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ROKHLIN DIMENSION: ABSORPTION OF MODEL ACTIONS

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We establish a connection between Rokhlin dimension and the absorption of certain model actions on strongly self-absorbing C*-algebras. Namely, as to be made precise in the paper, let G be a well-behaved locally compact group. If $\mathcal{D}$ is a strongly self-absorbing C*-algebra and $\alpha : G \curvearrowright A$ is an action on a separable, $\mathcal{D}$-absorbing C*-algebra that has finite Rokhlin dimension with commuting towers, then α tensorially absorbs every semi-strongly self-absorbing G-action on $\mathcal{D}$. In particular, this is the case when α satisfies any version of what is called the Rokhlin property, such as for $G = \mathbb{R}$ or $G = \mathbb{Z}^k$. This contains several existing results of similar nature as special cases. We will in fact prove a more general version of this theorem, which is intended for use in subsequent work. We will then discuss some nontrivial applications. Most notably it is shown that for any $k \geq 1$ and on any strongly self-absorbing Kirchberg algebra, there exists a unique $\mathbb{R}^k$-action having finite Rokhlin dimension with commuting towers up to (very strong) cocycle conjugacy.

Introduction

The present work is a continuation of the author's quest to study fine structure and classification of certain C*-dynamics by employing ideas related to tensorial absorption. In previous work, the theory of (semi-)strongly self-absorbing actions on C*-algebras [Szabó 2017b; 2018b; 2018c] was developed, closely following the important results established in the classical theory of strongly self-absorbing C*-algebras by Toms and Winter [2007] and others [Kirchberg 2006; Dadarlat and Winter 2009]. Strongly self-absorbing C*-algebras have historically emerged by example [Jiang and Su 1999], and now play a central role in the structure theory of simple nuclear C*-algebras; see for example [Kirchberg and

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Phillips 2000; Rørdam 2004; Elliott and Toms 2008; Winter and Zacharias 2010; Winter 2010; 2012; 2014; Matui and Sato 2012b; 2014a; Bosa et al. 2015; Castillejos et al. 2018]. Roughly speaking, a tensorial factorization of the form $A \cong A \otimes \mathcal{D}$ — for a given C*-algebra A and a strongly self-absorbing C*-algebra $\mathcal{D}$ — provides sufficient space to perform nontrivial manipulations on elements inside A, which often gives rise to structural properties of particular interest for classification. The underlying motivation behind [Szabó 2017b; 2018b; 2018c] is the idea that this kind of phenomenon should persist at the level of C*-dynamics if one is interested in classification of group actions up to cocycle conjugacy; in fact some much earlier work [Kishimoto 2001; 2002; Izumi and Matui 2010; 2012; Goldstein and Izumi 2011; Matui and Sato 2012a; 2014b] has (sometimes implicitly) used this idea to reasonable success. It was further demonstrated in [Szabó 2017b; 2018a] how this approach can indeed give rise to new insights about classification or rigidity of group actions on certain C*-algebras, in particular strongly self-absorbing ones.

Starting from Connes' groundbreaking work [1975; 1976; 1977] on injective factors, which involved classification of single automorphisms, the Rokhlin property in its various forms became a key tool to classify actions of amenable groups on von Neumann algebras [Jones 1980; Ocneanu 1985; Sutherland and Takesaki 1989; Kawahigashi et al. 1992; Katayama et al. 1998; Masuda 2007]. It did not take long for these ideas to reach the realm of C*-algebras. Initially appearing in [Herman and Jones 1982] and [Herman and Ocneanu 1984], the Rokhlin property for single automorphisms and its applications for classification were perfected in works of Kishimoto and various collaborators [Kishimoto 1995; 1996b; 1998a; 1998b; Bratteli et al. 1993; 1995; Evans and Kishimoto 1997; Elliott et al. 1998; Bratteli and Kishimoto 2000; Nakamura 2000]. Further work pushed these techniques to actions of infinite higher-rank groups as well [Nakamura 1999; Katsura and Matui 2008; Matui 2008; 2010; 2011; Izumi and Matui 2010; 2012; 2018]. The case of finite groups was treated in [Izumi 2004a; 2004b], where it was shown that such actions with the Rokhlin property have a particularly rigid theory; see also [Santiago 2015; Gardella and Santiago 2016; Gardella 2014a; 2014b; 2017; Barlak and Szabó 2016; Barlak et al. 2017]. In contrast to von Neumann algebras, however, the Rokhlin property for actions on C*-algebras has too many obstructions in general, ranging from obvious ones like lack of projections to more subtle ones of K-theoretic nature.

Rokhlin dimension is a notion of dimension for actions of certain groups on C*-algebras and was first introduced by Hirshberg, Winter and Zacharias [Hirshberg et al. 2015]. Several natural variants of Rokhlin dimension have been introduced, and all of them have in common that they generalize (to some degree) the Rokhlin property for actions of either finite groups or the integers. The theory has been extended and applied in many following works, such as [Szabó 2015; Hirshberg and Phillips 2015; Szabó et al. 2017; Gardella 2017; Hirshberg et al. 2017; Liao 2016; 2017; Brown et al. 2018; Gardella et al. 2017]. In short, the advantage of working with Rokhlin dimension is that it is both more prevalent and more flexible than the Rokhlin property, but is yet often strong enough to deduce interesting structural properties of the crossed product, such as finite nuclear dimension [Winter and Zacharias 2010].

A somewhat stronger version of Rokhlin dimension, namely with commuting towers, has been considered from the very beginning as a variant that was also compatible with respect to the absorption of strongly self-absorbing C*-algebras. Although the assumption of commuting towers initially only looked like a minor technical assumption, it was eventually discovered that it can make a major difference in some cases, such as for actions of finite groups [Hirshberg and Phillips 2015].

The purpose of this paper is to showcase a decisive connection between finite Rokhlin dimension with commuting towers and the absorption of semi-strongly self-absorbing model actions. The following describes a variant of the main result; see Theorem 4.4:

Theorem A. *Let G be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Suppose that the quotient G/N contains a discrete, normal, cocompact subgroup that is residually finite and has a box space with finite asymptotic dimension. Let A be a separable* C^**-algebra with an action $\alpha : G \curvearrowright A$. Let $\gamma : G \curvearrowright \mathcal{D}$ be a semi-strongly self-absorbing action that is unitarily regular. Suppose that $\alpha|_N$ is $\gamma|_N$-absorbing. If the Rokhlin dimension of α with commuting towers relative to N is finite, then it follows that α is γ-absorbing.*

Since many assumptions in this theorem are fairly technical at first glance, it may be helpful for the reader to keep in mind some special cases. For example, the above assumptions on the pair $N \subset G$ are satisfied when the quotient G/N above is isomorphic to either $\mathbb{R}$ or $\mathbb{Z}$. In this case, the theorem states that as long as the action α satisfies a suitable Rokhlin-type criterion relative to N, tensorial absorption of the G-action γ can be detected by restricting to the N-actions, even though this restriction procedure (a priori) comes with great loss of dynamical information. This is most apparent when the normal subgroup N is trivial, which is yet another important special case; see Corollary 5.1:

Corollary B. *Let G be a second-countable, locally compact group. Suppose that G contains a discrete, normal, cocompact subgroup that is residually finite and has a box space with finite asymptotic dimension. Let A be a separable* C^**-algebra with an action $\alpha : G \curvearrowright A$. Suppose that $\mathcal{D}$ is a strongly self-absorbing* C^**-algebra with $A \cong A \otimes \mathcal{D}$. If the Rokhlin dimension of α with commuting towers is finite, then it follows that α is γ-absorbing for every semi-strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{D}$.*

Here it may be useful to keep in mind that any version of what is called the Rokhlin property for $G = \mathbb{R}$ or $G = \mathbb{Z}^k$ will automatically imply finite Rokhlin dimension with commuting towers, and is therefore covered by Corollary B. This is in turn a generalization of [Hirshberg and Winter 2007, Theorem 1.1; Hirshberg et al. 2015, Theorems 5.8, 5.9; 2017, Theorem 5.3; Szabó et al. 2017, Theorem 9.6; Gardella and Lupini 2018, Theorem 4.50(2)]. We will in fact only apply the corollary within this paper, with a particular focus on the special case where the action is assumed to have the Rokhlin property. Some immediate applications of Corollary B will be discussed in Section 5. The main nontrivial application is pursued in Section 6, which is as follows; see Theorem 6.7 and Corollary 6.11:

Theorem C. *Let $\mathcal{D}$ be a strongly self-absorbing Kirchberg algebra. Then up to (very strong) cocycle conjugacy, there is a unique action $\gamma : \mathbb{R}^k \curvearrowright \mathcal{D}$ that has finite Rokhlin dimension with commuting towers.*

We note that a strongly self-absorbing C^*-algebra is a Kirchberg algebra precisely when it is traceless. Kirchberg algebras are (by convention) the separable, simple, nuclear, purely infinite C^*-algebras, whose celebrated classification is due to [Kirchberg and Phillips 2000; Phillips 2000; Kirchberg 2003] and which constitutes a prominent special case of the Elliott classification program. We note that all other strongly self-absorbing C*-algebras are conjectured to be quasidiagonal — see [Tikuisis et al. 2017, Corollary 6.7] — and so any Rokhlin flows on them would induce Rokhlin flows on the universal UHF algebra, which do not exist; see [Kishimoto 1996a, page 600; 1998a, page 289; Hirshberg et al. 2017, Section 2]. In particular, the underlying problem above is only interesting to consider in the purely infinite case.

Although the theorem above is not too far off from being a very special case of [Szabó 2017a] for ordinary flows, this result is entirely new for $k \geq 2$, and is in fact the first classification result for $\mathbb{R}^k$-actions on C*-algebras up to cocycle conjugacy.

The proof goes via induction in the number k of flows generating the action. In order to achieve a major part of the induction step, the corollary above is used in order to see that any two $\mathbb{R}^k$-actions as in the statement absorb each other tensorially. However, in order for this to make sense, it has to be at least established beforehand (as part of the induction step) that any such action has equivariantly approximately inner flip. This is achieved via a relative Kishimoto-type approximate cohomology-vanishing argument inspired by [Kishimoto 2002, Section 3], which combines arguments related to the Rokhlin property for $\mathbb{R}^k$-actions with arguments related to the structure theory of semi-strongly self-absorbing actions.

At this moment it seems unclear whether or not to expect a similarly rigid situation for Rokhlin $\mathbb{R}^k$-actions on general Kirchberg algebras, as is the case for $k = 1$ [Szabó 2017a]. In general, in order to implement a more general classification of this sort, it would require a technique for both constructing and manipulating cocycles for $\mathbb{R}^k$-actions (where $k \geq 2$) with the help of the Rokhlin property, which may potentially be much more complicated than for $k = 1$. In essence, our approach based on ideas related to strong self-absorption works because the main result allows one to bypass the need to bother with general cocycles for all of $\mathbb{R}^k$, but instead requires one only to consider individual copies of $\mathbb{R}$ inside $\mathbb{R}^k$ at a time (represented by the flows generating the $\mathbb{R}^k$-action), enabling an induction process.

In forthcoming work, the full force of the aforementioned main result of this paper (Theorem 4.4) will form the basis of further uniqueness results regarding actions of certain discrete amenable groups on strongly self-absorbing C*-algebras.

1. Preliminaries

Notation 1.1. Unless specified otherwise, we will stick to the following notational conventions:

- G denotes a locally compact Hausdorff group.
- A and B denote C*-algebras.
- The symbols α, β, γ are used to denote point-norm continuous actions on C*-algebras. Since continuity is always assumed in this context, we will simply refer to them as actions.
- If $\alpha : G \curvearrowright A$ is an action, then A^α denotes the fixed-point algebra of A.
- If F is a finite subset inside some set M, we often write $F \Subset M$.
- If (X, d) is some metric space with elements $a, b \in X$, then we write $a =_\varepsilon b$ as a shorthand for $d(a, b) \leq \varepsilon$.

We first recall some needed definitions and notation.

Definition 1.2 (cf. [Packer and Raeburn 1989, Definition 3.2] and [Szabó 2018b; 2017b, Section 1]). Let $\alpha : G \curvearrowright A$ be an action. Consider a strictly continuous map $w : G \to \mathcal{U}(\mathcal{M}(A))$:

(i) w is called an α-1-cocycle if one has $w_g \alpha_g(w_h) = w_{gh}$ for all $g, h \in G$. In this case, the map $\alpha^w : G \to \mathrm{Aut}(A)$ given by $\alpha^w_g = \mathrm{Ad}(w_g) \circ \alpha_g$ is again an action, and is called a cocycle perturbation of α. Two G-actions on A are called exterior equivalent if one of them is a cocycle perturbation of the other.

(ii) Assume that w is an α-1-cocycle. It is called an approximate coboundary if there exists a sequence of unitaries $x_n \in \mathcal{U}(\mathcal{M}(A))$ such that $x_n \alpha_g(x_n^*) \xrightarrow{\text{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two G-actions on A are called strongly exterior equivalent if one of them is a cocycle perturbation of the other via an approximate coboundary.

(iii) Assume w is an α-1-cocycle. It is called an asymptotic coboundary if there exists a strictly continuous map $x : [0, \infty) \to \mathcal{U}(\mathcal{M}(A))$ with $x_0 = \mathbf{1}$ and such that $x_t \alpha_g(x_t^*) \xrightarrow{\text{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two G-actions on A are called very strongly exterior equivalent if one of them is a cocycle perturbation of the other via an asymptotic coboundary.

(iv) Let $\beta : G \curvearrowright B$ be another action. The actions α and β are called cocycle conjugate, written $\alpha \simeq_{\mathrm{cc}} \beta$ if there exists an isomorphism $\psi : A \to B$ such that $\psi^{-1} \circ \beta \circ \psi$ and α are exterior equivalent. If ψ can be chosen such that $\psi^{-1} \circ \beta \circ \psi$ and α are strongly exterior equivalent, then α and β are called strongly cocycle conjugate, written $\alpha \simeq_{\mathrm{scc}} \beta$. If ψ can be chosen such that $\psi^{-1} \circ \beta \circ \psi$ and α are very strongly exterior equivalent, then α and β are called very strongly cocycle conjugate, written $\alpha \simeq_{\mathrm{vscc}} \beta$.

Note that for a cocycle w, the cocycle identity applied to $g = h = e$ yields $w_e = w_e^2$, and hence $w_e = \mathbf{1}$. This is implicitly used in many calculations without further mention.

Definition 1.3 (cf. [Kirchberg 2006, Definition 1.1] and [Szabó 2018b, Section 1]). Let A be a C*-algebra and let $\alpha : G \curvearrowright A$ be an action of a locally compact group:

(i) The sequence algebra of A is given as

$$A_\infty = \ell^\infty(\mathbb{N}, A) / \big\{ (x_n)_n \;\big|\; \lim_{n\to\infty} \|x_n\| = 0 \big\}.$$

There is a standard embedding of A into A_∞ by sending an element to its constant sequence. We shall always identify $A \subset A_\infty$ this way, unless specified otherwise.

(ii) Pointwise application of α on representing sequences defines a (not necessarily continuous) G-action α_∞ on A_∞. Let

$$A_{\infty,\alpha} = \{ x \in A_\infty \mid [g \mapsto \alpha_{\infty,g}(x)] \text{ is continuous} \}$$

be the continuous part of A_∞ with respect to α.

(iii) For some C*-subalgebra $B \subset A_\infty$, the (corrected) relative central sequence algebra is defined as

$$F(B, A_\infty) = (A_\infty \cap B') / \mathrm{Ann}(B, A_\infty).$$

(iv) If $B \subset A_\infty$ is α_∞-invariant, then the G-action α_∞ on A_∞ induces a (not necessarily continuous) G-action $\tilde{\alpha}_\infty$ on $F(B, A_\infty)$. Let

$$F_\alpha(B, A_\infty) = \{y \in F_\alpha(B, A_\infty) \mid [g \mapsto \tilde{\alpha}_{\infty,g}(y)] \text{ is continuous}\}$$

be the continuous part of $F(B, A_\infty)$ with respect to α.

(v) When $B = A$, we write $F(A, A_\infty) = F_\infty(A)$ and $F_\alpha(A, A_\infty) = F_{\infty,\alpha}(A)$.

Definition 1.4 [Barlak and Szabó 2016, Definition 3.3]. Let G be a second-countable, locally compact group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable C*-algebras. An equivariant $*$-homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ is called (equivariantly) sequentially split if there exists a $*$-homomorphism $\psi : (B, \beta) \to (A_{\infty,\alpha}, \alpha_\infty)$ such that $\psi(\varphi(a)) = a$ for all $a \in A$.

Definition 1.5. Let G be a second-countable, locally compact group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on unital C*-algebras. Let $\varphi_1, \varphi_2 : (A, \alpha) \to (B, \beta)$ be two unital and equivariant $*$-homomorphisms. We say that φ_1 and φ_2 are approximately G-unitarily equivalent if the following holds. For every $\mathcal{F} \Subset A$, $\varepsilon > 0$, and compact set $K \subseteq G$, there exists a unitary $v \in \mathcal{U}(B)$ such that

$$\max_{a \in \mathcal{F}} \|\varphi_2(a) - v\varphi_1(a)v^*\| \le \varepsilon, \quad \max_{g \in K} \|v - \beta_g(v)\| \le \varepsilon.$$

Definition 1.6 [Szabó 2018b, Definitions 3.1, 4.1]. Let $\mathcal{D}$ be a separable, unital C*-algebra and G a second-countable, locally compact group. Let $\gamma : G \curvearrowright \mathcal{D}$ be an action. We say that:

(i) γ is a strongly self-absorbing action if the equivariant first-factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G-unitarily equivalent to an isomorphism.

(ii) γ is semi-strongly self-absorbing if it is strongly cocycle conjugate to a strongly self-absorbing action.

Definition 1.7 [Szabó 2018c, Definition 2.17]. Let G be a second-countable, locally compact group. An action $\alpha : G \curvearrowright A$ on a unital C*-algebra is called unitarily regular if for every $\varepsilon > 0$ and compact set $K \subseteq G$, there exists $\delta > 0$ such that for every pair of unitaries

$$u, v \in \mathcal{U}(A) \quad \text{with } \max_{g \in K} \max\{\|\alpha_g(u) - u\|, \|\alpha_g(v) - v\|\} \le \delta,$$

there exists a continuous path of unitaries $w : [0, 1] \to \mathcal{U}(A)$ satisfying

$$w(0) = \mathbf{1}, \quad w(1) = uvu^*v^*, \quad \max_{0 \le t \le 1} \max_{g \in K} \|\alpha_g(w(t)) - w(t)\| \le \varepsilon.$$

Let us recall some of the main results from [Szabó 2017b; 2018b; 2018c], which we will use throughout. We will also use the perspective given in [Barlak and Szabó 2016, Section 4].

Theorem 1.8 (cf. [Szabó 2018b, Theorems 3.7, 4.7]). *Let G be a second-countable, locally compact group. Let A be a separable* C**-algebra and $\alpha : G \curvearrowright A$ an action. Let $\mathcal{D}$ be a separable, unital* C**-algebra and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing action. The following are equivalent*:

(i) *α and $\alpha \otimes \gamma$ are strongly cocycle conjugate.*

(ii) *α and $\alpha \otimes \gamma$ are cocycle conjugate.*

(iii) *There exists a unital, equivariant $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$.*

(iv) *The equivariant first-factor embedding $\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \to (A \otimes \mathcal{D}, \alpha \otimes \gamma)$ is sequentially split.*

If γ is moreover unitarily regular, then these statements are equivalent to

(v) *α and $\alpha \otimes \gamma$ are very strongly cocycle conjugate.*

Remark. For the rest of this paper, an action α satisfying condition (i) from above is called γ-absorbing or γ-stable. In the particular case that γ is the trivial G-action on a strongly self-absorbing C*-algebra $\mathcal{D}$, we will say that α is equivariantly $\mathcal{D}$-stable.

Remark 1.9. Unitary regularity for an action is a fairly mild technical assumption. It can be seen as the equivariant analog of the C*-algebraic property that the commutator subgroup inside the unitary group lies in the connected component of the unit. Unitary regularity holds automatically under equivariant $\mathcal{Z}$-stability, but also in other cases; see [Szabó 2018c, Proposition 2.19 and Example 6.4].

Theorem 1.10 [Szabó 2018c, Theorem 5.9]. *A semi-strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{D}$ is unitarily regular if and only if the class of all separable γ-absorbing G-C*-dynamical systems is closed under equivariant extensions.*

We will extensively use the following without much mention:

Proposition 1.11 [Brown 2000]. *Let G be a second-countable, locally compact group. Let A be a C*-algebra and $\alpha : G \curvearrowright A$ an action. Let $x \in A_{\infty,\alpha}$ and $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ be a bounded sequence representing x. Then $(x_n)_n$ is a continuous element with respect to the componentwise action of α on $\ell^\infty(\mathbb{N}, A)$.*

2. Box spaces and partitions of unity over groups

Definition 2.1. Let G be a second-countable, locally compact group. A residually compact approximation of G is a decreasing sequence $H_{n+1} \subseteq H_n \subseteq G$ of normal, discrete, cocompact subgroups in G with $\bigcap_{n\in\mathbb{N}} H_n = \{1\}$. If G is a discrete group, then the subgroups H_n will have finite index, in which case we call the sequence $(H_n)_n$ a residually finite approximation.

Remark 2.2. In the above setting, the sequence $(H_n)_n$ is automatically a residually finite approximation of the discrete group H_1.

Recall the definition of a box space; see [Roe 2003, Definition 10.24; Khukhro 2012].

Definition 2.3. Let Γ be a countable discrete group and $\mathcal{S} = (H_n)_n$ a residually finite approximation of Γ. Let d be a proper, right-invariant metric on Γ. For every $n \in \mathbb{N}$, denote by $\pi_n : \Gamma \to \Gamma/H_n$ the quotient map and by $\pi_{n*}(d)$ the push-forward metric on Γ/H_n that is induced by d. The box space of Γ along $\mathcal{S}$, denoted by $\square_{\mathcal{S}}\Gamma$, is the coarse disjoint union of the sequence of finite metric spaces $(\Gamma/H_n, \pi_{n*}(d))$.

The main purpose of this section will be to prove the following technical lemma:

Lemma 2.4. *Let G be a second-countable, locally compact group and $\mathcal{S} = (H_n)_n$ a residually compact approximation of G. Assume that the box space $\square_{\mathcal{S}} H_1$ has finite asymptotic dimension d. Then for every $\varepsilon > 0$ and compact set $K \subset G$, there exists $n \in \mathbb{N}$ and continuous, compactly supported functions $\mu^{(0)}, \dots, \mu^{(d)} : G \to [0, 1]$ satisfying:*

(a) *For every $l = 0, \dots, d$ and $h \in H_n \setminus \{1\}$, we have*

$$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)}) \cdot h = \varnothing.$$

(b) *For every $g \in G$, we have*

$$\sum_{l=0}^{d} \sum_{h \in H_n} \mu^{(l)}(gh) = 1.$$

(c) *For every $l = 0, \dots, d$ and $g \in K$, we have*

$$\|\mu^{(l)}(g \cdot \_) - \mu^{(l)}\|_\infty \leq \varepsilon.$$

Remark 2.5. In the case that $G = \Gamma$ is a discrete group and $\mathcal{S}$ is a residually finite approximation, this is precisely [Szabó et al. 2017, Lemma 2.13]. In order to prove Lemma 2.4, we shall convince ourselves that the desired functions can be constructed from finitely supported functions with similar properties on the cocompact subgroup H_1. For this, we first have to observe a slightly improved version of [Szabó et al. 2017, Lemma 2.13] in the discrete case.

Lemma 2.6. *Let Γ be a countable discrete group and $\mathcal{S} = (H_n)_n$ a residually finite approximation of Γ. Assume that the box space $\square_{\mathcal{S}} \Gamma$ has finite asymptotic dimension d. Then for every $\varepsilon > 0$ and finite set $F \Subset \Gamma$, there exists $n \in \mathbb{N}$ and finitely supported functions $\nu^{(0)}, \dots, \nu^{(d)} : \Gamma \to [0, 1]$ satisfying:*

(a) *For every $l = 0, \dots, d$ and $h \in H_n \setminus \{1\}$, we have*

$$g_1 h g_2^{-1} \notin F \quad \text{for all } g_1, g_2 \in \operatorname{supp}(\nu^{(l)}).$$

(b) *For every $g \in \Gamma$, we have*

$$\sum_{l=0}^{d} \sum_{h \in H_n} \nu^{(l)}(gh) = 1.$$

(c) *For every $l = 0, \dots, d$ and $g \in F$, we have*

$$\|\nu^{(l)}(g \cdot \_) - \nu^{(l)}\|_\infty \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $F \subset G$ be given. We apply [Szabó et al. 2017, Lemma 2.13] and choose some n and finitely supported functions $\theta^{(0)}, \dots, \theta^{(d)} : \Gamma \to [0, 1]$ satisfying

$$\operatorname{supp}(\theta^{(l)}) \cap \operatorname{supp}(\theta^{(l)}) \cdot h_n = \varnothing \quad \text{for all } h_n \in H_n \setminus \{1\}, \tag{2-1}$$

as well as properties (b) and (c). Combining property (2-1) and (c), we see that if $g_1, g_2 \in \operatorname{supp}(\theta^{(l)})$ and $h \in H_n \setminus \{1\}$ are such that $g_1 h g_2^{-1} = g_1(g_2 h^{-1}) \in F$, then we get

$$|\theta^{(l)}(g_1)| = |\theta^{(l)}(g_1 h g_2^{-1} \cdot g_2 h^{-1})| \overset{(c)}{\leq} \varepsilon + |\theta^{(l)}(g_2 h^{-1})| \overset{(2\text{-}1)}{=} \varepsilon. \tag{2-2}$$

Let us define new functions $\kappa^{(l)} : \Gamma \to [0, 1]$ via

$$\kappa^{(l)}(g) = (\theta^{(l)}(g) - \varepsilon)_+. \tag{2-3}$$

These new functions clearly still satisfy property (c). For any $g_1, g_2 \in \mathrm{supp}(\kappa^{(l)})$, we evidently have $g_1, g_2 \in \mathrm{supp}(\theta^{(l)})$, so assuming $g_1 h g_2^{-1} \in F$ for some $h \in H_n \setminus \{1\}$ would imply $\kappa^{(l)}(g_1) = 0$ by (2-2) and (2-3), a contradiction. In particular we obtain property (a) for these functions.

Lastly, note that property (a) implies that any sum as in (b) can have at most $d + 1$ nonvanishing summands, and thus we may estimate for all $g \in \Gamma$ that

$$1 = \sum_{l=0}^{d} \sum_{h \in H_n} \theta^{(l)}(gh) \geq \sum_{l=0}^{d} \sum_{h \in H_n} \kappa^{(l)}(gh) \geq \left(\sum_{l=0}^{d} \sum_{h \in H_n} \theta^{(l)}(gh) \right) - (d+1)\varepsilon = 1 - (d+1)\varepsilon.$$

So let us yet again define new functions $\nu^{(l)} : \Gamma \to [0, 1]$ via

$$\nu^{(l)}(g) = \left(\sum_{l=0}^{d} \sum_{h \in H_n} \kappa^{(l)}(gh) \right)^{-1} \kappa^{(l)}(g).$$

By our previous calculation, we have

$$\kappa^{(l)} \leq \nu^{(l)} \leq \frac{1}{1 - (d+1)\varepsilon} \kappa^{(l)}.$$

For these functions, property (a) will still hold, while property (b) holds by construction. Moreover property (c) holds with regard to the tolerance

$$\eta_\varepsilon := \varepsilon + \frac{2(d+1)\varepsilon}{1 - (d+1)\varepsilon}$$

in place of ε. Since $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$, this means that the functions $\nu^{(l)}$ will have the desired property after rescaling ε. This shows our claim. □

Lemma 2.7. *Let G be a locally compact group and $H \subset G$ a closed and cocompact subgroup. Let μ be a left-invariant Haar measure on H:*

(i) *There exists a compactly supported continuous function $C : G \to [0, \infty)$ satisfying the equation*

$$\int_H C(gh)\, d\mu(h) = 1 \quad \textit{for all } g \in G.$$

(ii) *Assume furthermore that G is amenable. Let $\varepsilon > 0$ and let $K \subset G$ be a compact subset. Then there exists a function C as above with the additional property that*

$$\|C(g \cdot \_) - C\|_\infty \leq \varepsilon$$

for all $g \in K$.

Proof. (i): As H is a cocompact subgroup, there exists some compact set $K_H \subset G$ such that $G = K_H \cdot H$. By Urysohn–Tietze, we may choose a compactly supported continuous function $c : G \to [0, 1]$ with $c|_{K_H} = 1$. Define the compact set $K_c \subset H$ via

$$K_c = (K_H^{-1} \cdot \mathrm{supp}(c)) \cap H.$$

Then for every $g \in G$, there is some $h_0 \in H$ with $gh_0 \in K_H$. We have

$$\operatorname{supp}(c(gh_0 \cdot \_)) \cap H = ((gh_0)^{-1} \cdot \operatorname{supp}(c)) \cap H \subset K_c.$$

Thus, we get that

$$0 < \int_H c(gh)\, d\mu(h) = \int_H c(gh_0 h)\, d\mu(h) \leq \mu(K_c) < \infty.$$

Note that by the properties of the Haar measure, the assignment

$$\mathcal{I} : G \to (0, \infty), \quad g \mapsto \int_H c(gh)\, d\mu(h),$$

is H-periodic. Then the above computation shows that this assignment yields a well-defined, continuous function on G, which by H-periodicity and cocompactness of H can be viewed as a continuous function on the compact space G/H. Thus the image of this function is compact. In particular, its (pointwise) multiplicative inverse is also bounded and continuous. Let us define

$$C : G \to [0, \infty), \quad g \mapsto \mathcal{I}(g)^{-1} c(g).$$

Then this again yields a continuous function on G with compact support, but with the property that

$$\int_H C(gh)\, d\mu(h) = 1 \quad \text{for all } g \in G. \tag{2-4}$$

(ii): Let us now additionally assume that G is amenable. Let $\varepsilon > 0$ and $K \subset G$ be given as in the statement. Let ρ^G denote a right-invariant Haar measure on G. It follows from [Emerson and Greenleaf 1967] that we may find some compact set $J \subset G$ with $\rho^G(J) > 0$ such that $\rho^G(J \Delta (J \cdot K)) \leq \varepsilon / \|C\|_\infty \cdot \rho^G(J)$. Define $C' : G \to [0, \infty)$ via

$$C'(g) = \frac{1}{\rho^G(J)} \cdot \int_J C(xg)\, d\rho^G(x).$$

Clearly C' is yet another continuous function with compact support contained in $J^{-1} \cdot \operatorname{supp}(C)$. Given any element $g \in G$, we compute

$$\begin{aligned}
\int_H C'(gh)\, d\mu(h) &= \int_H \frac{1}{\rho^G(J)} \Big(\int_J C(xgh)\, d\rho^G(x) \Big)\, d\mu(h) \\
&= \frac{1}{\rho^G(J)} \int_J \Big(\int_H C(xgh)\, d\mu(h) \Big)\, d\rho^G(x) \\
&\overset{\text{(2-4)}}{=} \frac{1}{\rho^G(J)} \int_J 1\, d\rho^G(x) = 1.
\end{aligned}$$

Furthermore, we have for any $g_K \in K$ and $g \in G$ that

$$\begin{aligned}
|C'(g_K g) - C'(g)| &= \frac{1}{\rho^G(J)} \cdot \Big| \int_J C(x g_K g)\, d\rho^G(x) - \int_J C(xg)\, d\rho^G(x) \Big| \\
&\leq \frac{1}{\rho^G(J)} \cdot \|C\|_\infty \cdot \rho^G(J \Delta J g_K) \\
&\leq \varepsilon.
\end{aligned}$$

This shows the last part of the claim. ☐

Proof of Lemma 2.4. We first remark that since the box space $\square_S H_1$ has finite asymptotic dimension, it also has property A, and therefore H_1 is amenable; see [Nowak and Yu 2012, Theorems 4.3.6 and 4.4.6; Roe 2003, Proposition 11.39]. As H_1 is a discrete cocompact normal subgroup in G, we also see that G is amenable.

Let $\varepsilon > 0$ and $K \subset G$ be given. Then there exists a function $C : G \to [0, \infty)$ as in Lemma 2.7 for H_1 in place of H, with the property that

$$\|C(g\cdot\_) - C\|_\infty \leq \varepsilon \quad \text{for all } g \in K. \tag{2-5}$$

Let us denote the support of C by $S = \operatorname{supp}(C)$. As H_1 is discrete in G and S is compact, there exists a finite set $F \subset H_1$ with

$$h_1 \in F \quad \text{whenever } h_1 \in H \text{ and } S \cap Sh_1 \neq \varnothing. \tag{2-6}$$

Applying Lemma 2.6, there exists some n and finitely supported functions $\nu^{(0)}, \dots, \nu^{(d)} : H_1 \to [0, 1]$ satisfying the following properties:[1]

$$h_1 h_n h_2^{-1} \notin F \quad \text{for all } h_1, h_2 \in \operatorname{supp}(\nu^{(l)}) \text{ and } h_n \in H_n \setminus \{1\}, \tag{2-7}$$

$$1 = \sum_{l=0}^{d} \sum_{h_n \in H_n} \nu^{(l)}(h_1 h_n) \quad \text{for all } h_1 \in H_1. \tag{2-8}$$

We define $\mu^{(l)} : G \to [0, \infty)$ for $l = 0, \dots, d$ via

$$\mu^{(l)}(g) = \sum_{h_1 \in H_1} C(gh_1^{-1})\nu^{(l)}(h_1).$$

Since $\nu^{(l)}$ is finitely supported on H_1, we see that $\mu^{(l)}$ is a finite sum of continuous functions with compact support, and hence $\mu^{(l)} \in \mathcal{C}_c(G)$.

We claim that these functions have the desired properties. Let us verify (a), which is equivalent to the statement that

$$\mu^{(l)}(g) \cdot \mu^{(l)}(gh_n^{-1}) = 0 \quad \text{for all } g \in G \text{ and } h_n \in H_n \setminus \{1\}.$$

Fix an element $h_n \in H_n \setminus \{1\}$ for the moment. We compute

$$\begin{aligned} \mu^{(l)}(g) \cdot \mu^{(l)}(gh_n^{-1}) &= \sum_{h_1, h_2 \in H_1} C(gh_1^{-1})C(gh_n^{-1}h_2^{-1})\nu^{(l)}(h_1)\nu^{(l)}(h_2) \\ &= \sum_{h_1, h_2 \in H_1} C(gh_1^{-1})C(gh_2^{-1})\nu^{(l)}(h_1)\nu^{(l)}(h_2h_n^{-1}). \end{aligned}$$

We claim that each individual summand is zero. Indeed, suppose $h_1, h_2 \in H_1$ are such that

$$\nu^{(l)}(h_1)\nu^{(l)}(h_2h_n^{-1}) > 0.$$

[1] Note that we will reserve the notation h_1, h_2 for elements in H_1, whereas h_n will denote an element in the smaller subgroup H_n for $n > 2$.

Then $h_1 \in \mathrm{supp}(\nu^{(l)})$ and $h_2 \in \mathrm{supp}(\nu^{(l)}) \cdot h_n$, which implies $h_1 h_2^{-1} \notin F$ by (2-7). By our choice of F, we obtain

$$\mathrm{supp}(C(\_ \cdot h_1^{-1})) \cap \mathrm{supp}(C(\_ \cdot h_2^{-1})) \subseteq Sh_1 \cap Sh_2 = (Sh_1 h_2^{-1} \cap S) \cdot h_2 \overset{(2\text{-}6)}{=} \varnothing,$$

and in particular $C(gh_1^{-1})C(gh_2^{-1}) = 0$. This finishes the proof that each summand of the above sum is zero and shows property (a).

Let us now show property (b). We calculate for every $g \in G$ that

$$\begin{aligned} \sum_{l=0}^{d} \sum_{h_n \in H_n} \mu^{(l)}(gh_n) &= \sum_{l=0}^{d} \sum_{h_n \in H_n} \sum_{h_1 \in H_1} C(gh_n h_1^{-1}) \nu^{(l)}(h_1) \\ &= \sum_{l=0}^{d} \sum_{h_n \in H_n} \sum_{h_1 \in H_1} C(gh_1^{-1}) \nu^{(l)}(h_1 h_n) \\ &= \sum_{h_1 \in H_1} C(gh_1^{-1}) \Big(\sum_{l=0}^{d} \sum_{h_n \in H_n} \nu^{(l)}(h_1 h_n) \Big) \\ &\overset{(2\text{-}8)}{=} \sum_{h_1 \in H_1} C(gh_1) \\ &\overset{\text{Lem. 2.7}}{=} 1. \end{aligned}$$

Let us now turn to (c). Given any $g \in G$ and $g_K \in K$, we compute

$$\begin{aligned} |\mu^{(l)}(g_K g) - \mu^{(l)}(g)| &= \Big| \sum_{h_1 \in H_1} (C(g_K g h_1^{-1}) - C(gh_1^{-1})) \nu^{(l)}(h_1) \Big| \\ &\overset{(2\text{-}8)}{\leq} \sup_{h_1 \in H_1} |C(g_K g h_1^{-1}) - C(gh_1^{-1})| \\ &\leq \|C(g_K \cdot \_) - C\|_\infty \overset{(2\text{-}6)}{\leq} \varepsilon. \end{aligned}$$

As $g \in G$ was arbitrary, this finishes the proof. □

Remark. Let G be a locally compact group and $H \subset G$ a closed, cocompact subgroup. For any C*-algebra A, we may naturally view $\mathcal{C}(G/H, A)$ as a C*-subalgebra of (right-)H-periodic functions inside $\mathcal{C}_b(G, A)$ by assigning a function f to the function f' given by $f'(g) = f(gH)$.

In what follows, we will briefly establish a technical result that allows one to perturb *approximately* H-periodic functions in $\mathcal{C}_b(G, A)$ to *exactly* H-periodic functions in a systematic way.

Lemma 2.8. *Let G be a locally compact group and $H \subset G$ a closed, cocompact subgroup. Let A be a* C**-algebra. Then there exists a conditional expectation $E : \mathcal{C}_b(G, A) \to \mathcal{C}(G/H, A)$ with the following property.*

For every $\varepsilon > 0$ and compact set $K \subset G$, there exists $\delta > 0$ and a compact set $J \subset H$ such that the following holds:

If $f \in \mathcal{C}_b(G, A)$ satisfies

$$\max_{g\in K} \max_{h\in J} \|f(g) - f(gh)\| \leq \delta,$$

then

$$\|f - E(f)\|_{\infty,K} \leq \varepsilon.$$

Proof. Let μ be a left-invariant Haar measure on H. Let $C \in \mathcal{C}_c(G)$ be a function as in Lemma 2.7. Then we define

$$E : \mathcal{C}_b(G, A) \to \mathcal{C}(G/H, A), \quad E(f)(gH) = \int_H C(gh) f(gh)\, d\mu(h).$$

Since C is compactly supported and the Haar measure μ is left-invariant, it is clear that E is well-defined and indeed a conditional expectation. Let $\varepsilon > 0$ and $K \subset G$ be given. Let S be the compact support of C. Then the set $J := (K^{-1}S) \cap H$ is compact in H with the property that

$$g \in K \text{ and } gh \in S \quad \Longrightarrow \quad h \in J \tag{2-9}$$

for all $h \in H$. Set

$$\delta = \frac{\varepsilon}{1 + \mu(J) \cdot \|C\|_\infty}.$$

For every $f \in \mathcal{C}_b(G, A)$ with

$$\max_{g\in K} \max_{h\in J} \|f(g) - f(gh)\| \leq \delta,$$

it follows for every $g \in K$ that

$$\begin{aligned}\|f(g) - E(f)(gH)\| &= \left\| \left(\int_H C(gh)\, d\mu(h) \right) f(g) - \int_H C(gh) f(gh)\, d\mu(h) \right\| \\ &\overset{(2\text{-}9)}{=} \left\| \int_J C(gh)(f(g) - f(gh))\, d\mu(h) \right\| \\ &\leq \mu(J) \cdot \|C\|_\infty \cdot \delta \leq \varepsilon.\end{aligned}$$

This shows our claim. □

Corollary 2.9. *Let G be a locally compact group and $H \subset G$ a closed, cocompact subgroup. Let A and B be two* C*-*algebras. Then for every $\varepsilon > 0$, $F \Subset B$ and compact set $K \subset G$, there exists $\delta > 0$ and a compact set $J \subset H$ such that the following holds*:

If $\Theta : B \to \mathcal{C}_b(G, A)$ is a c.p.c. map with

$$\max_{g\in K} \max_{h\in J} \|\Theta(b)(g) - \Theta(b)(gh)\| \leq \delta \quad \textit{for all } b \in F,$$

then there exists a c.p.c. map $\Psi : B \to \mathcal{C}(G/H, A)$ with

$$\max_{g\in K} \|\Psi(b)(gH) - \Theta(b)(g)\| \leq \varepsilon \quad \textit{for all } b \in F.$$

Proof. Let $E : \mathcal{C}_b(G, A) \to \mathcal{C}(G/H, A)$ be a conditional expectation as in Lemma 2.8. Given a triple (ε, F, K), choose $\delta > 0$ and $J \subset H$ so that the property in Lemma 2.8 holds for all $f \in \mathcal{C}_b(G, A)$ with respect to the pair (ε, K). Then we can directly conclude that if Θ is a map as in the statement, then $\Psi = E \circ \Theta$ has the desired property. □

3. Systems generated by order-zero maps with commuting ranges

The following notation and observations are [Hirshberg et al. 2017, Lemma 6.6] and originate in [Hirshberg et al. 2015, Section 5].

Notation 3.1. Let $D_1, \dots, D_n$ be finitely many unital C*-algebras. For $t \in [0, 1]$ and $j = 1, \dots, n$, we define

$$D_j^{(t)} := \begin{cases} D_j, & t > 0, \\ \mathbb{C} \cdot \mathbf{1}_{D_j}, & t = 0. \end{cases}$$

Given moreover a tuple $\vec{t} = (t_1, \dots, t_n) \in [0, 1]^n$, let us define

$$D^{(\vec{t})} := D_1^{(t_1)} \otimes_{\max} D_2^{(t_2)} \otimes_{\max} \cdots \otimes_{\max} D_n^{(t_n)}.$$

Consider the simplex

$$\Delta^{(n)} := \{\vec{t} \in [0, 1]^n \mid t_1 + \cdots + t_n = 1\}$$

and set

$$\mathcal{E}(D_1, \dots, D_n) := \{f \in \mathcal{C}(\Delta^{(n)}, D_1 \otimes_{\max} \cdots \otimes_{\max} D_n) | f(\vec{t}) \in D^{(\vec{t})}\}.$$

In the case that $D_j = D$ are all the same C*-algebra, we will write

$$\mathcal{E}(D_1, \dots, D_n) =: \mathcal{E}(D, n)$$

instead. For every $j = 1, \dots, n$, we will consider the canonical c.p.c. order-zero map

$$\eta_j : D_j \to \mathcal{E}(D_1, \dots, D_n)$$

given by

$$\eta_j(d_j)(\vec{t}) = t_j \cdot (\mathbf{1}_{D_1} \otimes \cdots \otimes \mathbf{1}_{D_{j-1}} \otimes d_j \otimes \mathbf{1}_{D_{j+1}} \otimes \cdots \otimes \mathbf{1}_{D_n}).$$

One easily checks that the ranges of the maps η_j generate $\mathcal{E}(D_1, \dots, D_n)$ as a C*-algebra.

Proposition 3.2. *Let $D_1, \dots, D_n$ be unital C*-algebras. Then the C*-algebra $\mathcal{E}(D_1, \dots, D_n)$, together with the c.p.c. order-zero maps $\eta_j : D_j \to \mathcal{E}(D_1, \dots, D_n)$, satisfies the following universal property:*

If B is any unital C-algebra and $\psi_j : D_j \to B$ for $j = 1, \dots, n$ are c.p.c. order-zero maps with pairwise commuting ranges and*

$$\psi_1(\mathbf{1}_{D_1}) + \cdots + \psi_n(\mathbf{1}_{D_n}) = \mathbf{1}_B,$$

then there exists a unique unital $*$-homomorphism $\Psi : \mathcal{E}(D_1, \dots, D_n) \to B$ such that $\Psi \circ \eta_j = \psi_j$ for all $j = 1, \dots, n$.

Notation 3.3. Let G be a second-countable, locally compact group. Let $D_1, \dots, D_n$ be unital C*-algebras with continuous actions $\alpha^{(j)} : G \curvearrowright D_j$ for $j = 1, \dots, n$. Then the G-action on $\mathcal{C}(\Delta^{(n)}, D_1 \otimes_{\max} \cdots \otimes_{\max} D_n)$ defined fibrewise by $\alpha^{(1)} \otimes_{\max} \cdots \otimes_{\max} \alpha^{(n)}$ restricts to a well-defined action

$$\mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)}) : G \curvearrowright \mathcal{E}(D_1, \dots, D_n).$$

We will again write $\mathcal{E}(\alpha, n) := \mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)})$ in the special case that all $(D_j, \alpha^{(j)}) = (D, \alpha)$ are the same C*-dynamical system.

Remark 3.4. By the universal property in Proposition 3.2, the G-action $\mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)})$ defined in Notation 3.3 is uniquely determined by the identity $\mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)})_g \circ \eta_j = \eta_j \circ \alpha_g^{(j)}$ for all $j = 1, \dots, n$ and $g \in G$.

This immediately allows us obtain the following equivariant version of Proposition 3.2 as a consequence: Let B be any unital C*-algebra with an action $\beta : G \curvearrowright B$. If $\psi_j : (D_j, \alpha^{(j)}) \to (B, \beta)$ are equivariant c.p.c. order-zero maps with pairwise commuting ranges and $\psi_1(\mathbf{1}_{D_1}) + \cdots + \psi_n(\mathbf{1}_{D_n}) = \mathbf{1}_B$, then there exists a unique unital equivariant $*$-homomorphism

$$\Psi : (\mathcal{E}(D_1, \dots, D_n), \mathcal{E}(\alpha^{(1)}, \dots, \alpha^{(n)})) \to (B, \beta)$$

satisfying $\Psi \circ \eta_j = \psi_j$ for all $j = 1, \dots, n$.

Remark 3.5. Let us now also convince ourselves of a different natural way to view the C*-algebras from Notation 3.1.

For this, let us first consider the case $n = 2$, so we have two unital C*-algebras D_1 and D_2. Notice that $[0, 1]$ is naturally homeomorphic to the simplex $\Delta^{(2)} = \{(t_1, t_2) \in [0, 1]^2 \mid t_1 + t_2 = 1\}$ via the assignment $t \mapsto (t, t - 1)$. In this way we may see that there is a natural isomorphism

$$\begin{aligned} \mathcal{E}(D_1, D_2) &\overset{\text{def}}{=} \{f \in \mathcal{C}(\Delta^{(2)}, D_1 \otimes_{\max} D_2) \mid f(0, 1) \in D_1 \otimes \mathbf{1},\ f(1, 0) \in \mathbf{1} \otimes D_2\} \\ &\cong \{f \in \mathcal{C}([0, 1], D_1 \otimes_{\max} D_2) \mid f(0) \in D_1 \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes D_2\} \\ &=: D_1 \star D_2. \end{aligned}$$

In particular, we see that the notation $\mathcal{E}(D_1, D_2)$ is consistent with [Szabó 2018c, Definition 5.1]. As pointed out in Remark 5.2 of the same paper, the assignment $(D_1, D_2) \mapsto \mathcal{E}(D_1, D_2)$ on pairs of unital C*-algebras therefore generalizes the join construction for pairs of compact spaces, which gives rise to the notation $D_1 \star D_2$.

Let now $n \geq 2$ and let $D_1, \dots, D_{n+1}$ be unital C*-algebras. The simplex $\Delta^{(n+1)}$ is homeomorphic to $[0, 1] \times \Delta^{(n)}$ via the assignment

$$(t_1, \vec{t}) \mapsto \begin{cases} (1, \vec{t}), & t_1 = 0, \\ (1 - t_1, \vec{t}/(1 - t_1)), & t_1 \neq 0 \end{cases}$$

for $(\vec{t}, t_{n+1}) \in \Delta^{(n+1)}$. Keeping this in mind, we see that there is a natural map

$$\Phi : D_1 \star \mathcal{E}(D_2, \dots, D_{n+1}) \to \mathcal{E}(D_1, \dots, D_{n+1})$$

given by[2]

$$\Phi(f)(t_1, \vec{t}) = \begin{cases} f(1)(\vec{t}), & t_1 = 0, \\ f(1 - t_1)(\vec{t}/(1 - t_1)), & t_1 \neq 0 \end{cases}$$

for $(t_1, \vec{t}) \in \Delta^{(n+1)}$. It is a simple exercise to see that this is a well-defined isomorphism. This shows that it makes sense to view the C*-algebra $\mathcal{E}(D_1, \dots, D_n)$ as the n-fold join $D_1 \star \cdots \star D_n$. We can also

[2]The reader should keep in mind that an element f in the domain is a continuous function on $[0, 1]$ whose values are in turn (certain) continuous functions from $\Delta^{(n)}$ to the tensor product $D_1 \otimes_{\max} \cdots \otimes_{\max} D_{n+1}$.

observe that this isomorphism is natural in each C*-algebra, and therefore becomes equivariant as soon as we equip each C*-algebra D_j with an action $\alpha^{(j)}$ of some group G.

Henceforth, we will in particular write

$$D^{\star n} := \mathcal{E}(D, n) \quad \text{and} \quad \alpha^{\star n} := \mathcal{E}(\alpha, n)$$

for a unital C*-algebra D and some group action $\alpha : G \curvearrowright D$.

Remark 3.6. By the definition of the join of two C*-algebras D_1 and D_2, there is a natural short exact sequence

$$0 \longrightarrow \mathcal{C}_0(0,1) \otimes D_1 \otimes_{\max} D_2 \longrightarrow D_1 \star D_2 \longrightarrow D_1 \oplus D_2 \longrightarrow 0.$$

Given some $n \geq 1$ and a unital C*-algebra D, we have $D^{\star n+1} \cong D \star (D^{\star n})$, and therefore a special case of the above yields the short exact sequence

$$0 \longrightarrow \mathcal{C}_0(0,1) \otimes D \otimes_{\max} D^{\star n} \longrightarrow D^{\star n+1} \longrightarrow D \oplus D^{\star n} \longrightarrow 0.$$

Again by naturality, we note that this short exact sequence is automatically equivariant if we additionally equip D with a group action.

We now come to the main observation about C*-dynamical systems arising in this fashion, which will be crucial in proving our main result:

Lemma 3.7. *Let G be a second-countable, locally compact group. Let A be a separable, unital C*-algebra with an action $\alpha : G \curvearrowright A$. Suppose that $\gamma : G \curvearrowright \mathcal{D}$ is a semi-strongly self-absorbing and unitarily regular action. If α is γ-absorbing, then so is the action $\alpha^{\star n} : G \curvearrowright A^{\star n}$ for all $n \geq 2$.*

Proof. This follows directly from Remark 3.6 and Theorem 1.10 by induction. □

Remark 3.8. It ought to be mentioned that Lemma 3.7 does not depend in any way on the fact that one considers the n-fold join over the same C*-algebra and the same action. The analogous statement is valid for more general joins of the form

$$\alpha^{(1)} \star \cdots \star \alpha^{(n)} : G \curvearrowright A_1 \star \cdots \star A_n$$

by virtually the same argument.

In fact, by putting in a bit more work, one could likely prove an equivariant version of [Hirshberg et al. 2007, Theorem 4.6] for $\mathcal{C}_0(X)$-G-C*-algebras with $\dim(X) < \infty$ whose fibres absorb a given semi-strongly self-absorbing and unitarily regular action. This would contain Lemma 3.7 as a special case since the G-C*-algebra $A_1 \star \cdots \star A_n$ is in fact a $\mathcal{C}(\Delta^{(n)})$-$G$-C*-algebra with each fibre being isomorphic to some finite tensor product of the A_j. We will never need this level of generality within this paper, however.

4. Rokhlin dimension with commuting towers

The following notion generalizes analogous definitions made in [Hirshberg et al. 2015; 2017; Szabó et al. 2017; Gardella 2017].

Definition 4.1 (cf. [Hirshberg et al. 2017, Definition 4.1]). Let G be a second-countable, locally compact group. Let $\alpha : G \curvearrowright A$ be an action on a separable C*-algebra:

(i) Let $H \subset G$ be a closed, cocompact subgroup. The Rokhlin dimension of α with commuting towers relative to H, denoted by $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, H)$, is the smallest natural number d such that there exist equivariant c.p.c. order-zero maps

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(G/H), G\text{-shift}) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$$

with pairwise commuting ranges such that $\mathbf{1} = \varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1})$.

(ii) If $\mathcal{S} = (G_k)_k$ denotes a decreasing sequence of closed, cocompact subgroups, then we define the Rokhlin dimension of α with commuting towers relative to $\mathcal{S}$ via

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, \mathcal{S}) = \sup_{k\in\mathbb{N}} \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, G_k).$$

(iii) Let $N \subset G$ be any closed, normal subgroup. The Rokhlin dimension of α with commuting towers relative to N is defined as

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, N) := \sup\{\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, H) \mid H \subseteq G \text{ closed, cocompact,} N \subseteq H\}.$$

(iv) Lastly, the Rokhlin dimension of α with commuting towers is defined as

$$\begin{aligned}\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) &:= \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, \{1\}) \\ &= \sup\{\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, H) \mid H \subseteq G \text{ closed, cocompact}\}.\end{aligned}$$

We note that, even though the second half of Definition 4.1 always makes sense, these concepts are not expected to be of any practical use when G (or G/N) is not assumed to have enough closed cocompact subgroups, or to admit at least some residually compact approximation.

Notation 4.2. Let G be a second-countable, locally compact group. Given a decreasing sequence $\mathcal{S} = (G_k)_k$ of closed, cocompact subgroups, we will define

$$G/\mathcal{S} = \varprojlim G/G_k.$$

This is a metrizable, compact space,[3] which carries a natural continuous G-action induced by the left G-shift on each building block G/G_k; in particular we will call the resulting action also just the G-shift and denote it by

$$\sigma^{\mathcal{S}} : G \curvearrowright G/\mathcal{S}.$$

In the sequel, we will adopt the perspective of the associated G-C*-dynamical system, which is given as the equivariant inductive limit

$$\mathcal{C}(G/\mathcal{S}) = \varinjlim \mathcal{C}(G/G_k).$$

[3]This construction generalizes the profinite completion of a discrete residually finite group along a chosen separating sequence of normal subgroups of finite index.

We will moreover consider $\mathcal{C}(G/\mathcal{S})^{\star n}$ for $n \geq 2$. With some abuse of terminology, we will use the term "G-shift" also to refer to the canonical action on this C*-algebra (or the underlying space) that is induced by the n-fold tensor products of the G-shift on each fibre.

Lemma 4.3. *Let G be a second-countable, locally compact group. Let $\alpha : G \curvearrowright A$ be an action on a separable* C*-*algebra. Let $\mathcal{S} = (G_k)_k$ be a decreasing sequence of closed, cocompact subgroups. Let $d \geq 0$ be some natural number. Then the following are equivalent*:

(i) $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, \mathcal{S}) \leq d$.

(ii) *There exist equivariant c.p.c. order-zero maps*

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(G/\mathcal{S}), G\text{-shift}) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$$

with pairwise commuting ranges such that $\mathbf{1} = \varphi^{(0)}(\mathbf{1}) + \cdots + \varphi^{(d)}(\mathbf{1})$.

(iii) *There exists a unital G-equivariant $*$-homomorphism*

$$(\mathcal{C}(G/\mathcal{S})^{\star(d+1)}, G\text{-shift}) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty).$$

(iv) *The first-factor embedding*

$$\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \to (A \otimes \mathcal{C}(G/\mathcal{S})^{\star(d+1)}, \alpha \otimes (G\text{-shift}))$$

is G-equivariantly sequentially split.

Proof. The equivalence (i)⇔(ii) follows from a standard reindexing trick such as Kirchberg's ε-test [2006, Lemma A.1], using the equivariant inductive limit structure of $\mathcal{C}(G/\mathcal{S})$ as pointed out in Notation 4.2. We will leave the details to the reader.

The equivalence (ii)⇔(iii) is a direct consequence of Proposition 3.2 and Remark 3.5, and the equivalence (iii)⇔(iv) is a direct consequence of [Barlak and Szabó 2016, Lemma 4.2]. □

The purpose of this section is to prove the following theorem, which can be regarded as the main result of the paper. Some of its nontrivial applications will be discussed in the subsequent sections. See in particular Corollary 5.1 for a possibly more accessible special case of this theorem.

Theorem 4.4. *Let G be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Denote by $\pi_N : G \to G/N$ the quotient map. Let $\mathcal{S}_1 = (H_k)_k$ be a residually compact approximation of G/N, and set $G_k = \pi_N^{-1}(H_k)$ for all $k \in \mathbb{N}$ and $\mathcal{S}_0 = (G_k)_k$. Let A be a separable* C*-*algebra and $\mathcal{D}$ a strongly self-absorbing* C*-*algebra. Let $\alpha : G \curvearrowright A$ be an action and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing, unitarily regular action. Suppose that for the restrictions to the N-actions, we have $\alpha|_N \simeq_{\mathrm{cc}} (\alpha \otimes \gamma)|_N$. If*

$$\mathrm{asdim}(\Box_{\mathcal{S}_1} H_1) < \infty \quad \textit{and} \quad \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, \mathcal{S}_0) < \infty,$$

then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.

We note that Theorem A is a direct consequence of this result. The hypothesis that G/N has some discrete, normal, residually finite, cocompact subgroup admitting a box space with finite asymptotic dimension means that there is choice for $\mathcal{S}_1$ as required by the above statement. The hypothesis that α has finite Rokhlin dimension with commuting towers means that the value $\dim^c_{\mathrm{Rok}}(\alpha, \mathcal{S}_0)$ has a finite uniform upper bound, for any possible choice of $\mathcal{S}_1$. Hence the statement of Theorem A follows.

The proof of Theorem 4.4 will occupy the rest of this section. The first and most difficult step is to convince ourselves of a very special case of Theorem 4.4, which involves the technical preparation below and from Section 2.

For convenience, we isolate the following lemma, which is a consequence of Proposition 1.11, the Winter–Zacharias structure theorem for order-zero maps, along with the Choi–Effros lifting theorem [1976]; see also [Winter and Zacharias 2009, Section 3].

Lemma 4.5. *Let G be a second-countable, locally compact group. Let A be a separable C^*-algebra and B a separable, unital and nuclear C^*-algebra. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions. Let $\kappa : (B, \beta) \to (A_{\infty,\alpha}, \alpha_\infty)$ be an equivariant c.p.c. order-zero map. Then κ can be represented by a sequence of c.p.c. maps $\kappa_n : B \to A$ satisfying*

(a) $\|\kappa_n(xy)\kappa(\mathbf{1}) - \kappa_n(x)\kappa_n(y)\| \to 0$,

(b) $\max_{g\in K} \|(\kappa_n \circ \gamma_g)(x) - (\alpha_g \circ \kappa_n)(x)\| \to 0$,

for all $x, y \in B$ and compact subsets $K \subset G$.

The proof of the following is based on a standard reindexing trick. In the short proof below, precise references are provided for completeness, although we note that this might not be the most elegant or direct way to show these statements.

Lemma 4.6. *Let G be a second-countable, locally compact group. Suppose that $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$, and $\gamma : G \curvearrowright \mathcal{D}$ are actions on separable C^*-algebras. Assume furthermore that $\mathcal{D}$ is unital, that γ is semi-strongly self-absorbing, and that $\beta \simeq_{\mathrm{cc}} \beta \otimes \gamma$:*

(i) *Suppose that there exists an equivariant $*$-homomorphism $(A, \alpha) \to (B, \beta)$ that is G-equivariantly sequentially split. Then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.*

(ii) *Suppose that B is unital and that there exists an equivariant and unital $*$-homomorphism from (B, β) to $(F_{\infty,\alpha}, \tilde{\alpha}_\infty)$. Then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.*

Proof. (i): By Theorem 1.8, the statement $\beta \simeq_{\mathrm{cc}} \beta \otimes \gamma$ is equivalent to the equivariant first-factor embedding

$$\mathrm{id}_B \otimes \mathbf{1} : (B, \beta) \to (B \otimes \mathcal{D}, \beta \otimes \gamma)$$

being sequentially split. Let $\varphi : (A, \alpha) \to (B, \beta)$ be sequentially split. By [Barlak and Szabó 2016, Proposition 3.7], the composition $\varphi \otimes \mathbf{1}_{\mathcal{D}} = (\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{D}}) \circ \varphi$ is also sequentially split. However, we also have

$$\varphi \otimes \mathbf{1}_{\mathcal{D}} = (\varphi \otimes \mathrm{id}_{\mathcal{D}}) \circ (\mathrm{id}_A \otimes \mathbf{1}_{\mathcal{D}}),$$

which implies that $\mathrm{id}_A \otimes \mathbf{1}_{\mathcal{D}}$ is also sequentially split. This implies the claim that $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \gamma$.

(ii): By [Barlak and Szabó 2016, Lemma 4.2], it follows that the embedding

$$\mathrm{id}_A \otimes \mathbf{1}_B : (A, \alpha) \to (A \otimes_{\max} B, \alpha \otimes \beta)$$

is sequentially split. Since we assumed that β is γ-absorbing, so is $\alpha \otimes \beta$, and so the claim arises as a special case of (i). □

The following is a special case of Theorem 4.4, as the process of tensorially stabilizing any action $\alpha : G \curvearrowright A$ with $(\mathcal{C}(G/\mathcal{S}), \sigma^{\mathcal{S}})$ causes the Rokhlin dimension relative to $\mathcal{S}$ to collapse to zero by definition. This explains why the statement below makes no explicit reference to Rokhlin dimension. Its proof is by far the most technical part of this paper:

Lemma 4.7. *Let G be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Denote by $\pi_N : G \to G/N$ the quotient map. Let $\mathcal{S}_1 = (H_k)_k$ be a residually compact approximation of G/N, and set $G_k = \pi_N^{-1}(H_k)$ for all $k \in \mathbb{N}$ and $\mathcal{S}_0 = (G_k)_k$. Let A be a separable* C**-algebra and $\mathcal{D}$ a strongly self-absorbing* C**-algebra. Let $\alpha : G \curvearrowright A$ be an action and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing, unitarily regular action. Suppose that for the restrictions to the N-actions, we have $\alpha|_N \simeq_{\mathrm{cc}} (\alpha \otimes \gamma)|_N$. If $\mathrm{asdim}(\square_{\mathcal{S}_1} H_1) < \infty$, then the G-action*

$$\sigma^{\mathcal{S}_0} \otimes \alpha : G \curvearrowright \mathcal{C}(G/\mathcal{S}_0) \otimes A$$

is γ-absorbing.

Proof. Set $d := \mathrm{asdim}(\square_{\mathcal{S}_1} H_1)$. Let

$$\tilde{\kappa} : (\mathcal{D}, \gamma|_N) \to (F_{\infty,\alpha|_N}(A), \tilde{\alpha}_\infty|_N)$$

be an N-equivariant, unital $*$-homomorphism. Using [Szabó 2018c, Example 4.4 and Proposition 4.5], we may choose an equivariant c.p.c. order-zero map

$$\kappa : (\mathcal{D}, \gamma|_N) \to (A_{\infty,\alpha|_N} \cap A', \alpha_\infty|_N)$$

that lifts $\tilde{\kappa}$.

Consider a sequence of c.p.c. maps $\kappa_n : B \to A$ lifting κ as in Lemma 4.5. Let us choose finitely many subsequences $\kappa_n^{(l)} : B \to A$ of the maps κ_n for $l = 0, \dots, d$ so that, using Lemma 4.5, each sequence $\kappa_n^{(l)}$ has the following properties for all $a \in A$, $b, b_1, b_2 \in \mathcal{D}$ and compact sets $L \subseteq N$:

$$\|[\kappa_n^{(l)}(b), a]\| \to 0, \tag{4-1}$$

$$\|\kappa_n^{(l)}(b_1 b_2)\kappa_n^{(l)}(\mathbf{1}) - \kappa_n^{(l)}(b_1)\kappa_n^{(l)}(b_2)\| \to 0, \tag{4-2}$$

$$\|(\kappa_n^{(l)}(\mathbf{1}) - \mathbf{1}) \cdot a\| \to 0, \tag{4-3}$$

$$\max_{r \in L} \|(\kappa_n^{(l)} \circ \gamma_r)(b) - (\alpha_r \circ \kappa_n^{(l)})(b)\| \to 0, \tag{4-4}$$

and additionally one has for every compact set $K \subseteq G$ that

$$\max_{g \in K} \|[\kappa_n^{(l_1)}(b_1), (\alpha_g \circ \kappa_n^{(l_2)})(b_2)]\| \to 0 \quad \text{for all } l_1 \neq l_2. \tag{4-5}$$

Let now $\varepsilon > 0$ be a fixed parameter and $1_G \in K \subseteq G$ a fixed compact set. Apply Lemma 2.4 and find k and compactly supported functions $\mu^{(0)}, \dots, \mu^{(d)} \in \mathcal{C}_c(G/N)$, so that for every $l = 0, \dots, d$ we have

$$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)}) \cdot h = \varnothing \quad \text{for all } h \in H_k \setminus \{1\}, \tag{4-6}$$

$$\sum_{l=0}^{d} \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) = 1 \quad \text{for all } g \in G, \tag{4-7}$$

$$\|\mu^{(l)}(\pi_N(g) \cdot \_) - \mu^{(l)}\|_\infty \le \varepsilon \quad \text{for all } g \in K \cup K^{-1}. \tag{4-8}$$

The group H_k is discrete, so we may choose a cross-section $\sigma : H_k \to G_k = \pi_N^{-1}(H_k) \subseteq G$. For each $l = 0, \dots, d$, consider the sequence of c.p.c. maps

$$\Theta_n^{(l)} : \mathcal{D} \to \mathcal{C}_b(G, A)$$

given by

$$\Theta_n^{(l)}(b)(g) = \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) \cdot (\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{g\sigma(h)}^{-1})(b). \tag{4-9}$$

This sum is well-defined because the compact support of the function $\mu^{(l)}$ on G/N meets a set of the form $\pi_N(g) \cdot H_k$ at most once according to (4-6).

We wish to show that given an element $b \in \mathcal{D}$, the functions $\Theta_n^{(l)}(b)$ are approximately G_k-periodic on large compact sets. This is so that we may apply Corollary 2.9 in order to approximate the maps $\Theta_n^{(l)}$ by other maps going into $\mathcal{C}(G/G_k, A)$.

Let $K_{H_k} \subseteq G_k$ and $K_G \subseteq G$ be two compact sets. As H_k is discrete, we observe two facts. First, there exists a compact set $K_N \subseteq N$ and a finite set $1 \in F_k \Subset H_k$ with

$$K_{H_k} \subset \sigma(F_k) \cdot K_N. \tag{4-10}$$

Second, by possibly enlarging F_k if necessary, we may assume by (4-6) that also

$$\mu^{(l)}(\pi_N(g)h) > 0 \quad \text{implies} \quad h \in F_k \qquad \text{for all } g \in K_G. \tag{4-11}$$

Define also

$$K_N' = \bigcup_{h_0, h \in F_k} \sigma(h_0) \cdot K_N \cdot \sigma(h_0^{-1}h)\sigma(h)^{-1} \subseteq N, \tag{4-12}$$

$$K_N'' = \bigcup_{h \in F_k} \sigma(h)^{-1} \cdot K_N' \cdot \sigma(h) \subseteq N. \tag{4-13}$$

As N is a normal subgroup and σ is a cross-section for the quotient map π_N, it follows that these are compact subsets in N.

We compute for all $l = 0, \dots, d$, $b \in \mathcal{D}$, $g \in K_G$, $h_0 \in F_k$ and $r \in K_N'$ that

$$\begin{aligned}
&\|(\alpha_{g\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{g\sigma(h_0)}^{-1})(b) - (\alpha_{gr\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{gr\sigma(h_0)}^{-1})(b)\| \\
&\quad = \|(\alpha_{\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) - (\alpha_{r\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{r\sigma(h_0)}^{-1})(\gamma_g^{-1}(b))\| \\
&\quad = \|(\alpha_{\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) - (\alpha_{\sigma(h_0)} \circ \alpha_{\sigma(h_0)^{-1}r\sigma(h_0)} \circ \kappa_n^{(l)} \circ \gamma_{\sigma(h_0)^{-1}r\sigma(h_0)}^{-1} \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b))\| \\
&\quad \overset{(4\text{-}13)}{\le} \max_{g \in K_G} \max_{s \in K_N''} \|(\alpha_s \circ \kappa_n^{(l)} \circ \gamma_s^{-1})(\gamma_{g\sigma(h_0)}^{-1}(b)) - \kappa_n^{(l)}(\gamma_{g\sigma(h_0)}^{-1}(b))\| \\
&\quad \overset{(4\text{-}4)}{\longrightarrow} 0 \quad (\text{uniformly on } K_G, K_N').
\end{aligned}$$

It thus follows for all $l=0,\dots,d,\ b\in\mathcal{D},\ g\in K_G,\ h_0\in F_k$ and $r\in K_N$ that

$$
\begin{aligned}
&\|\Theta_n^{(l)}(b)(g)-\Theta_n^{(l)}(b)(g\sigma(h_0)r)\|\\
&\overset{(4\text{-}9),\,(4\text{-}11)}{=}\Big\|\sum_{h_1\in F_k}\mu^{(l)}(\pi_N(g)h_1)\cdot(\alpha_{g\sigma(h_1)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_1)})(b)\\
&\qquad\qquad -\sum_{h_2\in h_0^{-1}F_k}\mu^{(l)}(\pi_N(g)h_0h_2)\cdot(\alpha_{g\sigma(h_0)r\sigma(h_2)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_0)r\sigma(h_2)})(b)\Big\|\\
&=\Big\|\sum_{h_1\in F_k}\mu^{(l)}(\pi_N(g)h_1)\cdot(\alpha_{g\sigma(h_1)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_1)})(b)\\
&\qquad\qquad -\sum_{h_2\in F_k}\mu^{(l)}(\pi_N(g)h_2)\cdot(\alpha_{g\sigma(h_0)r\sigma(h_0^{-1}h_2)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_0)r\sigma(h_0^{-1}h_2)})(b)\Big\|\\
&\overset{(4\text{-}6)}{=}\max_{h\in F_k}\|(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h)})(b)-(\alpha_{g\sigma(h_0)r\sigma(h_0^{-1}h)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)r\sigma(h_0^{-1}h)})(b)\|\\
&=\max_{h\in F_k}\|(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h)})(b)-(\alpha_{g\sigma(h_0)r\sigma(h_0^{-1}h)\sigma(h)^{-1}\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h_0)r\sigma(h_0^{-1}h)\sigma(h)^{-1}\sigma(h)})(b)\|\\
&\overset{(4\text{-}12)}{=}\max_{h\in F_k}\max_{s\in K_N'}\|(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{g\sigma(h)})(b)-(\alpha_{gs\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma^{-1}_{gs\sigma(h)})(b)\|\\
&\longrightarrow 0\quad(\text{uniformly on } K_G,\,K_N).
\end{aligned}
$$

Here we have used (4-6) in the third equality in the sense that $\mu^{(l)}(\pi_N(g)h)$ is nonzero for a unique element $h\in F_k$. By (4-10) we get for all $b\in\mathcal{D}$ that

$$\max_{g\in K_G}\max_{t\in K_{H_k}}\|\Theta_n^{(l)}(b)(g)-\Theta_n^{(l)}(b)(gt)\|\overset{(4\text{-}10)}{\le}\max_{g\in K_G}\max_{h_0\in F_k}\max_{r\in K_N}\|\Theta_n^{(l)}(b)(g)-\Theta_n^{(l)}(b)(g\sigma(h_0)r)\|\longrightarrow 0.$$

Since $K_G\subseteq G$ and $K_{H_k}\subseteq G_k$ were arbitrary compact sets, we are in the position to apply Corollary 2.9. As $\mathcal{D}$ is separable, it follows for every $l=0,\dots,d$ that there exists a sequence of c.p.c. maps

$$\Psi_n^{(l)}:B\to\mathcal{C}(G/G_k,A)$$

so that for every compact set $K_G\subseteq G$ and $b\in\mathcal{D}$, we have

$$\max_{g\in K_G}\|\Psi_n^{(l)}(b)(gG_k)-\Theta_n^{(l)}(b)(g)\|\to 0. \tag{4-14}$$

We now wish to show that these c.p.c. maps are approximately equivariant with regard to γ and $\sigma^{G_k}\otimes\alpha$, where σ^{G_k} is the G-action on $\mathcal{C}(G/G_k)$ induced by the left-translation of G on G/G_k.

Let us fix a compact set $K_G\subseteq G$ as above. Without loss of generality, let us assume that it is large enough so that the quotient map $G\to G/G_k$ is still surjective when restricted to K_G. Given $b\in\mathcal{D}$, set

$$\rho_n(b)=\max_{l=0,\dots,d}\max_{g\in K^{-1}K_G}\|\Psi_n^{(l)}(b)(gG_k)-\Theta_n^{(l)}(b)(g)\|. \tag{4-15}$$

Note that by an elementary compactness argument, it follows from (4-14) that for every compact set $J\subset\mathcal{D}$, we have

$$\max_{b\in J}\rho_n(b)\to 0. \tag{4-16}$$

Let $t \in K$, $g \in K_G$ and $b \in \mathcal{D}$ with $\|b\| \leq 1$. Then

$$\begin{aligned}(\sigma_t^{G_k} \otimes \alpha_t)((\Psi_n^{(l)})(b))(gG_k) &= \alpha_t(\Psi_n^{(l)}(b)(t^{-1}gG_k))\\ &\overset{(4\text{-}15)}{=}_{\rho_n(b)} \alpha_t(\Theta_n^{(l)}(b)(t^{-1}gG_k))\\ &\overset{(4\text{-}9)}{=} \sum_{h\in H_k} \mu^{(l)}(\pi_N(t^{-1}g)h)\cdot(\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{t^{-1}g\sigma(h)}^{-1})(b)\\ &\overset{(4\text{-}6),(4\text{-}8)}{=}_{\varepsilon} \sum_{h\in H_k} \mu^{(l)}(\pi_N(g)h)\cdot(\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{t^{-1}g\sigma(h)}^{-1})(b)\\ &\overset{(4\text{-}15),(4\text{-}9)}{=}_{\rho_n(\gamma_t(b))} \Psi_n^{(l)}(\gamma_t(b))(gG_k).\end{aligned}$$

Note that as K_G contains a representative for every G_k-orbit in G, these approximations carry over to the $\|\cdot\|_\infty$-norm of the involved functions. Using (4-16), we obtain for all $b \in \mathcal{D}$ with $\|b\| \leq 1$ that

$$\limsup_{n\to\infty} \max_{t\in K} \|(\sigma_t^{G_k} \otimes \alpha_t)(\Psi_n^{(l)})(b) - (\Psi_n^{(l)} \circ \gamma_t)(b)\| \leq \varepsilon. \tag{4-17}$$

Next, we wish to show that for $l_1 \neq l_2$, the c.p.c. maps $\Psi_n^{(l_1)}$ and $\Psi_n^{(l_2)}$ have approximately commuting ranges as $n \to \infty$. Let $g_1, g_2 \in K_G$ and $b \in \mathcal{D}$ with $\|b\| \leq 1$ be given. Then we compute

$$\begin{aligned}&\|[\Psi_n^{(l_1)}(b)(g_1G_k), \Psi_n^{(l_2)}(b)(g_2G_k)]\|\\ &\qquad\overset{(4\text{-}15)}{=}_{4\rho_n(b)} \|[\Theta_n^{(l_1)}(b)(g_1), \Theta_n^{(l_2)}(b)(g_2)]\|\\ &\qquad\overset{(4\text{-}6),(4\text{-}9),(4\text{-}11)}{\leq} \max_{h_1,h_2\in F_k} \|[(\alpha_{g_1\sigma(h_1)} \circ \kappa_n^{(l_1)} \circ \gamma_{g_1\sigma(h_1)}^{-1})(b), (\alpha_{g_2\sigma(h_2)} \circ \kappa_n^{(l_2)} \circ \gamma_{g_2\sigma(h_2)}^{-1})(b)]\|\\ &\qquad= \max_{h_1,h_2\in F_k} \|[(\kappa_n^{(l_1)} \circ \gamma_{g_1\sigma(h_1)}^{-1})(b), (\alpha_{\sigma(h_1)^{-1}g_1^{-1}g_2\sigma(h_2)} \circ \kappa_n^{(l_2)} \circ \gamma_{g_2\sigma(h_2)}^{-1})(b)]\|\end{aligned}$$

In particular, we obtain for every contraction $b \in \mathcal{D}$ that

$$\begin{aligned}&\max_{g_1,g_2\in K_G} \|[\Psi_n^{(l_1)}(b)(g_1G_k), \Psi_n^{(l_2)}(b)(g_2G_k)]\|\\ &\quad\leq \max_{g_1,g_2\in K_G} \max_{h_1,h_2\in F_k} \|[(\kappa_n^{(l_1)} \circ \gamma_{g_1\sigma(h_1)}^{-1})(b), (\alpha_{\sigma(h_1)^{-1}g_1^{-1}g_2\sigma(h_2)} \circ \kappa_n^{(l_2)} \circ \gamma_{g_2\sigma(h_2)}^{-1})(b)]\| + 4\rho_n(b)\\ &\overset{(4\text{-}16),(4\text{-}5)}{\longrightarrow} 0.\end{aligned} \tag{4-18}$$

Here we have used that the convergence in (4-5) automatically holds uniformly when quantifying over b_1, b_2 belonging to some compact subset in $\mathcal{D}$, in this case

$$b_1, b_2 \in \{\gamma_g^{-1}(b) \mid g \in K_G \cdot \sigma(F_k)\}.$$

In exactly the same fashion, one also computes

$$\|[\Psi_n^{(l)}(b), a]\| \longrightarrow 0 \tag{4-19}$$

for all $l = 0, \ldots, d$, $b \in \mathcal{D}$, and $a \in A$, by using (4-1) in place of (4-5).

Next, we wish to show that for each $l = 0, \ldots, d$, the c.p.c. maps $\Psi_n^{(l)}$ behave approximately like order-zero maps. Let $g \in K_G$. Choose the unique element $h_0 \in F_k$ with $\mu^{(l)}(\pi_N(g)h_0) > 0$. Then it follows for every $b_1, b_2 \in \mathcal{D}$ that

$$\begin{aligned}\Theta_n^{(l)}(b_1)(g)\cdot\Theta_n^{(l)}(b_2)(g) &= \mu^{(l)}(\pi_N(g)h_0)^2\cdot(\alpha_{g\sigma(h_0)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_1)\cdot(\alpha_{g\sigma(h_0)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_2)\\ &= \mu^{(l)}(\pi_N(g)h_0)^2\cdot\alpha_{g\sigma(h_0)}\big((\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_1)\cdot(\kappa_n^{(l)}\circ\gamma_{g\sigma(h_0)}^{-1})(b_1)\big).\end{aligned}$$

It follows from this calculation that

$$\begin{aligned}&\|\Theta_n^{(l)}(b_1)\cdot\Theta_n^{(l)}(b_2)-\Theta_n^{(l)}(b_1b_2)\cdot\Theta_n^{(l)}(\mathbf{1})\|_{\infty,K_G}\\ &\qquad\le\max_{s\in K_{G_k}\cdot\sigma(F_k)}\|(\kappa_n^{(l)}\circ\gamma_s^{-1})(b_1)\cdot(\kappa_n^{(l)}\circ\gamma_s^{-1})(b_1)-(\kappa_n^{(l)}\circ\gamma_s^{-1})(b_1b_2)\cdot(\kappa_n^{(l)}\circ\gamma_s^{-1})(\mathbf{1})\|\\ &\qquad\xrightarrow{(4\text{-}7),(4\text{-}8)} 0.\end{aligned}$$

As K_G contains a representative of every G_k-orbit in G, it follows from (4-14) that

$$\|\Psi_n^{(l)}(b_1)\cdot\Psi_n^{(l)}(b_2)-\Psi_n^{(l)}(b_1b_2)\cdot\Psi_n^{(l)}(\mathbf{1})\|\longrightarrow 0 \tag{4-20}$$

for every $b_1, b_2 \in \mathcal{D}$.

Next, we wish to show that the completely positive sum $\sum_{l=0}^d \Psi_n^{(l)}$ behaves approximately like a u.c.p. map upon multiplication with an element of $\mathbf{1}\otimes A$ as $n\to\infty$. Let $g\in K_G$. We have

$$\begin{aligned}\Theta_n^{(l)}(\mathbf{1})(g) &\overset{(4\text{-}9),(4\text{-}10)}{=} \sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)}\circ\gamma_{g\sigma(h)}^{-1})(\mathbf{1})\\ &= \sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\alpha_{g\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1}).\end{aligned}$$

It follows for all $a\in A$ that

$$\begin{aligned}&\max_{g\in K_G}\Big\|\Big(\mathbf{1}-\sum_{l=0}^d\Theta_n^{(l)}(\mathbf{1})(g)\Big)\cdot a\Big\|\\ &\quad\le\max_{g\in K_G}(d+1)\cdot\max_l\max_{h\in F_k}\|(\alpha_{\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1})-\kappa_n^{(l)}(\mathbf{1})\|+\Big\|\Big(\mathbf{1}-\sum_{l=0}^d\sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\alpha_g\circ\kappa_n^{(l)})(\mathbf{1})\Big)\cdot a\Big\|\\ &\quad\overset{(4\text{-}7)}{=}\max_{g\in K_G}(d+1)\cdot\max_l\max_{h\in F_k}\|(\alpha_{\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1})-\kappa_n^{(l)}(\mathbf{1})\|+\Big\|\Big(\sum_{l=0}^d\sum_{h\in F_k}\mu^{(l)}(\pi_N(g)h)\cdot(\mathbf{1}-(\alpha_g\circ\kappa_n^{(l)})(\mathbf{1}))\Big)\cdot a\Big\|\\ &\quad\overset{(4\text{-}6)}{\le}\max_{g\in K_G}\big((d+1)\cdot\max_l\max_{h\in F_k}\|(\alpha_{\sigma(h)}\circ\kappa_n^{(l)})(\mathbf{1})-\kappa_n^{(l)}(\mathbf{1})\|+(d+1)\cdot\max_l\|(\mathbf{1}-\kappa_n^{(l)}(\mathbf{1}))\cdot\alpha_g^{-1}(a)\|\big)\\ &\quad\xrightarrow{(4\text{-}3),(4\text{-}4)} 0.\end{aligned}$$

Since K_G contains every G_k-orbit in G, it follows from (4-14) that

$$\Big\|\Big(\mathbf{1}-\sum_{l=0}^d\Psi_n^{(l)}(\mathbf{1})\Big)\cdot(\mathbf{1}\otimes a)\Big\|\to 0\quad\text{for all } a\in A. \tag{4-21}$$

Let us now summarize everything we have obtained so far. The c.p.c. maps $\Psi_n^{(l)} : \mathcal{D} \to \mathcal{C}(G/G_k, A)$, for $l = 0, \dots, d$ and $n \in \mathbb{N}$ satisfy the following properties for all $b, b_1, b_2 \in \mathcal{D}$ and $a \in A$:

$$\|[\Psi_n^{(l)}(b), \mathbf{1} \otimes a]\| \longrightarrow 0, \tag{4-22}$$

$$\limsup_{n\to\infty} \max_{t \in K} \|((\sigma^{G_k} \otimes \alpha)_t \circ \Psi_n^{(l)})(b) - (\Psi_n^{(l)} \circ \gamma_t)(b)\| \le \varepsilon, \tag{4-23}$$

$$\|[\Psi_n^{(l_1)}(b), \Psi_n^{(l_2)}(b)]\| \longrightarrow 0 \quad \text{for all } l_1 \neq l_2, \tag{4-24}$$

$$\|\Psi_n^{(l)}(b_1) \cdot \Psi_n^{(l)}(b_2) - \Psi_n^{(l)}(b_1 b_2) \cdot \Psi_n^{(l)}(\mathbf{1})\| \longrightarrow 0, \tag{4-25}$$

$$\left\| \left(\mathbf{1} - \sum_{l=0}^{d} \Psi_n^{(l)}(\mathbf{1}) \right) \cdot \mathbf{1} \otimes a \right\| \longrightarrow 0. \tag{4-26}$$

Note that k, and thus the codomain of $\Psi_n^{(l)}$, had to be chosen depending on ε and $K \subseteq G$. However, we have canonical (equivariant) inclusions $\mathcal{C}(G/G_k, A) \subseteq \mathcal{C}(G/\mathcal{S}_0, A)$, which we may compose our maps with. It is then clear that the same properties as in (4-22) up to (4-26) hold, where we replace the action $\sigma^{G_k} : G \curvearrowright \mathcal{C}(G/G_k)$ by $\sigma^{\mathcal{S}_0} : G \curvearrowright \mathcal{C}(G/\mathcal{S}_0)$.

Since A and $\mathcal{D}$ are separable and G is second-countable, we can let the tolerance ε go to zero, let the set $K \subseteq G$ get larger and apply a diagonal sequence argument. Putting the appropriate choices of c.p.c. maps into a single sequence, we can thus obtain c.p.c. maps

$$\psi^{(l)} : B \to (\mathcal{C}(G/\mathcal{S}_0) \otimes A)_\infty, \quad l = 0, \dots, d,$$

that satisfy the following properties for all $g \in G$, $a \in A$, and $b, b_1, b_2 \in \mathcal{D}$:

$$[\psi^{(l)}(b), \mathbf{1} \otimes a] = 0, \tag{4-27}$$

$$(\sigma^{\mathcal{S}_0} \otimes \alpha)_g \circ \psi^{(l)} = \psi^{(l)} \circ \gamma_g, \tag{4-28}$$

$$[\psi^{(l_1)}(b), \psi^{(l_2)}(b)] = 0 \quad \text{for all } l_1 \neq l_2, \tag{4-29}$$

$$\psi^{(l)}(b_1) \cdot \psi^{(l)}(b_2) = \psi^{(l)}(b_1 b_2) \cdot \psi^{(l)}(\mathbf{1}), \tag{4-30}$$

$$\left(\mathbf{1} - \sum_{l=0}^{d} \psi^{(l)}(\mathbf{1}) \right) \cdot \mathbf{1} \otimes a = 0. \tag{4-31}$$

Since $\gamma : G \curvearrowright \mathcal{D}$ is point-norm continuous, (4-28) implies that the image of each map $\psi^{(l)}$ is in the continuous part $(\mathcal{C}(G/\mathcal{S}_0) \otimes A)_{\infty, \sigma^{\mathcal{S}_0} \otimes \alpha}$. In fact it is in the relative commutant of $\mathbf{1} \otimes A$ by (4-27), but then also automatically in the relative commutant of all of $\mathcal{C}(G/\mathcal{S}_0) \otimes A$. This allows us to define equivariant maps

$$\zeta^{(l)} : \mathcal{D} \to F_{\infty, \sigma^{\mathcal{S}_0} \otimes \alpha}(\mathcal{C}(G/\mathcal{S}_0) \otimes A), \quad \zeta^{(l)}(b) = \psi^{(l)}(b) + \mathrm{Ann}(\mathcal{C}(G/\mathcal{S}_0) \otimes A)$$

for all $l = 0, \dots, d$. Then (4-29) implies that these maps have commuting ranges, (4-30) implies that they are c.p.c. order-zero, and finally (4-31) implies the equation $\sum_{l=0}^{d} \zeta^{(l)}(\mathbf{1}) = \mathbf{1}$.

By virtue of Proposition 3.2 and Remark 3.5, this gives rise to a unital equivariant $*$-homomorphism

$$(\mathcal{D}^{\star(d+1)}, \gamma^{\star(d+1)}) \to \big(F_{\infty, \sigma^{\mathcal{S}_0} \otimes \alpha}(\mathcal{C}(G/\mathcal{S}_0) \otimes A), (\widetilde{\sigma^{\mathcal{S}_0} \otimes \alpha})_\infty \big).$$

As γ is unitarily regular, it follows from Lemma 3.7 that $\gamma^{\star(d+1)}$ is a γ-absorbing action. Applying Lemma 4.6 yields that $\sigma^{\mathcal{S}_0} \otimes \alpha$ is γ-absorbing, which finishes the proof. □

Now we are in a position to prove Theorem 4.4:

Proof of Theorem 4.4. Let $\alpha : G \curvearrowright A$ and $\gamma : G \curvearrowright \mathcal{D}$ be the two actions as in the assumption. Let also $N \subset G$, $H_k \subset G/N$, and $G_k \subset G$ be subgroups as specified in the statement, and denote by $\mathcal{S}_1 = (H_k)_k$ a sequence of subgroups in G/N, and by $\mathcal{S}_0 = (G_k)_k$ a sequence of subgroups in G.

Suppose $\operatorname{asdim}(\square_{\mathcal{S}_1} H_1) < \infty$ and $s := \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha, \mathcal{S}_0) < \infty$. Using the latter, Lemma 4.3(iv) implies that the equivariant embedding

$$\mathrm{id}_A \otimes \mathbf{1} : (A, \alpha) \to (A \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star(s+1)}, \alpha \otimes (G\text{-shift}))$$

is G-equivariantly sequentially split. By Lemma 4.6, in order to show that α is γ-absorbing, it suffices to show that the G-C*-algebra $A \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star(s+1)}$ is γ-absorbing. We will show this via induction on s.

For $s = 0$, the claim is precisely Lemma 4.7, and in particular it holds because we assumed that $\operatorname{asdim}(\square_{\mathcal{S}_1} H_1) < \infty$.

Given $s \geq 1$, assume that the claim holds for $s - 1$. It follows by Remark 3.6 that there is an extension of G-C*-algebras of the form

$$0 \longrightarrow J^{(s)} \longrightarrow A \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star(s+1)} \longrightarrow Q^{(s)} \longrightarrow 0,$$

where

$$J^{(s)} = A \otimes \mathcal{C}_0(0, 1) \otimes \mathcal{C}(G/\mathcal{S}_0) \otimes \mathcal{C}(G/\mathcal{S}_0)^{\star s},$$
$$Q^{(s)} = A \otimes (\mathcal{C}(G/\mathcal{S}_0) \oplus \mathcal{C}(G/\mathcal{S}_0)^{\star s}).$$

By the induction hypothesis, both the kernel and the quotient of this extension are γ-absorbing G-C*-algebras, and therefore so is the middle by Theorem 1.10. This finishes the induction step and the proof. □

Remark 4.8. We remark that the statement of the main result holds verbatim for cocycle actions instead of genuine actions. Note that the concept of Rokhlin dimension makes sense for cocycle actions with the same definition, since there is still a natural genuine action induced on the central sequence algebra. If $(\alpha, w) : G \curvearrowright A$ is a cocycle action on a separable C*-algebra, then $(\alpha \otimes \mathrm{id}_{\mathcal{K}}, w \otimes \mathbf{1}) : G \curvearrowright A \otimes \mathcal{K}$ is cocycle conjugate to a genuine action by the Packer–Raeburn stabilization trick [1989]. Since both Rokhlin dimension and absorption of a semi-strongly self-absorbing action are invariants under stable (cocycle) conjugacy, the statement of Theorem 4.4 follows for cocycle actions.

5. Some applications

Let us now discuss some immediate applications of the main result. First we wish to point out that the following result arises as a special case.

Corollary 5.1. *Let G be a second-countable, locally compact group. Let $\mathcal{S} = (H_n)_n$ be a residually compact approximation consisting of normal subgroups of G with*

$$\operatorname{asdim}(\square_{\mathcal{S}} H_1) < \infty.$$

Let A be a separable C^**-algebra and $\mathcal{D}$ a strongly self-absorbing* C^**-algebra with $A \cong A \otimes \mathcal{D}$. Let $\alpha : G \curvearrowright A$ be an action with*

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha, \mathcal{S}) < \infty.$$

Then $\alpha \simeq_{\mathrm{vscc}} \alpha \otimes \gamma$ for all semi-strongly self-absorbing actions $\gamma : G \curvearrowright \mathcal{D}$.

Proof. Let $\gamma : G \curvearrowright \mathcal{D}$ be a semi-strongly self-absorbing action. Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$ by [Winter 2011], we may replace γ with $\gamma \otimes \mathrm{id}_{\mathcal{Z}}$ for the purpose of showing the claim, as $\gamma \otimes \mathrm{id}_{\mathcal{Z}}$ is again semi-strongly self-absorbing and every $(\gamma \otimes \mathrm{id}_{\mathcal{Z}})$-absorbing action is γ-absorbing. So let us simply assume $\gamma \simeq_{\mathrm{cc}} \gamma \otimes \mathrm{id}_{\mathcal{Z}}$. By Remark 1.9, we may thus assume that γ is unitarily regular. The claim then follows directly from Theorem 4.4 applied to the case $N = \{1\}$. Note that one automatically has absorption with respect to very strong cocycle conjugacy by virtue of Theorem 1.8(v). □

Note that the results below in part refer to Rokhlin dimension without commuting towers, as defined in [Szabó et al. 2017, Section 5]. For the Rokhlin dimension-zero case, the commuting tower assumption is vacuous.

Example 5.2. Let $\mathcal{Q}$ denote the universal UHF algebra. Let Γ be a countable, discrete group and $H \subset \Gamma$ a normal subgroup with finite index. There exists a strongly self-absorbing action $\gamma : G \curvearrowright \mathcal{Q}$ with $\dim_{\mathrm{Rok}}(\gamma, H) = 0$.

Proof. Such an action is constructed as part of [Szabó et al. 2017, Remark 10.8]. Namely, consider the left-regular representation $\lambda^{G/H} : G/H \to \mathcal{U}(M_{|G:H|})$, consider the quotient map $\pi_H : G \to G/H$, and define

$$\gamma_g = \mathrm{id}_{\mathcal{Q}} \otimes \bigotimes_{\mathbb{N}} \mathrm{Ad}(\lambda^{G/H}(\pi_H(g)))$$

as an action on $\mathcal{Q} \cong \mathcal{Q} \otimes M_{[G:H]}^{\otimes\infty}$. As the diagonal embedding $\mathcal{C}(G/H) \subset M_{[G:H]}$ is equivariant, it follows that $\dim_{\mathrm{Rok}}(\gamma, H) = 0$. By [Szabó 2018c, Proposition 6.3], such an action is strongly self-absorbing. □

This in turn has the following consequence regarding the dimension-reducing effect of strongly self-absorbing C^*-algebras.

Corollary 5.3. *Let Γ be a countable, discrete, residually finite group that has some box space with finite asymptotic dimension. Let $\alpha : \Gamma \curvearrowright A$ be an action on a separable* C^**-algebra with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$:*

(1) *If $A \cong A \otimes \mathcal{Q}$, then $\dim_{\mathrm{Rok}}(\alpha) = 0$.*

(2) *If $A \cong A \otimes \mathcal{Z}$, then $\dim_{\mathrm{Rok}}(\alpha) \le 1$.*

Proof. (1): This follows directly from Example 5.2 and Corollary 5.1.

(2): We have $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{Z}}$, and there exist two c.p.c. order-zero maps $\psi_0, \psi_1 : \mathcal{Q} \to \mathcal{Z}_\infty \cap \mathcal{Z}'$ with $\psi_0(\mathbf{1}) + \psi_1(\mathbf{1}) = \mathbf{1}$; see [Matui and Sato 2014a, Section 5; Sato et al. 2015, Section 6]. Consider two sequences $\psi_{0,n}, \psi_{1,n} : \mathcal{Q} \to \mathcal{Z}$ of c.p.c. maps lifting ψ_0 and ψ_1.

By (1), $\alpha \otimes \mathrm{id}_{\mathcal{Q}}$ has Rokhlin dimension zero. Given any subgroup $H \subset \Gamma$ with finite index, we can find c.p.c. order-zero maps $\mathcal{C}(\Gamma/H) \to A \otimes \mathcal{Q}$ which are approximately equivariant, have approximately central image, and are such that the image of the unit acts approximately like a unit on finite sets. Once we compose such maps with $\mathrm{id}_A \otimes \psi_{i,n}$ for $i = 0, 1$ and large enough n, we may obtain two c.p.c. maps $\kappa_0, \kappa_1 : C(\Gamma/H) \to A \otimes \mathcal{Z}$, which are approximately equivariant, have approximately central image, and so that the element $\kappa_0(\mathbf{1}) + \kappa_1(\mathbf{1})$ approximately acts like a unit on a given finite set in $A \otimes \mathcal{Z}$. But this is what is required by $\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}}) \leq 1$; we leave the finer details to the reader as the proof is quite standard. □

Remark 5.4. The reason why the proof of Corollary 5.3(2) does not yield $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) \leq 1$ is due to the fact that the two order-zero maps $\psi_0, \psi_1 : \mathcal{Q} \to \mathcal{Z}_\infty$ can never have commuting ranges. Indeed, this would imply the existence of a unital $*$-homomorphism $\mathcal{Q} \to \mathcal{Z}_\infty$ via Lemma 3.7, so it is impossible. More concretely, [Hirshberg and Phillips 2017, Example 3.32] exhibits an example of a $\mathbb{Z}_2$-action α on a Kirchberg algebra with $\dim_{\mathrm{Rok}}(\alpha) = 1$ and $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = 2$.

Corollary 5.5. *Let Γ be a discrete, finitely generated, virtually nilpotent group. Let X be a compact metrizable space with finite covering dimension, and $\alpha : \Gamma \curvearrowright X$ a free action by homeomorphisms. Then one has*

$$\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Q}} : \Gamma \curvearrowright \mathcal{C}(X) \otimes \mathcal{Q}) = 0$$

and

$$\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}} : \Gamma \curvearrowright \mathcal{C}(X) \otimes \mathcal{Z}) \leq 1.$$

Proof. By [Szabó et al. 2017, Corollary 7.5], the action $\alpha : \Gamma \curvearrowright \mathcal{C}(X)$ has finite Rokhlin dimension.[4] Since the underlying C*-algebra is abelian, the claim follows from Corollary 5.3. □

6. Multiflows on strongly self-absorbing Kirchberg algebras

In this section, we shall study actions of $\mathbb{R}^k$ on certain C*-algebras satisfying an obvious notion of the Rokhlin property.

Notation 6.1. For $k \geq 2$, we will refer to a continuous action of $\mathbb{R}^k$ on a C*-algebra as a *multiflow*. Let $(e_j)_{1 \leq j \leq k}$ be the standard basis of $\mathbb{R}^k$. Given $\alpha : \mathbb{R}^k \curvearrowright A$, we will define the *generating flows* $\alpha^{(j)} : \mathbb{R} \curvearrowright A$, given by $\alpha^{(j)}_t = \alpha_{te_j}$, for $j = 1, \dots, k$. We then have

$$\alpha^{(j)}_{t_j} \circ \alpha^{(i)}_{t_i} = \alpha^{(i)}_{t_i} \circ \alpha^{(j)}_{t_j} \quad \text{for all } i, j = 1, \dots, k \text{ and all } t_i, t_j \in \mathbb{R}.$$

We will also denote by $\alpha^{(\not{j})} : \mathbb{R}^{k-1} \curvearrowright A$ the action generated by the flows $(\alpha^{(i)})_{i \neq j}$. We remark that $\alpha^{(j)}$ reduces naturally to a flow on the fixed point algebra $A^{\alpha^{(\not{j})}}$.

Definition 6.2. Let A be a separable C*-algebra and $\alpha : \mathbb{R}^k \curvearrowright A$ an action. We say that α has the Rokhlin property if $\dim_{\mathrm{Rok}}(\alpha, p\mathbb{Z}^k) = 0$ for all $p > 0$.

Remark 6.3. An obvious question regarding Definition 6.2 is whether this is the same as $\dim_{\mathrm{Rok}}(\alpha) = 0$ when $k \geq 2$, especially because this appears to be (a priori) much more difficult to check. Nevertheless, this turns out to be case. Instead of giving a detailed proof here, let us just roughly sketch the basic idea.

[4] Strictly speaking, only the nilpotent case is proved there. The virtually nilpotent case follows from independent work of Bartels [2017, Section 1].

The condition $\dim_{\mathrm{Rok}}(\alpha) = 0$ in the sense of Definition 4.1 amounts to checking $\dim_{\mathrm{Rok}}(\alpha, H) = 0$ for every closed cocompact subgroup $H \subset \mathbb{R}^k$, or in other words finding approximately equivariant unital embeddings from $\mathcal{C}(\mathbb{R}^k/H)$ into the central sequence algebra of A. This only gets easier when we make H smaller, so we may assume without loss of generality that H is discrete. Since H is a free abelian group and is cocompact in $\mathbb{R}^k$, it has a $\mathbb{Z}$-basis $e_1, \dots, e_k \in H$. We may approximate these elements by $f_1, \dots, f_k \in \mathbb{Q}^k$, which are linearly independent over $\mathbb{Q}$ and span another subgroup H'. By using for example Lemma 2.8 we can then obtain approximately multiplicative and equivariant u.c.p. maps $\mathcal{C}(\mathbb{R}^k/H) \to \mathcal{C}(\mathbb{R}^k/H')$. By the properties of central sequence algebras, we may thus assume without loss of generality that in fact $H \subseteq \mathbb{Q}^k$. Now the same argument as in [Szabó et al. 2017, Example 3.19] allows one to see that H contains a finite-index subgroup of the form $n\mathbb{Z}^k$ for some $n \in \mathbb{N}$. In summary, we obtain $\dim_{\mathrm{Rok}}(\alpha, H) = 0$ for arbitrary H when we assume the Rokhlin property in the sense of Definition 6.2.

Remark 6.4. In the case of flows, i.e., the case $k = 1$ above, Definition 6.2 coincides with the notion of the Rokhlin property from [Kishimoto 1996a]. Let us for now denote by $\sigma^T : \mathbb{R} \curvearrowright \mathcal{C}(\mathbb{R}/T\mathbb{Z})$ the action induced by the $\mathbb{R}$-shift.

Proposition 6.5. *Let A be a separable C^*-algebra and $\alpha : \mathbb{R}^k \curvearrowright A$ an action. The following are equivalent*:

(i) *α has the Rokhlin property.*

(ii) *For every $j = 1, \dots, k$ and every $p > 0$, there exists a unitary*
$$u \in F_{\infty,\alpha}(A)^{\tilde{\alpha}_\infty^{(\not j)}} \quad \textit{such that} \quad \tilde{\alpha}^{(j)}_{\infty,t}(u) = e^{ipt}u, \quad t \in \mathbb{R}.$$

(iii) *For every $j = 1, \dots, k$ and every $T > 0$, there exists an equivariant and unital $*$-homomorphism*
$$(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^T) \longrightarrow (F_{\infty,\alpha}(A)^{\tilde{\alpha}_\infty^{(\not j)}}, \tilde{\alpha}^{(j)}_\infty).$$

Proof. (i)⇔(iii): Let $T > 0$. One has a canonical equivariant isomorphism
$$(\mathcal{C}(\mathbb{R}^k/T\mathbb{Z}^k), \mathbb{R}^k\text{-shift}) \cong (\mathcal{C}(\mathbb{R}/T\mathbb{Z})^{\otimes k}, \sigma^{T,1} \otimes \cdots \otimes \sigma^{T,k}),$$
where $\sigma^{T,j}$ is the $\mathbb{R}^k$-action on $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ where only the j-th component acts by the $\mathbb{R}$-shift. By definition, α having the Rokhlin property means that for every $T > 0$ the dynamical system on the left embeds into $(F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$. So in particular, when (i) holds, one also obtains an embedding of $(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^{T,j})$ for every $j = 1, \dots, k$, which implies (iii). Conversely, when (iii) holds, for all $T > 0$ one has an embedding of $(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^{T,j})$ into $(F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$ for all $j = 1, \dots, k$. By applying a standard reindexing argument in the central sequence algebra, one may assume that these embeddings have pairwise commuting ranges for all $j = 1, \dots, k$. Therefore one obtains an embedding of the C^*-dynamical system given by the tensor product of all $(\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \sigma^{T,j})$, which we have seen to be the same as the dynamical system $(\mathcal{C}(\mathbb{R}^k/T\mathbb{Z}^k), \mathbb{R}^k\text{-shift})$. In particular this implies (i).

(ii)⇔(iii): This follows directly from functional calculus. A unitary u as in (ii) gives rise to a unital equivariant $*$-homomorphism
$$\varphi_u : \big(\mathcal{C}\big(\mathbb{R}/\tfrac{2\pi}{p}\mathbb{Z}\big), \sigma^{\frac{2\pi}{p}}\big) \longrightarrow (F_{\infty,\alpha}(A)^{\tilde{\alpha}_\infty^{(\not j)}}, \tilde{\alpha}^{(j)}_\infty), \quad \varphi_u(f) = f(u).$$

Conversely, whenever φ is an arbitrary homomorphism between these two dynamical systems, $u = \varphi\big(\big[t + \frac{2\pi}{p}\mathbb{Z} \mapsto e^{ipt}\big]\big)$ yields a unitary as required by (ii). $\square$

Remark 6.6. We note that for $G = \mathbb{R}^k$, the sequence $H_n = (n!) \cdot \mathbb{Z}^k$ yields a residually compact approximation in the sense of Definition 2.1. Now it is well known that $\square_{(H_n)_n}\mathbb{Z}^k$ has finite asymptotic dimension k; see either [Szabó et al. 2017, Sections 2–3] or better yet [Delabie and Tointon 2018]. In particular, Corollary 5.1 is applicable to $\mathbb{R}^k$-actions that have finite Rokhlin dimension with commuting towers, and more specifically it is applicable to $\mathbb{R}^k$-actions with the Rokhlin property.

The following is the main result of this section.

Theorem 6.7. *Let $\mathcal{D}$ be a strongly self-absorbing Kirchberg algebra. Let $k \geq 1$ be a given natural number. Then all continuous $\mathbb{R}^k$-actions on $\mathcal{D}$ with the Rokhlin property are semi-strongly self-absorbing and are mutually (very strongly) cocycle conjugate.*

The approach for proving this result, at least in the way presented here, uses the theory of semi-strongly self-absorbing actions in a crucial way. In such dynamical systems, one has a very strong control over certain (approximately central) unitary paths, which, together with the Rokhlin property, allows one to obtain a relative cohomology-vanishing-type statement. This will be used to deduce inductively that the actions in the statement of Theorem 6.7 have approximately $\mathbb{R}^k$-inner flip. The desired uniqueness for such actions is then achieved by combining this fact with Corollary 5.1, which is a special case of our main result, in a suitable way.

Example 6.8 (see [Bratteli et al. 2007]). Denote by $s_1, s_2, \dots$ the generators of the Cuntz algebra $\mathcal{O}_\infty$. Define a quasifree flow $\gamma^0 : \mathbb{R} \curvearrowright \mathcal{O}_\infty$ via

$$\gamma^0_t(s_1) = e^{2\pi i t}s_1, \quad \gamma^0_t(s_2) = e^{-2\pi i\sqrt{2}t}s_2, \quad \text{and} \quad \gamma^0_t(s_j) = s_j \quad \text{for } j \geq 3.$$

Then γ^0 has the Rokhlin property by [Bratteli et al. 2007, Theorem 1.1].

In particular, given $k \geq 1$ and any strongly self-absorbing Kirchberg algebra $\mathcal{D}$, the action

$$\mathrm{id}_{\mathcal{D}} \otimes (\underbrace{\gamma^0 \times \cdots \times \gamma^0}_{k \text{ times}}) : \mathbb{R}^k \curvearrowright \mathcal{D} \otimes \mathcal{O}_\infty^{\otimes k} \cong \mathcal{D}$$

is a (k-)multiflow with the Rokhlin property on $\mathcal{D}$, and is in fact (very strongly) cocycle conjugate to every other one by Theorem 6.7.

Let us now implement the strategy outlined above step by step. We begin with the aforementioned cohomology-vanishing-type statement, which involves minimal assumptions about the underlying C*-algebras but otherwise very strong assumptions about the existence of certain unitary paths, which will naturally appear in our intended setup later.

Lemma 6.9. *Let A be a separable unital* C*-*algebra. Let $k \geq 1$ and let $\alpha : \mathbb{R}^k \curvearrowright A$ be a continuous action with the Rokhlin property, and fix some $j \in \{1, \dots, k\}$.*

For every $\varepsilon > 0$, $L > 0$ and $\mathcal{F} \Subset A$, there exists a $T > 0$ and $\mathcal{G} \Subset A$ with the following property:

If $\{w_t\}_{t\in\mathbb{R}} \subset \mathcal{U}(A)$ *is any* $\alpha^{(j)}$*-1-cocycle satisfying*

$$\max_{a\in\mathcal{F}} \max_{0\le t\le T} \|[w_t, a]\| \le \varepsilon,$$

$$\max_{0\le t\le T} \max_{\vec{r}\in[0,1]^{k-1}} \|w_t - \alpha^{(\not{j})}_{\vec{r}}(w_t)\| \le \varepsilon,$$

and moreover there exists some continuous path of unitaries $u : [0, 1] \to \mathcal{U}(A)$ *with*

$$u(0) = \mathbf{1}, \quad u(1) = w_{-T}, \quad \ell(u) \le L,$$

$$\max_{0\le t\le 1} \max_{\vec{r}\in[0,1]^{k-1}} \|u(t) - \alpha^{(\not{j})}_{\vec{r}}(u(t))\| \le \varepsilon,$$

$$\max_{0\le t\le 1} \max_{a\in\mathcal{G}} \|[u(t), a]\| \le \varepsilon,$$

then there exists a unitary $v \in \mathcal{U}(A)$ *satisfying*

$$\max_{0\le t\le 1} \|w_t - v\alpha^{(j)}_t(v^*)\| \le 3\varepsilon,$$

$$\max_{a\in\mathcal{F}} \|[v, a]\| \le 3\varepsilon,$$

$$\max_{\vec{r}\in[0,1]^{k-1}} \|v - \alpha^{(\not{j})}_{\vec{r}}(v)\| \le 3\varepsilon.$$

Proof. Let $T > 0$ and note that we have fixed $j \in \{1, \dots, k\}$ by assumption. By some abuse of notation, let us view σ^T as the $\mathbb{R}^k$-action on $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ such that the j-th coordinate acts as the $\mathbb{R}$-shift and all the other components act trivially. In this way, any $*$-homomorphism as in Proposition 6.5(iii) can be viewed as an $\mathbb{R}^k$-equivariant $*$-homomorphism from $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ to $F_{\infty,\alpha}(A)$. In particular, denote such a homomorphism by θ. We can then obtain a commutative diagram of $\mathbb{R}^k$-equivariant $*$-homomorphisms via

$$\begin{array}{ccc} (A,\alpha) & \longrightarrow & (A_{\infty,\alpha}, \alpha_\infty) \\ & {\scriptstyle d\mapsto \mathbf{1}\otimes d}\searrow \quad \nearrow{\scriptstyle f\otimes d\mapsto \theta(f)\cdot d} & \\ & (\mathcal{C}(\mathbb{R}/T\mathbb{Z})\otimes A, \sigma^T\otimes\alpha) & \end{array} \tag{6-1}$$

We will keep this in mind for later.

Now let $\varepsilon > 0$, $L > 0$ and $\mathcal{F} \Subset A$ be as in the statement. Without loss of generality, we assume that $\mathcal{F}$ consists of contractions. We choose $T > L/\varepsilon$ and $\mathcal{G} \Subset A$ to be any finite set of contractions containing $\mathcal{F}$ that is $\varepsilon/2$-dense in the compact subset

$$\{\alpha^{(j)}_{-s}(a) \mid a \in \mathcal{F},\ 0 \le s \le T\}. \tag{6-2}$$

We claim that these do the trick. We note that the rest of the proof below is almost identical to the proofs of [Kishimoto 1996a, Theorem 2.1; Szabó 2017a, Lemma 3.4], respectively, except for some obvious modifications.

Assume that $\{w_t\}_{t\in\mathbb{R}} \subset \mathcal{U}(A)$ is an $\alpha^{(j)}$-1-cocycle satisfying

$$\max_{a\in\mathcal{F}} \max_{0\le t\le T} \|[w_t, a]\| \le \varepsilon, \tag{6-3}$$

$$\max_{0\le t\le T} \max_{\vec{r}\in[0,1]^{k-1}} \|w_t - \alpha^{(\not{j})}_{\vec{r}}(w_t)\| \le \varepsilon, \tag{6-4}$$

and moreover that there exists some continuous path of unitaries $u : [0,1] \to \mathcal{U}(A)$ with

$$u(0) = \mathbf{1}, \quad u(1) = w_{-T}, \quad \ell(u) \leq L, \tag{6-5}$$

$$\max_{0\leq t\leq 1} \max_{\vec{r}\in[0,1]^{k-1}} \|u(t) - \alpha_{\vec{r}}^{(\not{j})}(u(t))\| \leq \varepsilon, \tag{6-6}$$

$$\max_{0\leq t\leq 1} \max_{a\in\mathcal{G}} \|[u(t), a]\| \leq \varepsilon. \tag{6-7}$$

As $\ell(u) \leq L$, we may assume that u is L-Lipschitz by passing to the arc-length parametrization if necessary. We denote by $\kappa : [0,T] \to \mathcal{U}(A)$ the path given by $\kappa_s = u(s/T)$, which is then Lipschitz with respect to the constant $L/T \leq \varepsilon$. Let us define a continuous path of unitaries $v : [0,T] \to \mathcal{U}(A)$ via $v_s = w_s\alpha_s^{(j)}(\kappa_s)$. Then by (6-5) it follows that $v(0) = v(T) = \mathbf{1}$. In particular, we may view v as a unitary in $\mathcal{C}(\mathbb{R}/T\mathbb{Z}) \otimes A$.

We have

$$\begin{aligned}
\max_{a\in\mathcal{F}} \|[v, \mathbf{1}\otimes a]\| &= \max_{a\in\mathcal{F}} \max_{0\leq s\leq T} \|[w_s\alpha_s^{(j)}(\kappa_s), a]\| \\
&\leq \max_{a\in\mathcal{F}} \max_{0\leq s\leq T} \|[w_s, a]\| + \|[\kappa_s, \alpha_{-s}^{(j)}(a)]\| \\
&\overset{(6\text{-}3)}{\leq} \varepsilon + \max_{a\in\mathcal{F}} \max_{0\leq s\leq T} \|[\kappa_s, \alpha_{-s}^{(j)}(a)]\| \\
&\overset{(6\text{-}2)}{\leq} 3\varepsilon/2 + \max_{b\in\mathcal{G}} \|[\kappa_s, b]\| \\
&\overset{(6\text{-}7)}{\leq} 5\varepsilon/2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\max_{\vec{r}\in[0,1]^{k-1}} \|v - (\sigma^T\otimes\alpha)_{\vec{r}}^{(\not{j})}(v)\| &= \max_{\vec{r}\in[0,1]^{k-1}} \|v - (\mathrm{id}\otimes\alpha)_{\vec{r}}^{(\not{j})}(v)\| \\
&= \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\leq s\leq T} \|v_s - \alpha_{\vec{r}}^{(\not{j})}(v_s)\| \\
&= \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\leq s\leq T} \|w_s\alpha_s^{(j)}(\kappa_s) - \alpha_{\vec{r}}^{(\not{j})}(w_s\alpha_s^{(j)}(\kappa_s))\| \\
&= \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\leq s\leq T} \|w_s\alpha_s^{(j)}(\kappa_s) - \alpha_{\vec{r}}^{(\not{j})}(w_s)\cdot\alpha_s^{(j)}(\alpha_{\vec{r}}^{(\not{j})}(\kappa_s))\| \\
&\leq \max_{\vec{r}\in[0,1]^{k-1}} \max_{0\leq s\leq T} \|w_s - \alpha_{\vec{r}}^{(\not{j})}(w_s)\| + \|\kappa_s - \alpha_{\vec{r}}^{(\not{j})}(\kappa_s)\| \\
&\overset{(6\text{-}4),(6\text{-}6)}{\leq} 2\varepsilon.
\end{aligned}$$

Lastly, let us fix $t \in [0,1]$ and $s \in [0,T]$. If $s \geq t$, then we compute

$$\begin{aligned}
(v(\sigma^T\otimes\alpha)_t^{(j)}(v^*))(s) &= w_s\alpha_s^{(j)}(\kappa_s)\cdot\alpha_t^{(j)}(\alpha_{s-t}^{(j)}(\kappa_{s-t}^*)w_{s-t}^*) \\
&= w_s\cdot\alpha_s^{(j)}(\kappa_s\kappa_{s-t}^*)\alpha_t^{(j)}(w_{s-t}^*) \\
&\overset{(6\text{-}5)}{=_{\varepsilon}} w_s\alpha_t^{(j)}(w_{s-t}^*) = w_t.
\end{aligned}$$

On the other hand, if $s \le t$, then in particular $s \le 1$ and $T-1 \le T+s-t \le T$, and we compute

$$\begin{aligned}(v(\sigma^T \otimes \alpha)^{(j)}_t(v^*))(s) &= w_s \alpha^{(j)}_s(\kappa_s) \cdot \alpha^{(j)}_t(\alpha^{(j)}_{T+s-t}(\kappa^*_{T+s-t}) w^*_{T+s-t}) \\ &= w_s \alpha^{(j)}_s(\kappa_s) \cdot \alpha^{(j)}_{T+s}(\kappa^*_{T+s-t}) \alpha^{(j)}_t(w^*_{T+s-t}) \\ &\overset{(6\text{-}5)}{=}_{2\varepsilon} w_s \cdot \mathbf{1} \cdot \alpha^{(j)}_{T+s}(w^*_{-T}) \alpha^{(j)}_t(w^*_{T+s-t}) \\ &= w_s \alpha^{(j)}_s(w_T) \alpha^{(j)}_t(w^*_{T+s-t}) \\ &= w_{T+s} \alpha^{(j)}_t(w^*_{T+s-t}) w^*_t \cdot w_t = w_t.\end{aligned}$$

Let us summarize what we have accomplished so far. Starting from the existence of the $\alpha^{(j)}$-1-cocycle $\{w_t\}_{t\in\mathbb{R}}$ and the unitary path u with the prescribed properties, we have found a unitary $v \in \mathcal{U}(\mathcal{C}(\mathbb{R}/T\mathbb{Z}) \otimes A)$ satisfying

$$\max_{a\in\mathcal{F}} \|[v, \mathbf{1} \otimes a]\| \le 5\varepsilon/2, \tag{6-8}$$

$$\max_{\vec{r}\in[0,1]^{k-1}} \|v - (\sigma^T \otimes \alpha)^{(\not j)}_{\vec{r}}(v)\| \le 2\varepsilon, \tag{6-9}$$

$$\max_{0\le t\le 1} \|w_t - v(\sigma^T \otimes \alpha)^{(j)}_t(v^*)\| \le 2\varepsilon. \tag{6-10}$$

By using the commutative diagram (6-1), we may send v into the sequence algebra of A, represent the resulting unitary by a sequence of unitaries in A, and then select a member of this sequence so that it will satisfy the properties in the claim with respect to the parameter 3ε. □

Now record the following useful technical result about semi-strongly self-absorbing actions, which arises as a special case of [Szabó 2018c, Lemma 3.12]:

Lemma 6.10. *Let G be a second-countable, locally compact group. Let $\mathcal{D}$ be a separable, unital* C*-*algebra and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing action. For every $\varepsilon > 0$, $\mathcal{F} \Subset \mathcal{D}$ and compact set $K \subset G$, there exist $\delta > 0$ and $\mathcal{G} \Subset \mathcal{D}$ with the following property*:

Suppose that $u : [0, 1] \to \mathcal{U}(\mathcal{D})$ is a unitary path satisfying

$$u(0) = \mathbf{1}, \qquad \max_{0\le t\le 1} \max_{g\in K} \|u(t) - \gamma_g(u(t))\| \le \delta,$$

and

$$\max_{a\in\mathcal{G}} \|[u(1), a]\| \le \delta.$$

Then there exists a unitary path $w : [0, 1] \to \mathcal{U}(\mathcal{D})$ satisfying

$$w(0) = \mathbf{1}, \qquad w(1) = u(1),$$

$$\max_{g\in K} \|w(t) - \gamma_g(w(t))\| \le \varepsilon,$$

$$\max_{0\le t\le 1} \max_{a\in\mathcal{F}} \|[w(t), a]\| \le \varepsilon.$$

Moreover, we may choose w in such a way that

$$\|w(t_1) - w(t_2)\| \le \|u(t_1) - u(t_2)\| \quad \textit{for all } 0 \le t_1, t_2 \le 1.$$

We are now ready to prove the main result of this section:

Proof of Theorem 6.7. We will prove this via induction in k. For this purpose, we will include the case $k = 0$, where the claim is true for trivial reasons.

Now let $k \geq 1$ and assume that the claim is true for actions of $\mathbb{R}^{k-1}$. We will then show that the claim is also true for actions of $\mathbb{R}^k$.

Step 1: Let $\alpha : \mathbb{R}^k \curvearrowright \mathcal{D}$ be an action with the Rokhlin property. In a similar fashion as in [Kishimoto 2002, Proposition 3.5], we shall show that α has approximately $\mathbb{R}^k$-inner flip.

Set $B = \mathcal{D} \otimes \mathcal{D}$ and $\beta = \alpha \otimes \alpha$. Denote by Σ the flip automorphism on B, which is equivariant with regard to β. Note that β is still a $\mathbb{R}^k$-action on a strongly self-absorbing Kirchberg algebra with the Rokhlin property. The $\mathbb{R}^{k-1}$-action $\alpha^{(\not k)}$ is semi-strongly self-absorbing by the induction hypothesis. Applying [Szabó 2018c, Proposition 3.6], we find a sequence of unitaries $y_n, z_n \in \mathcal{U}(B)$ satisfying

$$\max_{\vec{r} \in [0,1]^{k-1}} \|y_n - \beta^{(\not k)}_{\vec{r}}(y_n)\| + \|z_n - \beta^{(\not k)}_{\vec{r}}(z_n)\| \xrightarrow{n \to \infty} 0 \tag{6-11}$$

and

$$\Sigma(b) = \lim_{n \to \infty} \mathrm{Ad}(y_n z_n y_n^* z_n^*)(b), \quad b \in B. \tag{6-12}$$

Let us set $Y = [(y_n)_n]$ and $Z = [(z_n)_n]$ with $Y, Z \in B_{\infty,\beta^{(\not k)}}^{\beta^{(\not k)}_\infty}$. Moreover set $X = YZY^*Z^*$. Note that since $\mathcal{D}$ is a Kirchberg algebra, Corollary 5.1 implies that β is equivariantly $\mathcal{O}_\infty$-absorbing. By [Szabó 2018c, Proposition 2.19(iii)], the unitary X is thus homotopic to the unit inside $B_{\infty,\beta^{(\not k)}}^{\beta^{(\not k)}_\infty}$. Write $X = \exp(iH_1) \cdots \exp(iH_r)$ for certain self-adjoint elements $H_1, \dots, H_r \in B_{\infty,\beta^{(\not k)}}^{\beta^{(\not k)}_\infty}$. Set $L' = \|H_1\| + \cdots + \|H_r\|$. For $l = 1, \dots, r$, represent H_l via a sequence of self-adjoint elements $h_{l,n} \in B$ with $\|h_{l,n}\| \leq \|H_l\|$. We define a sequence of continuous paths $x_n : [0,1] \to \mathcal{U}(B)$ via

$$x_n(t) = \exp(ith_{1,n}) \cdots \exp(ith_{r,n}).$$

Then each of these paths is L'-Lipschitz. By slight abuse of notation we write $X : [0,1] \to \mathcal{U}(B_{\infty,\beta^{(\not k)}}^{\beta^{(\not k)}_\infty})$ for $X(t) = [(x_n(t))_n]$, which is then continuous and satisfies $X(0) = \mathbf{1}$ and $X(1) = X$. Also define $x_n = x_n(1)$ for all n.

Since we have $\Sigma(b) = XbX^*$ for all $b \in B$ and β and $\Sigma \circ \beta^{(k)}_t = \beta^{(k)}_t \circ \Sigma$, one has $\Sigma(b) = \beta^{(k)}_{\infty,t}(X) b \beta^{(k)}_{\infty,t}(X^*)$ for all $t \in \mathbb{R}$. It follows that for all $t \in \mathbb{R}$, one has that the element $X\beta^{(k)}_{\infty,t}(X^*)$ commutes with all elements in $B \subset B_\infty$.

Let us for the moment fix some number $T > 0$. Define $u_n^T : [0,1] \to \mathcal{U}(B)$ via $u_n^T(t) = x_n(t)\beta^{(k)}_{-T}(x_n(t)^*)$. Then u_n^T is a unitary path starting at the unit and with Lipschitz constant $L \leq 2L'$. We have

$$\max_{0 \leq t \leq 1} \max_{\vec{r} \in [0,1]^{k-1}} \|u_n^T(t) - \beta^{(\not k)}_{\vec{r}}(u_n^T(t))\| \xrightarrow{n \to \infty} 0$$

as $\beta^{(k)}_{-T} \circ \beta^{(\not k)}_{\vec{r}} = \beta^{(\not k)}_{\vec{r}} \circ \beta^{(k)}_{-T}$ and the elements $x_n(t)$ are approximately $\beta^{(\not k)}$-invariant by construction, and

$$\|[u_n^T(1), b]\| = \|[x_n \beta^{(k)}_{-T}(x_n^*), b]\| \xrightarrow{n \to \infty} 0 \quad \text{for all } b \in B.$$

Due to Lemma 6.10, we may replace the unitary paths u_n^T by ones which become approximately central along the entire path and retain all the other properties. In other words, by changing the path u_n on $(0,1)$, we may in fact assume

$$\max_{0\le t\le 1}\|[u_n^T(t),b]\|\xrightarrow{n\to\infty}0\quad\text{for all } b\in B.$$

Let us consider the sequence of $\beta^{(k)}$-1-cocycles $\{w_t^{(n)}\}_{t\in\mathbb{R}}$ given by $w_t^{(n)}=x_n\beta_t^{(k)}(x_n^*)$. Then by what we have observed before, we have

$$\max_{0\le t\le T}\|[w_t^{(n)},b]\|\xrightarrow{n\to\infty}0,\qquad b\in B,$$

as well as

$$\max_{0\le t\le T}\max_{\vec{r}\in[0,1]^{k-1}}\|w_t^{(n)}-\beta_{\vec{r}}^{(\not k)}(w_t^{(n)})\|\le 2\cdot\max_{\vec{r}\in[0,1]^{k-1}}\|x_n-\beta_{\vec{r}}^{(\not k)}(x_n)\|\xrightarrow{n\to\infty}0.$$

This puts us into the position to apply Lemma 6.9. Given some small tolerance $\varepsilon>0$ and $\mathcal{F}\Subset\mathcal{D}$, we can choose $T>0$ and $\mathcal{G}\Subset\mathcal{D}$ with respect to the constant $L=2L'$ and with (B,β) in place of (A,α). Without loss of generality, we choose $\mathcal{F}$ in such a way that

$$\Sigma(\mathcal{F})=\mathcal{F}. \tag{6-13}$$

Then the cocycles $\{w_t^{(n)}\}_{t\in\mathbb{R}}$ and the unitary paths u_n^T (in place of $\{w_t\}_{t\in\mathbb{R}}$ and u in Lemma 6.9) will eventually satisfy the assumptions in Lemma 6.9 for large enough n. By the conclusion of the statement, one finds a unitary $v_n\in\mathcal{U}(B)$ such that

$$\max_{0\le t\le 1}\|w_t^{(n)}-v_n\beta_t^{(k)}(v_n^*)\|=\max_{0\le t\le 1}\|x_n\beta_t^{(k)}(x_n)^*-v_n\beta_t^{(k)}(v_n^*)\|\le 3\varepsilon, \tag{6-14}$$

$$\max_{b\in\mathcal{F}}\|[v_n,b]\|\le 3\varepsilon, \tag{6-15}$$

$$\max_{\vec{r}\in[0,1]^{k-1}}\|v_n-\alpha_{\vec{r}}^{(\not k)}(v_n)\|\le 3\varepsilon. \tag{6-16}$$

We set $\mathbb{U}_n=v_n^*x_n$, which is yet another sequence of unitaries in B. Note that (6-14) translates to

$$\max_{0\le t\le 1}\|\mathbb{U}_n-\beta_t^{(k)}(\mathbb{U}_n)\|\le 3\varepsilon.$$

Together with (6-16) and $X\in B_{\infty,\beta^{(\not k)}}^{\beta_\infty^{(\not k)}}$ this yields

$$\max_{\vec{r}\in[0,1]^k}\|\mathbb{U}_n-\beta_{\vec{r}}(\mathbb{U}_n)\|\le 7\varepsilon$$

for large enough n. Finally, if we combine (6-12), (6-13) and (6-15), we obtain

$$\max_{b\in\mathcal{F}}\|\Sigma(b)-\mathbb{U}_nb\mathbb{U}_n^*\|\le 4\varepsilon$$

for sufficiently large n. Since $\varepsilon>0$ was an arbitrary parameter and $\mathcal{F}\Subset B$ was arbitrary as well, we see that the flip automorphism Σ on B is indeed approximately $\mathbb{R}^k$-inner.

Step 2: Let $\alpha:\mathbb{R}^k\curvearrowright\mathcal{D}$ be an action with the Rokhlin property. Due to the first step, α has approximately $\mathbb{R}^k$-inner flip. By [Szabó 2018b, Proposition 3.3], it follows that the infinite tensor power action

$\alpha^{\otimes\infty} : \mathbb{R}^k \curvearrowright \mathcal{D}^{\otimes\infty}$ is strongly self-absorbing. In view of Remark 6.6, we may apply Corollary 5.1 to α and $\alpha^{\otimes\infty}$ in place of γ, and see that

$$\alpha \simeq_{\mathrm{scc}} \alpha \otimes \alpha^{\otimes\infty} \cong \alpha^{\otimes\infty},$$

which implies that α is semi-strongly self-absorbing.

Step 3: For $i = 0, 1$, let $\alpha^{(i)} : \mathbb{R}^k \curvearrowright \mathcal{D}$ be two actions with the Rokhlin property. By the previous step, they are semi-strongly self-absorbing. If we apply Corollary 5.1 to $\alpha^{(0)}$ in place of α and $\alpha^{(1)}$ in place of γ, then it follows that $\alpha^{(0)} \simeq_{\mathrm{vscc}} \alpha^{(0)} \otimes \alpha^{(1)}$. If we exchange the roles of $\alpha^{(0)}$ and $\alpha^{(1)}$ and repeat this argument, we conclude $\alpha^{(0)} \simeq_{\mathrm{vscc}} \alpha^{(1)}$.

This finishes the induction step and the proof. □

We observe the following consequence as a combination of all of our main results for $\mathbb{R}^k$-actions; this is new even for ordinary flows.

Corollary 6.11. *Let A be a separable* C*-*algebra with $A \cong A \otimes \mathcal{O}_\infty$. Suppose that $\alpha : \mathbb{R}^k \curvearrowright A$ is a multiflow. The following are equivalent*:

(i) *α has the Rokhlin property.*

(ii) *α has finite Rokhlin dimension with commuting towers.*

(iii) *$\alpha \simeq_{\mathrm{vscc}} \alpha \otimes \gamma$ for any multiflow $\gamma : \mathbb{R}^k \curvearrowright \mathcal{O}_\infty$ with the Rokhlin property.*

(iv) *$\alpha \simeq_{\mathrm{vscc}} \alpha \otimes \gamma$ for every multiflow $\gamma : \mathbb{R}^k \curvearrowright \mathcal{O}_\infty$ with the Rokhlin property.*

Proof. This follows directly from Theorem 6.7 and Corollary 5.1. □

Once we combine Corollary 6.11 and Theorem 6.7, we obtain Theorem C as a direct consequence.

The following remains open:

Question 6.12. Let $\alpha : \mathbb{R}^k \curvearrowright A$ be a multiflow on a Kirchberg algebra. Suppose that for every $\vec{r} \in \mathbb{R}^k$ the flow on A given by $t \mapsto \alpha_{t\vec{r}}$ has the Rokhlin property. Does it follow that α has the Rokhlin property?

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