > s_0$ and $\frac{r}{2} < s < r'$. Replacing s_0 with s is a loss (whose magnitude depends on the distance from (r_0^{-1}, s_0^{-1}) to the scaling line), but $r < s'$ is a gain in the sense that the corresponding linear extension estimate $\mathcal{E} : L^r \to L^{2s}$ is false.

Lee [2006] and Vargas [2005] independently used the bilinear extension estimate (1-2) to prove that

$$\|\mathcal{E} f\|_{2s} \lesssim \|f\|_{L^r} \tag{1-4}$$

for all $s > \frac{5}{3}$, $r > s'$, and f supported in the unit ball. Cho and Lee [2017] used the polynomial partitioning argument from [Guth 2016] to prove (1-4) for f supported in the unit ball and $2s = r > 3.25$; this was subsequently improved by Kim [2017] to the range $2s > 3.25$ and $r > s'$. Using these results and the discussion in the preceding paragraph, Theorem 1.1 immediately yields the following slight improvement on Kim's result.

Corollary 1.2. *For* $2s > 3.25$, *the extension operator* $\mathcal{E}$ *is bounded from* $L^{s'}$ *to* L^{2s}.

To the author's knowledge, this is the first scalable restriction estimate for a negatively curved hypersurface, beyond the Stein–Tomas range ($s = 2$).

Terminology. A constant will be said to be admissible if it depends only on s, r. The inequality $A \lesssim B$ means that $A \leq CB$ for some implicit, admissible constant C, and implicit constants will be allowed to change from line to line. A dyadic interval is an interval of the form $[m2^{-n}, (m+1)2^{-n}]$ for some $m, n \in \mathbb{Z}$, and $\mathcal{I}_n$ denotes the set of all dyadic intervals of length 2^{-n}. A tile is a product of two dyadic intervals, and $\mathcal{D}_{J,K}$ denotes the set of all $2^{-J} \times 2^{-K}$ tiles. We denote by π_1, π_2 the projections $\pi_j : \mathbb{R}^2 \to \mathbb{R}$, $\pi_j(x) = x_j$. We use $\mathcal{H}^1$ for the one-dimensional Hausdorff measure. Finally, we use log to denote the base-2 logarithm.

Outline of proof. To prove our restricted strong-type estimate, it suffices to bound the extension of a characteristic function. Our starting point is the bilinear-to-linear deduction of [Vargas 2005], which shows that, under the hypotheses of Theorem 1.1, the extension of the characteristic function of a set Ω with roughly constant (vertical) fiber length obeys the scalable restriction estimate $\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim |\Omega|^{\frac{1}{s'}}$. In [Vargas 2005], off-scaling estimates are obtained by subdividing a set Ω in the unit cube into subsets having constant fiber length. Off-scaling contributions from those subsets with very short fibers are small (because the sets themselves are small), and adding these amounts to summing a convergent geometric series.

We wish to remain on the sharp line, so we must be more careful. Our first step, taken in Section 2, is to understand when Vargas's constant fiber length estimate can be improved. To this end, we prove a dichotomy result: If Ω has constant fiber length, then either Ω is highly structured (more precisely, Ω is nearly a tile), or we have a better bound on the extension of χ_Ω. Roughly speaking, this reduces matters to controlling the extension of a union of tiles τ_k each having height 2^{-k}, which is the task of Section 3. We can estimate

$$\|\mathcal{E}\chi_{\bigcup \tau_k}\|_{2s} \lesssim \left(\sum \|\mathcal{E}\chi_{\tau_k}\|_{2s}^{2s}\right)^{\frac{1}{2s}} + \text{off-diagonal terms},$$

where the off-diagonal terms involve products $\mathcal{E}\chi_{\tau_k}\mathcal{E}\chi_{\tau_{k'}}$, with $|k-k'|$ large. Boundedness of the main term follows from Vargas's estimate and convexity ($2s > s'$). It remains to bound the off-diagonal terms, for which it suffices to prove a bilinear estimate with decay:

$$\|\mathcal{E}\chi_{\tau_k}\mathcal{E}\chi_{\tau_{k'}}\|_s \lesssim 2^{-c_0|k-k'|}\max\{|\tau_k|, |\tau_{k'}|\}^{\frac{1}{s'}},$$

and we prove this by combining the bilinear extension estimate for separated tiles with a further decomposition.

Of course, we have lied. In Section 2, our dichotomy is not that a constant fiber length set Ω is either a tile or has zero extension, and so we still have remainder terms that must be summed. To address this, we argue more quantitatively than has been suggested above: Any constant fiber length set can be approximated by a union of tiles, where the number of tiles and tightness of the approximation depends on the sharpness of our estimate $\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim |\Omega|^{\frac{1}{s'}}$; then we must bound extensions of sets $\bigcup_k \bigcup_{\tau\in\mathcal{T}_k} \tau$, where $\mathcal{T}_k \subseteq \mathcal{D}_{j(k),k}$ may be large (but fortunately, not too large).

2. An inverse problem related to Vargas's linear estimate

To prove Theorem 1.1, it suffices to prove that $\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim |\Omega|^{\frac{1}{s'}}$ for all measurable sets Ω. By scaling, it suffices to consider Ω contained in the unit cube $[-1,1]^2$. Vargas [2005] proved the following.

Theorem 2.1 [Vargas 2005]. *For each $K \geq 0$, let*

$$\Omega(K) := \{\xi \in \Omega : \mathcal{H}^1(\pi_1^{-1}(\xi_1)\cap\Omega) \sim 2^{-K}\}. \tag{2-1}$$

Then under the hypotheses of Theorem 1.1, for any measurable set $\Omega' \subseteq \Omega(K)$,

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim |\Omega(K)|^{\frac{1}{s'}}. \tag{2-2}$$

This version differs slightly from the one stated in [Vargas 2005], but it follows from the same proof. In proving the next proposition, we will review Vargas's argument, so the reader may verify the above-stated version below.

Our first step is to solve an inverse problem: Characterize those sets $\Omega = \Omega(K)$ for which the inequality in (2-2) can be reversed.

Proposition 2.2. *Assume that the hypotheses of Theorem 1.1 hold. Let $\Omega \subseteq [-1,1]^2$ be a measurable set, and assume that $\Omega = \Omega(K)$ for some integer $K \geq 0$. Choose a nonnegative integer J such that*

$|\pi_1(\Omega)| \sim 2^{-J}$, *and let* $\varepsilon \lesssim 1$ *denote the smallest dyadic number such that*

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \leq \varepsilon|\Omega|^{\frac{1}{s'}}$$

for every measurable $\Omega' \subseteq \Omega$. *Then* $\Omega = \bigcup_{0<\delta\leq\varepsilon} \Omega_\delta$, *with the union taken over dyadic* δ. *For each* δ, $\Omega_\delta \subseteq \bigcup_{\tau\in\mathcal{T}_\delta}\tau$, *where* $\mathcal{T}_\delta \subseteq \mathcal{D}_{J,K}$ *has cardinality at most* $O(\delta^{-C})$, *with* C *an admissible constant. For each subset* $\Omega' \subseteq \Omega_\delta$, $\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \delta|\Omega|^{\frac{1}{s'}}$.

Proof of Proposition 2.2. It suffices to produce a union that contains almost every point of Ω, as a set of measure zero makes no contribution to the extension. Our decomposition will be done in three stages. Our first decomposition will be of Ω into sets Ω^1_η, with $\pi_1(\Omega^1_\eta)$ nearly an interval, $I \in \mathcal{I}_J$. Our second decomposition will be of Ω^1_η into sets $\Omega^2_{\eta,\rho}$, $\rho \leq \eta$, each of which is nearly a product of I with a set of measure 2^{-K}. Our third decomposition will be of $\Omega^2_{\eta,\rho}$ into sets $\Omega^3_{\eta,\rho,\delta}$, $\delta \leq \rho$, each of which is nearly a product of I with an interval in $\mathcal{I}_K$. The product of two dyadic intervals is a tile, so we take $\Omega_\delta := \bigcup_{\rho\geq\delta}\bigcup_{\eta\geq\rho}\Omega^3_{\eta,\rho,\delta}$; the $(\log\delta^{-1})^2$ factor that arises from taking this union is harmless.

Let $S := \pi_1(\Omega)$. We know that $|S| \sim 2^{-J}$ and that $S \subseteq [-1, 1]$. Let $\xi_1 \in S$, and for each $0 < \eta < \varepsilon$, let $I_\eta(\xi_1)$ be the maximal dyadic interval $I \ni \xi_1$ satisfying $|I \cap S| \geq \eta^C|I|$, if such an interval exists. We record that $|I_\eta(\xi_1)| \leq \eta^{-C}2^{-J}$, and if ξ_1 is a Lebesgue point of S, then $|I_\eta(\xi_1)| > 0$. Let

$$T_\eta := \{\xi_1 \in S : |I_\eta(\xi_1)| \geq \eta^C 2^{-J}\},$$

and let $S_\varepsilon := T_\varepsilon$, $S_\eta := T_\eta \setminus T_{2\eta}$ for dyadic $0 < \eta < \varepsilon$. Then a.e. (indeed, every Lebesgue) point of S is contained in a unique S_η. We set $\Omega^1_\eta := \Omega \cap \pi_1^{-1}(S_\eta)$.

Lemma 2.3. *For each* $0 < \eta \leq \varepsilon$, S_η *is contained in a union of* $O(\eta^{-3C})$ *dyadic intervals* $I \in \mathcal{I}_J$, *and for each* $\eta < \varepsilon$ *and each subset* $\Omega' \subseteq \Omega^1_\eta$,

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \eta^2|\Omega|^{\frac{1}{s'}}. \tag{2-3}$$

Proof of Lemma 2.3. By construction, S_η is covered by dyadic intervals I of length $|I| \geq \eta^C|S|$, in which S has density $|I \cap S| \geq \eta^C|I|$. The density of each such interval in S is $|I \cap S| \geq \eta^{2C}|S|$, and so the collection of maximal (hence pairwise disjoint) dyadic intervals with these properties has cardinality at most η^{-2C}. Moreover, from the density estimate, we see that $|I| \leq \eta^{-C}2^{-J}$, so these intervals are covered by a total of η^{-3C} intervals in $\mathcal{I}_J$.

To establish (2-3), we will optimize Vargas's proof of Theorem 2.1. Performing a Whitney decomposition in each variable ξ_1, ξ_2 separately and applying the almost orthogonality lemma from [Tao, Vargas, and Vega 1998] (for which it is important that $s \leq 2$),

$$\|\mathcal{E}\chi_{\Omega'}\|^2_{2s} \lesssim \sum_{k,j}\biggl(\sum_{\tau\sim\tau'\in\mathcal{D}_{j,k}} \|\mathcal{E}\chi_{\Omega'\cap\tau}\mathcal{E}\chi_{\Omega'\cap\tau'}\|^s_s\biggr)^{\frac{1}{s}},$$

where we say that $\tau \sim \tau'$ if τ and τ' are 2^{-j} separated in the horizontal direction and 2^{-k} separated in the vertical direction.

By rescaling our hypothesis, (1-3), for f, g supported on tiles in $\mathcal{D}_{j,k}$ that are separated by a distance 2^{-k} in the vertical direction and 2^{-j} in the horizontal direction,

$$\|\mathcal{E} f \mathcal{E} g\|_s \lesssim 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \|f\|_r \|g\|_r. \tag{2-4}$$

Thus

$$\begin{aligned}\|\mathcal{E}\chi_{\Omega'}\|_{2s}^2 &\lesssim \sum_{k,j} 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \Bigl(\sum_{\tau\in\mathcal{D}_{j,k}} |\Omega'\cap\tau|^{\frac{2s}{r}}\Bigr)^{\frac{1}{s}} \\ &\lesssim \sum_{k,j} 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \max_{\tau\in\mathcal{D}_{j,k}} |\Omega'\cap\tau|^{\frac{2}{r}-\frac{1}{s}} |\Omega'|^{\frac{1}{s}}. \end{aligned} \tag{2-5}$$

Our hypotheses on r, s imply that all exponents in the above sum are positive. To bound this double sum, Vargas used the inequality

$$|\Omega'\cap\tau| \lesssim \min\{2^{-j}, 2^{-J}\} \min\{2^{-k}, 2^{-K}\}. \tag{2-6}$$

The definition of Ω_η^1 will allow us to improve on this bound.

For $I_j \in \mathcal{I}_j$, we trivially have $|I_j \cap S_\eta| \le \min\{|I_j|, |S_\eta|\} \le \min\{2^{-j}, 2^{-J}\}$, but when $|j-J| < \frac{C}{4}\log\eta^{-1}$, we can do rather better. Suppose that $|j-J| \le \frac{C}{4}\log\eta^{-1}$. Since

$$|I_j| = 2^{-j} \ge \eta^{\frac{C}{4}} 2^{-J} \ge (2\eta)^C 2^{-J}$$

(provided η is sufficiently small), $I_j \cap S_\eta \ne \varnothing$ implies that $I_j \cap S_\eta \not\subseteq T_{2\eta}$, whence

$$|I_j \cap S_\eta| \le |I_j \cap S| \le (2\eta)^C |I_j| = (2\eta)^C 2^{-j} \lesssim \eta^{\frac{3C}{4}} \min\{2^{-j}, 2^{-J}\},$$

where for the last inequality, we used $2^{-j} \le \eta^{-\frac{C}{4}} 2^{-J}$.

Inserting this gain and $|\Omega' \cap (I_j \times I_k)| \le |S_\eta \cap I_j| \min\{2^{-k}, 2^{-K}\}$ into (2-5), and summing the resulting geometric series gives

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \eta^{C'} 2^{-(J+K)(1-\frac{3}{2s})} |\Omega'|^{\frac{1}{2s}} \lesssim \eta^{C'} |\Omega|^{\frac{1}{s'}}$$

for $C' > 0$ some admissible constant dictated by C, r, s; we can reverse engineer C so that $C' = 2$. $\square$

We now turn to our second decomposition. Although $\pi_1(\Omega_\eta^1)$ may be (roughly) thought of as a union of a small number of intervals, an individual horizontal slice $\pi_2^{-1}(\xi_2) \cap \Omega_\eta^1$ might be much smaller. Our next step is to decompose into sets where the size of a nonempty slice is roughly comparable to the size of the projection of the whole. (Sets with this property are nearly products.)

Fix $0 < \eta \le \varepsilon$. For dyadic $0 < \rho \le \eta$, we define

$$V_\rho = \{\xi_2 \in \pi_2(\Omega_\eta^1) : \mathcal{H}^1(\pi_2^{-1}(\xi_2) \cap \Omega_\eta^1) \ge \rho^C 2^{-J}\},$$

and set $U_\eta := V_\eta,\ U_\rho := V_\rho \setminus V_{2\rho}$ for $\rho < \eta$. We define $\Omega_{\eta,\rho}^2 := \pi_2^{-1}(U_\rho) \cap \Omega_\eta^1$.

Lemma 2.4. *For each $0 < \rho < \eta \le \varepsilon$, and each subset $\Omega' \subseteq \Omega_{\eta,\rho}^2$, we have $\|\mathcal{E}\chi_{\Omega'}\|_{2s} \lesssim \rho^2 |\Omega|^{\frac{1}{s'}}$.*

Proof of Lemma 2.4. Let $\tau \in \mathcal{D}_{j,k}$ and $\Omega' \subseteq \Omega^2_{\eta,\rho}$. Then $\tau \cap \Omega'$ has vertical and horizontal fiber lengths at most

$$\int \chi_{\tau\cap\Omega'}(\xi_1,\xi_2)\,d\xi_2 \lesssim \min\{2^{-K},2^{-k}\}, \quad \int \chi_{\tau\cap\Omega'}(\xi_1,\xi_2)\,d\xi_1 \lesssim \min\{\rho^C 2^{-J},2^{-j}\},$$

respectively, and projections of size at most

$$|\pi_1(\tau\cap\Omega')| \lesssim \min\{2^{-J},2^{-j}\}, \quad |\pi_2(\tau\cap\Omega')| \lesssim 2^{-k}.$$

By Fubini, we can bound $|\tau\cap\Omega'|$ by the measure of the projection times the maximum fiber length, so

$$|\tau\cap\Omega'| \lesssim \min\{2^{-(J+K)},\ 2^{-(j+K)},\ 2^{-(j+k)},\ \rho^C 2^{-(J+k)}\}. \tag{2-7}$$

To utilize (2-7), we let $C' = \frac{C}{2}$ and subdivide $\mathbb{Z}^2 = R_1 \cup R_2 \cup R_3 \cup R_4$, where

$$\begin{aligned}
R_1 &:= \{(j,k) : J - C'\log\rho^{-1} \ge j,\ K \ge k\} \cup \{(j,k) : J \ge j,\ K - C'\log\rho^{-1} \ge k\},\\
R_2 &:= \{(j,k) : j \ge J + C'\log\rho^{-1},\ K \ge k\} \cup \{(j,k) : j \ge J,\ K - C'\log\rho^{-1} \ge k\},\\
R_3 &:= \{(j,k) : j \ge J + C'\log\rho^{-1},\ k \ge K\} \cup \{(j,k) : j \ge J,\ k \ge K + C'\log\rho^{-1}\},\\
R_4 &:= \{(j,k) : J + C'\log\rho^{-1} \ge j,\ k + C'\log\rho^{-1} \ge K\}.
\end{aligned}$$

Now we insert (2-7) into (2-5) to obtain

$$\begin{aligned}
\|\mathcal{E}\chi_{\Omega'}\|_{2s}^2 \lesssim{}& \sum_{R_1} 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} 2^{-\frac{2(J+K)}{r}} + \sum_{R_2} 2^{-j(2-\frac{3}{s})} 2^{k(\frac{2}{s}+\frac{2}{r}-2)} 2^{-\frac{J}{s}} 2^{-\frac{2K}{r}}\\
&+ \sum_{R_3} 2^{-(j+k)(2-\frac{3}{s})} 2^{-\frac{J+K}{s}} + \rho^{C(\frac{2}{r}-\frac{1}{s})} \sum_{R_4} 2^{-k(2-\frac{3}{s})} 2^{j(\frac{2}{s}+\frac{2}{r}-2)} 2^{-\frac{2J}{r}} 2^{-\frac{K}{s}}.
\end{aligned}$$

As $\frac{2}{s}+\frac{2}{r}-2$ and $2-\frac{3}{s}$ are both positive, it is a simple matter to sum each of these terms, obtaining

$$\|\mathcal{E}\chi_{\Omega'}\|_{2s}^2 \lesssim \bigl(\rho^{(\frac{2}{s}+\frac{2}{r}-2)C'} + \rho^{(2-\frac{3}{s})C'} + \rho^{C(\frac{2}{r}-\frac{1}{s})-C'(2-\frac{3}{s})-C'(\frac{2}{s}+\frac{2}{r}-2)}\bigr) 2^{-\frac{2(J+K)}{s'}}.$$

Since $\frac{2}{r}-\frac{1}{s} > 0$ and $|\Omega| \sim 2^{-(J+K)}$, we obtain the bound claimed in the lemma by choosing C sufficiently large. □

Now we turn to our third decomposition. A single $\Omega^2_{\eta,\rho}$ is "nearly" a product, but $\pi_2(\Omega^2_{\eta,\rho})$ might be far from an interval. However, we may simply perform the first decomposition, with the roles of the coordinate indices reversed. Indeed, our sets satisfy analogous hypotheses to those in Lemma 2.3 (i.e., the hypotheses of Proposition 2.2 with the indices reversed) when $\rho < \eta$, because $|\mathcal{H}^1(\pi_2^{-1}(\xi_2) \cap \Omega^2_{\eta,\rho})| \sim \rho^C 2^{-J}$ for all $\xi_2 \in \pi_2(\Omega^2_{\eta,\rho})$; when $\rho = \eta$, we may abuse notation slightly by decomposing $\Omega^2_{\eta,\eta}$ into $\log\eta^{-1}$ sets $\Omega^2_{\eta,\rho}$ with the same property.

We complete the proof of Proposition 2.2 by taking unions as described at the outset. The factors of η^2 and ρ^2 in Lemmas 2.3 and 2.4 (and the factor of δ^2 in the analogue for $\Omega^3_{\eta,\rho,\delta}$) mean that the resulting factor of $(\log\delta^{-1})^2$ is indeed harmless. □

3. Extensions of characteristic functions of near tiles

We recall the definition (2-1) of $\Omega(K)$. For each K, we define $J(K)$ to be an integer such that $|\Omega(K)| \sim 2^{-J(K)-K}$. Let $\mathcal{K}(\varepsilon)$ denote the collection of all $K \in \mathbb{Z}_{\geq 0}$ for which ε is the smallest dyadic number such that $\|\mathcal{E}\chi_{\Omega'}\|_{2s} \leq \varepsilon|\Omega(K)|^{\frac{1}{s'}}$ holds for all $\Omega' \subseteq \Omega(K)$. Then Proposition 2.2 gives us a decomposition $\Omega(K) = \bigcup_{0<\delta\leq\varepsilon} \Omega_\delta(K)$, where for each δ, we have $\Omega_\delta(K) \subseteq \bigcup_{\tau\in\mathcal{T}_\delta(K)} \tau$ for some $\mathcal{T}_\delta(K) \subseteq \mathcal{D}_{J(K),K}$ of cardinality $\#\mathcal{T}_\delta(K) \lesssim \delta^{-C}$.

Lemma 3.1. *For $0 < \delta \leq \varepsilon$, under the hypotheses of Theorem 1.1,*

$$\Big\| \sum_{K\in\mathcal{K}(\varepsilon)} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} \lesssim (\log \delta^{-1})^{4s} \sum_{K\in\mathcal{K}(\varepsilon)} \|\mathcal{E}\chi_{\Omega_\delta(K)}\|_{2s}^{2s} + \delta|\Omega|^{\frac{2s}{s'}}.$$

Proof of Lemma 3.1. To prove the lemma, it suffices to prove

$$\Big\| \sum_{K\in\mathcal{K}} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} \lesssim \sum_{K\in\mathcal{K}} \|\mathcal{E}\chi_{\Omega_\delta(K)}\|_{2s}^{2s} + \delta^2|\Omega|^{\frac{2s}{s'}},$$

with $\mathcal{K} \subseteq \mathcal{K}(\varepsilon)$ chosen so that $\mathcal{K}$ and $J(\mathcal{K})$ are both $A\log\delta^{-1}$-separated, with A a sufficiently large admissible constant. (A will be much larger than the constant C in Proposition 2.2.) Since $s < 2$, the triangle inequality gives

$$\Big\| \sum_{K\in\mathcal{K}} \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} = \int \Big| \sum_{\boldsymbol{K}\in\mathcal{K}^4} \prod_{i=1}^{4} \mathcal{E}\chi_{\Omega_\delta(K_i)} \Big|^{\frac{s}{2}} \lesssim \sum_{K\in\mathcal{K}} \Big\| \mathcal{E}\chi_{\Omega_\delta(K)} \Big\|_{2s}^{2s} + {\sum}' \Big\| \prod_{i=1}^{4} \mathcal{E}\chi_{\Omega_\delta(K_i)} \Big\|_{\frac{s}{2}}^{\frac{s}{2}},$$

where $\sum'$ indicates a sum taken on quadruples $\boldsymbol{K} = (K_1, K_2, K_3, K_4) \in \mathcal{K}^4$, with at least two entries distinct. The following lemma will be useful in controlling this sum.

Lemma 3.2. *If $K, K' \in \mathcal{K}$, and $J := J(K)$, $J' := J(K')$, then*

$$\|\mathcal{E}\chi_{\Omega_\delta(K)}\mathcal{E}\chi_{\Omega_\delta(K')}\|_s \lesssim 2^{-c_0|K-K'|} \max\{|\Omega(K)|, |\Omega(K')|\}^{\frac{2}{s'}} \tag{3-1}$$

for some admissible constant $c_0 > 0$.

Proof of Lemma 3.2. Inequality (3-1) is an immediate consequence of Cauchy–Schwarz and (2-2) whenever

$$|\Omega(K)|^{\frac{1}{s'}}\, |\Omega(K')|^{\frac{1}{s'}} \lesssim 2^{-\frac{|K-K'|}{s'}} \max\{|\Omega(K)|, |\Omega(K')|\}^{\frac{2}{s'}}.$$

This inequality holds whenever $K = K'$, $J = J'$, $J < J'$ and $K < K'$, or $J > J'$ and $K > K'$.

By symmetry, this leaves us to prove (3-1) when $K < K'$ and $J > J'$. By Proposition 2.2 and the separation condition on $\mathcal{K}$, it suffices to prove that

$$\|\mathcal{E}\chi_{\tau\cap\Omega_\delta(K)}\mathcal{E}\chi_{\tau'\cap\Omega_\delta(K')}\|_s \lesssim \delta^{-C} 2^{-c_0|K-K'|}\, |\Omega(K)|^{\frac{1}{s'}}\, |\Omega(K')|^{\frac{1}{s'}} \tag{3-2}$$

for tiles $\tau \in \mathcal{T}_\delta(K)$, $\tau' \in \mathcal{T}_\delta(K')$.

Note that our conditions on J, J', K, K' mean that τ is taller than τ', and τ' is wider than τ. By translating, we may assume that the y-axis forms the center line of τ and that the x-axis forms the center line of τ'. Now our tiles are contained in $[-2,2]^2$, and we decompose:

$$\tau = \bigcup_{k=0}^{K'} \tau_k, \quad \tau' = \bigcup_{j=0}^{J} \tau'_j,$$

where

$$\tau_k = \begin{cases} \tau \cap \{\xi : |\xi_2| \sim 2^{-k}\}, & k < K', \\ \tau \cap \{\xi : |\xi_2| \lesssim 2^{-K'}\}, & k = K' \end{cases} \quad \text{and} \quad \tau'_j = \begin{cases} \tau' \cap \{\xi : |\xi_1| \sim 2^{-j}\}, & j < J, \\ \tau' \cap \{\xi : |\xi_1| \lesssim 2^{-J}\}, & j = J. \end{cases}$$

By the (2-parameter) Littlewood–Paley square function estimate (the two-parameter version can be proved using Khintchine's inequality), the fact that $s < 2$, and the triangle inequality,

$$\begin{aligned} \|\mathcal{E}\chi_{\tau \cap \Omega_\delta(K)} \mathcal{E}\chi_{\tau' \cap \Omega_\delta(K')}\|_s^s &\lesssim \int \Bigg(\sum_{k=0}^{K'} \sum_{j=1}^{J} |\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}|^2 \Bigg)^{\frac{s}{2}} \\ &\lesssim \sum_{k=0}^{K'} \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s^s. \end{aligned} \tag{3-3}$$

We begin with the sum over those terms with $k = K'$. By Cauchy–Schwarz and (2-2),

$$\sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{K'} \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s^s \lesssim \sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{K'} \cap \Omega(K)}\|_{2s}^s \, \|\mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_{2s}^s \lesssim \sum_{j=0}^{J} |\tau_{K'}|^{\frac{s}{s'}} \, |\tau'_j|^{\frac{s}{s'}}.$$

Because of the way the τ'_j were defined, we have at most two nonempty τ'_j with $j \le J'$. This, combined with the bound $|\tau'_j| \le \min\{2^{-(j-J')}, 1\}|\tau'|$ gives $\sum_j |\tau'_j|^{\frac{s}{s'}} \lesssim |\tau'|^{\frac{s}{s'}}$ (despite the fact that $s < s'$). Since $|\tau_{K'}| \sim 2^{-(K'-K)}|\tau|$, $|\tau| \sim |\Omega(K)|$, and $|\tau'| \sim |\Omega(K')|$,

$$\sum_{j=0}^{J} \|\mathcal{E}\chi_{\tau_{K'} \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s^s \lesssim 2^{-(K'-K)\frac{s}{s'}} \, |\Omega(K)|^{\frac{s}{s'}} \, |\Omega(K')|^{\frac{s}{s'}}.$$

In the case $j = J$, a similar argument implies that

$$\sum_{k=0}^{K'} \|\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_J \cap \Omega(K')}\|_s^s \lesssim 2^{-(J-J')\frac{s}{s'}} \, |\Omega(K)|^{\frac{s}{s'}} \, |\Omega(K')|^{\frac{s}{s'}} \sim 2^{-(K'-K)\frac{s}{s'}} \, |\Omega(K)|^{\frac{2s}{s'}}.$$

In the cases $k < K'$ and $j < J$, we have a gain, due to our bilinear extension estimate. If $k < K'$ and $j < J$, then τ_k is a (subset of four) tile(s) in $\mathcal{D}_{J,\max\{k,K\}}$, τ_j is a (subset of four) tile(s) in $\mathcal{D}_{\max\{j,J'\},K'}$, and these tiles are separated by a distance 2^{-k} in the vertical direction 2^{-j} in the horizontal direction. These tiles are thus contained in separated tiles in $\mathcal{D}_{j,k}$, so by (2-4),

$$\|\mathcal{E}\chi_{\tau_k \cap \Omega(K)} \mathcal{E}\chi_{\tau'_j \cap \Omega(K')}\|_s \lesssim 2^{(j+k)(\frac{2}{s}+\frac{2}{r}-2)} \, |\tau_k \cap \Omega(K)|^{\frac{1}{r}} \, |\tau'_j \cap \Omega(K')|^{\frac{1}{r}}.$$

From our observation above that we have at most two values of j (resp. k) in our sum with $j \le J'$ (resp. $k \le K$), our assumption that $r < s'$ gives

$$\begin{aligned}\sum_{j=0}^{J}\sum_{k=0}^{K'} & 2^{(j+k)(2+\frac{2s}{r}-2s)}\,|\tau_k \cap \Omega(K)|^{\frac{s}{r}}\,|\tau_j' \cap \Omega(K')|^{\frac{s}{r}} \\ &\le \sum_{j=0}^{J}\sum_{k=0}^{K'} 2^{(j+k)(2+\frac{2s}{r}-2s)}\,|\tau_k|^{\frac{s}{r}}\,|\tau_j'|^{\frac{s}{r}} \\ &\lesssim 2^{(J'+K)(2+\frac{2s}{r}-2s)}|\tau|^{\frac{s}{r}}\,|\tau'|^{\frac{s}{r}} \sim \delta^{-C}2^{(J'+K)(2+\frac{2s}{r}-2s)}\,|\Omega(K)|^{\frac{s}{r}}\,|\Omega(K')|^{\frac{s}{r}} \\ &\lesssim \delta^{-C}2^{(J-J'+K'-K)(1+\frac{s}{r}-s)}\,|\Omega(K)|^{\frac{s}{s'}}\,|\Omega(K')|^{\frac{s}{s'}},\end{aligned}$$

which, by (3-3) and $\frac{1}{s}+\frac{1}{r}-1>0$, is stronger than (3-2). □

We return to the proof of Lemma 3.1.

Let $K_1, K_2, K_3, K_4 \in \mathcal{K}$, not all equal. Rearranging indices if needed, we may assume that $N_1 := K_1 + J(K_1)$ is minimal among all $N_i := K_i + J(K_i)$ and that $|K_1 - K_4| \ge \frac{1}{2}|K_i - K_j|$ for all i, j. Thus $|\Omega(K_1)|$ is maximal. By Hölder's inequality and Lemma 3.2,

$$\Bigl\|\prod_{i=1}^{4}\mathcal{E}\chi_{\Omega_\delta(K_i)}\Bigr\|_{\frac{s}{2}} \lesssim 2^{-c_0|K_1-K_4|}|\Omega(K_1)|^{\frac{4}{s'}}.$$

Therefore

$$\sum{}'\Bigl\|\prod_{i=1}^{4}\mathcal{E}\chi_{\Omega_\delta(K_i)}\Bigr\|_{\frac{s}{2}}^{\frac{s}{2}} \lesssim \sum_{K_1\in\mathcal{K}}\ \sum_{K_1\neq K_4\in\mathcal{K}} |K_4-K_1|^2\,2^{-c_0|K_4-K_1|}|\Omega(K_1)|^{\frac{2s}{s'}}.$$

Because $2s > s'$ and $\mathcal{K}$ is $A\log\delta^{-1}$-separated for some very large A, this error term is bounded by $\delta^{C}|\Omega|^{\frac{2s}{s'}}$. □

Proof of Theorem 1.1. We decompose Ω by fiber length as in (2-1), $\Omega = \bigcup \Omega(K)$, then decompose the fiber lengths according to the exactness of Vargas's estimate as at the beginning of Section 3, $\mathbb{Z}_{\ge 0} = \bigcup_{0\le\varepsilon\lesssim 1}\mathcal{K}(\varepsilon)$, and finally apply the decomposition in Proposition 2.2, $\Omega(K) = \bigcup_{0<\delta\le\varepsilon}\Omega_\delta(K)$. By the triangle inequality,

$$\|\mathcal{E}\chi_\Omega\|_{2s} \le \sum_{0<\varepsilon\lesssim 1}\ \sum_{0<\delta\le\varepsilon}\Bigl\|\sum_{K\in\mathcal{K}(\varepsilon)}\mathcal{E}\chi_{\Omega_\delta(K)}\Bigr\|_{2s}.$$

Thus by Lemma 3.1 and Proposition 2.2,

$$\begin{aligned}\|\mathcal{E}\chi_\Omega\|_{2s} &\lesssim \sum_{0<\varepsilon\lesssim 1}\ \sum_{0<\delta\le\varepsilon}\Bigl\{(\log\delta^{-1})^{4s}\sum_{K\in\mathcal{K}(\varepsilon)}\|\mathcal{E}\chi_{\Omega_\delta(K)}\|_{2s}^{2s} + \delta|\Omega|^{\frac{2s}{s'}}\Bigr\}^{\frac{1}{2s}} \\ &\lesssim \Bigl\{\sum_{0<\varepsilon\lesssim 1}\ \sum_{0<\delta\le\varepsilon}(\log\delta^{-1})^2\Bigl(\sum_{K\in\mathcal{K}(\varepsilon)}\delta^{2s}|\Omega(K)|^{\frac{2s}{s'}}\Bigr)^{\frac{1}{2s}}\Bigr\} + |\Omega|^{\frac{1}{s'}}.\end{aligned}$$

Since $2s > s'$ and the $\Omega(K)$ are disjoint, we may use the triangle inequality for $\ell^{\frac{2s}{s'}}$ to sum the volumes of the $\Omega(K)$ in the preceding, and, finally, we sum a geometric series to obtain

$$\|\mathcal{E}\chi_\Omega\|_{2s} \lesssim \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\leq\varepsilon} (\log \delta^{-1})^2 \delta |\Omega|^{\frac{1}{s'}} \lesssim |\Omega|^{\frac{1}{s'}}. \qquad \square$$

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References

[Cho and Lee 2017] C.-H. Cho and J. Lee, "Improved restriction estimate for hyperbolic surfaces in $\mathbb{R}^3$", *J. Funct. Anal.* **273**:3 (2017), 917–945. MR Zbl

[Guth 2016] L. Guth, "A restriction estimate using polynomial partitioning", *J. Amer. Math. Soc.* **29**:2 (2016), 371–413. MR Zbl

[Kim 2017] J. Kim, "Some remarks on Fourier restriction estimates", preprint, 2017. arXiv

[Lee 2006] S. Lee, "Bilinear restriction estimates for surfaces with curvatures of different signs", *Trans. Amer. Math. Soc.* **358**:8 (2006), 3511–3533. MR Zbl

[Tao, Vargas, and Vega 1998] T. Tao, A. Vargas, and L. Vega, "A bilinear approach to the restriction and Kakeya conjectures", *J. Amer. Math. Soc.* **11**:4 (1998), 967–1000. MR Zbl

[Tomas 1975] P. A. Tomas, "A restriction theorem for the Fourier transform", *Bull. Amer. Math. Soc.* **81** (1975), 477–478. MR Zbl

[Vargas 2005] A. Vargas, "Restriction theorems for a surface with negative curvature", *Math. Z.* **249**:1 (2005), 97–111. MR Zbl

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BETSY STOVALL: stovall@math.wisc.edu
Department of Mathematics, University of Wisconsin, Madison, WI, United States

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