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TRANSFORM**



# SPARSE BOUNDS FOR THE DISCRETE CUBIC HILBERT TRANSFORM

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Consider the discrete cubic Hilbert transform defined on finitely supported functions  $f$  on  $\mathbb{Z}$  by

$$H_3 f(n) = \sum_{m \neq 0} \frac{f(n - m^3)}{m}.$$

We prove that there exists  $r < 2$  and universal constant  $C$  such that for all finitely supported  $f, g$  on  $\mathbb{Z}$  there exists an  $(r, r)$ -sparse form  $\Lambda_{r,r}$  for which

$$|\langle H_3 f, g \rangle| \leq C \Lambda_{r,r}(f, g).$$

This is the first result of this type concerning discrete harmonic analytic operators. It immediately implies some weighted inequalities, which are also new in this setting.

## 1. Introduction

The purpose of this paper is to initiate a theory of sparse domination for discrete operators in harmonic analysis. We do so in the simplest nontrivial case; it will be clear that there is a much richer theory to be uncovered.

Our main result concerns the discrete cubic Hilbert transform, defined for finitely supported functions  $f$  on  $\mathbb{Z}$  by

$$H_3 f(x) = \sum_{n \neq 0} \frac{f(x - n^3)}{n}.$$

It is known [Stein and Wainger 1990; Ionescu and Wainger 2006] that this operator extends to a bounded linear operator on  $\ell^p(\mathbb{Z})$  to  $\ell^p(\mathbb{Z})$  for all  $1 < p < \infty$ . We prove a sparse bound, which in turn proves certain weighted inequalities. Both results are entirely new.

By an *interval* we mean a set  $I = \mathbb{Z} \cap [a, b]$  for  $a < b \in \mathbb{R}$ . For  $1 \leq r < \infty$ , set

$$\langle f \rangle_{I,r} := \left[ \frac{1}{|I|} \sum_{x \in I} |f(x)|^r \right]^{1/r}.$$

We say a collection of intervals  $\mathcal{S}$  is *sparse* if there are subsets  $E_S \subset S \subset \mathbb{Z}$  with (a)  $|E_S| > \frac{1}{4}|S|$ , uniformly in  $S \in \mathcal{S}$ , and (b) the sets  $\{E_S : S \in \mathcal{S}\}$  are pairwise disjoint. For sparse collections  $\mathcal{S}$ , consider sparse

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bisublinear forms

$$\Lambda_{\mathcal{S},r,s}(f,g) := \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,r} \langle g \rangle_{S,s}.$$

Frequently we will suppress the collection  $\mathcal{S}$ , and if  $r = s = 1$ , we will suppress this dependence as well.

The main result of this paper is the following theorem.

**Theorem 1.1.** *There is a choice of  $1 < r < 2$  and constant  $C > 0$  so that for all  $f, g$  that are finitely supported on  $\mathbb{Z}$  there is a sparse collection of intervals  $\mathcal{S}$  so that*

$$|\langle H_3 f, g \rangle| \leq C \Lambda_{\mathcal{S},r,r}(f, g).$$

The beauty of sparse operators is that they are both positive and highly localized operators. In particular, many of their mapping properties can be precisely analyzed. As an immediate corollary [Bernicot et al. 2016, §6] we obtain weighted inequalities, holding in an appropriate intersection of Muckenhoupt and reverse Hölder weight classes.

**Corollary 1.2.** *There exists  $1 < r < 2$  so that for all weights  $w^{-1}, w \in A_2 \cap RH_r$  we have*

$$\|H_3 : \ell^2(w) \mapsto \ell^2(w)\| \lesssim 1.$$

For instance, one can take

$$w(x) = [1 + |x|]^a \quad \text{for } -\frac{1}{2} < a < \frac{1}{2}.$$

The concept of a sparse bound originated in [Lerner 2013; Conde-Alonso and Rey 2016; Lacey 2017], so it is new, in absolute terms, as well as this area. On the other hand, the study of norm inequalities for discrete arithmetic operators has been under active investigation for over 30 years. However, *no weighted inequalities have ever been proved in this setting*.

The subject of discrete norm inequalities of this type began with the breakthrough work of Bourgain [1988a; 1988b] on arithmetic ergodic theorems. He proved, for instance, the following theorem.

**Theorem 1A.** *Let  $P$  be a polynomial on  $\mathbb{Z}$  which takes integer values. Then the maximal function  $M_P$  below maps  $\ell^p(\mathbb{Z})$  to  $\ell^p(\mathbb{Z})$  for all  $1 < p < \infty$ :*

$$M_P f(x) = \sup_N \frac{1}{N} \sum_{n=1}^N |f(x - p(n))|.$$

Subsequently, attention turned to a broader understanding of Bourgain's work, including its implications for singular integrals and Radon transforms [Ionescu et al. 2007; Stein and Wainger 1990]. The fine analysis needed to obtain results in all  $\ell^p$  spaces was developed by Ionescu and Wainger [2006]. This theme is ongoing, with recent contributions in [Mirek et al. 2015; 2017; 2018], while other variants of these questions can be found in [Krause and Lacey 2017; Pierce 2010].

Initiated by Lerner [2013] as a remarkably simple proof of the so-called  $A_2$  theorem, the study of sparse bounds for operators has recently been an active topic. The norm control provided in [Lerner 2013] was improved to a pointwise control for Calderón–Zygmund operators in [Lacey 2017; Conde-Alonso

and Rey 2016]. The paper [Culiuc et al. 2016] proved sparse bounds for the bilinear Hilbert transform, in the language of sparse forms, pointing to the applicability of sparse bounds outside the classical Calderón–Zygmund setting. That point of view is crucial for this paper.

Two papers [Lacey and Spencer 2017; Krause and Lacey 2018] have proved sparse bounds for *random* discrete operators, a much easier setting than the current one. A core technique of these papers reappears in Section 4. Sparse bounds continue to be explored in a variety of settings [Benea et al. 2017; Bernicot et al. 2016; Karagulyan 2016; Lacey and Mena Arias 2017; Hytönen et al. 2017].

We recall some aspects of known techniques in sparse bounds in Section 2. These arguments and results are formalized in a new notation, which makes the remaining quantitative proof more understandable. In particular, we define a “sparse norm” and formalize some of its properties. Our main theorem above is a sparse bound for a Fourier multiplier. In Section 3, we describe a decomposition of this Fourier multiplier, which has a familiar form within the discrete harmonic analysis literature. The multiplier is decomposed into “minor” and “major” arc components, which require dramatically different methods to control. Concerning the minor arcs, there is one novel aspect of the decomposition, a derivative condition which has a precursor in [Krause and Lacey 2017]. Using this additional feature, the minor arcs are controlled in Section 4 through a variant of an argument in [Lacey and Spencer 2017]. The major arcs are the heart of the matter, and are addressed in Section 5.

An expert in the subject of discrete harmonic analysis will recognize that there are many possible extensions of the main result of this paper. We have chosen to present the main techniques in the simplest nontrivial example. Many variants and extensions to our main theorem hold, but all the ones we are aware of are more complicated than this one.

## 2. Generalities

We collect some additional notation, beginning with the one term that is not standard, namely the sparse operators. Given an operator  $T$  acting on finitely supported functions on  $\mathbb{Z}$ , and index  $1 \leq r, s < \infty$ , we set

$$\|T : \text{Sparse}(r, s)\| \quad (2.1)$$

to be the infimum over constants  $C > 0$  so that for all finitely supported functions  $f, g$  on  $\mathbb{Z}$ ,

$$|\langle Tf, g \rangle| \leq C \sup \Lambda_{r,s}(f, g),$$

where the supremum is over all sparse forms. In particular, the “sparse norm” in (2.1) satisfies a triangle inequality:

$$\left\| \sum_j T_j : \text{Sparse}(r, s) \right\| \leq \sum_j \|T_j : \text{Sparse}(r, s)\|.$$

We collect some quantitative estimates for different operators; hence the notation. As the notation indicates, it suffices to exhibit a single sparse bound for  $\langle Tf, g \rangle$ .

It is known that the Hardy–Littlewood maximal function

$$M_{\text{HL}}f = \sup_N \frac{1}{2N+1} \sum_{j=-N}^N |f(x-j)|$$

satisfies a sparse bound. This is even a classical result.

**Theorem 2B.** *We have*

$$\|M_{\text{HL}} : \text{Sparse}(1, 1)\| \lesssim 1.$$

The following is a deep fact about sparse bounds that is at the core of our main theorem.

**Theorem 2C** [Conde-Alonso and Rey 2016; Lacey 2017]. *Let  $T_K$  be the convolution with any Calderón–Zygmund kernel. For a Hilbert space  $\mathcal{H}$ , and viewing  $T_K$  as acting on  $\mathcal{H}$ -valued functions, we have the sparse bound*

$$\|T_K : \text{Sparse}(1, 1)\| < \infty.$$

*We make the natural extension of the definition of the sparse form to vector-valued functions, namely  $\langle f \rangle_I = |I|^{-1} \sum_{x \in I} \|f\|_{\mathcal{H}}$ .*

Recall that  $K$  is a Calderón–Zygmund kernel on  $\mathbb{R}$  if  $K : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}$  satisfies

$$\sup_{x \in \mathbb{R} \setminus \{0\}} |xK(x)| + \left| x^2 \frac{d}{dx} K(x) \right| < \infty, \quad (2.2)$$

and  $T_K$  acts boundedly from  $L^2$  to  $L^2$ . The kernels that we will encounter are small perturbations of  $1/x$ . Restricting a Calderón–Zygmund kernel to the integers, we have a kernel which satisfies [Theorem 2C](#).

In a different direction, we will accumulate a range of sparse operator bounds at different points of our argument. Yet there is, in a sense, a unique maximal sparse operator, once a pair of functions  $f, g$  are specified. Thus we need not specify the exact sparse form which proves our main theorem.

**Lemma 2.3** [Lacey and Mena Arias 2017, Lemma 4.7]. *Given finitely supported functions  $f, g$  and choices of  $1 \leq r, s < \infty$ , there is a sparse form  $\Lambda_{r,s}^*$ , and constant  $C > 0$  so that for any other sparse form  $\Lambda_{r,s}$  we have*

$$\Lambda_{r,s}(f, g) \leq C \Lambda_{r,s}^*(f, g).$$

A couple of elementary estimates, which we will appeal to, are in this next proposition. The use of these inequalities in the sparse-bound setting appeared in [\[Lacey and Spencer 2017\]](#).

**Proposition 2.4.** *Let  $T_K f(x) = \sum_n K(n) f(x-n)$  be convolution with kernel  $K$ . Assuming that  $K$  is finitely supported on interval  $[-N, N]$  we have the inequalities*

$$\|T_K : \text{Sparse}(r, s)\| \lesssim N^{1/r+1/s-1} \|T_K : \ell^r \mapsto \ell^{s'}\|, \quad 1 \leq r, s < \infty. \quad (2.5)$$

The two instances of the above inequality we will use are  $(r, s) = (1, 1), (2, 2)$ . In the latter case, one should observe that the power of  $N$  above is zero.

*Proof.* Let  $\mathcal{I}$  be a partition of  $\mathbb{Z}$  into intervals of length  $2N$ . Assume that if  $I, I' \in \mathcal{I}$  with  $\text{dist}(I, I') \leq 1$ , then either  $f\mathbf{1}_I$  or  $f\mathbf{1}_{I'}$  are identically zero. Then,

$$\begin{aligned} |\langle T_K f, g \rangle| &\leq \sum_{I \in \mathcal{I}} \langle f\mathbf{1}_I, T_K^*(g\mathbf{1}_{3I}) \rangle \\ &\leq \|T_K : \ell^r \mapsto \ell^{s'}\| \sum_{I \in \mathcal{I}} \|f\mathbf{1}_{3I}\|_r \|g\mathbf{1}_{3I}\|_s \\ &\lesssim N^{1/r+1/s-1} \|T_K : \ell^r \mapsto \ell^{s'}\| \sum_{I \in \mathcal{I}} |3I| \cdot \langle f \rangle_{3I,r} \langle g \rangle_{3I,s}. \quad \square \end{aligned}$$

The definition of sparse collections has a useful variant. Let  $0 < \eta \leq \frac{1}{4}$ . We say a collection of intervals  $\mathcal{S}$  is  $\eta$ -sparse if there are subsets  $E_S \subset S \subset \mathbb{Z}$  with (a)  $|E_S| > \eta|S|$ , uniformly in  $S \in \mathcal{S}$ , and (b) the sets  $\{E_S : S \in \mathcal{S}\}$  are pairwise disjoint.

**Lemma 2.6.** *For each  $f, g$  there is a  $\frac{1}{2}$ -sparse form  $\Lambda$  so that for all  $\eta$ -sparse forms  $\Lambda^\eta$ , we have*

$$\Lambda^\eta(f, g) \lesssim \eta^{-1} \Lambda(f, g), \quad 0 < \eta < \frac{1}{4}.$$

*Proof.* Let  $\mathcal{S}^\eta$  be the sparse collection of intervals associated to  $\Lambda^\eta$ . Using shifted dyadic grids [Hytönen et al. 2013, Lemma 2.5], we can, without loss of generality, assume that  $\mathcal{S}^\eta$  consists of dyadic intervals. It follows that we have the uniform Carleson measure estimate

$$\sum_{J \in \mathcal{S}^\eta : J \subset I} |J| \lesssim \eta^{-1} |I|, \quad I \in \mathcal{S}^\eta.$$

Then, for an integer  $J \lesssim \eta^{-1}$ , we can decompose  $\mathcal{S}^\eta$  into subcollections  $\mathcal{S}_j$ , for  $1 \leq j \leq J$ , so that each collection  $\mathcal{S}_j$  is  $\frac{1}{2}$ -sparse.

Now, with  $f, g$  fixed, by Lemma 2.3, there is a single sparse operator  $\Lambda$  so that uniformly in  $1 \leq j \leq J$  we have

$$\Lambda_{\mathcal{S}_j}(f, g) \lesssim \Lambda(f, g). \quad \square$$

A variant of the sparse operator will appear, one with a “long tails” average. Define

$$\{f\}_S = \frac{1}{|S|} \sum_x \frac{|f(x)|}{(1 + \text{dist}(x, S)/|S|)^3}. \quad (2.7)$$

**Lemma 2.8.** *For all finitely supported  $f, g$ , there is a sparse operator  $\Lambda$  so that for any sparse collection  $\mathcal{S}_0$  there holds*

$$\sum_{S \in \mathcal{S}_0} |S| \{f\}_S \{g\}_S \lesssim \Lambda(f, g).$$

*Proof.* For integers  $t > 0$  let  $\mathcal{S}_t = \{2^t S : S \in \mathcal{S}\}$ . Assuming that  $\mathcal{S}_0$  is  $\frac{1}{2}$ -sparse, it follows that  $\mathcal{S}_t$  is  $2^{-t-1}$ -sparse, for  $t > 0$ . Appealing to the power decay in (2.7),

$$\sum_{S \in \mathcal{S}_0} |S| \{f\}_S \{g\}_S \lesssim \sum_{t=0}^{\infty} 2^{-2t} \Lambda_{\mathcal{S}_t}(f, g).$$

But by Lemma 2.6, there is a fixed  $\frac{1}{2}$ -sparse form  $\Lambda(f, g)$  so that

$$\Lambda_{S_t}(f, g) \lesssim 2^t \Lambda(f, g), \quad t > 0. \quad \square$$

Throughout,  $e(x) := e^{2\pi i x}$ , and  $\varepsilon > 0$  is a fixed small absolute constant. For a function  $f \in \ell^2(\mathbb{Z})$ , the (inverse) Fourier transform of  $f$  is defined as

$$\begin{aligned} \mathcal{F}f(\beta) &:= \sum_{n \in \mathbb{Z}} f(n) e(-\beta n), \\ \mathcal{F}^{-1}g(n) &= \int_{\mathbb{T}} g(\beta) e(\beta n) d\beta. \end{aligned}$$

We will define operators as Fourier multipliers. Namely, given a function  $M : \mathbb{T} \mapsto \mathbb{C}$ , we define the associated linear operator by

$$\mathcal{F}[\Phi_M f](\beta) = M(\beta) \mathcal{F}f(\beta). \quad (2.9)$$

The notation  $\mathcal{F}^{-1}M = \check{M}$  will be convenient. As above, for kernel  $K$ , the operator  $T_K$  will denote convolution with respect to  $K$ . Thus,  $\Phi_M = T_{\check{M}}$ .

### 3. The main decomposition

We prove the main result by decomposition of the Fourier multiplier

$$M(\beta) := \sum_{m \neq 0} \frac{e(-\beta m^3)}{m}. \quad (3.1)$$

In this section, we detail the decomposition, which is done in the standard way, with one new point needed.

**The kernel.** Let  $\{\psi_j\}_{j \geq 0}$  be a dyadic resolution of  $1/t$ , where  $\psi_j(x) = 2^{-j} \psi(2^{-j}x)$  is a smooth odd function satisfying  $|\psi(x)| \leq 1_{[1/4, 1]}(|x|)$ . In particular

$$\sum_{k \geq 0} \psi_k(t) = \frac{1}{t}, \quad |t| \geq 1. \quad (3.2)$$

**The major arcs.** The rationals in the torus are the union over  $s \in \mathbb{N}$  of the collections  $\mathcal{R}_s$  given by

$$\mathcal{R}_s := \{B/Q \in \mathbb{T} : (B, Q) = 1, 2^{s-1} \leq Q < 2^s\}. \quad (3.3)$$

Namely the denominator of the rationals is held approximately fixed. For all rationals  $B/Q \in \mathcal{R}_s$ , define the  $j$ -th major box at  $B/Q$  to be

$$\mathfrak{M}_j(B/Q) := \{\beta \in \mathbb{T} : |\beta - B/Q| \leq 2^{(\varepsilon-3)j}\}, \quad s \leq \varepsilon j.$$

Collect the major arcs, defining

$$\mathfrak{M}_j := \bigcup_{(B, Q)=1: Q \leq 2^{6j\varepsilon}} \mathfrak{M}_j(B/Q). \quad (3.4)$$

Note in particular that for a sufficiently small  $\varepsilon$ , in the union above no two distinct major arcs  $\mathfrak{M}_j(B/Q)$  intersect. That is, if  $B_1/Q_1 \neq B_2/Q_2$ , suppose that  $\beta \in \mathfrak{M}_j(B_1/Q_1) \cup \mathfrak{M}_j(B_2/Q_2)$ . Then

$$2^{-6j\varepsilon} \leq |B_1/Q_1 - B_2/Q_2| \leq |B_1/Q_1 - \beta| + |B_2/Q_2 - \beta| \leq 2^{(\varepsilon-3)j+1},$$

which is a contradiction for  $\varepsilon < \frac{2}{7}$ .

**Multipliers.** We use the notation below for the decomposition of the multiplier:

$$M_j(\beta) := \sum_{m \in \mathbb{Z}} e(-\beta m^3) \psi_j(m), \quad (3.5)$$

$$H_j(y) := \int_{\mathbb{R}} e(-yt^3) \psi_j(t) dt \quad (\text{continuous analog of } M_j), \quad (3.6)$$

$$S(B/Q) := \frac{1}{Q} \sum_{r=0}^{Q-1} e(-B/Q \cdot r^3) \quad (\text{Gauss sum}),$$

$$L_{j,s}(\beta) := \sum_{B/Q \in \mathcal{R}_s} S(B/Q) H_j(\beta - B/Q) \chi_s(\beta - B/Q), \quad (3.7)$$

where  $\chi$  is a smooth even bump function with  $\mathbb{1}_{[-1/10, 1/10]} \leq \chi \leq \mathbb{1}_{[-1/5, 1/5]}$  and  $\chi_s(t) = \chi(10^s t)$ ,

$$L_j(\beta) := \sum_{s \leq j\varepsilon} L_{j,s}(\beta), \quad j \geq 1, \quad (3.8)$$

$$L^s(\beta) := \sum_{j \geq s/\varepsilon} L_{j,s}(\beta), \quad s \geq 1, \quad (3.9)$$

$$L(\beta) := \sum_{s=1}^{\infty} L^s(\beta) = \sum_{j=1}^{\infty} L_j(\beta),$$

$$E_j(\beta) := M_j(\beta) - L_j(\beta), \quad j \geq 1, \quad (3.10)$$

$$E(\beta) := \sum_{j=1}^{\infty} E_j(\beta). \quad (3.11)$$

Therefore, by construction,  $M(\beta) = L(\beta) + E(\beta)$  for all  $\beta \in \mathbb{T}$ . Our motivation for introducing the above decomposition is that the discrete multiplier  $M_j$  is well-approximated by its continuous analog  $L_j$  on the major arcs in  $\mathfrak{M}_j$ . And off of the major arcs, the multiplier is otherwise small.

**Theorem 1.1** is proved by showing that there exist  $1 < r < 2$  and  $\kappa > 0$  such that

$$\|\Phi_{E_j} : \text{Sparse}(r, r)\| \lesssim 2^{-\kappa j}, \quad j \geq 1, \quad (3.12)$$

$$\|\Phi_{L^s} : \text{Sparse}(r, r)\| \lesssim 2^{-\kappa s}, \quad s \geq 1. \quad (3.13)$$

Indeed, from the above inequalities, it follows that

$$\|\Phi_L : \text{Sparse}(r, r)\| \leq \sum_{s=1}^{\infty} \|\Phi_{L^s} : \text{Sparse}(r, r)\| \lesssim \sum_{s=1}^{\infty} 2^{-\kappa s} \lesssim 1,$$



$$\|\Phi_E : \text{Sparse}(r, r)\| \leq \sum_{j=1}^{\infty} \|\Phi_{E_j} : \text{Sparse}(r, r)\| \lesssim \sum_{j=1}^{\infty} 2^{-\kappa j} \lesssim 1.$$

Therefore, our main theorem follows from

$$\|\Phi_M : \text{Sparse}(r, r)\| \leq \|\Phi_L : \text{Sparse}(r, r)\| + \|\Phi_E : \text{Sparse}(r, r)\| \lesssim 1.$$

We prove the “minor arcs” estimate (3.12) in Section 4 and the “major arcs” estimate (3.13) in Section 5.

The next theorem gives quantitative estimates for the Gauss sums (3.15) and the multipliers  $E_j$  defined in (3.10) that are essential to our proof of Theorem 1.1.

**Theorem 3.14.** *For absolute choices of  $\varepsilon > 0$ ,*

$$|S(B/Q)| \lesssim 2^{-\varepsilon s}, \quad B/Q \in \mathcal{R}_s, \quad s \geq 1, \quad (3.15)$$

$$\|E_j(\beta)\|_{\infty} \lesssim 2^{-\varepsilon j}, \quad j \geq 1, \quad (3.16)$$

$$\left\| \frac{d^2}{d\beta^2} E_j(\beta) \right\|_{\infty} \lesssim 2^{7j}, \quad j \geq 1. \quad (3.17)$$

The first two are well-known estimates. The estimate (3.15) is the Gauss sum bound, see [Hua 1982], while the estimate (3.16) is gotten by combining Lemmas 3.21 and 3.18. The only unfamiliar estimate is the derivative bound (3.17), but our claim is very weak and follows from elementary considerations.

The details of a proof of Theorem 3.14 are represented in the literature [Stein and Wainger 1990; Krause and Lacey 2017]. We indicate the details. A central lemma is this approximation of  $M_j$  defined in (3.5), in terms of  $L_j$  defined in (3.8).

**Lemma 3.18.** *For  $1 \leq s \leq \varepsilon j$ ,  $B/Q \in \mathcal{R}_s$ , we have the approximation*

$$M_j(\beta) = L_j(\beta) + O(2^{(2\varepsilon-1)j}), \quad \beta \in \mathfrak{M}_j(B/Q).$$

*Proof.* We closely follow the argument in [Krause and Lacey 2017]. There are two estimates to prove:

$$|M_j(\beta) - S(B/Q)H_j(\beta - B/Q)| \lesssim 2^{(2\varepsilon-1)j}, \quad (3.19)$$

$$|L_j(\beta) - S(B/Q)H_j(\beta - B/Q)| \lesssim 2^{(2\varepsilon-1)j}, \quad (3.20)$$

both estimates holding uniformly over  $\beta \in \mathfrak{M}_j(B/Q)$ , and  $B/Q \in \mathcal{R}_s$ .

For the second estimate (3.20), it follows from the definitions of  $L_j$  and  $L_{j,s}$  in (3.7), as well as the disjointness of the major arcs, that

$$\begin{aligned} |L_j(\beta) - S(B/Q)H_j(\beta - B/Q)| &= |L_{j,s}(\beta) - S(B/Q)H_j(\beta - B/Q)| \\ &\leq |S(B/Q)H_j(\beta - B/Q)|(\mathbf{1}_{\mathfrak{M}_j(B/Q)} - \chi(10^s(\beta - B/Q))) \\ &\lesssim \sup_{|\beta| > (1/2)10^{s-1}} |H_j(\beta)| \lesssim 10^{-s}. \end{aligned}$$

The last bound is a standard van der Corput estimate.

We turn to (3.19). Write  $\beta = B/Q + \eta$ , where  $|\eta| \leq 2^{(\varepsilon-3)j}$ . For all positive  $m$  in the support of  $\psi_j$ , decompose these integers into their residue classes mod  $Q$ , i.e.,  $m = pQ + r$ , where  $0 \leq r < Q \leq 2^{j\varepsilon}$  and the  $p$ -values are integers in  $[c, d]$ , with  $c = d/8 \simeq 2^j/Q$  to cover the support of  $\psi_j$ . The argument of the exponential in (3.1) is, modulo 1, given by

$$\beta(pQ + r)^3 = (B/Q + \eta)(pQ + r)^3 \equiv r^3 B/Q + (pQ)^3 \eta + O(2^{j(2\varepsilon-1)}).$$

Then the sum over all positive integers  $m$  in the support of  $\psi_j$  can be written as

$$\begin{aligned} \sum_{p \in [c, d]} \sum_{r=0}^{Q-1} [e(-r^3 B/Q - (pQ)^3 \eta) + O(2^{(2\varepsilon-1)j})] \psi_j(pQ + r) \\ = \sum_{r=0}^{Q-1} e(-r^3 \cdot B/Q) \times \sum_{p \in [c, d]} e(-\eta(pQ)^3) \psi_j(pQ) + O(2^{(2\varepsilon-1)j}) \\ = S(B, Q) \times Q \sum_{p \in [c, d]} e(-\eta(pQ)^3) \psi_j(pQ) + O(2^{(2\varepsilon-1)j}). \end{aligned}$$

For fixed  $p \in [c, d]$  and  $0 \leq t \leq Q$ , we have

$$\begin{aligned} |e(-\eta(pQ)^3) \psi_j(pQ) - e(-\eta(pQ + t)^3) \psi_j(pQ + t)| \\ \lesssim |e(-\eta(pQ)^3) - e(-\eta(pQ + t)^3)| 2^{-j} + |\psi_j(pQ) - \psi_j(pQ + t)| \\ \lesssim 2^{(2\varepsilon-2)j}. \end{aligned}$$

Therefore,

$$Q \sum_{p \in [c, d]} e(-\eta(pQ)^3) \psi_j(pQ) = \int_0^\infty e(-\eta t^3) \psi_j(t) dt + O(2^{(2\varepsilon-1)j}).$$

The analogous computation for negative values of  $m$  yields

$$\sum_{m < 0} e(-\beta m^3) \psi_j(m) = S(B, Q) \times \int_{-\infty}^0 e(-\eta t^3) \psi_j(t) dt + O(2^{(2\varepsilon-1)j}),$$

and combining the two estimates with the notation in (3.11) leads to the desired conclusion.  $\square$

We also need control of  $M_j$  and  $L_j$ , defined in (3.8) on the minor arcs, which are the open components of the complement of  $\mathfrak{M}_j$  defined in (3.4).

**Lemma 3.21.** *There is a  $\delta = \delta(\varepsilon)$  so that uniformly in  $j \geq 1$ ,*

$$|M_j(\beta)| + |L_j(\beta)| \lesssim 2^{-\delta j}, \quad \beta \notin \mathfrak{M}_j.$$

This estimate is essentially present in [Krause and Lacey 2017]. The bound  $|M_j(\beta)| \lesssim 2^{-\delta j}$  for  $\beta \notin \mathfrak{M}_j$  can be seen from [Bourgain 1989, Lemma 5.4], and is a consequence of a fundamental estimate of Weyl [Iwaniec and Kowalski 2004, Theorem 8.1]. The corresponding bound on  $L_j$  is an easy consequence of the van der Corput estimate  $|H_j(y)| \lesssim 2^{-j} |y|^{-1/3}$ .

#### 4. Minor arcs

Recalling the sparse-form notation (2.1) and the Fourier multiplier notation (2.9), we now proceed to the proof of the bound in (3.12).

**Lemma 4.1.** *There exists  $\kappa > 0$  and  $1 < r < 2$  such that*

$$\|\Phi_{E_j} : \text{Sparse}(r, r)\| \lesssim 2^{-\kappa j}, \quad j \geq 1.$$

*Proof.* We only need the  $L^\infty$  bound on  $E_j$  given in (3.16), and the derivative condition (3.17). In particular, these two conditions imply

$$|\mathcal{F}^{-1} E_j(m)| \lesssim \min \left\{ 2^{-\varepsilon j}, \frac{2^{7j}}{1+m^2} \right\}. \quad (4.2)$$

Write  $\mathcal{F}^{-1} E_j = \check{E}_{j,0} + \check{E}_{j,1}$ , where  $\check{E}_{j,0}(m) = [\mathcal{F}^{-1} E_j(m)] \mathbf{1}_{[-2^{10j}, 2^{10j}]}(m)$ . It follows immediately from (4.2) that

$$\|T_{\check{E}_{j,1}} : \ell^2 \mapsto \ell^2\| \lesssim \|\check{E}_{j,1}\|_1 \lesssim 2^{-3j}.$$

(Recall that  $T_K$  denotes convolution with respect to kernel  $K$ .) But, it follows that  $T_K f \lesssim M_{\text{HL}} f$ , where the latter is the maximal function. And so by Theorem 2B, we have

$$\|T_{\check{E}_{j,1}} : \text{Sparse}(1, 1)\| \lesssim 2^{-3j}.$$

It remains to provide a sparse bound for  $T_{\check{E}_{j,0}}$  (which is the interesting case). We are in a position to use (2.5), with  $N \simeq 2^{10j}$ . We have for  $1 < r < 2$

$$\|T_{\check{E}_{j,0}} : \text{Sparse}(r, r)\| \lesssim 2^{10j(2/r-1)} \|T_{\check{E}_{j,0}} : \ell^r \mapsto \ell^{r'}\|. \quad (4.3)$$

Notice that  $2/r - 1$  can be made arbitrarily small. We need to estimate the operator norm above. But, we have the two estimates

$$\|T_{\check{E}_{j,0}} : \ell^s \mapsto \ell^{s'}\| \lesssim 2^{-\varepsilon j}, \quad s = 1, 2.$$

The case of  $s = 1$  follows from (4.2), and the case of  $s = 2$  from Plancherel and (3.16). We therefore see that we have a uniformly small estimate on the norm of  $T_{\check{E}_{j,0}}$  from  $\ell^r \mapsto \ell^{r'}$  for  $1 < r < 2$ . For  $0 < 2 - r \ll \varepsilon$ , we have the desired bound in (4.3).  $\square$

#### 5. Major arcs

The following estimate is the core of the main result, Theorem 1.1. Recalling the definition of  $L^s$  in (3.9), the notation for Fourier multipliers (2.9) and the sparse norm notation (2.1), we have this, which verifies the bound in (3.13):

**Lemma 5.1.** *There exist  $\kappa > 0$  and  $1 < r < 2$  such that*

$$\|\Phi_{L^s} : \text{Sparse}(r, r)\| \lesssim 2^{-\kappa s}, \quad s \geq 1.$$

Combining the “major arcs” estimate in Lemma 5.1 with the “minor arcs” estimate in Lemma 4.1, the proof of Theorem 1.1 is complete.

The remainder of this section is taken up with the proof of the lemma. The central facts are (1) the Gauss sum bound (3.15); (2) the sparse bound for Hilbert-space-valued singular integrals Theorem 2C, which is applied to Fourier projections of  $f$  and  $g$  onto the major arcs; (3) an argument to pass from a sparse operator applied to the aforementioned Fourier projections to a sparse bound in terms of just  $f$  and  $g$ .

Step 1. We define our Hilbert-space-valued functions, where the Hilbert space will be the finite-dimensional space  $\ell^2(\mathcal{R}_s)$ . Recall that the rationals  $\mathcal{R}_s$  are defined in (3.3), and the functions  $\chi_s$  are defined in (3.7). Given  $f \in \ell^2$ , set

$$f_s = \{f_{s,B/Q} : B/Q \in \mathcal{R}_s\} := \{\chi_{s-1} * (\text{Mod}_{-B/Q} f) : B/Q \in \mathcal{R}_s\}. \quad (5.2)$$

Above,  $\text{Mod}_\lambda f(x) = e(\lambda x) f(x)$  is modulation by  $\lambda$ . The intervals

$$\{[B/Q - 10^{-s}, B/Q + 10^{-s}] : B/Q \in \mathcal{R}_s\} \quad (5.3)$$

are pairwise disjoint, so that by Bessel's theorem, we have

$$\|f_s\|_{\ell^2(\ell^2(\mathcal{R}_s))} = \|\{f_{s,B/Q} : B/Q \in \mathcal{R}_s\}\|_{\ell^2(\ell^2(\mathcal{R}_s))} \leq \|f\|_2.$$

Step 2. The inner product we are interested in can be viewed as one acting on  $\ell^2(\mathcal{R}_s)$  functions. Observe that the Fourier multiplier associated to  $L^s$  enjoys the equalities below. Beginning from (3.9) and (3.7),

$$\begin{aligned} \langle \Phi_{L^s} f, g \rangle &= \sum_{B/Q \in \mathcal{R}_s} \sum_{j \geq s/\varepsilon} S(B, Q) \cdot \langle H_j(\beta - B/Q) \chi_s(\beta - B/Q) \mathcal{F}f(\beta), \mathcal{F}g(\beta) \rangle \\ &= \sum_{B/Q \in \mathcal{R}_s} \sum_{j \geq s/\varepsilon} S(B, Q) \cdot \langle H_j(\beta) \chi_s(\beta) f(\beta + B/Q), \mathcal{F}g(\beta + B/Q) \rangle \\ &= \sum_{B/Q \in \mathcal{R}_s} \sum_{j \geq s/\varepsilon} S(B, Q) \cdot \langle H_j(\beta) \chi_s(\beta) \mathcal{F}f_{s,B/Q}(\beta), \mathcal{F}g_{s,B/Q}(\beta) \rangle. \end{aligned}$$

Crucially, above we have removed some modulation factors to get a fixed multiplier acting on a Hilbert-space-valued function. Continuing the equalities, we have

$$= \sum_{B/Q \in \mathcal{R}_s} S(B, Q) \langle \Phi_{H^s} f_{s,B/Q}, g_{s,B/Q} \rangle, \quad \text{where } H^s = \sum_{j \geq s/\varepsilon} H_j. \quad (5.4)$$

We address the Gauss sums  $S(B, Q)$  above. Recalling (3.15) and setting  $f'_s = \{\lambda_{B/Q} f_{s,B/Q}\}$  for appropriate choice of  $|\lambda_{B/Q}| = 1$ , we have

$$|\langle \Phi_{L^s} f_s, g_s \rangle| \lesssim 2^{-\varepsilon s} \langle \Phi_{H^s} f'_s, g_s \rangle. \quad (5.5)$$

Above we have gained a geometric decay in  $s$ .

On the right-hand side of (5.5), we have an operator acting on Hilbert-space-valued functions. Noting that  $\|f'_s\|_{\ell^2(\mathcal{R}_s)} = \|f_s\|_{\ell^2(\mathcal{R}_s)}$  pointwise, we are free to replace  $f'_s$  in (5.5) by simply  $f_s$ , as defined in (5.2). The remaining estimate to prove is that there is a choice of  $1 < r < 2$  and sparse operator  $\Lambda_{r,r}$  so that

$$|\langle \Phi_{H^s} f_s, g_s \rangle| \lesssim 2^{(\varepsilon/4)s} \Lambda_{r,r}(f, g). \quad (5.6)$$

Note in particular that we will allow small geometric growth in this estimate, which will be absorbed into the geometric decay in (5.5).

**Step 3.** The principal step is the application of the sparse bound in Theorem 2C. From the definitions in (3.6) and (5.4), we have

$$H^s(\beta) = \sum_{j \geq s/\varepsilon} H_j(\beta) = \sum_{j \geq s/\varepsilon} \int e(-\beta t^3) \psi_j(t) dt.$$

By the choice of  $\psi$  in (3.2), it follows that the integrand on the right equals  $e(-\beta t^3) dt/t$  for  $t > 2^{s/\varepsilon+1}$ . And, in particular,

$$H^s(\beta) = \frac{1}{3} \sum_{j \geq s/\varepsilon} \int e(-\beta s) \frac{\psi_j(s^{1/3})}{s^{2/3}} ds.$$

But  $\psi$  is odd; hence so is  $\psi_j(s^{1/3})/s^{2/3}$ . It follows that  $\check{H}^s$  is a Calderón–Zygmund kernel; that is, it meets the conditions in (2.2). Thus, the operator we are considering is convolution with respect to  $\check{H}^s$ , namely  $\Phi_{H^s} = T_{\check{H}^s}$ .

Therefore, from Theorem 2C, we have the following inequality for the expression in (5.4):

$$|\langle T_{\check{H}^s} f_s, g_s \rangle| \lesssim \Lambda_{1,1}(f_s, g_s). \quad (5.7)$$

There is one additional fact: all the intervals used in the definition of the sparse form in (5.7) above have length at least  $2^{3(s/\varepsilon-2)}$ . This is a simple consequence of  $\check{H}^s(x) \mathbf{1}_{[-2^{3(s/\varepsilon-2)}, 2^{3(s/\varepsilon-2)}]} \equiv 0$ .

**Step 4.** We should emphasize that (5.7) has a small abuse of notation: The sparse form is computed on the vector-valued functions  $f_s$  and  $g_s$ . That is, the implied averages have to be made relative to the  $\ell^2(\mathcal{R}_s)$ -norm. The last step is to remove the norm. Namely, we show that there is a choice of  $1 < r < 2$  and sparse form  $\Lambda_{r,r}$  so that

$$\Lambda_{1,1}(f_s, g_s) \lesssim 2^{(\varepsilon/4)s} \Lambda_{r,r}(f, g). \quad (5.8)$$

Combining this estimate with (5.7) proves (5.6), completing the proof.

The proof of (5.8) is reasonably routine. It will be crucial that we have the estimate  $\sharp \mathcal{R}_s \lesssim 2^{2s}$ . Let  $\mathcal{S}$  be the sparse collection of intervals associated with the sparse form  $\Lambda_{1,1}(f_s, g_s)$ . As noted, we are free to assume that for all  $S \in \mathcal{S}$ , we have  $|S| \geq 10^{s/4\varepsilon}$ . Recall the definition of  $f_s$  in (5.2). Write  $f_s = f_s^{S,0} + f_s^{S,1}$ , where

$$f_s^{S,0} := \{\chi_{s-1} * (\text{Mod}_{-B/Q}(f \mathbf{1}_{2S})) : B/Q \in \mathcal{R}_s\}.$$

Above, we have localized the support of  $f$  to the interval  $2S$ . The same decomposition is used on the functions  $g$  and  $g_s$ . By subadditivity, we have

$$\Lambda_{1,1}(f_s, g_s) \leq \Lambda_{1,1}(f_s^{S,0}, g_s^{S,0}) \quad (5.9)$$

$$+ \Lambda_{1,1}(f_s^{S,1}, g_s^{S,0}) + \Lambda_{1,1}(f_s^{S,0}, g_s^{S,1}) \quad (5.10)$$

$$+ \Lambda_{1,1}(f_s^{S,1}, g_s^{S,1}). \quad (5.11)$$



The crux of the matter is this estimate: For each interval  $S \in \mathcal{S}$ , we have

$$\langle f_s^{S,0} \rangle_S \lesssim 2^{s(2-r)/r} \langle f \rangle_{2S,r}, \quad 1 < r < 2. \quad (5.12)$$

And, the fraction  $(2-r)/r$  in the exponent can be made arbitrarily small by taking  $0 < 2-r$  very small. Indeed, using the disjointness of the intervals in (5.3), and Plancherel, we have

$$\langle f_s^{S,0} \rangle_{S,2} \lesssim \langle f \rangle_{2S,2}. \quad (5.13)$$

Second, it is trivial that

$$\langle \chi_{s-1} * (\text{Mod}_{-B/Q} f \mathbf{1}_{2S}) \rangle_S \lesssim \langle f \rangle_{2S}$$

and by simply summing over the bounded number of choices of  $B/Q \in \mathcal{R}_s$ , we have

$$\langle f_s^{S,0} \rangle_S \lesssim 2^{2s} \langle f \rangle_{2S}.$$

Interpolating between this and (5.13) proves (5.12). With that inequality in hand, we have, for  $0 < 2-r$  sufficiently small,

$$\sum_{S \in \mathcal{S}} |S| \langle f_s^{S,0} \rangle_S \langle g_s^{S,0} \rangle_S \lesssim 2^{s(\epsilon/4)} \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{2S,r} \langle g \rangle_{2S,r}.$$

If the family  $\mathcal{S}$  is  $\frac{1}{2}$ -sparse, then the family  $\{2S : S \in \mathcal{S}\}$  is  $\frac{1}{4}$ -sparse, so we have our desired bound for the term on the right in (5.9).

There are three more terms, in (5.10) and (5.11), which are all much smaller. Recall the notation  $\{f\}$  of (2.7). Since  $\chi$ , as chosen in (3.7), is smooth, and the length of  $S \in \mathcal{S}$  is much larger than  $10^s$ , we have

$$\langle \chi_{s-1} * (\text{Mod}_{-B/Q} f \mathbf{1}_{\mathbb{R} \setminus 2S}) \rangle_S \lesssim 2^{-100s} \{f\}_S, \quad B/Q \in \mathcal{R}_s.$$

Summing this estimate over all  $2^{2s}$  choices  $B/Q \in \mathcal{R}_s$ , we see that each of the three terms in (5.10) and (5.11) are at most

$$2^{-s} \sum_{S \in \mathcal{S}} |S| \{f\}_S \{g\}_S.$$

It remains to bound this last bilinear form, which is the task taken up in Lemma 2.8. This completes the argument for (5.8).

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