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LONG TIME BEHAVIOR OF THE MASTER EQUATION IN MEAN FIELD GAME THEORY

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Mean field game (MFG) systems describe equilibrium configurations in games with infinitely many interacting controllers. We are interested in the behavior of this system as the horizon becomes large, or as the discount factor tends to 0. We show that, in these two cases, the asymptotic behavior of the mean field game system is strongly related to the long time behavior of the so-called master equation and to the vanishing discount limit of the discounted master equation, respectively. Both equations are nonlinear transport equations in the space of measures. We prove the existence of a solution to an ergodic master equation, towards which the time-dependent master equation converges as the horizon becomes large, and towards which the discounted master equation converges as the discount factor tends to 0. The whole analysis is based on new estimates for the exponential rates of convergence of the time-dependent and the discounted MFG systems, respectively.

Given a terminal time T and an initial measure m_0, we consider the solution to the mean field game (MFG) system

$$\begin{cases} -\partial_t u^T - \Delta u^T + H(x, Du^T) = F(x, m^T) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m^T - \Delta m^T - \operatorname{div}(m^T H_p(x, Du^T)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m^T(0,\cdot) = m_0, \quad u^T(T,\cdot) = G(\cdot, m^T(T)) & \text{in } \mathbb{T}^d, \end{cases} \tag{1}$$

where $\mathbb{T}^d$ is the d-dimensional flat torus $\mathbb{R}^d/\mathbb{Z}^d$, F, G are functions defined on $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ (the space of probability measures on $\mathbb{T}^d$) and H is a function, defined on $\mathbb{T}^d \times \mathbb{R}^d$, which is convex in the second variable.

Let us recall that this system appears in mean field games theory, introduced by Lasry and Lions [2006a; 2006b; 2007] and by Huang, Caines and Malhamé [Huang et al. 2006]. Mean field games are dynamic

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games with infinitely many players. The first equation in (1) can be interpreted as the value function of a small player whose cost depends on the density $m(t)$ of the players, while the second equation describes the evolution in time of the density of the players. Note that the first equation is backward in time (and with a terminal condition) while the second one is forward, with the initial condition $m(0) = m_0$, m_0 being the initial repartition of the players.

The study of the long time average of the MFG system was initiated in [Lions 2010] and then discussed in several different contexts [Cardaliaguet et al. 2012; 2013; Cardaliaguet 2013; Cardaliaguet and Graber 2015; Gomes et al. 2010].

In [Cardaliaguet et al. 2013] the long time average of u^T is investigated when $H(x, p) = \frac{1}{2}|p|^2$ and $F(x, m)$, $G(x, m)$ satisfy suitable smoothing conditions with respect to the measure m. Then it is proved that there exists a constant $\bar\lambda \in \mathbb{R}$ such that the scaled function $(s, x) \to u^T(Ts, x)/T$ locally uniformly converges to the map $(s, x) \to -\bar\lambda s$ as $T \to \infty$ on $(0, 1) \times \mathbb{T}^d$, while the rescaled measure $(s, x) \to m^T(sT, x)$ converges to a time-invariant measure $\bar m$ in $L^1((0, 1) \times \mathbb{T}^d)$. The constant $\bar\lambda$ and the measure $\bar m$ are characterized as solutions of the ergodic MFG system; namely, there exists a unique triple $(\bar\lambda, \bar u, \bar m)$ which solves

$$\begin{cases} \bar\lambda - \Delta\bar u + H(x, D\bar u) = F(x, \bar m) & \text{in } \mathbb{T}^d, \\ -\Delta\bar m - \operatorname{div}(\bar m H_p(x, D\bar u)) = 0 & \text{in } \mathbb{T}^d, \\ \bar m \geq 0,\ \int_{\mathbb{T}^d} \bar m = 1, \quad \int_{\mathbb{T}^d} \bar u = 0 & \text{in } \mathbb{T}^d, \end{cases} \tag{2}$$

and $Du^T(sT, x)$ actually converges to $D\bar u(x)$. The result holds under a monotonicity condition on F and G:

$$\int_{\mathbb{T}^d} (F(x, m) - F(x, m'))(m - m')\, dx \geq 0, \qquad \int_{\mathbb{T}^d} (G(x, m) - G(x, m'))(m - m')\, dx \geq 0$$

for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$. Moreover it is proved in [Cardaliaguet et al. 2013] that the convergence holds with an exponential rate. Precisely, under some additional conditions on the smoothing properties of the coupling terms F and G, one has

$$\|m^T(t) - \bar m\|_{C^{2+\alpha}} + \|Du^T(t) - D\bar u\|_{C^{2+\alpha}} \leq C(e^{-\omega t} + e^{-\omega(T-t)})$$

for some constants $C, \omega > 0$ and $\alpha \in (0, 1)$.

This paper is devoted to the long time behavior of u^T, i.e., the convergence, as $T \to \infty$, of the map $(t, x) \to u^T(t, x) - \bar\lambda(T - t)$. This question is inspired by results of Fathi [1997a; 1997b], Roquejoffre [1998], Namah and Roquejoffre [1999] and Barles and Souganidis [2000] for Hamilton–Jacobi equations. In that framework, it is known that if u solves the (forward) Hamilton–Jacobi equation

$$\partial_t u - \Delta u + H(x, Du) = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^d,$$

with associated ergodic constant $\bar\lambda$, then $u(t, x) - \bar\lambda t$ converges, as $t \to +\infty$, to a solution $\bar u$ of the associated ergodic problem. One may wonder what remains of this result for the MFG system.

The convergence of the difference $u^T(t, \cdot) - \bar\lambda(T - t)$, as $T \to \infty$, has been an open (and puzzling) question since [Cardaliaguet et al. 2013]. We prove in this paper that the limit of $u^T(t, \cdot) - \bar\lambda(T - t)$

indeed exists, although it cannot be described just in terms of the $\bar{u}$-component of the MFG ergodic system (2). In order to describe this long-time behavior, we have to keep track of the initial measure m_0. To do so, we rely on the master equation, which is the following (backward) transport equation in the space of measures:

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - F(x,m) \\ \qquad - \displaystyle\int_{\mathbb{T}^d} \operatorname{div}(D_m U(t,x,m,y))\, dm(y) \\ \qquad\qquad + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot H_p(y, D_x U(t,y,m))\, dm(y) = 0 & \text{in } (-\infty,0) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(0,x,m) = G(x,m) & \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{cases} \tag{3}$$

In the above equation, the unknown $U = U(t,x,m)$ depends on time, space and the measure on the space; moreover, the notation $D_m U$ denotes a suitable derivative with respect to probability measures, which will be described in Section 1A. Note that, in contrast with the MFG system, the master equation is a classical evolution equation, so its long time behavior may be described in a usual form. We recall, see [Lions 2010; Gangbo and Święch 2015; Chassagneux et al. 2014; Cardaliaguet et al. 2019], that the master equation is well-posed under the monotonicity condition on F and G and that the MFG system (1) plays the role of characteristics for this equation. Namely, if (u^T, m^T) solves (1), then

$$U(-T,x,m_0) = u^T(0,x) \quad \text{for all } x \in \mathbb{T}^d.$$

Our main result (Theorem 5.1) states that $U(t,\cdot,\cdot) + \bar{\lambda} t$ has a limit $\chi = \chi(x,m)$ as $t \to -\infty$. This limit solves (in a weak sense) the ergodic master equation

$$\bar{\lambda} - \Delta_x \chi(x,m) + H(x, D_x \chi(x,m)) - \int_{\mathbb{T}^d} \operatorname{div}(D_m \chi(x,m,y))\, dm(y) \\ + \int_{\mathbb{T}^d} D_m \chi(x,m,y) \cdot H_p(y, D_x \chi(y,m))\, dm(y) = F(x,m) \quad \text{in } T^d \times \mathcal{P}(\mathbb{T}^d). \tag{4}$$

As a consequence, the limit $u^T(0,\cdot) - \bar{\lambda} T$ exists as $T \to \infty$ and is equal to $\chi(\cdot, m_0)$. Note that, in general, $u^T(0,\cdot) - \bar{\lambda} T$ does not converge to $\bar{u}$, since it is not always true that $\chi(\cdot, m_0) = \bar{u}$ (even up to an additive constant); this is however the case if $m_0 = \bar{m}$.

We are also interested in the infinite-horizon MFG system

$$\begin{cases} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = F(x, m^\delta(t)) & \text{in } (0,+\infty) \times \mathbb{T}^d, \\ \partial_t m^\delta - \Delta m^\delta - \operatorname{div}(m^\delta H_p(x, Du^\delta)) = 0 & \text{in } (0,+\infty) \times \mathbb{T}^d, \\ m^\delta(0,\cdot) = m_0 \quad \text{in } (0,+\infty) \times \mathbb{T}^d, \qquad u^\delta \text{ bounded.} \end{cases} \tag{5}$$

In the first-order stationary Hamilton–Jacobi (HJ) setting, where the equation reads

$$\delta u^\delta + H(x, Du^\delta) = 0 \quad \text{in } \mathbb{T}^d,$$

Gomes [2008] and Davini, Fathi, Iturriaga and Zavidovique [2016] have proved the convergence of $u^\delta - \delta^{-1}\bar{\lambda}$ as δ tends to 0 and characterized the limit. The result has been generalized to the second-order

HJ setting by Mitake and Tran [2017]; see also [Le et al. 2017; Ishii et al. 2017]. In the viscous case, the result is that, if u^δ solves the infinite-horizon problem

$$\delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = 0 \quad \text{in } \mathbb{T}^d,$$

then $u^\delta - \delta^{-1}\bar\lambda$ converges as $\delta \to 0$ to the unique solution $\bar u$ of the ergodic cell problem

$$-\Delta \bar u + H(x, D\bar u) = 0 \quad \text{in } \mathbb{T}^d$$

such that $\int_{\mathbb{T}^d} \bar u \bar m = 0$, where $\bar m$ solves

$$-\Delta \bar m - \operatorname{div}(\bar m H_p(x, D\bar u)) = 0 \quad \text{in } \mathbb{T}^d, \quad \bar m \geq 0, \ \int_{\mathbb{T}^d} \bar m = 1.$$

Here again, one may wonder if such a result remains true for the infinite-horizon MFG system (5) (which, in contrast with the Hamilton–Jacobi case, is time-dependent). As for the time-evolution MFG problem, we rely on a master equation. Following [Cardaliaguet et al. 2019], this infinite-horizon master equation takes the form[1]

$$\begin{aligned} \delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - \int_{\mathbb{T}^d} \operatorname{div}_y(D_m U^\delta(x, m, y))\, dm(y) \\ + \int_{\mathbb{T}^d} D_m U^\delta(x, m, y) \cdot H_p(y, D_x U^\delta(y, m))\, dm(y) = F(x, m) \quad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{aligned} \tag{6}$$

Our second main result (Theorem 6.1) is that $U^\delta - \delta^{-1}\bar\lambda$ converges to the unique solution χ of the master ergodic problem (4) satisfying $\chi(x, \bar m) = \bar u$, where $\bar u$ is the unique solution of the ergodic MFG system (2) for which the following (new) linearized ergodic MFG system has a solution $(\bar v, \bar\mu)$:

$$\begin{cases} \bar u - \Delta \bar v + H_p(x, D\bar u) \cdot D\bar v = \dfrac{\delta F}{\delta m}(x, \bar m)(\bar\mu) & \text{in } \mathbb{T}^d, \\ -\Delta \bar\mu - \operatorname{div}(\bar\mu H_p(x, D\bar u)) - \operatorname{div}(\bar m H_{pp}(x, D\bar u) D\bar v) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \bar\mu = \int_{\mathbb{T}^d} \bar v = 0, \end{cases}$$

(the definition of the derivative $\delta F/\delta m$ is explained in Section 1). This implies the convergence of $u^\delta(0, \cdot) - \delta^{-1}\bar\lambda$ to $\chi(\cdot, m_0)$ as δ tends to 0. Note that if $F \equiv 0$, i.e., in the Hamilton–Jacobi case, one recovers the condition $\int_{\mathbb{T}^d} \bar u \bar m = 0$ by integrating the $\bar v$-equation against the measure $\bar m$. The MFG setting is more subtle since it keeps track of the coupling between the equations.

Let us now say a few words about the method of proofs. As in the Hamilton–Jacobi setting, the argument relies on compactness arguments and, therefore, on the regularity (Lipschitz estimates) for the solution U of the master equation (3) and for the solution U^δ of the infinite-horizon master equation (6). The main difficulty comes from the fact that these equations *do not satisfy a comparison principle* (in contrast to the HJ equation). Moreover, as can be seen plainly from (3) and (6), the equations do not provide easy bounds on the derivatives with respect to m of U and U^δ.

[1] See in particular the comments in the introduction of [Cardaliaguet et al. 2019], which explain that the approach of that work also applies to get the existence and uniqueness of solutions to this equation.

The key Lipschitz estimates come from the fact that the characteristics (1) and (5) of these master equations stabilize exponentially fast in time to the solution of the ergodic MFG system (2) and, respectively, to the solution of the time-invariant infinite-horizon problem

$$\begin{cases} \delta\bar{u}^\delta - \Delta\bar{u}^\delta + H(x, D\bar{u}^\delta) = F(x, \bar{m}^\delta) & \text{in } \mathbb{T}^d, \\ -\Delta\bar{m}^\delta - \operatorname{div}(\bar{m}^\delta H_p(x, D\bar{u}^\delta)) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \bar{m}^\delta = 1; \end{cases} \tag{7}$$

see Theorems 2.6 and 3.7 respectively. These exponential convergence rates were only known for system (1) when $H(x, p) = |p|^2$, see [Cardaliaguet et al. 2013], where the argument relied on some commutation properties which do not hold for general Hamiltonians. To prove the exponential convergence in our setting, we use a technique developed by one of us with E. Zuazua [Porretta and Zuazua 2013] to establish the so-called turnpike property for optimal control problems. The exponential rate for the infinite-horizon MFG system is new, but uses similar ideas.

The starting point of this analysis consists in studying the linearized MFG systems. For simplicity, let us explain this idea for the time-dependent problem, i.e., for U. In this framework, the MFG linearized system reads

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, Du) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m)(\mu(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, Du)) - \operatorname{div}(m H_{pp}(x, Du) Dv) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0, \qquad v(T, x) = \dfrac{\delta G}{\delta m}(x, m)(\mu(T)) \quad \text{in } \mathbb{T}^d, \end{cases}$$

where (u, m) is the solution of (1) and μ_0 is given. When $(u, m) = (\bar{u}, \bar{m})$, the analysis of the above system (the exponential decay of the solutions) provides an exponential convergence of the solution of the MFG system to $(\bar{u}, \bar{m})$ — at least, this holds true for the m-component. A very interesting point is that this linearized system turns out to be also strongly related to the derivative of U with respect to m: indeed, as explained in [Cardaliaguet et al. 2019], we have

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(0, x, m_0, y)\mu_0(y)\, dy = v(0, x) \quad \text{for all } x \in \mathbb{T}^d.$$

Thus controlling v allows us to control the variations of U with respect to m. Once the Lipschitz estimates for U and for U^δ are obtained, the construction of a corrector χ (solution of the ergodic master equation (4)) follows in a standard way; see Theorem 4.2.

However, the convergence of the solution of the time-dependent master equation (3) requires new ideas since, in contrast with the Hamilton–Jacobi setting, see [Fathi 2008; Barles and Souganidis 2000], there is no obvious quantity which is monotone in time; the reason is that the master equation does not satisfy a comparison principle. To overcome this issue, we rely again on the exponential convergence rate from which we derive a suitable convergence of the solution of the master equation when evaluated at $\bar{m}$ as time tends to $-\infty$ (see Proposition 2.7). Then we obtain the convergence of the map U by a compactness argument and using again the convergence of the characteristics.

The convergence of U^δ is more subtle: the key point is that two solutions of the ergodic master equation differ only by a constant. Thus we only have to show that $U^\delta(\cdot\,, m) - \delta^{-1}\bar\lambda$ has a limit for some m. The good choice turns out to be $m = \bar m^\delta$, where $(\bar u^\delta, \bar m^\delta)$ solves (7); indeed, we have then $U^\delta(\cdot\,, \bar m^\delta) = \bar u^\delta$ and we expect $(\bar u^\delta, \bar m^\delta)$ to be close to $(\bar u, \bar m)$ in some sense, where $(\bar u, \bar m)$ satisfies (2). Actually a formal expansion yields $(\bar u^\delta, \bar m^\delta) = (\delta^{-1}\bar\lambda + \bar u + \bar\theta + \delta\bar v, \bar m + \delta\bar\mu)$, where $(\bar\theta, \bar v, \bar\mu)$ solves

$$\begin{cases} \bar u + \bar\theta - \Delta\bar v + H_p(x, D\bar u)\cdot D\bar v = \dfrac{\delta F}{\delta m}(x, \bar m)(\bar\mu) & \text{in } \mathbb{T}^d, \\ -\Delta\bar\mu - \operatorname{div}(\bar\mu H_p(x, D\bar u)) - \operatorname{div}(\bar m H_{pp}(x, D\bar u) D\bar v) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \bar\mu = \int_{\mathbb{T}^d} \bar v = 0. \end{cases}$$

The rigorous justification is given in Proposition 6.5.

The paper is organized in the following way. In Section 1 we recall the notion of derivative in the space of measures and state our main assumptions. We also recall some decay and regularity estimates which hold separately for the two equations of the system and we provide the basic fundamental estimates for (1) which are independent of the horizon T. Section 2 is devoted to the exponential convergence rate, as $T \to \infty$, of solutions of (1) towards the pair $(\bar u, \bar m)$, a solution of (2). For this purpose, first we develop decay estimates in L^2 for the linearized system, and then we export the estimates (in stronger norms) to $(u^T - \bar u, m^T - \bar m)$ by using a fixed-point argument. A similar strategy is used in Section 3 for the infinite-horizon discounted problem (5); in this case we prove the exponential convergence as $t \to \infty$ towards the stationary pair $(\bar u^\delta, \bar m^\delta)$, a solution of (7). In both Sections 2 and 3, the analysis of the linearized systems is a crucial step, and this will also play a key role in the study of the master equations, both the time-dependent (3) and the stationary one (6), respectively. This is the content of Sections 4–6. More precisely, in Section 4 we prove the existence of a solution to the ergodic master equation, obtained as the limit, when $\delta \to 0$, of a subsequence of solutions of (6). The long-time behavior of the time-dependent master equation (3) is addressed in Section 5. Finally, the limit of the whole sequence of solutions of (6) is proved in Section 6.

1. Notation, assumptions and preliminary estimates

1A. ***Notation and assumptions.*** Throughout the paper we work on the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$; this means that all equations are $\mathbb{Z}^d$-periodic in space. This assumption is standard in the framework of the long time behavior. We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed with the Monge–Kantorovich distance $\boldsymbol{d}_1$

$$\boldsymbol{d}_1(m, m') = \sup_\phi \int_{\mathbb{T}^d} \phi\, d(m - m') \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d),$$

where the supremum is taken over all 1-Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$.

For $\alpha \in [0, 1]$, we denote by $C^\alpha([0, T], \mathcal{P}(\mathbb{T}^d))$ the set of maps $m : [0, T] \to \mathcal{P}(\mathbb{T}^d)$ which are α-Hölder continuous if $\alpha \in (0, 1)$ and continuous if $\alpha = 0$.

Next we recall the notion of derivative of a map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ as introduced in [Cardaliaguet et al. 2019]. We say that U is C^1 if there exists a continuous map $\delta U/\delta m : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-t)m + tm', y)\, d(m'-m)(y)\, dt \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d).$$

We observe that if U can be extended to $L^2(\mathbb{T}^d)$ then $y \mapsto (\delta U/\delta m)(m, y)$ is nothing but the representation in L^2 of the Gâteaux derivative of U computed at m. The fact that U is defined on probability measures, i.e., with the constraint of mass 1, lets $(\delta U/\delta m)(m, y)$ be defined up to a constant. We normalize the derivative by the condition

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m, y)\, dm(y) = 0 \quad \text{for all } m \in \mathcal{P}(\mathbb{T}^d). \tag{8}$$

We write interchangeably $(\delta U/\delta m)(m)(\mu)$ and $\int_{\mathbb{T}^d} (\delta U/\delta m)(m, y)\, d\mu(y)$ for a signed measure μ with finite mass.

When the map $\delta U/\delta m = (\delta U/\delta m)(m, y)$ is differentiable with respect to the last variable, we denote by $D_m U(m, y)$ its gradient:

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$

Let us recall [Cardaliaguet et al. 2019] that $D_m U$ can be used to estimate the Lipschitz regularity of U in the m-variable:

$$|U(m) - U(m')| \le \boldsymbol{d}_1(m, m')\Big[\sup_{m'' \in \mathcal{P}(\mathbb{T}^d),\, y \in \mathbb{T}^d} |D_m U(m'', y)|\Big] \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d).$$

For $p = 1, 2, \infty$, we denote by $\|\cdot\|_{L^p}$ the L^p norm of a map on $\mathbb{T}^d$ (we often use the notation $\|\cdot\|_\infty$ for $\|\cdot\|_{L^\infty}$). For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, we denote by $\|\cdot\|_{C^k}$ and $\|\cdot\|_{C^{k+\alpha}}$ the standard norm on the set of maps defined on $\mathbb{T}^d$ and which are, respectively, of class C^k and of class C^k with a k-th derivative which is α-Hölder continuous. By $\|\cdot\|_{(C^{k+\alpha})'}$ we mean the norm in the dual space:

$$\|\phi\|_{(C^{k+\alpha})'} := \sup\left\{\int_{\mathbb{T}^d} \phi\psi,\ \|\psi\|_{C^{k+\alpha}} \le 1\right\}.$$

For a map ϕ depending of two spatial variables, we denote by $\|\phi(\cdot,\cdot)\|_{k+\alpha, k'+\alpha}$ the supremum of the α-Hölder norm of the partial derivatives of order $l \le k$ and $l' \le k'$ respectively of the map ϕ.

Finally, if $\phi = \phi(x)$, we systematically denote by $\langle\phi\rangle := \int_{\mathbb{T}^d} \phi(x)\, dx$ the average of ϕ.

If $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ is a sufficiently smooth map, we denote by $Du(t, x)$ and $\Delta u(t, x)$ its spatial gradient and spatial Laplacian and by $\partial_t u(t, x)$ its partial derivative with respect to the time variable. We will also use the classical parabolic Hölder spaces: for $\alpha \in (0, 1)$, we denote by $C^{\alpha/2,\alpha}$ the set of maps which are α-Hölder in space and $\alpha/2$-Hölder in time and by $C^{1+\alpha/2,2+\alpha}$ the set of maps u such that $\partial_t u$ and $D^2 u$ are in $C^{\alpha/2,\alpha}$.

Assumptions. The following assumptions are in force throughout the paper.

(H) The Hamiltonian $H = H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is of class C^2 and the function $p \mapsto D^2_{pp}H(x, p)$ is Lipschitz continuous, uniformly with respect to x, and satisfies the growth condition

$$C^{-1}I_d \leq D^2_{pp}H(x, p) \leq CI_d \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{9}$$

Moreover we suppose that there exist $\theta \in (0, 1)$ and $C > 0$ such that

$$|D_{xx}H(x, p)| \leq C(1 + |p|)^{1+\theta}, \quad |D_{xp}H(x, p)| \leq C(1 + |p|)^{\theta} \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{10}$$

This latter assumption is a little awkward, since it requires the quadratic part of H to be independent of the space variable, but we actually need it in order to ensure uniform Lipschitz regularity of a solution u^T of (1) and of a solution u^δ of (5) independently of T and δ: see Lemmas 1.5 and 3.6. If the same bounds were available with different arguments, then we could get rid of this condition, since in the rest of the paper we do not use it at all.

(FG) The coupling functions $F, G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are assumed to be of class C^1 and their first derivatives satisfy the following Lipschitz conditions:

(FGa) F, G are twice differentiable in the x-variable and $F_{xx}(x, m)$, $G_{xx}(x, m)$ are bounded uniformly in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.

(FGb) $(\delta F/\delta m)(x, m, y)$, $(\delta G/\delta m)(x, m, y)$ are differentiable with respect to (x, y) and Lipschitz continuous in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d$ (i.e., globally Lipschitz in the three variables).

Even if this will not be strictly needed, an extra regularity condition is assumed in order to get to smooth solutions of the master equation as stated in [Cardaliaguet et al. 2019]. Namely we assume that:

(FGc) For any $\alpha \in (0, 1)$, $F(\cdot, m)$ and $(\delta F/\delta m)(\cdot, m, \cdot)$ are of class $C^{2+\alpha}$ in all space variables, uniformly in m, and $\delta F/\delta m$ is Lipschitz continuous in m with respect to $C^{2+\alpha}$ in space. The same holds for G in norm $C^{3+\alpha}$.

(FGd) The maps F and G are assumed to be monotone: for any $m \in \mathcal{P}(\mathbb{T}^d)$ and for any centered Radon measure μ,

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\delta F}{\delta m}(x, m, y)\mu(x)\mu(y)\,dx\,dy \geq 0, \quad \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\delta G}{\delta m}(x, m, y)\mu(x)\mu(y)\,dx\,dy \geq 0. \tag{11}$$

Let us comment upon our assumptions.

The regularity of H as well as the uniform convexity with respect to the second variable are standard in MFG theory. Here these assumptions are all the more important because we make systematic use of the duality inequality, see [Lasry and Lions 2007], which provides uniqueness and quantified stability for the MFG system under this strong convexity assumption.

The regularity assumption on $\delta F/\delta m$ (and on $\delta G/\delta m$) allows for instance inequalities of the form

$$\left\| \frac{\delta F}{\delta m}(\cdot, m)(\mu) \right\|_{C^2} \leq C\|\mu\|_{(C^2)'}$$

for any $m \in \mathcal{P}(\mathbb{T}^d)$ and any distribution μ on $\mathbb{T}^d$.

The monotonicity assumption (11) implies (and, under our regularity assumptions, is equivalent to) the more standard one

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))\, d(m-m')(x) \geq 0, \qquad \int_{\mathbb{T}^d} (G(x,m) - G(x,m'))\, d(m-m')(x) \geq 0$$

for any measures $m, m' \in \mathcal{P}(\mathbb{T}^d)$. This condition ensures the well-posedness of the MFG system (1) for large time intervals and the well-posedness of the ergodic MFG system (2). Without this assumption, these MFG systems may have several solutions and the long time average (and a fortiori the long time behavior) of the MFG system (1) is not known.

Let us illustrate our assumptions by examples. The Hamiltonian functions we have in mind are for instance of the form

$$H(x,p) = \tfrac{1}{2}|p|^2 + V(x)\cdot p + g(x),$$

where $V : \mathbb{T}^d \to \mathbb{R}^d$ is a smooth vector field and $g : \mathbb{T}^d \to \mathbb{R}$ is a smooth map. Typical examples of coupling maps F and G satisfying our conditions take the form

$$\Phi(x,m) = [\phi(\,\cdot\,, (\rho \star m)(\,\cdot\,)) \star \rho](x),$$

where $\star$ denotes the usual convolution product in $\mathbb{R}^d$, $\phi : \mathbb{R}^2 \to \mathbb{R}$ is smooth and nondecreasing with respect to the second variable and $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth, even function with compact support; see for instance Example 2.3.1. in [Cardaliaguet et al. 2019].

Let us stress that, in the following, we will denote generically by C possibly different constants appearing in the estimates which depend on the data F, G and H through the above assumptions. In particular, those constants will depend on the sup-norm of F_{xx}, G_{xx} (which are bounded uniformly with respect to x and m from (FGa)), the Lipschitz constants of $\delta F/\delta m$, $\delta G/\delta m$ and the conditions (9)–(10), respectively. Actually, those constants will also depend on the unique solution $\bar{\lambda}$, $\bar{u}$, $\bar{m}$ of (2), but this triple is also meant as (uniquely) depending on the data F, G and H, so we will not mention this kind of dependence otherwise.

1B. *Preliminary estimates.* We will use throughout the text the following estimates on linear equations which are independent of the time horizon. The first one is about linear equations in divergence form; see [Cardaliaguet et al. 2013, Lemmas 7.1 and 7.6].

Lemma 1.1. *Let V be a bounded vector field on $(0,T) \times \mathbb{T}^d$, let $B \in L^2((0,T) \times \mathbb{T}^d)$ and let μ be the solution to*

$$\begin{cases} \partial_t \mu - \Delta\mu + \operatorname{div}(\mu V) = \operatorname{div}(B) & \text{in } (0,T) \times \mathbb{T}^d, \\ \mu(0) = \mu_0 & \text{in } \mathbb{T}^d, \end{cases} \tag{12}$$

with $\int_{\mathbb{T}^d} \mu_0 = 0$.

There exist constants $\omega > 0$ and $C > 0$, depending only on $\|V\|_\infty$, such that

$$\|\mu(t)\|_{L^2} \leq Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^t \|B(s)\|_{L^2}^2\, ds\right]^{1/2}.$$

If $B \equiv 0$, we also have, for any $\tau > 0$,

$$\|\mu(t)\|_\infty \le C_\tau e^{-\omega t}\|\mu_0\|_{L^1} \quad \textit{for all } t \ge \tau,$$

where the constant C_τ depends on τ and $\|V\|_\infty$ only.

The second lemma is about a viscous transport equation; see [Cardaliaguet et al. 2013, Lemmas 7.4 and 7.5].

Lemma 1.2. *Let V be a bounded vector field, $A \in L^2((0,T)\times\mathbb{T}^d)$ and v be the solution to the backward equation*

$$-\partial_t v - \Delta v + V\cdot Dv = A \quad \textit{in } (0,T)\times\mathbb{T}^d. \tag{13}$$

There exist constants $\omega > 0$ and $C > 0$, depending only on $\|V\|_\infty$, such that

$$\|v(t)-\langle v(t)\rangle\|_{L^2} \le Ce^{-\omega(T-t)}\|v(T)-\langle v(T)\rangle\|_{L^2} + C\int_t^T e^{-\omega(s-t)}\|A(s)\|_{L^2}\,ds$$

and, if $A \in L^\infty((0,T)\times\mathbb{T}^d)$,

$$\|v(t)-\langle v(t)\rangle\|_{L^\infty} \le Ce^{-\omega(T-t)}\|v(T)-\langle v(T)\rangle\|_{L^\infty} + C\int_t^T e^{-\omega(s-t)}\|A(s)\|_{L^\infty}\,ds,$$

where $\langle\phi\rangle = \int_{\mathbb{T}^d}\phi$ for any map ϕ. Moreover, for any $0 \le t < t_0 \le T$,

$$(t_0-t)\|Dv(t)\|_{L^2} \le C(t_0-t+1)\big(\|v(t_0)-\langle v(t_0)\rangle\|_{L^2} + \|A\|_{L^2((t,t_0)\times\mathbb{T}^d)} + \|v-\langle v\rangle\|_{L^2((t,t_0)\times\mathbb{T}^d)}\big).$$

We note for later use a simple consequence of Lemma 1.1:

Corollary 1.3. *Let V and B be (time-independent) vector fields. Then any L^2 solution of*

$$-\Delta\mu + \operatorname{div}(\mu V) = \operatorname{div}(B) \quad \textit{in } \mathbb{T}^d,$$

with $\int_{\mathbb{T}^d}\mu = 0$, satisfies

$$\|\mu\|_{H^1} \le C\|B\|_{L^2},$$

where C depends only on $\|V\|_\infty$.

Proof. It is enough to apply Lemma 1.1:

$$\|\mu\|_{L^2} \le Ce^{-\omega t}\|\mu\|_{L^2} + C\|B\|_{L^2}t^{1/2}.$$

Choosing t large enough, this gives

$$\|\mu\|_{L^2} \le C\|B\|_{L^2}.$$

Then, multiplying the equation by μ, the standard energy estimate gives

$$\|D\mu\|_{L^2} \le [\|V\|_\infty\|\mu\|_{L^2} + \|B\|_{L^2}],$$

which gives the result. □

We conclude this section with a further bound for the solutions of the Fokker–Planck equation.

Lemma 1.4. *Let V be a bounded vector field on $(0, T) \times \mathbb{T}^d$ with bounded space derivatives and μ be a weak solution to* (12) *with $B \equiv 0$. Then, for any $\tau > 0$,*

$$\|\mu(t)\|_\infty \le C_\tau e^{-\omega t}\|\mu_0\|_{(C^{2+\alpha})'} \quad \textit{for all } t \ge \tau,$$

where ω is given by Lemma 1.1, $\alpha \in (0, 1)$ and $C_\tau > 0$ depends on $\|V\|_{L^\infty}$, $\|DV\|_{L^\infty}$ and τ.

Proof. Let $\tau > 0$ and v be the solution to the transport equation

$$\begin{cases} -\partial_t v - \Delta v + V \cdot Dv = 0 & \text{in } (0, \tau) \times \mathbb{T}^d, \\ v(\tau, x) = v_\tau(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{14}$$

where v_τ is in $C^\infty(\mathbb{T}^d)$. One easily checks that

$$\sup_t \|v(t)\|_{L^2} + \|Dv\|_{L^2((0,\tau)\times\mathbb{T}^d)} \le C\|v_\tau\|_{L^2},$$

where C depends on $\|V\|_\infty$ and τ only. Standard parabolic regularity [Ladyženskaja et al. 1968, Theorem III.11.1] then implies

$$\|Dv\|_{C^{\alpha/2,\alpha}([0,\tau/2]\times\mathbb{T}^d)} \le C\|v_\tau\|_{L^2}$$

for some α and C depending on $\|V\|_\infty$ and τ only. For any $i \in \{1, \dots, d\}$, the derivative v_{x_i} solves

$$-\partial_t v_{x_i} - \Delta v_{x_i} + V \cdot Dv_{x_i} + V_{x_i} \cdot Dv = 0 \quad \text{in } (0, \tau/2) \times \mathbb{T}^d.$$

By parabolic regularity [Ladyženskaja et al. 1968, Theorem III.11.1], we infer that

$$\|D^2 v\|_{C^{\alpha/2,\alpha}([0,\tau/4]\times\mathbb{T}^d)} \le C\|Dv\|_{L^\infty((0,\tau/2)\times\mathbb{T}^d)} \le C\|v_\tau\|_{L^2}$$

for some α and C depending on $\|V\|_\infty$, $\|DV\|_\infty$ and τ only. We have, since (14) is the dual equation of (13),

$$\int_{\mathbb{T}^d} v_\tau \mu(\tau) = \int_{\mathbb{T}^d} v(0)\, d\mu_0(x).$$

So taking the supremum over v_τ such that $\|v_\tau\|_{L^2} \le 1$, we infer that

$$\|\mu(\tau)\|_{L^2} \le C_\tau\|\mu_0\|_{(C^{2+\alpha})'} \quad \text{for all } \tau > 0.$$

We can then derive the conclusion by Lemma 1.1. □

1C. ***Regularity of the MFG system.*** The aim of this section is to provide additional basic estimates on the solution to the MFG system

$$\begin{cases} -\partial_t u + \bar{\lambda} - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ m(0, \cdot) = m_0, \quad u(T, \cdot) = g & \text{in } \mathbb{T}^d, \end{cases} \tag{15}$$

where $m_0 \in \mathcal{P}(\mathbb{T}^d)$. Let us recall that $\bar{\lambda} \in \mathbb{R}$ is the unique ergodic constant and $(\bar{u}, \bar{m})$ the unique solution to the ergodic MFG system (2).

The following estimates have been mostly well known since [Cardaliaguet et al. 2013], but we collect them for the sake of completeness. The whole point is to get estimates which are independent of the time horizon or of the discount rate. For this purpose we rely on conditions (9)–(10), as well as on the smoothing assumption (FGa) for the couplings.

Lemma 1.5. *For any $M > 0$, there exists a constant $C > 0$ such that for any horizon $T > 0$, if (u, m) is a solution to the MFG system* (15) *and $\|g\|_{C^2(\mathbb{T}^d)} \le M$, then*

$$\|Du\|_\infty \le C.$$

Proof. As in Lemma 3.2 in [Cardaliaguet et al. 2013], the proof relies on the uniform semiconcavity of the solution. Let us recall that, for any smooth map $\phi \in C^\infty(\mathbb{T}^d)$, we have

$$\|D\phi\|_\infty \le d^{1/2} \sup_{x\in\mathbb{T}^d,\ |z|\le 1} (D^2\phi(x)z\cdot z)_+. \tag{16}$$

Let ξ with $|\xi| \le 1$ be a direction for which $C_0 := \sup_{(t,x)} D^2u(t,x)\xi\cdot\xi$ is maximal (and thus nonnegative). We set $w(t,x) = D^2u(t,x)\xi\cdot\xi = u_{\xi\xi}(t,x)$. Then w solves

$$-\partial_t w - \Delta w + H_{\xi\xi}(x, Du) + 2H_{\xi p}(x, Du)\cdot Du_\xi + H_{pp}(x, Du)Du_\xi\cdot Du_\xi + H_p(x, Du)\cdot Dw = F_{\xi\xi}(x, m(t)).$$

If the maximum of w is reached at T, then

$$C_0 \le \max_{x\in\mathbb{T}^d} D^2g(x)\xi\cdot\xi \le M.$$

Otherwise, one has at the maximum point (t, x) of w:

$$H_{\xi\xi}(x, Du) + 2H_{\xi p}(x, Du)\cdot Du_\xi + H_{pp}(x, Du)Du_\xi\cdot Du_\xi \le F_{\xi\xi}(x, m(t)),$$

where by our standing assumptions on H we have

$$H_{\xi\xi}(x, Du) \ge -C(1+|Du|)^{1+\theta},$$
$$H_{pp}(x, Du)Du_\xi\cdot Du_\xi + 2H_{\xi p}(x, Du)\cdot Du_\xi \ge C^{-1}|Du_\xi|^2 - C(1+|Du|)^{2\theta}.$$

Since (16) implies $\|Du\|_\infty \le d^{1/2}C_0$, we deduce that

$$-C(1+C_0)^{1+\theta} - C(1+C_0)^{2\theta} + C^{-1}|Du_\xi|^2 \le C$$

and since $|Du_\xi| \ge C_0$ at the maximum point of $w(t,x)$, because $\theta < 1$ we conclude that C_0 is bounded. By (16), we infer the Lipschitz estimate for u. □

Remark 1.6. Thanks to Lemma 1.5, the drift $H_p(x, Du)$ in the Fokker–Planck equation is uniformly bounded. As a consequence, as it is well known (see, e.g., in [Cardaliaguet 2010, Lemma 3.4]), the solution m satisfies the following Hölder continuity estimate in time:

$$\boldsymbol{d}_1(m(t), m(s)) \le C|t-s|^{1/2} \quad \text{for all } t, s \in (0, T) \text{ such that } |t-s| \le 1, \tag{17}$$

for some constant C independent of T.

Next result exploits the stability of the system which stems from the monotonicity of F and the convexity of H; see [Lasry and Lions 2007]. In particular, whenever H is uniformly convex, as is assumed in (9), the following estimate holds for any pair of solutions (u_1, m_1) and (u_2, m_2) of the system (15):

$$C^{-1}\int_{\mathbb{T}^d}(m_1+m_2)|D(u_1-u_2)|^2 \le -\frac{d}{dt}\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2). \tag{18}$$

Lemma 1.7. *For any $\varepsilon > 0$ and $M > 0$, there exist times $\widehat{T} > \tau > 0$ (depending only on ε, M and the data of the problem) such that, if $T \ge \widehat{T}$ and if (u, m) is a solution to the MFG system (15) and $\|g\|_{C^2(\mathbb{T}^d)} \le M$, we have, for some $\alpha \in (0, 1)$,*

$$\|m(t)-\bar m\|_{C^\alpha} + \|Du(t) - D\bar u\|_{C^\alpha} \le \varepsilon \quad \textit{for all } t \in [\tau, T-\tau].$$

Proof. We follow closely the argument of Lemma 3.5 of [Cardaliaguet et al. 2013] (in the case $H = |p|^2$) and, for this reason, we only sketch the proof. By Lemma 1.5, u is uniformly Lipschitz continuous in space, with a Lipschitz constant depending only on the regularity of H, F and on $\|Dg\|_\infty + \|D^2 g\|_\infty$. So, by Lemma 1.1, we have

$$\sup_{t\ge 1}\|m(t)\|_\infty \le C,$$

where C depends only on $\|H_p(\,\cdot\,, Du(\cdot))\|_\infty$, and thus only on the data. Applying (18) to (u, m) and $(\bar u, \bar m)$, and using $\bar m > 0$ in $\mathbb{T}^d$, we have

$$C^{-1}\int_{t_1}^{t_2}\|D(u(t)-\bar u)\|_{L^2}^2\,dt \le -\left[\int_{\mathbb{T}^d}(u(t)-\bar u)(m(t)-\bar m)\right]_{t_1}^{t_2}. \tag{19}$$

Thus

$$\int_0^T \|D(u(t)-\bar u)\|_{L^2}^2\,dt \le C,$$

because u is uniformly Lipschitz continuous in space and $m(t)$ and $\bar m$ are probability measures. In particular, if $T \ge 3\varepsilon^{-1}$, there exist times $t_1 \in [1, \varepsilon^{-1}]$, $t_2 \in [T-\varepsilon^{-1}, T]$ such that

$$\|D(u(t_i)-\bar u)\|_{L^2} \le C\varepsilon^{1/2} \quad \text{for } i = 1, 2. \tag{20}$$

Coming back to (19), we infer by Poincaré's inequality that

$$\begin{aligned} C^{-1}\int_{1/\varepsilon}^{T-1/\varepsilon}\|D(u(t)-\bar u)\|_{L^2}^2\,dt &\le C^{-1}\int_{t_1}^{t_2}\|D(u(t)-\bar u)\|_{L^2}^2\,dt \\ &\le \|D(u(t_1)-\bar u)\|_{L^2}\|m(t_1)-\bar m\|_{L^2} + \|D(u(t_2)-\bar u)\|_{L^2}\|m(t_2)-\bar m\|_{L^2} \\ &\le C\varepsilon^{1/2}. \end{aligned}$$

As $\mu := m - \bar m$ satisfies

$$\partial_t\mu - \Delta\mu - \operatorname{div}(\mu H_p(x, Du)) = -\operatorname{div}(\bar m(H_p(x, D\bar u) - H_p(x, Du))), \tag{21}$$

and still using the fact that Du is bounded, we have from Lemma 1.1 that, for any $t \in [1/\varepsilon, T - 1/\varepsilon]$,

$$\begin{aligned}\|m(t) - \bar{m}\|_{L^2} &\le Ce^{-\omega(t-1/\varepsilon)}\|m(1/\varepsilon) - \bar{m}\|_{L^2} + C\left[\int_{1/\varepsilon}^{T-1/\varepsilon} \|D(u(t) - \bar{u})\|_{L^2}^2\, dt\right]^{1/2} \\ &\le C(e^{-\omega(t-1/\varepsilon)} + \varepsilon^{1/4}).\end{aligned}$$

So we can choose τ large enough (depending only on ε, on the data and on M) such that the right-hand side is less than $C\varepsilon^{1/4}$ if $t \in [\tau - 1, T - 1/\varepsilon]$.

Let us now upgrade this inequality into an L^∞ estimate for the interval $[\tau, T - 1/\varepsilon]$. For this, we recall from (21) that μ solves a parabolic equation of the type

$$\partial_t \mu - \Delta\mu - \operatorname{div}(\mu b + B) = 0,$$

where b is bounded in L^∞ and B is bounded in L^p for any $p \ge 2$ since

$$\int_{1/\varepsilon}^{T-1/\varepsilon} \|B(t)\|_{L^p}^p \le C\int_{1/\varepsilon}^{T-1/\varepsilon}\int_{\mathbb{T}^d} |D(u(t) - \bar{u})|^p \le C\int_{1/\varepsilon}^{T-1/\varepsilon}\int_{\mathbb{T}^d} |D(u(t) - \bar{u})|^2 \le C\varepsilon^{1/2},$$

where we used the global bound for $Du(t)$. Since we already know that $\|\mu(t)\|_{L^2} \le C\varepsilon^{1/4}$, by choosing p sufficiently large we deduce (see, e.g., [Ladyženskaja et al. 1968, Theorem III.8.1, p. 196]) that μ is bounded in $C^{\alpha/2,\alpha}$ for some $\alpha \in (0, 1)$ and

$$\|\mu(t)\|_{C^\alpha} \le C\Big(\sup_{s\in(\tau-1,T-1/\varepsilon)} \|\mu(s)\|_{L^2} + \|B\|_{L^p((1/\varepsilon,T-1/\varepsilon)\times\mathbb{T}^d)}\Big) \le C(\varepsilon^{1/4} + \varepsilon^{1/(2p)})$$

for any $t \in [\tau, T - 1/\varepsilon]$. This concludes the bound for $\|m(t) - \bar{m}\|_{C^\alpha}$. In order to prove the estimate for u, let us note that $v = u - \bar{u}$ satisfies

$$-\partial_t v - \Delta v + V \cdot Dv = F(x, m(t)) - F(x, \bar{m}),$$

where V is the bounded vector field

$$V(t, x) = \int_0^1 H_p(x, \lambda Du(t, x) + (1 - \lambda)D\bar{u}(x))\, d\lambda.$$

By Lemma 1.2 we have, for $t \in [1/\varepsilon, T - 1/\varepsilon]$,

$$\begin{aligned}&\|v(t) - \langle v(t)\rangle\|_\infty \\ &\quad\le \|v(T - 1/\varepsilon) - \langle v(T - 1/\varepsilon)\rangle\|_\infty e^{-\omega(T-1/\varepsilon-t)} + C\int_t^{T-1/\varepsilon} e^{-\omega(s-t)}\|F(x, m(t)) - F(x, \bar{m})\|_\infty\, ds \\ &\quad\le C(e^{-\omega(T-1/\varepsilon-t)} + \varepsilon^{1/(2p)}).\end{aligned}$$

Choosing $\tau > 1/\varepsilon$ large enough then implies

$$\|v(t) - \langle v(t)\rangle\|_\infty \le C\varepsilon^{1/(2p)} \quad \text{for all } t \in [\tau, T - \tau].$$

Finally, we can replace the left-hand side by $\|Dv(t)\|_{C^\alpha}$ by using again Lemma 1.2. Indeed, whenever v satisfies

$$-\partial_t v - \Delta v + V \cdot Dv = A$$

with V, A bounded, we estimate, for any interval $[t, t+1]$,

$$\begin{aligned}\|Dv(t)\|_{C^\alpha} &\le C \sup_{s\in(t,t+1/2)} [\|v(s) - \langle v(s)\rangle\|_\infty + \|A(s)\|_\infty + \|Dv(s)\|_{L^2}] \\ &\le C \sup_{s\in(t,t+1)} [\|v(s) - \langle v(s)\rangle\|_\infty + \|A(s)\|_\infty].\end{aligned}$$

Since $A = F(x, m(t)) - F(x, \bar m)$, the previous estimates give the conclusion. □

2. Exponential rate of convergence for the finite-horizon MFG system

In this section we provide several convergence results with an exponential rate of convergence for finite-horizon MFG systems. The results of this section extend to general Hamiltonians the main results of [Cardaliaguet et al. 2013] (though requiring slightly stronger assumptions on the coupling F). Although the results are interesting themselves, they are nevertheless motivated by the rest of the paper, in which they play a central role.

The method of proof for these exponential rates differs completely from [Cardaliaguet et al. 2013], where it relied on an algebraic structure of the linearized system. We start with the linearized systems and first get a crude estimate on the solution. Using the monotonicity assumption, the duality method shows that a suitable quantity is monotone in time and bounded (thanks to the rough estimate). A compactness argument, borrowed from [Porretta and Zuazua 2013], then shows that the limit of this quantity must vanish. We then use the linearity property of the system to get an exponential rate of convergence. The nonlinear equations are treated as perturbations of the linear ones. Note that the key argument is inspired by [Porretta and Zuazua 2013], where the long time behavior of optimality systems is analyzed by using the stabilizing properties of the Riccati feedback operator. However, in contrast with that paper, our system does not come from an optimal control problem in general, which makes a substantial difference.

2A. ***Estimates for the linearized system.*** We now study the linearized MFG system around the stationary ergodic solution $(\bar u, \bar m)$: namely, given $\mu_0, v_T : \mathbb{T}^d \to \mathbb{R}$ smooth with $\int_{\mathbb{T}^d} \mu_0 = 0$, we consider a solution (v, μ) to

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, D\bar u)\cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar m)(\mu(t)) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar u)) - \operatorname{div}(\bar m H_{pp}(x, D\bar u)Dv) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ \mu(0,\cdot) = \mu_0, \quad v(T,x) = \dfrac{\delta G}{\delta m}(x, \bar m)(\mu(T)) + v_T(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{22}$$

Thanks to the assumptions made upon $\delta F/\delta m$ and $\delta G/\delta m$, and to the smoothness of $(\bar u, \bar m)$, problem (22) can be considered in a standard framework of weak solutions with finite energy, i.e., $v, m \in L^2((0,T); H^1(\mathbb{T}^d))$. Solutions will eventually be more regular, but we are not considering this issue here; our main purpose, which is the following result, is to show the L^2 decay estimates for μ and Dv, assuming the same regularity on the initial-terminal conditions.

Proposition 2.1. *There exist $C_0 > 0$, $\lambda > 0$ such that, if (v, μ) is a solution to the MFG linearized system* (22) *with $\int_{\mathbb{T}^d} \mu_0 = 0$, then we have*

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C_0(e^{-\lambda t} + e^{-\lambda(T-t)})(\|\mu_0\|_{L^2} + \|Dv_T\|_{L^2}) \quad \text{for all } t \in [0, T].$$

Let us start the proof with a lemma which explains that the solution is uniformly bounded, with a bound depending on $\|\mu_0\|_{L^2}$ only.

Lemma 2.2. *There is a constant $C_0 > 0$, depending only on the data H, F and G, but not on T, such that, if (v, μ) is a solution of the linearized problem* (22), *then*

$$\int_0^T \|Dv\|_{L^2}^2 + \sup_{t\in[0,T]} (\|\mu(t)\|_{L^2}^2 + \|Dv(t)\|_{L^2}^2) \le C_0(\|\mu_0\|_{L^2}^2 + \|Dv_T\|_{L^2}^2). \tag{23}$$

Proof. Note that $\int_{\mathbb{T}^d} \mu(t) = 0$ for any t. Multiplying the equation for v by μ and the equation for μ by v, integrating in time and space and adding the resulting relations, we have, for any $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{T}^d\times\mathbb{T}^d} \frac{\delta F}{\delta m}(x, \bar m, y)\mu(t, y)\mu(t, x)\, dy\, dx\, dt \\ + \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \bar m H_{pp}(x, D\bar u(x))Dv(t, x)\cdot Dv(t, x)\ dx\, dt = -\left[\int_{\mathbb{T}^d} v\mu\right]_{t_1}^{t_2},\end{aligned} \tag{24}$$

so, by the monotonicity of F and G, see assumption (11),

$$\begin{aligned}C^{-1}\int_0^T \|Dv(t)\|_{L^2}^2\, dt &\le \int_{\mathbb{T}^d}(v(0) - \langle v(0)\rangle)\mu_0 - \int_{\mathbb{T}^d}(v_T - \langle v_T\rangle)\mu(T) \\ &\le C(\|Dv(0)\|_{L^2}\|\mu_0\|_{L^2} + \|Dv_T\|_{L^2}\|\mu(T)\|_{L^2}),\end{aligned} \tag{25}$$

thanks to Poincaré's inequality. Using Lemma 1.1, we have

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^t \|\bar m H_{pp}(\cdot, D\bar u)Dv\|_{L^2}^2\right]^{1/2} \le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^T \|Dv\|_{L^2}^2\right]^{1/2} \\ &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + C(\|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}^{1/2}\|\mu(T)\|_{L^2}^{1/2}).\end{aligned}$$

For $t = T$, we get, after simplification,

$$\|\mu(T)\|_{L^2} \le C(\|\mu_0\|_{L^2} + \|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}),$$

from which we deduce that

$$\sup_{t\in[0,T]} \|\mu(t)\|_{L^2} \le C(\|\mu_0\|_{L^2} + \|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}). \tag{26}$$

Note that the derivative v_{x_i} of v satisfies

$$\begin{cases} -\partial_t v_{x_i} - \Delta v_{x_i} + H_p \cdot Dv_{x_i} + D_{x_i}[H_p]\cdot Dv = D_{x_i}\dfrac{\delta F}{\delta m}(x, \bar m)(\mu(t)) & \text{in } (0, T)\times\mathbb{T}^d, \\ v_{x_i}(T, x) = D_{x_i}\dfrac{\delta G}{\delta m}(x, \bar m)(\mu(T)) + D_{x_i}v_T(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{27}$$

where, to simplify the notation, we have set $H_p = H_p(x, D\bar{u})$, etc. Then Lemma 1.2 gives, in view of our assumptions on $\delta F/\delta m$ and $\delta G/\delta m$,

$$\begin{aligned}
\|v_{x_i}(t)\|_{L^2} &\le Ce^{-\omega(T-t)}\Big(\|Dv_T\|_{L^2} + \Big\|D_{x_i}\frac{\delta G}{\delta m}(\cdot,\bar{m})(\mu(T))\Big\|_{L^2}\Big) \\
&\qquad + C\int_t^T e^{-\omega(s-t)}\Big(\|D_{x_i}[H_p]\cdot Dv\|_{L^2} + \Big\|D_{x_i}\frac{\delta F}{\delta m}(\cdot,\bar{m})(\mu(t))\Big\|_{L^2}\Big)\,ds \\
&\le Ce^{-\omega(T-t)}(\|Dv_T\|_{L^2} + \|\mu(T)\|_{L^2}) + C\int_t^T e^{-\omega(s-t)}(\|Dv\|_{L^2} + \|\mu(t)\|_{L^2})\,ds \\
&\le Ce^{-\omega(T-t)}\|Dv_T\|_{L^2} + C\Big(\int_t^T \|Dv\|_{L^2}^2\Big)^{1/2} + C\sup_{s\ge t}\|\mu(s)\|_{L^2}. \qquad (28)
\end{aligned}$$

Combining this with (25) and with the estimate for μ in (26), we find, for any $t \in [0, T]$,

$$\|Dv(t)\|_{L^2} \le C(\|\mu_0\|_{L^2} + \|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}).$$

In particular, for $t = 0$, we get, after simplification,

$$\|Dv(0)\|_{L^2} \le C(\|\mu_0\|_{L^2} + \|Dv_T\|_{L^2}),$$

which jointly with (25) and (26) gives the desired statement. □

Remark 2.3. The above lemma also provides an argument for proving the existence of a solution (v, μ) to (22). Indeed, the a priori estimate (23) allows for a standard application of Schaefer's fixed-point theorem by freezing μ in the right-hand side as well as in the final value of the equation of v.

Proof of Proposition 2.1. For $\tau \ge 0$, let us set

$$\rho(\tau) = \sup_{(T,t,\mu_0,v_T)\in S(\tau)}\Big|\int_{\mathbb{T}^d}\mu(t)v(t)\Big|,$$

where the supremum is taken over the set $S(\tau)$ defined as

$$S(\tau) := \{(T, t, \mu_0, v_T) : T \ge 2\tau,\ t \in [\tau, T-\tau],\ \|\mu_0\|_{L^2} \le 1 \text{ and } \|Dv_T\|_{L^2} \le 1\},$$

the pair (v, μ) being a solution to (22). According to Lemma 2.2, $\rho(\tau)$ is bounded for any τ, since, using that μ has zero average, one has for any t

$$\Big|\int_{\mathbb{T}^d}\mu(t)v(t)\Big| \le C\|\mu(t)\|_{L^2}\|Dv(t)\|_{L^2}$$

by Poincaré's inequality. By definition, the map ρ is nonincreasing, since $S(\tau) \subseteq S(\tau')$ if $\tau > \tau'$. Let us denote by ρ_∞ the limit of $\rho(\tau)$ as $\tau \to +\infty$. The key step consists in proving that $\rho_\infty = 0$.

Let $\tau_n \to +\infty$, $T_n \ge 2\tau_n$, $t_n \in [\tau_n, T_n - \tau_n]$, μ_0^n with $\|\mu_0^n\|_{L^2} \le 1$ and v_T^n with $\|Dv_T^n\|_{L^2} \le 1$ be such that

$$\Big|\int_{\mathbb{T}^d}\mu^n(t_n)v^n(t_n)\Big| \ge \rho_\infty - \frac{1}{n}.$$

We set

$$\tilde{\mu}^n(t,x) = \mu^n(t_n+t,x), \quad \tilde{v}^n(t,x) = v^n(t_n+t,x) - \langle v^n(t_n)\rangle \quad \text{for all } t\in[-t_n, T_n-t_n],\ x\in\mathbb{T}^d.$$

By the estimates of Lemma 2.2, the $(\tilde{v}^n, \tilde{\mu}^n)$ are locally bounded in L^2. By parabolic regularity (from [Ladyženskaja et al. 1968, Theorem III.8.1, p. 196] combined with Theorem III.10.1, p. 204, and Theorem III.11.1, p. 211, of the same work), the $\tilde{v}^n$ and $D\tilde{v}^n$ are locally bounded in $C^{\alpha/2,\alpha}$, while the $\tilde{\mu}^n$ are bounded in $C^{\alpha/2,\alpha}$ for some $\alpha\in(0,1)$. So the pair $(\tilde{v}^n, \tilde{\mu}^n)$ locally uniformly converges to some (v,μ) which satisfies the linearized MFG system on $\mathbb{R}\times\mathbb{T}^d$. Moreover, we have

$$\left|\int_{\mathbb{T}^d}\mu(0)v(0)\right| = \lim_n \left|\int_{\mathbb{T}^d}\mu^n(t_n)v^n(t_n)\right| = \rho_\infty.$$

On the other hand, for any $t\in\mathbb{R}$ and for n large enough, we have that $t_n+t\in[\tau_n-|t|, T_n-(\tau_n-|t|)]$, so

$$\left|\int_{\mathbb{T}^d}\mu(t)v(t)\right| = \lim_n \left|\int_{\mathbb{T}^d}\mu^n(t_n+t)v^n(t_n+t)\right| \le \lim_n \rho(\tau_n-|t|) = \rho_\infty.$$

The duality equality (24) implies that, for any $t_1\le t_2$, we have

$$C^{-1}\int_{t_1}^{t_2}\|Dv\|_{L^2}^2 \le -\left[\int_{\mathbb{T}^d}\mu v\right]_{t_1}^{t_2}. \tag{29}$$

Therefore the map $t\to\int_{\mathbb{T}^d}\mu(t)v(t)$ is nonincreasing, with a derivative bounded above by $-\|Dv(0)\|_{L^2}^2$ at $t=0$, while the map $t\to\left|\int_{\mathbb{T}^d}\mu(t)v(t)\right|$ has a maximum ρ_∞ at $t=0$; this implies $Dv(0)=0$. As $\int_{\mathbb{T}^d}v(0)=0$, we can infer that

$$\rho_\infty = \left|\int_{\mathbb{T}^d}\mu(0)v(0)\right| = 0.$$

We now prove that $\rho(t)$ converges to 0 with an exponential rate. Let $T>0$ and (v,μ) be a solution of the MFG linearized system with $\|\mu(0)\|_{L^2}\le 1$ and $\|Dv_T\|_{L^2}\le 1$. Using Lemma 1.1 and (29), we have, for $\tau\ge 0$ and $t\in[\tau, T-\tau]$

$$\|\mu(t)\|_{L^2} \le Ce^{-\omega(t-\tau/2)}\|\mu(\tau/2)\|_{L^2} + C\left(-\left[\int_{\mathbb{T}^d}\mu v\right]_{\tau/2}^{t}\right)^{1/2} \le Ce^{-\omega\tau/2} + C[2\rho(\tau/2)]^{1/2},$$

because μ is uniformly bounded in L^2 (Lemma 2.2). Thus

$$\sup_{t\in[\tau,T-\tau]}\|\mu(t)\|_{L^2} \le C(e^{-\omega\tau/2} + (\rho(\tau/2))^{1/2}). \tag{30}$$

Coming back to (28), we have, for all $t\in[2\tau, T-2\tau]$,

$$\begin{aligned}\|Dv(t)\|_{L^2} &\le Ce^{-\omega(T-\tau-t)}\|Dv(T-\tau)\|_{L^2} + C\left(\int_t^{T-\tau}\|Dv\|_{L^2}^2\right)^{1/2} + C\sup_{s\in[t,T-\tau]}\|\mu(s)\|_{L^2}\\ &\le Ce^{-\omega\tau} + C\left(-\left[\int_{\mathbb{T}^d}\mu(s)v(s)\right]_t^{T-\tau}\right)^{1/2} + C\sup_{s\in[t,T-\tau]}\|\mu(s)\|_{L^2}\\ &\le Ce^{-\omega\tau} + C\rho^{1/2}(\tau) + C(e^{-\omega\tau/2} + (\rho(\tau/2))^{1/2}),\end{aligned} \tag{31}$$

because Dv is uniformly bounded in L^2 (Lemma 2.2). In view of (30) and (31), we can fix $\tau > 0$ large enough so that, for any $T \geq 4\tau$ and any (v, μ) as above, one has

$$\sup_{t\in[2\tau,T-2\tau]} (\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2}) \leq \varepsilon,$$

where $\varepsilon \in \left(0, \frac{1}{4}\right)$ is to be chosen below. Notice that, by the definition of ρ and by Poincaré's inequality, this also implies $\rho(2\tau) \leq C\varepsilon \leq \frac{1}{4}$ for a suitable choice of ε. Now we can iterate the previous estimate. Indeed, for $T \geq 4\tau$, the restriction to $[2\tau, T-2\tau]$ of (v, μ) is a solution of the linearized MFG system (22) on $[2\tau, T-2\tau]$ with boundary conditions $\|\mu(2\tau)\|_{L^2} \leq \frac{1}{2}$ and $\|Dv(T-2\tau)\|_{L^2} \leq \frac{1}{2}$. As the problem is invariant by time translation, we deduce that

$$\sup_{t\in[4\tau,T-4\tau]} (\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2}) \leq \tfrac{1}{4},$$

(and similarly $\rho(4\tau) \leq 1/4^2$). By a standard iteration, this shows that there exists λ such that

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C(e^{-\lambda t} + e^{-\lambda(T-t)}) \quad \text{for all } t \in [0, T]. \qquad \square$$

Proposition 2.4. *Let λ be as in Proposition 2.1. There exists C_1 such that, if $B = B(t, x)$ satisfies*

$$\|B(t)\|_{L^2} \leq e^{-\lambda t} + e^{-\lambda(T-t)}, \tag{32}$$

and if (v, μ) is a solution to the MFG linearized system

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, D\bar{u}) \cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv) = \operatorname{div}(B) & \text{in } (0, T) \times \mathbb{T}^d \\ \mu(0, \cdot) = 0, \quad v(T, x) = 0 & \text{in } \mathbb{T}^d, \end{cases} \tag{33}$$

then

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C_1((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}) \quad \textit{for all } t \in [0, T].$$

Proof. Let us first prove that (v, μ) is bounded. Multiplying the equation for v by μ and the equation for μ by v, integrating in time and space and adding the resulting relations gives, for any $0 \leq t_1 \leq t_2 \leq T$,

$$C^{-1} \int_{t_1}^{t_2} \|Dv\|_{L^2}^2 \, dt \leq -\left[\int_{\mathbb{T}^d} v\mu\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{T}^d} B \cdot Dv.$$

Thus, by Young's inequality,

$$C^{-1} \int_{t_1}^{t_2} \|Dv\|_{L^2}^2 \, dt \leq -\left[\int_{\mathbb{T}^d} v\mu\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \|B\|_{L^2}^2 \, ds.$$

Using the homogeneous boundary conditions at $t = 0$, $t = T$, we obtain the bound

$$\int_0^T \|Dv\|_{L^2}^2 \, dt \leq C \int_0^T \|B\|_{L^2}^2 \, ds.$$

This implies, with the same arguments as in Lemma 2.2,

$$\sup_{t\in[0,T]} \|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C\left[\int_0^T \|B\|_{L^2}^2\right]^{1/2} \leq C,$$

where the last inequality comes from (32).

For $\tau \geq 0$, we set

$$\rho(\tau) = \sup_{T,t,B} (\|\mu(t)\|_2 + \|Dv(t)\|_{L^2}), \tag{34}$$

where the supremum is taken over any $T \geq 2\tau,\ t \in [\tau, T-\tau]$ and any B satisfying (32), the pair (v, μ) being a solution to (33). In view of the previous discussion, $\rho(\tau)$ is bounded for any τ.

The restriction $(\tilde{v}, \tilde{\mu})$ of (v, μ) to $[\tau, T-\tau]$ can be written as

$$(\tilde{v}, \tilde{\mu}) = (\tilde{v}_1, \tilde{\mu}_1) + (\tilde{v}_2, \tilde{\mu}_2),$$

where $(\tilde{v}_1, \tilde{\mu}_1)$ solves the homogeneous MFG linearized system (22) with boundary conditions $\tilde{v}_1(T-\tau) = v(T-\tau)$ and $\tilde{\mu}_1(\tau) = \mu(\tau)$, while $(\tilde{v}_2, \tilde{\mu}_2)$ solves the linearized MFG system (33) on the time interval $[\tau, T-\tau]$ with homogeneous boundary conditions.

From Proposition 2.1, we have, for any $t \in [\tau, T-\tau]$,

$$\begin{aligned}\|\tilde{\mu}_1(t)\|_{L^2} + \|D\tilde{v}_1(t)\|_{L^2} &\leq C_0(e^{-\lambda(t-\tau)} + e^{-\lambda(T-\tau-t)})(\|\mu(\tau)\|_{L^2} + \|Dv(T-\tau)\|_{L^2}) \\ &\leq C(e^{-\lambda(t-\tau)} + e^{-\lambda(T-\tau-t)}).\end{aligned}$$

Note that the restriction of B to $[\tau, T-\tau]$ satisfies

$$\|B(t)\|_{L^2} \leq e^{-\lambda\tau}[e^{-\lambda(t-\tau)} + e^{-\lambda(T-\tau-t)}].$$

So by the linearity and the invariance in time of the equation, we get

$$\|\tilde{\mu}_2(t)\|_2 + \|D\tilde{v}_2(t)\|_{L^2} \leq e^{-\lambda\tau}\rho(t-\tau) \quad \text{for all } t \in [\tau, T-\tau].$$

Putting together the estimates of $(\tilde{v}_1, \tilde{\mu}_1)$ and $(\tilde{v}_2, \tilde{\mu}_2)$, we obtain, for any $t \geq \tau$,

$$\begin{aligned}\sup_{s\in[t+\tau, T-\tau-t]} (\|\mu(s)\|_{L^2} + \|Dv(s)\|_{L^2}) &\leq \sup_{s\in[t+\tau, T-\tau-t]} C(e^{-\lambda(s-\tau)} + e^{-\lambda(T-\tau-s)}) + e^{-\lambda\tau}\rho(s-\tau) \\ &\leq Ce^{-\lambda t} + e^{-\lambda\tau}\rho(t).\end{aligned}$$

Taking the supremum over (v, μ) and multiplying by $e^{\lambda(t+\tau)}$ gives

$$e^{\lambda(t+\tau)}\rho(t+\tau) \leq Ce^{\lambda\tau} + e^{\lambda t}\rho(t),$$

from which we infer that

$$\rho(t) \leq C(1+t)e^{-\lambda t}.$$

By the definition of ρ in (34), this implies the conclusion when choosing $\tau = t$ if $t \in [0, T/2]$ and $\tau = T - t$ otherwise. □

Collecting the above propositions we finally obtain:

Theorem 2.5. *Let λ be as in Proposition 2.1. There exists $C_0 > 0$ such that, if $A = A(t, x)$ and $B = B(t, x)$ satisfy*

$$\|A(t)\|_{L^2} + \|B(t)\|_{L^2} \leq M(e^{-\lambda t} + e^{-\lambda(T-t)}), \tag{35}$$

and if (v, μ) *is a solution to the MFG linearized system*

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, D\bar{u}) \cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) + A(t, x) & \textit{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv) = \operatorname{div}(B) & \textit{in } (0, T) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0, \quad v(T, x) = \dfrac{\delta G}{\delta m}(x, \bar{m})(\mu(T)) + v_T(x) & \textit{in } \mathbb{T}^d, \end{cases} \tag{36}$$

with $\int_{\mathbb{T}^d} \mu_0 = 0$, *we have*

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C_0((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)})(\|Dv_T\|_{L^2} + \|\mu_0\|_{L^2} + M)$$

for any $t \in [0, T]$.

Proof. Let $\tilde{v}$ be the solution to

$$\begin{cases} -\partial_t \tilde{v} - \Delta \tilde{v} + H_p(x, D\bar{u}) \cdot D\tilde{v} = A(t, x) & \text{in } (0, T) \times \mathbb{T}^d, \\ \tilde{v}(T, x) = 0 & \text{in } \mathbb{T}^d. \end{cases}$$

Note for later use that, assuming $\lambda < \omega$, we have

$$\|D\tilde{v}(t)\|_{L^2} \le CM(e^{-\lambda t} + e^{-\lambda(T-t)}). \tag{37}$$

Indeed, using Lemma 1.2, we have

$$\|\tilde{v}(t) - \langle \tilde{v}(t) \rangle\|_{L^2} \le C \int_t^T e^{-\omega(s-t)} \|A(s)\|_{L^2}\, ds \le CM(e^{-\lambda t} + e^{-\lambda(T-t)}).$$

Then the regularizing property of the equation leads to (37).

The pair $(v_1, \mu_1) := (v - \tilde{v}, \mu)$ solves

$$\begin{cases} -\partial_t v - \Delta v_1 + H_p(x, D\bar{u}) \cdot Dv_1 = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu_1 - \Delta \mu_1 - \operatorname{div}(\mu_1 H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv_1) = \operatorname{div}(B + \bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}) & \text{in } (0, T) \times \mathbb{T}^d, \\ \mu_1(0, \cdot) = \mu_0, \quad v_1(T, x) = \dfrac{\delta G}{\delta m}(x, \bar{m})(\mu_1(T)) + v_T(x) & \text{in } \mathbb{T}^d, \end{cases}$$

where, by (35) and (37),

$$\|B(t) + \bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}(t)\|_{L^2} \le CM(e^{-\lambda t} + e^{-\lambda(T-t)}).$$

Using Propositions 2.1 and 2.4, we get

$$\|\mu_1(t)\|_{L^2} + \|Dv_1(t)\|_{L^2} \le C((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)})(\|Dv_T\|_{L^2} + \|\mu_0\|_{L^2} + M)$$

for any $t \in [0, T]$. Recalling the definition of (v_1, μ_1) and using again inequality (37) gives the result. □

2B. ***Estimates for the nonlinear system.*** Now we consider the nonlinear MFG systems. For the finite-horizon problem, we have:

Theorem 2.6. *There exists $\gamma > 0$ and $C > 0$ such that, if (u, m) is a solution of the MFG system with initial condition $m_0 \in \mathcal{P}(\mathbb{T}^d)$*

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ m(0, \cdot) = m_0, \quad u(T, x) = G(x, m(T)) & \text{in } \mathbb{T}^d, \end{cases} \tag{38}$$

then, for some $\alpha \in (0, 1)$,

$$\|Du(t) - D\bar{u}\|_{C^{1+\alpha}} \le C(e^{-\gamma t} + e^{-\gamma(T-t)}) \quad \text{for all } t \in [0, T],$$
$$\|m(t) - \bar{m}\|_{C^{\alpha}} \le C(e^{-\gamma t} + e^{-\gamma(T-t)}) \quad \text{for all } t \in [1, T].$$

In particular,

$$\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} |u(t, x) - \bar{u}(x) - \bar{\lambda}(T - t)| \le C.$$

Proof. We use a fixed-point argument. Let us start with the proof for initial and terminal conditions which are sufficiently close to $\bar{m}$ and $\bar{u}$ respectively. Let $\widehat{K} > 0$ be small enough and $\gamma \in (\lambda/2, \lambda)$, where λ is given by Proposition 2.1. Let E be the set of continuous maps (v, μ) on $[0, T] \times \mathbb{T}^d$ such that Dv is also continuous and

$$\|Dv(t)\|_{L^\infty} + \|\mu(t)\|_{L^\infty} \le \widehat{K}(e^{-\gamma t} + e^{-\gamma(T-t)}).$$

We suppose that $\widehat{K}$ is such that

$$\bar{m}(x) > \widehat{K} \quad \text{for all } x \in \mathbb{T}^d.$$

We also assume that the initial condition m_0 and the terminal condition u_T are close to $\bar{m}$ and $\bar{u}$ (plus a constant) respectively, namely that $\mu_0 := m_0 - \bar{m}$ and $v_T := u_T - \bar{u}$ satisfy

$$\|\mu_0\|_{L^\infty} + \|Dv_T\|_\infty \le \widehat{K}^2. \tag{39}$$

We may suppose further that μ_0 and Dv_T belong to $C^\alpha(\mathbb{T}^d)$ for some $\alpha \in (0, 1)$.

For $(v, \mu) \in E$, we consider the solution $(\tilde{v}, \tilde{\mu})$ to the linearized system

$$\begin{cases} -\partial_t \tilde{v} - \Delta \tilde{v} + H_p(x, D\bar{u}) \cdot D\tilde{v} = \dfrac{\delta F}{\delta m}(x, \bar{m})(\tilde{\mu}(t)) + A(t, x) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \tilde{\mu} - \Delta \tilde{\mu} - \operatorname{div}(\tilde{\mu} H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}) = \operatorname{div}(B) & \text{in } (0, T) \times \mathbb{T}^d, \\ \tilde{\mu}(0, \cdot) = \mu_0, \quad \tilde{v}(T, x) = v_T(x) & \text{in } \mathbb{T}^d, \end{cases}$$

with

$$A(t, x) = -H(x, D(\bar{u} + v)) + H(x, D\bar{u}) + H_p(x, D\bar{u}) \cdot Dv + F(x, \bar{m} + \mu) - F(x, \bar{m}) - \frac{\delta F}{\delta m}(x, \bar{m})(\mu)$$

and

$$B(t, x) = (\bar{m} + \mu) H_p(x, D(\bar{u} + v)) - \bar{m} H_p(x, D\bar{u}) - \mu H_p(x, D\bar{u}) - \bar{m} H_{pp}(x, D\bar{u}) Dv.$$

We note that $\bar{m} + \mu \ge 0$ on $[0, T] \times \mathbb{T}^d$ and

$$\|A(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} \le C\widehat{K}^2(e^{-2\gamma t} + e^{-2\gamma(T-t)}).$$

Here we used that $m \mapsto (\delta F/\delta m)(x, m, y)$ is Lipschitz (uniformly with respect to (x, y)), and H_{pp} is Lipschitz as well.

From Theorem 2.5 we have, as $\gamma \in (\lambda/2, \lambda)$,

$$\|\tilde{\mu}(t)\|_{L^2} + \|D\tilde{v}(t)\|_{L^2} \leq C\widehat{K}^2((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}).$$

We upgrade the previous estimates to L^∞ norms with our usual arguments: from Lemma 1.2 we have

$$\begin{aligned}
&\|\tilde{v}(t) - \langle \tilde{v}(t)\rangle\|_{L^\infty} \\
&\quad \leq Ce^{-\omega(T-t)}\|v(T) - \langle v(T)\rangle\|_{L^\infty} + C\int_t^T e^{-\omega(s-t)}\Big(\Big\|\frac{\delta F}{\delta m}(x, \bar{m})(\tilde{\mu}(s))\Big\|_{L^\infty} + \|A(s)\|_{L^\infty}\Big)\,ds \\
&\quad \leq C\widehat{K}^2((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}).
\end{aligned}$$

Then, in any interval $[t, t+1]$, we have, by using the uniform parabolicity of the equation,

$$\|D\tilde{v}(t)\|_\infty \leq C \sup_{s\in(t,t+1)} \Big[\|\tilde{v}(s) - \langle \tilde{v}(s)\rangle\|_\infty + \Big\|\frac{\delta F}{\delta m}(x, \bar{m})(\tilde{\mu}(s))\Big\|_{L^\infty} + \|A(s)\|_\infty\Big],$$

and this concludes the estimate for $\|D\tilde{v}(t)\|_{L^\infty}$. Now, using the bound for $D\tilde{v}$ and B, we have

$$\|\tilde{\mu}(t)\|_\infty \leq C \sup_{s\in(t-1,t)} [\|\tilde{\mu}(s)\|_{L^2} + \|D\tilde{v}(s)\|_{L^\infty} + \|B(s)\|_\infty]$$

and we conclude the estimate for $\|\tilde{\mu}(t)\|_\infty$. Notice that the above bounds hold up to $t = 0$ and $t = T$ by using the condition (39) assumed on μ_0 and v_T. Eventually, we obtain that

$$\|\tilde{\mu}(t)\|_{L^\infty} + \|D\tilde{v}(t)\|_{L^\infty} \leq C\widehat{K}^2((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}).$$

Since $\gamma < \lambda$, for $\widehat{K}$ small enough we infer that

$$\|\tilde{\mu}(t)\|_{L^\infty} + \|D\tilde{v}(t)\|_{L^\infty} \leq \widehat{K}(e^{-\gamma t} + e^{-\gamma(T-t)})$$

and $(\tilde{v}, \tilde{\mu})$ belongs to E. In addition, $\tilde{\mu}$ and $\tilde{v} - \langle\tilde{v}\rangle$ solve linear parabolic equations with bounded coefficients, so classical parabolic estimates [Ladyženskaja et al. 1968, Theorems III.8.1, III.10.1 and III.11.1, p. 196] imply that $\tilde{\mu}$ and $D\tilde{v}$ are locally bounded in $C^{\alpha/2,\alpha}$ for some $\alpha \in (0, 1)$, with bounds that only depend on the L^∞ norm of the coefficients. In particular, the map $(v, \mu) \to (\tilde{v}, \tilde{\mu})$ is compact and it has a fixed point (v, μ). Then $(u, m) := (\bar{u}, \bar{m}) + (v, \mu)$ is a solution to (38) with terminal condition u_T and which satisfies the decay

$$\|Du(t) - D\bar{u}(t)\|_{C^\alpha} + \|m(t) - \bar{m}\|_{C^\alpha} \leq \widehat{K}(e^{-\gamma t} + e^{-\gamma(T-t)}).$$

We now remove the smallness and regularity assumptions on the initial condition m_0 and the terminal condition u_T. Let (u, m) be the solution to (38). From Lemma 1.7 there exists $0 < \tau < \widehat{T}$ such that, if $T \geq \widehat{T}$, then the solution to (38) satisfies, again for some $\alpha \in (0, 1)$,

$$\|m(t) - \bar{m}\|_{C^\alpha} + \|Du(t) - D\bar{u}\|_{C^\alpha} \leq \widehat{K}^2 \quad \text{for all } t \in [\tau, T-\tau]. \tag{40}$$

From the first step we conclude that

$$\|m(t)-\bar{m}\|_{C^\alpha}+\|Du(t)-D\bar{u}\|_{C^\alpha}\leq \widehat{K}(e^{-\gamma(t-\tau)}+e^{-\gamma(T-\tau-t)})\quad \text{for all } t\in[\tau,T-\tau].$$

Using Lemma 1.1 and changing the constant if necessary, we can extend this inequality for m to the time interval $[1,T]$. Moreover, $Du(t)-D\bar{u}$ also satisfies a parabolic equation with uniformly bounded coefficients. Thus it is bounded in $C^{1+\alpha/2,1+\alpha}$ (for some possibly different α, depending on the data only) and we can improve the above inequality for u into

$$\|Du(t)-D\bar{u}\|_{C^{1+\alpha}}\leq C(e^{-\gamma t}+e^{-\gamma(T-t)})\quad \text{for all } t\in[0,T].$$

We finally prove the last bound on $v:=u-\bar{u}-\bar{\lambda}(T-t)$. Note that v satisfies

$$-\partial_t v-\Delta v=A(t,x),$$

where

$$A(t,x)=-(H(x,Du)-H(x,D\bar{u}))+F(x,m(t))-F(x,\bar{m}),$$

so

$$\|A(t)\|_{L^\infty}\leq C(e^{-\gamma t}+e^{-\gamma(T-t)})\quad \text{for all } t\in[0,T].$$

Thus, by a standard heat estimate,

$$\|v(t)\|_{L^\infty}\leq Ce^{-\omega(T-t)}\|v(T)\|_{L^\infty}+C\int_t^T e^{-\omega(s-t)}\|A(s)\|_{L^\infty}\,ds\leq C. \qquad \square$$

Let us stress that the above proof provides an explicit smallness estimate on $D(u-\bar{u})$ and $m-\bar{m}$ for initial-terminal data which are correspondingly small. This allows us to derive the convergence of $u^T(0,x)$ as the time horizon tends to infinity, for the special case with initial measure $m_0=\bar{m}$. This result is a first key argument in the analysis of the long time behavior of the general MFG system and of the master equation (Theorem 5.1 and Corollary 5.2).

Proposition 2.7. *For any $T>0$, let (u^T,m^T) be a solution to*

$$\begin{cases}-\partial_t u^T-\Delta u^T+H(x,Du^T)=F(x,m^T(t)) & \text{in } (0,T)\times\mathbb{T}^d,\\ \partial_t m^T-\Delta m^T-\operatorname{div}(m^T H_p(x,Du^T))=0 & \text{in } (0,T)\times\mathbb{T}^d,\\ m^T(0,\cdot)=\bar{m},\quad u^T(T,x)=G(x,m(T)) & \text{in } \mathbb{T}^d.\end{cases} \tag{41}$$

Then there exists a constant $\bar{c}$ such that

$$\lim_{T\to+\infty} u^T(0,x)-\bar{\lambda}T=\bar{u}(x)+\bar{c},$$

where the limit is uniform in $x\in\mathbb{T}^d$.

Proof. The proof consists in showing that the quantity $u^T(0,x)-\bar{\lambda}T-\bar{u}(x)$ is Cauchy in T in the uniform topology and converges to a constant. In a first step, we show that there exists $\tau>0$ large enough such that $u^T(T-\tau)$ and $u^{T'}(T'-\tau)$ are close in L^∞ for $T,T'\geq 2\tau$. Then we use Theorem 2.6 (and its proof) to extend this proximity up to time $t=0$.

Let us fix $\varepsilon > 0$ small. Theorem 2.6 states that

$$\|Du^T(t) - D\bar{u}\|_{C^{1+\alpha}} + \|m^T(t) - \bar{m}\|_{L^\infty} \le C(e^{-\gamma t} + e^{-\gamma(T-t)}) \quad \text{for all } t \in [1, T], \tag{42}$$

for some constant C independent of T. Fix τ large enough and let $T, T' \ge 2\tau$. If we consider $(\hat{u}^T, \hat{m}^T)(t, x) := (\hat{u}^T, \hat{m}^T)(t + T, x)$ and $(\hat{u}^{T'}, \hat{m}^{T'})(t, x) := (\hat{u}^{T'}, \hat{m}^{T'})(t + T', x)$, which are both solutions of the MFG system in $(-\tau, 0)$, the energy inequality gives

$$\begin{aligned} C^{-1}\int_{-\tau}^0 \int_{\mathbb{T}^d} (\hat{m}^T + \hat{m}^{T'})|D(\hat{u}^T - \hat{u}^{T'})|^2 &\le -\left[\int_{\mathbb{T}^d} (\hat{u}^T(t) - \hat{u}^{T'}(t))(\hat{m}^T(t) - \hat{m}^{T'}(t))\right]_{-\tau}^0 \\ &\le \int_{\mathbb{T}^d} (\hat{u}^T(-\tau) - \hat{u}^{T'}(-\tau))(\hat{m}^T(-\tau) - \hat{m}^{T'}(-\tau)), \end{aligned}$$

where we used that $(\hat{u}^T - \hat{u}^{T'})(0) = G(x, \hat{m}^T(0)) - G(x, \hat{m}^{T'}(0))$ and the monotonicity of G. Using (42) and the fact that $T, T' \ge 2\tau$ we deduce that

$$\int_{-\tau}^0 \int_{\mathbb{T}^d} (\hat{m}^T + \hat{m}^{T'})|D(\hat{u}^T - \hat{u}^{T'})|^2 \le Ce^{-2\gamma\tau},$$

where C is independent of T, T'. Now we apply Lemma 1.1 and (42) to $\hat{m}^T - \hat{m}^{T'}$ in the interval $(-\tau, 0)$ and we get

$$\|\hat{m}^T(t) - \hat{m}^{T'}(t)\|_{L^2} \le C\|\hat{m}^T(-\tau) - \hat{m}^{T'}(-\tau)\|_{L^2} + C\left(\int_{-\tau}^0 \int_{\mathbb{T}^d} (\hat{m}^{T'})^2|D(\hat{u}^T - \hat{u}^{T'})|^2\, dt\right)^{1/2} \le Ce^{-\gamma\tau}.$$

In particular, by the assumptions on F, G, there exists $C > 0$ such that

$$\sup_{t\in(-\tau,0)} \|F(x, \hat{m}^T(t)) - F(x, \hat{m}^{T'}(t))\|_{L^\infty} + \|G(x, \hat{m}^T(0)) - G(x, \hat{m}^{T'}(0))\|_{L^\infty} \le Ce^{-\gamma\tau}.$$

By the comparison principle between $\hat{u}^T$ and $\hat{u}^{T'}$ in $(-\tau, 0)$, we conclude that

$$\|\hat{u}^T(-\tau) - \hat{u}^{T'}(-\tau)\|_\infty \le C(1 + \tau)e^{-\gamma\tau}.$$

Hence we can choose τ sufficiently large such that

$$\|u^T(T - \tau) - u^{T'}(T' - \tau)\|_\infty \le \varepsilon \tag{43}$$

for any T, T' large enough.

Now we extend the proximity of u^T and $u^{T'}$ up to time $t = 0$. Recalling that, by (42),

$$\|Du^T(T - \tau) - D\bar{u}\|_\infty \le \varepsilon$$

for any T large enough, there exists $\bar{c}_0(T)$ such that

$$\|u^T(T - \tau) - \bar{u} - \bar{c}_0(T)\|_\infty \le C\varepsilon. \tag{44}$$

Note that (43) implies that $(\bar{c}_0(T))$ is Cauchy as $T \to +\infty$ and thus converges to a limit $\bar{c}$. Let $\gamma > 0$ be defined in the first step of the proof of Theorem 2.6; since (u^T, m^T) satisfy (39) with $\widehat{K} = \varepsilon^{1/2}$, we can choose ε small enough so that the fixed-point argument of Theorem 2.6 applies. Then, the restriction of (u^T, m^T) to $[0, T-\tau]$ satisfies

$$\|Du^T(t) - D\bar{u}\|_{L^\infty} + \|m^T(t) - \bar{m}\|_\infty \le \varepsilon^{1/2}(e^{-\gamma t} + e^{-\gamma(T-\tau-t)}) \quad \text{for all } t \in [0, T-\tau]. \tag{45}$$

Integrating in space the equation satisfied by $u^T - \bar{u}$, we get

$$\left| \int_{\mathbb{T}^d} (u^T(0) - u^T(T-\tau)) - \bar{\lambda}(T-\tau) \right| \le \int_0^{T-\tau}\int_{\mathbb{T}^d} |H(x, Du^T) - H(x, D\bar{u})| + |F(x, m^T(t)) - F(x, \bar{m})| \, dx\, dt \le C\varepsilon^{1/2}.$$

Using (45) (at time $t = 0$ and at time $t = T - \tau$) and Poincaré's inequality, we infer therefore that

$$\|u^T(0) - u^T(T-\tau) - \bar{\lambda}(T-\tau)\|_\infty \le C\varepsilon^{1/2}. \tag{46}$$

Combining (43), (44) and (46), we conclude that, for any T, T' large enough,

$$\|u^T(0) - \bar{u} - \bar{c}_0(T) - \bar{\lambda}(T-\tau)\|_\infty \le C\varepsilon^{1/2}.$$

From this we can deduce that $(u^T(0, x) - \bar{\lambda}T)$ converges uniformly to $\bar{u}(x) + \bar{c}$ as T tends to ∞. □

We also deduce from Theorem 2.6 crucial estimates for the linearized system around *any* solution (u, m) of (38).

Corollary 2.8. *There exists $\gamma > 0$ and $C > 0$ such that, if (u, m) is a solution of the MFG system* (38), *and if (v, μ) is a solution to the linearized MFG system*

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, Du)\cdot Dv = \dfrac{\delta F}{\delta m}(x, m)(\mu) & \text{in } (0, T)\times\mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, Du)) - \operatorname{div}(m H_{pp}(x, Du) Dv) = 0 & \text{in } (0, T)\times\mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0, \quad v(T, \cdot) = \dfrac{\delta G}{\delta m}(x, m(T))(\mu(T)) & \text{in } \mathbb{T}^d, \end{cases}$$

with $\int_{\mathbb{T}^d}\mu_0 = 0$, we have

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C(e^{-\gamma t} + e^{-\gamma(T-t)})\|\mu_0\|_{L^2} \tag{47}$$

and, for some $\alpha \in (0, 1)$ depending only on the data,

$$\sup_{t\in[0,T]} \|v\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}. \tag{48}$$

Proof. We first need a priori estimates on (v, μ). To this end we assume that $\mu_0 \in L^2(\mathbb{T}^d)$, and we proceed exactly as in Lemma 2.2 obtaining

$$\int_0^T\int_{\mathbb{T}^d} m|Dv|^2 + \sup_{t\in[0,T]} (\|\mu(t)\|_{L^2}^2 + \|Dv(t)\|_{L^2}^2) \le C_0\|\mu_0\|_{L^2}^2. \tag{49}$$

Next we note that (v, μ) is the solution to (36) with

$$A = -(H_p(x, Du) - H_p(x, D\bar{u})) \cdot Dv + \frac{\delta F}{\delta m}(x, m(t))(\mu(t)) - \frac{\delta F}{\delta m}(x, \bar{m})(\mu(t)),$$
$$B = \mu(H_p(x, Du) - H_p(x, D\bar{u})) + (mH_{pp}(x, Du) - \bar{m}H_{pp}(x, D\bar{u}))Dv$$

and

$$v_T(x) = \frac{\delta G}{\delta m}(x, m(T))(\mu(T)) - \frac{\delta G}{\delta m}(x, \bar{m})(\mu(T)).$$

Note that

$$\|A(t)\|_{L^2} \leq C\|Du - D\bar{u}\|_\infty \|Dv\|_{L^2} + C\boldsymbol{d}_1(m(t), \bar{m})\|\mu(t)\|_{L^2},$$

while

$$\|B(t)\|_{L^2} \leq C\|Du - D\bar{u}\|_\infty \|\mu(t)\|_{L^2} + C(\boldsymbol{d}_1(m(t), \bar{m}) + \|Du(t) - D\bar{u}\|_\infty)\|Dv(t)\|_{L^2}$$

and

$$\|v_T\| \leq C\boldsymbol{d}_1(m(T), \bar{m})\|\mu(T)\|_{L^2}.$$

Here we used once more that $m \mapsto (\delta F/\delta m)(x, m, y)$, $m \mapsto (\delta G/\delta m)(x, m, y)$ and $p \mapsto H_{pp}(x, p)$ are Lipschitz.

Using Theorem 2.6 and (49), we deduce

$$\|A(t)\|_{L^2} + \|B(t)\|_{L^2} \leq C\|\mu_0\|_{L^2}(e^{-\gamma t} + e^{-\gamma(T-t)}).$$

Then Theorem 2.5 (used with $\lambda = \gamma$) and the bound (49) imply

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C((1+t)e^{-\gamma t} + (1+T)e^{-\gamma(T-t)})\|\mu_0\|_{L^2}.$$

So we deduce (47), possibly for a smaller value of γ.

Now we upgrade the above estimate by using weaker norms for μ_0 and stronger norms for v. For this, we use Lemma 2.9 below, which states that

$$\|\mu(1)\|_{L^2} \leq C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Applying our previous estimate (47) to the time interval $[1, T]$, we find that, for any $t \geq 1$,

$$\begin{aligned}\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} &\leq C(e^{-\gamma(t-1)} + e^{-\gamma(T-(t-1))})\|\mu(1)\|_{L^2} \\ &\leq C(e^{-\gamma t} + e^{-\gamma(T-t)})\|\mu_0\|_{(C^{2+\alpha})'}.\end{aligned}$$

Lemma 2.9 also states that

$$\sup_{t\in[0,T]} \|v(t) - \langle v(t)\rangle\|_{C^{2+\alpha}} + \sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} \leq C\|\mu_0\|_{(C^{2+\alpha})'}, \tag{50}$$

so we also have

$$\sup_{t\in[0,1]} \|Dv(t)\|_{L^2} + \sup_{t\in[0,1]} \|\mu(t)\|_{(C^{2+\alpha})'} \leq C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Integrating in space the equation for v and using the above bounds on Dv and μ then implies

$$|\langle v(t)\rangle| \leq C\|\mu_0\|_{(C^{2+\alpha})'} \quad \text{for all } t \in [0, T].$$

We can then deduce (48) from (50) and the above inequality. □

Lemma 2.9. *Under the assumptions of Corollary 2.8, there exists a constant $C > 0$ (independent of T, m_0 and μ_0) such that*

$$\sup_{t\in[0,T]} \|v(t) - \langle v(t)\rangle\|_{C^{2+\alpha}} + \sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} + \|\mu(1)\|_{L^2} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Proof. The estimate (49) gives

$$c\int_0^t\int_{\mathbb{T}^d} m|Dv|^2 \le \int_0^T\int_{\mathbb{T}^d} mH_{pp}(x, Du)Dv\cdot Dv \le \int_{\mathbb{T}^d} v(0)\mu_0, \tag{51}$$

where we used that $(\delta G/\delta m)(x, m(T))$ is a nonnegative operator. By duality, we also have

$$\int_{\mathbb{T}^d} \mu(t)\xi = -\int_0^t\int_{\mathbb{T}^d} mH_{pp}(Du)Dv\cdot D\psi + \int_{\mathbb{T}^d} \psi(0)\mu_0,$$

where ψ solves (for some smooth terminal condition ξ at time t)

$$\begin{cases} -\partial_t\psi - \Delta\psi + H_p(x, Du)\cdot D\psi = 0 & \text{in } (0, t)\times\mathbb{T}^d,\\ \psi(t, \cdot) = \xi & \text{in } \mathbb{T}^d.\end{cases}$$

Since, by Lemma 1.2, $\|\psi(s) - \langle\psi(s)\rangle\|_{L^2} \le ce^{-\omega(t-s)}\|\xi\|_{L^2}$, we have by standard estimates

$$\int_0^t\int_{\mathbb{T}^d} |D\psi|^2 \le \|\xi\|_2^2 + C\int_0^t\int_{\mathbb{T}^d} |\psi - \langle\psi\rangle|^2 \le C\|\xi\|_{L^2}^2.$$

Therefore,

$$\int_{\mathbb{T}^d} \mu(t)\xi \le C\left(\int_0^t\int_{\mathbb{T}^d} m|Dv|^2\right)^{1/2} \|\xi\|_{L^2} + \|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}}\|\mu_0\|_{(C^{2+\alpha})'}.$$

From (51) we deduce

$$\int_{\mathbb{T}^d} \mu(t)\xi \le C\bigl(\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}\|\mu_0\|_{(C^{2+\alpha})'}\bigr)^{1/2}\|\xi\|_{L^2} + \|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}}\|\mu_0\|_{(C^{2+\alpha})'}. \tag{52}$$

To estimate last term, we note that, if $t \le 1$, we have by Schauder estimates that

$$\|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}} \le C\|\xi\|_{C^{2+\alpha}},$$

while, if $t \ge 1$, we have, by Schauder interior estimates

$$\|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}} \le C\|\psi(1) - \langle\psi(1)\rangle\|_{L^2} \le C\|\xi\|_{L^2} \le C\|\xi\|_{C^{2+\alpha}}. \tag{53}$$

Coming back to (52) and taking the supremum over the ξ with $\|\xi\|_{C^{2+\alpha}} \le 1$, this implies

$$\sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} \le C\bigl(\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}^{1/2}\|\mu_0\|_{(C^{2+\alpha})'}^{1/2} + \|\mu_0\|_{(C^{2+\alpha})'}\bigr). \tag{54}$$

Similarly, from (52) and (53) we also estimate

$$\|\mu(1)\|_{L^2} \le C\bigl(\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}^{1/2}\|\mu_0\|_{(C^{2+\alpha})'}^{1/2} + \|\mu_0\|_{(C^{2+\alpha})'}\bigr). \tag{55}$$

We now have to estimate $v(0) - \langle v(0)\rangle$. First we have, by Lemma 1.2, that for any $t \in [0, T]$

$$\begin{aligned} \|v(t) - \langle v(t)\rangle\|_\infty &\le e^{-\omega(T-t)} \Big\| \frac{\delta G}{\delta m}(x, m(T))\mu(T)\Big\|_\infty + \int_t^T e^{-\omega(s-t)} \Big\| \frac{\delta F}{\delta m}(\,\cdot\,, m(s))(\mu(s))\Big\|_\infty ds \\ &\le C \sup_{[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'}, \end{aligned} \tag{56}$$

where we used that $\delta F/\delta m, \delta G/\delta m$ are $C^{2+\alpha}$ with respect to y. We also estimate Dv in L^2 in terms of the same quantity due to Lemma 1.2. Next, the regularizing property of the equation for $v - \langle v\rangle$ [Ladyženskaja et al. 1968, Theorem IV.9.1] implies that, for any $t \in \big[0, T - \frac{1}{2}\big]$ and any $\beta \in (0, 1)$,

$$\begin{aligned} \|v(t) - \langle v(t)\rangle\|_{C^{1+\beta}} &\le \big\|v\big(t + \tfrac{1}{2}\big) - \big\langle v\big(t + \tfrac{1}{2}\big)\big\rangle\big\|_\infty + C \sup_{s\in[t,t+1/2]} \|\mu(s)\|_{(C^{2+\alpha})'} \\ &\le C \sup_{[0,T]} \|\mu(s)\|_{(C^{2+\alpha})'}, \end{aligned}$$

(where the constant depends on β). Then considering the equation for v_{x_i} (for $i \in \{1, \dots, d\}$) and using the uniform C^2 regularity of u as well as the C^2 regularity of $D_x(\delta F/\delta m)$ in the y-variable as in (56), we obtain in the same way, for any $t \in [0, T - 1]$

$$\begin{aligned} \|v_{x_i}(t)\|_{C^{1+\beta}} &\le \big\|v_{x_i}\big(t + \tfrac{1}{2}\big)\big\|_\infty + C\Big(\sup_{s\in[t,t+1/2]} \|Dv(s)\|_\infty + \sup_{s\in[t,t+1/2]} \|\mu(s)\|_{(C^{2+\alpha})'}\Big) \\ &\le C \sup_{[0,T]} \|\mu(s)\|_{(C^{2+\alpha})'}. \end{aligned}$$

Choosing $\beta = \alpha$, we have proved therefore that

$$\sup_{s\in[0,T-1]} \|v(s) - \langle v(s)\rangle\|_{C^{2+\alpha}} \le C \sup_{s\in[0,T]} \|\mu(s)\|_{(C^{2+\alpha})'}.$$

Using this inequality for $\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}$ in (54) then gives

$$\sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} \le C\|\mu_0\|_{(C^{2+\alpha})'},$$

which in turn implies

$$\sup_{t\in[0,T-1]} \|v(s) - \langle v(s)\rangle\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Note that we can extend this inequality to the time interval $[T-1, T]$ by using the regularity of the equation satisfied by v on this interval, the regularity of the terminal condition and the bound on $\|\mu(t)\|_{(C^{2+\alpha})'}$.

In the same way, from (55) we obtain

$$\|\mu(1)\|_{L^2} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

□

Remark 2.10. In order to estimate v in the C^2 norm, we have used in Lemma 2.9 the regularity condition (FGc) on the couplings. However, by only using condition (FGb), we could similarly obtain a milder estimate as

$$\sup_{t\in[0,T]} \|v(t) - \langle v(t)\rangle\|_{C^1} + \sup_{t\in[0,T]} \|\mu(t)\|_{(C^1)'} \le C\|\mu_0\|_{(C^1)'}. \tag{57}$$

Indeed, an estimate similar to (56) would hold in terms of $\|\mu(t)\|_{(C^1)'}$ by using condition (FGb) since

$$\begin{aligned}\|v(t)-\langle v(t)\rangle\|_\infty &\le e^{-\omega(T-t)}\Big\|\frac{\delta G}{\delta m}(x,m(T))\mu(T)\Big\|_\infty+\int_t^T e^{-\omega(s-t)}\Big\|\frac{\delta F}{\delta m}(\,\cdot\,,m(s))(\mu(s))\Big\|_\infty ds\\ &\le C\sup_{[0,T]}\|\mu(t)\|_{(C^1)'},\end{aligned}$$

where we only used that $\delta F/\delta m$, $\delta G/\delta m$ are C^1 and globally Lipschitz with respect to y. Under the same condition the estimate for Dv in L^∞ would follow. Eventually, with the same strategy as in the above proof, by using C^1 rather than $C^{2+\alpha}$ and using estimates on v, we would get (57).

3. Exponential rate of convergence for the infinite-horizon MFG system

We now study the infinite-horizon discounted problem and show an exponential convergence towards a stationary solution. The existence of this solution is new, as well as the convergence rate towards this solution. The method of proof is close to the one employed in the previous section for the finite horizon.

3A. *The stationary solution of the infinite-horizon problem.*

Proposition 3.1. *There exists $\delta_0>0$ such that, if $\delta\in(0,\delta_0)$, there is a unique solution $(\bar u^\delta,\bar m^\delta)$ to the problem* (7). *Moreover, for any $\delta\in(0,\delta_0)$,*

$$\|D\bar u^\delta\|_\infty+\delta\|\bar u^\delta\|_\infty+\|\bar m^\delta\|_\infty\le C \qquad \text{and} \qquad \bar m^\delta(x)\ge C^{-1}\quad \text{for all } x\in\mathbb{T}^d,$$

for some constant $C>0$.

Proof. The existence of a solution can be achieved by a standard fixed-point argument, so we omit it. In the same way, the regularity of $\bar u^\delta$ and $\bar m^\delta$ is standard. The strong maximum principle implies that m^δ is bounded below by a constant independent of δ. For proving the uniqueness, we argue as usual by duality, see [Lasry and Lions 2007]: Let (u_1,m_1) and (u_2,m_2) be two solutions. We multiply the equation for u_1-u_2 by m_1-m_2 and the equation for m_1-m_2 by u_1-u_2, we integrate in time and space and add the resulting quantities to obtain, by Poincaré's inequality,

$$C^{-1}\|D(u_1-u_2)\|_{L^2}^2\le\delta\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)\le C\delta\|D(u_1-u_2)\|_{L^2}\|m_1-m_2\|_{L^2}.$$

Thus

$$\|D(u_1-u_2)\|_{L^2}\le C\delta\|m_1-m_2\|_{L^2}. \tag{58}$$

On another hand, by Corollary 1.3, we have

$$\|m_1-m_2\|_{L^2}\le C\|H_p(\,\cdot\,,Du_1)-H_p(\,\cdot\,,Du_2)\|_{L^2}\le C\|D(u_1-u_2)\|_{L^2} \tag{59}$$

For δ small enough, we deduce from (58)–(59) that $m_1=m_2$ and $Du_1=Du_2$, whence $u_1=u_2$. □

We now note that the solution $(\bar u^\delta,\bar m^\delta)$ is close to $(\bar u,\bar m)$, where $(\bar\lambda,\bar u,\bar m)$ is the solution of the ergodic problem (2):

Proposition 3.2. *We have*

$$\|\delta\bar{u}^\delta - \bar{\lambda}\|_\infty + \|D(\bar{u}^\delta - \bar{u})\|_{L^2} + \|\bar{m}^\delta - \bar{m}\|_{L^2} \le C\delta^{1/2}.$$

Proof. We use again the duality argument (consisting in multiplying the equation for $u^\delta - \bar{u}$ by $m^\delta - \bar{m}$ and the equation for $m^\delta - \bar{m}$ by $u^\delta - \bar{u}$, integrating in space and adding the resulting quantities) to get

$$C^{-1}\|D(\bar{u}^\delta - \bar{u})\|_{L^2}^2 \le \int_{\mathbb{T}^d} (\delta\bar{u}^\delta - \bar{\lambda})(\bar{m}^\delta - \bar{m}) \le C\delta\|D\bar{u}^\delta\|_\infty \le C\delta.$$

Thus

$$\|D(\bar{u}^\delta - \bar{u})\|_{L^2} \le C\delta^{1/2}.$$

By Corollary 1.3, we have

$$\|\bar{m}^\delta - \bar{m}\|_{L^2} \le C\|D(\bar{u}^\delta - \bar{u})\|_{L^2} \le C\delta^{1/2}.$$

The estimate between $\delta\bar{u}^\delta$ and $\bar{\lambda}$ then comes from the comparison principle. □

3B. ***Exponential rate for the linearized system.*** Let $(\bar{u}^\delta, \bar{m}^\delta)$ be the solution to (7). We consider the linearized discounted problem around this solution

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, D\bar{u}^\delta)\cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu(t)) & \text{in } (0,+\infty)\times\mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u}^\delta)) - \operatorname{div}(\bar{m}^\delta H_{pp}(x, D\bar{u}^\delta)Dv) = 0 & \text{in } (0,+\infty)\times\mathbb{T}^d, \\ \mu(0,\cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded}, \end{cases} \tag{60}$$

with $\int_{\mathbb{T}^d} \mu_0 = 0$. As in Section 2A, the existence of a solution to (60) can be proved for $\mu_0 \in L^2(\mathbb{T}^d)$ by using fixed-point arguments and relying on the conditions enjoyed by $\delta F/\delta m$ and the smoothness of $(\bar{u}^\delta, \bar{m}^\delta)$. In particular, one can first solve the system in a finite horizon $t \in (0, n)$ with terminal condition $v(n) = 0$, and then obtain a solution to (60) by letting $n \to \infty$. Since $\delta > 0$, here $\|\delta F/\delta m\|_\infty \delta^{-1}$ is a uniform bound with respect to n and leads to a bounded v in (60).

In the rest of this section, we are going to show that v actually enjoys a bound which is uniform in δ and that μ, Dv decay exponentially in L^2 as $t \to \infty$, uniformly with respect to δ.

Lemma 3.3. *Let* (v, μ) *be a solution to* (60). *Then we have*

$$\int_{\mathbb{T}^d} \mu(t)v(t) \ge 0 \quad \textit{for all } t \ge 0$$

and there exists a constant $C_0 > 0$, *independent of* μ_0 *and* δ, *such that*, *for any* $t \ge 0$,

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C_0\|\mu_0\|_{L^2} e^{\delta t/2}.$$

Proof. We consider the duality between $e^{-\delta t}v$ and μ (i.e, we multiply the equation for $e^{-\delta t}v$ by μ and the equation for μ by $e^{-\delta t}v$, we integrate in time and space and we add the resulting quantities); using properties of $(\bar{u}_\delta, \bar{m}_\delta)$ from Proposition 3.1 we get

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta t}\|Dv(t)\|_{L^2}^2\,dt \le -\left[e^{-\delta t}\int_{\mathbb{T}^d} v(t)\mu(t)\right]_{t_1}^{t_2}. \tag{61}$$

Next we claim that

$$C^{-1}\int_0^\infty e^{-\delta t}\|Dv(t)\|_{L^2}^2\,dt \le \int_{\mathbb{T}^d}\mu_0 v(0) \le C\|\mu_0\|_{L^2}\|v(0)-\langle v(0)\rangle\|_{L^2}. \tag{62}$$

This inequality is obvious from (61) if we know that the limit $e^{-\delta t}\int_{\mathbb{T}^d} v(t)\mu(t)$ vanishes as $t\to+\infty$. For this we need a first rough bound on μ. By Lemma 1.1 we have

$$\|\mu(t)\|_{L^2} \le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^t \|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}.$$

By (61), we get

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\left[\int_0^t e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\|v\|_\infty^{1/2}[\|\mu_0\|_{L^2}^{1/2} + e^{-\delta t/2}\|\mu(t)\|_{L^2}^{1/2}],\end{aligned}$$

so

$$\|\mu(t)\|_{L^2} \le C_\delta e^{\delta t/2},$$

where C_δ depends on μ_0 and δ. This inequality then implies

$$\lim_{t\to+\infty} e^{-\delta t}\int_{\mathbb{T}^d}\mu(t)v(t) = 0$$

and (62) holds. Note that (61) implies that the map $t\to e^{-\delta t}\int_{\mathbb{T}^d}\mu(t)v(t)$ is nonincreasing, and we just proved that it has limit 0 as $t\to+\infty$. Thus it is nonnegative.

In light of (62) we revisit the estimate of μ. We have

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\left[\int_0^t e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}.\end{aligned}$$

We plug this inequality into the usual estimate for v (Lemma 1.2): for any $0\le t\le t_1$,

$$\begin{aligned}&\|v(t)-\langle v(t)\rangle\|_{L^2}\\ &\quad\le Ce^{-\omega(t_1-t)}\|v(t_1)-\langle v(t_1)\rangle\|_{L^2}+C\int_t^{t_1} e^{-\omega(s-t)}\|\mu(s)\|_{L^2}\,ds\\ &\quad\le Ce^{-\omega(t_1-t)}\|v(t_1)-\langle v(t_1)\rangle\|_{L^2}+C\int_t^{t_1} e^{-\omega(s-t)}\big(e^{-\omega s}\|\mu_0\|_{L^2}+Ce^{\delta s/2}\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}\big)\,ds\\ &\quad\le Ce^{-\omega(t_1-t)}\|v(t_1)-\langle v(t_1)\rangle\|_{L^2}+C\|\mu_0\|_{L^2}e^{-\omega t}+C\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}e^{\delta t/2}.\end{aligned}$$

Letting $t_1\to+\infty$ gives

$$\|v(t)-\langle v(t)\rangle\|_{L^2} \le C\|\mu_0\|_{L^2}e^{-\omega t} + C\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}e^{\delta t/2}.$$

Choosing $t=0$ and rearranging we find

$$\|v(0)-\langle v(0)\rangle\|_{L^2} \le C\|\mu_0\|_{L^2}.$$

So we have for any $t\ge 0$

$$\|\mu(t)\|_{L^2} + \|v(t)-\langle v(t)\rangle\|_{L^2} \le C\|\mu_0\|_{L^2}e^{\delta t/2}.$$

We can then conclude by Lemma 1.2. □

Proposition 3.4. *Let* $(\bar u^\delta, \bar m^\delta)$ *be the solution to* (7). *There exist* $\delta_0, C_0, \lambda > 0$ *such that, if* (v, μ) *is the solution to* (60) *associated with* $(\bar u^\delta, \bar m^\delta)$ *and* $\int_{\mathbb{T}^d} \mu_0 = 0$, *and if* $\delta \in (0, \delta_0)$, *then*

$$\|Dv(t)\|_{L^2} + \|\mu(t)\|_{L^2} \le C_0 \|\mu_0\|_{L^2} e^{-\lambda t} \quad \textit{for all } t \ge 0.$$

In particular,

$$\|v\|_{L^\infty} \le C.$$

Proof. Let us set

$$\rho^\delta(t) := \sup_{\mu_0} e^{-\delta t} \int_{\mathbb{T}^d} \mu(t) v(t),$$

where the supremum is taken over $\|\mu_0\|_{L^2} \le 1$ and where (v, μ) is the solution to (60) with initial condition $\mu(0) = \mu_0$. In view of the inequality (61), the map ρ^δ is nonincreasing. Moreover, Lemma 3.3 states that $\rho^\delta(t)$ is bounded independently of δ and nonnegative. Then we set

$$\rho(t) = \limsup_{\delta \to 0} \rho^\delta(t).$$

Note that ρ is also nonincreasing, nonnegative and bounded. We denote by ρ_∞ its limit as $t \to +\infty$. We claim that $\rho_\infty = 0$.

Indeed, let $t_n \to +\infty$, $\delta_n \to 0$, and μ_0^n with $\|\mu_0^n\|_{L^2} \le 1$ be such that

$$e^{-\delta_n t_n} \int_{\mathbb{T}^d} \mu^n(t_n) v^n(t_n) \ge \rho_\infty - \frac{1}{n}.$$

We let, for $s \in [-t_n, +\infty)$,

$$\tilde v^n(s) = e^{-\delta_n t_n/2}(v^n(t_n + s) - \langle v^n(t_n)\rangle), \quad \tilde\mu^n(s) = e^{-\delta_n t_n/2} \mu^n(t_n + s).$$

From Lemma 3.3 we know that $\tilde v^n$, $D\tilde v^n$ and $\tilde \mu^n$ are locally bounded in L^2. As the pair $(\tilde v^n, \tilde \mu^n)$ satisfies an equation of the form (60), standard regularity estimates for parabolic equations with bounded coefficients [Ladyženskaja et al. 1968, Theorem III.10.1] imply that $\tilde v^n$, $D\tilde v^n$ and $\tilde\mu^n$ are locally bounded in $C^{\beta/2,\beta}$ for some $\beta \in (0, 1)$. Therefore, up to a subsequence, denoted in the same way, $(\tilde v^n)$ converges to $\tilde v$ and $(\tilde \mu^n)$ converges to $\tilde\mu$ locally uniformly, where by linearity $(\tilde v, \tilde \mu)$ solves

$$\begin{cases} -\partial_t \tilde v - \Delta \tilde v + H_p(x, D\bar u^\delta) \cdot D\tilde v = \dfrac{\delta F}{\delta m}(x, \bar m^\delta)(\tilde\mu(t)) & \text{in } (-\infty, 0) \times \mathbb{T}^d, \\ \partial_t \tilde\mu - \Delta\tilde\mu - \operatorname{div}(\tilde\mu H_p(x, D\bar u^\delta)) - \operatorname{div}(\bar m^\delta H_{pp}(x, D\bar u^\delta) D\tilde v) = 0 & \text{in } (-\infty, 0) \times \mathbb{T}^d. \end{cases}$$

For any $s \le 0$ and any $\tau \ge 0$, we have, for n large enough,

$$\int_{\mathbb{T}^d} \tilde\mu^n(s)\tilde v^n(s) = e^{-\delta_n t_n} \int_{\mathbb{T}^d} \mu^n(t_n + s) v^n(t_n + s) \le e^{\delta_n s} \rho^{\delta_n}(t_n + s) \le e^{\delta_n s}\rho^{\delta_n}(\tau),$$

so

$$\int_{\mathbb{T}^d} \tilde\mu(s)\tilde v(s) \le \rho(\tau).$$

Letting $\tau \to +\infty$, we find therefore

$$\int_{\mathbb{T}^d} \tilde{\mu}(s)\tilde{v}(s) \le \rho_\infty = \int_{\mathbb{T}^d} \tilde{\mu}(0)\tilde{v}(0) \quad \text{for all } s \le 0.$$

However $\int_{\mathbb{T}^d} \tilde{\mu}(s)\tilde{v}(s)$ is nonincreasing, so we also have the reverse inequality, and we deduce that this quantity must be constant in $(-\infty, 0]$. The duality relation (consisting as usual in multiplying the equation for $\tilde{v}$ by $\tilde{\mu}$ and the equation for $\tilde{\mu}$ by $\tilde{v}$, integrating in time and space and adding the resulting quantities) then implies $D\tilde{v} = 0$ for any $t \le 0$, which gives $\rho_\infty = 0$.

Next we claim that there exist $\gamma > 0$, $C > 0$ and $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, one has

$$\rho^\delta(t) \le Ce^{-\gamma t} \quad \text{for all } t \ge 0. \tag{63}$$

Indeed, let $\varepsilon > 0$ small to be chosen later and let $T_0 > 0$, $\delta_0 > 0$ be such that

$$\rho^\delta(t) \le \varepsilon \quad \text{for all } t \ge T_0,\ \delta \in (0, \delta_0). \tag{64}$$

Fix $\delta \in (0, \delta_0)$ and let (v, μ) be a solution to (60). Inequalities (61) (combined with the fact that $\int_{\mathbb{T}^d} v\mu$ is nonnegative) and (64) imply

$$\int_{t_1}^{t_2} e^{-\delta s} \|Dv(s)\|_{L^2}^2\, ds \le C\varepsilon \|\mu_0\|_{L^2}^2 \quad \text{for all } t_1, t_2 \ge T_0,\ \delta \in (0, \delta_0).$$

Revisiting the estimate for μ, we have, for any $t_1 \ge 0$,

$$\|\mu(T_0 + t_1)\|_{L^2} \le Ce^{-\omega t_1} \|\mu(T_0)\|_{L^2} + C\left[\int_{T_0}^{T_0+t_1} \|Dv(s)\|_{L^2}^2\, ds\right]^{1/2},$$

so, using Lemma 3.3 and the above estimate on Dv,

$$\begin{aligned}\|\mu(T_0 + t_1)\|_{L^2} &\le Ce^{-\omega t_1 + \delta T_0/2} \|\mu_0\|_{L^2} + Ce^{\delta(T_0+t_1)/2}\left[\int_{T_0}^{T_0+t_1} e^{-\delta s}\|Dv(s)\|_{L^2}^2\, ds\right]^{1/2} \\ &\le C\|\mu_0\|_{L^2} e^{\delta(T_0+t_1)/2}(e^{-(\omega+\delta/2)t_1} + \varepsilon^{1/2}).\end{aligned}$$

We choose t_1 large enough (independently of ε and $\delta \in (0, \omega)$) so that $Ce^{-\omega t_1} \le \frac{1}{4}$ and ε so small that $C\varepsilon^{1/2} \le \frac{1}{4}$. Setting $\tau := T_0 + t_1$, this yields

$$\|\mu(\tau)\|_{L^2} \le \tfrac{1}{2} \|\mu_0\|_{L^2} e^{\delta\tau/2}. \tag{65}$$

Fix (v, μ) a solution to (60). The pair $(\tilde{v}, \tilde{\mu}) := (v(\tau + \cdot), \mu(\tau + \cdot))$ is also a solution of (60) with initial condition $\tilde{\mu}(0) = \mu(\tau)$. Since the equation is linear in μ_0 and the quantity $\int_{\mathbb{T}^d} \mu(t)v(t)$is homogeneous of degree 2, we have therefore

$$e^{-\delta t} \int_{\mathbb{T}^d} \tilde{\mu}(t)\tilde{v}(t) \le \|\mu(\tau)\|_{L^2}^2 \rho^\delta(t) \quad \text{for all } t \ge 0,$$

where

$$e^{-\delta t} \int_{\mathbb{T}^d} \tilde{\mu}(t)\tilde{v}(t) = e^{\delta\tau} e^{-\delta(t+\tau)} \int_{\mathbb{T}^d} \mu(t + \tau)v(t + \tau).$$

This implies

$$e^{-\delta(t+\tau)}\int_{\mathbb{T}^d} \mu(t+\tau)v(t+\tau) \le e^{-\delta\tau}\|\mu(\tau)\|_{L^2}^2\rho^\delta(t).$$

Recalling estimate (65) and taking the supremum over $\|\mu_0\|_{L^2} \le 1$, we find

$$\rho^\delta(t+\tau) \le \tfrac{1}{2}\rho^\delta(t) \quad \text{for all } t \ge 0.$$

This easily implies (63).

We can now come back to the estimates of μ and v for a given solution (v, μ) of (60) with $\delta \in (0, \delta_0)$. For $t > 0$, we have, using Lemma 3.3, (61) and (63) successively,

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t/2}\|\mu(t/2)\|_{L^2} + C\left[\int_{t/2}^t \|Dv(s)\|_{L^2}^2\, ds\right]^{1/2}\\ &\le Ce^{-\omega t/2+\delta t/2}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\left[\int_{t/2}^t e^{-\delta s}\|Dv(s)\|_{L^2}^2\, ds\right]^{1/2}\\ &\le C\|\mu_0\|_{L^2}(e^{-\omega t/2+\delta t/2} + e^{\delta t/2-\gamma t/4}).\end{aligned}$$

For δ small enough, this implies

$$\|\mu(t)\|_{L^2} \le C\|\mu_0\|_{L^2}e^{-\lambda t} \quad \text{for all } t \ge 0,$$

for some $\lambda \in (0, \omega)$. Thus, by Lemma 3.3 applied on the time-interval $[t/2, +\infty)$,

$$\|Dv(t)\|_{L^2} \le C\|\mu(t/2)\|_{L^2}e^{\delta t/4} \le C\|\mu_0\|_{L^2}e^{-\lambda t}$$

for some possibly different $\lambda > 0$. The bound on $\|v\|_\infty$ follows directly from the equation for v and our regularity assumption on $\delta F/\delta m$, which implies

$$\left\|\frac{\delta F}{\delta m}(x, m^\delta)(\mu(t))\right\|_\infty \le C\|\mu(t)\|_{L^2} \le C\|\mu_0\|_{L^2}e^{-\lambda t} \quad \text{for all } t \ge 0. \qquad \square$$

In the next step we study a perturbed discounted linearized problem.

Proposition 3.5. *Let* (v, μ) *solve*

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, D\bar{u}^\delta)\cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu(t)) + A(t,x) & \text{in } (0,+\infty)\times\mathbb{T}^d,\\ \partial_t\mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u}^\delta)) - \operatorname{div}(\bar{m}^\delta H_{pp}(x, D\bar{u}^\delta)Dv) = \operatorname{div}(B(t,x)) & \text{in } (0,+\infty)\times\mathbb{T}^d,\\ \mu(0,\cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded}, \end{cases} \tag{66}$$

with $\int_{\mathbb{T}^d}\mu_0 = 0$, $\|\mu_0\|_{L^2} \le 1$ *and assume that, for some* $\gamma > 0$,

$$\|A(t)\|_{L^2} + \|B(t)\|_{L^2} \le e^{-\gamma t} \quad \text{for all } t \ge 0. \tag{67}$$

If $\delta \in (0, \delta_0)$, *then*

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C(1+t)e^{-\theta t}, \tag{68}$$

where $\theta := \gamma \wedge \lambda$ *and* $\delta_0, \lambda > 0$ *are defined in Proposition 3.4.*

Proof. Using Proposition 3.4 and the linearity of the equation, we can assume, without loss of generality, that $\mu_0 = 0$. We first assume that $A \equiv 0$. Throughout the proof, the constant C can depend on γ.

Let us start with preliminary estimates. The duality identity (i.e., the equality obtained by multiplying the equation for $e^{-\delta t}v$ by μ and the equation for μ by $e^{-\delta t}v$, integrating in time and space and adding the resulting quantities) here implies

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds \le -\left[e^{-\delta s}\int_{\mathbb{T}^d} v(s)\mu(s)\right]_{t_1}^{t_2} + C\int_{t_1}^{t_2} e^{-\delta s}\|B(s)\|_{L^2}^2\,ds. \tag{69}$$

One can check, exactly as for the proof of Lemma 3.3, that

$$\lim_{t\to+\infty} e^{-\delta t}\int_{\mathbb{T}^d}\mu(t)v(t) = 0.$$

Then the inequality (69) and our assumption (67) on B imply

$$\int_0^{+\infty} e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds \le C.$$

Arguing as before, we derive for μ that

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le C\left[\int_0^t \|Dv(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{\delta t/2}\left[\int_0^t e^{-\delta s}(\|Dv(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2)\,ds\right]^{1/2} \le Ce^{\delta t/2}.\end{aligned}$$

Thus, applying Lemma 1.2 (with $T\to\infty$) to $e^{-\delta t}v$, we deduce

$$e^{-\delta t}\|v(t)-\langle v(t)\rangle\|_{L^2} \le C\int_t^{+\infty} e^{-\omega(s-t)}\|\mu(s)\|_{L^2}e^{-\delta s}\,ds \le Ce^{-\delta t/2},$$

which gives

$$\|v(t)-\langle v(t)\rangle\|_{L^2} \le Ce^{\delta t/2}.$$

We set

$$\rho^\delta(t) = \sup_B[e^{-\delta t}(\|\mu(t)\|_{L^2} + \|v(t)-\langle v(t)\rangle\|_{L^2})],$$

where the supremum is taken over the B that satisfy (67) and where (v,μ) solves (66) (with $A\equiv 0$ and $\mu_0 = 0$). Fix a solution (v,μ) to (66) with $A\equiv 0$ and $\mu_0 = 0$ and let us consider its restriction to a time interval $[\tau,+\infty)$. We can write

$$(v,\mu) = (v_1,\mu_1) + (v_2,\mu_2),$$

where (v_1,μ_1) solves on $[\tau,+\infty)$ the homogeneous equation (60) with initial condition $\mu_1(\tau) = \mu(\tau)$ and (v_2,μ_2) solves on $[\tau,+\infty)$ the inhomogeneous equation (66) with $\mu_2(\tau) = 0$ and $A\equiv 0$. By Proposition 3.4 we have, for $\delta\in(0,\delta_0)$,

$$\|\mu_1(\tau+t)\|_{L^2} + \|Dv_1(\tau+t)\|_{L^2} \le C_0e^{-\lambda t}\|\mu(\tau)\|_{L^2} \le C_0e^{-\lambda t}e^{\delta\tau/2} \quad \text{for all } t\ge 0,$$

while, as the restriction of B to $[\tau, +\infty)$ satisfies

$$\|B(\tau+t)\|_{L^2} \le e^{-\gamma\tau}e^{-\gamma t} \quad \text{for all } t \ge 0,$$

we have

$$\|\mu_2(\tau+t)\|_{L^2} + \|v_2(\tau+t) - \langle v_2(\tau+t)\rangle\|_{L^2} \le e^{-\gamma\tau}\rho^\delta(t)e^{\delta t} \quad \text{for all } t \ge 0.$$

So

$$\|\mu(\tau+t)\|_{L^2} + \|v(\tau+t) - \langle v(\tau+t)\rangle\|_{L^2} \le Ce^{-\lambda t}e^{\delta\tau/2} + e^{-\gamma\tau}\rho^\delta(t)e^{\delta t}.$$

Multiplying by $e^{-\delta(t+\tau)}$ and taking the supremum over B leads to

$$\rho^\delta(\tau+t) \le Ce^{-(\lambda+\delta)t} + e^{-(\gamma+\delta)\tau}\rho^\delta(t).$$

Setting $\theta := \gamma \wedge \lambda$ and considering the inequality satisfied by $e^{(\theta+\delta)t}\rho^\delta(t)$, we then obtain the exponential decay of ρ^δ

$$\rho^\delta(t) \le C(1+t)e^{-(\theta+\delta)t},$$

which implies, by the definition of $\rho^\delta(t)$, that

$$\sup_B(\|\mu(t)\|_{L^2} + \|v(t) - \langle v(t)\rangle\|_{L^2}) \le C(1+t)e^{-\theta t}.$$

Once more we observe that, by Lemma 1.2, we can estimate $\|Dv(t)\|_{L^2}$ in terms of $\|\mu(t)\|_{L^2}$ and $\|v(t) - \langle v(t)\rangle\|_{L^2}$. Hence (68) is proved when $A = 0$.

It remains to consider the case where $A \not\equiv 0$. Let v_1 be the unique bounded solution to

$$-\partial_t v_1 + \delta v_1 - \Delta v_1 + H_p(x, D\bar{u}^\delta) \cdot Dv_1 = A(t, x) \quad \text{in } (0, +\infty) \times \mathbb{T}^d.$$

Using as before Lemma 1.2 for $e^{-\delta t}v_1$ and with $T \to \infty$, we estimate

$$\|v_1(t) - \langle v_1(t)\rangle\|_{L^2} \le C\int_t^\infty e^{-(\omega+\delta)(s-t)}\|A(s)\|_{L^2}\, ds \le Ce^{-\gamma t}.$$

Finally, using again Lemma 1.2 gives

$$\|Dv_1(t)\|_{L^2} \le Ce^{-\gamma t}.$$

Note that, if (v, μ) is the solution to (66), then $(v - v_1, \mu)$ solves (66) with $A \equiv 0$ and $B' = B + \bar{m}^\delta H_{pp}Dv_1$, so, applying the above estimate gives

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C(1+t)e^{-\theta t},$$

where $\theta := \gamma \wedge \lambda$. □

3C. ***Exponential rate for the nonlinear system.*** We now consider the infinite-horizon discounted nonlinear MFG system (5). Let us recall that this system is well-posed and that we have Lipschitz estimates:

Lemma 3.6. *Under our standing assumptions, for any $\delta \in (0, 1)$ there exists a unique solution (u^δ, m^δ) to (5). Moreover, for any $\alpha \in (0, 1)$, there exists a constant $C > 0$, independent of δ, such that*

$$\|Du^\delta\|_{C^{(1+\alpha)/2, 1+\alpha}} + \sup_{t\in[1,\infty)} \|m^\delta(t)\|_\infty \le C.$$

Proof. Existence and uniqueness of the solution rely on standard arguments, discussed for instance in [Lions 2010]. In particular, the unique solution can be obtained as limit of solutions in horizons $T_n \to \infty$ with the terminal condition $u(T_n) = 0$; this way one can prove, exactly as in Lemma 1.5, that Du^δ is uniformly bounded, and one also has a uniform bound for $\|\delta u^\delta\|_\infty$. As a consequence, m^δ is uniformly bounded in $[1, +\infty)$ thanks to Lemma 1.1 and is (uniformly) Hölder continuous in time with values in $\mathcal{P}(\mathbb{T}^d)$; see estimate (17). Finally, by considering the equation of $(u^\delta)_{x_i}$, namely

$$-\partial_t (u^\delta)_{x_i} + \delta (u^\delta)_{x_i} - \Delta (u^\delta)_{x_i} + H_{x_i} + H_p \cdot D(u^\delta)_{x_i} = F_{x_i},$$

the parabolic regularity applied in any interval $(t, t+1)$, jointly with the uniform bound already found for $\|(u^\delta)_{x_i}\|_\infty$, implies the desired estimate upon Du^δ. More precisely, by only using that $F_x(x, m)$ is uniformly bounded, and the bound on H_x and H_p, we deduce a bound for $(u^\delta)_{x_i}$ in $C^{(1+\alpha)/2, 1+\alpha}$ for any $\alpha \in (0, 1)$. □

The main result of this part is the following exponential convergence of the discounted problem.

Theorem 3.7. *Let (u^δ, m^δ) be the solution to the discounted MFG system* (5). *There exist $\gamma, \delta_0 > 0$ and $C > 0$ such that, if $\delta \in (0, \delta_0)$, then*

$$\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^\infty} \le C e^{-\gamma t} \quad \text{for all } t \ge 0, \tag{70}$$

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^\infty} \le C e^{-\gamma t} \quad \text{for all } t \ge 1. \tag{71}$$

Proof. The proof is very close to the proof of Theorem 2.6. Let

$$E := \{(v, \mu),\ \|Dv(t)\|_{L^\infty} + \|\mu(t)\|_{L^\infty} \le \widehat{K} e^{-\gamma t}\},$$

where $\widehat{K} > 0$ and $\gamma > 0$ are to be chosen below. We assume that $\widehat{K}$ is small enough so that

$$\bar{m}^\delta > \widehat{K} \quad \text{in } \mathbb{T}^d.$$

We also assume that the initial condition is close to $\bar{m}^\delta$, namely $\mu_0 := m_0 - \bar{m}^\delta$ satisfies

$$\|\mu_0\|_{L^\infty} \le \widehat{K}^2.$$

We consider the solution $(\tilde{v}, \tilde{\mu})$ to (66) with initial condition $\tilde{\mu}(0) = \mu_0$,

$$A(t,x) = -H(x, D(\bar{u}^\delta + v)) + H(x, D\bar{u}^\delta) + H_p(x, D\bar{u}^\delta)\cdot Dv + F(x, \bar{m}^\delta + \mu) - F(x, \bar{m}^\delta) - \frac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu),$$

$$B(t,x) = (\bar{m}^\delta + \mu) H_p(x, D(\bar{u}^\delta + v)) - \bar{m}^\delta H_p(x, D\bar{u}^\delta) - \mu H_p(x, D\bar{u}^\delta) - \bar{m}^\delta H_{pp}(x, D\bar{u}^\delta) Dv.$$

We note that

$$\|A(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} \le C \widehat{K}^2 e^{-2\gamma t}.$$

From Proposition 3.5 we have

$$\|\tilde{\mu}(t)\|_{L^2} + \|D\tilde{v}(t)\|_{L^2} \le C \widehat{K}^2 (1+t) e^{-\theta t},$$

where $\theta := 2\gamma \wedge \lambda$. Using the smoothing properties of $\delta F/\delta m$ and the parabolic regularity of the equation satisfied by $\tilde{v} - \langle \tilde{v} \rangle$, exactly as in Theorem 2.6 we can upgrade the above estimate to

$$\|\tilde{\mu}(t)\|_\infty + \|D\tilde{v}(t)\|_\infty \le C\widehat{K}^2(1+t)e^{-\theta t}.$$

So if one chooses $\gamma \in (0, \lambda)$, we infer that

$$\|\tilde{\mu}(t)\|_{L^\infty} + \|D\tilde{v}(t)\|_{L^\infty} \le C\widehat{K}^2 e^{-\gamma t}.$$

For $\widehat{K}$ small enough, this implies that $(\tilde{v}, \tilde{\mu})$ belongs to E. Note that $\tilde{v}$, $D\tilde{v}$ and $\tilde{\mu}$ are bounded in $C^{\alpha/2,\alpha}$ because they solve parabolic equations with bounded coefficients. So the map $(v, \mu) \to (\tilde{v}, \tilde{\mu})$ is compact (say in $W^{1,\infty} \times L^\infty$) and thus has a fixed point (v^δ, μ^δ). Then $(u^\delta, m^\delta) := (\bar{u}^\delta, \bar{m}^\delta) + (v^\delta, \mu^\delta)$ is a solution to (5) which satisfies the decay

$$\|m^\delta(t) - \bar{m}^\delta\|_\infty + \|D(u^\delta(t) - \bar{u}^\delta)\|_\infty \le Ce^{-\gamma t} \quad \text{for all } t \ge 0.$$

It remains to remove the assumption on the initial condition m_0. For this we only need to show that there exists a time $T > 0$ such that, for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, the solution (u^δ, m^δ) of (5) satisfies $\|m^\delta(T) - \bar{m}^\delta\|_\infty \le \widehat{K}^2$. Indeed, we can then apply the previous result to the restriction of (u^δ, m^δ) to the time interval $[T, +\infty)$.

By the duality relation (consisting here in multiplying the equation for $u^\delta - \bar{u}^\delta$ by $m^\delta - \bar{m}^\delta$ and the equation for $m^\delta - \bar{m}^\delta$ by $u^\delta - \bar{u}^\delta$, integrating in time and space and adding the resulting quantities), we have

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta t}\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^2}^2\, dt \le -\left[e^{-\delta t}\int_{\mathbb{T}^d}(u^\delta(t) - \bar{u}^\delta)(m^\delta(t) - \bar{m}^\delta)\right]_{t_1}^{t_2}. \tag{72}$$

Thus

$$C^{-1}\int_0^{+\infty} e^{-\delta t}\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^2}^2\, dt \le \int_{\mathbb{T}^d}(u^\delta(0) - \bar{u}^\delta)(m_0 - \bar{m}^\delta) \le C \tag{73}$$

because u^δ is uniformly Lipschitz continuous in space (see Lemma 3.6). As $\mu^\delta := m^\delta - \bar{m}^\delta$ satisfies

$$\partial_t \mu^\delta - \Delta\mu^\delta - \operatorname{div}(\mu^\delta H_p(x, Du^\delta)) = \operatorname{div}(\bar{m}^\delta(H_p(x, D\bar{u}^\delta) - H_p(x, Du^\delta))),$$

and still using the fact that Du^δ is bounded, Lemma 1.1 implies that, for any $t \ge 1$,

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^2} \le Ce^{-\omega(t-1)}\|m^\delta(1) - \bar{m}^\delta\|_{L^2} + Ce^{\delta t/2}\left[\int_1^t e^{-\delta s}\|D(u^\delta(s) - \bar{u}^\delta)\|_{L^2}^2\, ds\right]^{1/2}.$$

Recalling that m^δ is bounded in L^∞ (Lemma 1.1), we find

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^2} \le Ce^{\delta t/2} \quad \text{for all } t \ge 1.$$

Let $T \ge 2$ to be chosen below. Coming back to (73), there exist $t_1 \in [1, T]$ and $t_2 \in [3T+1, 4T]$ such that

$$e^{-\delta t_i}\|D(u^\delta(t_i) - \bar{u}^\delta)\|_{L^2}^2 \le \frac{C}{T}.$$

Then from (72) we deduce

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta t}\|D(u^\delta(t)-\bar u^\delta)\|_{L^2}^2\,dt$$
$$\le e^{-\delta t_1}\|D(u^\delta(t_1)-\bar u^\delta)\|_{L^2}\|m^\delta(t_1)-\bar m^\delta\|_{L^2}+e^{-\delta t_2}\|D(u^\delta(t_2)-\bar u^\delta)\|_{L^2}\|m^\delta(t_2)-\bar m^\delta\|_{L^2}\le CT^{-1/2}.$$

Then, as $t_1 \le T \le 3T+1 \le t_2 \le 4T$, we have, for any $t \in [2T, t_2]$,

$$\|m^\delta(t)-\bar m^\delta\|_{L^2}\le Ce^{-\omega(2T-t_1)}\|m^\delta(t_1)-\bar m^\delta\|_{L^2}+Ce^{\delta t_2/2}\left[\int_{t_1}^{t_2} e^{-\delta t}\|D(u^\delta(t)-\bar u^\delta)\|_{L^2}^2\,dt\right]^{1/2}$$
$$\le Ce^{-\omega T}e^{\delta T/2}+Ce^{2\delta T}T^{-1/4}. \tag{74}$$

Notice that, by choosing T large, and then δ small, the above inequality implies that $m^\delta(t)-\bar m^\delta$ is sufficiently small for any $t\in[2T,3T]$. In order to conclude, we only need to upgrade this estimate to the L^∞ norm.

To this end, recall that $w^\delta := u^\delta - \bar u^\delta$ solves the equation

$$-\partial_t w^\delta+\delta w^\delta-\Delta w^\delta+V^\delta\cdot Dw^\delta = F(x,m^\delta(t))-F(x,\bar m^\delta),$$

where $V^\delta=\int_0^1 H_p(x,D\bar u^\delta+sD(u^\delta-\bar u^\delta))\,ds$ is uniformly bounded. Since we have, by Poincaré's inequality,

$$e^{-\delta t_2}\|w^\delta(t_2)-\langle w^\delta(t_2)\rangle\|_{L^2}^2\le Ce^{-\delta t_2}\|Dw^\delta(t_2)\|_{L^2}^2\le \frac{C}{T},$$

applying Lemma 1.2 to $e^{-\delta t}w^\delta$ we deduce that, for $t\in[2T,2T+2]$,

$$\|w^\delta(t)-\langle w^\delta(t)\rangle\|_{L^2}\le Ce^{-\omega(t_2-t)}\|w^\delta(t_2)-\langle w^\delta(t_2)\rangle\|_{L^2}e^{\delta(t-t_2)}+C\int_t^{t_2}e^{-\omega(s-t)}\|m^\delta(s)-\bar m^\delta\|_{L^2}e^{\delta(t-s)}\,ds$$
$$\le Ce^{-\omega(t_2-t)}\frac{e^{\delta(t-t_2/2)}}{T^{1/2}}+C(e^{-\omega T}e^{\delta T/2}+e^{2\delta T}T^{-1/4})\int_t^{t_2}e^{-\omega(s-t)}e^{\delta(t-s)}\,ds,$$

where we also used (74). Recalling that $t\in[2T,2T+2]$ and $t_2\in[3T+1,4T]$, we have $t-t_2/2\ge 0$, so if δ is small enough compared to ω we conclude that

$$\|w^\delta(t)-\langle w^\delta(t)\rangle\|_{L^2}\le C(e^{-\omega T/2}+e^{2\delta T}T^{-1/4}).$$

We apply once more Lemma 1.2 to estimate $Dw^\delta(t)$ in $(2T,2T+1)$: we deduce that

$$\|D(u^\delta(t)-\bar u^\delta)\|_{L^2}\le C(e^{-\omega T/2}+e^{2\delta T}T^{-1/4})$$

for every $t\in(2T,2T+1)$. In fact, since $D(u^\delta(t)-\bar u^\delta)$ is bounded, a similar estimate actually holds in L^p for all $p<\infty$:

$$\|D(u^\delta(t)-\bar u^\delta)\|_{L^p}\le C(e^{-\omega T/p}+e^{4\delta T/p}T^{-1/(2p)}).$$

Recalling the estimate (74), by parabolic regularity used for the equation of μ^δ in the interval $(2T,2T+1)$, we conclude that the L^∞ norm of μ^δ satisfies a similar estimate for, say, $t\in\left(2T+\frac12,2T+1\right)$. In particular, we can fix T large and $\delta_0>0$ small such that in this interval we have $\|m^\delta(t)-\bar m^\delta\|_{L^\infty}\le \widehat K^2$

for any $\delta \in (0, \delta_0)$. We notice that the choice of T (and so δ_0) only depends on $\widehat{K}$, which is only dependent on the data. This means that the estimates (70) and (71) have been proved to hold for $t \geq T_{\widehat{K}}$, for some $T_{\widehat{K}}$ only depending on the data. On the other hand, the global gradient bound implies

$$\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^\infty} \leq \widehat{C} e^{-\gamma T_{\widehat{K}}}$$

for some constant $\widehat{C} > 0$ and for every $t \in [0, T_{\widehat{K}}]$ and a similar estimate holds for $\|m^\delta(t) - \bar{m}^\delta\|_{L^\infty}$ for $t \in [1, T_{\widehat{K}}]$. Hence (70) and (71) are proved in the whole time range. □

Let us underline the following consequence of our estimates on the solution to the linearized system

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, Du^\delta) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m^\delta(t))(\mu(t)) \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, Du^\delta)) - \operatorname{div}(m^\delta H_{pp}(x, Du^\delta) Dv) = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded.} \end{cases} \tag{75}$$

Notice that the system has been now linearized around the pair (u^δ, m^δ) which solves the discounted MFG system (5).

Corollary 3.8. *There exist $\theta, \delta_0 > 0$ and a constant $C > 0$ such that, if $\delta \in (0, \delta_0)$, then the solution (v, μ) to (75) with $\int_{\mathbb{T}^d} \mu_0 = 0$ satisfies*

$$\|Dv(t)\|_{L^2} \leq Ce^{-\theta t} \|\mu_0\|_{L^2} \quad \textit{for all } t \geq 0,$$
$$\|\mu(t)\|_{L^2} \leq Ce^{-\theta t} \|\mu_0\|_{L^2} \quad \textit{for all } t \geq 1.$$

In addition, for any $\alpha \in (0, 1)$, there is a constant C (independent of $\delta \in (0, \delta_0)$) such that

$$\sup_{t \geq 0} \|v(t)\|_{C^{2+\alpha}} \leq C \|\mu_0\|_{(C^{2+\alpha})'}.$$

Proof. As in the proof of Lemma 3.3, we have a preliminary estimate:

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C_0 \|\mu_0\|_{L^2} e^{\delta t/2}.$$

We rewrite system (75) in the form (66) with

$$A(t, x) := -(H_p(x, Du^\delta) - H_p(x, D\bar{u}^\delta)) \cdot Dv + \frac{\delta F}{\delta m}(x, m^\delta(t))(\mu(t)) - \frac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu(t)),$$
$$B(t, x) := -\mu(H_p(x, Du^\delta) - H_p(x, D\bar{u}^\delta)) - (m^\delta H_{pp}(x, Du^\delta) - \bar{m}^\delta H_{pp}(x, D\bar{u}^\delta)) Dv.$$

From Theorem 3.7, we have, for δ small enough,

$$\|A(t)\|_{L^2} \leq Ce^{-\gamma t}(\|Dv\|_{L^2} + \|\mu(t)\|_{L^2}) \leq Ce^{-(\gamma - \delta)t} \|\mu_0\|_{L^2} \leq Ce^{-\gamma t/2} \|\mu_0\|_{L^2}.$$

In the same way,

$$\|B(t)\|_{L^2} \leq Ce^{-\gamma t/2} \|\mu_0\|_{L^2}.$$

Then Proposition 3.5 implies

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C(1 + t)e^{-\gamma t/2} \|\mu_0\|_{L^2}.$$

The above estimates combined with the maximum principle imply that v is bounded in L^∞ by

$$\sup_{t\in[0,T]} \|v(t)\|_\infty \le C\|\mu_0\|_{L^2}.$$

In order to change the left-hand side $\|v(t)\|_\infty$ into $\|v(t)\|_{C^{2+\alpha}}$ and the right-hand side $\|\mu_0\|_{L^2}$ into $\|\mu_0\|_{(C^{2+\alpha})'}$, one can proceed as in Corollary 2.8. □

4. The master cell problem

In this section we study the master cell problem:

$$\lambda - \Delta_x \chi(x,m) + H(x, D_x\chi(x,m)) - \int_{\mathbb{T}^d} \operatorname{div}_y(D_m\chi(x,m,y))\,dm(y) \\ + \int_{\mathbb{T}^d} D_m\chi(x,m,y)\cdot H_p(y, D_x\chi(y,m))\,dm(y) = F(x,m) \quad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \tag{76}$$

We prove that this equation is well-defined in a suitable sense: there is a unique constant $\bar\lambda$ for which the master cell problem has a "weak" solution in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$. Moreover we prove that $\bar\lambda$ is also the unique constant for which the ergodic mean field game system (2) has a solution $(\bar\lambda, \bar u, \bar m)$.

Let us stress that a weak solution of (76), according to our next definition, is not necessarily C^1 with respect to m, so (76) is not formulated classically. Instead, the equation is interpreted as is often done with transport equations, by requiring somehow that the value of the solution is obtained through the characteristic curves. By considering weak solutions, we avoid some lengthy and involved estimates which are needed to achieve the C^1 character with respect to m. The reader is referred to [Cardaliaguet et al. 2019] for this issue. For our purposes, the context of weak solutions is enough to characterize the ergodic limit.

Definition 4.1. We say that the pair (λ, χ), with $\lambda \in \mathbb{R}$ and $\chi : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ a map, is a weak solution to the master cell problem (76) if χ and $D_x\chi$ are globally Lipschitz continuous in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ and if χ satisfies the two conditions

(i) χ is monotone, i.e.,

$$\int_{\mathbb{T}^d} (\chi(x,m) - \chi(x,m'))\,d(m-m')(x) \ge 0 \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d),$$

(ii) for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, and any $T > 0$, whenever we consider the unique solution (u, m) to

$$\begin{cases} -\partial_t u + \lambda - \Delta u + H(x, Du) = F(x,m) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(mH_p(x, Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = m_0, \quad u(T,\cdot) = \chi(x, m(T)) & \text{in } \mathbb{T}^d, \end{cases} \tag{77}$$

then we have $\chi(x, m_0) = u(0, x)$ for any $x \in \mathbb{T}^d$.

Let us make some comments about the above definition. Firstly, the monotonicity condition on χ ensures the uniqueness of the solution (u, m) to (77). Secondly, if $\chi = \chi(x,m)$ is a weak solution, then χ is actually C^2 in the space variable x because so is the solution u of (77) at time $t = 0$. Thirdly, condition (ii) implies that in (77) one actually has $\chi(x, m(t)) = u(t, x)$ for any $(t, x) \in [0, T] \times \mathbb{T}^d$, so m solves the McKean–Vlasov equation

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D\chi(x, m(t)))) = 0, \quad m(0, \cdot) = m_0. \tag{78}$$

The Lipschitz continuity of $D_x \chi$ ensures that this equation has a unique solution.

Theorem 4.2. *There is a unique constant $\bar{\lambda} \in \mathbb{R}$ for which the master cell problem* (76) *has a weak solution. The constant $\bar{\lambda}$ is also the unique constant for which the ergodic MFG problem* (2) *has a solution. Besides, if χ is a solution to* (76), *then $\chi(\cdot, m)$ is of class C^2 for any $m \in \mathcal{P}(\mathbb{T}^d)$ and*

$$D_x \chi(x, \bar{m}) = D\bar{u}(x) \quad \textit{for all } x \in \mathbb{T}^d,$$

where $(\bar{u}, \bar{m})$ is a solution to (2).

The proof requires several steps. As usual, we build the solution through the discounted problem, for which we have to show uniform regularity estimates (independent of the discount factor).

4A. ***Estimates for the discounted master equation.*** In order to build a solution to the cell problem, we consider, for $\delta > 0$, the discounted master equation (6). Let us recall, see [Cardaliaguet et al. 2019], that U^δ can be built as follows: for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let (u^δ, m^δ) be the solution to (5). Then

$$U^\delta(x, m_0) = u^\delta(0, x). \tag{79}$$

The next lemma collects standard estimates on U^δ.

Lemma 4.3. *Let U^δ be the solution to* (6). *Then, for any $\alpha \in (0, 1)$, there is a constant C, independent of m_0 and δ, such that*

$$\|\delta U^\delta(\cdot, m)\|_\infty + \|D_x U^\delta(\cdot, m)\|_{C^{1+\alpha}} \le C \quad \textit{for all } m \in \mathcal{P}(\mathbb{T}^d).$$

Proof. Let (u^δ, m^δ) be a solution to (5). As u^δ is a bounded solution to the first equation in (5), it is well known that

$$\sup_{(t,x)\in[0,+\infty)\times\mathbb{T}^d} |\delta u^\delta(t, x)| \le \sup_{x\in\mathbb{T}^d} |H(x, 0)| + \sup_{(x,m)\in\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)} |F(x, m)|.$$

This yields the uniform estimate on $\|\delta U^\delta\|_\infty$. From Lemma 3.6, we know that Du^δ is bounded in $C^{(1+\alpha)/2,1+\alpha}$ for any $\alpha \in (0, 1)$; this implies the same bound on $D_x U^\delta$. □

The next result states that U^δ is uniformly Lipschitz continuous with respect to m.

Proposition 4.4. *Let U^δ be the solution to* (6). *Then, for any $\alpha \in (0, 1)$, there exists a constant C, depending on α and on the data only, such that*

$$\|D_m U^\delta(\cdot, m, \cdot)\|_{2+\alpha,1+\alpha} \le C. \tag{80}$$

In particular, $U^\delta(\cdot, \cdot)$ and $D_x U^\delta(\cdot, \cdot)$ are uniformly Lipschitz continuous.

Proof. Let us fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$, and let (u^δ, m^δ) be the solution to (5). We use the following representation formula, see [Cardaliaguet et al. 2019]: for any smooth map μ_0, we have

$$\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(x, m_0, y)\mu_0(y)\, dy = v(0, x), \tag{81}$$

where (v, μ) is the unique solution to the linearized system

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, Du^\delta) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m^\delta(t))(\mu(t)) & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, Du^\delta)) - \operatorname{div}(m^\delta H_{pp}(x, Du^\delta) Dv) = 0 & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded.} \end{cases} \tag{82}$$

If we suppose that $\int_{\mathbb{T}^d} \mu_0 = 0$, Corollary 3.8 states that

$$\sup_{t\geq 0} \|v(t)\|_{C^{2+\alpha}} \leq C\|\mu_0\|_{(C^{2+\alpha})'}$$

for any $\alpha > 0$. By (81) and

$$D_y \frac{\delta U^\delta}{\delta m}(x, m_0, y) = D_m U^\delta(x, m_0, y),$$

we infer exactly as in [Cardaliaguet et al. 2019] that

$$\|D_m U^\delta(\cdot, m_0, \cdot)\|_{2+\alpha,1+\alpha} \leq C. \qquad \square$$

Remark 4.5. We stress that the uniform Lipschitz continuity of $U^\delta(\cdot, \cdot)$ and $D_x U^\delta(\cdot, \cdot)$ would require milder assumptions than those needed to prove (80). Indeed, by only using condition (FGb) on the couplings, we can replace the conclusion of Corollary 3.8 with the estimate

$$\sup_{t\geq 0} \|v(t)\|_{C^1} \leq C\|\mu_0\|_{(C^1)'},$$

which would follow as explained in Remark 2.10. With this latter estimate in hand, using (81) with $\mu_0 = D_y\psi(y)$ (for ψ smooth), it follows that

$$\int_{\mathbb{T}^d} D_y D_x \frac{\delta U^\delta}{\delta m}(x, m_0, y)\psi(y)\, dy \leq C\|\mu_0\|_{(C^1)'} \leq C\|\psi\|_{L^1},$$

which yields

$$\|D_m D_x U^\delta(x, m_0)\|_\infty \leq C.$$

Since $D^2_{xx} U^\delta(x, m)$ is estimated from Lemma 4.3, this would imply the Lipschitz uniform bound for $D_x U^\delta(\cdot, \cdot)$.

In the following, we actually only use this information in order to prove the existence of a weak solution to the master equation and the convergence of the ergodic limit.

We finally establish that U^δ is monotone:

Lemma 4.6. *For any $\delta > 0$ the map U^δ is monotone.*

Proof. Fix $m_0, m_0' \in \mathcal{P}(\mathbb{T}^d)$. Let us recall that $U^\delta(x, m_0) = u^\delta(0, x)$, where the pair (u^δ, m^δ) solves (5) with initial condition m_0. We denote by (u', m') the solution of (5) with initial condition m_0'. Then by duality (consisting here in multiplying the equation for $u^\delta - u'$ by $m^\delta - m'$ and the equation by $m^\delta - m'$ by $u^\delta - u'$, integrating in time and space and adding the resulting quantities), we have

$$\frac{d}{dt} e^{-\delta t} \int_{\mathbb{T}^d} (u^\delta(t, x) - u'(t, x))(m^\delta(t, x) - m'(t, x))\, dx \le 0,$$

where, as u^δ and u' are bounded and m^δ and m' are probability measures,

$$\lim_{t\to+\infty} e^{-\delta t} \int_{\mathbb{T}^d} (u^\delta(t, x) - u'(t, x))(m^\delta(t, x) - m'(t, x))\, dx = 0.$$

This proves that

$$\int_{\mathbb{T}^d} (U^\delta(x, m_0) - U^\delta(x, m_0'))\, d(m_0 - m_0')(x) = \int_{\mathbb{T}^d} (u^\delta(0, x) - u'(0, x))\, d(m_0 - m_0')(x) \ge 0. \qquad \square$$

4B. ***Existence of a solution for the master cell problem.***

Proof of Theorem 4.2. Let us start with the proof of the existence of the solution to the master cell problem. The proof of the uniqueness of the ergodic constant is given in Proposition 4.7 below.

For $\delta > 0$, let U^δ be the solution to the discounted master equation (6). We have seen in Lemma 4.3 and Proposition 4.4 that U^δ and $D_x U^\delta$ are uniformly Lipschitz continuous and that δU^δ is bounded. We set $W^\delta(x, m) = U^\delta(x, m) - U^\delta(0, \bar{m})$. Then W^δ is bounded and uniformly Lipschitz continuous on the compact space $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$, so it converges, up to a subsequence, to a continuous map $\chi : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. Since $D_x W^\delta$ is also bounded in Lipschitz norm, we deduce that $D_x \chi$ is Lipschitz continuous (in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$). Moreover $(\delta U^\delta(0, \bar{m}))$ converges (along the same subsequence, without loss of generality) to some constant λ.

Next we prove that χ is a weak solution to (76). We already know that χ and $D_x \chi$ are Lipschitz continuous with respect to both variables. In addition, χ is monotone thanks to Lemma 4.6. Let $T > 0$, $m_0 \in \mathcal{P}(\mathbb{T}^d)$ with a smooth density and (w^δ, m^δ) be the solution to

$$\begin{cases} -\partial_t w^\delta + \delta w^\delta + \delta U^\delta(0, \bar{m}) - \Delta w^\delta + H(x, Dw^\delta) = F(x, m^\delta) & \text{on } (0, T) \times \mathbb{T}^d, \\ \partial_t m^\delta - \Delta m^\delta - \operatorname{div}(m^\delta H_p(x, Dw^\delta)) = 0 & \text{on } (0, T) \times \mathbb{T}^d, \\ m^\delta(0, \cdot) = m_0, \quad w^\delta(T, \cdot) = W^\delta(x, m^\delta(T)) & \text{on } \mathbb{T}^d. \end{cases}$$

By definition we have $W^\delta(x, m^\delta(T)) = U^\delta(x, m^\delta(T)) - U^\delta(0, \bar{m})$ and we know that $U^\delta(x, m^\delta(t)) = u^\delta(t, x)$ for all t, where u^δ is the solution to (5). Hence we deduce that

$$w^\delta(t, x) = u^\delta(t, x) - U^\delta(0, \bar{m}) = W^\delta(x, m^\delta(t))$$

for all $(t, x) \in (0, T) \times \mathbb{T}^d$. In particular, by Lemma 3.6, w^δ is uniformly bounded in $C^{1+\alpha/2, 2+\alpha}$ for some $\alpha \in (0, 1)$, while m^δ is uniformly bounded and uniformly continuous on $[0, T]$ with values in $\mathcal{P}(\mathbb{T}^d)$. So there exists a subsequence, still denoted for simplicity by (w^δ, m^δ), such that w^δ converges in $C^{1,2}$ to a map w and m^δ converges in $C^0([0,T],\mathcal{P}(\mathbb{T}^d))$ to a map m. The pair (w,m) is a solution to

$$\begin{cases} -\partial_t w + \lambda - \Delta w + H(x, Dw) = F(x,m) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Dw)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = m_0, \quad w(T,\cdot) = \chi(x, m(T)) & \text{in } \mathbb{T}^d. \end{cases}$$

As the solution to this equation is unique (because χ is monotone), we derive that (w,m) is the unique solution to (77). Moreover, as $w^\delta(0,x) = W^\delta(x,m_0)$, we also have at the limit $w(0,x) = \chi(x,m_0)$. This proves that χ is a weak solution to (76). □

Let us now come back to the ergodic MFG problem (2). We denote by $(\bar\lambda, \bar u, \bar m)$ the solution to this equation.

Proposition 4.7. *Let (λ,χ) be a solution of the ergodic master equation. Then we have $\lambda = \bar\lambda$ and $D_x\chi(x,\bar m) = D\bar u(x)$.*

Proof. Let us fix $T>0$ and let (u,m) be the solution to

$$\begin{cases} -\partial_t u + \lambda - \Delta u + H(x, Du) = F(x,m) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = \bar m, \quad u(T,\cdot) = \chi(x, m(T)) & \text{in } \mathbb{T}^d. \end{cases} \tag{83}$$

We have already noticed that m is the solution to the McKean–Vlasov equation

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D_x\chi(x, m(t)))) = 0, \quad m(0,\cdot) = \bar m,$$

which has a unique solution because $D_x\chi$ is Lipschitz continuous. This means that m is defined independently of the horizon T. As we know that $u(t,x) = \chi(x,m(t))$, the same holds for u. Then, from the usual energy inequality applied to $(u-\bar u, m-\bar m)$, we have, for any $0\le t_1\le t_2\le T$,

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^d} \frac{m+\bar m}{2}|Du - D\bar u|^2 \le -C\left[\int_{\mathbb{T}^d}(u-\bar u)(m-\bar m)\right]_{t_1}^{t_2}. \tag{84}$$

The right-hand side is bounded because $u(t,\cdot) = \chi(\cdot, m(t))$ and $\bar u$ are bounded, so

$$\int_0^T\int_{\mathbb{T}^d} \bar m |Du - D\bar u|^2 \le C. \tag{85}$$

By Lemma 1.4 we have

$$\sup_{t\in[0,T]} \|m(t)-\bar m\|_{L^2} \le C. \tag{86}$$

As $\bar m$ is bounded below, (85) implies that there exists $t_T\in[T/2,T]$ such that $\int_{\mathbb{T}^d}|Du(t_T) - D\bar u|^2 \le 2C/T$. In particular, for T large enough, we have, by (84) applied with $t_1 = 0$ and $t_2 = t_T$,

$$\begin{aligned} \int_0^1\int_{\mathbb{T}^d}|Du - D\bar u|^2 &\le \int_0^{t_T}\int_{\mathbb{T}^d}|Du - D\bar u|^2 \le -C\int_{\mathbb{T}^d}(u(t_T)-\bar u)(m(t_T)-\bar m) \\ &\le -C\int_{\mathbb{T}^d}(u(t_T) - \bar u - \langle u(t_T)-\bar u\rangle)(m(t_T)-\bar m) \\ &\le C\|Du(t_T) - D\bar u\|_{L^2} \le CT^{-1/2}, \end{aligned}$$

by Poincaré's inequality, (86) and our choice of t_T. Letting $T \to \infty$ we can conclude that $Du = D\bar{u}$ on $[0, 1] \times \mathbb{T}^d$. Therefore, m satisfies

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D\bar{u}(x))) = 0 \quad \text{on } (0, 1) \times \mathbb{T}^d, \quad m(0, \cdot) = \bar{m}.$$

But this equation has $\bar{m}$ as a unique solution, which shows that $m(t, x) = \bar{m}(x)$ on $[0, 1] \times \mathbb{T}^d$. Since the McKean–Vlasov equation (78) is autonomous, we finally have $m(t) = \bar{m}$ and $Du(t, x) = D_x\chi(x, \bar{m}) = D\bar{u}(x)$ for any $(t, x) \in [0, T] \times \mathbb{T}^d$ and, as a consequence, $\lambda = \bar{\lambda}$. □

5. The long time behavior

We now fix a solution χ to the master cell problem and, given a terminal condition $G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ satisfying our standing assumptions (see Section 1A), we consider the solution to the backward equation

$$\begin{cases} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + H(x, D_x U(t,x,m)) \\ \quad - \displaystyle\int_{\mathbb{T}^d} \operatorname{div}(D_m U(t,x,m,y))\, dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot H_p(y, D_x U(t,y,m))\, dm(y) = F(x,m) & \text{in } (-\infty, 0) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(0,x,m) = G(x,m) & \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{cases} \tag{87}$$

We recall that the existence of a unique classical solution to (87) was proved in [Cardaliaguet et al. 2019]. Here is our main convergence result.

Theorem 5.1. *Let χ be a weak solution to the master cell problem* (76). *Then, there exists a constant $c \in \mathbb{R}$ such that*

$$\lim_{t \to -\infty} U(t, x, m) + \bar{\lambda} t = \chi(x, m) + c,$$

uniformly with respect to $(x, m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.

Moreover, we also have that $D_x U(t, x, m) \to D_x \chi(x, m)$ as $T \to \infty$, uniformly with respect to (x, m).

Theorem 5.1 implies the convergence of the solution of the MFG system as $T \to +\infty$.

Corollary 5.2. *Let c be the constant given in Theorem 5.1. For $T > 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let (u^T, m^T) be the solution to* (1). *Then, for any $t \geq 0$,*

$$\lim_{T \to +\infty} u^T(t, x) - \bar{\lambda}(T - t) = \chi(x, m(t)) + c,$$

where the convergence is uniform in x and m solves

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D_x \chi(x, m))) = 0, \quad m(0) = m_0. \tag{88}$$

Moreover, for any $\delta \in (0, 1)$,

$$\lim_{T \to +\infty} u^T(\delta T, x) - (1 - \delta)\bar{\lambda} T = \chi(x, \bar{m}) + c,$$

where $(\bar{u}, \bar{m})$ solves (2) *and where the convergence is uniform in x.*

In particular, when $t = 0$, we get

$$\lim_{T\to+\infty} u^T(0,x) - \bar{\lambda}T = \chi(x, m_0) + c.$$

Proof of Corollary 5.2. We know that $u^T(t,x) = U(t-T, x, m^T(t))$ and that m^T solves the McKean–Vlasov equation

$$\partial_t m^T - \Delta m^T - \operatorname{div}(m^T H_p(x, D_x U(t-T, x, m))) = 0, \quad m^T(0) = m_0.$$

As $x \to D_x U(t,x,m)$ is bounded in C^1 (see Proposition 5.3 below), we know from Theorem 5.1 that, as $T \to +\infty$, $(D_x U(t-T,\cdot,\cdot))$ converges uniformly to $D_x\chi$. So, for any $t \geq 0$, m^T converges in $C^0([0,t], \mathcal{P}(\mathbb{T}^d))$ towards a solution m of (88). Then again by Theorem 5.1, we have

$$\lim_{T\to+\infty} u^T(t,x) + \bar{\lambda}(t-T) = \lim_{T\to+\infty} U(t-T, x, m^T(t)) + \bar{\lambda}(t-T) = \chi(x, m(t)) + c.$$

Let us now fix $\delta > 0$. From Theorem 2.6, we have that $m^T(\delta T)$ converges (exponentially fast) to $\bar{m}$. Hence, by Theorem 5.1 again, we have

$$\lim_{T\to+\infty} u^T(\delta T, x) - (1-\delta)\bar{\lambda}T = \lim_{T\to+\infty} U(-(1-\delta)T, x, m^T(\delta T)) - (1-\delta)\bar{\lambda}T = \chi(x,\bar{m}) + c. \quad \square$$

The proof of Theorem 5.1 relies on estimates on $U(t,\cdot,\cdot)$ (independent of t) developed in the next section.

5A. ***Lipschitz estimates of the solution U.*** We collect here the main estimates satisfied by the solution of (87). They actually follow from the estimates developed in Section 2B for the solution (u,m) of the MFG system.

Proposition 5.3. *Let U be the solution to the master equation* (87). *Then there exists a constant C such that*

$$\sup_{t\leq 0,\ m\in\mathcal{P}(\mathbb{T}^d)} \|U(t,\cdot,m) + \bar{\lambda}t\|_{C^{2+\alpha}} + \|D_m U(t,\cdot,m,\cdot)\|_{2+\alpha,1+\alpha} \leq C, \tag{89}$$

while

$$\sup_{(x,m)\in\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)} |U(t,x,m) - U(s,x,m)| \leq C|t-s|^{1/2} \quad \textit{for all } s,t \leq 0,\ |s-t| \leq 1.$$

Proof. Let us recall that, for any $t_0 \leq 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, one has $U(t_0, x, m_0) = u(t_0, x)$, where (u,m) is the solution to the MFG system

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x,m) & \text{in } (t_0,0)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (t_0,0)\times\mathbb{T}^d, \\ m(t_0,\cdot) = m_0, \quad u(0,\cdot) = G(x, m(0)) & \text{in } \mathbb{T}^d. \end{cases}$$

By Lemma 1.5, we have the Lipschitz bound $\|Du\|_\infty \leq C$, uniform with respect to the horizon t_0. This proves that $\|D_x U\|_\infty \leq C$ and, in turn, that m is uniformly Hölder continuous in time with values in

$\mathcal{P}(\mathbb{T}^d)$; see (17). Furthermore, from Theorem 2.6 we get an estimate for $U(t-T, x, m)$ at time $t=0$; namely, that there exists a constant C, independent of T, such that

$$\|D_x U(-T, \cdot, m_0)\|_{C^{1+\alpha}} \leq C,$$
$$\|U(-T, x, m_0) - \bar{\lambda} T\|_\infty \leq C.$$

Therefore, we deduce that

$$\sup_{t \leq 0,\ m \in \mathcal{P}(\mathbb{T}^d)} \|U(t, \cdot, m) + \bar{\lambda} t\|_{C^{2+\alpha}} \leq C.$$

Following [Cardaliaguet et al. 2019], the derivative of U with respect to m can be represented as

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(t_0, x, m_0, y) \mu_0(y)\, dy = v(t_0, x), \tag{90}$$

where, for any smooth map $\mu_0 : \mathbb{T}^d \to \mathbb{R}$, (v, μ) solves the linearized problem

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, Du) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m)(\mu) & \text{in } (t_0, 0) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, Du)) - \operatorname{div}(m H_{pp}(x, Du) Dv) = 0 & \text{in } (t_0, 0) \times \mathbb{T}^d, \\ \mu(t_0, \cdot) = \mu_0, \quad v(0, \cdot) = \dfrac{\delta G}{\delta m}(x, m(0))(\mu(0)) & \text{in } \mathbb{T}^d. \end{cases}$$

Our aim is to provide estimates on v in order to show the uniform Lipschitz regularity of U with respect to m. We assume that $\int_{\mathbb{T}^d} \mu_0 = 0$ since we are only interested in $D_m U = D_y(\delta U / \delta m)$. Then Corollary 2.8 states that

$$\sup_{t \in [0,T]} \|v(t)\|_{C^{2+\alpha}} \leq C \|\mu_0\|_{(C^{2+\alpha})'}.$$

This proves that

$$\left\| \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(t_0, \cdot, m_0, y) \mu_0(y)\, dy \right\|_{C^{2+\alpha}} \leq C \|\mu_0\|_{(C^{2+\alpha})'}$$

for any smooth map μ_0 with $\int_{\mathbb{T}^d} \mu_0 = 0$. Therefore, as in [Cardaliaguet et al. 2019], we obtain

$$\|D_m U(t_0, \cdot, m_0, \cdot)\|_{2+\alpha, 1+\alpha} \leq C. \tag{91}$$

It remains to check the time regularity of U. For this, let us first check that u is globally $\frac{1}{2}$-Holder in time. Let us recall that u is globally Lipschitz continuous in space. So, integrating in space the equation for u, the map $t \to \langle u(t) \rangle$ is globally Lipschitz continuous. Then the map $(t, x) \to u(t, x) - \langle u(t) \rangle$ is globally bounded in L^∞, is globally Lipschitz continuous in space and solves a heat equation with bounded right-hand side; therefore it is $\frac{1}{2}$-Holder continuous in time. This implies the global Holder continuity in time for u. As $U(t, x, m(t)) = u(t, x)$ and U is uniformly Lipschitz continuous in m, we have, for $t_0 \leq s \leq t_0 + 1$,

$$\begin{aligned} |U(s, x, m_0) - U(t_0, x, m_0)| &\leq |U(s, x, m_0) - U(s, x, m(s))| + |U(s, x, m(s)) - U(t_0, x, m_0)| \\ &\leq C \boldsymbol{d}_1(m_0, m(s)) + |u(s, x) - u(t_0, x)| \\ &\leq C|s - t_0|^{1/2} + |u(s, x) - u(t_0, x)| \leq C|s - t_0|^{1/2}, \end{aligned}$$

where we used the uniform regularity of m in time (since $H_p(\cdot, Du)$ is bounded, see Remark 1.6) for the second inequality, and the uniform Holder regularity in time of u in the last one. □

Remark 5.4. We stress that if we only use the regularity condition (FGb) on the couplings, then we can replace the conclusion of Corollary 2.8 with the first-order estimate (57) and obtain, rather than (91), the milder estimate $\|D_m D_x U(t,x,m)\|_\infty \le C$. This is actually enough to conclude with the uniform Lipschitz bound for U and $D_x U$, which is what is only needed in the proof of Theorem 5.1.

5B. ***Proof of Theorem 5.1.*** We are now ready to prove our main result.

Proof of Theorem 5.1. Let χ be a weak solution to the master cell problem (76). For $T > 0$, let us consider

$$U^T(t,x,m) = U(t-T,x,m) \quad \text{for } (t,x,m) \in (-\infty, T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

Then U^T solves

$$\begin{cases} -\partial_t U^T - \Delta_x U^T + H(x, D_x U) \\ \quad -\displaystyle\int_{\mathbb{T}^d} \operatorname{div}(D_m U^T(t,x,m,y))\,dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m U^T(t,x,m,y)\cdot H_p(D_x U(t,y,m,y))\,dm(y) = F(x,m) & \text{in } (-\infty,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d), \\ U^T(T,x,m) = G(x,m) & \text{in } \mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d). \end{cases}$$

By the Lipschitz regularity of U and $D_x U$ and the bound in (89) (Proposition 5.3), the family $\{U^T(\cdot,\cdot,\cdot)+\bar\lambda(\cdot - T)\}_T$ is relatively compact in $C^0(\mathbb{R}\times\mathbb{T}\times\mathcal{P}(\mathbb{T}^d))$. Let $T_n \to +\infty$ be any sequence such that $(t,x,m) \to U^{T_n}(t,x,m) + \bar\lambda(t-T_n)$ locally uniformly converges to some $V(t,x,m)$. Then V is a weak solution to

$$-\partial_t V + \bar\lambda - \Delta_x V + H(x, D_x V) - \int_{\mathbb{T}^d} \operatorname{div}(D_m V(t,x,m,y))\,dm(y) \\ + \int_{\mathbb{T}^d} D_m V(t,x,m,y)\cdot H_p(y, D_x U(t,y,m,y))\,dm(y) = F(x,m) \quad \text{in } \mathbb{R}\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d) \tag{92}$$

in the sense that V satisfies similar requirements to those in Definition 4.1. Namely, V and $D_x V$ are uniformly Lipschitz continuous in x and m and $\frac{1}{2}$-Hölder continuous in the time variable, V is monotone in m and satisfies that, for any $t_1 \le t_2$ and if (u,m) solves the MFG system

$$\begin{cases} -\partial_t u + \bar\lambda - \Delta u + H(x, Du) = F(x,m) & \text{in } (t_1,t_2)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (t_1,t_2)\times\mathbb{T}^d, \\ m(t_1,\cdot) = m_0, \quad u(t_2,\cdot) = V(t_2,x,m(t_2)) & \text{in } \mathbb{T}^d, \end{cases} \tag{93}$$

we have $V(t_1,x,m_0) = u(t_1,x)$ (and so $V(t,x,m(t)) = u(t,x)$ for any $t \in [t_1,t_2]$).

Our goal is to show that $V(t,x,m) - \chi(x,m)$ is constant. Let us recall that Proposition 2.7 implies that $U^T(0,x,\bar m) - \bar\lambda T - \bar u$ converges to a constant $\bar c$ as $T \to +\infty$. Hence $V(0,x,\bar m) = \bar u(x) + \bar c$. Since $\chi(x,\bar m) = \bar u$, this shows that, if $V(t,x,m) - \chi(x,m)$ is proved to be constant, then this constant will be equal to $\bar c$, and independent of the subsequence (T_n).

Let us fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$. Let $T > 0$ be large and (u, m) be the solution to the MFG system (93) with $t_1 = 0$ and $t_2 = T$. We note that m is the unique solution to the McKean–Vlasov equation

$$\begin{cases} \partial_t m - \Delta m - \operatorname{div}(m H_p(x, D_x V(t, x, m))) = 0 & \text{on } [0, T] \times \mathbb{T}^d, \\ m(0) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{94}$$

In particular, since V and $D_x V$ are globally Lipschitz in m, this implies that m and u are defined independently of the horizon T (meaning that, for $t \in [0, T]$, $u(t, \cdot) := V(t, \cdot, m(t))$ and $m(t, \cdot)$ do not depend on T).

In the same way we define $(\tilde{u}, \tilde{m})$ to be the solution to the MFG system

$$\begin{cases} -\partial_t \tilde{u} + \bar{\lambda} - \Delta \tilde{u} + H(x, D\tilde{u}) = F(x, \tilde{m}) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \tilde{m} - \Delta \tilde{m} - \operatorname{div}(\tilde{m} H_p(x, D\tilde{u})) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ \tilde{m}(0, \cdot) = m_0, \quad \tilde{u}(T, \cdot) = \chi(x, \tilde{m}(T)) & \text{in } \mathbb{T}^d. \end{cases}$$

As before we note that $(\tilde{u}, \tilde{m})$ does not depend on the horizon T, that $\tilde{u}(t, x) = \chi(x, \tilde{m}(t))$ for any $t \in [0, T]$ and that $\tilde{m}$ is the unique solution to the McKean–Vlasov equation

$$\partial_t \tilde{m} - \Delta \tilde{m} - \operatorname{div}(m H_p(x, D_x \chi(x, \tilde{m})) = 0 \quad \text{on } [0, T],\ \tilde{m}(0) = m_0. \tag{95}$$

Using the result of Theorem 2.6 with both $G(x, \cdot) = V(T, x, \cdot)$ and $G = \chi(x, \cdot)$, we have (changing u into $u + \bar{\lambda}(T - t)$ and $\tilde{u}$ into $\tilde{u} + \bar{\lambda}(T - t)$),

$$\|m(t) - \bar{m}\|_\infty + \|\tilde{m}(t) - \bar{m}\|_\infty \leq C(e^{-\gamma t} + e^{-\gamma(T-t)}), \quad t \in [1, T],$$

where $(\bar{u}, \bar{m})$ is the solution to the ergodic MFG system (2). But since m and $\tilde{m}$ do not depend on the horizon T, here we can let first $T \to \infty$, and then $t \to \infty$, so we conclude that both $m(t)$ and $\tilde{m}(t)$ converge to $\bar{m}$ as $t \to +\infty$.

Applying once more the standard estimates on the MFG systems, we have

$$\int_0^T \int_{\mathbb{T}^d} (m + \tilde{m})|Du - D\tilde{u}|^2 \leq -C\left[\int_{\mathbb{T}^d} (u - \tilde{u})(m - \tilde{m})\right]_0^T = -C\int_{\mathbb{T}^d} (u(T) - \tilde{u}(T))(m(T) - \tilde{m}(T))$$

since $m(0) = \tilde{m}(0) = m_0$. As u and $\tilde{u}$ are uniformly Lipschitz continuous in space and $m(T)$ and $\tilde{m}(T)$ have the same limit $\bar{m}$ as $T \to +\infty$, we deduce that

$$\lim_{T \to +\infty} \int_0^T \int_{\mathbb{T}^d} (m + \tilde{m})|Du - D\tilde{u}|^2 = 0.$$

In particular, as m (and $\tilde{m}$) are regular and bounded below by a positive constant on intervals of the form $[\varepsilon, T]$ with $\varepsilon > 0$, we deduce that $Du = D\tilde{u}$ on $[\varepsilon, T]$ and thus on $[0, T]$. Therefore m and $\tilde{m}$ solve the same equation, which implies $m(t) = \tilde{m}(t)$ for any $t \geq 0$. Coming back to the equations satisfied by u and $\tilde{u}$ gives $\partial_t u = \partial_t \tilde{u}$, so there is a constant c such that $u(t, x) = \tilde{u}(t, x) + c$. In other words

$$V(t, x, m(t)) = \chi(x, m(t)) + c \quad \text{for all } t \geq 0.$$

Notice that the above conclusion holds for any given $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and the constant c could depend on m_0 at this stage. But we are going to show that this is actually not the case.

Indeed, let us choose $m_0 = \bar{m}$. Then Proposition 4.7 says that $m(t) = \tilde{m}(t) = \bar{m}$. We denote by $\bar{c}$ the constant found above, i.e., $u(t,x) = \tilde{u}(t,x) + \bar{c}$. By definition, this implies $V(t,x,\bar{m}) = \chi(x,\bar{m}) + \bar{c}$. Now, for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, we recall that the solution $m(t) = \tilde{m}(t)$ converges to $\bar{m}$ as $t \to +\infty$. By the uniform Lipschitz continuity of χ and V with respect to m (uniform in (t,x)), this implies

$$|V(t,x,m(t)) - V(t,x,\bar{m})| + |\chi(x,m(t)) - \chi(x,\bar{m})| \to 0 \quad \text{as } t \to \infty.$$

Since

$$|c - \bar{c}| = |V(t,x,m(t)) - \chi(x,m(t)) - (V(t,x,\bar{m}) - \chi(x,\bar{m}))|,$$

by letting $t \to \infty$ we deduce that $c = \bar{c}$. In particular, we have proved that

$$V(0,x,m_0) = \chi(x,m_0) + \bar{c} \quad \text{for all } m_0 \in \mathcal{P}(\mathbb{T}^d).$$

Finally, we can apply the above reasoning to the translation $V(\cdot + t_0, x, m)$ for any $t_0 \in \mathbb{R}$. It turns out that $\bar{c} = \lim_{t\to\infty} V(t+t_0,x,m(t)) - \chi(x,m(t))$, which is clearly independent of t_0. Therefore we conclude that

$$V(t_0,x,m_0) = \chi(x,m_0) + \bar{c} \quad \text{for all } (t_0,x,m_0) \in \mathbb{R} \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \qquad \square$$

Let us point out that any weak solution of the ergodic master equation solves (92). So the above proof actually shows that two solutions of the ergodic master equation differ only by a constant:

Corollary 5.5. *If χ_1 and χ_2 are weak solutions of the ergodic master equation* (76), *then there exists a constant $\bar{c}$ such that*

$$\chi_2(x,m) = \chi_1(x,m) + \bar{c} \quad \textit{for all } (x,m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

6. The discounted problem

We now investigate the behavior, as $\delta \to 0^+$, of the solution U^δ of the discounted master equation (6). Our main result is:

Theorem 6.1. *Let U^δ be the solution to the discounted master equation* (6) *and $(\bar{\lambda}, \bar{u}, \bar{m})$ the solution of the ergodic problem* (2). *Then, as $\delta \to 0^+$, $U^\delta - \bar{\lambda}/\delta$ converges uniformly to the solution χ to the master cell problem* (76) *such that $\chi(x,\bar{m}) = \bar{u}(x) + \bar{\theta}$, where $\bar{\theta}$ is the unique constant for which the following linearized ergodic problem has a solution $(\bar{v}, \bar{\mu})$:*

$$\begin{cases} \bar{u} + \bar{\theta} - \Delta\bar{v} + H_p(x,D\bar{u}) \cdot D\bar{v} = \dfrac{\delta F}{\delta m}(x,\bar{m})(\bar{\mu}) & \textit{in } \mathbb{T}^d, \\ -\Delta\bar{\mu} - \operatorname{div}(\bar{\mu} H_p(x,D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x,D\bar{u}) D\bar{v}) = 0 & \textit{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \bar{\mu} = \int_{\mathbb{T}^d} \bar{v} = 0. \end{cases} \tag{96}$$

Let us comment a bit more on the normalization condition $\chi(x,\bar{m}) = \bar{u}(x) + \bar{\theta}$ which selects the unique limit of the discounted master equation (6), according to the above result. As we shall see in the next section, given any (not necessarily normalized with zero average) solution $\bar{u}$ to

$$\bar{\lambda} - \Delta\bar{u} + H(x,D\bar{u}) = F(x,\bar{m}) \text{in } \mathbb{T}^d, \tag{97}$$

there is a unique constant $\bar{\theta}$ for which (96) admits a solution. However, since $\bar{u}$ is unique up to addition of a constant, the sum $\bar{u}+\bar{\theta}$ will be uniquely determined. Indeed, by changing $\bar{u}$ through the addition of a constant, the value $\bar{\theta}$ will be translated accordingly. In other words, one can say that the limit of $U^\delta - \bar{\lambda}/\delta$ is the solution χ of the master cell problem (76) such that $\chi(x,\bar{m})$ coincides with the unique solution of (97) for which the constant $\bar{\theta}$ vanishes.

Exactly as for the time-dependent problem, we can infer from Theorem 6.1 the limit behavior of the solution of the discounted MFG system:

Corollary 6.2. *Let $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and, for $\delta > 0$, let (u^δ, m^δ) be the solution to the discounted MFG system* (5). *Then*

$$\lim_{\delta\to 0} u^\delta(0,x) - \bar{\lambda}/\delta = \chi(x,m_0),$$

uniformly with respect to x, where χ is the solution of the ergodic cell problem (76) *given in Theorem 6.1.*

6A. ***An additional ergodic system.*** Given a solution $\bar{u}$ of the MFG ergodic problem (2), we investigate the ergodic problem (96). The heuristic justification of (96) is that we expect the solution $(\bar{u}^\delta, \bar{m}^\delta)$ of (7) to be of the form

$$\bar{u}^\delta \sim \frac{\bar{\lambda}}{\delta} + \bar{u} + \bar{\theta} + \delta\bar{v}, \quad \bar{m}^\delta \sim \bar{m} + \delta\bar{\mu}, \tag{98}$$

and, in view of (7), the equation satisfied by $(\bar{\theta}, \bar{v}, \bar{\mu})$ should be (96).

We start the proof of the existence for (96) as usual, by a discounted problem:

Lemma 6.3. *Let $A, B \in L^\infty(\mathbb{T}^d)$. For $\delta > 0$ small, there is a unique solution $(v^\delta, \mu^\delta) \in W^{1,\infty}(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d)$ to the discounted system*

$$\begin{cases} \bar{u} + \delta v^\delta - \Delta v^\delta + H_p(x, D\bar{u})\cdot Dv^\delta = \dfrac{\delta F}{\delta m}(x,\bar{m})(\mu^\delta) + A & \text{in } \mathbb{T}^d, \\ -\Delta\mu^\delta - \operatorname{div}(\mu^\delta H_p(x,D\bar{u})) - \operatorname{div}(\bar{m}H_{pp}(x,D\bar{u})Dv^\delta) = \operatorname{div}(B) & \text{in } \mathbb{T}^d, \end{cases} \tag{99}$$

with $\int_{\mathbb{T}^d}\mu^\delta = 0$. Moreover, there is a constant $C > 0$ (independent of δ, A and B) such that

$$\|\delta v^\delta\|_\infty + \|Dv^\delta\|_\infty + \|\mu^\delta\|_\infty \le C(1 + \|A\|_\infty + \|B\|_\infty).$$

Proof. Existence of a solution runs with a standard fixed point, so we omit it. The duality relation (here between v^δ and μ^δ) gives (using Poincaré's inequality)

$$\begin{aligned} C^{-1}\|Dv^\delta\|_{L^2}^2 &\le \int_{\mathbb{T}^d} (\bar{u} + \delta v^\delta - A)\mu^\delta + B\cdot Dv^\delta \\ &\le (\|D\bar{u}\|_{L^2} + \delta\|Dv^\delta\|_{L^2} + \|A\|_{L^2})\|\mu^\delta\|_{L^2} + \|B\|_{L^2}\|Dv^\delta\|_{L^2}, \end{aligned}$$

so

$$\|Dv^\delta\|_{L^2} \le C\big((\|D\bar{u}\|_{L^2}^{1/2} + \|A\|_{L^2}^{1/2})\|\mu^\delta\|_{L^2}^{1/2} + \delta\|\mu^\delta\|_{L^2} + \|B\|_{L^2}\big).$$

By Corollary 1.3, we have

$$\|\mu^\delta\|_{L^2} \le C(\|Dv^\delta\|_{L^2} + \|B\|_{L^2}) \le C\big((\|D\bar{u}\|_{L^2}^{1/2} + \|A\|_{L^2}^{1/2})\|\mu^\delta\|_{L^2}^{1/2} + \delta\|\mu^\delta\|_{L^2} + \|B\|_{L^2}\big).$$

So, for $\delta > 0$ small enough, we obtain

$$\|\mu^\delta\|_{L^2} \le C(\|D\bar{u}\|_{L^2} + \|A\|_{L^2} + \|B\|_{L^2}).$$

This implies the same bound for Dv^δ and, by the maximum principle, the estimate

$$\|\delta v^\delta\|_\infty \le C(\|\bar{u}\|_{L^\infty} + \|D\bar{u}\|_{L^2} + \|B\|_{L^2} + \|A\|_{L^\infty}).$$

Moreover, considering the equation satisfied by $w := v^\delta - \langle v^\delta\rangle$, we have by local regularity for weak solutions [Gilbarg and Trudinger 1977, Theorem 8.17] and Poincaré's inequality

$$\|v^\delta - \langle v^\delta\rangle\|_\infty \le C(1 + \|v^\delta - \langle v^\delta\rangle\|_{L^2}) \le C(1 + \|Dv^\delta\|_{L^2}) \le C(1 + \|\bar{u}\|_{W^{1,\infty}} + \|A\|_{L^\infty} + \|B\|_{L^2}).$$

Then by classical elliptic regularity [Gilbarg and Trudinger 1977, Theorem 8.32], we have, for any $\alpha \in (0, 1)$,

$$\|v^\delta - \langle v^\delta\rangle\|_{C^{1+\alpha}} \le C(1 + \|\bar{u}\|_{W^{1,\infty}} + \|A\|_{L^\infty} + \|B\|_{L^2}).$$

We can now apply the local regularity for weak solutions to μ^δ [Gilbarg and Trudinger 1977, Theorem 8.17]) and infer that

$$\|\mu^\delta\|_{C^\alpha} \le C(\|Dv^\delta\|_\infty + \|B\|_\infty) \le C(\|\bar{u}\|_{W^{1,\infty}} + \|A\|_{L^\infty} + \|B\|_{L^\infty}). \qquad \square$$

Proposition 6.4. *Let $(\bar{\lambda}, \bar{u}, \bar{m})$ be a solution of the ergodic system* (2) *and (v^δ, μ^δ) be the solution to* (99) *for A and B satisfying*

$$\|A\|_\infty + \|B\|_\infty \le C\delta$$

for some constant C. Then, as $\delta \to 0^+$,

$$\delta\langle v^\delta\rangle \longrightarrow \bar{\theta}, \quad (v^\delta - \langle v^\delta\rangle) \xrightarrow{L^\infty} \bar{v}, \quad \mu^\delta \xrightarrow{L^\infty} \bar{\mu},$$

where $(\bar{\theta}, \bar{v}, \bar{\mu})$ is the unique solution to (96).

Proof of Proposition 6.4. Passing to the limit in (99) (up to a subsequence) provides a constant $\bar{\theta}$ (limit of $\delta\langle v^\delta\rangle$), a map $\bar{v} \in W^{1,\infty}$ (limit of $v^\delta - \langle v^\delta\rangle$) and a map $\bar{\mu} \in L^\infty$ (limit of μ^δ) which solve (96). The uniqueness of $D\bar{v}$ (and hence of $\bar{v}$) and of $\bar{\mu}$ can be established by the standard duality argument of [Lasry and Lions 2007]. Then $\bar{\theta}$ is unique by the equation. The full convergence of $(\delta\langle v^\delta\rangle, v^\delta - \langle v^\delta\rangle, \mu^\delta)$ holds by uniqueness of the limit. $\square$

6B. ***Proof of Theorem 6.1.*** The proof of Theorem 6.1 consists mostly in showing that the heuristic relation (98) holds.

Proposition 6.5. *Let $(\bar{\lambda}, \bar{u}, \bar{m})$, $(\bar{u}^\delta, \bar{m}^\delta)$ and $(\bar{\theta}, \bar{v}, \bar{\mu})$ be respectively solutions to* (2), (7) *and* (96). *Then*

$$\lim_{\delta\to 0^+} \left\| \bar{u}^\delta - \frac{\bar{\lambda}}{\delta} - \bar{u} - \bar{\theta} \right\|_\infty + \|\bar{m}^\delta - \bar{m}\|_\infty = 0.$$

Proof. The argument is very close to the proof of the exponential rate (see Theorem 2.6). Let

$$E = \big\{(v, \mu) \in W^{1,\infty}(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d) : \|\delta v\|_\infty + \|Dv\|_\infty + \|\mu\|_\infty \le \widehat{C}\big\},$$

where $\widehat{C}$ is to be chosen below. For $(v, \mu) \in E$, we consider the solution $(\hat{v}, \hat{\mu})$ to (99) with

$$A(x) := \delta^{-1}\Big(-(H(x, D(\bar{u}+\delta v)) - H(x, D\bar{u}) - \delta H_p(x, D\bar{u}) \cdot Dv) \\ + F(x, \bar{m}+\delta\mu) - F(x, \bar{m}) - \delta \frac{\delta F}{\delta m}(x, \bar{m})(\mu)\Big),$$

$$B(x) := \delta^{-1}\big((\bar{m}+\delta\mu)H_p(x, D(\bar{u}+\delta v)) - \bar{m}H_p(x, D\bar{u}) - \delta\mu H_p(x, D\bar{u}) - \delta\bar{m}H_{pp}(x, \bar{m})Dv\big).$$

As

$$\|A\|_\infty + \|B\|_\infty \le C\widehat{C}^2\delta,$$

we have, by Lemma 6.3 (and for δ small enough),

$$\|\delta\hat{v}\|_\infty + \|D\hat{v}\|_\infty + \|\hat{\mu}\|_\infty \le C(1 + \|A\|_\infty + \|B\|_\infty) \le C(1 + \widehat{C}^2\delta).$$

We can choose $\widehat{C}$ such that, for δ small enough, the right-hand side is less than $\widehat{C}$. Then we can easily conclude that the map $(v, \mu) \to (\hat{v}, \hat{\mu})$ has a fixed point (v^δ, μ^δ). Note that $(\bar{\lambda}/\delta + \bar{u} + \delta v^\delta, \bar{m} + \delta\mu^\delta)$ solves (7) and therefore is equal to $(\bar{u}^\delta, \bar{m}^\delta)$. Hence, by Proposition 6.4, we deduce

$$\Big\|\bar{u}^\delta - \frac{\bar{\lambda}}{\delta} - \bar{u} - \bar{\theta}\Big\|_\infty = \|\delta v^\delta - \bar{\theta}\|_\infty \le \|\delta(v^\delta - \langle v^\delta\rangle)\|_\infty + |\delta\langle v^\delta\rangle - \bar{\theta}| \to 0 \quad \text{as } \delta \to 0,$$

which completes the proof. □

Proof of Theorem 6.1. Recall that we have uniform Lipschitz estimates on U^δ and on D_xU^δ (Lemma 4.3 and Proposition 4.4) and that any converging subsequence is a weak solution of the ergodic master equation (proof of Theorem 4.2). Therefore, we only need to show that $U^\delta - \delta^{-1}\bar{\lambda}$ has a limit when evaluated at some value. For this, let $(\bar{u}^\delta, \bar{m}^\delta)$ be the solution to (7). As $(\bar{u}^\delta, \bar{m}^\delta)$ is also a stationary solution to (5), we have

$$U^\delta(x, \bar{m}^\delta) = \bar{u}^\delta(x) \quad \text{for all } x \in \mathbb{T}^d.$$

We have seen in Proposition 6.5 that, as $\delta \to 0$, $\bar{m}^\delta$ converges to $\bar{m}$, while $\bar{u}^\delta - \delta^{-1}\bar{\lambda}$ converges to $\bar{u} + \bar{\theta}$. □

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ON THE COST OF OBSERVABILITY IN SMALL TIMES FOR THE ONE-DIMENSIONAL HEAT EQUATION

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We aim at presenting a new estimate on the cost of observability in small times of the one-dimensional heat equation, which also provides a new proof of observability for the one-dimensional heat equation. Our proof combines several tools. First, it uses a Carleman-type estimate borrowed from our previous work (*SIAM J. Control Optim.* **56**:3 (2018), 1692–1715), in which the weight function is derived from the heat kernel and which is therefore particularly easy. We also use explicit computations in the Fourier domain to compute the high-frequency part of the solution in terms of the observations. Finally, we use the Phragmén–Lindelöf principle to estimate the low-frequency part of the solution. This last step is done carefully with precise estimations coming from conformal mappings.

1. Introduction

Setting. The goal of this work is to analyze the cost of observability in small times of the one-dimensional heat equation. To fix the ideas, let $L, T > 0$ and consider the following heat equation, set in the bounded interval $(-L, L)$ and among some time interval $(0, T)$:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,T)\times(-L,L), \\ u(t,-L) = u(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (-L,L). \end{cases} \tag{1-1}$$

In (1-1), the state $u = u(t,x)$ satisfies a heat equation, with initial datum $u_0 \in H_0^1(-L,L)$.

Our main goal is to study the cost of observability in small times T of the problem (1-1) observed from both sides $x = -L$ and $x = +L$. To be more precise, let us recall that it is by now well known that there exists $C_0(T,L)$ such that all solutions u of (1-1) with initial datum $u_0 \in H_0^1(-L,L)$ satisfy

$$\|u(T)\|_{L^2(-L,L)} \leqslant C_0(T,L)\big(\|\partial_x u(t,-L)\|_{L^2(0,T)} + \|\partial_x u(t,L)\|_{L^2(0,T)}\big). \tag{1-2}$$

In fact, the existence of the constant $C_0(T,L)$ is a consequence of the null controllability results in small times obtained by [Egorov 1963; Fattorini and Russell 1971] in the one-dimensional case. From now on, we denote by $C_0(T,L)$ the best constant in the observability inequality (1-2).

A precise description of the constant $C_0(T,L)$ as $T \to 0$ is still missing, despite several contributions in this direction, which we would like to briefly recall here. First, [Seidman 1984] showed that

$$\limsup_{T\to 0} T \log C_0(T,L) < \infty, \tag{1-3}$$

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while [Güichal 1985] proved that

$$\liminf_{T\to 0} T \log C_0(T,L) > 0. \tag{1-4}$$

Besides, due to the scaling of the equation, $C_0(T,L)$ depends only on the ratio L^2/T. Therefore, the quantity $T\log C_0(T,L)$ should be compared to L^2. We list below several contributions:

$$\liminf_{T\to 0} T\log C_0(T,L) \geqslant \tfrac{1}{4}L^2 \qquad \text{[Miller 2004]},$$

$$\liminf_{T\to 0} T\log C_0(T,L) \geqslant \tfrac{1}{2}L^2 \qquad \text{[Lissy 2015]},$$

$$\limsup_{T\to 0} T\log C_0(T,L) \leqslant 2\big(\tfrac{36}{37}\big)^2 L^2 \qquad \text{[Miller 2006]},$$

$$\limsup_{T\to 0} T\log C_0(T,L) \leqslant \tfrac{3}{4}L^2 \qquad \text{[Tenenbaum and Tucsnak 2007]}.$$

Main result. Our contribution comes in this context. Namely we prove the following result:

Theorem 1.1. *Setting*

$$K_0 = \frac{1}{4} + \frac{\Gamma\big(\frac{1}{4}\big)^2}{8\sqrt{2}\pi^2}\sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)} \frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)} \quad (K_0 \simeq 0.6966), \tag{1-5}$$

where Γ *denotes the gamma function, we have*

$$\limsup_{T\to 0} T\log C_0(T,L) \leqslant K_0 L^2. \tag{1-6}$$

In fact, for all $K > K_0$, *there exists a constant* $C > 0$ *such that for all* $T \in (0,1]$, *for all solutions* u *of* (1-1) *with initial datum* $u_0 \in H_0^1(-L,L)$,

$$\left\| u(T)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)} \leqslant C\exp\left(\frac{KL^2}{T}\right)\big(\|\partial_x u(t,-L)\|_{L^2(0,T)} + \|\partial_x u(t,L)\|_{L^2(0,T)}\big). \tag{1-7}$$

Remark 1.2. The constant K_0 in (1-5) can alternatively be written as

$$K_0 = \frac{1}{4} + \frac{2}{\pi}\frac{\int_0^{\frac{\pi}{2}} \ln\big(\cot\big(\frac{t}{2}\big)\big)\sqrt{\cos(t)}\,dt}{\int_0^{\frac{\pi}{2}} \sqrt{\cos(t)}\,dt}; \tag{1-8}$$

see Proposition 2.3 in Section 2.

Theorem 1.1 slightly improves the cost of observability in small times when compared to [Tenenbaum and Tucsnak 2007]. However, we do not claim that this bound is sharp, and this remains, to our knowledge, an open problem. In particular, we shall comment in Section 4F a possible path to improve the estimates given in Theorem 1.1.

In fact, we believe that Theorem 1.1 is interesting mostly by its proof, presented in Section 2, which combines several arguments. In particular, it uses a Carleman-type estimate, which was already used in [Dardé and Ervedoza 2018] to derive a good description of the reachable set for the one-dimensional heat equation in terms of domains of holomorphic extension of the states. This Carleman-type estimate is used to reduce the problem of observability to an estimate of the low-frequency part of the solution of (1-1). Then, we shall use Fourier analysis on the conjugated heat equation to get an exact formula for the high-frequency part of the solution of (1-1) in terms of the observations. The last part of the argument is a complex analysis argument based on the Phragmén–Lindelöf principle. We refer to Sections 2 and 3 for the detailed proof of Theorem 1.1.

Let us also mention that Theorem 1.1 is strongly connected to control theory. Indeed, let us consider the following null-controllability problem: given $T > 0$ and $y_0 \in L^2(-L, L)$, find control functions $v_-, v_+ \in L^2(0, T)$ such that the solution y of

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{in } (0,T) \times (-L, L), \\ y(t, -L) = v_-(t) & \text{in } (0,T), \\ y(t, +L) = v_+(t) & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (-L, L), \end{cases} \tag{1-9}$$

satisfies

$$y(T, x) = 0 \quad \text{in } (-L, L). \tag{1-10}$$

It is well known, see, e.g., [Egorov 1963; Fattorini and Russell 1971], that for any $T > 0$, one can find controls v_-, v_+ of minimal $(L^2(0,T))^2$ norm, depending linearly on $y_0 \in L^2(-L, L)$, such that the controlled trajectory, i.e., the solution of (1-9), satisfies (1-10). Besides, the $\mathscr{L}(L^2(-L, L); (L^2(0,T))^2)$-norm of the linear map $y_0 \mapsto (v_-, v_+)$ is precisely $C_0(T, L)$. In other words, $C_0(T, L)$ also characterizes the cost of controllability for the one-dimensional heat equation.

We emphasize that Theorem 1.1 also allows us to tackle some multidimensional settings. Namely, as a consequence of Theorem 1.1 and the control transmutation method, see [Miller 2006], one gets the following corollary:

Corollary 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$, and let Γ_0 be an open subset of $\partial\Omega$. Let $a = a(x) \in L^\infty(\Omega; M_d(\mathbb{R}))$ and $\rho \in L^\infty(\Omega; \mathbb{R})$ be such that there exist strictly positive numbers ρ_-, ρ_+, a_- and a_+ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$,*

$$a_-|\xi|^2 \leqslant a(x)\xi \cdot \xi \leqslant a_+|\xi|^2, \quad \rho_- \leqslant \rho(x) \leqslant \rho_+.$$

Further assume that there exist a time $S_0 > 0$ and a constant $C > 0$ such that, for any $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution w of

$$\begin{cases} \rho(x)\, \partial_{ss} w - \operatorname{div}(a(x)\nabla w) = 0 & \text{in } (0,S) \times \Omega, \\ w(s,x) = 0 & \text{on } (0,S) \times \partial\Omega, \\ (w(0,x), \partial_s w(0,x)) = (w_0(x), w_1(x)) & \text{in } \Omega \end{cases} \tag{1-11}$$

satisfies $a(x)\nabla w \cdot n \in L^2((0, S_0) \times \Gamma_0)$ and

$$\|(w_0, w_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \, \|a(x)\nabla w \cdot n\|_{L^2((0,S_0)\times\Gamma_0)}. \tag{1-12}$$

We define $C_0(T,\Omega,\Gamma_0)$ as the best constant in the following observability inequality: for all $u_0 \in H_0^1(M)$, the solution u of

$$\begin{cases} \rho(x)\,\partial_t u - \operatorname{div}(a(x)\nabla u) = 0 & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases} \tag{1-13}$$

satisfies

$$\|u(T)\|_{L^2(M)} \leqslant C_0(T,\Omega,\Gamma_0)\|a(x)\nabla u\cdot n\|_{L^2((0,T)\times\Gamma_0)}. \tag{1-14}$$

Then we have

$$\limsup_{T\to 0} T\log C_0(T,\Omega,\Gamma_0) \leqslant K_0 S_0^2. \tag{1-15}$$

Corollary 1.3 uses the transmutation method and therefore the observability of the corresponding wave equation (1-11), which has been well-studied in the literature. In particular, if the coefficients ρ and a are $C^2(\overline{\Omega})$, according to [Bardos et al. 1988; 1992; Burq and Gérard 1997], the wave equation (1-11) satisfies the observability inequality (1-12) if and only if all the rays of geometric optics meet Γ_0 in a nondiffractive point in time less than S_0. In the case of coefficients ρ and a which are less regular, let us quote [Fanelli and Zuazua 2015] in the one-dimensional case with ρ and a in the Zygmund class, and [Dehman and Ervedoza 2017] in the multidimensional case for coefficients $\rho \in C^0(\overline{\Omega})$ and $a = 1$, with ρ satisfying a multiplier-type condition similar to the one in [Ho 1986; Lions 1988] in the sense of distributions (and ρ locally C^1 close to the boundary, see [Dehman and Ervedoza 2017, Section 4.2]).

Let us emphasize that Corollary 1.3 can be applied in the one-dimensional case as well for coefficients in the Zygmund class [Fanelli and Zuazua 2015], thus allowing a more general class of coefficients than in the analysis of [Miller 2004; Tenenbaum and Tucsnak 2007], which is done for $\rho = 1$ and $a \in C^2$ (and, possibly, a continuous potential). But even in the case $\Omega = (-L, L)$, $\Gamma_0 = \{-L, L\}$, $\rho(x) = 1$, $a(x) = 1$, we get $S_0 = 2L$ and thus we obtain an estimate on the cost of observability of the form

$$\limsup_{T\to 0} T\log C_0(T,(-L,L),\{-L,L\}) \leqslant 4K_0L^2,$$

instead of (1-6). In other words, we have a loss of a factor 4, so that the results in [Miller 2004; Tenenbaum and Tucsnak 2007] are better than ours in the one-dimensional case for a coefficient a in (1-13) which belongs to C^2. Therefore, we shall also explain how Theorem 1.1 can be extended to a multidimensional case directly when the observation is performed on the whole boundary; see Theorems 4.1–4.2.

Let us mention that the proofs of the observability inequality of the heat equation for general smooth bounded domains Ω and observation in an open subset Γ_0 of the boundary in [Fursikov and Imanuvilov 1996; Lebeau and Robbiano 1995] yield that

$$\limsup_{T\to 0} T\log C_0(T,\Omega,\Gamma_0) < \infty,$$

while on the other hand, [Miller 2004] proves

$$\liminf_{T\to 0} T\log C_0(T,\Omega,\Gamma_0) \geqslant \tfrac{1}{4}\sup_{\Omega} d(x,\Gamma_0)^2.$$

To our knowledge, getting more intrinsic geometric upper estimates on the cost of observability in small times in such general settings is still out of reach. In fact [Laurent and Léautaud 2018] shows that upper estimates on the cost of observability in small times cannot be linked only to the maximal distance to the control set and are deeply related to the geometry of the domain and of the observation set; see Remark 4.3. However, in geometrical cases which can be obtained by tensorization, some estimates can be obtained; see [Miller 2005] and Section 4B for more details.

We shall also mention that estimating the observability constant in small times for the heat equation in the one-dimensional case is related to the uniform controllability of viscous approximations of the transport equation; see [Coron and Guerrero 2005; Glass 2010; Lissy 2012; 2015]. We refer in particular to Section 4G for a more precise discussion on this problem. In particular, the proof in [Lissy 2012], when combined with Theorem 1.1, easily yields an improvement of the results known on this problem; see Section 4G and Theorem 4.10 for more details.

As we have seen in the above discussion, there are still some open questions on the observability of the one-dimensional constant-coefficient parabolic equations, despite the efficiency and robustness of the approach based on Carleman estimates [Fursikov and Imanuvilov 1996; Lebeau and Robbiano 1995]. This has justified the development of new manners to derive controllability of parabolic equations, and we shall in particular quote the flatness method developed in [Martin et al. 2014; 2016], a heat packet decomposition [Gimperlein and Waters 2017] and the backstepping approach [Coron and Nguyen 2017]. Our method comes in this context and provides what seems to be another approach to obtain observability results for the heat equation.

Outline. Section 2 presents the main strategy of the proof of Theorem 1.1 using several technical results that will be proved afterwards, in Section 3 for the ones using new arguments and in the Appendix for a Carleman-type estimate (Theorem 2.1) which can be found also in [Dardé and Ervedoza 2018] in a slightly different form. Section 4 provides several comments on Theorem 2.1 and its generalization, including a discussion on what can be done in the multidimensional setting (Section 4A), when the geometry has a tensorized form (Section 4B), or when the observation is on one side of the domain (Section 4C) or on some distributed open subset (Section 4D). We also present in Section 4E an alternative proof of a weaker version of Theorem 1.1 based on the uncertainty principles of [Landau and Pollak 1961] and the result in [Fuchs 1964], recovering the result of [Tenenbaum and Tucsnak 2007]. This will lead us to discuss the possibility of improving the estimate of the cost of observability in small times in Theorem 1.1 by using a better bound than the one provided by the use of Phragmén–Lindelöf principle for entire functions; see Section 4F for more details. We end up in Section 4G by giving a consequence of our result on the problem of uniform controllability of viscous approximations of transport equations. The Appendix gives the detailed proof of a rather easy Carleman estimate which is one of the building blocks of our analysis.

2. Proof of Theorem 1.1: main steps

As said in the Introduction, the proof of Theorem 1.1 relies on several steps.

The first step is the following Carleman-type estimate.

Theorem 2.1. *For any smooth solution u of* (1-1), *setting*

$$z(t,x)=u(t,x)\exp\left(\frac{x^2-L^2}{4t}\right),\quad (t,x)\in(0,T)\times(-L,L), \tag{2-1}$$

we have the inequality

$$\int_{-L}^{L}|\partial_x z(T,x)|^2\,dx-\frac{L^2}{4T^2}\int_{-L}^{L}|z(T,x)|^2\,dx\leqslant\frac{L}{T^2}\int_0^T t(|\partial_x u(t,-L)|^2+|\partial_x u(t,L)|^2)\,dt. \tag{2-2}$$

Theorem 2.1 is based on the study of the equation satisfied by z in (2-1). As u satisfies the heat equation (1-1), the function z in (2-1) satisfies

$$\begin{cases}\partial_t z+\dfrac{x}{t}\partial_x z+\dfrac{1}{2t}z-\partial_x^2 z-\dfrac{L^2}{4t^2}z=0, & (t,x)\in(0,\infty)\times(-L,L),\\ z(t,-L)=z(t,L)=0, & t\in(0,\infty),\\ z(0,x)=0, & x\in(-L,L).\end{cases} \tag{2-3}$$

One can therefore perform energy estimates on (2-3), which will eventually lead to (2-2). In the Appendix, we prove a slightly more general result, encompassing also some multidimensional settings, see Proposition A.1, from which one immediately derives Theorem 2.1 by setting $\Omega=(-L,L)$ and $g\equiv 0$.

Note that Theorem 2.1 was used in [Dardé and Ervedoza 2018] in time $T>L^2/\pi$ in order to describe the reachable set of the one-dimensional heat equation. Estimate (2-2) is somehow a Carleman estimate even if here no parameter appears in the proof. In fact, it rather corresponds to a *limiting Carleman estimate* as the conjugated operator (2-3) does not satisfy the usual strict pseudoconvexity conditions of [Hörmander 1985]. We refer in particular to [Dos Santos Ferreira et al. 2009] for other instances of limiting Carleman weights in another context, namely elliptic operators.

The second step of our analysis amounts to realizing that the solutions z of (2-3) could be explicitly solved using Fourier analysis if one extends the solution z of (2-3) by zero outside the space interval $(-L,L)$. We therefore introduce, for $t\in(0,T]$,

$$w(t,x)=\begin{cases}z(t,x)\quad\left(=u(t,x)\exp\left(\dfrac{x^2-L^2}{4t}\right)\right) & \text{for } x\in(-L,L),\\ 0 & \text{for } x\notin(-L,L).\end{cases} \tag{2-4}$$

In view of the above definition, it is then natural to set $w(0,\cdot)=0$, since it is consistent with the above definition when taking the limit $t\to 0$. This function w satisfies

$$\begin{cases}\partial_t w+\dfrac{x}{t}\partial_x w+\dfrac{1}{2t}w-\partial_x^2 w-\dfrac{L^2}{4t^2}w=\partial_x u(t,L)\delta_L-\partial_x u(t,-L)\delta_{-L}, & (t,x)\in(0,\infty)\times\mathbb{R},\\ w(0,x)=0, & x\in\mathbb{R}.\end{cases} \tag{2-5}$$

Using Fourier transform, one can then compute explicitly

$$\hat{w}(T,\xi)=\int_{\mathbb{R}}w(T,x)e^{-\imath\xi x}\,dx,$$

at least for some frequency $\xi\in\mathbb{C}$:

Proposition 2.2. *For $\alpha \geqslant 0$, define the sets (see Figure 1)*

$$\mathscr{C}_\alpha = \{\xi = a + \imath b \in \mathbb{C} : (a, b) \in \mathbb{R}^2 \text{ with } |a| \geqslant |b| + \alpha\}. \tag{2-6}$$

Let w be given by (2-4) corresponding to some smooth solution u of (1-1).

Then, for any $\xi \in \mathscr{C}_{L/(2T)}$,

$$\hat{w}(T, \xi) = \int_0^T \sqrt{\frac{T}{t}} \big(-\partial_x u(t, -L) e^{\imath \frac{\xi L T}{t}} + \partial_x u(t, L) e^{-\imath \frac{\xi L T}{t}}\big) e^{-(\xi^2 T^2 - \frac{L^2}{4})(\frac{1}{t} - \frac{1}{T})}\, dt. \tag{2-7}$$

In particular, for any $\alpha > L/(2T)$, setting

$$C_\alpha(T) = \frac{1}{\sqrt{L(\alpha - L/(2T))}}, \tag{2-8}$$

for all $\xi \in \mathscr{C}_\alpha$, we have

$$|\hat{w}(T, \xi)| \leqslant C_\alpha(T) \sqrt{T} e^{|\Im(\xi)| L} \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\big). \tag{2-9}$$

The proof of Proposition 2.2 is done in Section 3A and relies on explicit computations. In particular, it gives a precise L^∞ bound on the high-frequency component of $w(T)$ given by (2-4) corresponding to a smooth solution u of (1-1).

The third step of our analysis consists in the recovery of the low-frequency part of w given by (2-4). In order to do that, we recall that $\hat{w}(T, \cdot)$ is the Fourier transform of a function supported in $[-L, L]$. Therefore, its growth as $|\Im(\xi)| \to \infty$ is known, while $\hat{w}(T, \cdot)$ is holomorphic in the whole complex plane $\mathbb{C}$. Combined with the fact that we have nice estimates on $\hat{w}(T, \cdot)$ in $\mathscr{C}_\alpha$ for $\alpha > L^2/(2T)$, we are in the position to use Phragmén–Lindelöf principles to estimate $\hat{w}(T, \cdot)$ everywhere in the complex plane, but more importantly on the real axis $\mathbb{R}$.

Proposition 2.3. *Let $L > 0$, $\alpha > 0$ and f be a holomorphic function on $\mathscr{O}_\alpha = \mathbb{C} \setminus \mathscr{C}_\alpha$ (see Figure 1) such that:*

- *There exists a constant C_0 such that*

$$\text{for all } \xi \in \partial\mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp(|\Im(\xi)| L). \tag{2-10}$$

- *There exists a constant C_1 such that*

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_1 \exp(|\Im(\xi)| L). \tag{2-11}$$

Defining

$$\widetilde{\mathscr{O}}_1 = \{(a, b) \in \mathbb{R}^2 : |a| < |b| + 1\},$$

there exists a unique function $\tilde{\varphi}$ satisfying

$$\begin{cases} \Delta\tilde{\varphi} = -2\delta_{(-1,1)\times\{0\}} & \text{in } \widetilde{\mathscr{O}}_1, \\ \tilde{\varphi} = 0 & \text{on } \partial\widetilde{\mathscr{O}}_1, \\ \lim_{|b|\to\infty} \sup_{a\in(-|b|-1,|b|+1)} |\tilde{\varphi}(a, b)| = 0, \end{cases} \tag{2-12}$$

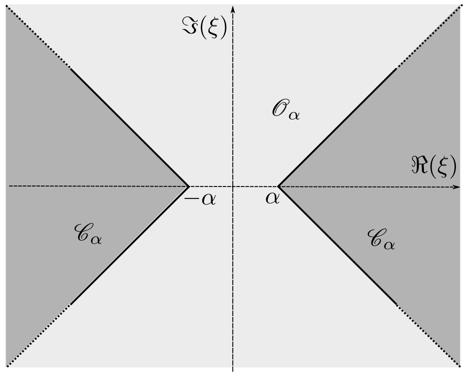

Figure 1. The complex plane, with domains $\mathscr{C}_\alpha$ and $\mathscr{O}_\alpha$.

and we define the function φ *on* $\mathscr{O}_1$ *by*

$$\varphi(\xi) = \tilde{\varphi}(\Re(\xi), \Im(\xi)), \quad \xi \in \mathscr{O}_1. \tag{2-13}$$

Then we have the bound

$$\textit{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp(|\Im(\xi)|L) \exp\left(L\alpha\varphi\left(\frac{\xi}{\alpha}\right)\right). \tag{2-14}$$

Besides, *the maximum of* φ *on* $\mathscr{O}_1$ *is attained in* 0:

$$\sup_{\mathscr{O}_1} \varphi = \varphi(0) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2}\pi^2} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)} \quad (\simeq 0.893204), \tag{2-15}$$

which can be alternatively written as

$$\varphi(0) = \frac{2}{\pi} \frac{\int_0^{\frac{\pi}{2}} \ln\left(\cot\left(\frac{t}{2}\right)\right)\sqrt{\cos(t)}\,dt}{\int_0^{\frac{\pi}{2}} \sqrt{\cos(t)}\,dt}. \tag{2-16}$$

Proposition 2.3 mainly reduces to the application of Phragmén–Lindelöf principle for holomorphic functions. In fact, the main point in Proposition 2.3 is that the maximum of the harmonic function $\tilde{\varphi}$ can be explicitly computed. This is done using conformal maps to link the solution of the Laplace equation in the domain $\widetilde{\mathscr{O}}_1$ with solutions of the Laplace operator in the half-strip, in which explicit solutions can be computed using Fourier decomposition techniques. We refer to Section 3B for the proof of Proposition 2.3.

Of course, we shall apply Proposition 2.3 to the function $f = \hat{w}(T, \cdot)$, which, according to (2-9), satisfies (2-10) for any $\alpha > L/(2T)$ with

$$C_0 = C_\alpha(T)\sqrt{T}\left(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\right),$$

while (2-11) holds with

$$C_1 = \|w(T)\|_{L^1(-L,L)} \leqslant \sqrt{2L}\|u(T)\|_{L^2(-L,L)} \leqslant \sqrt{2L}\|u_0\|_{L^2(-L,L)}.$$

We then immediately deduce the following corollary.

Corollary 2.4. *Let w be given by* (2-4) *corresponding to some smooth solution u of* (1-1). *Then, for any $\alpha > L/(2T)$,*

$$\text{for all } \xi \in \mathscr{O}_\alpha \cap \mathbb{R}, \quad |\hat{w}(T,\xi)| \leqslant C_\alpha(T)\sqrt{T} e^{L\alpha\varphi(0)} \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\big), \tag{2-17}$$

where $C_\alpha(T)$ denotes the constant in (2-8).

End of the proof of Theorem 1.1. Let $\varepsilon > 0$, and choose $\alpha = (1+\varepsilon)L/(2T)$. Combining (2-17) and (2-9), we see that

$$\text{for all } \xi \in \mathbb{R}, \quad |\hat{w}(T,\xi)| \leqslant \sqrt{\frac{2}{\varepsilon}\frac{T}{L}} \exp\bigg((1+\varepsilon)\frac{L^2}{2T}\varphi(0)\bigg) \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\big). \tag{2-18}$$

Then, using Theorem 2.1 and the identity

$$\int_{-L}^{L} |\partial_x z(T,x)|^2 \, dx - \frac{L^2}{4T^2} \int_{-L}^{L} |z(T,x)|^2 \, dx = \int_{\mathbb{R}} \bigg(|\xi|^2 - \frac{L^2}{4T^2}\bigg) |\hat{w}(T,\xi)|^2 \, d\xi$$

we have

$$\frac{3L^2}{4T^2} \int_{|\xi| > L/T} |\hat{w}(T,\xi)|^2 \, d\xi \leqslant \frac{L}{T} \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)}^2 + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}^2\big) + \frac{L^2}{4T^2} \int_{|\xi| < L/(2T)} |\hat{w}(T,\xi)|^2 \, d\xi.$$

Combined with (2-18), we obtain

$$\int_{|\xi| > L/T} |\hat{w}(T,\xi)|^2 \, d\xi \leqslant \bigg(\frac{4T}{3L} + \frac{4T}{3L\varepsilon} \exp\bigg((1+\varepsilon)\frac{L^2}{T}\varphi(0)\bigg)\bigg) \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)}^2 + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}^2\big) \tag{2-19}$$

and

$$\int_{|\xi| < L/T} |\hat{w}(T,\xi)|^2 \, d\xi \leqslant \frac{8T}{\varepsilon L} \exp\bigg((1+\varepsilon)\frac{L^2}{T}\varphi(0)\bigg) \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)}^2 + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}^2\big). \tag{2-20}$$

Using Parseval's identity and the explicit form of w in (2-4), we easily get, for some constant $C_\varepsilon(T)$ that goes to zero as $T \to 0$, that

$$\bigg\| u(T,x) \exp\bigg(\frac{x^2 - L^2}{4T}\bigg) \bigg\|_{L^2(-L,L)} \leqslant C_\varepsilon(T) \exp\bigg(\frac{L^2}{2T}(1+\varepsilon)\varphi(0)\bigg) \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\big),$$

which we rewrite as

$$\left\| u(T,x)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)} \leqslant C_\varepsilon(T)\exp\left(\frac{L^2}{T}\left(\tfrac14+\tfrac12(1+\varepsilon)\varphi(0)\right)\right)\left(\|\partial_x u(\cdot,L)\|_{L^2(0,T)}+\|\partial_x u(\cdot,-L)\|_{L^2(0,T)}\right). \tag{2-21}$$

This concludes the proof of Theorem 1.1, as $C_\varepsilon(T)\leqslant C_\varepsilon(1)=C_\varepsilon$ for T small enough, for some C_ε independent of T. □

Remark 2.5. Note that the constant C_ε in the above proof blows up as ε goes to zero. If it were not the case, one could pass to the limit $\varepsilon\to 0$ in (2-21), so that one could choose $K=K_0$ in the observability inequality (1-7). So far, we do not know if this choice is allowed in the observability inequality (1-7) or not.

We have thus reduced the proof of Theorem 1.1 to the proofs of Theorem 2.1 and Propositions 2.2 and 2.3. The proof of Theorem 2.1 is postponed to the Appendix in which a slightly more general result is proved (Proposition A.1), while the proofs of Propositions 2.2 and 2.3 are detailed in Section 3.

Remark 2.6. The above approach allows us in fact to recover an explicit formula to compute $\hat{w}(T)$ in terms of the observations. Namely, for $\xi\in\mathbb{R}$ with $|\xi|\geqslant L/(2T)$, formula (2-7) yields

$$\hat{w}(T,\xi)=\int_0^T\sqrt{\frac{T}{t}}\left(-\partial_x u(t,-L)e^{\imath\frac{\xi LT}{t}}+\partial_x u(t,L)e^{-\imath\frac{\xi LT}{t}}\right)e^{-(\xi^2T^2-\frac{L^2}{4})(\frac1t-\frac1T)}\,dt. \tag{2-22}$$

On the other hand, combining the formula (2-7) and Remark 3.2 allowing us to get an explicit expression under the assumptions of Proposition 2.3, we get: for all $\alpha_*>\alpha>L/(2T)$, for all $\xi\in\mathbb{R}$ with $|\xi|<L/(2T)$,

$$\begin{aligned}\hat{w}(T,\xi)=&-\int_0^T\sqrt{\frac{T}{t}}\partial_x u(t,-L)\frac{1}{2\imath\pi}\int_{\gamma_\alpha}\frac{e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}}{\zeta-\xi}e^{\imath\frac{\zeta LT}{t}}e^{-(\zeta^2T^2-\frac{L^2}{4})(\frac1t-\frac1T)}\,d\zeta\,dt\\&+\int_0^T\sqrt{\frac{T}{t}}\partial_x u(t,L)\frac{1}{2\imath\pi}\int_{\gamma_\alpha}\frac{e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}}{\zeta-\xi}e^{-\imath\frac{\zeta LT}{t}}e^{-(\zeta^2T^2-\frac{L^2}{4})(\frac1t-\frac1T)}\,d\zeta\,dt,\end{aligned} \tag{2-23}$$

where ϕ is a holomorphic function on $\mathscr{O}_1$ such that $\Re(\phi(\xi))=\varphi(\xi)+|\Im(\xi)|$ for all $\xi\in\mathscr{O}_1$ (see Section 3B2 for the existence of such function ϕ), and γ_α is the union of the two connected components of $\partial\mathscr{O}_\alpha$ oriented counterclockwise. But this formula does not seem easy to deal with as the kernels

$$K_\mp(t,\xi)=\frac{1}{2\imath\pi}\int_{\gamma_\alpha}\frac{e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}}{\zeta-\xi}e^{\pm\imath\frac{\zeta LT}{t}}e^{-(\zeta^2T^2-\frac{L^2}{4})(\frac1t-\frac1T)}\,d\zeta,\quad (t,\xi)\in(0,T)\times\left(-\frac{L}{2T},\frac{L}{2T}\right),$$

are difficult to estimate directly.

3. Proof of Theorem 1.1: intermediate results

3A. ***Proof of Proposition 2.2.*** Let w be as in Proposition 2.2. Then w satisfies (2-5). When taking its Fourier transform in the space variable, we easily check that

$$\hat{w}(t,\xi)=\int_{\mathbb{R}}w(t,x)e^{-\imath\xi x}\,dx,\quad (t,\xi)\in[0,T]\times\mathbb{R},$$

solves the equation

$$\begin{cases} \partial_t \hat{w} - \frac{\xi}{t}\partial_\xi \hat{w} - \frac{1}{2t} w + \xi^2 \hat{w} - \frac{L^2}{4t^2}\hat{w} = \partial_x u(t,L) e^{-\imath \xi L} - \partial_x u(t,-L) e^{\imath \xi L}, & (t,\xi) \in (0,\infty)\times\mathbb{R}, \\ \hat{w}(0,\xi) = 0, & \xi \in \mathbb{R}. \end{cases} \tag{3-1}$$

We are thus back to the study of a transport equation. For each $\xi_0 \in \mathbb{R}$, we therefore introduce the characteristics $\xi(t,\xi_0)$ reaching ξ_0 at time T,

$$\frac{d\xi}{dt}(t,\xi_0) = -\frac{\xi(t,\xi_0)}{t}, \quad t \in (0,T], \qquad \xi(T,\xi_0) = \xi_0, \tag{3-2}$$

which is explicitly given by

$$\xi(t,\xi_0) = \frac{\xi_0 T}{t}, \quad t \in (0,T].$$

We can thus write, for all $t \in (0,T]$,

$$\frac{d}{dt}\left(\hat{w}\left(t, \frac{\xi_0 T}{t}\right)\right) + \left(\frac{1}{t^2}\left(\xi_0^2 T^2 - \frac{L^2}{4}\right) - \frac{1}{2t}\right)\hat{w}\left(t, \frac{\xi_0 T}{t}\right) = \partial_x u(t,L) e^{-\imath \frac{\xi_0 L T}{t}} - \partial_x u(t,-L) e^{\imath \frac{\xi_0 L T}{t}}.$$

This yields the formula

$$\frac{d}{dt}\left(\hat{w}\left(t, \frac{\xi_0 T}{t}\right) t^{-\frac{1}{2}} e^{-(\xi_0^2 T^2 - \frac{L^2}{4})/t}\right) = \left(\partial_x u(t,L) e^{-\imath \frac{\xi_0 L T}{t}} - \partial_x u(t,-L) e^{\imath \frac{\xi_0 L T}{t}}\right) t^{-\frac{1}{2}} e^{-(\xi_0^2 T^2 - \frac{L^2}{4})/t}.$$

For any $\eta > 0$, we can integrate this formula between η and T to get

$$\hat{w}(T,\xi_0) T^{\frac{1}{2}} e^{-(\xi_0^2 T^2 - \frac{L^2}{4})/T} - \hat{w}(\eta,\xi_0) \eta^{\frac{1}{2}} e^{-(\xi_0^2 T^2 - \frac{L^2}{4})/\eta}$$
$$= \int_\eta^T t^{-\frac{1}{2}} \left(\partial_x u(t,L) e^{-\imath \frac{\xi_0 L T}{t}} - \partial_x u(t,-L) e^{\imath \frac{\xi_0 L T}{t}}\right) e^{-(\xi_0^2 T^2 - \frac{L^2}{4})/t}\, dt.$$

It is not difficult to check that for $\xi_0 \in \mathbb{R}$ with $|\xi_0| > L/(2T)$, the integral on the right-hand side converges when η goes to zero, and

$$\lim_{\eta \to 0} \hat{w}(\eta,\xi_0) \eta^{-\frac{1}{2}} e^{-(\xi_0^2 T^2 - \frac{L^2}{4})/\eta} = 0.$$

Therefore, provided $\xi_0 \in \mathbb{R}$ satisfies $|\xi_0| > L/(2T)$, one gets the formula

$$\hat{w}(T,\xi_0) = \int_0^T \sqrt{\frac{T}{t}} \left(\partial_x u(t,L) e^{-\imath \frac{L\xi_0 T}{t}} - \partial_x u(t,-L) e^{\imath \frac{L \xi_0 T}{t}}\right) e^{-(\xi_0^2 T^2 - \frac{L^2}{4})(\frac{1}{t} - \frac{1}{T})}\, dt. \tag{3-3}$$

This formula coincides with the one in (2-7) for $\xi_0 \in \mathscr{C}_{L^+/2T} \cap \mathbb{R}$ (here, we use the notation L^+ to denote any constant strictly larger than L). As $\hat{w}(T,\cdot)$ is holomorphic on $\mathbb{C}$, we only have to check that the right-hand side of formula (3-3) can be extended holomorphically to $\mathscr{C}_{L^+/2T}$. In fact, writing $\xi = a + \imath b$ with $(a,b) \in \mathbb{R}^2$, the right-hand side of (3-3) can be extended holomorphically in the domain in which

$$\begin{cases} \Re\left(+\imath \xi L T - \left(\xi^2 T^2 - \frac{L^2}{4}\right)\right) = -bLT - \left((a^2 - b^2)T^2 - \frac{L^2}{4}\right) < 0, \\ \Re\left(-\imath \xi L T - \left(\xi^2 T^2 - \frac{L^2}{4}\right)\right) = +bLT - \left((a^2 - b^2)T^2 - \frac{L^2}{4}\right) < 0, \end{cases}$$

which is equivalent to

$$|a| > |b| + \frac{L}{2T},$$

i.e., $\xi \in \mathscr{C}_{L+/(2T)}$. We have thus proved that for all $\xi \in \mathscr{C}_{L+/(2T)}$, $\hat{w}(T,\xi)$ is given by the formula (2-7). In fact, by continuity, this formula also holds for $\xi \in \mathscr{C}_{L/2T}$.

In order to deduce (2-9), we start from the formula (2-7) and we use a Cauchy–Schwarz estimate: for $\xi \in \mathscr{C}_\alpha$ with $\alpha > L/(2T)$,

$$\begin{aligned}|\hat{w}(T,\xi)| \leqslant \sqrt{T}\|\partial_x u(t,L)\|_{L^2(0,T)}&\left\| t^{-\frac12}\exp\left(-\frac{i\xi LT}{t}-\left(\xi^2T^2-\frac{L^2}{4}\right)\left(\frac1t-\frac1T\right)\right)\right\|_{L^2(0,T)}\\ &+\sqrt{T}\|\partial_x u(t,-L)\|_{L^2(0,T)}\left\| t^{-\frac12}\exp\left(+\frac{i\xi LT}{t}-\left(\xi^2T^2-\frac{L^2}{4}\right)\left(\frac1t-\frac1T\right)\right)\right\|_{L^2(0,T)}. \qquad (3\text{-}4)\end{aligned}$$

Writing $\xi \in \mathscr{C}_\alpha$ for $\alpha > L/(2T)$ as $\xi = a + \imath b$ with $(a,b) \in \mathbb{R}^2$ and using the fact that

$$\begin{aligned}\Re\left(\mp\imath\xi LT - \left(\xi^2T^2-\frac{L^2}{4}\right)\right) &\leqslant |b|LT - \left((a^2-b^2)T^2-\frac{L^2}{4}\right)\\ &\leqslant -T^2\left(a^2-\left(|b|+\frac{L}{2T}\right)^2\right)\\ &\leqslant -T^2\left(|a|-\left(|b|+\frac{L}{2T}\right)\right)\left(|a|+|b|+\frac{L}{2T}\right)\leqslant -\frac{LT}{2}\left(\alpha-\frac{L}{2T}\right),\end{aligned}$$

we have the estimates, for $s \in \{-1,1\}$,

$$\begin{aligned}&\left\| t^{-\frac12}\exp\left(s\frac{\imath\xi LT}{t}-\left(\xi^2T^2-\frac{L^2}{4}\right)\left(\frac1t-\frac1T\right)\right)\right\|_{L^2(0,T)}\\ &\qquad\leqslant \left\| t^{-\frac12}\exp\left(|b|L+\left(|b|LT-\left((a^2-b^2)T^2-\frac{L^2}{4}\right)\left(\frac1t-\frac1T\right)\right)\right)\right\|_{L^2(0,T)}\\ &\qquad\leqslant e^{|b|L}\left\| t^{-\frac12}\exp\left(-\frac{LT}{2}\left(\alpha-\frac{L}{2T}\right)\left(\frac1t-\frac1T\right)\right)\right\|_{L^2(0,T)}.\end{aligned}$$

Now, doing the change of variable $\mu = LT\left(\alpha-\frac{L}{2T}\right)\left(\frac1t-\frac1T\right)$, we easily get, for all $\xi \in \mathscr{C}_\alpha$,

$$\begin{aligned}\left\| t^{-\frac12}\exp\left(-\frac{LT}{2}\left(\alpha-\frac{L}{2T}\right)\left(\frac1t-\frac1T\right)\right)\right\|^2_{L^2(0,T)} &= \int_0^\infty e^{-\mu}\frac{d\mu}{\mu+L(\alpha-L/(2T))}\\ &\leqslant \frac{1}{L(\alpha-L/(2T))}.\end{aligned}$$

Combining (3-4) and this last estimate, we easily conclude estimate (2-9).

3B. ***Proof of Proposition 2.3.*** We shall start the proof of Proposition 2.3 by proving the existence of a function $\tilde{\varphi}$ satisfying (2-12), and we will then explain how it can be used to derive the bound in (2-14).

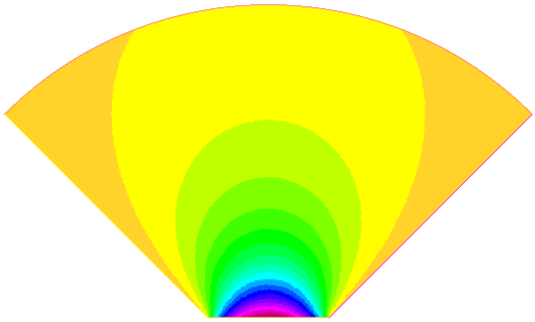

Figure 2. Approximation of $\tilde{\varphi}$ solving (3-5), obtained by a finite-element approach (using FreeFem++ [Hecht 2012]).

Notation. In the following arguments, to avoid ambiguities, we will write differently complex sets and their identification as a part of $\mathbb{R}^2$; for instance we write $\mathscr{O}_1 = \{\xi \in \mathbb{C} : |\Re(\xi)| < |\Im(\xi)| + 1\}$ and $\widetilde{\mathscr{O}}_1 = \{(a,b) \in \mathbb{R}^2 : |a| < |b| + 1\}$ as in Proposition 2.3. To be consistent with this notation, we will also distinguish functions of the complex variable ξ from the corresponding ones considered as functions of the real variables (a,b) using a tilde notation for the function viewed as depending on real variables, as in (2-13).

3B1. *Existence and uniqueness of a function $\tilde{\varphi}$ satisfying* (2-12). The first remark is that the uniqueness of a function $\tilde{\varphi}$ satisfying (2-12) is rather easy to prove. Indeed, if two functions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ satisfy (2-12), then their difference $\tilde{\varphi}_2 - \tilde{\varphi}_1$ is harmonic in $\mathscr{O}_1$ and vanishes on $\partial\widetilde{\mathscr{O}}_1$ as well as at infinity. Therefore, the minimum and maximum of $\tilde{\varphi}_2 - \tilde{\varphi}_1$ is zero, and $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ coincide.

Thus, we will focus on the existence of a function $\tilde{\varphi}$ as in (2-12). In fact, by uniqueness, we see that necessarily $\tilde{\varphi}(a,b) = \tilde{\varphi}(a,|b|)$ for all $(a,b) \in \mathscr{O}_1$. We will thus only look for a solution $\tilde{\varphi}$ in $\widetilde{\mathscr{O}}_1^+ = \widetilde{\mathscr{O}}_1 \cap (\mathbb{R} \times \mathbb{R}_+^*)$ of the problem

$$\begin{cases} \Delta\tilde{\varphi} = 0 & \text{in } \widetilde{\mathscr{O}}_1^+, \\ \tilde{\varphi} = 0 & \text{on } \partial\widetilde{\mathscr{O}}_1^+ \setminus (-1,1), \\ \partial_b\tilde{\varphi}(a,0) = -1 & \text{for } a \in (-1,1), \end{cases} \tag{3-5}$$

with the condition at infinity

$$\lim_{b\to\infty} \sup_{a\in(-|b|-1,|b|+1)} |\tilde{\varphi}(a,b)| = 0. \tag{3-6}$$

Let us introduce

$$\begin{aligned} \Gamma_\ell &= \{\xi \in \mathbb{C} : \Im(\xi) > 0 \text{ and } -\Re(\xi) = 1 + \Im(\xi)\}, \\ \Gamma_r &:= \{\xi \in \mathbb{C} : \Im(\xi) > 0 \text{ and } \Re(\xi) = 1 + \Im(\xi)\}, \\ \Gamma_b &:= \{\xi \in \mathbb{C} : (\Re(\xi), \Im(\xi)) \in [-1,1] \times \{0\}\}, \end{aligned}$$

the three components of the boundary of $\mathscr{O}_1^+ = \mathscr{O}_1 \cap \{\Im(\xi) > 0\}$.

Our goal is to show the existence of a function $\tilde{\varphi}$ satisfying (3-5). In order to do so, we will rely on two Schwarz–Christoffel conformal mappings [Henrici 1974, Chapter 5.12].

The first one, $F_{\frac{3}{4}}$, is defined for all $\zeta \in \mathbb{C}^+ = \{\zeta \in \mathbb{C} : \Im(\zeta) \geqslant 0\}$ by

$$F_{\frac{3}{4}}(\zeta) = \frac{2}{K_{\frac{3}{4}}} \int_{-1}^{\zeta} (1-z^2)^{-\frac{1}{4}}\, dz - 1, \quad \text{with } K_{\frac{3}{4}} = \int_{-1}^{1} (1-x^2)^{-\frac{1}{4}}\, dx = \sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)},$$

where the path integration is arbitrary in $\mathbb{C}^+$.

The map $F_{\frac{3}{4}}$ conformally maps $\mathbb{C}^+$ into $\overline{\mathscr{O}}_1^+$, and satisfies the properties

$$F_{\frac{3}{4}}(-1) = -1, \quad F_{\frac{3}{4}}(0) = 0, \quad F_{\frac{3}{4}}(1) = 1,$$

and

$$F_{\frac{3}{4}}((-\infty,-1)) = \Gamma_\ell, \quad F_{\frac{3}{4}}((-1,1)) = \Gamma_b, \quad F_{\frac{3}{4}}((1,\infty)) = \Gamma_r, \quad F_{\frac{3}{4}}(\imath\mathbb{R}^+) = \imath\mathbb{R}^+.$$

The second conformal mapping we will use is defined, for any $\zeta \in \mathbb{C}^+$, by

$$F_{\frac{1}{2}}(\zeta) = \frac{2}{\pi}\arcsin(\zeta) = \frac{2}{\pi}\int_{-1}^{\zeta} (1-z^2)^{-\frac{1}{2}}\, dz - 1,$$

which conformally maps $\mathbb{C}^+$ into the closure of the half strip $\mathscr{S}_1^+ = \{\Xi = A + \imath B : A \in (-1,1),\ B > 0\}$ with the properties

$$F_{\frac{1}{2}}(-1) = -1, \quad F_{\frac{1}{2}}(0) = 0, \quad F_{\frac{1}{2}}(1) = 1,$$

and

$$F_{\frac{1}{2}}((-\infty,-1]) = -1 + \imath\mathbb{R}^+, \quad F_{\frac{1}{2}}((-1,1)) = (-1,1), \quad F_{\frac{1}{2}}([1,\infty)) = 1 + \imath\mathbb{R}^+, \quad F_{\frac{1}{2}}(\imath\mathbb{R}^+) = \imath\mathbb{R}^+.$$

Finally, we define the conformal mapping

$$F = F_{\frac{1}{2}} \circ F_{\frac{3}{4}}^{-1},$$

which maps $\mathscr{O}_1^+$ into $\mathscr{S}_1^+$.

For any $\xi = a + \imath b \in \mathscr{O}_1^+$, we define $\Xi = A + \imath B = F(\xi)$. Using a standard computation from conformal transplantation [Henrici 1974, Chapter 5.6], we see that $\tilde{\varphi}$ solves (3-5) in $\widetilde{\mathscr{O}}_1^+$ if and only if $\widetilde{\Phi}$ given by $\widetilde{\Phi}(A,B) = \tilde{\varphi}(a,b)$ for $A + \imath B = F(a + \imath b)$ solves the following problem posed in the half-strip $\widetilde{\mathscr{S}}_1^+$:

$$\begin{cases} \Delta_{A,B}\widetilde{\Phi} = 0 & \text{for } A \in (-1,1),\ B > 0, \\ \widetilde{\Phi}(-1,B) = \widetilde{\Phi}(1,B) = 0 & \text{for } B > 0, \\ \partial_B \widetilde{\Phi}(A,0) = -\dfrac{\pi}{K_{\frac{3}{4}}}\sqrt{\cos\left(\frac{\pi}{2}A\right)} & \text{for } A \in (-1,1). \end{cases} \tag{3-7}$$

If the first two equations are standard, the last one deserves additional details. In fact, it comes from the identity [Henrici 1974, Theorem 5.6a]

$$\mathrm{grd}_\xi\, \varphi(\xi) = \mathrm{grd}_\Xi\, \Phi(F(\xi))\overline{F'(\xi)}, \tag{3-8}$$

applied to $\xi = a \in (-1,1)$, (implying $F(\xi) = A \in (-1,1)$), where grd is the complex gradient: for $\xi = a + \imath b$, $\mathrm{grd}_\xi\, \varphi(\xi) = \partial_a \tilde{\varphi}(a,b) + \imath\, \partial_b \tilde{\varphi}(a,b)$ and for $\Xi = A + \imath B$, $\mathrm{grd}_\Xi\, \Phi(\Xi) = \partial_A \widetilde{\Phi}(A,B) + \imath\, \partial_B \widetilde{\Phi}(A,B)$.

We therefore have to compute $F'(\xi) = (F_{\frac{1}{2}} \circ F_{\frac{3}{4}}^{-1})'(\xi) = F'_{\frac{1}{2}}(F_{\frac{3}{4}}^{-1}(\xi))(F_{\frac{3}{4}}^{-1})'(\xi)$. To do so, let us define $\zeta = F_{\frac{3}{4}}^{-1}(\xi) \in \mathbb{C}^+$. By definition,

$$F'_{\frac{1}{2}}(F_{\frac{3}{4}}^{-1}(\xi)) = F'_{\frac{1}{2}}(\zeta) = \frac{2}{\pi}\frac{1}{\sqrt{1-\zeta^2}},$$

whereas

$$(F_{\frac{3}{4}}^{-1})'(\xi) = (F_{\frac{3}{4}}^{-1})'(F_{\frac{3}{4}}(\zeta)) = \frac{1}{F'_{\frac{3}{4}}(\zeta)} = \frac{K_{\frac{3}{4}}}{2}\sqrt[4]{1-\zeta^2}.$$

Therefore,

$$F'(\xi) = \frac{K_{\frac{3}{4}}}{\pi}\frac{1}{\sqrt[4]{1-\zeta^2}},$$

with $\zeta = F_{\frac{3}{4}}^{-1}(\xi)$. In particular, for $\xi = a \in (-1,1)$, we have $\zeta \in (-1,1)$ and therefore $F'(\xi) \in \mathbb{R}$ and

$$\partial_B \tilde{\Phi}(A,0) = \partial_b \tilde{\varphi}(a,0)\frac{1}{F'(a)} = -\frac{\pi}{K_{\frac{3}{4}}}\sqrt[4]{1-\zeta^2}, \quad \text{with } \zeta = F_{\frac{3}{4}}^{-1}(a).$$

To conclude, we just note that $\zeta = F_{\frac{1}{2}}^{-1}(A)$ if and only if $\zeta = \sin(A\pi/2)$, and the third identity in (3-7) follows.

Problem (3-7) has the advantage of being explicitly solvable. Indeed, as $\widetilde{\Phi}$ is harmonic in $(-1,1) \times (0,\infty)$, and satisfies $\widetilde{\Phi}(-1,B) = \widetilde{\Phi}(1,B) = 0$ for all $B > 0$, it necessarily has the decomposition

$$\widetilde{\Phi}(A,B) = \sum_{k\geqslant 1}(\alpha_k e^{-k\frac{\pi}{2}B} + a_k e^{k\frac{\pi}{2}B})\sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr), \quad (A,B) \in \widetilde{\mathscr{S}}_1^+.$$

Recalling condition (3-6) on $\tilde{\varphi}$, we wish to have $\widetilde{\Phi}$ going to zero as $B \to \infty$. We thus choose $a_k = 0$ for all $k \geqslant 1$, so that $\widetilde{\Phi}$ can be written as

$$\widetilde{\Phi}(A,B) = \sum_{k\geqslant 1}\alpha_k e^{-k\frac{\pi}{2}B}\sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr), \quad (A,B) \in \widetilde{\mathscr{S}}_1^+.$$

But the boundary condition on $B = 0$ is equivalent to

$$\frac{\pi}{2}\sum_{k\geqslant 1}k\alpha_k\sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr) = \frac{\pi}{K_{\frac{3}{4}}}\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)},$$

which explicitly yields the coefficients α_k:

$$\text{for all } k \in \mathbb{N}, \quad \alpha_k = \frac{2}{k}\frac{1}{K_{\frac{3}{4}}}\int_{-1}^{1}\sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr)\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)}\,dA.$$

As $\sqrt{\cos(A\pi/2)}$ is an even function and $\sin(k\pi(A+1)/2)$ is an odd function for all even k, we have $\alpha_k = 0$ for all even k. On the other hand, we have for any $n \in \mathbb{N}$, see [Gradshteyn and Ryzhik 2007, equation 3.631.9],

$$\begin{aligned}\int_{-1}^{1}\sin\bigl((2n+1)\tfrac{\pi}{2}(A+1)\bigr)\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)}\,dA &= (-1)^n\int_{-1}^{1}\cos\bigl((2n+1)\tfrac{\pi}{2}A\bigr)\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)}\,dA \\ &= (-1)^n\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos((2n+1)t)\sqrt{\cos(t)}\,dt = \frac{1}{2\sqrt{\pi}}\frac{\Gamma\bigl(n+\frac{1}{4}\bigr)}{\Gamma\bigl(n+\frac{7}{4}\bigr)},\end{aligned}$$

where $\Gamma(\cdot)$ stands for the Gamma function, so in the end we obtain

$$\alpha_{2n+1} = \frac{1}{\pi}\frac{1}{2n+1}\frac{\Gamma\big(\frac{5}{4}\big)}{\Gamma\big(\frac{3}{4}\big)}\frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)},$$

which can be slightly simplified using that $\Gamma\big(\frac{5}{4}\big) = \Gamma\big(\frac{1}{4}\big)/4$ and $\Gamma\big(\frac{3}{4}\big) = \sqrt{2}\pi/\Gamma\big(\frac{1}{4}\big)$, giving

$$\alpha_{2n+1} = \frac{\Gamma\big(\frac{1}{4}\big)^2}{4\sqrt{2}\pi^2}\frac{1}{(2n+1)}\frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)}.$$

So finally, we have

$$\widetilde{\Phi}(A,B) = \frac{\Gamma\big(\frac{1}{4}\big)^2}{4\sqrt{2}\pi^2}\sum_{n\in\mathbb{N}}\frac{1}{(2n+1)}\frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)}e^{-(2n+1)\frac{\pi}{2}B}\sin\big((2n+1)\tfrac{\pi}{2}(A+1)\big), \quad (A,B)\in\mathscr{S}_1^+, \tag{3-9}$$

and

$$\widetilde{\Phi}(0,0) = \frac{\Gamma\big(\frac{1}{4}\big)^2}{4\sqrt{2}\pi^2}\sum_{n\in\mathbb{N}}\frac{(-1)^n}{(2n+1)}\frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)}. \tag{3-10}$$

Note that, according to [Lebedev 1972, (1.4.25)],

$$\frac{1}{2n+1}\frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)} \underset{n\to\infty}{\simeq} \frac{1}{2n^{\frac{5}{2}}};$$

hence the above series are well-defined. In particular, the identity (3-9) can be understood pointwise and $\widetilde{\Phi}(\cdot\,,B)$ goes to zero as $B\to\infty$:

$$\sup_{A\in(-1,1)}\{|\widetilde{\Phi}(A,B)| + |\partial_A\widetilde{\Phi}(A,B)|\} \leqslant C\exp\Big(-\frac{\pi B}{2}\Big), \quad B\geqslant 0. \tag{3-11}$$

Let us also note that, because $\widetilde{\Phi}(0,0)$ is defined through a converging alternating series, we have

$$\widetilde{\Phi}(0,0) < \frac{\Gamma\big(\frac{1}{4}\big)^2}{4\sqrt{2}\pi^2}\sum_{n=0}^{2}\frac{(-1)^n}{(2n+1)}\frac{\Gamma\big(n+\frac{1}{4}\big)}{\Gamma\big(n+\frac{7}{4}\big)} < \tfrac{9}{10}.$$

Computing the 100th partial sum of the series using Octave [Eaton et al. 2014], we obtain

$$\widetilde{\Phi}(0,0) \sim 0.893204.$$

A different expression for $\widetilde{\Phi}(0,0)$ is

$$\widetilde{\Phi}(0,0) = \frac{2}{\pi}\frac{\int_0^{\frac{\pi}{2}}\ln\big(\cot\big(\frac{t}{2}\big)\big)\sqrt{\cos(t)}\,dt}{\int_0^{\frac{\pi}{2}}\sqrt{\cos(t)}\,dt}, \tag{3-12}$$

which easily comes from the equality $\widetilde{\Phi}(0,0) = \sum_{n\in\mathbb{N}}(-1)^n\alpha_{2n+1}$, the fact that

$$\alpha_{2n+1} = (-1)^n\frac{8}{(2n+1)\pi}\frac{1}{K_{\frac{3}{4}}}\int_0^{\frac{\pi}{2}}\cos((2n+1)t)\sqrt{\cos(t)}\,dt,$$

the definition of $K_{\frac{3}{4}}$ and the identity, see [Gradshteyn and Ryzhik 2007, identity 1.442.2 p. 46],

$$\sum_{n\in\mathbb{N}} \frac{\cos((2n+1)t)}{2n+1} = \tfrac{1}{2}\ln\big(\cot\big(\tfrac{t}{2}\big)\big).$$

Note in particular that under the form (3-12), one immediately checks that

$$\widetilde{\Phi}(0,0) > 0. \tag{3-13}$$

In agreement with Figure 2, we then show that the maximum of $\widetilde{\Phi}$ is attained at $(A, B) = (0,0)$. We first note that the function $\widetilde{\Phi}$ given by (3-9) is positive in the strip $\widetilde{\mathscr{S}}_1^+$. Indeed, since $\widetilde{\Phi}$ is harmonic in the half strip $\widetilde{\mathscr{S}}_1^+$ and is not constant, its minimum is attained at the boundary $\widetilde{\mathscr{S}}_1^+$ or at infinity [Gilbarg and Trudinger 1998, Lemma 3.4, Theorem 3.5]. The boundary conditions on $\partial\widetilde{\mathscr{S}}_1^+$ and the behavior of $\widetilde{\Phi}$ as $B \to \infty$ in (3-11) implies that the minimum value of $\widetilde{\Phi}$ is 0 and is attained on the lateral boundaries $\{-1,1\}\times\mathbb{R}_+$ of the half strip. Consequently, the function $\widetilde{\Phi}$ is positive in $\widetilde{\mathscr{S}}_1^+$, and its minimal value is 0.

Besides, as $\widetilde{\Phi}$ vanishes on the lateral boundaries $\{-1,1\}\times\mathbb{R}^+$ of the half strip, $\partial_A\widetilde{\Phi}(1,\cdot)$ is strictly negative by the Hopf maximum principle [Protter and Weinberger 1984, Chapter 2, Theorem 7]. We then consider the function $\widetilde{\Phi}_A = \partial_A\widetilde{\Phi}$. Formula (3-9) easily yields that $\widetilde{\Phi}_A(0, B) = 0$ for $B > 0$, so that $\widetilde{\Phi}_A$ satisfies

$$\begin{cases} \Delta\widetilde{\Phi}_A = 0 & \text{in } \widetilde{\mathscr{S}}_1^+ \cap \{A > 0\}, \\ \widetilde{\Phi}_A(0, B) = 0 & \text{for } B > 0, \\ \widetilde{\Phi}_A(1, B) < 0 & \text{for } B > 0, \\ \partial_B\widetilde{\Phi}_A(A, 0) \geqslant 0 & \text{for } A \in (0,1), \\ \lim_{|B|\to\infty} \sup_{A\in(0,1)} |\widetilde{\Phi}_A(A, B)| = 0. \end{cases}$$

It easily follows that the maximum of $\widetilde{\Phi}_A$ is necessarily nonpositive in $\widetilde{\mathscr{S}}_1^+ \cap \{A > 0\}$ by the application of the maximum principle.

Finally, as $\widetilde{\Phi}$ is harmonic in the half-strip $\widetilde{\mathscr{S}}_1^+$ and is strictly positive in $(0,0)$, see (3-13), the maximum of $\widetilde{\Phi}$ on the half strip $\widetilde{\mathscr{S}}_1^+$ is necessarily attained on the boundary of the half-strip or at infinity, and therefore on $(-1,1)\times\{0\}$ according to the boundary conditions satisfied by $\widetilde{\Phi}$ in (3-7) and the conditions (3-11) as $B \to \infty$. Now, $\partial_A\widetilde{\Phi}$ is nonpositive in $\widetilde{\mathscr{S}}_1^+ \cap \{A > 0\}$ and $\widetilde{\Phi}(A, B) = \widetilde{\Phi}(|A|, B)$ in the half-strip $\widetilde{\mathscr{S}}_1^+$ according to (3-9), so the maximum of $\widetilde{\Phi}$ is necessarily attained in $(A, B) = (0,0)$.

We then come back to the problem (3-5)–(3-6) and check that the function $\tilde{\varphi}$ given by

$$\tilde{\varphi}(a,b) = \widetilde{\Phi}(A, B) \quad \text{for } A + \imath B = F(a + \imath b), \ \ (a,b) \in \widetilde{\mathscr{O}}_1^+, \tag{3-14}$$

with $\widetilde{\Phi}$ as in (3-9), satisfies (3-5)–(3-6).

By construction, $\tilde{\varphi}$ automatically satisfies (3-5) and its maximum is attained in $(a,b) = (0,0)$ and takes value $\tilde{\varphi}(0,0) = \widetilde{\Phi}(0,0)$. We thus only have to check the condition (3-6). In order to do that, let us introduce the real functions $\widetilde{A} = \widetilde{A}(a,b)$ and $\widetilde{B} = \widetilde{B}(a,b)$ given for $(a,b) \in \widetilde{\mathscr{O}}_1^+$ by

$$F(a + \imath b) = \widetilde{A}(a,b) + \imath\widetilde{B}(a,b), \tag{3-15}$$

and let us check that

$$\lim_{b\to\infty} \inf_{|a|<b+1} \widetilde{B}(a,b) = +\infty. \tag{3-16}$$

Indeed, if it were not the case, we could find real sequences $(a_n, b_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} b_n = +\infty, \qquad \text{for all } n\in\mathbb{N},\ \ |a_n| \leqslant b_n+1, \qquad \text{and} \qquad \sup_n \widetilde{B}(a_n,b_n) < \infty. \tag{3-17}$$

Then, if we set $\zeta_n = F_{\frac{3}{4}}^{-1}(a_n + \imath b_n)$, by construction,

$$F_{\frac{1}{2}}(\zeta_n) = \widetilde{A}(a_n,b_n) + \imath \widetilde{B}(a_n,b_n).$$

Therefore, according to the definition of $F_{\frac{1}{2}}$,

$$\zeta_n = \sin\bigl(\tfrac{\pi}{2}(\widetilde{A}(a_n,b_n) + \imath\widetilde{B}(a_n,b_n))\bigr),$$

so that the sequence (ζ_n) is uniformly bounded in $\mathbb{C}$ as $n\to\infty$. Then the sequence (a_n, b_n) is given by $a_n + \imath b_n = F_{\frac{3}{4}}(\zeta_n)$. But $F_{\frac{3}{4}}$ maps bounded sets of $\mathbb{C}$ into bounded sets of $\mathbb{C}$, so this is in contradiction with (3-17), and the property (3-16) holds.

We can thus use (3-11) to get that for all $b\geqslant 0$,

$$\sup_{|a|<b+1} \{|\tilde{\varphi}(a,b)|\} \leqslant C\exp\Bigl(-\frac{\pi}{2}\inf_{|a|<b+1}\widetilde{B}(a,b)\Bigr),$$

which, according to (3-16), implies (3-6).

Remark 3.1. Another approach to obtain information on $\tilde{\varphi}$, the solution of (3-5), is through integral equations. More precisely, for $((a,b),(a_0,b_0)) \in (\widetilde{\mathscr{O}}_1^+)^2$, we define $\mathcal{G}$ as

$$\widetilde{\mathcal{G}}(a,b,a_0,b_0) = \frac{1}{4\pi}\ln\left(\frac{\bigl((a-a_0)^2+(b-b_0)^2\bigr)\bigl((a+a_0)^2+(b+b_0+2)^2\bigr)}{\bigl((a+b_0+1)^2+(b+a_0+1)^2\bigr)\bigl((a-b_0-1)^2+(a_0-b-1)^2\bigr)}\right).$$

It is readily verified that for any $(a_0,b_0)\in\widetilde{\mathscr{O}}_1^+$, $\widetilde{\mathcal{G}}(\cdot,\cdot,a_0,b_0)$ satisfies

$$\begin{cases} \Delta_{a,b}\widetilde{\mathcal{G}}(\cdot,\cdot,a_0,b_0) = \delta_{(a_0,b_0)} & \text{in } \widetilde{\mathscr{O}}_1^+, \\ \widetilde{\mathcal{G}}(a,b,a_0,b_0) = 0 & \text{for } (a,b) \text{ such that } |a| = |b|+1. \end{cases}$$

Indeed, this comes from the fact that $\widetilde{\mathcal{G}}$ is the suitable combination of the fundamental solution of the Laplace operator in the sectors $\{(a,b)\in\mathbb{R}^2 : b = |a|-1\}$ and $\{(a,b)\in\mathbb{R}^2 : b = 1-|a|\}$.

Then, standard computations show that $\tilde{\varphi}$ is a solution of (3-5) if and only if it satisfies the integral equation

$$\tilde{\varphi}(a_0,b_0) = -\int_{-1}^{1}\partial_b\widetilde{\mathcal{G}}(a,0,a_0,b_0)\,\tilde{\varphi}(a,0)\,da + \int_{-1}^{1}\widetilde{\mathcal{G}}(a,0,a_0,b_0)\,da \quad \text{for all } (a_0,b_0)\in\widetilde{\mathscr{O}}_1^+. \tag{3-18}$$

We then introduce $\widetilde{\mathscr{G}}$ defined by

$$\widetilde{\mathscr{G}}(a,a_0,b_0) = -\partial_b\widetilde{\mathcal{G}}(a,0,a_0,b_0) - \frac{1}{2\pi}\frac{b_0}{b_0^2+(a-a_0)^2}.$$

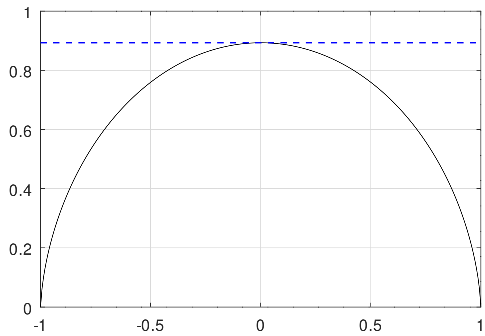

Figure 3. The solid line shows $\tilde{\varphi}(a_0, 0)$ for $a_0 \in (-1, 1)$, obtained by discretization of (3-19). The dashed line shows $\widetilde{\Phi}(0, 0) = \tilde{\varphi}(0, 0)$.

It is easily seen that for any $a_0 \in (-1, 1)$,

$$\lim_{b_0 \to 0} \int_{-1}^{1} \widetilde{\mathscr{G}}(a, a_0, b_0)\, \tilde{\varphi}(a, 0)\, da = \int_{-1}^{1} \widetilde{\mathscr{G}}(a, a_0, 0)\, \tilde{\varphi}(a, 0)\, da,$$

$$\lim_{b_0 \to 0} \int_{-1}^{1} \widetilde{\mathcal{G}}(a, 0, a_0, b_0)\, da = \int_{-1}^{1} \widetilde{\mathcal{G}}(a, 0, a_0, 0)\, da,$$

whereas

$$\lim_{b_0 \to 0} \frac{1}{2\pi} \int_{-1}^{1} \frac{b_0}{b_0^2 + (a - a_0)^2}\, \tilde{\varphi}(a, 0)\, da = \tfrac{1}{2} \tilde{\varphi}(a_0, 0).$$

Therefore, choosing $a_0 \in (-1, 1)$ and taking the limit $b_0 \to 0$ in (3-18) leads to the integral equation

$$\tfrac{1}{2} \tilde{\varphi}(a_0, 0) = \int_{-1}^{1} \widetilde{\mathscr{G}}(a, a_0, 0)\, \tilde{\varphi}(a, 0)\, da + \int_{-1}^{1} \widetilde{\mathcal{G}}(a, 0, a_0, 0)\, da. \tag{3-19}$$

Discretizing (3-19), we can obtain a good approximation of $\tilde{\varphi}(a_0, 0)$ for $a_0 \in (-1, 1)$ (see Figure 3).

3B2. *Phragmén–Lindelöf principle.* With $\tilde{\varphi}$ as in (2-12), the function $(a, b) \mapsto \tilde{\varphi}(a, b) + |b|$ is harmonic in $\widetilde{\mathscr{O}}_1$, and it is therefore the real part of some holomorphic function ϕ in $\mathscr{O}_1$:

$$\text{for all } (a, b) \in \widetilde{\mathscr{O}}_1, \quad \Re(\phi(a + \imath b)) = \tilde{\varphi}(a, b) + |b|,$$

or, equivalently, for all $\xi \in \mathscr{O}_1$, $\Re(\phi(\xi)) = \varphi(\xi) + |\Im(\xi)|$.

For each $\alpha_* > \alpha$, we consider the function g_{α_*} defined for $\xi \in \mathscr{O}_\alpha$ by

$$g_{\alpha_*}(\xi) = f(\xi) \exp\left(-L\alpha_* \phi\left(\frac{\xi}{\alpha}\right)\right). \tag{3-20}$$

By construction, g_{α_*} is holomorphic in $\mathscr{O}_\alpha$ and satisfies

$$\text{for all } \xi \in \partial\mathscr{O}_\alpha, \quad |g_{\alpha_*}(\xi)| \leqslant C_0, \quad \text{and} \quad \lim_{|\Im(\xi)| \to \infty} \Big(\sup_{|\Re(\xi)| < |\Im(\xi)| + \alpha} |g_{\alpha_*}(\xi)| \Big) = 0.$$

Therefore, g_{α_*} attains its maximum on $\partial\mathscr{O}_\alpha$, so that

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp\left(\frac{\alpha_*}{\alpha}|\Im(\xi)|L\right)\exp\left(L\alpha_*\varphi\left(\frac{\xi}{\alpha}\right)\right).$$

Taking the limit $\alpha_* \to \alpha$, we immediately have

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp(|\Im(\xi)|L)\exp\left(L\alpha\varphi\left(\frac{\xi}{\alpha}\right)\right), \tag{3-21}$$

that is, (2-14).

Remark 3.2. Let us remark that we can obtain from the above proof an explicit formula for f. Indeed, for $\alpha_* > \alpha > L/(2T)$, we can use the Cauchy formula for the function g_{α_*} in (3-20) on the contour given by

$$\gamma_{\alpha,R} = \partial(\mathscr{O}_\alpha \cap \{\Im(\xi) < R\}) \quad (\text{with } R > 0)$$

oriented in a counterclockwise manner, which yields, for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$g_{\alpha_*}(\xi) = \frac{1}{2\imath\pi}\int_{\gamma_{\alpha,R}} \frac{g_{\alpha_*}(\zeta)}{\zeta - \xi}\,d\zeta.$$

Now, due to the decay of g_{α_*} at infinity, one can pass to the limit in the above formula as $R \to \infty$: for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$g_{\alpha_*}(\xi) = \frac{1}{2\imath\pi}\int_{\gamma_\alpha} \frac{g_{\alpha_*}(\zeta)}{\zeta - \xi}\,d\zeta,$$

where γ_α is the union of the two connected components of $\partial\mathscr{O}_\alpha$ oriented counterclockwise. Recalling the definition of g_{α_*}, we end up with the following formula: for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$f(\xi) = \frac{1}{2\imath\pi}\int_{\gamma_\alpha} e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}\frac{f(\zeta)}{\zeta - \xi}\,d\zeta. \tag{3-22}$$

4. Further comments

4A. *Higher-dimensional settings.* The method developed above applies also to the cost of observability of the heat equation in multidimensional balls. More precisely, we consider the following heat equation, set in the ball of radius $L > 0$ of $\mathbb{R}^d$ $(d \geqslant 1)$, denoted by $\mathcal{B}_L$ in the following, and in the time interval $(0, T)$:

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } (0,T)\times\mathcal{B}_L,\\ u(t,x) = 0 & \text{in } (0,T)\times\partial\mathcal{B}_L,\\ u(0,x) = u_0(x) & \text{in } \mathcal{B}_L, \end{cases} \tag{4-1}$$

where the initial datum u_0 belongs to $H_0^1(\mathcal{B}_L)$. In that setting, we have the following result:

Theorem 4.1. *Setting K_0 as in Theorem 1.1, for any $K > K_0$, there exists a constant $C > 0$ such that for all $T \in (0,1]$, for all solutions u of (4-1) with initial datum $u_0 \in H_0^1(\mathcal{B}_L)$,*

$$\left\| u(T)\exp\left(\frac{|x|^2}{4T}\right)\right\|_{L^2(\mathcal{B}_L)} \leqslant C\exp\left(\frac{KL^2}{T}\right)\|\partial_\nu u\|_{L^2((0,T)\times\partial\mathcal{B}_L)}. \tag{4-2}$$

Here and in the following, $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$. The proof of Theorem 4.1 closely follows the one of Theorem 1.1; therefore we only sketch its proof, explaining the main differences with the proof of Theorem 1.1.

Sketch of the proof of Theorem 4.1. We start by considering a smooth solution u of (4-1), and define

$$z(t,x) = u(t,x)\exp\left(\frac{|x|^2 - L^2}{4t}\right), \quad (t,x) \in (0,T) \times \mathcal{B}_L,$$

which satisfies

$$\begin{cases} \partial_t z + \dfrac{x}{t}\cdot\nabla_x z + \dfrac{d}{2t}z - \Delta_x z - \dfrac{L^2}{4t^2}z = 0 & \text{in } (0,\infty)\times\mathcal{B}_L, \\ z(t,x) = 0 & \text{in } (0,T)\times\partial\mathcal{B}_L, \\ z(0,x) = 0 & \text{in } \mathcal{B}_L, \end{cases}$$

Proposition A.1 with $\Omega = \mathcal{B}_L$ and $g \equiv 0$ implies directly the following estimate for z:

$$\int_{\mathcal{B}_L} |\nabla_x z(T,x)|^2\,dx - \frac{L^2}{4T^2}\int_{\mathcal{B}_L} |z(T,x)|^2\,dx \leqslant \frac{L}{T^2}\int_0^T\int_{\partial\mathcal{B}_L} t|\nabla_x z(t,x)\cdot\nu|^2\,ds(x)\,ds. \tag{4-3}$$

We define w as the extension of z by 0 outside $\mathcal{B}_L$: w satisfies the equations

$$\begin{cases} \partial_t w + \dfrac{x}{t}\cdot\nabla_x w + \dfrac{d}{2t}w - \Delta_x w - \dfrac{L^2}{4t}w = \nabla_x u(t,x)\cdot\nu\delta_{\partial\mathcal{B}_L} & \text{in } (0,\infty)\times\mathbb{R}^d, \\ w(0,x) = 0, & x\in\mathbb{R}^d. \end{cases}$$

Thus, its Fourier transform, defined for $(t,\xi)\in(0,T)\times\mathbb{C}^d$ by

$$\hat{w}(t,\xi) = \int_{\mathbb{R}^d} w(t,x)e^{-\imath\xi\cdot x}\,dx$$

satisfies

$$\begin{cases} \partial_t\hat{w} - \dfrac{\xi}{t}\cdot\nabla_\xi\hat{w} - \dfrac{d}{2t}\hat{w} + \xi^2\hat{w} - \dfrac{L^2}{4t^2}\hat{w} = \displaystyle\int_{\partial\mathcal{B}_L}\nabla_x u(t,x)\cdot\nu e^{-\imath\xi\cdot x}\,ds(x), & (t,\xi)\in(0,\infty)\times\mathbb{R}^d, \\ \hat{w}(0,\xi) = 0, & \xi\in\mathbb{R}^d. \end{cases} \tag{4-4}$$

As in the one-dimensional case, (4-3) gives a high-frequency ($|\xi| > L/(2T)$) L^2-estimate of $w(T,\cdot)$ depending on the observation and the low-frequency ($|\xi| \leqslant L/(2T)$) L^2-norm of $w(T,\cdot)$, on which we focus from now. To do so, much as in Section 3A, we solve the transport equation (4-4), and obtain, for $\xi_0\in\mathbb{R}^d$ such that $|\xi_0| > L/(2T)$,

$$\hat{w}(T,\xi_0) = \int_0^T\left(\frac{T}{t}\right)^{\frac{d}{2}}\int_{\partial\mathcal{B}_L}\nabla_x u(t,x)\cdot\nu e^{-\imath\frac{x\cdot\xi_0 T}{t} - (\xi_0^2T^2 - \frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,ds(x)\,dt, \tag{4-5}$$

with $\xi_0^2 = \xi_0\cdot\xi_0$.

Once here, we consider $\xi_0 = (\xi_1,\tilde{\xi})$, with $\tilde{\xi}\in\mathbb{R}^{d-1}$ fixed, and $\xi_1 = a + \imath b$, $a,b\in\mathbb{R}$, and define $f(\xi_1) = \hat{w}(T,\xi_1,\tilde{\xi})$, which is an entire function satisfying (2-11). Besides, with computations similar to those in Section 3A, it is easy to obtain that for all $\alpha > L^2/(2T)$, there exists $C_\alpha(T) > 0$, which may blow up polynomially in T as $T\to 0$ (contrarily to what happens in the one-dimensional setting, the constant $C_\alpha(T)$ may now blow up as $T \to 0$, but only polynomially in T, so that it will not significantly affect the cost of observability in small times in (4-2), which blows up as an exponential of $1/T$ as $T \to 0$), such that for all $\xi_1 \in \mathscr{C}_\alpha$ as in (2-6), we have

$$|f(\xi_1)| \leqslant C_\alpha e^{|\Im(\xi_1)|L} \|\partial_\nu u\|_{L^2((0,T)\times\partial\mathcal{B}_L)}.$$

From that, we end the proof of Theorem 4.1 exactly as in the one-dimensional case, with the use of Proposition 2.3. □

Actually, the method developed above works not only for balls, but also for any bounded domain $\Omega \subset \mathbb{R}^d$. More precisely:

Theorem 4.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$. If we set*

$$L_\Omega = \inf_{x\in\Omega} \sup_{y\in\partial\Omega} |x-y|,$$

and we choose $\bar{x} \in \overline{\Omega}$ such that

$$\sup_{y\in\partial\Omega} |\bar{x}-y| = L_\Omega,$$

then for any $K > K_0$, there exists $C > 0$ such that any smooth solution u of

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4-6}$$

satisfying

$$\left\| u(T)\exp\left(\frac{|x-\bar{x}|^2}{4T}\right)\right\|_{L^2(\Omega)} \leqslant C\exp\left(\frac{KL_\Omega^2}{T}\right)\|\partial_\nu u\|_{L^2((0,T)\times\partial\Omega)}.$$

Note that this is a geometrical setting in which Corollary 1.3 applies but yields a different estimate on the cost of observability. Indeed, when the observation is done on the whole boundary, one easily checks that the choice $S_0 = S_\Omega^+$, where

$$S_\Omega = \sup\{\text{length of segments included in } \Omega\},$$

is suitable for the application of Corollary 1.3. In particular, when Ω is convex, $L_\Omega \leqslant S_\Omega \leqslant 2L_\Omega$ and Theorem 4.2 always yields at least the estimate given by Corollary 1.3 when the observation is done on the whole boundary of Ω, and a better one in general (as in the case of a ball discussed in Theorem 4.1).

Remark 4.3. The above discussion, and Theorem 4.2 in particular, might suggest that the cost of observability in small times is linked only to the maximal distance to the control set. This is not the case, as it is strongly underlined by [Laurent and Léautaud 2018]. There, among other results, an analysis of the observability constant $C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))$ for the heat equation is done when the domain of interest is $\Omega = \mathbb{B}(0,1) \subset \mathbb{R}^2$, the unit ball of the plane, and the observation set is $\mathbb{B}(0,r)$. To be more precise, $C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))$ is the best constant in the following estimate: for any solution u of (4-6) with $\Omega = \mathbb{B}(0,1)$ with initial datum $u_0 \in H_0^1(\Omega)$,

$$\|u(T)\|_{L^2(\Omega)} \leqslant C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))\|u\|_{L^2((0,T)\times\mathbb{B}(0,r)))}.$$

The work [Laurent and Léautaud 2018] then shows the following result: there exist $C > 0$ and $r_0 < 1$ such that for all $r \in (0, r_0)$

$$\liminf_{T\to 0} T \log(C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))) \geqslant C \log(r)^2. \tag{4-7}$$

This shows that the behavior of the cost of observability in small times is in fact strongly linked to the geometry under consideration. Indeed, Theorem 4.2 in fact corresponds to a geometrical setting in which the wave equation is observable in small times, while the result (4-7) proved in [Laurent and Léautaud 2018] focuses on a case in which the geometric control condition for the observability of the wave equation fails due to whispering gallery phenomena.

4B. ***Tensorized equations.*** Another application of our method concerns the cost of observability of the heat equation on a tensorized domain. More precisely, we consider the heat equation set in a tensorized spatial domain $\Omega = \Omega_x \times \Omega_y$, and want to know the cost of observability in small time when the solution is observed on $\partial\Omega_x \times \Omega_y$. Note that the answer is already known: the cost is the same as the one for the heat equation set on Ω_x only, when the observation is done on the whole boundary $\partial\Omega_x$ [Miller 2005, Theorem 1.5]. Our purpose is therefore just to underline that our approach also applies in that context and allows us to retrieve easily this result.

To fix ideas, we focus on the case $\Omega_x = (-L, L)$ (when Ω_x is a multidimensional domain, similar arguments can be developed, under appropriate geometric conditions, by using Theorem 4.2 instead of Theorem 1.1). Hence we are interested in the following heat equation, set in the domain $\Omega = (-L, L) \times \Omega_y$, with $L > 0$ and Ω_y a smooth bounded domain of $\mathbb{R}^{d_y}$, in some time interval $(0, T)$, $T > 0$:

$$\begin{cases} \partial_t u - \partial_x^2 u - \Delta_y u = 0 & \text{for } (t,x,y) \in (0,T) \times (-L,L) \times \Omega_y, \\ u(t,L,y) = u(t,-L,y) = 0 & \text{for } (t,y) \in (0,T) \times \Omega_y, \\ u(t,x,y) = 0 & \text{for } (t,x,y) \in (0,T) \times (-L,L) \times \partial\Omega_y, \\ u(0,x,y) = u_0(x,y) & \text{in } (-L,L) \times \Omega_y. \end{cases} \tag{4-8}$$

As usual, the initial datum u_0 belongs to $H_0^1((-L, L) \times \Omega_y)$. We have the following:

Theorem 4.4. *Setting K_0 as in Theorem 1.1, for any $K > K_0$, there exists a constant $C > 0$ such that for all $T \in (0, 1]$, for all solutions u of* (4-8),

$$\begin{aligned} &\left\| u(T,x,y) \exp\left(\frac{x^2}{4T}\right) \right\|_{L^2((-L,L)\times\Omega_y)} \\ &\qquad \leqslant C \exp\left(\frac{KL^2}{T}\right) \big(\|\partial_x u(t,-L,y)\|_{L^2((0,T)\times\Omega_y)} + \|\partial_x u(t,L,y)\|_{L^2((0,T)\times\Omega_y)} \big). \end{aligned} \tag{4-9}$$

Sketch of the proof of Theorem 4.4. Let us denote by (v_n, λ_n^2) the family of normalized eigenfunctions and eigenvalues of the Dirichlet–Laplace operator set in Ω_y, that is,

$$\begin{cases} -\Delta_y v_n = \lambda_n^2 v_n & \text{in } \Omega_y, \\ v_n = 0 & \text{on } \partial\Omega_y, \\ \|v_n\|_{L^2(\Omega_y)} = 1. \end{cases}$$

Expanding u, a solution of (4-8), on the $L^2(\Omega_y)$ Hilbert basis (v_n), that is,

$$u(t,x,y)=\sum_{n\in\mathbb{N}} u_n(t,x)v_n(y),$$

we see that each u_n solves a one-dimensional heat equation with potential λ_n^2 set in $(0,T)\times(-L,L)$:

$$\begin{cases}\partial_t u_n-\partial_x^2 u_n+\lambda_n^2 u_n=0 & \text{in } (0,T)\times(-L,L),\\ u_n(t,-L)=u_n(t,L)=0 & \text{in } (0,T),\\ u_n(0,x)=u_{n,0}(x) & \text{in } (-L,L),\end{cases} \tag{4-10}$$

with

$$u_{n,0}(x)=\int_\Omega u_0(x,y)\,v_n(y)\,dy.$$

To prove Theorem 4.4, it is sufficient to prove that each u_n satisfies the observability inequality

$$\left\| u_n(T,x)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)} \leqslant C\exp\left(\frac{KL^2}{T}\right)\big(\|\partial_x u_n(t,-L)\|_{L^2(0,T)}+\|\partial_x u_n(t,L)\|_{L^2(0,T)}\big), \tag{4-11}$$

with a constant C independent of n. To do so, we consider $\tilde{u}_n=u_n e^{\lambda_n^2 t}$, which satisfies

$$\begin{cases}\partial_t \tilde{u}_n-\partial_x^2 \tilde{u}_n=0 & \text{in } (0,T)\times(-L,L),\\ \tilde{u}_n(t,-L)=\tilde{u}_n(t,L)=0 & \text{in } (0,T),\\ \tilde{u}_n(0,x)=u_{n,0}(x) & \text{in } (-L,L).\end{cases}$$

Applying Theorem 1.1, we get

$$\left\| \tilde{u}_n(T,x)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)} \leqslant C\exp\left(\frac{KL^2}{T}\right)\big(\|\partial_x \tilde{u}_n(t,-L)\|_{L^2(0,T)}+\|\partial_x \tilde{u}_n(t,L)\|_{L^2(0,T)}\big),$$

which directly gives (4-11) as $e^{\lambda_n^2(t-T)}\leqslant 1$ for all $t\in(0,T)$, and therefore ends the proof. □

4C. ***Observation from one side of the domain: symmetrization argument.*** In this section, we are interested in the cost of observability for the one-dimensional heat equation when observed on one side of the domain. In other words, for $L,T>0$ and $u_0\in H_0^1(0,L)$, we consider the system

$$\begin{cases}\partial_t u-\partial_x^2 u=0 & \text{in } (0,T)\times(0,L),\\ u(t,0)=u(t,L)=0 & \text{in } (0,T),\\ u(0,x)=u_0(x) & \text{in } (0,L).\end{cases} \tag{4-12}$$

We have the following:

Theorem 4.5. *Setting K_0 as in Theorem 1.1, for any $K>K_0$, there exists a constant $C>0$ such that for all $T\in(0,1]$, for all solutions u of* (4-12) *with $u_0\in H_0^1(0,L)$,*

$$\left\| u(T)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(0,L)} \leqslant C\exp\left(\frac{KL^2}{T}\right)\|\partial_x u(t,L)\|_{L^2(0,T)}. \tag{4-13}$$

Proof. The proof is based on a classical symmetrization argument: for u a solution of (4-12), we define

$$u_s(t,x) = \begin{cases} u(t,x) & \text{for } (t,x) \in (0,T) \times (0,L), \\ -u(t,-x) & \text{for } (t,x) \in (0,T) \times (-L,0). \end{cases}$$

It is readily seen that u_s satisfies system (1-1). Therefore, Theorem 1.1 gives

$$\left\| u_s(T) \exp\left(\frac{x^2}{4T}\right) \right\|_{L^2(-L,L)} \leqslant C \exp\left(\frac{KL^2}{T}\right) \big(\|\partial_x u_s(t,-L)\|_{L^2(0,T)} + \|\partial_x u_s(t,L)\|_{L^2(0,T)} \big).$$

The result follows easily, as $\partial_x u_s(t,-L) = \partial_x u_s(t,L) = \partial_x u(t,L)$ for all $t \in (0,T)$. □

4D. *Distributed observations.* One is sometimes interested in distributed observations, in which case the corresponding observability inequality reads

$$\|u(T)\|_{L^2(0,L)} \leqslant C(T,L,a,b) \|u\|_{L^2((0,T)\times(a,b))} \tag{4-14}$$

for smooth solutions u of (4-12), where $a, b \in \mathbb{R}$ are such that $(a,b) \subset (0,L)$ and $a < b$.

We claim the following:

Theorem 4.6. *Let $0 \leqslant a < b \leqslant L$. Setting K_0 as in Theorem 1.1, for any $K > K_0$, there exists a constant $C > 0$ such that for all $T \in (0,1]$, for all solutions u of* (4-12),

$$\|u(T)\|_{L^2(0,L)} \leqslant C \exp\left(\frac{K \min\{a^2, (L-b)^2\}}{T}\right) \|u\|_{L^2(0,T;H^1(a,b))}. \tag{4-15}$$

Proof. As in the proof of Theorem 4.5, we start by symmetrizing the function u, and we call u_s its symmetric extension. We then take $\varepsilon > 0$ small enough to have $a + 2\varepsilon < b$ and we choose an even cut-off function ρ taking value 1 on $(-a-\varepsilon, a+\varepsilon)$ and vanishing for $|x| > a + 2\varepsilon$. Then the function

$$z(t,x) = \begin{cases} \rho(x)\, u_s(t,x) \exp\left(\dfrac{x^2 - (a+2\varepsilon)^2}{4t}\right) & \text{for } |x| < a + 2\varepsilon, \\ 0 & \text{for } |x| > a + 2\varepsilon \end{cases}$$

satisfies, much as in (2-3),

$$\begin{cases} \partial_t z + \dfrac{x}{t} \partial_x z + \dfrac{1}{2t} z - \partial_x^2 z - \dfrac{(a+2\varepsilon)^2}{4t^2} z = g, & (t,x) \in (0,\infty) \times (-a-2\varepsilon, a+2\varepsilon), \\ z(t,-a-2\varepsilon) = z(t,a+2\varepsilon) = 0, & t \in (0,\infty), \\ z(0,x) = 0, & x \in (-a-2\varepsilon, a+2\varepsilon), \end{cases} \tag{4-16}$$

where

$$g(t,x) = \exp\left(\frac{x^2 - (a+2\varepsilon)^2}{4t}\right) (2\partial_x \rho\, \partial_x u(t,x) + \partial_{xx} \rho u(t,x)).$$

One can then follow the approach developed in Section 2 (using Proposition A.1 instead of Theorem 2.1 and the fact that $\partial_x z(t,-a-2\varepsilon) = \partial_x z(t,a+2\varepsilon) = 0$) to show that for all $K_1 > K_0$, there exists C such that for all $T \in (0,1]$,

$$\left\| u_s(T) \exp\left(\frac{x^2}{4T}\right) \right\|_{L^2(-a-2\varepsilon,a+2\varepsilon)} \leqslant C \exp\left(\frac{K_1(a+2\varepsilon)^2}{T}\right) \|g\|_{L^2((0,T)\times(-a-2\varepsilon,a+2\varepsilon))}.$$

Using the definition of g, one easily gets

$$\|u(T)\|_{L^2(0,a+\varepsilon)} \leqslant C \exp\left(\frac{K_1(a+2\varepsilon)^2}{T}\right) \|u\|_{L^2(0,T;H^1(a,a+2\varepsilon))}.$$

Similarly, one can obtain

$$\|u(T)\|_{L^2(b-\varepsilon,L)} \leqslant C \exp\left(\frac{K_1(L-b+2\varepsilon)^2}{T}\right) \|u\|_{L^2(0,T;H^1(b-2\varepsilon,b))}.$$

It is straightforward to show that

$$\|u(T)\|_{L^2(a+\varepsilon,b-\varepsilon)} \leqslant C \|u\|_{L^2(0,T;H^1(a,b))},$$

for instance by looking at $v(t,x) = \eta(t)\, u(t,x)\, \rho_0(x)$, where $\eta = \eta(t)$ is a smooth function of time taking value 0 at $t = 0$ and 1 at $t = T$, and $\rho_0 = \rho_0(x)$ taking value 1 on $(a+\varepsilon, b-\varepsilon)$ and vanishing for $x \notin (a,b)$, and doing energy estimates.

Combining the three above estimates, we easily conclude (4-15) by taking $K_1 \in (K_0, K)$ and $\varepsilon > 0$ small enough. □

Note that the above argument is only based on suitable cut-off arguments. It can therefore be applied as well in multidimensional settings, provided some geometric assumptions compatible with Theorem 4.2 are satisfied, namely if the distributed observation set is a neighborhood of the whole boundary.

4E. ***Related uncertainty principles.*** One key point to obtain Theorem 1.1 is the complex analysis argument developed in Section 3B, based principally on the Schwarz–Christoffel conformal mapping and the Phragmén–Lindelöf principle. It is nevertheless possible to develop a purely *real analysis* argument, but it only allows us to retrieve the cost of observability for the one-dimensional heat equation known since [Tenenbaum and Tucsnak 2007]:

Theorem 4.7. *For all $K > \frac{3}{4}$, there exists a constant $C > 0$ such that for all $T \in (0,1]$, all solutions u of* (1-1) *with initial datum $u_0 \in H^1_0(-L,L)$ satisfy* (1-2).

The proof of Theorem 4.7 is based on the following *uncertainty principle result*.

Proposition 4.8 [Landau and Pollak 1961; Fuchs 1964]. *Let $A, B > 0$. Let $f \in L^2(\mathbb{R})$ be supported in $[-A,A]$ and $\hat{f}$ its Fourier transform. Then*

$$\int_{-B}^{B} |\hat{f}(\xi)|^2 \, d\xi \leqslant \lambda_0 \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi, \tag{4-17}$$

where $\lambda_0 = \lambda_0(AB)$ satisfies $0 < \lambda_0 < 1$ and

$$\lambda_0 = 1 - 4\sqrt{\pi}\sqrt{AB}e^{-2AB}(1+\varepsilon_{AB}), \tag{4-18}$$

where $\varepsilon_{AB} \to 0$ as $AB \to \infty$.

Relation (4-17) is a particular case of [Landau and Pollak 1961, Theorem p. 68], whereas the proof of the asymptotic behavior of λ_0 can be found in [Fuchs 1964, Theorem 1, p. 319].

Proof of Theorem 4.7. We start from formula (2-7), which we recall: for any $\xi_0 \in \mathbb{R}$ such that $|\xi_0| > L/(2T)$, we have

$$\hat{w}(T,\xi_0) = -\int_0^T \sqrt{\frac{T}{t}}\partial_x u(t,-L) e^{\imath \frac{L\xi_0 T}{t} - (\xi_0^2 T^2 - \frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,dt \\ + \int_0^T \sqrt{\frac{T}{t}}\partial_x u(t,L) e^{-\imath \frac{L\xi_0 T}{t} - (\xi_0^2 T^2 - \frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,dt.$$

Therefore, we directly obtain, for $\xi_0 \in \mathbb{R}$ with $|\xi_0| > L/(2T)$,

$$|\hat{w}(T,\xi_0)|^2 \leqslant T\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big)\int_0^T e^{-2T^2(\xi_0^2 - \frac{L^2}{4T^2})(\frac{1}{t}-\frac{1}{T})}\,\frac{dt}{t}.$$

For $\eta > 1$, we choose $\xi_0 \in \mathbb{R}$ with $|\xi_0| \geqslant \eta L/(2T)$, which implies

$$\xi_0^2 - \frac{L^2}{4T^2} \geqslant \frac{\eta^2-1}{\eta^2}\xi_0^2$$

and

$$\int_0^T e^{-2T^2(\xi_0^2 - \frac{L^2}{4T^2})(\frac{1}{t}-\frac{1}{T})}\,\frac{dt}{t} \leqslant \int_0^T e^{-2T^2\frac{\eta^2-1}{\eta^2}\xi_0^2(\frac{1}{t}-\frac{1}{T})}\,\frac{dt}{t} \leqslant \frac{\eta^2}{2T(\eta^2-1)\xi_0^2}.$$

Hence we obtain, for $\xi_0 \in \mathbb{R}$ with $|\xi_0| > \eta L/(2T)$,

$$|\hat{w}(T,\xi_0)|^2 \leqslant \frac{\eta^2}{2(\eta^2-1)\xi_0^2}\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big)$$

and

$$\int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \frac{2T\eta}{(\eta^2-1)L}\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big).$$

Now, from (4-17) applied to $f = \hat{w}(T)$ with $A = L$, $B = \eta L/(2T)$ and $\lambda_0 = \lambda_0(\eta L^2/(2T))$, we have

$$\int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 = \int_{|\xi_0|<\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 + \int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \\ \leqslant \lambda_0 \int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 + \int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0,$$

and thus

$$\int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \frac{1}{1-\lambda_0}\int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0.$$

We have therefore obtained

$$\int_{-L}^{L} |w(T,x)|^2\,dx = \int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \frac{1}{1-\lambda_0}\frac{2T\eta}{(\eta^2-1)L}\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big),$$

which implies from Proposition 4.8 and (4-18) the existence of a constant C such that for T small enough

$$\|w(T)\|_{L^2(-L,L)} \leqslant C e^{\eta\frac{L^2}{2T}}\big(\|\partial_x u(t,-L)\|_{L^2(0,T)} + \|\partial_x u(t,L)\|_{L^2(0,T)}\big).$$

The result of Theorem 4.7 follows from the definition of w. □

4F. ***On a possible improvement of Theorem 1.1.*** As we said in the Introduction, we do not know if the estimate on the cost of observability in small times given by Theorem 1.1 is sharp or not. In fact, when looking at the main steps of the proof of Theorem 1.1 given in Section 2, it seems that one step in which our estimates are not sharp may be the one using Phragmén–Lindelöf principles, i.e., Proposition 2.3.

Indeed, introducing the class

$$\mathscr{E}_\alpha = \{f \in \mathrm{Hol}(\mathcal{O}_\alpha) : f(\xi)e^{-|\Im(\xi)|} \in L^\infty(\mathscr{O}_\alpha) \text{ and for all } \xi \in \partial\mathcal{O}_\alpha,\ |f(\xi)| \leqslant e^{|\Im(\xi)|}\},$$

Proposition 2.3 shows that for all $\alpha \in \mathbb{R}_+^*$,

$$\sup_{f \in \mathscr{E}_\alpha} \Big(\sup_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) \leqslant \exp(\alpha\varphi(0)), \tag{4-19}$$

where $\varphi(0)$ is given by (2-15). Besides, this estimate is sharp as we can construct a holomorphic function ϕ in $\mathscr{O}_1$ whose real part coincides with $\varphi(\xi) + |\Im(\xi)|$ given by (2-12)–(2-13) and check that $f_\phi(\xi) = \exp(\alpha\phi(\xi/\alpha))$ belongs to $\mathscr{E}_\alpha$ and saturates the estimate (4-19), so that for all $\alpha \in \mathbb{R}_+^*$,

$$\max_{f \in \mathscr{E}_\alpha} \Big(\max_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) = \exp(\alpha\varphi(0)). \tag{4-20}$$

Now, in our approach (in the case $L = 1$, which can always be assumed by a scaling argument), we apply estimate (4-19) to the function $f = \hat{w}(T,\cdot)/\|\hat{w}(T,\xi)e^{-|\Im(\xi)|}\|_{L^\infty(\mathscr{C}_\alpha)}$, which in fact belongs to a smaller class

$$\mathscr{E}_\alpha^* = \{f \in \mathrm{Hol}(\mathbb{C}) : f(\xi)e^{-|\Im(\xi)|} \in L^\infty(\mathbb{C}) \text{ and for all } \xi \in \mathscr{C}_\alpha, |f(\xi)| \leqslant e^{|\Im(\xi)|}\}.$$

Therefore, our proof requires an estimate on the constant

$$C^*(\alpha) = \sup_{f \in \mathscr{E}_\alpha^*} \Big(\sup_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) \tag{4-21}$$

in the asymptotics $\alpha \to \infty$. It is clear that

$$C^*(\alpha) \leqslant \exp(\alpha\varphi(0)), \tag{4-22}$$

which is precisely the estimate we use, but there is no evidence to support the idea that this estimate gives the good asymptotics as $\alpha \to \infty$.

Let us in particular point out that:

- The function f_ϕ given above to show that estimate (4-19) is sharp does not belong to the class $\mathscr{E}_\alpha^*$.
- The constant $C^*(\alpha)$ in (4-21) blows up at least like $\exp(\alpha/2)$ as $\alpha \to \infty$, as otherwise the proof given in Section 2 would yield a cost of observability smaller than $\exp(L^2/2T)$ in small times, which is known to be false due to [Lissy 2015].
- Looking at the 2-parameter family of functions of the form

$$f_{A,\gamma}(\xi) = \cos(A\sqrt{\xi^2 - \gamma^2})$$

for parameters $A \in [0, 1]$ and $\gamma \in [0, \alpha]$, we find that

$$\sup_{f \in \{f_{A,\gamma}\} \cap \mathscr{E}_\alpha^*} \Big(\sup_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) = \cosh\Big(\frac{\alpha}{2}\Big),$$

and is achieved when taking $A = 1/\sqrt{2}$ and $\gamma = \alpha/\sqrt{2}$, i.e.,

$$f(\xi) = \cos\bigg(\frac{1}{\sqrt{2}}\sqrt{\xi^2 - \frac{\alpha^2}{2}}\bigg).$$

This function yields evidence of the fact that

$$\liminf_{\alpha \to \infty} \frac{1}{\alpha} \log(C^*(\alpha)) \geqslant \frac{1}{2}.$$

Let us finally emphasize that if we were able to show that

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha} \log(C^*(\alpha)) \leqslant \frac{1}{2},$$

the proof given in Section 2 would yield a cost of observability in small times $C_0(T, L)$ satisfying

$$\limsup_{T \to 0} T \log(C_0(T, L)) \leqslant \frac{L^2}{2}.$$

Combined with [Lissy 2015], this would give that

$$\lim_{T \to 0} T \log(C_0(T, L)) = \frac{L^2}{2}.$$

4G. ***Uniform controllability of viscous approximations of the transport equation.*** The problem we considered in this article is intimately related to the question of uniform controllability of viscous approximations of the transport equation raised in [Coron and Guerrero 2005]. Namely, for all $\varepsilon > 0$, one considers the following viscous approximation of the transport equation at velocity $M \in \mathbb{R}$:

$$\begin{cases} \partial_t y_\varepsilon - \varepsilon\, \partial_x^2 y_\varepsilon + M\, \partial_x y_\varepsilon = 0, & (t, x) \in (0, T) \times (0, L), \\ y_\varepsilon(t, 0) = v_\varepsilon(t), & t \in (0, T), \\ y_\varepsilon(t, L) = 0, & t \in (0, T), \\ y_\varepsilon(0, \cdot) = y_0(x), & x \in (0, L). \end{cases} \tag{4-23}$$

For each $\varepsilon > 0$, the equation (4-23) is null-controllable in any time $T > 0$, and the map $\mathcal{V}_{\varepsilon,T} : y_0 \to v_\varepsilon$ which to any $y_0 \in L^2(0, L)$ associates the control v_ε of minimal $L^2(0, T)$-norm is linear. The problem raised in [Coron and Guerrero 2005] is the following one: give conditions on the time T guaranteeing that

$$\limsup_{\varepsilon \to 0} \|\mathcal{V}_{\varepsilon,T}\|_{\mathscr{L}(L^2(0,L);L^2(0,T))} < \infty. \tag{4-24}$$

It is clear that if $|M|T < L$, (4-24) cannot happen, as otherwise the convergence of (4-23) as $\varepsilon \to 0$ would imply the null-controllability of the transport equation in a time which is not enough to make the characteristics go out of the domain.

Several conditions on the time T ensuring (4-24) were then proposed in the literature, namely in [Coron and Guerrero 2005; Glass 2010; Lissy 2012]. In fact, to our knowledge, the best results are the ones obtained in [Lissy 2012], which we recall now:

Theorem 4.9 [Lissy 2012]. *If $M \neq 0$ and*

$$|M|T > L(2\sqrt{3} + 1 - \operatorname{sign}(M)) \quad (2\sqrt{3} \approx 3.4641),$$

where $\operatorname{sign}(M) = 1$ *if* $M > 0$ *and* $= -1$ *if* $M < 0$, *we have*

$$\limsup_{\varepsilon \to 0} \|\mathcal{V}_{\varepsilon,T}\|_{\mathscr{L}(L^2(0,L);L^2(0,T))} = 0.$$

These results are based on the knowledge of the cost of observability of the one-dimensional heat equation in small time obtained in [Tenenbaum and Tucsnak 2007]. Therefore, as Theorem 4.5 improves the one in that paper, following the proof of [Lissy 2012] immediately improves the known result on the uniform controllability of the viscous approximations (4-23) of the transport equation:

Theorem 4.10. *Let K_0 as in* (1-5). *Then, if $M \neq 0$ and*

$$|M|T > L(4\sqrt{K_0} + 1 - \operatorname{sign}(M)) \quad (4\sqrt{K_0} \approx 3.3385),$$

we have

$$\limsup_{\varepsilon \to 0} \|\mathcal{V}_{\varepsilon,T}\|_{\mathscr{L}(L^2(0,L);L^2(0,T))} = 0. \tag{4-25}$$

As the proof of Theorem 4.10 follows line to line the one of [Lissy 2012], as it is explained in Section 3, item (i) of that paper, it is left to the reader.

We are currently investigating if one can do better than the combination of the cost of observability of the one-dimensional heat equation in small times and of the arguments in [Lissy 2012] to obtain better sufficient conditions on the ratio $|M|T/L$ to guarantee (4-25). We believe that a direct approach following the strategy in Section 2 could help in improving Theorem 4.10.

Appendix: Carleman-type estimate

We consider the equation

$$\begin{cases} \partial_t z - \Delta_x z + \dfrac{1}{2t}(2x \cdot \nabla_x z + dz) - \dfrac{L^2}{4t^2} z = g & \text{in } (0,T) \times \Omega, \\ z(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ \lim_{t \to 0} \|z(t)\|_{L^2(\Omega)} = 0, & \\ \lim_{t \to 0} t\|\nabla z(t)\|_{L^2(\Omega)} = 0, & \end{cases} \tag{A-1}$$

with $T > 0$, Ω a bounded domain of $\mathbb{R}^d$, $d \geqslant 1$,

$$L = \sup_{x \in \Omega} |x| \tag{A-2}$$

and

$$g \in L^2((0,T) \times \Omega).$$

We then have the following result:

Proposition A.1. *Any smooth solution z of* (A-1) *with $g \in L^2((0,T)\times\Omega)$ satisfies the estimate*

$$\int_\Omega \left(|\nabla_x z(T)|^2 - \frac{L^2}{4T^2}|z(T)|^2 \right) dx \leqslant \frac{L}{T^2}\int_0^T \left(t \int_{\Gamma_+} |\nabla_x z(t,x)\cdot\nu|^2 \, ds(x) \right) dt + \frac{1}{T^2}\int_0^T\int_\Omega t^2 |g|^2 \, dx\, dt, \tag{A-3}$$

with $\Gamma_+ = \{x \in \partial\Omega : x\cdot\nu > 0\}$, and L is given by (A-2).

Proof. We define the spatial operators

$$Sz = -\Delta_x z - \frac{L^2}{4t^2} z, \quad Az = \frac{1}{2t}(2x\cdot\nabla_x z + d\, z),$$

so that z is a solution of (A-1) satisfying

$$\partial_t z + Sz + Az = g \quad \text{in } (0,T)\times\Omega.$$

Note that S and A respectively correspond to the symmetric and skew-symmetric parts of the operator in (A-1).

We then consider

$$D(t) := \int_\Omega \left(|\nabla_x z(t,x)|^2 - \frac{L^2}{4t^2}|z(t,x)|^2 \right) dx = \int_\Omega (Sz)(t,x) z(t,x)\, dx.$$

A direct calculation shows that

$$\begin{aligned} D'(t) &= \frac{L^2}{2t^3}\int_\Omega |z|^2\, dx + 2\int_\Omega Sz\, \partial_t z\, dx \\ &= \frac{L^2}{2t^3}\int_\Omega |z|^2\, dx - 2\int_\Omega |Sz|^2\, dx - 2\int_\Omega Sz Az\, dx + 2\int_\Omega Sz\, g\, dx. \end{aligned}$$

Furthermore, as A is a skew-symmetric operator, we have

$$-2\int_\Omega Sz\, Az\, dx = 2\int_\Omega \Delta_x z\, Az\, dx = \frac{1}{t}\int_\Omega \Delta_x z (2x\cdot\nabla_x z + d\, z)\, dx.$$

On one hand, we obviously have

$$\int_\Omega \Delta_x z\, d\, z\, dx = -d\int_\Omega |\nabla_x z|^2\, dx.$$

On the other hand, we note that

$$\begin{aligned} \int_\Omega \Delta_x z\, 2x\cdot\nabla_x z\, dx &= 2\int_{\partial\Omega} (\nabla_x z\cdot\nu)(x\cdot\nabla_x z)\, ds(x) - 2\int_\Omega \nabla_x z\cdot\nabla_x(x\cdot\nabla_x z)\, dx \\ &= 2\int_{\partial\Omega} (x\cdot\nu)|\nabla_x z\cdot\nu|^2\, ds(x) - 2\int_\Omega \nabla_x z\cdot\nabla_x(x\cdot\nabla_x z)\, dx. \end{aligned}$$

Here, we have used that as $z=0$ on $\partial\Omega$, $\nabla_x z = (\nabla_x z\cdot\nu)\nu$ on $\partial\Omega$. As

$$\nabla_x z\cdot\nabla_x(x\cdot\nabla_x z) = |\nabla_x z|^2 + \frac{x}{2}\cdot\nabla_x(|\nabla_x z|^2),$$

we have

$$\begin{aligned}\int_\Omega \nabla_x z\cdot\nabla_x(x\cdot\nabla_x z)\,dx &= \int_\Omega |\nabla_x z|^2\,dx + \int_\Omega \frac{x}{2}\cdot\nabla_x(|\nabla_x z|^2)\,dx\\ &= \int_\Omega |\nabla_x z|^2\,dx + \frac12\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z|^2\,ds(x) - \frac{d}{2}\int_\Omega |\nabla_x z|^2\,dx\\ &= \int_\Omega |\nabla_x z|^2\,dx + \frac12\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) - \frac{d}{2}\int_\Omega |\nabla_x z|^2\,dx.\end{aligned}$$

Gathering the above computations, we get

$$\begin{aligned}D'(t)+2\int_\Omega |Sz|^2\,dx &= \frac{L^2}{2t^3}\int_\Omega |z|^2\,dx - \frac{2}{t}\int_\Omega |\nabla_x z|^2\,dx + \frac1t\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) + 2\int_\Omega Sz\,g\,dx\\ &\leqslant -\frac{2}{t}D(t) + \frac1t\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) + \int_\Omega |Sz|^2\,dx + \int_\Omega |g|^2\,dx,\end{aligned}$$

which implies in particular

$$(t^2D(t))' \leqslant t\int_{\Gamma_+}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) + t^2\int_\Omega |g|^2\,dx. \tag{A-4}$$

Using the assumption on z in the third and fourth lines of (A-1), one easily checks $\lim_{t\to 0} t^2D(t)=0$; hence we can integrate (A-4) between 0 and T, which gives (A-3), as $|(x\cdot\nu)|\leqslant L$ for all $x\in\overline{\Omega}$. □

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ZEROS OF REPEATED DERIVATIVES OF RANDOM POLYNOMIALS

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It has been shown that zeros of Kac polynomials $K_n(z)$ of degree n cluster asymptotically near the unit circle as $n \to \infty$ under some assumptions. This property remains unchanged for the l-th derivative of the Kac polynomials $K_n^{(l)}(z)$ for any fixed order l. So it's natural to study the situation when the number of the derivatives we take depends on n, i.e., $l = N_n$. We will show that the limiting behavior of zeros of $K_n^{(N_n)}(z)$ depends on the limit of the ratio N_n/n. In particular, we prove that when the limit of the ratio is strictly positive, the property of the uniform clustering around the unit circle fails; when the ratio is close to 1, the zeros have some rescaling phenomenon. Then we study such problem for random polynomials with more general coefficients. But things, especially the rescaling phenomenon, become very complicated for the general case when $N_n/n \to 1$, where we compute the case of the random elliptic polynomials to illustrate this.

1. Introduction

There are many well-known results regarding the nontrivial relations between zeros and critical points of polynomials. The classical Gauss–Lucas theorem states that all the critical points of a polynomial are in the convex hull of its zeros; in particular, if all the zeros are real, then so are the zeros of the derivative. Differentiating a polynomial which has only real zeros will even out zero spacings [Farmer and Rhoades 2005]; in the case of random trigonometric polynomials, it's proved in [Farmer and Yerrington 2006] that the repeated differentiation causes the roots of the function to approach equal spacing, which can be viewed as a toy model of crystallization in one dimension. For random polynomials under some mild assumptions, the distribution of critical points and the distribution of its zeros are asymptotically the same as the degree tends to infinity. This is because, roughly speaking, the coefficients of the derivative of a random polynomial are not changed dramatically. Actually, such result holds for any fixed number of derivatives [Feng $\geq$ 2019; Kabluchko and Zaporozhets 2014]. In this article, we are primarily interested in the case when the number of the derivatives we take for the random polynomials is not fixed but grows to infinity with the degree.

Our starting point is the classical Kac polynomials. Let $\xi_0, \xi_1, \ldots$ be nondegenerate, independent and identically distributed (i.i.d.) complex random variables. The Kac polynomials are defined as

$$K_n(z) = \sum_{k=0}^{n} \xi_k z^k. \tag{1}$$

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The Kac polynomials have degree n almost surely by assuming

$$\mathbb{P}(\xi_0 = 0) = 0. \tag{2}$$

The distribution of zeros of Kac polynomials has been studied for decades; we refer to [Bloom and Shiffman 2007; Hough et al. 2009; Ibragimov and Zeitouni 1997; Ibragimov and Zaporozhets 2013; Kac 1943; Kabluchko and Zaporozhets 2013; 2014; Sodin and Tsirelson 2004; Shepp and Vanderbei 1995]. It's proved that if

$$\mathbb{E}\log(1+|\xi_0|) < \infty, \tag{3}$$

then with probability 1, the empirical measure of zeros of Kac polynomials converges weakly to the uniform probability measure on the unit circle as n tends to infinity [Ibragimov and Zeitouni 1997; Ibragimov and Zaporozhets 2013; Kabluchko and Zaporozhets 2013; 2014; Shepp and Vanderbei 1995]. If the assumption (3) is removed, then zeros of $K_n(z)$ may not concentrate around the unit circle; see [Ibragimov and Zaporozhets 2013; Kabluchko and Zaporozhets 2013] for the case when $|\xi_0|$ has some logarithmic tails.

The property of clustering around the unit circle remains unchanged for the l-th derivative of the Kac polynomials $K_n^{(l)}(z)$ for any fixed l as n tends to infinity [Feng $\geq$ 2019; Kabluchko and Zaporozhets 2014]. But things become interesting if the number of the derivatives we take depends on n, e.g., $l = N_n$. For the extreme case when $N_n = n$, there is no zero for $K_n^{(n)}$ almost surely. Hence, some natural questions are: What is the critical growth order of N_n so that the property of clustering around the unit circle for the Kac polynomials $K_n^{(N_n)}$ fails? When it fails, what is the distribution of zeros of $K_n^{(N_n)}$? And how does the distribution depend on the growth order of N_n? In this article, we will answer these questions for the Kac polynomials completely. The estimates we derive for the Kac case can be applied to the general random polynomials. But there are some issues for the general random polynomials, where we will compute the case of the random elliptic polynomials to illustrate this.

1.1. ***Notation.*** Before we state our main results, we need to introduce some notation. We denote by

$$p_n(z) = \sum_{k=0}^{n} \xi_k p_{k,n} z^k \tag{4}$$

the random polynomials of degree n with general coefficients, where $p_{k,n}$ are deterministic coefficients and ξ_k are nondegenerate i.i.d. complex random variables. Throughout the article, we assume the random variable ξ_0 satisfies the conditions (2) and (3).

We denote by $p_n^{(N_n)}(z)$ the N_n-th derivative of $p_n(z)$ with the degree

$$D_n = n - N_n. \tag{5}$$

Without loss of generality, we may assume the convergence of

$$\frac{N_n}{n} \to a \in [0, 1]. \tag{6}$$

The random measure of zeros of $p_n(z)$ is denoted by

$$\mu_n = \sum_{z:p_n(z)=0} \delta_z, \tag{7}$$

and we use the notation

$$\mu_{D_n} = \sum_{z:p_n^{(N_n)}(z)=0} \delta_z \tag{8}$$

for the random measure of zeros of $p_n^{(N_n)}(z)$ of degree D_n.

Similarly, we denote by μ_n^K and $\mu_{D_n}^K$ the random measures of zeros of $K_n(z)$ and $K_n^{(N_n)}(z)$ for the Kac polynomials, respectively, and we denote by μ_n^E and $\mu_{D_n}^E$ the random elliptic polynomials. We denote by $\mathbb{D}_r$ the open disk of radius r centered at the origin in the complex plane. The convergence of the random measures ν_n to ν in probability (or in distribution) means the convergence in probability (or in distribution) in the weak sense, i.e., $\int_X \phi\nu_n(dx) \to \int_X \phi\nu(dx)$ in probability (or in distribution) for any smooth test function ϕ with compact support. Given a measure ν on the complex plane, we define the scaling operator $(\mathcal{S}_h\nu)(B) = \nu(B/h)$ for $h > 0$ where B is any Borel set in $\mathbb{C}$. In the end, we set $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ and set $\log 0 = -\infty$.

1.2. ***Kabluchko–Zaporozhets theorem.*** There are many well-known results regarding the global distribution of zeros of some special Gaussian random analytic functions where the ensembles are usually invariant under some group action, such as the Gaussian elliptic polynomials and Gaussian hyperbolic analytic functions [Hough et al. 2009; Sodin and Tsirelson 2004]. Recently, a remarkable result proved in [Kabluchko and Zaporozhets 2014] deals with more general random analytic functions. Kabluchko and Zaporozhets [2014] proved that under certain assumptions on the coefficients of the random analytic functions, the distribution of zeros will converge to a deterministic rotationally invariant measure on a domain of the complex plane. Such measure can be explicitly characterized in terms of the coefficients. To be more precise, let's consider the random analytic function in the form of

$$F_n(z) = \sum_{k=0}^{\infty} \xi_k p_{k,n} z^k, \tag{9}$$

where ξ_k are nondegenerate i.i.d. complex random variables satisfying condition (3) and the coefficients $p_{k,n}$ satisfy the following assumptions.

Assumptions 1. Assume there are a function $p : [0, \infty) \to [0, \infty)$ and a number $T_0 \in (0, \infty]$ such that

(1) $p(t) > 0$ for $t < T_0$ and $p(t) = 0$ for $t > T_0$,

(2) p is continuous on $[0, T_0)$, and in the case $T_0 < \infty$, left continuous at T_0,

(3) $\lim_{n\to\infty} \sup_{k\in[0,An]} ||p_{k,n}|^{1/n} - p(k/n)| = 0$ for every $A > 0$, and

(4) $R_0 = \liminf_{t\to\infty} p(t)^{-1/t} \in (0, \infty]$, $\liminf_{k\to\infty} |p_{k,n}|^{-1/k} \geq R_0$ for every fixed $n \in \mathbb{N}$ and additionally, $\liminf_{n,k/n\to\infty} |p_{k,n}|^{-1/k} \geq R_0$.

Roughly speaking, the major assumption is that the coefficients $p_{k,n}$ are approximately $e^{n \log p(k/n)}$ for some p, which is positive on some interval $[0, T_0)$, continuous in $[0, T_0]$, and equal to 0 in (T_0, ∞).

Theorem 1 [Kabluchko and Zaporozhets 2014]. *Under Assumptions 1 and* (3), *let $I(s)$ be the Legendre–Fenchel transform of $-\log p$, i.e., $I(s) = \sup_{t\geq 0}(st + \log p(t))$; then the random measure $(1/n)\mu_{F_n}$*

of zeros of $F_n(z)$ converges in probability to a deterministic measure μ in $\mathbb{D}_{R_0}$, which is rotationally invariant and satisfies

$$\mu(\mathbb{D}_r) = I'(\log r), \quad r \in (0, R_0).$$

As a convention, I' is the left derivative of I. A typical example to which to apply the Kabluchko–Zaporozhets theorem is the Kac polynomials where we have

$$p_{k,n} = 1_{k \le n}, \qquad p(t) = 1_{t \le 1}, \qquad T_0 = 1. \tag{10}$$

By some computations, we have $I(s) = s \vee 0$ and thus the limiting distribution satisfies

$$\mu(\mathbb{D}_r) = \begin{cases} 0, & 0 \le r \le 1, \\ 1, & r > 1, \end{cases} \tag{11}$$

i.e., the uniform probability measure on the unit circle.

But we cannot apply the Kabluchko–Zaporozhets theorem directly in our case to derive the distribution of zeros of $K_n^{(N_n)}$ or that of the general random polynomials $p_n^{(N_n)}$. For example, if $N_n = n - \lfloor \log n \rfloor$, then the degree of $p_n^{(N_n)}$ is $D_n = \lfloor \log n \rfloor$; therefore, one cannot find some A so that Assumption 1(3) is satisfied. We need to modify their theorem to deal with our situation more conveniently. We consider the random polynomials in the form of

$$F_n(z) = \sum_{k=0}^{(T_0-\delta_n)L_n} \xi_k p_{k,n} z^k, \tag{12}$$

where $(T_0 - \delta_n)L_n$ is an integer and we assume that $F_n(z)$ satisfies the following assumptions:

Assumptions 2. There exist a function $p : [0, \infty) \to [0, \infty)$, a positive number $T_0 \in (0, \infty)$, a sequence of positive integers L_n going to ∞ as $n \to \infty$, and a sequence of numbers $\delta_n \in (-T_0, T_0)$ (not necessarily positive) that goes to 0 as $n \to \infty$ such that

(1) $p(t) > 0$ for $t \in [0, T_0)$ and $p(t) = 0$ for $t > T_0$,

(2) p is continuous in $[0, T_0]$, and

(3) $\lim_{n\to\infty} \sup_{0 \le k \le (T_0-\delta_n)L_n} ||p_{k,n}|^{1/L_n} - p((k/L_n) \wedge T_0)| = 0$.

Then we have the following theorem whose proof is sketched in the Appendix.

Theorem 2. *For random polynomials $F_n(z)$ in the form of* (12) *which satisfy Assumptions 2, let $I(s)$ be the Legendre–Fenchel transform of $-\log p$; then the random measure $(1/L_n)\mu_{F_n}$ of zeros will converge in probability to a deterministic rotationally invariant measure μ where*

$$\mu(\mathbb{D}_r) = I'(\log r), \quad r > 0. \tag{13}$$

Throughout the article, we often make use of the estimate

$$\lim_{n\to\infty} \sup_{0 \le k \le (T_0-\delta_n)L_n} \left| \frac{1}{L_n} \log|p_{k,n}| - \log p\left(\frac{k}{L_n} \wedge T_0\right) \right| = 0. \tag{14}$$

This estimate implies the main Assumption 2(3), which is the direct consequence of the inequality

$$|x-y| \le (x \wedge y) e^{|\log x - \log y|} |\log x - \log y|$$

for any $x, y > 0$. The main advantage of (14) is the convenience in computations.

1.3. ***Main results.*** We first state our main results for the Kac polynomials, which will answer the questions we raised at the beginning of the article.

Kac polynomials. The main result is that the limiting behavior of the distribution of zeros of $K_n^{(N_n)}$ will depend on the limit of the ratio N_n/n. We will divide our discussions into two categories: D_n goes to infinity and D_n remains a fixed number, where $D_n = n - N_n$ is the degree of the random polynomials $K_n^{(N_n)}$. Without loss of generality, we consider the four different cases ① $N_n/n \to 0$, ② $N_n/n \to a \in (0, 1)$, ③ $N_n/n \to 1$ and $D_n \to \infty$, e.g., $N_n = n - \lfloor \log n \rfloor$ and $D_n = \lfloor \log n \rfloor$, and ④ $N_n/n \to 1$ but $D_n = m < \infty$, i.e., $K_n^{(N_n)}$ has a fixed degree m.

In the cases of ①–③ where $D_n \to \infty$, we will show that the coefficients of $K_n^{(N_n)}$ or its rescaling will satisfy Assumptions 2 with different choices of L_n, δ_n, T_0, and p; then we apply Theorem 2 to prove:

Theorem 3. *Assume $D_n \to \infty$ as $n \to \infty$; we have the following results regarding the empirical measure of zeros of derivatives of Kac polynomials $K_n^{(N_n)}$:*

(1) *If* $\lim_{n\to\infty} N_n/n = 0$, *then* $(1/D_n)\mu_{D_n}^K$ *converges in probability to the uniform probability measure on the unit circle*, *i.e.*, *the measure defined in* (11).

(2) *If* $\lim_{n\to\infty} N_n/n = a \in (0, 1)$, *then* $(1/D_n)\mu_{D_n}^K$ *converges in probability to a rotationally invariant measure* μ_a^K *on* $\mathbb{C}$ *defined by*

$$\mu_a^K(\mathbb{D}_r) = \begin{cases} ar/((1-a)(1-r)), & 0 < r < 1-a, \\ 1, & r \ge 1-a. \end{cases} \tag{15}$$

(3) *If* $\lim_{n\to\infty} N_n/n = 1$, *then globally we have the convergence in probability*

$$\frac{1}{D_n} \mu_{D_n}^K \to \delta_0. \tag{16}$$

If we set $R_n = n/D_n$ as the quotient of the degrees of K_n and $K_n^{(N_n)}$ and consider the rescaling Kac polynomials $\widetilde{K}_n(z) := K_n^{(N_n)}(z/R_n)$, then the empirical measure $(1/D_n)\mu_{D_n}^{\widetilde{K}}$ which is the same as $(1/D_n)\mathcal{S}_{R_n}(\mu_{D_n}^K)$ converges in probability to a rotationally invariant measure $\tilde{\mu}^K$ where

$$\tilde{\mu}^K(\mathbb{D}_r) = \begin{cases} r, & r < 1, \\ 1, & r \ge 1. \end{cases} \tag{17}$$

In particular, *the density for the measure $\tilde{\mu}^K$ is*

$$\tilde{d}^K(z) = \frac{1}{2\pi|z|} 1_{|z| \le 1}. \tag{18}$$

In the case ④ when D_n remains a fixed number, we will show that the measure of zeros of the rescaling polynomials $K_n^{(N_n)}(z/n)$ will converge to some random measure. The main tool to prove this result is Rouché's theorem in complex analysis. Our result is as follows.

Theorem 4. *Suppose* $\lim_{n\to\infty} N_n/n = 1$ *and* $D_n = m$ *for all* n; *then globally*

$$\frac{1}{m}\mu_{D_n}^K \to \delta_0, \tag{19}$$

where the convergence is in probability. Furthermore, we have the rescaling limit

$$\mathcal{S}_n(\mu_{D_n}^K) \to \mu_{f_m^K}, \tag{20}$$

where the convergence is in distribution and $\mu_{f_m^K}$ *is the random measure of zeros of the random polynomial*

$$f_m^K(z) = \sum_{k=0}^{m} \frac{\xi_k}{k!} z^k. \tag{21}$$

Remark. The relationship between the results in Theorems 3(3) and 4 has an intuitive explanation. Consider the case in Theorem 3(3). We can zoom in zeros of $K_n^{(N_n)}(z)$ in two steps. First we zoom in the zeros of $K_n^{(N_n)}(z)$ by a factor of n; then by Theorem 4 (treating D_n as fixed for this moment) the scaled zeros will be close to the zeros of $f_{D_n}^K(z)$. Here $f_{D_n}^K(z)$ is just the function in (21) with m replaced by D_n. If we then zoom out zeros of $f_{D_n}^K$ by a factor of D_n (which is the degree of the polynomial $f_{D_n}^K$), then as a whole we get something close to zooming in the zeros of $K_n^{(N_n)}(z)$ by a factor of n/D_n. Taking n to infinity we should get the limit in Theorem 3(3). This is in accordance with the fact that (17) is also the limit of the empirical measure of zeros of $f_{D_n}^K(D_n z)$ as $m \to \infty$, as shown in Theorem 2.3 of [Kabluchko and Zaporozhets 2014]. Note that in the zooming out process, we can also replace $\sum_{k=0}^{D_n}(\xi_k/k!)(D_n z)^k$ by $\sum_{k=0}^{\infty}(\xi_k/k!)(D_n z)^k$ since Theorem 2.1 of [Kabluchko and Zaporozhets 2014] shows the empirical measure of $\sum_{k=0}^{\infty}(\xi_k/k!)(D_n z)^k$ restricted to unit disk also converges to the measure in (17).

As a summary, we show that the clustering property of zeros around the unit circle for the derivatives of Kac polynomials holds if and only if $N_n/n \to 0$; the conclusion (3) in Theorem 3 together with Theorem 4 imply that, if $N_n/n \to 1$, zeros will converge to the origin with the average decay rate D_n/n which is the quotient of the degrees of $K_n^{(N_n)}$ and K_n. Thus we will completely answer the questions we proposed at the beginning of the article.

General random polynomials. We can extend the above results for the Kac polynomials to the general random polynomials where the coefficients satisfy Assumptions 1 in the Kabluchko–Zaporozhets theorem.

Theorem 5. *Suppose the random polynomial* $p_n(z)$ *of* (4) *satisfies Assumptions 1 with some function* $p(t)$; *then regarding the zeros of* $p_n^{(N_n)}$, *we have*:

(1) *If* $\lim_{n\to\infty} N_n/n = 0$, *let* $I(s)$ *be the Legendre–Fenchel transform of* $-\log p$; *then* $(1/D_n)\mu_{D_n}$ *converges in probability to a rotationally invariant measure* μ *given by*

$$\mu(\mathbb{D}_r) = I'(\log r), \quad r > 0.$$

That is, $(1/D_n)\mu_{D_n}$ *has the same limit as* $(1/n)\mu_n$.

(2) *If* $\lim_{n\to\infty} N_n/n = a \in (0,1)$, *let* $\log u_a = \log p(t+a) + (t+a)\log(t+a) - t\log t + (1-a)\log(1-a)$ *if* $0 \le t \le 1-a$ *and* $-\infty$ *if* $t > 1-a$. *Let* $I_a(s)$ *be the Legendre–Fenchel transform of* $-\log u_a$; *then* $(1/D_n)\mu_{D_n}$ *converges in probability to a rotationally invariant measure* μ_a *given by*

$$\mu_a(\mathbb{D}_r) = \frac{1}{1-a} I_a'(\log r), \quad r > 0.$$

Compared with Theorems 3 and 4 for the Kac case, things become complicated for the general random polynomials when the ratio N_n/n tends to 1. First, one cannot conclude that $(1/D_n)\mu_{D_n}$ converges in probability to δ_0. To see this, let's consider the following example where the coefficients of the random polynomials p_n are

$$p_{k,n} = \begin{cases} 1, & 0 \le k < N_n, \\ n!\,(k-N_n)!/(k!\,D_n!), & N_n \le k \le n, \end{cases}$$

where

$$D_n = \lfloor \log n \rfloor \quad \text{and} \quad N_n = n - D_n.$$

We let

$$p(t) = 1_{0 \le t \le 1}.$$

We claim that $p_{k,n}$ and p satisfy Assumptions 1. Indeed, when $0 \le k < N_n$, we have

$$p_{k,n}^{1/n} = p\Big(\frac{k}{n}\Big).$$

Therefore, it remains to prove

$$\lim_{n\to\infty} \sup_{N_n \le k \le n} |p_{k,n}^{1/n} - 1| = 0.$$

By (14), it's enough to show

$$\lim_{n\to\infty} \sup_{N_n \le k \le n} \left|\frac{1}{n}\log p_{k,n}\right| = 0. \tag{22}$$

For $N_n \le k \le n$, we have $1 \le n!\,(k-N_n)!/(k!\,D_n!) \le n!/k!$; then

$$\sup_{N_n \le k \le n} \left|\frac{1}{n}\log p_{k,n}\right| \le \sup_{N_n \le k \le n} \frac{1}{n}\log\frac{n!}{k!} \le \frac{1}{n}\log n^{D_n} \le \frac{\log^2 n}{n},$$

where (22) follows as $n \to \infty$, which completes the proof of the claim. But the N_n-th derivative of p_n is

$$p_n^{(N_n)} = \frac{n!}{D_n!} \sum_{k=0}^{D_n} \xi_{k+N_n} z^k,$$

which is in the form of Kac polynomials; thus, the empirical measure of zeros will converge to the uniform probability measure on the circle instead of the delta function at the origin.

Secondly, even if zeros converge to δ_0, one cannot easily find the rescaling limit of the empirical measure of zeros if there exists one. The rescaling property should highly depend on the properties of coefficients, such as the convergent rate of $p_{n,k}$ to $p(t)$ and the monotonicity of $p_{k,n}$ for each fixed n. The following results regarding the elliptic polynomials provide such an example.

Random elliptic polynomials. The random elliptic polynomials are in the form of

$$E_n(z) = \sum_{k=0}^{n} \xi_k \sqrt{\binom{n}{k}} z^k. \tag{23}$$

If ξ_k are i.i.d. complex Gaussian random variables, then the random elliptic polynomials are also called Gaussian SU(2) polynomials. The Gaussian SU(2) polynomials can be viewed as meromorphic functions defined on the complex projective space $\mathbb{CP}^1 \cong S^2$, and a basic fact is that the distribution of its zeros is invariant under the SU(2) action. The Gaussian SU(2) polynomial is the standard model when one tries to generalize the random polynomials to random holomorphic sections on the complex manifolds [Bleher et al. 2000; Hough et al. 2009].

One can show that the coefficients of the random elliptic polynomials satisfy all of Assumptions 1 with the associated function (see also [Kabluchko and Zaporozhets 2014])

$$\log p^E(t) = -\tfrac{1}{2} t \log t - \tfrac{1}{2}(1-t)\log(1-t) \quad \text{for } 0 \le t \le 1. \tag{24}$$

Theorem 6. *For the random elliptic polynomials $E_n(z)$ defined in* (23), *we have*:

(1) *The conclusions in Theorem 5 hold for $(1/D_n)\mu^E_{D_n}$ with p replaced by p^E defined in* (24).

(2) *If* $\lim_{n\to\infty} N_n/n = 1$, *then we have the global convergence in probability*

$$\frac{1}{D_n}\mu^E_{D_n} \to \delta_0.$$

Furthermore, if $D_n \to \infty$, then in probability, we have

$$\frac{1}{D_n}\mathcal{S}_{\sqrt{R_n}}(\mu^E_{D_n}) \to \mu,$$

where $R_n = n/D_n$ as before and μ is the rotationally invariant probability measure defined as

$$\mu(\mathbb{D}_r) = \frac{r(\sqrt{4+r^2}-r)}{2}, \quad r \in (0,\infty). \tag{25}$$

If $D_n = m < \infty$, then the following rescaling limit holds in distribution:

$$\mathcal{S}_{\sqrt{n}}(\mu^E_{D_n}) \to \mu_{f^E_m},$$

where $\mu_{f^E_m}$ is the random measure of zeros of $f^E_m = \sum_{k=0}^{m}(\xi_k/(k!\sqrt{(m-k)!}))z^k$.

1.4. ***Further remarks.*** Let's compare Theorem 6 with Theorems 3(3) and 4 for the case when $N_n/n \to 1$. Both the empirical measures of zeros of derivatives tend to the point mass at the origin, but the interesting result is that they converge with different decay rates. Zeros converge to the origin with the average decay rate D_n/n for the Kac case and $\sqrt{D_n/n}$ for the elliptic case, which indicates that Assumptions 1 is not enough to extract the complete information about the convergence of zeros of the N_n-th derivative of general random polynomials; i.e., the main assumption $\lim_{n\to\infty}\sup_{k\in[0,An]}||p_{k,n}|^{1/n} - p(k/n)| = 0$ for every $A > 0$ is not enough. It seems that we need to impose additional assumptions on the rate of the convergence of $p_{k,n}$ to p for $N_n \le k \le n$ and the growth order of $p_{k,n}$. As in (14), we may alternatively consider the quantities

$$\eta_n := \sup_{N_n \le k \le n} \left| \frac{1}{n} \log|p_{k,n}| - \log p\left(\frac{k}{n}\right) \right| \tag{26}$$

and

$$b_n := \sup_{N_n \le k \le n} |p_{k,n}|. \tag{27}$$

The asymptotic properties of η_n and b_n may play important roles in the case when $N_n/n \to 1$. Note that η_n is identical to 0 for the Kac polynomials and asymptotic to $(\log D_n)/(4n) + O(1/n)$ for the random elliptic polynomials. Two questions are raised: What are the asymptotic properties of η_n and b_n so that zeros of $p_n^{(N_n)}$ tend to the origin? And if zeros tend to the origin, how does the decay rate depend on η_n and b_n? We postpone these two problems for further investigation.

Along with Kac polynomials, there is another important type of random polynomial defined via the orthogonal polynomials. Given a bounded simply connected domain Ω in the complex plane with analytic boundary C of length L and a positive weight function $w(z)$, we define the inner product

$$\langle f, g \rangle = \frac{1}{L} \int_C f(z)\overline{g(z)} w(z)|dz|. \tag{28}$$

Then we can find an orthonormal basis $\{p_n^w(z)\}$ with respect to this inner product, where $p_n^w(z)$ is a polynomial of degree n in which the coefficient of z^n is real and positive. Shiffman and Zelditch [2003] prove that the empirical measure of zeros of

$$P_n(z) = \sum_{k=0}^{n} \xi_k p_k^w(z), \tag{29}$$

where ξ_k are i.i.d. standard complex Gaussian random variables, tends to the equilibrium measure of Ω as n tends to infinity. Such result is then generalized by Bloom and Shiffman [2007] to higher dimensions where they get rid of the analytic assumption and replace it by the Bernstein–Markov condition. In [Feng $\ge$ 2019], the author further studied zeros of the l-th derivative of $P_n^{(l)}$ for any fixed l as $n \to \infty$, and proved that zeros of derivatives of any fixed order also tend to the equilibrium measure. The method used in [Feng $\ge$ 2019; Shiffman and Zelditch 2003] is quite different from that of [Kabluchko and Zaporozhets 2014]. One needs to apply the classical Szegő theorem [1975] on orthogonal polynomials together with the conformal transformation between the bounded domain and the unit disk. Then it's a natural problem to study the behavior of zeros of derivatives of $P_n^{(N_n)}$. As indicated by Theorem 3 for the Kac polynomials, it seems that zeros will still converge to the equilibrium measure if $N_n/n \to 0$, but the results for the case when $N_n/n \to a \in (0, 1]$ are quite hard to predict. One may prove the results with the aid of the conformal transformation, but the strategy is unclear to the authors.

The paper is organized as follows. We will prove Theorems 3 and 4 for the Kac polynomials in great details in Section 2. The estimates for the Kac case can be applied to prove Theorem 5 for the general random polynomials in Section 3. In the end, we will prove Theorem 6 for the random elliptic polynomials. In the Appendix, we will sketch the proof of Theorem 2.

2. Kac polynomials

In this section, we will prove Theorems 3 and 4 for the Kac polynomials.

Let $K_n^{(N_n)}$ be the N_n-th derivative of the Kac polynomials. Since we want to prove the empirical measure of zeros converges to a deterministic limit, it suffices to prove the convergence in distribution. By the fact that ξ_k are i.i.d., it's equivalent to consider

$$K_n^{(N_n)}(z)=\sum_{k=0}^{D_n}\xi_k(k+1)\cdots(k+N_n)z^k. \tag{30}$$

Observing that the random measure of zeros is invariant by the dilation, i.e., $\mu_{cf}=\mu_f$ for any nonzero c, we can alternatively consider the following normalized random polynomial so that the leading-order term is $\xi_{D_n}z^{D_n}$:

$$K_n^{(N_n)}(z)=\sum_{k=0}^{D_n}\xi_k f_{k,n}z^k, \tag{31}$$

where throughout the article, we set

$$f_{k,n}:=\frac{(k+N_n)!\,D_n!}{k!\,n!}. \tag{32}$$

Stirling's formula reads

$$k!=c_k\sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k, \tag{33}$$

where c_k is a sequence of positive numbers tending to 1 as k tends to ∞ and hence uniformly bounded. Then we have

$$\begin{aligned}\frac{1}{L_n}\log f_{k,n}=\frac{1}{L_n}\bigg[&(k+N_n)\log(k+N_n)-(k+N_n)+\frac{\log(k+N_n)}{2}+D_n\log D_n-D_n+\frac{\log D_n}{2}\\&-\Big(k\log k-k+\frac{\log k}{2}+n\log n-n+\frac{\log n}{2}\Big)\bigg]\\&+\frac{1}{L_n}(\log c_{k+N_n}+\log c_{D_n}-\log c_k-\log c_n)\\=\frac{1}{L_n}&[(k+N_n)\log(k+N_n)+D_n\log D_n-n\log n-k\log k]\\&+\frac{1}{2L_n}(\log(k+N_n)+\log D_n-\log n-\log k)+\frac{1}{L_n}(\log c_{k+N_n}+\log c_{D_n}-\log c_k-\log c_n)\\:=I_1&(k,n)+I_2(k,n)+I_3(k,n).\end{aligned} \tag{34}$$

When $k=0$, we set $c_k=1$ and set $I_1(0,n)=(1/L_n)(N_n\log N_n+D_n\log D_n-n\log n)$, $I_2(0,n)=(1/(2L_n))(\log N_n+\log D_n-\log n)$, and $I_3(0,n)=(1/L_n)(\log c_{N_n}+\log c_{D_n}-\log c_n)$ to be consistent with the definitions. The expressions of I_j are different according to the choices of L_n (only differ by the front factor L_n), but we use the same notation I_j for different cases throughout the article to reduce the notation we use.

In the following computations, we will let $L_n \to \infty$ (although we choose different L_n for different cases); hence, $I_3(k, n)$ will tend to 0 uniformly by the uniform bound of c_k, which means the third term $I_3(k, n)$ is always negligible.

2.1. ***Case* ①.** Let's first consider the case ① when

$$\lim_{n\to\infty} \frac{N_n}{n} = 0. \tag{35}$$

For this case, we need to choose $L_n = n$ in (34). We first simply have

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} |I_2(k, n)| \le \lim_{n\to\infty} \frac{2}{n} \log n = 0. \tag{36}$$

For $I_1(k, n)$, we observe that for each fixed n, $I_1(k, n)$ is increasing with respect to k by considering the function $I(x) = (x + N_n) \log(x + N_n) - x \log x$ where $I'(x) = \log(x + N_n) - \log x \ge 0$. We combine this with the fact that $I_1(D_n, n) = (1/n)((D_n + N_n) \log(D_n + N_n) + D_n \log D_n - n \log n - D_n \log D_n) = 0$; we first have

$$\sup_{0\le k\le D_n} |I_1(k, n)| \le |I_1(0, n)| \vee |I_1(D_n, n)| = |I_1(0, n)|,$$

which further reads

$$\begin{aligned}\sup_{0\le k\le D_n} |I_1(k, n)| &\le \frac{1}{n}|n \log n - N_n \log N_n - D_n \log D_n| \\ &= \frac{1}{n}|N_n \log n + D_n \log n - N_n \log N_n - D_n \log D_n| \\ &= \left|-\frac{N_n}{n} \log\left(\frac{N_n}{n}\right) - \frac{D_n}{n} \log\left(\frac{D_n}{n}\right)\right|.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} |I_1(k, n)| = 0, \tag{37}$$

since $N_n/n \to 0$ and $D_n/n = 1 - N_n/n \to 1$ as $n \to \infty$.

Combining (36)–(37) and the fact that I_3 always tends to 0, we get

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} \left|\frac{1}{n} \log f_{k,n}\right| = 0. \tag{38}$$

Hence, the coefficients $f_{k,n}$ satisfy Assumptions 2 with $L_n = n$, $T_0 = 1$, and $\delta_n = N_n/n$ so that $(1-\delta_n)L_n = D_n$ and $\log f(t) = 0$ for $0 \le t \le 1$ and $\log f = -\infty$ for $t > 1$. Therefore, zeros of $K_n^{(N_n)}$ will have the same distribution as the Kac polynomials by computations in (10) and (11) as $n \to \infty$.

2.2. ***Case* ②.** Let's consider the case when

$$\lim_{n\to\infty} \frac{N_n}{n} = a \in (0, 1). \tag{39}$$

Let's choose $L_n = n$ in (34) again. By the same arguments as in Case ①, I_2 and I_3 converge to 0 uniformly for $0 \le k \le D_n$ as $n \to \infty$. Therefore, it remains to estimate I_1. Let's put $N_n/n = a + \delta_n$ where $\delta_n \to 0$. Assume n is large enough so that

$$|\delta_n| \le \frac{1-a}{2} \wedge \frac{a}{2}. \tag{40}$$

For $k \ge 1$, we rewrite

$$\begin{aligned}
I_1 &= \frac{1}{n}[(k+N_n)\log(k+N_n) + D_n \log D_n - n\log n - k\log k] \\
&= \frac{1}{n}[(n - D_n + k)\log(k+N_n) - n\log n - k\log k + D_n\log k - D_n \log k + D_n \log D_n] \\
&= \frac{1}{n}\left[n\log\left(\frac{k}{n} + \frac{N_n}{n}\right) + (k - D_n)\log(k+N_n) - (k-D_n)\log k + D_n \log\left(\frac{D_n}{k}\right)\right] \\
&= \log\left(\frac{k}{n} + \frac{N_n}{n}\right) + \left[\left(\frac{k}{n} - \frac{D_n}{n}\right)\log\left(1 + \frac{N_n}{k}\right) + \frac{D_n}{n}\log\left(\frac{D_n}{k}\right)\right] \\
&:= I_4 + I_5.
\end{aligned}$$

To estimate I_4 and I_5, we will make use of the following inequality which is the direct consequence of the intermediate value theorem:

$$0 \le \log y - \log x \le \frac{1}{c}(y - x) \quad \text{for } 0 < c \le x \le y. \tag{41}$$

We can rewrite I_4 as $\log(k/n + a + \delta_n)$; by (40)–(41), we have

$$\left| I_4 - \log\left(\frac{k}{n} + a\right)\right| \le \left|\frac{2\delta_n}{a}\right|$$

for all $1 \le k \le D_n$. So we have

$$\lim_{n\to\infty} \sup_{1\le k\le D_n} \left| I_4 - \log\left(\frac{k}{n} + a\right)\right| = 0. \tag{42}$$

For I_5, since $N_n/n = a + \delta_n$ and $D_n/n = 1 - a - \delta_n$, we can rewrite it as

$$\begin{aligned}
I_5 &= \left(\frac{k}{n} - (1-a) + \delta_n\right)\log\left(1 + \frac{(a+\delta_n)n}{k}\right) + (1 - a - \delta_n)\log\frac{(1-a-\delta_n)n}{k} \\
&= \frac{k}{n}\log\left(1 + a\frac{n}{k} + \delta_n\frac{n}{k}\right) + (1-a-\delta_n)\log\left(-1 + \frac{n+k}{k + (a+\delta_n)n}\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left| I_5 - \left[\frac{k}{n}\log\left(1 + a\frac{n}{k}\right) + (1-a)\log\left(-1 + \frac{n+k}{k+an}\right)\right]\right| \\
&\quad \le \frac{k}{n}\left|\log\left(1 + a\frac{n}{k} + \delta_n\frac{n}{k}\right) - \log\left(1 + a\frac{n}{k}\right)\right| \\
&\qquad + (1-a)\left|\log\left(-1 + \frac{n+k}{k+(a+\delta_n)n}\right) - \log\left(-1 + \frac{n+k}{k+an}\right)\right| + |\delta_n|\left|\log\left(-1 + \frac{n+k}{k+(a+\delta_n)n}\right)\right| \\
&\quad := I_6 + I_7 + I_8.
\end{aligned}$$

By (40)–(41) again, we have

$$I_6 \le \frac{k}{n}\frac{1}{1+(a/2)(n/k)}|\delta_n|\frac{n}{k} \le |\delta_n| \to 0.$$

For I_7, since $|\delta_n| \le (1-a)/2$, we know $k+(a+\delta_n)n \le k+((1+a)/2)n$. Therefore,

$$-1+\frac{n+k}{k+(a+\delta_n)n} \ge -1+\frac{n+k}{k+((1+a)/2)n} = \frac{(1-a)n/2}{((1+a)/2)n+k} \ge \frac{(1-a)n/2}{((1+a)/2)n+n} = \frac{1-a}{3+a}. \quad (43)$$

We also have

$$-1+\frac{k+n}{k+an} \ge \frac{1-a}{3+a}.$$

Thus, by (41), we have

$$\begin{aligned} I_7 &= (1-a)\left|\log\left(-1+\frac{n+k}{k+(a+\delta_n)n}\right)-\log\left(-1+\frac{n+k}{k+an}\right)\right| \\ &\le (1-a)\frac{3+a}{1-a}\left|\frac{k+n}{k+(a+\delta_n)n}-\frac{k+n}{k+an}\right| \\ &\le (3+a)\frac{(k+n)|\delta_n|n}{(k+an/2)^2} \le (3+a)\frac{(n+n)n|\delta_n|}{(an/2)^2} \le \frac{8(3+a)|\delta_n|}{a^2} \to 0. \end{aligned}$$

For I_8, taking into account (40) and (43), we have

$$\frac{1-a}{3+a} \le -1+\frac{k+n}{k+(a+\delta_n)n} = \frac{(1-a-\delta_n)n}{k+(a+\delta_n)n} \le \frac{[(1-a)+(1-a)/2]n}{(a-a/2)n} \le \frac{3(1-a)}{a};$$

it follows that

$$I_8 \le \left(\left|\log\left(\frac{3(1-a)}{a}\right)\right| \vee \left|\log\left(\frac{1-a}{3+a}\right)\right|\right)|\delta_n| \to 0.$$

If we combine the estimates of I_6, I_7, and I_8, we conclude that

$$\lim_{n\to\infty}\sup_{1\le k\le D_n}\left|I_5-\left[\frac{k}{n}\log\left(1+a\frac{n}{k}\right)+(1-a)\log\left(-1+\frac{n+k}{k+an}\right)\right]\right| = 0. \quad (44)$$

If we set

$$\begin{aligned} \log f_1(t) &= \log(t+a)+t\log\left(1+\frac{a}{t}\right)+(1-a)\log\left(\frac{1-a}{t+a}\right) \\ &= (t+a)\log(t+a)-t\log t+(1-a)\log(1-a), \qquad t>0, \end{aligned} \quad (45)$$

then the estimates (42) and (44) for I_4 and I_5 imply

$$\lim_{n\to\infty}\sup_{1\le k\le D_n}\left|I_1-\log f_1\left(\frac{k}{n}\right)\right| = 0. \quad (46)$$

The estimate of I_1 in the case $k=0$ can be achieved by the same way, and actually (46) holds with the supremum taken over $0\le k\le D_n$.

Let's set $f = f_1$ for $0 \le t \le 1-a$ and 0 for $t > 1-a$. Let's set

$$\Delta(b) = \sup_{1-a\le t\le s\le 1,\, s-t\le b} |\log f_1(t) - \log f_1(s)|;$$

then we have

$$\sup_{0\le k\le D_n}\left|I_1-\log f\Big(\frac{k}{n}\wedge(1-a)\Big)\right|\le\sup_{0\le k\le D_n}\left|I_1-\log f_1\Big(\frac{k}{n}\Big)\right|+\Delta(|\delta_n|).$$

Observing that $\log f_1$ is uniformly continuous on $[1-a, 1]$, combining (46) and the fact that $\delta_n\to 0$, then we have

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|I_1-\log f\Big(\frac{k}{n}\wedge(1-a)\Big)\right|=0.$$

Therefore, if we combine the estimates of I_1, I_2, and I_3 we derived above for Case ②, we have

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|\frac{1}{n}f_{k,n}-\log f\Big(\frac{k}{n}\wedge(1-a)\Big)\right|=0. \tag{47}$$

As a summary, in the case when $N_n/n\to a\in(0,1)$, by defining $f(t)$ above, the coefficients $f_{k,n}$ will satisfy Assumptions 2 with $T_0=1-a$, $L_n=n$, and $\delta_n=N_n/n-a$ (note that $D_n=(T_0-\delta_n)L_n$ again). The Legendre–Fenchel transform of $-\log f$ is

$$I(s)=\begin{cases}a\log(a/(e^{-s}-1)+a)+(1-a)\log(1-a), & s<\log(1-a),\\ s(1-a), & s\ge\log(1-a).\end{cases}$$

Therefore, by Theorem 2, the limiting measure for the sequence of the random measure $(1/L_n)\mu^K_{D_n}$ (which is $(1/n)\mu^K_{D_n}$) satisfies

$$\hat{\mu}(\mathbb{D}_r)=\begin{cases}ar/(1-r), & 0<r<1-a,\\ 1-a, & r\ge 1-a.\end{cases}$$

Since $D_n/n\to 1-a$, the limit of the empirical measure $(1/D_n)\mu^K_{D_n}$ will thus be $(1/(1-a))\hat{\mu}(\mathbb{D}_r)$, which is (15).

2.3. *Case* ③. In the case when

$$\lim_{n\to\infty}\frac{N_n}{n}=1\quad\text{and}\quad D_n\to\infty, \tag{48}$$

we only prove (17), which implies (16). To prove (17), we need to consider

$$\widetilde{K}_n(z):=R_n^{D_n}K_n^{(N_n)}\Big(\frac{z}{R_n}\Big)=\sum_{k=0}^{D_n}\xi_k\tilde{f}_{k,n}z^k, \tag{49}$$

where

$$\tilde{f}_{k,n}=f_{k,n}R_n^{D_n-k}\quad\text{and}\quad R_n=\frac{n}{D_n}. \tag{50}$$

It's enough to study $\widetilde{K}_n(z)$ since it has the same zeros as $K_n^{(N_n)}(z/R_n)$.

In this case, we need to choose $L_n=D_n$ in (34) with the decomposition

$$\frac{1}{D_n}\log f_{k,n}:=I_1(k,n)+I_2(k,n)+I_3(k,n).$$

Thus, we have the decomposition

$$\begin{aligned}\frac{1}{D_n}\log\tilde{f}_{k,n} &= \left(1-\frac{k}{D_n}\right)\log R_n + \frac{1}{D_n}\log f_{k,n}\\ &= \left[\left(1-\frac{k}{D_n}\right)\log R_n + I_1\right] + I_2 + I_3.\end{aligned} \tag{51}$$

As before, I_3 goes to 0 uniformly again since $D_n \to \infty$ as $n \to \infty$.

We note that

$$I_2(k,n) = \frac{1}{2D_n}(\log(k+N_n) + \log D_n - \log n - \log k)$$

is decreasing with respect to $k \geq 1$ for fixed N_n, D_n, and n; thus, we simply have $\sup_{0\leq k\leq D_n}|I_2(k,n)| = |I_2(1,n)| \vee |I_2(D_n,n)|$. Since $I_2(D_n,n) = 0$, we further have

$$\sup_{0\leq k\leq D_n}|I_2(k,n)| = |I_2(1,n)| = \frac{1}{2D_n}|\log(N_n+1) + \log D_n - \log n|.$$

By assumption (48), we can choose n large enough so that $N_n \geq \frac{1}{2}n$; thus, we have

$$\sup_{0\leq k\leq D_n}|I_2(k,n)| \leq \frac{1}{2D_n}\left(\log\left(\frac{n}{N_n}\right) + \log D_n\right) \leq \frac{\log 2}{2D_n} + \frac{\log D_n}{2D_n} \to 0,$$

since $D_n \to \infty$ as $n \to \infty$.

For I_1, we rewrite it as

$$\begin{aligned}I_1 &= \frac{1}{D_n}((k+n-D_n)\log(k+N_n) - n\log n + D_n\log D_n - k\log k)\\ &= \frac{1}{D_n}\left(n\log\frac{k+N_n}{n} + (k-D_n)\log(k+N_n) + D_n\log D_n - k\log k\right)\\ &= \frac{1}{D_n}\left(n\log\left(\frac{k+N_n}{n}\right) + (k-D_n)\log\left(\frac{k+N_n}{n}\right) + (k-D_n)\log n + D_n\log D_n - k\log D_n - k\log\left(\frac{k}{D_n}\right)\right)\\ &= \frac{n}{D_n}\log\left(\frac{n+k-D_n}{n}\right) - \frac{k}{D_n}\log\left(\frac{k}{D_n}\right) + \left(\frac{k}{D_n}-1\right)(\log n - \log D_n) + \left(\frac{k}{D_n}-1\right)\log\left(\frac{k+N_n}{n}\right).\end{aligned}$$

Thus, we can rewrite

$$\begin{aligned}\tilde{I}_1 :&= \left(1-\frac{k}{D_n}\right)\log R_n + I_1\\ &= \frac{n}{D_n}\log\left(\frac{n+k-D_n}{n}\right) - \frac{k}{D_n}\log\left(\frac{k}{D_n}\right) + \left(\frac{k}{D_n}-1\right)\log\left(\frac{k+N_n}{n}\right).\end{aligned}$$

Now we put

$$\log\tilde{f} = t - 1 - t\log t \quad \text{for } 0 \leq t \leq 1 \qquad \text{and} \qquad \log\tilde{f} = -\infty \quad \text{for } t > 1. \tag{52}$$

Then we can write $\tilde{I}_1$ as

$$\tilde{I}_1 = \log\tilde{f}\left(\frac{k}{D_n}\right) + I_9, \tag{53}$$

where

$$I_9 = \frac{n}{D_n}\left[\log\left(1+\frac{k-D_n}{n}\right)-\frac{k-D_n}{n}\right]+\left(\frac{k}{D_n}-1\right)\log\left(\frac{k+N_n}{n}\right).$$

Since $|\log(1+x)| \le |x|$ and $|\log(1+x)-x| \le x^2$ when $|x|$ is small, then we have the uniform estimate

$$\left|\log\left(1+\frac{k-D_n}{n}\right)-\frac{k-D_n}{n}\right| \le \left(\frac{k-D_n}{n}\right)^2 \le \left(\frac{D_n}{n}\right)^2$$

as n becomes large enough, which implies the first term in I_9 tends to 0.

Note that $1 \ge (k+N_n)/n \ge N_n/n$; thus, $|\log((k+N_n)/n)| \le |\log(N_n/n)| = |\log(1-D_n/n)| \le D_n/n$. If we combine this with the fact $|k/D_n - 1| \le 1$, we prove that the second term in I_9 also tends to 0. Hence, $I_9 \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} \left|\tilde{I}_1 - \log \tilde{f}\left(\frac{k}{D_n}\right)\right| = 0.$$

If we combine the estimates of $\tilde{I}_1$, I_2, and I_3 above, we have proved

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} \left|\frac{1}{D_n}\log \tilde{f}_{k,n} - \log \tilde{f}\left(\frac{k}{D_n}\right)\right| = 0. \tag{54}$$

As a summary, the coefficients $\tilde{f}_{k,n}$ satisfy Assumptions 2 with $L_n = D_n$, $T_0 = 1$, $\delta_n = 0$, and $\tilde{f}$. The Legendre–Fenchel transform $I(s) = \sup_{0\le t\le 1}(st + \log \tilde{f}(t))$ is

$$I(s) = \begin{cases} e^s - 1, & s < 0, \\ s, & s \ge 0. \end{cases}$$

Thus, the explicit expression (17) of the limiting measure $\tilde{\mu}^K$ follows by Theorem 2.

2.4. ***Case*** **④.** Now we prove Theorem 4 for the case where D_n remains a fixed positive integer m. The proof makes use of Rouché's theorem. We start with the following proposition regarding the convergence of zeros of a sequence of deterministic polynomials.

Proposition 7. *Let $G = \sum_{k=0}^{m} g_k z^k$, where $\{g_k\}$ are deterministic constants and $g_m \ne 0$. Let $G_n = \sum_{k=0}^{m} g_{k,n} z^k$, where $\{g_{k,n}\}$ are also deterministic. Assume $g_{k,n}$ converges to g_k for each fixed k. Then the measure of zeros μ_{G_n} will converge to μ_G in the sense of distribution.*

Proof. Let's choose ϕ as the smooth test function with compact support and pick $\epsilon > 0$ small enough. We first claim that for each zero z_0 of G with multiplicity α_0, for n large enough, G_n has exactly α_0 zeros in $\mathbb{D}(z_0, \epsilon)$, the open disc centered at z_0 with radius ϵ. Once this is done, since G has m zeros (m is a finite number), we can pick a common N_0 such that when $n > N_0$, G_n will have exactly α_i zeros in $\mathbb{D}(z_i, \epsilon)$ for any z_i in the zero set of G with multiplicity α_i. This means that we can make an appropriate ordering of the zero set of G (denoted by z_i, $1 \le i \le m$) and the zero set of G_n (denoted by $z_{i,n}$, $1 \le i \le m$) such that $|z_i - z_{i,n}| \le \epsilon$ for all i. Then we have

$$|\mu_{G_n}(\phi) - \mu_G(\phi)| \le \sum_{1\le i\le m} |\phi(z_i) - \phi(z_{i,n})| \le mK\epsilon, \tag{55}$$

where K is the sup norm of the derivative of ϕ. Since ϵ is arbitrarily small, this implies the weak convergence of μ_{G_n}. All the rest is to prove the claim.

Let's choose $\epsilon < 1$ small enough such that z_0 is the only zero of G with multiplicity $\alpha \geq 1$ in the closure of $\mathbb{D}(z_0, \epsilon)$. Assume $|z_0| + 1 \leq R$ for some R. For any $z \in \overline{\mathbb{D}(z_0, \epsilon)}$, we have

$$|G_n - G| \leq \sum_{k=0}^{m} |g_{n,k} - g_k| R^k. \tag{56}$$

Let's set

$$\eta(\epsilon) = \min_{z \in \partial\mathbb{D}(z_0,\epsilon)} |G(z)|;$$

then as n becomes large enough, we have

$$\sum_{k=0}^{m} |g_{n,k} - g_k| R^k < \eta(\epsilon),$$

which implies that

$$|G_n(z) - G(z)| < |G(z)| \quad \text{for any } z \in \partial\mathbb{D}(z_0, \epsilon).$$

Hence, G_n and G have the same number of zeros in $\mathbb{D}(z_0, \epsilon)$ by Rouché's theorem. This completes the proof of the claim and hence Proposition 7. □

Let's apply Proposition 7 to prove Theorem 4. In the case of $D_n = m$ and $N_n = n - m$, (31) reads

$$K_n^{(n-m)}(z) = \sum_{k=0}^{m} \xi_k f_{k,n} z^k.$$

To study the limiting behavior of zeros of $K_n^{(n-m)}(z/n)$, we may alternatively consider the random polynomials $G_n(z) = n^m K_n^{(n-m)}(z/n)$. The coefficients of G_n are

$$g_{k,n} = n^{m-k} f_{k,n} = \frac{m!}{k!} \frac{n^{m-k}}{n(n-1)\cdots(n-(m-k)+1)}.$$

Since k and m are both fixed when $n \to \infty$, we have

$$\lim_{n\to\infty} g_{k,n} = \frac{m!}{k!}.$$

By Proposition 7, the measure of zeros μ_{G_n} will converge to $\mu_{f_m^K}$ almost surely, where $\mu_{f_m^K}$ is the random measure of zeros of $f_m^K(z) = \sum_{k=0}^{m} (\xi_k/k!) z^k$. The limit (20) follows from this since $K_n^{(n-m)}(z/n)$ have the same zeros as G_n. In particular, the empirical measure of zeros of $K_n^{(n-m)}$ will converge to δ_0.

3. General random polynomials

In this section, we will apply the estimates we derived for the Kac polynomials in Section 2 to prove Theorem 5 for the general random polynomials.

Let p_n be the general random polynomials of degree n defined in (4). Let's assume that the coefficients $p_{k,n}$ satisfy Assumptions 1 with the associated continuous function p that is positive on [0, 1) and

$$\lim_{n\to\infty}\sup_{k\in[0,n]}\left||p_{k,n}|^{1/n}-p\left(\frac{k}{n}\right)\right|=0. \tag{57}$$

The N_n-th derivative of p_n is

$$p_n^{(N_n)}=\sum_{k=0}^{D_n}\xi_{k+N_n}p_{k+N_n,n}f_{k,n}z^k, \tag{58}$$

where $f_{k,n}$ is defined in (32). Since ξ_k are i.i.d., it's equivalent to consider the random polynomials

$$p_n^{(N_n)}=\sum_{k=0}^{D_n}\xi_k p_{k+N_n,n}f_{k,n}z^k, \tag{59}$$

where (58) and (59) have the same distribution of zeros. We set

$$u_{k,n}=p_{k+N_n,n}f_{k,n};$$

then we rewrite

$$p_n^{(N_n)}=\sum_{k=0}^{D_n}\xi_k u_{k,n}z^k.$$

We now verify that $u_{k,n}$ satisfy Assumptions 2 with some associated function u.

3.1. ***Case 1.*** ($N_n/n\to 0$). As in Case ① of Kac polynomials, we take $L_n=n$, $\delta_n=N_n/n$, and $T_0=1$. For fixed n, $f_{k,n}$ is increasing with k since

$$\frac{f_{k+1,n}}{f_{k,n}}=\frac{k+1+N_n}{k+1}>1.$$

Since $f_{D_n,n}=1$, it follows that $f_{k,n}\le 1$ for all n and $0\le k\le D_n$. By Assumptions 1, p is continuous on [0, 1] and therefore is bounded by C. Hence,

$$\begin{aligned}
&\sup_{0\le k\le D_n}\left||u_{k,n}|^{1/n}-p\left(\frac{k}{n}\right)\right|\\
&\quad\le \sup_{0\le k\le D_n}\left||p_{k+N_n,n}|^{1/n}-p\left(\frac{k}{n}\right)\right||f_{k,n}|^{1/n}+\sup_{0\le k\le D_n}\left||f_{k,n}|^{1/n}-1\right|p\left(\frac{k}{n}\right)\\
&\quad\le \sup_{0\le k\le D_n}\left||p_{k+N_n,n}|^{1/n}-p\left(\frac{k+N_n}{n}\right)\right|+\sup_{0\le k\le D_n}\left|p\left(\frac{k+N_n}{n}\right)-p\left(\frac{k}{n}\right)\right|+C\sup_{0\le k\le D_n}||f_{k,n}|^{1/n}-1|\\
&\quad:=J_1+J_2+J_3.
\end{aligned}$$

Our assumption (57) implies that J_1 converges to 0. J_2 converges to 0 since p is uniformly continuous on [0, 1] and N_n/n converges to 0 under the definition of the case. J_3 also converges to 0 by the estimate (38) which we have already proved for the Kac polynomials. Hence, the coefficients $u_{k,n}$ satisfy Assumptions 2 with $L_n=n$, $\delta_n=N_n/n$, $T_0=1$, and the associated function p. The conclusion (1) of Theorem 5 then follows.

3.2. ***Case 2.*** $(N_n/n \to a \in (0,1))$. As in Case ② of Section 2.2, we set $L_n = n$, $\delta_n = N_n/n - a$, and $T_0 = 1-a$; then $(T_0-\delta_n)L_n = D_n$. Let's choose f_1 as in (45) and set that f coincides with f_1 in $[0, 1-a]$ and equals 0 in $[1-a, \infty)$ as in the Kac case. Proceeding like Case 1 above, we have

$$\begin{aligned}
&\sup_{0\le k\le D_n}\left||u_{k,n}|^{1/n} - p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big) f\Big(\frac{k}{n}\wedge T_0\Big)\right| \\
&\quad\le \sup_{0\le k\le D_n}|f_{k,n}|^{1/n}\left||p_{k+N_n,n}|^{1/n} - p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\right| + \sup_{0\le k\le D_n} p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\left||f_{k,n}|^{1/n} - f\Big(\frac{k}{n}\wedge T_0\Big)\right| \\
&\quad\le \sup_{0\le k\le D_n}|f_{k,n}|^{1/n}\left||p_{k+N_n,n}|^{1/n} - p\Big(\frac{n+N_n}{n}\Big)\right| + \sup_{0\le k\le D_n}|f_{k,n}|^{1/n}\left|p\Big(\frac{k+N_n}{n}\Big) - p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\right| \\
&\qquad\qquad + \sup_{0\le k\le D_n} p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\left||f_{k,n}|^{1/n} - f\Big(\frac{k}{n}\wedge T_0\Big)\right| \\
&\quad := J_1 + J_2 + J_3.
\end{aligned}$$

As in Case 1, our assumptions of p imply that J_1 converges to 0; J_3 converges to 0, which is equivalent to (47) as in the Kac case. Again using the boundedness of $f_{k,n}$ and the uniform continuity of p together with the fact that

$$\sup_{0\le k\le D_n}\left|\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big) - \frac{k+N_n}{n}\right| \le |\delta_n|,$$

we have $J_2 \to 0$ since $\delta_n \to 0$. Hence, the coefficients $u_{k,n}$ satisfy Assumptions 2 with $u^a(t) = f(t)p(t+a)$; this will complete the proof of Theorem 5(2).

4. Random elliptic polynomials

In this section, we will prove Theorem 6 for the random elliptic polynomials E_n defined in (23). Let's denote by

$$p^E_{k,n} = \sqrt{\binom{n}{k}}$$

the coefficients. By Stirling's formula, one can prove that the coefficients $p^E_{k,n}$ satisfy Assumptions 1 with the associated function p^E given in (24). Thus, Theorem 6(1) is the direct consequence of Theorem 5. Now let's prove Theorem 6(2), which is the interesting part, and the nontrivial ingredient is to find the rescaling factor.

As in (59), the N_n-th derivative of E_n is equivalent to

$$E_n^{(N_n)} = \sum_{k=0}^{D_n} \xi_k p^E_{k+N_n,n} f_{k,n} z^k := \sum_{k=0}^{D_n} \xi_k u^E_{k,n} z^k. \tag{60}$$

Let's first consider the case when $N_n/n \to 1$ and $D_n \to \infty$. By discarding a negligible lower-order term and by Stirling's formula, we have

$$\begin{aligned}\frac{1}{D_n}\log p^E_{k+N_n,n} &\sim \frac{1}{2D_n}(n\log n-(k+N_n)\log(k+N_n)-(D_n-k)\log(D_n-k))\\ &=\frac{1}{2}\left(\frac{k+N_n}{D_n}\log\left(\frac{n}{k+N_n}\right)+\frac{D_n-k}{D_n}\log\left(\frac{n}{D_n-k}\right)\right)\\ &=\frac{1}{2}\left(-\frac{n+k-D_n}{D_n}\log\left(\frac{n-D_n+k}{n}\right)-\frac{D_n-k}{D_n}\log\left(\frac{D_n-k}{D_n}\right)+\frac{D_n-k}{D_n}\log\left(\frac{n}{D_n}\right)\right)\\ &=I_{1,1}+I_{1,2}+I_{1,3}. \end{aligned}\tag{61}$$

By $|\log(1+x)-x|\le x^2$ when $|x|$ is small, we can get the uniform estimate

$$\left|I_{1,1}-\frac{1}{2}\left(-\frac{n+k-D_n}{D_n}\frac{-D_n+k}{n}\right)\right|\le\frac{n}{2D_n}\left(\frac{-D_n+k}{n}\right)^2\le\frac{n}{2D_n}\left(\frac{D_n}{n}\right)^2\to 0.$$

We also have the uniform estimate

$$\left|\frac{1}{2}\left(-\frac{n+k-D_n}{D_n}\frac{-D_n+k}{n}\right)-\frac{D_n-k}{2D_n}\right|=\frac{(D_n-k)^2}{2nD_n}\le\frac{D_n}{2n}\to 0;$$

it follows that if we define

$$h_1=\tfrac{1}{2}(1-t),$$

then

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|I_{1,1}-h_1\left(\frac{k}{D_n}\right)\right|=0.\tag{62}$$

Let's put

$$h_2=-\tfrac{1}{2}(1-t)\log(1-t);$$

then we can rewrite

$$I_{1,2}=h_2\left(\frac{k}{D_n}\right).\tag{63}$$

The trick now is to eliminate $I_{1,3}$ by a rescaling factor. To be more explicit, let's put $R_n=n/D_n$ again and put

$$\tilde{p}^E_{k+N_n,n}=p^E_{k+N_n,n}R_n^{-(D_n-k)/2}.\tag{64}$$

By defining in this way, we note that

$$\frac{1}{D_n}\log R_n^{-(D_n-k)/2}=-I_{1,3};\tag{65}$$

hence, if we combine (61)–(65) and define the function

$$\log\tilde{p}^E(x)=h_1+h_2=\tfrac{1}{2}(1-t)-\tfrac{1}{2}(1-t)\log(1-t),\tag{66}$$

then we have proved

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|\frac{1}{D_n}\log\tilde{p}^E_{k+N_n,n}-\log\tilde{p}^E\left(\frac{k}{D_n}\right)\right|=0.\tag{67}$$

Let's further recall (50) in the proof of Case ③ for the Kac case where

$$\tilde{f}_{k,n}=f_{k,n}R_n^{D_n-k};\tag{68}$$

then we can rewrite (60) as

$$E_n^{(N_n)}(z) = \sum_{k=0}^{D_n} \xi_k \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n} z^k R_n^{-(D_n-k)/2}.$$

Therefore, the rescaling random polynomials read

$$E_n^{(N_n)}\left(\frac{z}{\sqrt{R_n}}\right) = R_n^{-D_n/2} \sum_{k=0}^{D_n} \xi_k \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n} z^k. \tag{69}$$

Let's define

$$\widetilde{E}_n^{(N_n)}(z) := \sum_{k=0}^{D_n} \xi_k \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n} z^k.$$

Let's derive the limit of the empirical measure of zeros of $E_n^{(N_n)}(z/\sqrt{R_n})$, which is the same as $\widetilde{E}_n^{(N_n)}(z)$. To do this, let's define the coefficients $\tilde{u}^E_{k,n} := \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n}$; then the estimates (54) and (67) imply that $\tilde{u}^E_{k,n}$ satisfy Assumptions 2 with $L_n = D_n$, $\delta_n = 0$, and $T_0 = 1$ and the associated function $\tilde{u}^E$ is given by $\log \tilde{u}^E = \log \tilde{p}^E + \log \tilde{f}$. By (52) and (66), we have

$$\log \tilde{u}^E(t) = \begin{cases} \frac{1}{2}(t-1) - \frac{1}{2}(1-t)\log(1-t) - t\log t, & 0 \le t \le 1, \\ -\infty, & t > 1. \end{cases}$$

Therefore, $(1/D_n)\mu_{D_n}^{\widetilde{E}}$, or equivalently $(1/D_n)\mathcal{S}_{\sqrt{R_n}}(\mu^E_{D_n})$, converges in probability to a deterministic measure. To find out the limit, we compute the Legendre–Fenchel transform of $-\log \tilde{u}^E$ as

$$I(s) = \sup_{0\le t\le 1} (st + \log \tilde{u}(t)) = \tfrac{1}{2}(t_s - 1) - \tfrac{1}{2}\log(1-t_s),$$

where $t_s = (-1+\sqrt{1+4e^{-2s}})/(2e^{-2s})$. Therefore, (25) follows by Theorem 2.

The analysis for the case when D_n remains a fixed number m follows exactly the same approach as in Section 2.4 for the Kac case. Recall the definition of $u^E_{k,n}$ in (60); if we replace $D_n = m$ and $N_n = n - m$, then we can rewrite

$$u^E_{k,n} = \left(\frac{n!}{(k+n-m)!\,(m-k)!}\right)^{1/2} \frac{(k+n-m)!\,m!}{k!\,n!} = \frac{m!}{k!}\left(\frac{(n-m+k)!}{n!\,(m-k)!}\right)^{1/2}.$$

Now we consider the rescaling random polynomials

$$\widetilde{E}_n^m(z) := n^{m/2} E_n^{(n-m)}\left(\frac{z}{\sqrt{n}}\right) = \sum_{k=0}^{m} \tilde{u}^E_{k,n} \xi_k z^k,$$

where $\tilde{u}^E_{k,n} = u^E_{k,n} n^{(m-k)/2}$. Since m and k are both fixed when $n \to \infty$, we get

$$\lim_{n\to\infty} \tilde{u}^E_{k,n} = \frac{m!}{k!\,((m-k)!)^{1/2}}.$$

Therefore, since $\widetilde{E}_n^m(z)$ have the same zeros as $E_n^{(n-m)}(z/\sqrt{n})$, then by Proposition 7, the limiting measure $\mathcal{S}_{\sqrt{n}}(\mu^E_{D_n})$ when $D_n = m$ will tend to the random zeros of

$$f_m^E = \sum_{k=0}^{m} \frac{1}{k!\,((m-k)!)^{1/2}} \xi_k z^k$$

in distribution, which completes the proof of Theorem 6.

Appendix: Proof of Theorem 2

Now we sketch the proof of Theorem 2 by modifying the one in [Kabluchko and Zaporozhets 2014].

Let's first recall the proof of Theorem 1 in [Kabluchko and Zaporozhets 2014]. For random analytic functions $F(z)$ defined in (9) where the coefficients satisfy Assumptions 1, if one establishes the convergence in probability

$$\frac{1}{n} \log|F_n(z)| \to I(\log|z|) \tag{70}$$

as $n \to \infty$, then Theorem 1 follows by the classical Poincaré–Lelong formula. Kabluchko and Zaporozhets proved (70) by establishing some appropriate upper and lower bounds for $|F_n(z)|$; see estimates (22) and (27) in [Kabluchko and Zaporozhets 2014].

Under Assumptions 2, the convergence radius is automatically infinity because we are now dealing with a finite sum for any fixed n. Given random polynomials F_n in the form of (12) satisfying Assumptions 2, to prove Theorem 2, it's enough to derive the analogue convergence

$$\frac{1}{L_n} \log|F_n(z)| \to I(\log|z|) \tag{71}$$

as $n \to \infty$, where the convergence is also in probability. To prove this, we need the same upper and lower bounds as in [Kabluchko and Zaporozhets 2014].

For the upper bound, for any $\epsilon > 0$, we have

$$|F_n(z)| \le M e^{L_n(I(\log|z|)+3\epsilon+\delta_n^-(\log|z|)^+)} \quad \text{for } n \text{ large enough,} \tag{72}$$

where M is an almost surely finite random variable depending on ϵ. Here we use the convention that for any real number w, w^+ and w^- are the positive and negative parts of w, i.e., $w^+ = w \vee 0$ and $w^- = (-w) \vee 0$.

We also need to show the lower bound estimate

$$\mathbb{P}(|F_n(z)| < e^{L_n(I(\log|z|)-4\epsilon)}) = O\left(\frac{1}{\sqrt{L_n}}\right) \quad \text{as } n \to \infty. \tag{73}$$

Recall Lemma 4.4 in [Kabluchko and Zaporozhets 2014]; we know that for any $A > 0$, there exists an almost surely finite random variable M' such that $|\xi_k| \le M' e^{Ak}$ for all k with probability 1. If we set $A = \epsilon/(2T_0)$, then for all $0 \le k \le (T_0 - \delta_n)L_n$, we have

$$|\xi_k| \le M' e^{\epsilon k/(2T_0)} \le M' e^{\epsilon L_n}. \tag{74}$$

To prove (72), if we apply the bound (74) together with Assumptions 2, for n large enough and δ small enough, we have

$$\begin{aligned}|F_n(z)| &= \Big|\sum_{0\le k\le (T_0-\delta_n)L_n} \xi_k p_{k,n} z^k\Big| \le \sum_{0\le k\le (T_0-\delta_n)L_n} |\xi_k||p_{k,n}||z|^k \\ &\le M' e^{\epsilon L_n}\Big(\sum_{0\le k\le (T_0-\delta_n^+)L_n} |p_{k,n}||z|^k + \sum_{T_0L_n<k\le (T_0+\delta_n^-)L_n} |p_{k,n}||z|^k\Big) \\ &\le M' e^{\epsilon L_n} \sum_{0\le k\le (T_0-\delta_n^+)L_n} (e^{(k/L_n)\log|z|+\log p(k/L_n)} + \delta|z|^{k/L_n})^{L_n} \\ &\qquad + M' e^{\epsilon L_n} \sum_{T_0L_n<k\le (T_0+\delta_n^-)L_n} (e^{(k/L_n-T_0)\log|z|+(T_0\log|z|+\log p(T_0))} + \delta|z|^{k/L_n})^{L_n}.\end{aligned}$$

By the definition of the Legendre–Fenchel transform, we further have

$$\begin{aligned}|F_n(z)| &\le M' e^{2\epsilon L_n}(e^{I(\log|z|)} + \delta(1\vee|z|^{T_0}))^{L_n} + M' e^{2\epsilon L_n} e^{\delta_n^-(\log|z|)^+ L_n}(e^{I(\log|z|)} + \delta(1\vee|z|^{2T_0}))^{L_n} \\ &\le M'' e^{L_n(I(\log|z|)+3\epsilon+\delta_n^-(\log|z|)^+)},\end{aligned}$$

where M'' is another almost surely finite random variable, which completes the proof of the upper bound.

For the lower bound (73), if we choose the set J as the one in the proof of (27) in [Kabluchko and Zaporozhets 2014], then the assumptions $L_n \to \infty$ and $\delta_n \to 0$ imply that the set $\{k : 0 \le k \le (T_0-\delta_n)L_n,\ k/L_n \in J\}$ has cardinality bounded below by $(|J|/2)L_n$. The rest proof follows the one in [Kabluchko and Zaporozhets 2014] by replacing n by L_n and hence the lower bound follows.

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GROSS–PITAEVSKII DYNAMICS FOR BOSE–EINSTEIN CONDENSATES

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We study the time-evolution of initially trapped Bose–Einstein condensates in the Gross–Pitaevskii regime. We show that condensation is preserved by the many-body evolution and that the dynamics of the condensate wave function can be described by the time-dependent Gross–Pitaevskii equation. With respect to previous works, we provide optimal bounds on the rate of condensation (i.e., on the number of excitations of the Bose–Einstein condensate). To reach this goal, we combine the method of Lewin, Nam and Schlein (2015), who analyzed fluctuations around the Hartree dynamics for N-particle initial data in the mean-field regime, with ideas of Benedikter, de Oliveira and Schlein (2015), who considered the evolution of Fock-space initial data in the Gross–Pitaevskii regime.

1. Introduction and main results

Trapped gases of N bosons in the Gross–Pitaevskii regime can be described by the Hamilton operator

$$H_N^{\text{trap}} = \sum_{j=1}^{N} [-\Delta_{x_j} + V_{\text{ext}}(x_j)] + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{1-1}$$

acting on the Hilbert space $L^2_s(\mathbb{R}^{3N})$, the subspace of $L^2(\mathbb{R}^{3N})$ consisting of functions that are symmetric with respect to permutations of the N particles. Here, V_{ext} is a confining external potential. As for the interaction potential V, we assume it to be pointwise nonnegative, spherically symmetric and compactly supported (but our results could be easily extended to potentials decaying sufficiently fast at infinity).

Characteristically for the Gross–Pitaevskii regime, the interaction $N^2 V(N\,\cdot\,)$ appearing in (1-9) scales with N so that its scattering length is of the order N^{-1}. The scattering length a_0 of the unscaled potential V is defined by the condition that the solution of the zero-energy scattering equation

$$\big[-\Delta + \tfrac{1}{2} V(x)\big] f(x) = 0, \tag{1-2}$$

with the boundary condition $f(x) \to 1$ for $|x| \to \infty$, has the form

$$f(x) = 1 - \frac{a_0}{|x|} \tag{1-3}$$

outside the support of V. Equivalently, a_0 is determined by

$$8\pi a_0 = \int V(x) f(x)\, dx. \tag{1-4}$$

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By scaling, (1-2) also implies that

$$\big[-\Delta + \tfrac{1}{2}N^2 V(Nx)\big] f(Nx) = 0,$$

with $f(Nx) \to 1$ for $|x| \to \infty$. In particular, this means that the rescaled potential $N^2 V(N\,\cdot)$ in (1-9) has scattering length a_0/N.

It was shown in [Lieb et al. 2000], and more recently in [Nam et al. 2016], that the ground state energy E_N of the Hamilton operator (1-1) is such that

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\substack{\varphi\in L^2(\mathbb{R}^3)\\ \|\varphi\|_2=1}} \mathcal{E}_{\mathrm{GP}}^{\mathrm{trap}}(\varphi), \tag{1-5}$$

with the Gross–Pitaevskii energy functional

$$\mathcal{E}_{\mathrm{GP}}^{\mathrm{trap}}(\varphi) = \int \big[|\nabla\varphi(x)|^2 + V_{\mathrm{ext}}(x)|\varphi(x)|^2 + 4\pi a_0 |\varphi(x)|^4 \big]\, dx. \tag{1-6}$$

Furthermore, Bose–Einstein condensation in the ground state of (1-1) was established in [Lieb and Seiringer 2002]. More precisely, it was also shown in that paper that if $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}\, |\psi_N\rangle\langle\psi_N|$ denotes the one-particle reduced density associated with the ground state of (1-1), then

$$\gamma_N^{(1)} \to |\phi_{\mathrm{GP}}\rangle\langle\phi_{\mathrm{GP}}|, \tag{1-7}$$

where $\phi_{\mathrm{GP}} \in L^2(\mathbb{R}^3)$ is the unique nonnegative minimizer of (1-6), among all $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. The interpretation of (1-7) is straightforward: in the ground state of (1-1), all particles, up to a fraction vanishing in the limit of large N, are in the same one-particle state ϕ_{GP}.

In typical experiments, one observes the time-evolution of trapped Bose gases prepared in (or close to) their ground state, resulting from a change of the external fields. As an example, consider the situation in which the trapping potential is switched off at time $t = 0$. In this case, the dynamics is described, at the microscopic level, by the many-body Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \tag{1-8}$$

with the translation-invariant Hamilton operator

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \sum_{i<j}^N N^2 V(N(x_i - x_j)) \tag{1-9}$$

and with the ground state of (1-1) as initial data. The next theorem shows how the solution of (1-8) can be described in terms of the time-dependent Gross–Pitaevskii equation.

Theorem 1.1. *Let $V_{\mathrm{ext}} : \mathbb{R}^3 \to \mathbb{R}$ be locally bounded with $V_{\mathrm{ext}}(x) \to \infty$ as $|x| \to \infty$. Let $V \in L^3(\mathbb{R}^3)$ be nonnegative ($V(x) \geq 0$ for almost every $x \in \mathbb{R}^3$), compactly supported and spherically symmetric. Let ψ_N be a sequence in $L^2_s(\mathbb{R}^{3N})$, with one-particle reduced density $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}\, |\psi_N\rangle\langle\psi_N|$. We assume that, as $N \to \infty$,*

$$\begin{aligned} a_N &= 1 - \langle \phi_{\mathrm{GP}}, \gamma_N^{(1)} \phi_{\mathrm{GP}} \rangle \to 0, \\ b_N &= |N^{-1} \langle \psi_N, H_N^{\mathrm{trap}} \psi_N \rangle - \mathcal{E}_{\mathrm{GP}}^{\mathrm{trap}}(\phi_{\mathrm{GP}})| \to 0, \end{aligned} \tag{1-10}$$

where $\phi_{\mathrm{GP}} \in H^4(\mathbb{R}^3)$ *is the unique nonnegative minimizer of the Gross–Pitaevskii energy functional* (1-6). *Let* $\psi_{N,t} = e^{-iH_N t}\psi_N$ *be the solution of* (1-8) *with initial data* ψ_N *and let* $\gamma_{N,t}^{(1)}$ *be the one-particle reduced density associated with* $\psi_{N,t}$. *Then there are constants* $C, c > 0$ *such that*

$$1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t \rangle \le C[a_N + b_N + N^{-1}] \exp(c \exp(c|t|)) \tag{1-11}$$

for all $t \in \mathbb{R}$. *Here* φ_t *is the solution of the time-dependent Gross–Pitaevskii equation*

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t, \tag{1-12}$$

with the initial data $\varphi_{t=0} = \phi_{\mathrm{GP}}$.

Remarks. (1) The condition $a_N = 1 - \langle \phi_{\mathrm{GP}}, \gamma_N^{(1)} \phi_{\mathrm{GP}} \rangle \to 0$ is equivalent to $\gamma_N^{(1)} \to |\phi_{\mathrm{GP}}\rangle\langle\phi_{\mathrm{GP}}|$. Similarly, the bound (1-11) implies that $\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t|$. More precisely, using the fact that $|\varphi_t\rangle\langle\varphi_t|$ is a rank-one projection, it follows from (1-11) that

$$\begin{aligned} \operatorname{tr} |\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|| &\le 2\|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\|_{\mathrm{HS}} \\ &\le 2^{3/2}[1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t \rangle]^{1/2} \\ &\le C[a_N + b_N + N^{-1}]^{1/2} \exp(c \exp(c|t|)). \end{aligned}$$

Hence, (1-11) is a statement about the stability of Bose–Einstein condensation with respect to the many-body Schrödinger equation (1-8).

(2) Existence, uniqueness and decay of the minimizer ϕ_{GP} of the Gross–Pitaevskii energy functional (1-6) were established in [Lieb et al. 2000]. In Theorem 1.1 we additionally assume that $\phi_{\mathrm{GP}} \in H^4(\mathbb{R}^3)$. This condition follows from elliptic regularity and from the results of [Gagelman and Yserentant 2012] (establishing decay of the derivatives of ϕ_{GP}), under suitable assumptions on V_{ext} (for example, if $V_{\mathrm{ext}} \in C^2(\mathbb{R}^3)$ and its derivatives grow at most exponentially at infinity).

(3) As discussed above, it follows from [Lieb et al. 2000; Lieb and Seiringer 2002] that the assumptions (1-10) are satisfied if we take ψ_N as the ground state of (1-1). In this case, we expect both a_N and b_N to be of the order N^{-1}; indeed, $a_N, b_N \simeq N^{-1}$ was recently shown in [Boccato et al. 2018b] for systems of bosons trapped in a box with volume 1 (with periodic boundary conditions), interacting through a sufficiently small potential; in fact, the limit of Na_N, Nb_N was computed precisely in [Boccato et al. 2018a]. In this case, (1-11) implies that

$$1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t \rangle \le CN^{-1} \exp(c \exp(c|t|)) \tag{1-13}$$

and therefore that, for every fixed time $t \in \mathbb{R}$, Bose–Einstein condensation holds with the optimal rate N^{-1} (meaning that the number of excitations of the condensate remains bounded, uniformly in N[†]).

(4) To keep the notation as simple as possible, we consider the time-evolution (1-8) generated by the translation-invariant Hamiltonian (1-9). With the same techniques we use to prove Theorem 1.1, we could also have included in (1-9) an external potential W_{ext}, at least if the difference $W_{\mathrm{ext}} - V_{\mathrm{ext}}$ is bounded

[†] If $N[1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t \rangle] \to 0$, as $N \to \infty$, the expectation of the number of excitations of the condensate would tend to zero and thus $\psi_{N,t}$ could be approximated, in norm, by the factorized wave function $\varphi_t^{\otimes N}$; this cannot be true.

below. Under this assumption, the convergence (1-11) remains true, of course provided we introduce the external potential W_{ext} also in the time-dependent Gross–Pitaevskii equation (1-12). The external potential may also depend on time, under reasonable assumptions on the time-dependence (for example, if the time-derivative of W_{ext} is bounded). Physically, this would describe experiments where the system prepared at equilibrium (in the ground state) is perturbed by a change of the external potential, rather than by switching it off (we could also consider the situation where the external potential depends on time).

Theorem 1.1 is meant to describe the time-evolution of data prepared in the ground state of the trapped Hamilton operator (1-1). This is the reason why, in (1-10), we assumed ψ_N to exhibit Bose–Einstein condensation in the minimizer of the Gross–Pitaevskii energy functional (1-6). From the mathematical point of view, one may ask more generally whether it is possible to show that the evolution of an initial data exhibiting Bose–Einstein condensate in an arbitrary one-particle wave function $\varphi \in H^1(\mathbb{R}^3)$ (not necessarily minimizing the Gross–Pitaevskii functional (1-6)) continues to exhibit condensation in the solution of (1-12) with initial data $\varphi_{t=0} = \varphi$, also for $t \neq 0$. In the next theorem we show that the answer to this question is positive; the only difference with respect to (1-11) is the fact that, to get the same rate of convergence at time t, we need a stronger bound on the condensation of the initial data.

Theorem 1.2. *Assume that $V \in L^3(\mathbb{R}^3)$ is nonnegative ($V(x) \geq 0$ for almost every $x \in \mathbb{R}^3$), compactly supported and spherically symmetric. Let ψ_N be a sequence in $L^2_s(\mathbb{R}^{3N})$, with one-particle reduced density $\gamma_N^{(1)} = \operatorname{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$. Assume that, for a $\varphi \in H^4(\mathbb{R}^3)$,*

$$\begin{aligned} \tilde{a}_N &= \operatorname{tr} |\gamma_N^{(1)} - |\varphi\rangle\langle\varphi|| \to 0, \\ \tilde{b}_N &= |N^{-1}\langle\psi_N, H_N\psi_N\rangle - \mathcal{E}_{\text{GP}}(\varphi)| \to 0 \end{aligned} \tag{1-14}$$

as $N \to \infty$. Here $\mathcal{E}_{\text{GP}}$ is the translation-invariant Gross–Pitaevskii functional

$$\mathcal{E}_{\text{GP}}(\varphi) = \int [|\nabla\varphi|^2 + 4\pi a_0 |\varphi|^4]\, dx. \tag{1-15}$$

Let $\psi_{N,t} = e^{-iH_N t}\psi_N$ be the solution of the Schrödinger equation (1-8) with initial data ψ_N and let $\gamma_{N,t}^{(1)}$ denote the one-particle reduced density associated with $\psi_{N,t}$. Then

$$1 - \langle\varphi_t, \gamma_{N,t}^{(1)}\varphi_t\rangle \leq C[\tilde{a}_N + \tilde{b}_N + N^{-1}] \exp(c\exp(c|t|)), \tag{1-16}$$

where φ_t denotes the solution of the time-dependent Gross–Pitaevskii equation (1-12), with initial data $\varphi_0 = \varphi$.

A first proof of the convergence of the reduced density associated with the solution of the Schrödinger equation (1-8) towards the orthogonal projection onto the solution of the time-dependent Gross–Pitaevskii equation (1-12) was obtained in [Erdős et al. 2002; 2007; 2009b; 2010]; part of the proof was later simplified in [Chen et al. 2015], using also ideas from [Klainerman and Machedon 2008]. In these works, convergence was established with no control on its rate. A new proof of the convergence towards the Gross–Pitaevskii dynamics was later given in [Pickl 2015]; in this case, convergence was shown to hold with a rate $N^{-\eta}$, for a nonoptimal $\eta > 0$, whose value could be explicitly determined following the proof; this approach was adapted to two-dimensional systems in [Jeblick et al. 2016], to systems with magnetic fields in [Olgiati 2017] and to pseudospinor condensates in [Michelangeli and Olgiati 2017]. More recently, convergence with a rate similar to (1-11), (1-16) was proven to hold in [Benedikter et al. 2015] for a class of Fock space initial data. The novelty of (1-11), (1-16) is the fact that convergence is shown with an optimal rate determined by the properties of the N-particle initial data.

More results are available about quantum dynamics in the mean-field regime. In this case, the evolution of the Bose gas is generated by a Hamilton operator of the form

$$H_N^{\mathrm{mf}} = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N V(x_i - x_j). \tag{1-17}$$

In the limit $N \to \infty$, the solution of the Schrödinger equation $\psi_{N,t} = e^{-iH_N^{\mathrm{mf}}t}\psi_N$, for initial data ψ_N exhibiting Bose–Einstein condensation in a one-particle wave function $\varphi \in L^2(\mathbb{R}^3)$, can be approximated by products of the solution of the nonlinear Hartree equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + (V * |\varphi_t|^2)\varphi_t. \tag{1-18}$$

Convergence towards Hartree dynamics has been established in different settings, using different methods in several works, including [Adami et al. 2007; Ammari and Breteaux 2012; Ammari et al. 2016; Anapolitanos and Hott 2016; Ammari and Nier 2009; Bardos et al. 2000; Chen and Holmer 2017; Elgart and Schlein 2007; Erdős and Yau 2001; Fröhlich et al. 2007; 2009; Ginibre and Velo 1979a; 1979b; Hepp 1974; Knowles and Pickl 2010; Rodnianski and Schlein 2009; Spohn 1980]. In the mean-field regime, it is also possible to find a norm approximation of the many-body evolution by taking into account fluctuations around the Hartree dynamics (1-18); see, for example, [Ben Arous et al. 2013; Chen 2012; Grillakis et al. 2010; 2011; Kirkpatrick et al. 2011; Lewin et al. 2015a; Mitrouskas et al. 2016].

It is also interesting to consider the many-body evolution in scaling limits interpolating between the mean-field regime described by the Hamilton operator (1-17) and the Gross–Pitaevskii regime described by (1-9). A norm-approximation of the time-evolution in these intermediate regimes was recently obtained in [Boccato et al. 2017; Grillakis and Machedon 2013; Kuz 2017; Nam and Napiórkowski 2016; 2017].

To prove Theorem 1.1 and Theorem 1.2 we will combine the strategies used in [Benedikter et al. 2015] and [Lewin et al. 2015a]. Let us briefly recall the main ideas of these papers. In [Benedikter et al. 2015], the Bose gas was described on the Fock space $\mathcal{F} = \bigoplus_{n\geq 0} L_s^2(\mathbb{R}^{3n})$ by the Hamilton operator

$$\mathcal{H}_N = \int \nabla_x a_x^* \nabla_x a_x \, dx + \frac{1}{2}\int N^2 V(N(x-y))\, a_x^* a_y^* a_y a_x \, dx\, dy,$$

where a_x^*, a_x are the usual operator-valued distributions, creating and, respectively, annihilating a particle at the point $x \in \mathbb{R}^3$. Notice that $\mathcal{H}_N$ commutes with the number of particles operator $\mathcal{N} = \int a_x^* a_x \, dx$, and that its restriction to the sector of $\mathcal{F}$ with exactly N particles coincides with (1-9).

On the Fock space $\mathcal{F}$, a Bose–Einstein condensate can be described by a coherent state of the form $W(\sqrt{N}\varphi)\Omega$, where $\Omega = \{1, 0, 0, \dots\}$ is the vacuum vector, $\varphi \in L^2(\mathbb{R}^3)$ is a normalized one-particle orbital, and where, for every $f \in L^2(\mathbb{R}^3)$,

$$W(f) = \exp(a^*(f) - a(f))$$

is a Weyl operator with wave function f. Here, we denoted by

$$a^*(f) = \int f(x)\, a_x^*\, dx \quad \text{and} \quad a(f) = \int \bar{f}(x)\, a_x\, dx$$

the usual creation and annihilation operators on $\mathcal{F}$, creating and annihilating a particle with wave function f. A simple computation shows that

$$W(\sqrt{N}\varphi)\Omega = e^{-N/2}\left\{1, N^{1/2}\varphi, \dots, \frac{N^{n/2}\varphi^{\otimes n}}{\sqrt{n!}}, \dots\right\}.$$

In the coherent state $W(\sqrt{N}\varphi)\Omega$, the number of particles is Poisson distributed, with mean and variance equal to N.

On the Fock space $\mathcal{F}$, it is interesting to study the dynamics of approximately coherent initial states. In the Gross–Pitaevskii regime, however (in contrast with the mean-field limit), we cannot expect the evolution of approximately coherent initial data to remain approximately coherent. On every sector of $\mathcal{F}$ with a fixed number of particles, the coherent state $W(\sqrt{N}\varphi)\Omega$ is factorized; it describes therefore uncorrelated particles. On the other hand, already from [Erdős et al. 2009a; 2010] and more recently also from [Chen and Holmer 2016], we know that, in the Gross–Pitaevskii regime, particles develop substantial correlations. To provide a better approximation of the many-body dynamics, Weyl operators were combined in [Benedikter et al. 2015] with appropriate Bogoliubov transformations, leading to so-called squeezed coherent states. To be more precise, let f denote the solution of the zero-energy scattering equation (1-2) and $w = 1 - f$ (keep in mind that, for $|x| \gg 1$, $w(x) = a_0/|x|$). Using w, we define

$$k_{N,t}(x; y) = -N w(N(x-y))\varphi_t(x)\varphi_t(y), \tag{1-19}$$

where φ_t is the solution of the time-dependent Gross–Pitaevskii equation (1-12). In fact, in [Benedikter et al. 2015] and also later in the present paper, it is more convenient to replace φ_t with the solution of the slightly modified, N-dependent, Gross–Pitaevskii equation (4-8); to simplify the presentation, we neglect these technical details in this introduction. With (1-19), it is easy to check that $k_{N,t} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, with $\|k_{N,t}\|_2$ bounded, uniformly in N and in t. This implies that (1-19) is the integral kernel of a Hilbert–Schmidt operator. Hence, we can define, on $\mathcal{F}$, the unitary Bogoliubov transformation

$$T_t = \exp\left[\frac{1}{2}\int dx\, dy\, (k_{N,t}(x; y)\, a_x^* a_y^* - \text{h.c.})\right], \tag{1-20}$$

whose action on creation and annihilation operators is explicitly given by

$$T_t^* a^*(g) T_t = a^*(\cosh_{k_{N,t}}(g)) + a(\sinh_{k_{N,t}}(\bar{g})) \tag{1-21}$$

for all $g \in L^2(\mathbb{R}^3)$. Here $\cosh_{k_{N,t}}$ and $\sinh_{k_{N,t}}$ are the bounded operators ($\sinh_{k_{N,t}}$ is even Hilbert–Schmidt) defined by the convergent series

$$\cosh_{k_{N,t}} = \sum_{n=0}^{\infty} \frac{(k_{N,t}\bar{k}_{N,t})^n}{(2n)!} \quad \text{and} \quad \sinh_{k_{N,t}} = \sum_{n=0}^{\infty} \frac{(k_{N,t}\bar{k}_{N,t})^n k_{N,t}}{(2n+1)!}. \tag{1-22}$$

Using the Bogoliubov transformation T_t to generate correlations at time t, it makes sense to study the time-evolution of initial data close to the squeezed coherent state $W(\sqrt{N}\varphi)T_0\Omega$, and to approximate it with a Fock-space vector of the same form. More precisely, for $\xi_N \in \mathcal{F}$ close to the vacuum (in a sense to be made precise later), we may consider the time-evolution

$$e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)T_0\xi_N = W(\sqrt{N}\varphi_t)T_t\xi_{N,t}, \tag{1-23}$$

where we defined $\xi_{N,t} = \mathcal{U}_N(t)\xi_N$ and the fluctuation dynamics

$$\mathcal{U}_N(t) = T_t^* W^*(\sqrt{N}\varphi_t)e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi_0)T_0. \tag{1-24}$$

In order to show that the one-particle reduced density $\gamma_{N,t}^{(1)}$ associated with the left-hand side of (1-23) is close to the orthogonal projection onto the solution of the Gross–Pitaevskii equation (4-8), it is enough to prove that the expectation of the number of particles in $\xi_{N,t}$ is small, compared with the total number of particles N (assuming this is true for ξ_N, at time $t=0$). In other words, the problem of proving convergence towards the Gross–Pitaevskii dynamics reduces to the problem of showing that the expectation of the number of particles remains approximately preserved by the fluctuation dynamics (1-24). In [Benedikter et al. 2015], this strategy was used to show that the one-particle reduced density $\gamma_{N,t}^{(1)}$ associated with $\Psi_{N,t} = e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)T_0\xi_N$ is such that

$$\|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\,\|_{\mathrm{HS}} \le CN^{-1/2}\exp(c\exp(c|t|))$$

for any $\xi_N \in \mathcal{F}$ with $\|\xi_N\| = 1$ and such that

$$\langle \xi_N, [\mathcal{N} + \mathcal{N}^2/N + \mathcal{H}_N]\xi_N\rangle \le C$$

uniformly in N.

While the method of [Benedikter et al. 2015] works well to show convergence towards the Gross–Pitaevskii dynamics for the evolution of Fock-space data of the form $W(\sqrt{N}\varphi)T_0\xi_N$, it is difficult to apply it to N-particle initial data in $L^2_s(\mathbb{R}^{3N})$ (a special class of N-particle states for which this is indeed possible is discussed in Appendix C of that paper). An alternative approach, tailored on N-particle initial data, was proposed in [Lewin et al. 2015a] for bosons in the mean-field limit. An important observation in that paper (and already in [Lewin et al. 2015b]) is the fact that, for a fixed normalized $\varphi \in L^2(\mathbb{R}^3)$, every $\psi_N \in L^2_s(\mathbb{R}^{3N})$ can be uniquely represented as

$$\psi_N = \sum_{n=0}^{N} \psi_N^{(n)} \otimes_s \varphi^{\otimes(N-n)} \tag{1-25}$$

for a sequence $\{\psi_N^{(n)}\}_{n=0}^N$ with $\psi_N^{(n)} \in L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s n}$, the symmetric tensor product of n copies of the orthogonal complement of φ in $L^2(\mathbb{R}^3)$.

This remark allows us to define a unitary map

$$U(\varphi) : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}_{\perp\varphi}^{\le N} \quad \text{through} \quad U(\varphi)\psi_N = \{\psi_N^{(0)}, \psi_N^{(1)}, \dots, \psi_N^{(N)}\}. \tag{1-26}$$

Here $\mathcal{F}_{\perp\varphi}^{\le N} = \bigoplus_{n=0}^N L^2_\perp(\mathbb{R}^3)^{\otimes_s n}$ is the Fock space constructed on the orthogonal complement $L^2_{\perp\varphi}(\mathbb{R}^3)$ of φ, truncated to have at most N particles. The map $U(\varphi)$ factors out the condensate described by the one-particle wave function φ and allows us to focus on its orthogonal excitations. Notice that a similar idea (but with no second quantization) was used in [Pickl 2015; Mitrouskas et al. 2016] to identify excitations of the condensate. Using the unitary map (1-26), we can introduce, for the mean-field dynamics generated by (1-17), a fluctuation dynamics

$$\mathcal{W}^{\text{mf}}_{N,t} = U(\varphi_t) e^{-i H_N^{\text{mf}} t} U^*(\varphi) : \mathcal{F}^{\le N}_{\perp\varphi} \to \mathcal{F}^{\le N}_{\perp\varphi_t}, \tag{1-27}$$

where φ_t is the solution of the time-dependent Hartree equation (1-18). Much as above, to prove convergence towards Hartree dynamics, it is enough to control the growth of the expectation of the number of particles operator with respect to $\mathcal{W}^{\text{mf}}_{N,t}$. This strategy was used in [Lewin et al. 2015a] to find a norm-approximation for the many-body evolution in the mean-field regime.

It is natural to ask whether the techniques developed in [Lewin et al. 2015a] to study the time-evolution of bosonic systems in the mean-field regime can also be used to study the dynamics in the Gross–Pitaevskii limit. Much as above, where we argued that coherent states are not a good ansatz to describe the evolution of Fock space initial data, we cannot expect here that factorized N-particles states of the form $U^*_{\varphi_t}\Omega = \varphi_t^{\otimes N}$ provide a good approximation for the solution of the Schrödinger equation (1-8) in the Gross–Pitaevskii regime. Instead, much as in [Benedikter et al. 2015], we need to modify the ansatz to take into account correlations developed by the many-body evolution. As explained above, in that paper correlations were modeled by means of Bogoliubov transformations of the form (1-20). Unfortunately, since they do not preserve the number of particles, these Bogoliubov transformations do not leave the space $\mathcal{F}^{\le N}_{\perp\varphi_t}$, where excitations of the Bose–Einstein condensate are described, invariant. For this reason, to adapt the techniques of [Lewin et al. 2015a] to the Gross–Pitaevskii regime that we are considering here, we are going to introduce on $\mathcal{F}^{\le N}_{+}$ modified creation and annihilation operators, defined by

$$b^*(f) = a^*(f)\sqrt{\frac{N-\mathcal{N}}{N}} \quad \text{and} \quad b(f) = \sqrt{\frac{N-\mathcal{N}}{N}}\, a(f) \tag{1-28}$$

for all $f \in L^2_{\perp\varphi_t}(\mathbb{R}^3)$. As we will discuss in the next section, these new fields create and, respectively, annihilate excitations of the Bose–Einstein condensate leaving, at the same time, the total number of particles invariant. We will use the modified creation and annihilation operators to define a generalized Bogoliubov transformation having the form

$$S_t = \exp\left[\frac{1}{2}\int dx\, dy\, (\eta_t(x;y) b^*_x b^*_y - \text{h.c.})\right] \tag{1-29}$$

for a kernel $\eta_t \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, orthogonal to φ_t in both its variables. Compared with the standard Bogoliubov transformations in (1-20), (1-29) has an important advantage: it maps $\mathcal{F}^{\le N}_{\perp\varphi_t}$ back into itself.

With (1-29), we can therefore define the modified fluctuation dynamics

$$\mathcal{W}_{N,t} = S_t^* U(\varphi_t) e^{-i H_N t} U^*(\varphi_0) S_0 : \mathcal{F}^{\le N}_{+} \to \mathcal{F}^{\le N}_{+}, \tag{1-30}$$

which plays a role similar to that played by (1-24) in [Benedikter et al. 2015], describing the time-evolution of excitations of the Bose–Einstein condensate. To prove Theorems 1.1 and 1.2 it will then be enough to show a bound for the growth of the expectation of the number of particles with respect to $\mathcal{W}_{N,t}$. To achieve this goal, we will establish several properties of the generator

$$\mathcal{G}_{N,t} = (i\partial_t S_t^*)S_t + S_t^*[(i\partial_t U(\varphi_t))U^*(\varphi_t) + U(\varphi_t)H_N U^*(\varphi_t)]S_t$$

of (1-30), which is defined so that

$$i\partial_t \mathcal{W}_{N,t} = \mathcal{G}_{N,t}\mathcal{W}_{N,t}.$$

Technically, the main challenge we will have to face is the fact that, in contrast with (1-21), there is no explicit formula for the action of the generalized Bogoliubov transformation (1-29) on creation and annihilation operators. For this reason, we will have to expand expressions like $S_t^* b(g) S_t$ in absolutely convergent infinite series, and we will have to control the contribution of several different terms. The main tool to control these expansions is Lemma 3.2 below.

2. Fock space

In this section, we introduce some notation and we discuss some basic properties of operators on Fock spaces. Let

$$\mathcal{F} = \bigoplus_{n\geq 0} L_s^2(\mathbb{R}^{3n}) = \bigoplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s n}$$

denote the bosonic Fock space over the one-particle space $L^2(\mathbb{R}^3)$. Here $L_s^2(\mathbb{R}^{3n})$ is the subspace of $L^2(\mathbb{R}^{3n})$ consisting of all $\psi \in L^2(\mathbb{R}^{3n})$ with

$$\psi(x_{\pi 1}, x_{\pi 2}, \dots, x_{\pi n}) = \psi(x_1, \dots, x_n)$$

for all permutations $\pi \in S_n$. We use the notation $\Omega = \{1, 0, \dots\} \in \mathcal{F}$ for the vacuum vector, describing a state with no particles.

On $\mathcal{F}$, it is convenient to introduce creation and annihilation operators. For $g \in L^2(\mathbb{R}^3)$, we define the creation operator $a^*(g)$ and the annihilation operator $a(g)$ by

$$(a^*(g)\Psi)^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n g(x_j)\Psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n),$$

$$(a(g)\Psi)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int \bar{g}(x)\Psi^{(n+1)}(x, x_1, \dots, x_n).$$

Notice that creation operators are linear in their argument, and annihilation operators are antilinear. Creation and annihilation operators can be extended to closed unbounded operators on $\mathcal{F}$; $a^*(g)$ is the adjoint of $a(g)$. They satisfy canonical commutation relations

$$[a(g), a^*(h)] = \langle g, h\rangle, \quad [a(g), a(h)] = [a^*(g), a^*(h)] = 0 \tag{2-1}$$

for all $g, h \in L^2(\mathbb{R}^3)$ (here $\langle g, h\rangle$ denotes the usual inner product on $L^2(\mathbb{R}^3)$). It is also convenient to introduce operator-valued distributions a_x, a_x^*, formally creating and annihilating a particle at $x \in \mathbb{R}$. They are such that

$$a(f)=\int \bar{f}(x)a_x\,dx, \quad a^*(f)=\int f(x)a_x^*\,dx$$

and satisfy the commutation relations

$$[a_x,a_y^*]=\delta(x-y), \quad [a_x,a_y]=[a_x^*,a_y^*]=0.$$

It is also useful to introduce on $\mathcal{F}$ the number of particles operator, defined by $(\mathcal{N}\Psi)^{(n)}=n\Psi^{(n)}$. In terms of operator-valued distributions, $\mathcal{N}$ can be written as

$$\mathcal{N}=\int a_x^*a_x\,dx.$$

Creation and annihilation operators are bounded by the square root of the number of particles operator; i.e., we have

$$\|a(f)\Psi\|\le\|f\|_2\,\|\mathcal{N}^{1/2}\Psi\|, \quad \|a^*(f)\Psi\|\le\|f\|_2\,\|(\mathcal{N}+1)^{1/2}\Psi\| \tag{2-2}$$

for every $f\in L^2(\mathbb{R}^3)$.

For a one-particle operator $B:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ we define $d\Gamma(B):\mathcal{F}\to\mathcal{F}$ through $(d\Gamma(B)\Psi)^{(n)}=\sum_{j=1}^n B_j\psi^{(n)}$ for any $\Psi=\{\psi^{(n)}\}_{n\in\mathbb{N}}\in\mathcal{F}$. Here $B_j=1\otimes\cdots\otimes B\otimes\cdots\otimes 1$ acts as B on the j-th particles and as the identity on all other particles. If B has the integral kernel $B(x;y)$, we can write

$$d\Gamma(B)=\int B(x;y)a_x^*a_y\,dx\,dy.$$

If B is a bounded operator on the one-particle space $L^2(\mathbb{R}^3)$, then $d\Gamma(B)$ can be bounded with respect to the number of particles operator, i.e., we have the operator inequality

$$\pm d\Gamma(B)\le\|B\|_{\mathrm{op}}\,\mathcal{N} \tag{2-3}$$

and (since $d\Gamma(B)$ commutes with $\mathcal{N}$) also

$$\|d\Gamma(B)\Psi\|\le\|B\|_{\mathrm{op}}\,\|\mathcal{N}\Psi\|.$$

We will also need bounds for operators on the Fock space, quadratic in creation and annihilation operators, that do not necessarily preserve the number of particles. For $j\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$, we introduce the notation

$$A_{\sharp_1,\sharp_2}(j)=\int a^{\sharp_1}(j_x)a_x^{\sharp_2}\,dx=\int j^{\bar{\sharp}_1}(x;y)a_y^{\sharp_1}a_x^{\sharp_2}\,dx\,dy, \tag{2-4}$$

where $j_x(y):=j(x;y)$, $\sharp_1,\sharp_2\in\{\cdot,*\}$, $\bar{\sharp}_1=\cdot$ if $\sharp_1=*$ and $\bar{\sharp}_1=*$ if $\sharp_1=\cdot$, and where we use the notation $a^\sharp=a$ if $\sharp=\cdot$, $a^\sharp=a^*$ if $\sharp=*$ and, similarly, $j^\sharp=j$ if $\sharp=\cdot$ and $j^\sharp=\bar{j}$ if $\sharp=*$. If $\sharp_1=\cdot$ and $\sharp_2=*$ (i.e., if a creation operator lies on the right of an annihilation operator), in order to define $A_{\sharp_1,\sharp_2}(j)$ we also require that $x\to j(x;x)$ is integrable. In the next lemma, which follows easily from (2-2), we show how to bound these operators through the number of particles operator $\mathcal{N}$.

Lemma 2.1. *Let $j \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Then for any $\Psi \in \mathcal{F}$,*

$$\|A^{\flat}_{\sharp_1,\sharp_2}(j)\Psi\| \le \sqrt{2}\|(\mathcal{N}+1)\Psi\| \begin{cases} \|j\|_2 + \int |j(x;x)|\,dx & \text{if } \sharp_1 = \cdot\,,\ \sharp_2 = *, \\ \|j\|_2 & \text{otherwise.} \end{cases}$$

We will work on certain subspaces of $\mathcal{F}$. For a fixed $\varphi \in L^2(\mathbb{R}^3)$ (φ will later be the condensate wave function), we use the notation $L^2_{\perp\varphi}(\mathbb{R}^3)$ for the orthogonal complement of the one-dimensional space spanned by φ in $L^2(\mathbb{R}^3)$. We denote by

$$\mathcal{F}_{\perp\varphi} = \bigoplus_{n\ge 0} L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s n}$$

the Fock space constructed over $L^2_{\perp\varphi}(\mathbb{R}^3)$. A vector $\Psi = \{\psi^{(0)}, \psi^{(1)}, \dots\} \in \mathcal{F}$ lies in $\mathcal{F}_{\perp\varphi}$ if $\psi^{(n)}$ is orthogonal to φ, in each of its coordinates, for all $n \ge 1$, i.e., if

$$\int \bar{\varphi}(x)\,\psi^{(n)}(x, y_1, \dots, y_{n-1})\,dx = 0$$

for all $n \ge 1$. We will also need Fock spaces with a truncated number of particles. For $N \in \mathbb{N}\backslash\{0\}$, we define

$$\mathcal{F}^{\le N} = \bigoplus_{n=0}^{N} L^2(\mathbb{R}^3)^{\otimes_s n} \quad \text{and} \quad \mathcal{F}^{\le N}_{\perp\varphi} = \bigoplus_{n=0}^{N} L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s n}$$

as the Fock spaces over $L^2(\mathbb{R}^3)$ and over $L^2_{\perp\varphi}(\mathbb{R}^3)$ consisting of states with at most N particles. As already explained in the Introduction (but see Section 4 for more details), on the space $\mathcal{F}^{\le N}_{\perp\varphi}$ we will describe orthogonal fluctuations around a condensate with wave function $\varphi \in L^2(\mathbb{R}^3)$.

On $\mathcal{F}^{\le N}$ and $\mathcal{F}^{\le N}_{\perp\varphi}$, we introduce modified creation and annihilation operators. For $f \in L^2(\mathbb{R}^3)$, we define

$$b(f) = \sqrt{\frac{N-\mathcal{N}}{N}}\,a(f) \quad \text{and} \quad b^*(f) = a^*(f)\sqrt{\frac{N-\mathcal{N}}{N}}. \tag{2-5}$$

We clearly have $b(f), b^*(f) : \mathcal{F}^{\le N} \to \mathcal{F}^{\le N}$. If moreover $f \perp \varphi$ we also have $b(f), b^*(f) : \mathcal{F}^{\le N}_{\perp\varphi} \to \mathcal{F}^{\le N}_{\perp\varphi}$. As we will discuss in the next section, the importance of these fields arises from the application of the map $U(\varphi)$, defined in (1-25), since

$$\begin{aligned} U(\varphi)a^*(f)a(\varphi)U^*(\varphi) &= a^*(f)\sqrt{N-\mathcal{N}} = \sqrt{N}b^*(f), \\ U(\varphi)a^*(\varphi)a(f)U^*(\varphi) &= \sqrt{N-\mathcal{N}}\,a(f) = \sqrt{N}\,b(f). \end{aligned} \tag{2-6}$$

If φ is the condensate wave function and $f \perp \varphi$, the operator $b^*(f)$ excites a particle from the condensate to its orthogonal complement, while $b(f)$ annihilates an excitation back into the condensate. On states exhibiting Bose–Einstein condensation, we expect $a(\varphi), a^*(\varphi) \simeq \sqrt{N}$ and thus that the action of modified b^*- and b-fields is close to the action of the original creation and annihilation operators.

It is also convenient to introduce operator-valued distributions

$$b_x = \sqrt{\frac{N-\mathcal{N}}{N}}\,a_x \quad \text{and} \quad b^*_x = a^*_x\sqrt{\frac{N-\mathcal{N}}{N}}$$

so that

$$b(f) = \int \bar{f}(x)\, b_x \, dx \quad \text{and} \quad b^*(f) = \int f(x) b^*_x \, dx.$$

We find the modified canonical commutation relations

$$[b_x, b^*_y] = \left(1 - \frac{\mathcal{N}}{N}\right)\delta(x-y) - \frac{1}{N} a^*_y a_x, \quad [b_x, b_y] = [b^*_x, b^*_y] = 0. \tag{2-7}$$

Furthermore

$$[b_x, a^*_y a_z] = \delta(x-y) b_z, \quad [b^*_x, a^*_y a_z] = -\delta(x-z) b^*_y, \tag{2-8}$$

which leads to $[b_x, \mathcal{N}] = b_x$ and $[b^*_x, \mathcal{N}] = -b^*_x$. From (2-2), we immediately obtain the following bounds for the b-fields.

Lemma 2.2. *Let $f \in L^2(\mathbb{R}^3)$. For any $\xi \in \mathcal{F}^{\le N}$, we have*

$$\|b(f)\xi\| \le \|f\|_2 \left\| \mathcal{N}^{1/2} \left(\frac{N - \mathcal{N} + 1}{N} \right)^{1/2} \xi \right\| \le \|f\|_2 \|\mathcal{N}^{1/2} \xi\|,$$

$$\|b^*(f)\xi\| \le \|f\|_2 \left\| (\mathcal{N}+1)^{1/2} \left(\frac{N - \mathcal{N}}{N} \right)^{1/2} \xi \right\| \le \|f\|_2 \|(\mathcal{N}+1)^{1/2} \xi\|.$$

Notice, moreover, that since $\mathcal{N} \le N$ on $\mathcal{F}^{\le N}$, the operators $b(f), b^*(f) : \mathcal{F}^{\le N} \to \mathcal{F}^{\le N}$ are bounded with $\|b(f)\|, \|b^*(f)\| \le (N+1)^{1/2} \|f\|_2$.

We will also consider quadratic expressions in the b-fields. For an integral kernel $j \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, we define, similarly to (2-4),

$$B_{\sharp_1, \sharp_2}(j) = \int b^{\sharp_1}(j_x) b^{\sharp_2}_x \, dx = \int j^{\bar{\sharp}_1}(x; y) b^{\sharp_1}_y b^{\sharp_2}_x \, dx \, dy. \tag{2-9}$$

If $\sharp_1 = \cdot$ and $\sharp_2 = *$, we also require that $x \to j(x; x)$ is integrable. From Lemma 2.1, we obtain the following bounds.

Lemma 2.3. *Let $j \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Then*

$$\frac{\|B_{\sharp_1,\sharp_2}(j)\Psi\|}{\|(\mathcal{N}+1)((N - \mathcal{N} + 2)/N)\Psi\|} \le \sqrt{2} \begin{cases} \|j\|_2 + \int |j(x; x)| \, dx & \textit{if } \sharp_1 = \cdot,\ \sharp_2 = *, \\ \|j\|_2 & \textit{otherwise} \end{cases}$$

for all $\Psi \in \mathcal{F}^{\le N}$. Since $\mathcal{N} \le N$ on $\mathcal{F}^{\le N}$, the operator $B_{\sharp_1,\sharp_2}(j)$ is bounded, with

$$\|B_{\sharp_1,\sharp_2}(j)\| \le \sqrt{2} N \begin{cases} \|j\|_2 + \int |j(x; x)| \, dx & \textit{if } \sharp_1 = \cdot,\ \sharp_2 = *, \\ \|j\|_2 & \textit{otherwise.} \end{cases}$$

Remark. For $\varphi \in L^2(\mathbb{R}^3)$, let $q_\varphi = 1 - |\varphi\rangle\langle\varphi|$ be the orthogonal projection onto $L^2_{\perp\varphi}(\mathbb{R}^3)$. If $j \in (q_{\varphi^{\bar{\sharp}_1}} \otimes q_{\varphi^{\bar{\sharp}_2}})(L^2(\mathbb{R}^3 \times \mathbb{R}^3))$, we have $B_{\sharp_1,\sharp_2}(j) : \mathcal{F}^{\le N}_{\perp\varphi} \to \mathcal{F}^{\le N}_{\perp\varphi}$ (here we use the notation $\bar{\sharp} = *$ if $\sharp = \cdot$ and $\bar{\sharp} = \cdot$ if $\sharp = *$, and $\varphi^\sharp = \varphi$ if $\sharp = *$, $\varphi^\sharp = \bar{\varphi}$ if $\sharp = \cdot$).

We will consider products of several creation and annihilation operators, as well. In particular, two types of monomials in creation and annihilation operators will play an important role in our analysis. We define

$$\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)=\int b^{\flat_0}_{x_1}a^{\sharp_1}_{y_1}a^{\flat_1}_{x_2}a^{\sharp_2}_{y_2}a^{\flat_2}_{x_3}\cdots a^{\sharp_{n-1}}_{y_{n-1}}a^{\flat_{n-1}}_{x_n}b^{\sharp_n}_{y_n}\prod_{\ell=1}^{n}j_\ell(x_\ell;y_\ell)\,dx_\ell\,dy_\ell, \tag{2-10}$$

where $j_k\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for $k=1,\dots,n$ and where $\sharp=(\sharp_1,\dots,\sharp_n)$, $\flat=(\flat_0,\dots,\flat_{n-1})\in\{\cdot,*\}^n$. In other words, for every index $i\in\{1,\dots,n\}$, we have either $\sharp_i=\cdot$ (meaning that $a^{\sharp_i}=a$ or $b^{\sharp_i}=b$) or $\sharp_i=*$ (meaning that $a^{\sharp_i}=a^*$ or $b^{\sharp_i}=b^*$) and analogously for $\flat_i$, if $i\in\{0,\dots,n-1\}$. Furthermore, for $\ell=1,\dots,n-1$, we impose the condition that either $\sharp_\ell=\cdot$ and $\flat_\ell=*$ or $\sharp_\ell=*$ and $\flat_\ell=\cdot$ (so that the product $a^{\sharp_\ell}_{y_\ell}a^{\flat_\ell}_{x_{\ell+1}}$ always preserves the number of particles). If $\flat_{i-1}=\cdot$ and $\sharp_i=*$ (i.e., if the product $a^{\flat_{i-1}}_{x_i}a^{\sharp_i}_{y_i}$ for $i=2,\dots,n$, or the product $b^{\flat_0}_{x_1}a^{\sharp_1}_{y_1}$ for $i=1$, is not normally ordered) we require additionally $x\to j_i(x;x)$ to be integrable. An operator of the form (2-10), with all the properties listed above, will be called a $\Pi^{(2)}$-operator of order n.

Next, we define

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)=\int b^{\flat_0}_{x_1}a^{\sharp_1}_{y_1}a^{\flat_1}_{x_2}a^{\sharp_2}_{y_2}a^{\flat_2}_{x_3}\cdots a^{\sharp_{n-1}}_{y_{n-1}}a^{\flat_{n-1}}_{x_n}a^{\sharp_n}_{y_n}a^{\flat_n}(f)\prod_{\ell=1}^{n}j_\ell(x_\ell;y_\ell)\,dx_\ell\,dy_\ell, \tag{2-11}$$

where $f\in L^2(\mathbb{R}^3)$, $j_k\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for all $k=1,\dots,n$, $\sharp=(\sharp_1,\dots,\sharp_n)\in\{\cdot,*\}^n$, $\flat=(\flat_0,\dots,\flat_n)\in\{\cdot,*\}^{n+1}$ with the condition that, for all $\ell=1,\dots,n$, we either have $\sharp_\ell=\cdot$ and $\flat_\ell=*$ or $\sharp_\ell=*$ and $\flat_\ell=\cdot$. Additionally, we assume that $x\to j_i(x;x)$ is integrable if $\flat_{i-1}=\cdot$ and $\sharp_i=*$ for an $i=1,\dots,n$. An operator of the form (2-11) will be called a $\Pi^{(1)}$-operator of order n. Operators of the form $b(f)$, $b^*(f)$, for an $f\in L^2(\mathbb{R}^3)$, will be called $\Pi^{(1)}$-operators of order zero. It will also be useful to consider

$$\widetilde{\Pi}^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)=\int a^{\flat_0}(f)\,a^{\sharp_0}_{x_1}a^{\flat_1}_{y_1}a^{\sharp_1}_{x_2}a^{\flat_2}_{y_2}a^{\sharp_2}_{x_3}\cdots a^{\flat_{n-1}}_{y_{n-1}}a^{\sharp_{n-1}}_{x_n}b^{\flat_n}_{y_n}\prod_{\ell=1}^{n}j_\ell(x_\ell;y_\ell)\,dx_\ell\,dy_\ell, \tag{2-12}$$

where $f\in L^2(\mathbb{R}^3)$, $j_k\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for all $k=1,\dots,n$, $\sharp=(\sharp_0,\dots,\sharp_{n-1})\in\{\cdot,*\}^n$, $\flat=(\flat_0,\dots,\flat_n)\in\{\cdot,*\}^{n+1}$ with the condition that, for every $\ell\in\{0,\dots,n-1\}$, either $\sharp_\ell=\cdot$ and $\flat_\ell=*$ or $\sharp_\ell=*$ and $\flat_\ell=\cdot$. As above, we also assume that $x\to j_i(x;x)$ is integrable if $\flat_{i-1}=\cdot$ and $\sharp_i=*$ for $i=1,\dots,n$. Observe that

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)^*=\widetilde{\Pi}^{(1)}_{\sharp',\flat'}(j_n,\dots,j_1;f),$$

with $\flat'=(\bar\flat_n,\dots,\bar\flat_0)$, $\sharp'=(\bar\sharp_n,\dots,\bar\sharp_1)$, where $\bar\flat=\cdot$ if $\flat=*$ and $\bar\flat=*$ if $\flat=\cdot$ (and similarly for $\bar\sharp$).

Notice that $\Pi^{(2)}$-operators involve two b-operators and therefore may create or annihilate up to two excitations of the condensate (depending on the choice of $\flat_0$ and $\sharp_n$, they may also leave the number of excitations invariant). $\Pi^{(1)}$- and $\widetilde{\Pi}^{(1)}$-operators, on the other hand, create or annihilate exactly one excitation. The conditions on the number of creation and annihilation operators guarantee that $\Pi^{(2)}$-, $\Pi^{(1)}$- and $\widetilde{\Pi}^{(1)}$-operators always map $\mathcal{F}^{\leq N}$ back into itself. In the next lemma we collect bounds that we are going to use to control these operators.

Lemma 2.4. *Let $n\in\mathbb{N}$, $f\in L^2(\mathbb{R}^3)$, $j_1,\dots,j_n\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$. We assume the operators $\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)$ and $\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)$ are defined as in* (2-10), (2-11). *Then we have the bounds*

$$\begin{aligned}\|\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\xi\| &\le 6^n\prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell}\left\|(\mathcal{N}+1)^n\left(1-\frac{\mathcal{N}-2}{N}\right)\xi\right\|,\\ \|\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)\xi\| &\le 6^n\|f\|\prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell}\left\|(\mathcal{N}+1)^{n+1/2}\left(1-\frac{\mathcal{N}-2}{N}\right)^{1/2}\xi\right\|,\end{aligned}\tag{2-13}$$

where

$$K_\ell^{\flat_{\ell-1},\sharp_\ell}=\begin{cases}\|j_\ell\|_2+\int|j_\ell(x;x)|\,dx & \text{if } \flat_{\ell-1}=\cdot \text{ and } \sharp_\ell=*,\\ \|j_\ell\|_2 & \text{otherwise.}\end{cases}$$

Since $\mathcal{N}\le N$ on $\mathcal{F}^{\le N}$, it follows that $\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)$, $\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)$ are bounded operators on $\mathcal{F}^{\le N}$, with

$$\begin{aligned}\|\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\| &\le (12N)^n\prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell},\\ \|\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)\| &\le (12N)^n\sqrt{N}\|f\|_2\prod_{\ell=1}^n K_\ell^{\flat_{\ell-1},\sharp_\ell}.\end{aligned}$$

Remark. If $j_i\in(q_{\varphi^{\bar\flat_{i-1}}}\otimes q_{\varphi^{\bar\sharp_i}})L^2(\mathbb{R}^3\times\mathbb{R}^3)$ for all $i=1,\dots,,n$ and if $f\in L^2_\perp(\mathbb{R}^3)$, then $\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)$ and $\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)$ map $\mathcal{F}^{\le N}_{\perp\varphi}$ into itself.

Proof. We consider operators of the form (2-10). Let us assume, for example, that $\flat_0=\cdot$ and $\sharp_n=\cdot$. Then we have, writing $b_{x_1}=a_{x_1}(1-\mathcal{N}/N)^{1/2}$ and $b_{y_n}=a_{y_n}(1-\mathcal{N}/N)^{1/2}$ and using the pull-through formula $g(\mathcal{N})a_x=a_xg(\mathcal{N}-1)$,

$$\begin{aligned}\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n) &= \int a_{x_1}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2}a^{\sharp_1}_{y_1}\cdots a^{\sharp_{n-1}}_{y_{n-1}}a^{\flat_{n-1}}_{x_n}a_{y_n}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2}\prod_{\ell=1}^n j_\ell(x_\ell;y_\ell)\,dx_\ell\,dy_\ell\\ &= \int a_{x_1}a^{\sharp_1}_{y_1}\cdots a^{\sharp_{n-1}}_{y_{n-1}}a^{\flat_{n-1}}_{x_n}a_{y_n}\left(\frac{N-\mathcal{N}+1}{N}\right)^{1/2}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2}\prod_{\ell=1}^n j_\ell(x_\ell;y_\ell)\,dx_\ell\,dy_\ell\\ &= \prod_{\ell=1}^n A^{\flat_{\ell-1},\sharp_\ell}(j_\ell)\left(\frac{N-\mathcal{N}+1}{N}\right)^{1/2}\left(\frac{N-\mathcal{N}}{N}\right)^{1/2},\end{aligned}$$

where we used the definition (2-4). The first bound in (2-13) follows therefore from Lemma 2.1. The other estimates can be shown similarly. □

3. Generalized Bogoliubov transformations

For a kernel $\eta\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ with $\eta(x;y)=\eta(y;x)$, we define

$$B(\eta)=\frac{1}{2}\int[\eta(x;y)b^*_xb^*_y-\bar\eta(x;y)b_xb_y]\,dx\,dy.\tag{3-1}$$

Observe that, with the notation introduced in (2-9),

$$B(\eta)=\tfrac{1}{2}[B_{*,*}(\eta)-B^*_{*,*}(\eta)]=-\tfrac{1}{2}[B_{\cdot,\cdot}(\eta)-B^*_{\cdot,\cdot}(\eta)].$$

Generalized Bogoliubov transformations are unitary operators having the form

$$e^{B(\eta)}=\exp\biggl[\frac{1}{2}\int(\eta(x;y)b^*_xb^*_y-\bar{\eta}(x;y)b_xb_y)\biggr]. \tag{3-2}$$

It is clear that $B(\eta), e^{B(\eta)}:\mathcal{F}^{\le N}\to\mathcal{F}^{\le N}$. Furthermore, if $\eta\in(q_\varphi\otimes q_\varphi)L^2(\mathbb{R}^3\times\mathbb{R}^3)$ then we have $B(\eta), e^{B(\eta)}:\mathcal{F}^{\le N}_{\perp\varphi}\to\mathcal{F}^{\le N}_{\perp\varphi}$ for any normalized $\varphi\in L^2(\mathbb{R}^3)$ (as above, $q_\varphi=1-|\varphi\rangle\langle\varphi|$ is the projection into the orthogonal complement of φ). It may be helpful to observe that, with the unitary operator $U(\varphi)$ defined in (1-26), we can write, according to (2-6),

$$B(\eta)=\frac{1}{2}U(\varphi)\int dx\,dy\biggl[\eta(x;y)a^*_xa^*_y\frac{a(\varphi)a(\varphi)}{N}-\bar{\eta}(x;y)\frac{a^*(\varphi)a^*(\varphi)}{N}a_xa_y\biggr]U^*(\varphi). \tag{3-3}$$

On states exhibiting Bose–Einstein condensation in φ (so that $a(\varphi), a^*(\varphi)\simeq\sqrt{N}$), we can therefore expect the generalized Bogoliubov transformation (3-2) to be close to the standard Bogoliubov transformation

$$e^{\widetilde{B}(\eta)}=\exp\biggl[\frac{1}{2}\int(\eta(x;y)a^*_xa^*_y-\bar{\eta}(x;y)a_xa_y)\biggr], \tag{3-4}$$

whose action on creation and annihilation operators is explicitly given by

$$e^{-\widetilde{B}(\eta)}a(f)e^{\widetilde{B}(\eta)}=a(\cosh_\eta(f))+a^*(\sinh_\eta(\bar{f})), \tag{3-5}$$

with the operators $\cosh_\eta$, $\sinh_\eta$ defined as in (1-22). Standard Bogoliubov transformations of the form (3-4) were used in [Benedikter et al. 2015] to model correlations in the Gross–Pitaevskii regime for approximately coherent Fock space initial data. In the present paper, since (3-4) does not map $\mathcal{F}^{\le N}_{\perp\varphi}$ into itself (it does not respect the truncation $\mathcal{N}\le N$), we prefer to work with generalized Bogoliubov transformations of the form (3-2). The price that we have to pay is the fact that, in contrast to (3-5), the action of $\exp(B(\eta))$ on creation and annihilation operators is not explicit. Let us remark here that generalized Bogoliubov transformations of the form $\exp(B(\eta))$ were already used in [Seiringer 2011; Grech and Seiringer 2013] to study the excitation spectrum in the mean-field regime. Here we will need more detailed information on the action of these operators; the rest of this section is therefore devoted to the study of the properties of generalized Bogoliubov transformations.

First of all, we need the following generalization of Lemma 4.3 of [Benedikter et al. 2015]; a similar result was also proven in [Seiringer 2011].

Lemma 3.1. *Let $\eta\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$. Let $B(\eta)$ be the antisymmetric operator defined in* (3-1). *For every $n_1,n_2\in\mathbb{Z}$ there exists a constant $C=C(n_1,n_2,\|\eta\|_2)$ such that*

$$e^{-B(\eta)}(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}e^{B(\varphi)}\le C(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}$$

as operator inequality on $\mathcal{F}^{\le N}$.

Proof. We use Gronwall's inequality. For a fixed $\xi \in \mathcal{F}^{\leq N}$ and $s \in [0; 1]$, let

$$f(s) = \langle \xi, e^{-sB(\eta)}(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2} e^{sB(\eta)}\xi\rangle.$$

We compute

$$\begin{aligned} f'(s) &= \langle \xi, e^{-sB(\eta)}[(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle \\ &= \langle e^{sB(\eta)}\xi, \{(\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)] + [(\mathcal{N}+1)^{n_1}, B(\eta)](N+1-\mathcal{N})^{n_2}\} e^{sB(\eta)}\xi\rangle. \end{aligned} \tag{3-6}$$

From the pull-through formula $\mathcal{N} b^* = b^*(\mathcal{N}+1)$, we conclude that

$$\begin{aligned} [(N+1-\mathcal{N})^{n_2}, B(\eta)] &= \tfrac{1}{2} B_{*,*}(\eta)[(N-1-\mathcal{N})^{n_2} - (N+1-\mathcal{N})^{n_2}] + \text{h.c.}, \\ [(\mathcal{N}+1)^{n_1}, B(\eta)] &= \tfrac{1}{2} B_{*,*}(\eta)[(\mathcal{N}+3)^{n_1} - (\mathcal{N}+1)^{n_1}] + \text{h.c.} \end{aligned}$$

By the mean value theorem, we can find functions $\theta_1, \theta_2 : \mathbb{N} \to (0; 2)$ (depending also on N, n_1, n_2) such that

$$\begin{aligned} (N-j+1)^{n_2} - (N-j-1)^{n_2} &= 2n_2(N+1-j-\theta_1(j))^{n_2-1}, \\ (j+3)^{n_1} - (j+1)^{n_1} &= 2n_1(j+1+\theta_2(j)). \end{aligned}$$

Hence, the first term on the right-hand side of (3-6) can be written as

$$\begin{aligned} &\langle e^{sB(\eta)}\xi, (\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle \\ &= \tfrac{1}{2}\big\langle (\mathcal{N}+1)^{n_1} e^{sB(\eta)}\xi, \big(B_{*,*}(\eta)(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2-1} + \text{h.c.}\big) e^{sB(\eta)}\xi\big\rangle \\ &= \tfrac{1}{2}\big\langle (\mathcal{N}+1)^{n_1/2}(N+3-\mathcal{N}-\theta_1(\mathcal{N}-2))^{n_2/2} e^{sB(\eta)}\xi, B_{*,*}(\eta)(\mathcal{N}+3)^{n_1/2}(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2/2-1} e^{sB(\eta)}\xi\big\rangle \\ &\quad + \tfrac{1}{2}\big\langle (\mathcal{N}+1)^{n_1/2}(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2/2} e^{sB(\eta)}\xi, B_{\cdot,\cdot}(\eta)(\mathcal{N}-1)^{n_1/2}(N+3-\mathcal{N}-\theta_1(\mathcal{N}-2))^{n_2/2-1} e^{sB(\eta)}\xi\big\rangle. \end{aligned}$$

The Cauchy–Schwarz inequality implies with Lemma 2.3

$$\begin{aligned} &|\langle e^{sB(\eta)}\xi, (\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle| \\ &\leq C\|(\mathcal{N}+1)^{n_1/2}(N+3-\mathcal{N}-\theta_1(\mathcal{N}-2))^{n_2/2} e^{sB(\eta)}\xi\| \, \|(\mathcal{N}+3)^{n_1/2+1}(N+1-\mathcal{N}-\theta_1(\mathcal{N}))^{n_2} N^{-1} e^{sB(\eta)}\xi\|, \end{aligned}$$

with a constant C depending on $\|\eta\|_2$. Since on $\mathcal{F}^{\leq N}$ we have $\mathcal{N} \leq N$ and since $0 \leq \theta_1(n) \leq 2$ for all $n \in \mathbb{N}$, we conclude that

$$|\langle e^{sB(\eta)}\xi, (\mathcal{N}+1)^{n_1}[(N+1-\mathcal{N})^{n_2}, B(\eta)]e^{sB(\eta)}\xi\rangle| \leq C f(s)$$

for a constant C depending on $\|\eta\|_2, n_1, n_2$. The second term on the right-hand side of (3-6) can be bounded similarly. We infer that $f'(s) \leq C f(s)$. Gronwall's inequality implies that $f(s) \leq e^{Cs} f(0)$. Hence, taking $s = 1$, and renaming the constant C, we obtain

$$\langle \xi, e^{-B(\eta)}(\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2} e^{B(\eta)}\xi\rangle \leq C\langle \xi, (\mathcal{N}+1)^{n_1}(N+1-\mathcal{N})^{n_2}\xi\rangle,$$

which concludes the proof of the lemma. □

We will need to express the action of the generalized Bogoliubov transformation $e^{B(\eta)}$ on the b-fields by means of a convergent series of nested commutators. To this end, we start by noticing that, for $f \in L^2(\mathbb{R}^3)$,

$$\begin{aligned} e^{-B(\eta)} b(f) e^{B(\eta)} &= b(f) + \int_0^1 ds\, \frac{d}{ds} e^{-sB(\eta)} b(f) e^{sB(\eta)} \\ &= b(f) - \int_0^1 ds\, e^{-sB(\eta)} [B(\eta), b(f)] e^{sB(\eta)} \\ &= b(f) - [B(\eta), b(f)] + \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 B(\eta)} [B(\eta), [B(\eta), b(f)]] e^{s_2 B(\eta)}. \end{aligned}$$

Iterating m times, we obtain

$$\begin{aligned} &e^{-B(\eta)} b(f) e^{B(\eta)} \\ &\quad = \sum_{n=1}^{m-1} (-1)^n \frac{\mathrm{ad}^{(n)}_{B(\eta)}(b(f))}{n!} + \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m\, e^{-s_m B(\eta)} \mathrm{ad}^{(m)}_{B(\eta)}(b(f)) e^{s_m B(\eta)}, \end{aligned} \tag{3-7}$$

where we introduced the notation $\mathrm{ad}^{(n)}_{B(\eta)}(A)$ defined recursively by

$$\mathrm{ad}^{(0)}_{B(\eta)}(A) = A \quad \text{and} \quad \mathrm{ad}^{(n)}_{B(\eta)}(A) = [B(\eta), \mathrm{ad}^{(n-1)}_{B(\eta)}(A)].$$

We will show later that, under suitable assumptions on η, the error term on the right-hand side of (3-7) is negligible in the limit $m \to \infty$. This means that the action of the generalized Bogoliubov transformation $B(\eta)$ on $b(f)$ and similarly on $b^*(f)$ can be described in terms of the nested commutators $\mathrm{ad}_{B(\eta)}(A)$ for $A = b(f)$ or $A = b^*(f)$. In the next lemma, we give a detailed analysis of these terms.

For a kernel $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, we will use the notation

$$\eta^{(n)} = \begin{cases} 1 & \text{for } n = 0, \\ (\eta\bar{\eta})^{\ell} & \text{if } n = 2\ell,\ \ell \in \mathbb{N}\backslash\{0\}, \\ (\eta\bar{\eta})^{\ell}\eta & \text{if } n = 2\ell+1,\ \ell \in \mathbb{N}. \end{cases} \tag{3-8}$$

Here we, identify $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ with the Hilbert–Schmidt operator acting on $L^2(\mathbb{R}^3)$, having integral kernel η. To avoid keeping track of complex conjugations of η-kernels, we also introduce the following notation. For $\natural \in \{\cdot, *\}$ we write $\eta_\natural = \eta$ if $\natural = \cdot$ and $\eta_\natural = \bar{\eta}$ if $\natural = *$. More generally, for $n \in \mathbb{N}$, and $(\natural_1, \dots, \natural_n) \in \{\cdot, *\}^n$, we will use the notation $\eta^{(n)}_\natural = \eta_{\natural_1}\eta_{\natural_2} \cdots \eta_{\natural_n}$, in the sense of products of operators. Also for a function $f \in L^2(\mathbb{R}^3)$, we use the notation $f_\natural = f$ if $\natural = \cdot$ and $f_\natural = \bar{f}$ if $\natural = *$.

Lemma 3.2. *Let $\eta \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ be such that $\eta(x; y) = \eta(y; x)$ for all $x, y \in \mathbb{R}^3$. Let $B(\eta)$ be defined as in* (3-1). *Let $n \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^3)$. Then the nested commutators $\mathrm{ad}^{(n)}_{B(\eta)}(b(f))$ can be written as the sum of exactly $2^n n!$ terms, with the following properties*:

(i) *Possibly up to a sign, each term has the form*

$$\Lambda_1 \Lambda_2 \cdots \Lambda_i \frac{1}{N^k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1}, \dots, \eta^{(j_k)}_{\natural_k}; \eta^{(s)}_{\natural}(f_\Diamond)) \tag{3-9}$$

for some $i, k, s \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$, $\lozenge \in \{\cdot, *\}$, $\sharp \in \{\cdot, *\}^k$, $\flat \in \{\cdot, *\}^{k+1}$, $\natural_v \in \{\cdot, *\}^{j_v}$ *for all* $v = 1, \dots, k$ *and* $\natural \in \{\cdot, *\}^s$. *In* (3-9), *each operator* $\Lambda_w : \mathcal{F}^{\le N} \to \mathcal{F}^{\le N}$ *is a factor* $(N-\mathcal{N})/N$, *a factor* $(N+1-\mathcal{N})/N$ *or an operator of the form*

$$\frac{1}{N^p}\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1}, \eta^{(m_2)}_{\natural_2}, \dots, \eta^{(m_p)}_{\natural_p}) \tag{3-10}$$

for some $p, m_1, \dots, m_p \in \mathbb{N}\backslash\{0\}$, $\sharp, \flat \in \{\cdot, *\}^p$, $\natural_v \in \{\cdot, *\}^{m_v}$ *for all* $v = 1, \dots, p$.

(ii) *If a term of the form* (3-9) *contains* $m \in \mathbb{N}$ *factors* $(N-\mathcal{N})/N$ *or* $(N+1-\mathcal{N})/N$ *and* $j \in \mathbb{N}$ *factors of the form* (3-10) *with* $\Pi^{(2)}$*-operators of order* $p_1, \dots, p_j \in \mathbb{N}\backslash\{0\}$, *then we have*

$$m + (p_1+1) + \dots + (p_j+1) + (k+1) = n+1. \tag{3-11}$$

(iii) *If a term of the form* (3-9) *contains* (*considering all* Λ*-operators and the* $\Pi^{(1)}$*-operator*) *the kernels* $\eta^{(i_1)}_{\natural_1}, \dots, \eta^{(i_m)}_{\natural_m}$ *and the wave function* $\eta^{(s)}_{\natural}(f_\lozenge)$ *for some* $m, s \in \mathbb{N}$, $i_1, \dots, i_m \in \mathbb{N}\backslash\{0\}$, $\natural_r \in \{\cdot, *\}^{i_r}$ *for all* $r = 1, \dots, m$, $\natural \in \{\cdot, *\}^s$ *then*

$$i_1 + \dots + i_m + s = n.$$

(iv) *There is exactly one term having the form*

$$\left(\frac{N-\mathcal{N}}{N}\right)^{n/2}\left(\frac{N+1-\mathcal{N}}{N}\right)^{n/2} b(\eta^{(n)}(f)) \tag{3-12}$$

if n is even, *and*

$$-\left(\frac{N-\mathcal{N}}{N}\right)^{(n+1)/2}\left(\frac{N-\mathcal{N}+1}{N}\right)^{(n-1)/2} b^*(\eta^{(n)}(\bar{f})) \tag{3-13}$$

if n is odd.

(v) *If the* $\Pi^{(1)}$*-operator in* (3-9) *is of order* $k \in \mathbb{N}\backslash\{0\}$, *it has either the form*

$$\int b^{\flat_0}_{x_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} a^*_{y_k} a(\eta^{(2r)}(f)) \prod_{i=1}^{k} \eta^{(j_i)}_{\natural_i}(x_i; y_i)\, dx_i\, dy_i$$

or the form

$$\int b^{\flat_0}_{x_1} \prod_{i=1}^{k-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} a_{y_k} a^*(\eta^{(2r+1)}(\bar{f})) \prod_{i=1}^{k} \eta^{(j_i)}_{\natural_i}(x_i; y_i)\, dx_i\, dy_i$$

for some $r \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$. *If it is of order* $k = 0$, *then it is either given by* $b(\eta^{(2r)}_{\natural}(f_\lozenge))$ *or by* $b^*(\eta^{(2r+1)}_{\natural}(f_\lozenge))$, *for some* $r \in \mathbb{N}$.

(vi) *For every nonnormally ordered term of the form*

$$\int dx\,dy\,\eta^{(i)}_{\natural}(x;y)\,a_x a^*_y, \quad \int dx\,dy\,\eta^{(i)}_{\natural}(x;y)\,b_x a^*_y, \quad \int dx\,dy\,\eta^{(i)}_{\natural}(x;y)\,a_x b^*_y, \quad \text{or} \quad \int dx\,dy\,\eta^{(i)}_{\natural}(x;y)\,b_x b^*_y$$

appearing either in the Λ*-operators or in the* $\Pi^{(1)}$*-operator in* (3-9), *we have* $i \ge 2$.

Remark. Similarly, the nested commutator $\mathrm{ad}^{(n)}(b^*(f))$ can be written as the sum of $2^n n!$ terms of the form

$$\frac{1}{N^k}\widetilde{\Pi}^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(\ell)}_{\natural_{k+1}}(f_\Diamond))\Lambda_1\Lambda_2\cdots\Lambda_i$$

satisfying properties analogous to those listed in (i)–(vi).

Proof. We prove the lemma by induction. For $n=0$ all claims are trivially satisfied. For the induction step from n to $n+1$ we first compute, using (2-7) and (2-8), the commutators

$$\begin{aligned}
[B(\eta),b_z] &= -\frac{N-\mathcal{N}}{N}b^*(\eta_z)+\frac{1}{N}\int dx\,dy\,\eta(x;y)b^*_x a^*_y a_z\\
&= -b^*(\eta_z)\frac{N+1-\mathcal{N}}{N}+\frac{1}{N}\int dx\,dy\,\eta(x;y)a_z a^*_y b^*_x,\\
[B(\eta),b^*_z] &= -b(\eta_z)\frac{N-\mathcal{N}}{N}+\frac{1}{N}\int dx\,dy\,\bar\eta(x;y)a^*_z a_y b_x\\
&= -\frac{N+1-\mathcal{N}}{N}b(\eta_z)+\frac{1}{N}\int dx\,dy\,\bar\eta(x;y)b_x a_y a^*_z,\\
[B(\eta),a^*_z a_w] &= [B(\eta),a_w a^*_z] = -b^*_z b^*(\eta_w)-b(\eta_z)b_w,\\
[B(\eta),N-\mathcal{N}] &= [B(\eta),N+1-\mathcal{N}] = \int dx\,dy\,(\eta(x,y)b^*_x b^*_y+\bar\eta(x;y)b_x b_y).
\end{aligned}\tag{3-14}$$

From $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f))=[B(\eta),\mathrm{ad}^{(n)}_{B(\eta)}(b(f))]$ and by linearity, it is enough to analyze

$$[B(\eta),\Lambda_1\Lambda_2\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(\ell)}_{\natural_{k+1}}(f_\Diamond))],\tag{3-15}$$

with the operator $\Lambda_1\Lambda_2\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond))$ satisfying properties (i)–(vi). Applying the Leibniz rule $[A,BC]=[A,B]C+B[A,C]$, the commutator (3-15) is given by a sum of terms, where $B(\eta)$ is either commuted with a Λ-operator, or with the $\Pi^{(1)}$-operator.

Let's consider first the case that $B(\eta)$ is commuted with a Λ-operator, assuming further that Λ is either the operator $(N-\mathcal{N})/N$ or the operator $(N+1-\mathcal{N})/N$. The last line in (3-14) implies that such an operator Λ is replaced, after commutation with $B(\eta)$, by the sum

$$N^{-1}\Pi^{(2)}_{*,*}(\eta)+N^{-1}\Pi^{(2)}_{\cdot,\cdot}(\bar\eta).\tag{3-16}$$

With this replacement, we generate two terms contributing to $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f))$. Let us check that these new terms satisfy the properties (i)–(vi) (of course, with n replaced by $(n+1)$). Property (i) is obviously true. Also (ii) remains valid, because replacing a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ by one of the two summands in (3-16), the index m will decrease by 1, but there will be an additional factor of 2 because we added a $\Pi^{(2)}$-operator of order 1. Since exactly one additional kernel $\eta_\natural$ is inserted, also (iii) continues to hold true. The factor $\Pi^{(1)}$ is not affected by the replacement; hence the new terms will continue to satisfy (v). Furthermore, since both terms in (3-16) are normally ordered, also (vi) remains valid, by the induction assumption. We observe, finally, that the two terms we generated here do not have the form appearing in (iv).

Next, we consider the commutator of $B(\eta)$ with a Λ-operator of the form $\Lambda = N^{-p}\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_p)}_{\natural_p})$ for a $p\in\mathbb{N}$ ($p\le n$ by (ii)). By definition,

$$\Lambda = N^{-p}\int b^{\flat_0}_{x_1}\prod_{i=1}^{p-1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} b^{\sharp_p}_{y_p}\prod_{i=1}^{p}\eta^{(m_i)}_{\natural_i}(x_i;y_i)\,dx_i\,dy_i. \tag{3-17}$$

If $[B(\eta),\cdot\,]$ hits $b^{\flat_0}_{x_1}$, the first two relations in (3-14) imply that Λ is replaced by a sum of two operators, the first one being either

$$\begin{aligned}&-\frac{N-\mathcal{N}}{N}N^{-p}\Pi^{(2)}_{\sharp,\tilde\flat}(\eta^{(m_1+1)}_{\natural_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_p)}_{\natural_p}) \quad\text{or}\\ &-\frac{N+1-\mathcal{N}}{N}N^{-p}\Pi^{(2)}_{\sharp,\tilde\flat}(\eta^{(m_1+1)}_{\natural_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_p)}_{\natural_p})\end{aligned} \tag{3-18}$$

depending on whether $\flat_0=\cdot$ or $\flat_0=*$ (here $\tilde\flat=(\bar\flat_0,\flat_1,\dots,\flat_{p-1})$ with $\bar\flat_0=\cdot$ if $\flat_0=*$ and $\bar\flat_0=*$ if $\flat_0=\cdot$). The second operator emerging when $[B(\eta),\cdot\,]$ hits $b^{\flat_0}_{x_1}$ is a $\Pi^{(2)}$-operator of order $p+1$ given by

$$N^{-(p+1)}\Pi^{(2)}_{\tilde\sharp,\tilde\flat}(\eta_{\natural_0},\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_p)}_{\natural_p}), \tag{3-19}$$

where $\tilde\sharp=(\bar\flat_0,\sharp_1,\dots,\sharp_p)$, $\tilde\flat=(\bar\flat_0,\flat_0,\dots,\flat_{p-1})$ and $\natural_0=\flat_0$.

For both terms (3-18) and (3-19), (i) is clearly correct and also (ii) remains true (when we replace (3-17) with (3-18), the number of $(N-\mathcal{N})/N$- or $(N-\mathcal{N}+1)/N$-operators increases by 1, while everything else remains unchanged; similarly, when we replace (3-17) with (3-19), the order of the $\Pi^{(2)}$-operator increases by 1, while the rest remains unchanged). Property (iii) remains true as well, since, in (3-18), the power m_1+1 of the first η-kernel is increased by one unit and, in (3-19), there is one additional factor η, compared with (3-17). Property (v) remains valid, since the $\Pi^{(1)}$-operator on the right is not affected by this commutator. Property (vi) remains true in (3-18), because $m_1+1\ge 2$. It remains true also in (3-19). In fact, according to (3-14), when switching from (3-17) to (3-19), we are effectively replacing $b\to b^*a^*a$ or $b^*\to baa^*$. Hence, the first pair of operators in (3-19) is always normally ordered. As for the second pair of creation and annihilation operators (the one associated with the kernel $\eta^{(m_1)}_{\natural_1}$ in (3-19)), the first field is of the same type as the original b-field appearing in (3-17); hence nonnormally ordered pairs cannot be created. Finally, we remark that the terms we generated here are certainly not of the form in (iv) (because for terms as in (iv) all Λ-factors must be either $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$, and this is not the case, for terms containing (3-18) or (3-19)).

The same arguments can be applied if $B(\eta)$ hits the factor $b^{\sharp_p}_{y_p}$ on the right of (3-17) (in this case, we use the identities for the first two commutators in (3-14) having the b-field to the left of the factors $(N+1-\mathcal{N})/N$ and $(N-\mathcal{N})/N$ and to the right of the $a_z a^*_y$ and $a^*_z a_y$-operators).

If now $B(\eta)$ hits a term $a^*_{y_r}a_{x_{r+1}}$ or $a_{y_r}a^*_{x_{r+1}}$ in (3-17), for an $r=1,\dots,p-1$, then (3-14) implies that $\Lambda = N^{-p}\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_p)}_{\natural_p})$ is replaced by the sum of the two terms, given by

$$-[N^{-r}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_r+1)}_{\natural'_r})][N^{-(p-r)}\Pi^{(2)}_{\sharp'',\flat''}(\eta^{(m_{r+1})}_{\natural_{r+1}},\dots,\eta^{(m_p)}_{\natural_p})] \tag{3-20}$$

and by

$$-[N^{-r}\Pi^{(2)}_{\sharp''',\flat'}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_r)}_{\natural'_r})][N^{-(p-r)}\Pi^{(2)}_{\sharp'',\flat'''}(\eta^{(m_{r+1}+1)}_{\natural'_{r+1}},\dots,\eta^{(m_p)}_{\natural_p})], \tag{3-21}$$

with $\flat'=(\flat_0,\dots,\flat_{r-1})$, $\flat''=(\flat_r,\dots,\flat_{p-1})$, $\flat'''=(\bar\flat_r,\flat_{r+1},\dots,\flat_{p-1})$ and with $\sharp'=(\sharp_1,\dots,\sharp_{r-1},\bar\sharp_r)$, $\sharp''=(\sharp_{r+1},\dots,\sharp_p)$, $\sharp'''=(\sharp_1,\dots,\sharp_r)$ (here, we set $\bar\sharp_r=*$ if $\sharp_r=\cdot$ and $\bar\sharp_r=\cdot$ if $\sharp_r=*$, and similarly for $\bar\flat_{r-1}$). The precise forms of $\natural'_r$ and $\natural'_{r+1}$ do not play an important role (they are given by $\natural'_r=(\natural_r,\sharp_r)$ and $\natural'_{r+1}=(\natural_{r+1},\flat_r)$). The new terms containing (3-20) and (3-21) clearly satisfy (i). Furthermore, (ii) remains true because the contribution of the original Λ to the sum in (3-11), which was given by $p+1$ is now replaced by $(r+1)+(p-r+1)=p+2$. Clearly, (iii) remains true as well, since, for both terms (3-20) and (3-21), the total powers of the η-kernels is increased exactly by 1. As before, the terms we generated do not have the form (iv). Property (v) continues to hold true, because the $\Pi^{(1)}$-term is unaffected. As for (vi), we observe that nonnormally ordered pairs can only be created where $\sharp_r$ is changed to $\bar\sharp_r$ (in the term where $\sharp'$ appears) or where $\flat_r$ is changed to $\bar\flat_r$ (in the term where $\flat'''$ appears). In both cases, however, the change $\sharp_r\to\bar\sharp_r$ and $\flat_r\to\bar\flat_r$ comes together with an increase in the power of the corresponding η-kernel (i.e., $\eta^{(m_r)}_{\natural_r}$ is changed to $\eta^{(m_r+1)}_{\natural'_r}$ in the first case, while $\eta^{(m_{r+1})}_{\natural_{r+1}}$ is changed to $\eta^{(m_{r+1}+1)}_{\natural'_{r+1}}$ in the second case). Since $m_r+1, m_{r+1}+1\ge 2$, even if nonnormally ordered terms are created, they still satisfy (vi).

Next, let us consider the terms arising from commuting $B(\eta)$ with the operator

$$N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond)) = N^{-k}\int b^{\flat_0}_{x_1}\prod_{i=1}^{k-1} a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}a^{\sharp_k}_{y_k}a^{\flat_k}(\eta^{(s)}_{\natural}(f_\Diamond))\prod_{i=1}^{k}\eta^{(j_i)}_{\natural_i}(x_i;y_i)\,dx_i\,dy_i. \tag{3-22}$$

We argue similarly to the case in which $B(\eta)$ hits a $\Pi^{(2)}$-operator like (3-17). In particular, if $B(\eta)$ hits the operator $b^{\flat_0}_{x_1}$, the operator (3-22) is replaced by the sum of two terms, the first one being

$$-\frac{N-\mathcal{N}}{N}N^{-p}\Pi^{(1)}_{\sharp,\tilde\flat}(\eta^{(m_1+1)}_{\natural'_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond)) \quad\text{or}\quad -\frac{N+1-\mathcal{N}}{N}N^{-p}\Pi^{(1)}_{\sharp,\tilde\flat}(\eta^{(m_1+1)}_{\natural'_1},\eta^{(m_2)}_{\natural_2},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond)),$$

depending on whether $\flat_0=\cdot$ or $\flat_0=*$ (with $\tilde\flat=(\bar\flat_0,\flat_1,\dots,\flat_{k-1})$) and the second one being

$$N^{-(k+1)}\Pi^{(1)}_{\tilde\sharp,\tilde\flat}(\eta,\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_k)}_{\natural_k},\eta^{(s)}_{\natural}(f_\Diamond)),$$

with $\tilde\sharp=(\bar\flat_0,\sharp_1,\dots,\sharp_k)$ and $\tilde\flat=(\bar\flat_0,\flat_1,\dots,\flat_k)$. As we did in the analysis of (3-18) and (3-19), one can show that both these terms satisfy all properties (i), (ii), (iii), (v), (vi) (we will discuss (iv) below).

If instead $B(\eta)$ hits one of the factors $a^{\sharp_r}_{y_r}a^{\flat_r}_{x_{r+1}}$ for an $r=1,\dots,k-1$, the resulting two terms will have the form

$$-[N^{-r}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_r+1)}_{\natural'_r})][N^{-(k-r)}\Pi^{(1)}_{\sharp'',\flat''}(\eta^{(m_{r+1})}_{\natural_{r+1}},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond))] \tag{3-23}$$

or

$$-[N^{-r}\Pi^{(2)}_{\sharp''',\flat'}(\eta^{(m_1)}_{\natural_1},\dots,\eta^{(m_r)}_{\natural'_r})][N^{-(k-r)}\Pi^{(1)}_{\sharp'',\flat'''}(\eta^{(m_{r+1}+1)}_{\natural'_{r+1}},\dots,\eta^{(m_k)}_{\natural_k};\eta^{(s)}_{\natural}(f_\Diamond))], \tag{3-24}$$

with $\sharp', \sharp'', \sharp'''$ and $\flat', \flat'', \flat'''$ as defined after (3-21). Proceeding much as we did in (3-21), we can show that these terms satisfy (i), (ii), (iii), (v) and (vi).

Let us now consider the case that (3-22) is commuted with the last pair of operators appearing in (3-22). From the induction assumption, we know that this pair can only be $a^*_{y_k} a(\eta^{(2r)}(f))$ or $a_{y_k} a^*(\eta^{(2r+1)}(\bar f))$. In the first case, (3-22) is replaced by

$$-\Pi^{(2)}_{\sharp,\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k})\, b^*(\eta^{(2r+1)}(\bar f)) - \Pi^{(2)}_{\sharp',\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_{k-1})}_{\natural_{k-1}},\eta^{(j_k+1)}_{\natural'_k})\, b(\eta^{(2r)}(f)). \tag{3-25}$$

In the second case, it is replaced by

$$-\Pi^{(2)}_{\sharp',\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_{k-1})}_{\natural_{k-1}},\eta^{(j_k+1)}_{\natural'_k}) b^*(\eta^{(2r+1)}(\bar f)) - \Pi^{(2)}_{\sharp',\flat'}(\eta^{(j_1)}_{\natural_1},\dots,\eta^{(j_k)}_{\natural_k}) b(\eta^{(2r+2)}(f)). \tag{3-26}$$

In (3-25), (3-26), we used the notation $\flat' = (\flat_0,\dots,\flat_{k-1})$, $\sharp' = (\sharp_1,\dots,\bar\sharp_k)$ (as usual, the precise form of $\natural'_k$ is not important). From the expressions (3-25), (3-26), we see that also in this case, (i), (ii), (iii), (v) and (vi) are satisfied.

As for (iv), from the induction assumption we know that there is exactly one term in the expansion for $\mathrm{ad}^{(n)}_{B(\eta)}(b(f))$ given by (3-12) if n is even and by (3-13) if n is odd. Let us take, for example, (3-12). If we commute the zero-order $\Pi^{(1)}$-operator $b(\eta^{(n)}(f))$ in (3-12) with $B(\eta)$, we obtain exactly the term in (3-13), with n replaced by $n+1$ (together with a second term, containing a $\Pi^{(1)}$-operator of order 1). Similarly, if we take (3-13) and we commute the $\Pi^{(1)}$-operator $b^*(\eta^{(n)}(\bar f))$ with $B(\eta)$, we get (3-12), with n replaced by $n+1$. Looking at the terms above, it is clear that there can be only one term with this form. This shows that also in the expansion for $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f))$, there is exactly one term of the form given in (iv).

Finally, let us count the number of terms in the expansion for $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f))$. By the inductive assumption, the expansion for $\mathrm{ad}^{(n)}_{B(\eta)}(b(f))$ contains exactly $2^n n!$ terms. By (ii), each of these terms is a product of exactly $n+1$ operators, each of them being $(N-\mathcal{N})$, $(N+1-\mathcal{N})$, a field operator $b^\sharp_x$ or a quadratic factor $a^\sharp_y a^\flat_x$ commuting with the number of particles operator. By (3-14), the commutator of $B(\eta)$ with each such factor gives a sum of two terms. Therefore, by the product rule, $\mathrm{ad}^{(n+1)}_{B(\eta)}(b(f))$ contains $2^n(n!)\times 2(n+1) = 2^{(n+1)}(n+1)!$ summands. This concludes the proof of the lemma. □

From Lemma 3.2, we immediately obtain a convergent series expansion for the conjugation of the fields $b(f)$ and $b^*(f)$ with the unitary operator $\exp(B(\eta))$.

Lemma 3.3. *Let $\eta \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ be symmetric, with $\|\eta\|_2$ sufficiently small. Then we have*

$$\begin{aligned} e^{-B(\eta)} b(f) e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mathrm{ad}^{(n)}_{B(\eta)}(b(f)),\\ e^{-B(\eta)} b^*(f) e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mathrm{ad}^{(n)}_{B(\eta)}(b^*(f)), \end{aligned} \tag{3-27}$$

where the series on the right-hand sides are absolutely convergent.

Proof. From (3-7) we have

$$e^{-B(\eta)}b(f)e^{B(\eta)} = \sum_{n=1}^{m-1}(-1)^n \frac{\mathrm{ad}^{(n)}_{B(\eta)}(b(f))}{n!} + \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \, e^{-s_m B(\eta)} \mathrm{ad}^{(m)}_{B(\eta)}(b(f)) e^{s_m B(\eta)}. \quad (3\text{-}28)$$

To prove (3-27), we show that the norm of the error term converges to zero as $m \to \infty$. By Lemma 3.2, $\mathrm{ad}^{(n)}_{B(\eta)}(b(f)$ is given by a sum of $2^n n!$ terms of the form

$$\Lambda_1 \cdots \Lambda_i \, \frac{1}{N^k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1}, \dots, \eta^{(j_k)}_{\natural_k}; \eta^{(\ell)}(f_\lozenge)), \quad (3\text{-}29)$$

with $i, k, \ell \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$ and where each Λ_r is $(N-\mathcal{N})/N$, $(N+1-\mathcal{N})/N$ or an operator of the form

$$\frac{1}{N^p} \Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1}, \dots, \eta^{(m_p)}_{\natural_p}).$$

On $\mathcal{F}^{\le N}$, we have the bounds $\|(N-\mathcal{N})/N\| \le 1$ and $\|(N+1-\mathcal{N})/N\| \le 2$. Lemma 2.4 implies that

$$N^{-p} \|\Pi^{(2)}_{\sharp,\flat}(\eta^{(m_1)}_{\natural_1}, \dots, \eta^{(m_p)}_{\natural_p})\| \le (12)^p (2\|\eta\|_2)^{m_1+\dots+m_p}$$

and that

$$N^{-k} \|\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1}, \dots, \eta^{(j_k)}_{\natural_k}; \eta^{(\ell)}(f_\lozenge))\| \le (12)^k \sqrt{N} \|f\|_2 (2\|\eta\|_2)^{\ell+j_1+\dots+j_k}.$$

Here we used the fact that, if a kernel $\eta^{(j)}$ is associated with a normally ordered pair of creation and annihilation operators, then $\|\eta^{(j)}\|_{\mathrm{HS}} \le \|\eta\|^j_{\mathrm{HS}}$. If instead $\eta^{(j)}$ is associated with a nonnormally ordered pair, then point (vi) in Lemma 3.2 implies that $j \ge 2$. Hence,

$$\begin{aligned} \int \left| \eta^{(j)}(x;x) \right| dx &= \int \left| \int \eta(x;y) \eta^{(j-1)}(y;x) \, dy \right| dx \\ &\le \left(\int |\eta(x;y)|^2 \, dx \, dy \right)^{1/2} \left(\int |\eta^{(j-1)}(x;y)|^2 \, dx \, dy \right)^{1/2} \\ &\le \|\eta\|_2 \|\eta^{(j-1)}\|_2 \le \|\eta\|_2^j. \end{aligned}$$

Therefore, if the term (3-29) contains $\Pi^{(2)}$-operators of order $p_1, \dots, p_j \in \mathbb{N}\backslash\{0\}$, we can bound

$$\left\| \Lambda_1 \cdots \Lambda_i \frac{1}{N^k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{\natural_1}, \dots, \eta^{(j_k)}_{\natural_k}; \eta^{(\ell)}(f_\lozenge)) \right\| \le 12^{p_1+\dots+p_j+k} \sqrt{N} (2\|\eta\|_2)^m \le \sqrt{N} \|f\|_2 C^m \|\eta\|^m$$

and therefore, since $\mathrm{ad}^{(m)}_{B(\eta)}(b(f))$ is the sum of $2^m m!$ terms,

$$\|\mathrm{ad}^{(m)}_{B(\eta)}(b(f))\| \le \sqrt{N} \|f\|_2 (2C\|\eta\|_2)^m m!. \quad (3\text{-}30)$$

This proves, first of all, that the series on the right-hand side of (3-27) converges absolutely, if $\|\eta\|_2 \le (4C)^{-1}$. Under this condition, (3-30) also implies that the error term on the right-hand side of (3-28) converges to zero, as $m \to \infty$, since

$$\left\| \int_0^1 ds_1 \cdots \int_0^{s_{m-1}} ds_m \, e^{-s_m B(\eta)} \mathrm{ad}_{B(\eta)}(b(f)) e^{s_m B(\eta)} \right\| \le \sqrt{N} \|f\|_2 (2C\|\eta\|)^m. \qquad \square$$

4. Fluctuation dynamics

In this section, we are going to define the fluctuation dynamics describing the evolution of orthogonal excitations of the Bose–Einstein condensate.

Instead of comparing the solution of the many-body Schrödinger equation (1-8) directly with the solution of the Gross–Pitaevskii equation (1-12), it is convenient to introduce a modified, N-dependent, Gross–Pitaevskii equation. To this end, we fix $\ell > 0$ and we consider the ground state f_ℓ of the Neumann problem

$$\big(-\Delta + \tfrac{1}{2}V\big) f_\ell = \lambda_\ell f_\ell \tag{4-1}$$

on the ball $|x| \le N\ell$, such that the radial derivative $\partial_r f_\ell(x)$ is zero for $|x| = N\ell$ (we omit the N-dependence in the notation for f_ℓ and for λ_ℓ; notice that λ_ℓ scales as N^{-3}). The solution f_ℓ is radial, and we can normalize it so that $f_\ell(x) = 1$ for $|x| = N\ell$. We extend f_ℓ to $\mathbb{R}^3$ by setting $f_\ell(x) = 1$ for all $|x| > N\ell$. We also define $w_\ell = 1 - f_\ell$ (so that $w_\ell(x) = 0$ if $|x| > N\ell$). By scaling, we observe that $f_\ell(N\,\cdot\,)$ satisfies the equation

$$\big(-\Delta + \tfrac{1}{2}N^2 V(N\,\cdot\,)\big) f_\ell(N\,\cdot\,) = N^2 \lambda_\ell f_\ell(N\,\cdot\,) \tag{4-2}$$

on the ball $|x| \le \ell$ ($\ell > 0$ will be kept fixed, independent of N). With this choice, we expect that f_ℓ will be close, in the limit of large N, to the solution of the zero-energy scattering equation (1-2). This is confirmed by the next lemma, where we collect some important properties of f_ℓ. Most of the these results are taken from Lemma A.1 of [Erdős et al. 2006].

Lemma 4.1. *Let $V \in L^3(\mathbb{R}^3)$ be a nonnegative, spherically symmetric potential with $V(x) = 0$ for all $|x| > R$. Fix $\ell > 0$ and let f_ℓ denote the solution of* (4-1):

(i) *We have*

$$\lambda_\ell = \frac{3a_0}{N^3\ell^3}\left(1 + \mathcal{O}\left(\frac{a_0}{N\ell}\right)\right).$$

(ii) *We have $0 \le f_\ell, w_\ell \le 1$ and*

$$\int dx\, V(x) f_\ell(x) = 8\pi a_0 + \mathcal{O}(N^{-1}). \tag{4-3}$$

(iii) *There exists a constant $C > 0$, depending on the potential V, such that*

$$w_\ell(x) \le \frac{C}{|x|+1} \quad \text{and} \quad |\nabla w_\ell(x)| \le \frac{C}{|x|^2+1} \tag{4-4}$$

for all $|x| \le N\ell$.

Proof. Statement (i), the fact that $0 \le f_\ell, w_\ell \le 1$, and statement (iii) follow from Lemma A.1 in [Erdős et al. 2006]. We have to show (4-3). To this end, we adapt the proof of Lemma 5.1(iv) of [Erdős et al. 2010]. With $r = |x|$, we may write $m(r) = r f_\ell(r)$. We find that, for all $r \in (R, N\ell]$,

$$m(r) = \lambda_\ell^{-1/2} \sin(\lambda_\ell^{1/2}(r - N\ell)) + N\ell \cos(\lambda_\ell^{1/2}(r - N\ell)). \tag{4-5}$$

By expanding up to the order $\mathcal{O}(\lambda_\ell^2)$ we obtain

$$m(r) = r - a_0 + \mathcal{O}(N^{-1}), \quad m'(r) = 1 + \mathcal{O}(N^{-1}). \tag{4-6}$$

Hence

$$\begin{aligned}\int dx\, V(x) f_\ell(x) &= 4\pi \int_0^R dr\, r V(r) m(r) \\ &= 8\pi \int_0^R dr\, (r m''(r) + \lambda_\ell r^2 f_\ell(r)) \\ &= 8\pi \int_0^R dr\, r m''(r) + O(N^{-3}) \\ &= 8\pi (Rm'(R) - m(R)) + \mathcal{O}(N^{-1}) = 8\pi a_0 + \mathcal{O}(N^{-1}),\end{aligned} \tag{4-7}$$

completing the proof. □

Next, we introduce the modified Gross–Pitaevskii equation†

$$i\partial_t \tilde{\varphi}_t = -\Delta \tilde{\varphi}_t + (N^3 V(N\,\cdot\,) f_\ell(N\,\cdot\,) * |\tilde{\varphi}_t|^2)\tilde{\varphi}_t, \tag{4-8}$$

with initial data $\tilde{\varphi}_{t=0} = \varphi$ describing the Bose–Einstein condensate at time $t = 0$. While in Theorem 1.2 the notation φ is already used to indicate the initial condensate wave function, in the proof of Theorem 1.1 we will choose $\varphi = \phi_{\text{GP}}$ to be the minimizer of the Gross–Pitaevskii functional (1-6). In both cases, we assume that $\varphi \in H^4(\mathbb{R}^3)$.

Notice that, in contrast with the initial data φ, the solution $\tilde{\varphi}_t$ depends on N. With (4-3), one can show that $\tilde{\varphi}_t$ converges towards the solution of the original Gross–Pitaevskii equation (1-12) as $N \to \infty$. This fact and some other important properties of the solutions of (1-12) and (4-8) are listed in the next proposition, whose proof can be found in Theorem 3.1 of [Benedikter et al. 2015], with the only difference that, in that paper, the modified Gross–Pitaevskii equation was defined through the solution f of the zero-energy scattering equation, while here we work with the Neumann ground state f_ℓ. The only relevant consequence is the fact that, here, the integral of f_ℓ against V is not exactly equal to $8\pi a_0$; the error, however, is of order N^{-1} by (4-3).

Proposition 4.2. *Let $V \in L^3(\mathbb{R}^3)$ be a nonnegative, spherically symmetric, compactly supported potential. Let $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$:*

(i) *Well-posedness: For any $\varphi \in H^1(\mathbb{R}^3)$, with $\|\varphi\|_2 = 1$, there exist unique global solutions $t \to \varphi_t$ and $t \to \tilde{\varphi}_t$ in $C(\mathbb{R}, H^1(\mathbb{R}^3))$ of the Gross–Pitaevskii equation* (1-12) *and, respectively, of the modified Gross–Pitaevskii equation* (4-8) *with initial datum φ. We have $\|\varphi_t\|_2 = \|\tilde{\varphi}_t\|_2 = 1$ for all $t \in \mathbb{R}$. Furthermore, there exists a constant $C > 0$ such that*

$$\|\varphi_t\|_{H^1}, \|\tilde{\varphi}_t\|_{H^1} \le C.$$

(ii) *Propagation of higher regularity: If $\varphi \in H^m(\mathbb{R}^3)$ for some $m \ge 2$, then $\varphi_t, \tilde{\varphi}_t \in H^m(\mathbb{R}^3)$ for every $t \in \mathbb{R}$. Moreover, there exist constants $C > 0$, depending on m and on $\|\varphi\|_{H^m}$, and $c > 0$, depending on m*

†It is convenient to work with this modified equation, rather than directly with the Gross–Pitaevskii equation (1-12), to obtain a cleaner cancellation between the contributions (5-47) and (5-100) to the generator of the fluctuation dynamics (4-24).

and on $\|\varphi\|_{H^1}$, *such that, for all* $t \in \mathbb{R}$,

$$\|\varphi_t\|_{H^m}, \|\tilde{\varphi}\|_{H^m} \le Ce^{c|t|}. \tag{4-9}$$

(iii) Regularity of time-derivatives: *Suppose* $\varphi \in H^4(\mathbb{R}^3)$. *Then there exist* $C > 0$, *depending on* $\|\varphi\|_{H^4}$, *and* $c > 0$, *depending on* $\|\varphi\|_{H^1}$, *such that, for all* $t \in \mathbb{R}$,

$$\|\dot{\tilde{\varphi}}\|_{H^2}, \|\ddot{\tilde{\varphi}}\|_{H^2} \le Ce^{c|t|}.$$

(iv) Comparison of dynamics: *Suppose* $\varphi \in H^2(\mathbb{R}^3)$. *Then there exists a constant* $c > 0$, *depending on* $\|\varphi\|_{H^2}$, *such that, for all* $t \in \mathbb{R}$,

$$\|\varphi_t - \tilde{\varphi}_t\|_2 \le CN^{-1}\exp(c\exp(c|t|)). \tag{4-10}$$

To compare the many-body evolution $\psi_{N,t}$ with products of the solution $\tilde{\varphi}_t$ of the modified Gross–Pitaevskii equation (1-12), we are going to define a unitary map (already discussed in Section 1, after (1-25)) that was first introduced in [Lewin et al. 2015a; 2015b] in the mean-field setting. To this end, we remark that every $\psi_N \in L^2_s(\mathbb{R}^{3N})$ has a unique representation of the form

$$\psi_N = \sum_{n=0}^{N} \psi_N^{(n)} \otimes_s \tilde{\varphi}_t^{\otimes(N-n)}, \tag{4-11}$$

where $\psi_N^{(n)} \in L^2_{\perp\tilde{\varphi}_t}(\mathbb{R}^3)^{\otimes_s n}$ is symmetric with respect to permutations and orthogonal to $\tilde{\varphi}_t$, in each of its coordinates, and where, for $\psi_N^{(n)} \in L^2_\perp(\mathbb{R}^3)^{\otimes_s n}$ and $\psi_N^{(k)} \in L^2_\perp(\mathbb{R}^3)^{\otimes_s k}$, $\psi_N^{(n)} \otimes_s \psi_N^{(k)}$ denotes the symmetrized product defined by

$$\psi_N^{(k)} \otimes_s \psi_N^{(n)}(x_1, \dots, x_{k+n}) = \frac{1}{\sqrt{k!\,n!\,(k+n)!}} \sum_{\sigma \in S_{k+n}} \psi_N^{(k)}(x_{\sigma(1)}, \dots, x_{\sigma(k)})\psi_N^{(n)}(x_{\sigma(k+1)}, \dots, x_{\sigma(k+n)}). \tag{4-12}$$

Using the representation (4-11), we define $U_{N,t} : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\le N}_{\perp\tilde{\varphi}_t}$ by setting

$$U_{N,t}\psi_N = \{\psi_N^{(0)}, \psi_N^{(1)}, \dots, \psi_N^{(N)}\}. \tag{4-13}$$

In terms of creation and annihilation operators, the map $U_{N,t}$ is given by

$$U_{N,t}\psi_N = \bigoplus_{n=0}^{N} (1 - |\tilde{\varphi}_t\rangle\langle\tilde{\varphi}_t|)^{\otimes n} \frac{a(\tilde{\varphi}_t)^{N-n}}{\sqrt{(N-n)!}} \psi_N.$$

Here, and frequently in the sequel, we identify the wave function $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with the Fock-space vector $\{0, \dots, 0, \psi_N, 0, \dots\} \in \mathcal{F}$. From (4-11) and by the requirement of orthogonality, it is easy to check that $\|\psi_N\|^2 = \sum_{n=0}^{N} \|\psi_N^{(n)}\|^2$. Hence, $U_{N,t} : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\le N}_{\perp\tilde{\varphi}_t}$ is a unitary map, with inverse

$$U^*_{N,t}\{\psi_N^{(0)}, \psi_N^{(1)}, \dots, \psi_N^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\tilde{\varphi}_t)^{N-n}}{\sqrt{(N-n)!}} \psi_N^{(n)}.$$

The action of $U_{N,t}$ on creation and annihilation operators is determined by the following rules, see [Lewin et al. 2015a; 2015b]:

$$\begin{aligned} U_{N,t}\, a^*(\tilde\varphi_t) a(\tilde\varphi_t) U^*_{N,t} &= N - \mathcal{N}, \\ U_{N,t}\, a^*(f) a(\tilde\varphi_t) U^*_{N,t} &= a^*(f)\sqrt{N-\mathcal{N}} = \sqrt{N}\, b^*(f), \\ U_{N,t}\, a^*(\tilde\varphi_t) a(g) U^*_{N,t} &= \sqrt{N-\mathcal{N}} a(g) = \sqrt{N}\, b(g), \\ U_{N,t}\, a^*(f) a(g) U^*_{N,t} &= a^*(f) a(g) \end{aligned} \tag{4-14}$$

for all $f, g \in L^2_{\perp\tilde\varphi_t}(\mathbb{R}^3)$. Here we used modified creation and annihilation operators, as defined in (2-5).

With $U_{N,t}$ we factor out the condensate and we focus on its orthogonal excitations. Observe, however, that $U_{N,t}$ does not remove correlations, which are known to play a crucial role in the Gross–Pitaevskii regime; see, for example, [Erdős et al. 2009a; 2010; 2016]. To remove correlations from the excitation vectors, we are going to use a generalized Bogoliubov transformation, as introduced in Section 3. We define

$$k_t(x;y) = -N w_\ell(N(x-y))\tilde\varphi_t(x)\tilde\varphi_t(y). \tag{4-15}$$

From Lemma 4.1, it follows that $k_t \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$, with L^2-norm bounded uniformly in N. Hence, k_t is the integral kernel of a Hilbert–Schmidt operator on $L^2(\mathbb{R}^3)$, which we denote again with k_t. We define a new Hilbert–Schmidt operator setting

$$\eta_t = (1 - |\tilde\varphi_t\rangle\langle\tilde\varphi_t|)\, k_t\, (1 - |\bar{\tilde\varphi}_t\rangle\langle\bar{\tilde\varphi}_t|). \tag{4-16}$$

Also in this case, we will denote by η_t both the Hilbert–Schmidt operator defined in (4-16) and its integral kernel. Note that $\eta_t \in (q_{\tilde\varphi_t}\otimes q_{\tilde\varphi_t})L^2(\mathbb{R}^3\times\mathbb{R}^3)$, where $q_{\tilde\varphi_t} = 1 - |\tilde\varphi_t\rangle\langle\tilde\varphi_t|$. Let us write $\eta_t = k_t + \mu_t$, with the Hilbert–Schmidt operator

$$\mu_t = |\tilde\varphi_t\rangle\langle\tilde\varphi_t|\, k_t\, |\bar{\tilde\varphi}_t\rangle\langle\bar{\tilde\varphi}_t| - |\tilde\varphi_t\rangle\langle\tilde\varphi_t|\, k_t - k_t|\bar{\tilde\varphi}_t\rangle\langle\bar{\tilde\varphi}_t|. \tag{4-17}$$

In the next lemma we collect some important properties of the operators η_t, k_t, μ_t.

Lemma 4.3. *Let $\tilde\varphi_t$ be the solution of* (4-8) *with initial datum $\varphi \in H^4(\mathbb{R})$. Let $w_\ell = 1 - f_\ell$, with f_ℓ the ground state solution of the Neumann problem* (4-1). *Let k_t, η_t, μ_t be defined as in* (4-15), (4-16), (4-17). *Then there exist constants $C, c > 0$ depending only on $\|\varphi\|_{H^4}$ (in many cases, these constants actually depend only on lower Sobolev norms of φ) and on V such that the following bounds hold true for all $t \in \mathbb{R}$:*

(i) *We have*

$$\|\eta_t\|_2 \le C, \quad \|\eta_t^{(n)}\|_2 \le \|\eta_t\|_2^n \le C^n \quad \text{and} \quad \lim_{\ell\to 0}\ \sup_{t\in\mathbb{R},\, N\in\mathbb{N}} \|\eta_t\|_2 = 0 \tag{4-18}$$

and also

$$\|\nabla_j\eta_t\|_2 \le C\sqrt{N}, \quad \|\nabla_j\mu_t\|_2 \le C, \quad \|\nabla_j\eta_t^{(n)}\|_2 \le C\|\eta_t\|_2^{n-2}, \quad \|\Delta_j\eta_t^{(n)}\|_2 \le C\|\eta_t\|_2^{n-2}$$

for $j = 1, 2$ and for all $n \ge 2$. Here $\nabla_1\eta_t$ and $\nabla_2\eta_t$ denote the kernels $\nabla_x\eta_t(x;y)$ and $\nabla_y\eta_t(x;y)$ ($\Delta_1\eta_t$ and $\Delta_2\eta_t$ are defined similarly). Decomposing $\cosh_{\eta_t} = 1 + p_{\eta_t}$ and $\sinh_{\eta_t} = \eta_t + r_{\eta_t}$, we obtain

$$\|\sinh_{\eta_t}\|_2,\ \|p_{\eta_t}\|_2,\ \|r_{\eta_t}\|_2,\ \|\nabla_j p_{\eta_t}\|_2,\ \|\nabla_j r_{\eta_t}\|_2 \le C. \tag{4-19}$$

(ii) *For a.e. $x, y \in \mathbb{R}^3$ and $n \in \mathbb{N}$, $n \geq 2$, we have the pointwise bounds*

$$
\begin{aligned}
|\eta_t(x; y)| &\leq \frac{C}{|x-y| + N^{-1}} |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|, \\
|\eta_t^{(n)}(x; y)| &\leq C \|\eta_t\|_2^{n-2} |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|, \\
|\mu_t(x; y)|, \ |p_{\eta_t}(x; y)|, \ |r_{\eta_t}(x; y)| &\leq C |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)|.
\end{aligned}
\tag{4-20}
$$

(iii) *We have*

$$\sup_x \int |\eta_t(x; y)|^2 \, dy, \quad \sup_x \int |k_t(x; y)|^2 \, dy, \quad \sup_x \int |\mu_t(x; y)|^2 \, dy \leq C \|\tilde{\varphi}_t\|_{H^2} \leq C e^{c|t|}$$

and

$$\sup_x \int |\eta_t^{(n)}(x; y)|^2 \, dy \leq C \|\eta_t\|_2^{n-2} \|\tilde{\varphi}_t\|_{H^2} \leq C \|\eta_t\|_2^{n-2} e^{c|t|}$$

for all $n \geq 2$. Therefore

$$\sup_x \int |p_{\eta_t}(x; y)|^2 \, dy, \quad \sup_x \int |r_{\eta_t}(x; y)|^2 \, dy, \quad \sup_x \int |\sinh_{\eta_t}(x; y)|^2 \, dy \leq C e^{c|t|}.$$

(iv) *For $j = 1, 2$ and $n \geq 2$, we have*

$$\|\partial_t \eta_t\|_2, \ \|\partial_t^2 \eta_t\|_2 \leq C e^{c|t|}, \quad \|\partial_t \eta_t^{(n)}\|_2 \leq C n e^{c|t|} \|\eta_t\|_2^{n-1}$$

and also

$$\|\partial_t \nabla_j \eta_t\|_2 \leq C \sqrt{N} e^{c|t|}, \quad \|\partial_t \nabla_j \mu_t\|_2 \leq C e^{c|t|}, \quad \|\partial_t \nabla_j \eta_t^{(n)}\|_2 \leq C n \|\eta_t\|^{n-2} e^{c|t|}.$$

Therefore

$$\|\partial_t p_{\eta_t}\|_2, \ \|\partial_t r_{\eta_t}\|_2, \ \|\partial_t \sinh_{\eta_t}\|_2, \ \|\nabla_j \partial_t p_{\eta_t}\|_2, \ \|\nabla_j \partial_t r_{\eta_t}\|_2 \leq C e^{c|t|}.$$

(v) *For a.e. $x, y \in \mathbb{R}^3$, we have the pointwise bounds*

$$|\partial_t \eta_t(x; y)| \leq C \left[1 + \frac{1}{|x-y| + N^{-1}} \right] \big[|\dot{\tilde{\varphi}}_t(x)| |\tilde{\varphi}_t(y)| + |\tilde{\varphi}_t(x)| |\dot{\tilde{\varphi}}_t(y)| + |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)| \big].$$

Moreover, for $n \geq 2$, we have

$$|\partial_t \eta_t^{(n)}(x; y)| \leq C n e^{c|t|} \|\eta_t\|_2^{n-2} \big[|\dot{\tilde{\varphi}}_t(x)| |\tilde{\varphi}_t(y)| + |\tilde{\varphi}_t(x)| |\dot{\tilde{\varphi}}_t(y)| + |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)| \big].$$

Therefore

$$|\partial_t \mu_t(x; y)|, \ |\partial_t r_{\eta_t}(x; y)|, \ |\partial_t p_{\eta_t}(x; y)| \leq C e^{c|t|} \big[|\dot{\tilde{\varphi}}_t(x)| |\tilde{\varphi}_t(y)| + |\tilde{\varphi}_t(x)| |\dot{\tilde{\varphi}}_t(y)| + |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)| \big].$$

(vi) *Finally, we find*

$$\sup_x \int |\partial_t \eta_t(x; y)|^2 \, dy, \quad \sup_x \int |\partial_t k_t(x; y)|^2 \, dy, \quad \sup_x \int |\partial \mu_t(x; y)|^2 \, dy \leq C e^{c|t|}.$$

Furthermore, for all $n \geq 2$,

$$\sup_x \int |\partial_t \eta_t^{(n)}(x; y)| \, dy \leq C n e^{c|t|} \|\eta_t\|_2^{n-2}$$

and therefore

$$\sup_x \int |\partial_t p_{\eta_t}(x;y)|^2\,dy,\quad \sup_x \int |\partial_t r_{\eta_t}(x;y)|^2\,dy,\quad \sup_x \int |\partial_t \sinh_{\eta_t}(x;y)|^2\,dy \le Ce^{c|t|}.$$

Proof. To prove (4-18) observe that, using Lemma 4.1 and Young's inequality,

$$\|\eta_t\|_2^2 \le \|k_t\|_2^2 \le C \int \frac{\chi(|x-y|\le \ell)}{|x-y|^2} |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2\,dx\,dy \le C\ell \|\tilde{\varphi}_t\|_4^2 \le C\ell \|\tilde{\varphi}_t\|_{H^1}^2 \le C\ell$$

uniformly in $N \in \mathbb{N}$ and in $t \in \mathbb{R}$. The proof of the other bounds is a simple generalization of the proof of Lemmas 3.3 and 3.4 in [Benedikter et al. 2015]; we omit the details. □

We model correlations in the solution $\psi_{N,t}$ of the many-body Schrödinger equation (1-8) by means of the generalized Bogoliubov transformation $\exp(B(\eta_t)) : \mathcal{F}^{\le N}_{\perp \tilde{\varphi}_t} \to \mathcal{F}^{\le N}_{\perp \tilde{\varphi}_t}$ with the integral kernel $\eta_t \in (q_{\tilde{\varphi}_t} \otimes q_{\tilde{\varphi}_t}) L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ defined in (4-16). We define therefore the fluctuation dynamics

$$\mathcal{W}_{N,t} = e^{-B(\eta_t)}\, U_{N,t}\, e^{-iH_N t}\, U^*_{N,0}\, e^{B(\eta_0)}. \tag{4-21}$$

Then $\mathcal{W}_{N,t} : \mathcal{F}^{\le N}_{\perp \varphi} \to \mathcal{F}^{\le N}_{\perp \tilde{\varphi}_t}$ is a unitary operator. Clearly, $\mathcal{W}_{N,t}$ depends on the length parameter ℓ (the radius of the ball in (4-1)), through the modified Gross–Pitaevskii equation (4-8) and also through the kernel η_t defined in (4-15), (4-16). While $\mathcal{W}_{N,t}$ is well-defined for any value of $\ell > 0$, we will have to choose $\ell > 0$ small to make sure that $\|\eta_t\|_2$ is sufficiently small; this will allow us to expand the action of the generalized Bogoliubov transformation $\exp(B(\eta_t))$ appearing in (4-21) using the series expansion (3-27) (because, by (4-18), smallness of ℓ implies that $\|\eta_t\|_2$ is small, uniformly in t).

For $\xi \in \mathcal{F}^{\le N}_{\perp \varphi}$, the operator $\mathcal{W}_{N,t}$ is defined so that

$$e^{-iH_N t}\, U^*_{N,0}\, e^{B(\eta_0)} \xi = U^*_{N,t}\, e^{B(\eta_t)}\, \mathcal{W}_{N,t} \xi.$$

It allows us to describe the many-body evolution of initial data of the form

$$\psi_N = U^*_{N,0} e^{B(\eta_0)} \xi, \tag{4-22}$$

and to express the evolved state again in the form

$$\psi_{N,t} = e^{-iH_N t} \psi_N = U^*_{N,t} e^{B(\eta_t)} \xi_t, \tag{4-23}$$

where $\xi_t = \mathcal{W}_{N,t}\, \xi$. As we will see below, a vector of the form (4-22) exhibits Bose–Einstein condensation in the one-particle state φ if and only if the expectation of the number of particles operator $\langle \xi, \mathcal{N} \xi \rangle$ is small, compared with the total number of particles N. Hence, to prove Theorems 1.1 and 1.2, we will have to show first that every initial $\psi_N \in L^2_s(\mathbb{R}^{3N})$ satisfying (1-10) can be written in the form (4-22) for a $\xi \in \mathcal{F}^{\le N}_{\perp \varphi}$ with $\langle \xi, \mathcal{N} \xi \rangle \ll N$ and then that the bound on the expectation of the number of particles is approximately preserved by $\mathcal{W}_{N,t}$. In fact, it turns out that to control the growth of the expectation of $\mathcal{N}$ along the fluctuation dynamics, it is not enough to have a bound on $\langle \xi, \mathcal{N} \xi \rangle$; instead, we will also need a bound on the energy of ξ (this is why we need to assume $b_N \to 0$ in (1-10)).

To control the growth of the number of particles with respect to the fluctuation dynamics it is important to compute the generator of $\mathcal{W}_{N,t}$. A simple computation shows that

$$i\partial_t \mathcal{W}_{N,t} = \mathcal{G}_{N,t}\mathcal{W}_{N,t},$$

with the time-dependent generator

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + e^{-B(\eta_t)}[(i\partial_t U_{N,t})U_{N,t}^* + U_{N,t} H_N U_{N,t}^*]e^{B(\eta_t)}. \tag{4-24}$$

Notice, that $\mathcal{G}_{N,t}$ maps $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$ into $\mathcal{F}^{\leq N}$, but not into $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$. This is due to the fact that the space $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$ depends on time (and thus $\mathcal{G}_{N,t}$ must have a component which allows $\mathcal{W}_{N,t}$ to move to different spaces). We will mostly be interested in the expectation of $\mathcal{G}_{N,t}$ for states in $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$, but at some point (when we will consider the variation of the expectation of $\mathcal{G}_{N,t}$) it will be important to remember the component of $\mathcal{G}_{N,t}$ mapping out of $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$.

In the next proposition, we collect important properties of the generator $\mathcal{G}_{N,t}$.

Theorem 4.4. *Let $V \in L^3(\mathbb{R}^3)$ be nonnegative, spherically symmetric and compactly supported. Let $\mathcal{W}_{N,t}$ be defined as in* (4-21) *with the length parameter $\ell > 0$ sufficiently small and using the solution of the modified Gross–Pitaevskii equation* (4-8), *with an initial data $\varphi \in H^4(\mathbb{R}^3)$. Let*

$$\begin{aligned} C_{N,t} = \frac{1}{2}\langle \tilde{\varphi}_t, \big([N^3 V(N\,\cdot)(N-1-2Nf_\ell(N\,\cdot))] * |\tilde{\varphi}_t|^2\big)\tilde{\varphi}_t \rangle \\ + \int dx\,dy\,|\nabla_x k_t(x;y)|^2 + \frac{1}{2}\int dx\,dy\,N^2 V(N(x-y))|k_t(x;y)|^2 \\ + \operatorname{Re}\int dx\,dy\,N^3 V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)k_t(x;y). \end{aligned} \tag{4-25}$$

Then there exist constants $C, c > 0$ such that, in the sense of quadratic forms on $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$,

$$\begin{aligned} \tfrac{1}{2}\mathcal{H}_N - Ce^{c|t|}(\mathcal{N}+1) \leq (\mathcal{G}_{N,t} - C_{N,t}) \leq 2\mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1), \\ \pm i[\mathcal{N}, \mathcal{G}_{N,t}] \leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1), \\ \pm\partial_t(\mathcal{G}_{N,t} - C_{N,t}) \leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1), \\ \pm\operatorname{Re}[a^*(\partial_t\tilde{\varphi}_t)a(\tilde{\varphi}_t), \mathcal{G}_{N,t}] \leq \mathcal{H}_N + Ce^{c|t|}(\mathcal{N}+1), \end{aligned} \tag{4-26}$$

where $\mathcal{H}_N$ is the Fock-space Hamiltonian

$$\mathcal{H}_N = \int dx\,\nabla_x a_x^* \nabla_x a_x + \frac{1}{2}\int dx\,dy\,N^2 V(N(x-y))\,a_x^* a_y^* a_y a_x. \tag{4-27}$$

Note that, on $\mathcal{F}_{\perp\tilde{\varphi}_t}^{\leq N}$, we have

$$[a^*(\partial_t\tilde{\varphi}_t)a(\tilde{\varphi}_t), \mathcal{G}_{N,t}] = a^*(\partial_t\tilde{\varphi}_t)a(\tilde{\varphi}_t)\mathcal{G}_{N,t}.$$

The proof of Theorem 4.4 is given in the next section. From the technical point of view, it represents the main part of our paper. In Section 6, we show then how to use the properties of $\mathcal{G}_{N,t}$ established in Theorem 4.4 to complete the proof of Theorems 1.1 and 1.2.

5. Analysis of the generator of fluctuation dynamics

In this section we study the properties of the generator

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + e^{-B(\eta_t)}[(i\partial_t U_{N,t})U^*_{N,t} + U_{N,t} H_N U^*_{N,t}]e^{B(\eta_t)} \tag{5-1}$$

of the fluctuation dynamics (4-21); the goal is to prove Theorem 4.4.

As forms on $\mathcal{F}^{\leq N}_{\perp\tilde{\varphi}_t} \times \mathcal{F}^{\leq N}_{\perp\tilde{\varphi}_t}$, we find, see Lemma 6 in [Lewin et al. 2015a],

$$(i\partial_t U_{N,t})U^*_{N,t} = -\langle i\partial_t\tilde{\varphi}_t, \tilde{\varphi}_t\rangle(N-\mathcal{N}) - \sqrt{N}[b(i\partial_t\tilde{\varphi}_t) + b^*(i\partial_t\tilde{\varphi}_t)]. \tag{5-2}$$

Using (4-14) to compute $U_{N,t}H_N U^*_{N,t}$, a lengthy but straightforward computation, see Appendix B of [Lewin et al. 2015a], shows that

$$(i\partial_t U_{N,t})U^*_{N,t} + U_{N,t}H_N U^*_{N,t} = \sum_{j=0}^{4} \mathcal{L}^{(j)}_{N,t},$$

where

$$\begin{aligned}
\mathcal{L}^{(0)}_{N,t} &= \tfrac{1}{2}\big\langle\tilde{\varphi}_t, [N^3V(N\,\cdot)(1-2f_\ell(N\,\cdot))*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t\big\rangle(N-\mathcal{N}) - \tfrac{1}{2}\big\langle\tilde{\varphi}_t, [N^3V(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t\big\rangle(\mathcal{N}+1)\frac{(N-\mathcal{N})}{N},\\
\mathcal{L}^{(1)}_{N,t} &= \sqrt{N}\, b([N^3V(N\,\cdot)w_\ell(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t) - \frac{\mathcal{N}+1}{\sqrt{N}} b([N^3V(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t) + \text{h.c.},\\
\mathcal{L}^{(2)}_{N,t} &= \int dx\, \nabla_x a^*_x \nabla_x a_x + \int dx\,dy\, N^3V(N(x-y))|\tilde{\varphi}_t(y)|^2\Big(b^*_x b_x - \frac{1}{N}a^*_x a_x\Big)\\
&\quad + \int dx\,dy\, N^3V(N(x-y))\tilde{\varphi}_t(x)\bar{\tilde{\varphi}}_t(y)\Big(b^*_x b_y - \frac{1}{N}a^*_x a_y\Big)\\
&\quad + \frac{1}{2}\Big[\int dx\,dy\, N^3V(N(x-y))\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) b^*_x b^*_y + \text{h.c.}\Big],\\
\mathcal{L}^{(3)}_{N,t} &= \int dx\,dy\, N^{5/2}V(N(x-y))\tilde{\varphi}_t(y) b^*_x a^*_y a_x + \text{h.c.},\\
\mathcal{L}^{(4)}_{N,t} &= \frac{1}{2}\int dx\,dy\, N^2V(N(x-y)) a^*_x a^*_y a_y a_x.
\end{aligned} \tag{5-3}$$

The generator (5-1) of the fluctuation dynamics is therefore given by

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + \sum_{j=0}^{4} e^{-B(\eta_t)}\mathcal{L}^{(j)}_{N,t} e^{B(\eta_t)}.$$

In the next subsections, we will study separately the six terms contributing to $\mathcal{G}_{N,t}$. Before doing so, however, we collect some preliminary results, which will be useful for our analysis.

Notation and conventions. For the rest of this section we employ the short-hand notation η_x, k_x, μ_x for the wave functions $\eta_x(y) = \eta_t(x;y)$, $k_x(y) = k_t(x;y)$ and $\mu_x(y) = \mu_t(x;y)$. We will always assume that $\sup_{t\in\mathbb{R}} \|\eta_t\|_2$ is sufficiently small, so that we can use the expansions obtained in Lemma 3.3. Finally, by C and c we denote generic constants which only depend on fixed parameters, but not on N or t, and which may vary from one line to the next.

5A. *Preliminary results.* In this subsection we show some simple but important auxiliary results which will be used throughout the rest of Section 5. Recall the operators

$$\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)=\int b^{\flat_0}_{x_1}\prod_{i=1}^{n-1}a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}b^{\sharp_n}_{y_n}\prod_{i=1}^{n}j_i(x_i;y_i)\,dx_i\,dy_i,$$

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)=\int b^{\flat_0}_{x_1}\prod_{i=1}^{n-1}a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}a^{\sharp_n}_{y_n}a^{\flat_n}(f)\prod_{i=1}^{n}j_i(x_i;y_i)\,dx_i\,dy_i$$

introduced in Section 2. For each $i\in\{1,\dots,n\}$, we recall in particular the condition that either $\sharp_i=*$ and $\flat_i=\cdot$ or $\sharp_i=\cdot$ and $\flat_i=*$.

In the next lemma, we consider commutators of these operators with the number of particles operator $\mathcal{N}$ and with operators of the form $a^*(g_1)a(g_2)$.

Lemma 5.1. *Let $n\in\mathbb{N}$, $f,g_1,g_2\in L^2(\mathbb{R}^3)$, $j_1,\dots,j_n\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$:*

(i) *We have*

$$[\mathcal{N},\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)]=\kappa_{\flat_0,\sharp_n}\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)\quad\text{for all }\sharp,\flat\in\{\cdot,*\}^n,$$
$$[\mathcal{N},\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)]=\nu_{\flat_0}\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)\quad\text{for all }\sharp\in\{\cdot,*\}^n,\flat\in\{\cdot,*\}^{n+1}.$$

Here $\kappa_{\flat_0,\sharp_n}=2$ if $\flat_0=\sharp_n=*$, $\kappa_{\flat_0,\sharp_n}=-2$ if $\flat_0=\sharp_n=\cdot$, and $\kappa_{\flat_0,\sharp_n}=0$ otherwise, while $\nu_{\flat_0}=1$ if $\flat_0=*$ and $\nu_{\flat_0}=-1$ if $\flat_0=\cdot$.

(ii) *The commutator*

$$[a^*(g_1)a(g_2),\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)]$$

can be written as the sum of $2n$ terms, all having the form

$$\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_{i-1},h_i,j_{i+1},\dots,j_n)$$

for some $i\in\{1,\dots,n\}$. Here $h_i\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ has (up to a possible sign) one of the forms

$$h_i(x;y)=g_1(x)j_i(\bar g_2)(y),\qquad h_i(x;y)=g_1(y)j_i(\bar g_2)(y)\tag{5-4}$$

or one of those forms with g_1 and $\bar g_2$ exchanged. Here $j_i(g)(x)=\int j_i(x;z)g(z)\,dz$. Notice that

$$\|h_i\|_2\le\|g_1\|_2\,\|g_2\|_2\,\|j_i\|_2\tag{5-5}$$

and

$$|h_i(x;y)|\le\max\big\{|g_1(x)|\,\|j_i(\cdot\,;y)\|_2\,\|g_2\|_2,\ |g_1(y)|\,\|j_i(x;.)\|_2\,\|g_2\|_2,\ |g_2(x)|\,\|j_i(\cdot\,;y)\|_2\,\|g_1\|_2,\ |g_2(y)|\,\|j_i(x;.)\|_2\,\|g_1\|_2\big\}.\tag{5-6}$$

(iii) *The commutator*

$$[a^*(g_1)a(g_2),\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;f)]\tag{5-7}$$

can be written as the sum of $2n+1$ *terms*; $2n$ *of them have the form*

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_{i-1},h_i,j_{i+1},\dots,j_n;\,f),$$

where h_i *is* (*up to a possible sign*) *one of the kernels appearing in* (5-4) (*or the same with* g_1 *and* $\bar g_2$ *exchanged*), *and satisfying the bounds in* (5-5), (5-6). *The remaining term in the expansion for* (5-7) *has the form*

$$\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;\,k), \tag{5-8}$$

where $k\in L^2(\mathbb{R}^3)$ *is* (*up to a possible sign*) *one of the functions*

$$k(x)=\langle g_1,f\rangle\,g_2(x),\qquad k(x)=\langle g_2,f\rangle\,g_1(x) \tag{5-9}$$

or one of their complex conjugated functions. In any event, *we have*

$$\|k\|_2\le\|g_1\|_2\,\|g_2\|_2\,\|f\|_2$$

and

$$|k(x)|\le\|f\|_2\max\{\|g_1\|_2\,|g_2(x)|,\ \|g_2\|_2\,|g_1(x)|\}.$$

(iv) *If* $f\in L^2(\mathbb{R}^3)$ *and/or* $j_1,\dots,j_n\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ *depend on time* $t\in\mathbb{R}$, *we have*

$$\partial_t\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)=\sum_{i=1}^n\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_{i-1},\partial_t j_i,j_{i+1},\dots,j_n),$$

$$\partial_t\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;\,f)=\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_n;\,\partial_t f)+\sum_{i=1}^n\Pi^{(1)}_{\sharp,\flat}(j_1,\dots,j_{i-1},\partial_t j_i,j_{i+1},\dots,j_n;\,f).$$

Proof. Part (i) follows from $(\mathcal{N}+1)b_x=b_x\mathcal{N}$ and $\mathcal{N}b^*_x=b^*_x(\mathcal{N}+1)$. Part (iv) follows easily from the Leibniz rule. To prove part (ii), we apply the Leibniz rule:

$$\begin{aligned}
&[a^*(g_1)a(g_2),\,\Pi^{(2)}_{\sharp,\flat}(j_1,\dots,j_n)]\\
&\quad=\int[a^*(g_1)a(g_2),\,b^{\flat_0}_{x_1}]\prod_{i=1}^n a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}b^{\sharp_n}_{y_n}\prod_{i=1}^n j_i(x_i;y_i)\,dx_i\,dy_i\\
&\qquad+\sum_{m=1}^{n-1}\int b^{\flat_0}_{x_1}\prod_{i=1}^{m-1}a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}[a^*(g_1)a(g_2),\,a^{\sharp_m}_{y_m}a^{\flat_m}_{x_{m+1}}]\prod_{i=m+1}^{n-1}a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}b^{\sharp_n}_{y_n}\prod_{i=1}^n j_i(x_i;y_i)\,dx_i\,dy_i\\
&\qquad+\int b^{\flat_0}_{x_1}\prod_{i=1}^n a^{\sharp_i}_{y_i}a^{\flat_i}_{x_{i+1}}[a^*(g_1)a(g_2),\,b^{\sharp_n}_{y_n}]\prod_{i=1}^n j_i(x_i;y_i)\,dx_i\,dy_i.
\end{aligned}\tag{5-10}$$

Using the commutation relations

$$\begin{aligned}
[a^*(g_1)a(g_2),\,b_x]&=-g_1(x)b(g_2),\\
[a^*(g_1)a(g_2),\,b^*_x]&=\bar g_2(x)b^*(g_1),\\
[a^*(g_1)a(g_2),\,a^*_xa_y]=[a^*(g_1)a(g_2),\,a_ya^*_x]&=\bar g_2(x)a^*(g_1)a_y-g_1(y)a^*_xa(g_2),
\end{aligned}\tag{5-11}$$

we conclude that on the right-hand side of (5-10) we have $2n$ terms, each of them a $\Pi^{(2)}$-operator (with the same indices $\sharp, \flat$ as the $\Pi^{(2)}$-operator on the left-hand side of (5-10)). Furthermore, from (5-11) it is clear that for each $\Pi^{(2)}$-operator on the right-hand side of (5-10), only one j-kernel will differ from the j-kernels of the $\Pi^{(2)}$-operator on the left-hand side of (5-10). In the first term on the right-hand side of (5-10), we only have to replace the j_1-kernel (either with $g_1(x_1) j_1(\bar{g}_2)(y_1)$ or with $\bar{g}_2(x_1) j_1(g_1)(y_1)$, depending on $\flat_0 \in \{\cdot, *\}$). Similarly, in the last term on the right-hand side of (5-10), only the j_n-kernel has to be changed. In the m-th term in the sum, on the other hand, the commutator leads to the sum of two $\Pi^{(2)}$-operators, one where the kernel j_m is changed and one where the kernel j_{m+1} is replaced. From (5-11), it is easy to check that the new kernel can only have one of the forms listed in (5-4). The bounds (5-5), (5-6) follow easily from the explicit formula in (5-4). Part (iii) can be shown similarly; the only difference is that, in this case, the commutator can hit the last pair $a^{\sharp_n}_{y_n} a^{\flat_n}(f)$ instead of the $b^{\sharp_n}_{y_n}$ appearing in the $\Pi^{(2)}$-operator. □

It follows from Lemma 5.1 that

$$[\mathcal{N}, e^{-B(\eta)} b(f) e^{B(\eta)}] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} [\mathcal{N}, \mathrm{ad}^{(n)}_{B(\eta)}(b(f))],$$
$$[a^*(g_1) a(g_2), e^{-B(\eta)} b(f) e^{B(\eta)}] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} [a^*(g_1) a(g_2), \mathrm{ad}^{(n)}_{B(\eta)}(b(f))], \tag{5-12}$$
$$\partial_t (e^{-B(\eta)} b(f) e^{B(\eta)}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \partial_t \mathrm{ad}^{(n)}_{B(\eta)}(b(f)),$$

where the series on the right-hand sides are absolutely convergent.

In the next subsections we are going to study what happens to the operators $\mathcal{L}^{(j)}_{N,t}$ defined in (5-3) when they are conjugated with the generalized Bogoliubov transformation $e^{B(\eta_t)}$. The general strategy is to expand $e^{-B(\eta_t)} \mathcal{L}^{(j)}_{N,t} e^{B(\eta_t)}$ using (3-27), and then use Lemma 3.2 to express all nested commutators. For this reason, we will have to bound the action of operators of the form

$$\Lambda_1 \cdots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}(g)). \tag{5-13}$$

To this end, we will use the next lemma.

Lemma 5.2. *Let $g \in L^2(\mathbb{R}^3)$, $n, i_1, i_2, k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ and $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \in \mathbb{N} \backslash \{0\}$. For $s = 1, \dots, i_1$, $s' = 1, \dots, i_2$, we denote by each of Λ_s, $\Lambda'_{s'}$ a factor $(N - \mathcal{N})/N$ or a factor $(N - \mathcal{N} + 1)/N$ or an operator of the form*

$$N^{-p}\, \Pi^{(2)}_{\sharp,\flat}(\eta^{(q_1)}_{t,\natural_1}, \dots, \eta^{(q_p)}_{t,\natural_p}). \tag{5-14}$$

(i) *Assume that the operator*

$$\Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(\ell_1)}_{t,\diamondsuit}(g))$$

appears in the expansion of $\mathrm{ad}^{(n)}_{B(\eta_t)}(b(g))$ discussed in Lemma 3.2. Then

$$\|(\mathcal{N}+1)^{-1/2} \Lambda_1 \cdots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(\ell_1)}_{t,\diamondsuit}(g)) \xi\| \le C^n \|\eta_t\|^n \|g\| \|\xi\|.$$

If at least one of the Λ_s-operators has the form (5-14) *or if $k \geq 1$, we also have*

$$\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \leq C^n N^{-1/2}\|\eta_t\|^n\,\|g\|\,\|(\mathcal{N}+1)^{1/2}\xi\|. \tag{5-15}$$

(ii) *Let $r : L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ be a bounded linear operator. We use the notation*

$$(\eta^{(s)}r)_x(y) := (\eta^{(s)}r)(x;y)$$

(*if $s=0$, $(\eta^{(s)}r)_x(y) = r_x(y) = r(x;y)$ as a distribution*). *Assume that the operator*

$$\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\diamondsuit}r)_x)$$

appears in the expansion of $\mathrm{ad}^{(n)}_{B(\eta_t)}(b(r_x))$ *discussed in Lemma 3.2. Then*

$$\|\Lambda_1\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};(\eta^{(\ell_1)}_{t,\diamondsuit}r)_x)\xi\| \leq \begin{cases} C^n\|\eta_t\|^{n-1}\|(\eta_t r)_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_1\geq 1,\\ C^n\|\eta_t\|^n\,\|a(r_x)\xi\| & \text{if } \ell_1 = 0.\end{cases} \tag{5-16}$$

Proof. Let us start with part (i). If Λ_1 is either the operator $(N-\mathcal{N})/N$ or $(N-\mathcal{N}+1)/N$, then, on $\mathcal{F}^{\leq N}$,

$$\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \leq 2\|(\mathcal{N}+1)^{-1/2}\Lambda_2\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\|. \tag{5-17}$$

If instead Λ_1 has the form (5-14) for a $p\geq 1$, we apply Lemma 2.4 and we find, using Lemma 3.2(vi),

$$\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1} N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \leq C^p\|\eta_t\|^{\bar p}\,\|(\mathcal{N}+1)^{-1/2}\Lambda_2\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\|, \tag{5-18}$$

where we used the notation $\bar p = q_1+\cdots+q_p$ for the total number of η_t-kernels appearing in (5-14). Iterating the bounds (5-17) and (5-18), we conclude that

$$\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1} N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \leq C^{r+p_1+\cdots+p_s}\|\eta_t\|^{\bar p_1+\cdots+\bar p_s}\|(\mathcal{N}+1)^{1/2}N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \tag{5-19}$$

if r of the operators $\Lambda_1,\dots,\Lambda_{i_1}$ have either the form $(N-\mathcal{N})/N$ or the form $(N-\mathcal{N}+1)/N$, and the other $s = i_1 - r$ are $\Pi^{(2)}$-operators of the form (5-14) of order $p_1,\dots,p_s$, containing $\bar p_1,\dots,\bar p_s$ η_t-kernels. Again with Lemma 2.4, we obtain

$$\begin{aligned}\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| &\leq C^{r+p_1+\cdots+p_s+j_1+\cdots+j_{k_1}+l_1}\|\eta_t\|^{\bar p_1+\cdots+\bar p_s+j_1+\cdots+j_{k_1}+l_1}\|g\|\,\|\xi\|\\ &\leq C^n\|\eta_t\|^n\,\|g\|\,\|\xi\|.\end{aligned} \tag{5-20}$$

This shows the first bound in part (i). Now, assume that at least one of the Λ_m-operators, for $m \in \{1, \dots, i_1\}$, has the form (5-14). Since, for $\Psi \in \mathcal{F}^{\le N}$,

$$\begin{aligned}\|(\mathcal{N}+1)^{-1/2} N^{-p} \Pi^{(2)}_{\sharp,\flat}(\eta^{(q_1)}_{t,\natural_1}, \dots, \eta^{(q_p)}_{t,\natural_p})\Psi\| &\le C^p \|\eta_t\|^{q_1+\dots+q_p} N^{-p} \|(\mathcal{N}+1)^{p-1/2}\Psi\| \\ &\le C^p \|\eta_t\|^{q_1+\dots+q_p} N^{-1/2} \|\Psi\|\end{aligned}$$

for any $p \ge 1$, in this case we can improve (5-20) to

$$\|(\mathcal{N}+1)^{-1/2} \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1)}_{t,\diamondsuit}(g))\xi\| \le C^n N^{-1/2} \|\eta_t\|^n \, \|g\| \, \|(\mathcal{N}+1)^{1/2}\xi\|.$$

Similarly, if $k_1 \ge 1$, we have by Lemma 2.4,

$$\begin{aligned}N^{-k_1} \|(\mathcal{N}+1)^{-1/2} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_k}; \eta^{(\ell_1)}_{t,\natural_{k+1}}(g))\xi\| &\le N^{-k_1} C^{k_1} \|\eta_t\|^{j_1+\dots+j_{k_1}+\ell_1} \|g\| \, \|(N+1)^{k_1-1/2}\xi\| \\ &\le C^k N^{-1/2} \|\eta_t\|^{j_1+\dots+j_{k_1}+\ell_1} \|g\| \, \|(N+1)^{1/2}\xi\|.\end{aligned}$$

Hence, also in this case, the bound (5-15) holds true. If $\ell_1 \ge 1$, part (ii) can be proven similarly to part (i), noticing that

$$\|(\eta^{(\ell_1)}_{t,\diamondsuit} r)_x\| \le \|\eta_t\|^{\ell_1 - 1} \|(\eta_t r)_x\|.$$

If instead $\ell_1 = 0$, it follows from Lemma 3.2(v) that the field operator associated with $(\eta^{(\ell_1)}_{t,\diamondsuit} r)_x = r_x$ (the one appearing on the right of $\Pi^{(1)}$) is an annihilation operator (acting directly on ξ). Hence, (5-16) holds true also in this case. □

Often, we will also have to bound the action of products of operators of the form (5-13). In this case, the next lemma will be useful.

Lemma 5.3. *Let $g \in L^2(\mathbb{R}^3)$, $n, i_1, i_2, k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ and $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \in \mathbb{N}\backslash\{0\}$. For $s = 1, \dots, i_1$, $s' = 1, \dots, i_2$, we denote by each of Λ_s, $\Lambda'_{s'}$ a factor $(N-\mathcal{N})/N$ or a factor $(N-\mathcal{N}+1)/N$ or an operator of the form* (5-14). *Assume that the operators*

$$\begin{aligned}&\Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x), \\ &\Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\end{aligned} \tag{5-21}$$

appear in the expansions of $\mathrm{ad}^{(n)}_{B(\eta_t)}(b((\eta_t r)_x))$ *and* $\mathrm{ad}^{(k)}_{B(\eta_t)}(b_x)$ *respectively for some $n, k \in \mathbb{N}$, $x \in \mathbb{R}^3$. Then*

$$\begin{aligned}\Big\|(\mathcal{N}+1)^{-1/2} \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x) \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi\Big\| \\ \le \begin{cases} C^{n+k} \|\eta_t\|^{n+k-1} \|(\eta_t r)_x\| \, \|\eta_x\| \, \|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_2 > 0, \\ C^{n+k} \|\eta_t\|^{n+k} \|(\eta_t r)_x\| \, \|a_x \xi\| & \text{if } \ell_2 = 0. \end{cases}\end{aligned} \tag{5-22}$$

Similarly, if the operators

$$\begin{aligned}&\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\lozenge}\partial_t\eta_{t,\tilde\lozenge})_x),\\ &\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\lozenge'})\end{aligned}$$

appear in the expansions of $\mathrm{ad}^{(n)}_{B(\eta_t)}(b(\partial_t\eta_t))$ *and* $\mathrm{ad}^{(k)}_{B(\eta_t)}(b_x)$ *respectively for some* $n,k\in\mathbb{N}$, $x\in\mathbb{R}^3$, *we have*

$$\begin{aligned}\big\|(\mathcal{N}+1)^{-1/2}&\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\lozenge}\partial_t\eta_{t,\tilde\lozenge})_x)\\ &\times\times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\lozenge'})\xi\big\|\\ &\qquad\le\begin{cases}C^{n+k}\|\eta_t\|^{n+k-1}\|(\partial_t\eta_t)_x\|\,\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_2>0,\\ C^{n+k}\|\eta_t\|^{n+k}\|(\partial_t\eta_t)_x\|\,\|a_x\xi\| & \text{if } \ell_2=0.\end{cases}\end{aligned}\tag{5-23}$$

Proof. We can bound, first of all

$$\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1+1)}_t r)_x)\Psi\|\le C^n\|\eta_t\|^n\,\|(\eta_t r)_x\|\,\|\Psi\|$$

and

$$\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\lozenge}\partial_t\eta_{t,\tilde\lozenge})_x)\Psi\|\le C^n\|\eta_t\|^n\,\|(\partial_t\eta_t)_x\|\,\|\Psi\|.$$

Choosing now

$$\Psi=\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\lozenge'})\xi,$$

and proceeding as in Lemma 5.2(ii), distinguishing the cases $\ell_2\ge 1$ and $\ell_2=0$, we obtain (5-22) and (5-23). □

Finally, the next lemma will be important to bound products of operators of the form (5-13), with arguments labeled by different positions $x,y\in\mathbb{R}^3$ (as opposed to (5-21), where both operators are labeled by the same $x\in\mathbb{R}^3$).

Lemma 5.4. *Let* $g\in L^2(\mathbb{R}^3)$, $n,i_1,i_2,k_1,k_2,\ell_1,\ell_2\in\mathbb{N}$ *and* $j_1,\dots,j_{k_1},m_1,\dots,m_{k_2}\in\mathbb{N}\backslash\{0\}$. *For* $s=1,\dots,i_1$, $s'=1,\dots,i_2$, *we denote by* Λ_s, $\Lambda'_{s'}$ *a factor* $(N-\mathcal{N})/N$ *or a factor* $(N-\mathcal{N}+1)/N$ *or an operator of the form* (5-14). *Assume that the operators*

$$\begin{aligned}&\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{y,\lozenge}),\\ &\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\lozenge'})\end{aligned}$$

appear in the expansions of $\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)$ *and* $\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)$ *respectively for some* $n,k\in\mathbb{N}$. *For* $\alpha\in\mathbb{N}$, $t\in\mathbb{R}$, *we define*

$$\begin{aligned}\mathrm{D}_{x,y}=\big\|(\mathcal{N}+1)^{(\alpha-1)/2}&\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{y,\lozenge})\\ &\times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_1}};\eta^{(\ell_2)}_{x,\lozenge'})\xi\big\|\end{aligned}$$

for all $x, y \in \mathbb{R}^3$. *Then, if* $\ell_1 > 0$, *we have*

$$\mathrm{D}_{x,y} \le \begin{cases} C^{n+k} \|\eta_t\|^{n+k-2} \|\eta_x\| \, \|\eta_y\| \, \|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| & \text{if } \ell_2 \ge 1, \\ C^{n+k} \|\eta_t\|^{n+k-1} \|\eta_y\| \, \|a_x(\mathcal{N}+1)^{\alpha/2}\xi\| & \text{if } \ell_2 = 0 \end{cases} \tag{5-24}$$

for all $x, y \in \mathbb{R}^3$, $t \in \mathbb{R}$. *If instead* $\ell_1 = 0$, *we distinguish three cases. For* $\ell_2 > 1$, *we obtain*

$$\mathrm{D}_{x,y} \le C^{n+k} \|\eta_t\|^{n+k-2} \big\{ \|\eta_y\| \, \|\eta_x\| (\|(\mathcal{N}+1)^{(\alpha-1)/2}\xi\| + n/N \|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\|) + \|\eta_t\| \, \|\eta_x\| \, \|a_y(\mathcal{N}+1)^{\alpha/2}\xi\| \big\} \tag{5-25}$$

for all $x, y \in \mathbb{R}^3$, $t \in \mathbb{R}$. *If* $\ell_1 = 0$ *and* $\ell_2 = 1$, *we find*

$$\mathrm{D}_{x,y} \le C^{n+k} \|\eta_t\|^{n+k-2} \big\{ \big[n \|\eta_x\| \, \|\eta_y\| + \|\eta_t\| \, |\eta_t(x;y)| \big] \|(\mathcal{N}+1)^{(\alpha-1)/2}\xi\| + \|\eta_t\| \, \|\eta_x\| \, \|a_y(\mathcal{N}+1)^{\alpha/2}\xi\| \big\} \tag{5-26}$$

for all $x, y \in \mathbb{R}^3$, $t \in \mathbb{R}$. *If* $\ell_1 = 0$ *and* $\ell_2 = 1$ *and we additionally assume that* $k + n \ge 2$ (*since* $\ell_1 \le k$, $\ell_2 \le n$ *from Lemma 3.2, this assumption only excludes the case* $k = \ell_1 = 0$, $n = \ell_2 = 1$), *we find the improved estimate*

$$\mathrm{D}_{x,y} \le C^{n+k} \|\eta_t\|^{n+k-2} \big\{ N^{-1} \big[n \|\eta_x\| \, \|\eta_y\| + \|\eta_t\| \, |\eta_t(x;y)| \big] \|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| + \|\eta_t\| \, \|\eta_x\| \, \|a_y(\mathcal{N}+1)^{\alpha/2}\xi\| \big\} \tag{5-27}$$

for all $x, y \in \mathbb{R}^3$, $t \in \mathbb{R}$. *Finally, let* $\ell_1 = \ell_2 = 0$. *Then*

$$\mathrm{D}_{x,y} \le C^{n+k} \|\eta_t\|^{n+k-1} \{ n N^{-1} \|\eta_y\| \, \|a_x(\mathcal{N}+1)^{\alpha/2}\xi\| + \|\eta_t\| \, \|a_x a_y (\mathcal{N}+1)^{(\alpha-1)/2}\xi\| \} \tag{5-28}$$

for all $x, y \in \mathbb{R}^3$, $t \in \mathbb{R}$. *If, however,* $\ell_1 = \ell_2 = 0$ *and, additionally,* $k + n \ge 1$ (*excluding the case* $n = \ell_1 = k = \ell_2 = 0$), *we find the improved bound*

$$\mathrm{D}_{x,y} \le C^{n+k} \|\eta_t\|^{n+k-1} \{ n N^{-1} \|\eta_y\| \, \|a_x \xi\| + N^{-1/2} \|\eta_t\| \, \|a_x a_y (\mathcal{N}+1)^{\alpha/2}\xi\| \} \tag{5-29}$$

again for all $x, y \in \mathbb{R}^3$, $t \in \mathbb{R}$.

Proof. If $\ell_1 > 0$, we can proceed as in the proof of Lemma 5.3 to show (5-24). So, let us focus on the case $\ell_1 = 0$. In this case, the field operator on the right of the first $\Pi^{(1)}$-operator (the one on the left) is an annihilation operator, a_y. To estimate $\mathrm{D}_{x,y}$, we need to commute a_y to the right, until it hits ξ. To commute a_y through factors of $\mathcal{N}$, we just use the pull-through formula $a_y \mathcal{N} = (\mathcal{N}+1) a_y$. When we commute a_y through a pair of creation and/or annihilation operators associated with a kernel $\eta_t^{(j)}$ for a $j \ge 1$ (as the ones appearing in the $\Pi^{(2)}$-operators of the form (5-14) or in the operator $\Pi^{(1)}$-operator), we generate a creation or an annihilation operator with argument $\eta_y^{(j)}$ whose L^2-norm is uniformly bounded. At the same time, we spare a factor N^{-1}. For example, we have

$$\left[a_y, \int a_{x_i}^* a_{y_i} \eta^{(j)}(x_i; y_i) \, dx_i \, dy_i \right] = a(\bar{\eta}_y^{(j)}).$$

At the end, we have to commute a_y through the field operator with argument $\eta_{x,\lozenge'}^{(\ell_2)}$. The commutator is trivial if ℓ_2 is even (because then the corresponding field operator is an annihilation operator; see

Lemma 3.2(v)). It is given by

$$[a_y, a^*(\eta^{(\ell_2)}_{x,\diamondsuit'})] = \eta^{(\ell_2)}_{t,\diamondsuit'}(x; y) \tag{5-30}$$

if ℓ_2 is odd. If $\ell_2 \geq 2$, we can bound

$$|\eta^{(\ell_2)}_{t,\diamondsuit'}(x; y)| \leq \|\eta_t\|^{\ell_2-2}\|\eta_x\|\,\|\eta_y\|$$

and we obtain (taking into account the fact that there are at most n pairs of fields with which a_y has to be commuted)

$$\begin{aligned}\mathrm{D}_{x,y} \leq C^{k+n}\|\eta_t\|^{k+n-2}\big\{&nN^{-1}\|\eta_y\|\,\|\eta_x\|\,\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| \\ &+\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)^{(\alpha-1)/2}\xi\| + \|\eta_t\|\,\|\eta_x\|\,\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\|\big\}.\end{aligned}$$

If instead $\ell_2 = 1$, the right-hand side of (5-30) blows up as $N \to \infty$. To make up for this singularity, we use the additional assumption $k + n \geq 2$. Combining this information with $\ell_1 = 0,\ \ell_2 = 1$, we conclude that either $k_1 > 0$ or $k_2 > 0$ or there exists $i \in \mathbb{N}$ such that either Λ_i or Λ'_i is a $\Pi^{(2)}$-operator of the form (5-14) with $p \geq 1$. This factor allows us to gain a factor $(\mathcal{N}+1)/N$ in the estimate for the term arising from the commutator (5-30). We conclude that, in this case,

$$\begin{aligned}\mathrm{D}_{x,y} \leq C^{k+n}\|\eta_t\|^{k+n-2}\big\{&nN^{-1}\|\eta_y\|\,\|\eta_x\|\,\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| + N^{-1}|\eta_t(x; y)|\,\|(\mathcal{N}+1)^{(\alpha+1)/2}\xi\| \\ &+\|\eta_t\|\,\|\eta_x\|\,\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\|\big\}.\end{aligned}$$

Finally, let us consider the case $\ell_2 = 0$. Here we proceed as before, commuting a_y to the right. The commutator produces at most n factors, whose norm can be bounded much as before. We easily conclude that

$$\mathrm{D}_{x,y} \leq C^{k+n}\|\eta_t\|^{k+n-1}\{nN^{-1}\|\eta_x\|\,\|a_y(\mathcal{N}+1)^{\alpha/2}\xi\| + \|\eta_t\|\,\|a_x a_y(\mathcal{N}+1)^{(\alpha-1)/2}\xi\|\}.$$

If we impose the additional condition $k + n \geq 1$, we deduce that either $k_1 > 0$ or $k_2 > 0$ or there exists $i \in \mathbb{N}$ such that either Λ_i or Λ'_i is a $\Pi^{(2)}$-operator of the form (5-14) with $p \geq 1$. Much as we argued in the case $\ell_2 = 1$, when estimating the contribution with the two annihilation operators a_x, a_y acting on ξ, we can therefore extract an additional factor $(\mathcal{N}+1)/N$. Under this additional condition, we obtain

$$\mathrm{D}_{x,y} \leq C^{k+n}\|\eta_t\|^{k+n-1}\{nN^{-1}\|\eta_x\|\,\|a_y\xi\| + N^{-1/2}\|\eta_t\|\,\|a_x a_y(\mathcal{N}+1)^{(\alpha-1)/2}\xi\|\},$$

which proves (5-29). □

5B. *Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(0)}_{N,t}e^{B(\eta_t)}$.* From the definition (5-3), we can write

$$\begin{aligned}\mathcal{L}^{(0)}_{N,t} = C_{N,t} - \langle\tilde\varphi_t, [N^3V(N\,\cdot)w_\ell(N\,\cdot) * |\tilde\varphi_t|^2]\tilde\varphi_t\rangle\mathcal{N} \\ + \frac{1}{2N}\langle\tilde\varphi_t, [N^3V(N\,\cdot) * |\tilde\varphi_t|^2]\tilde\varphi_t\rangle\mathcal{N} + \frac{1}{2N}\langle\tilde\varphi_t, [N^3V(N\,\cdot) * |\tilde\varphi_t|^2]\tilde\varphi_t\rangle\mathcal{N}^2,\end{aligned}$$

with

$$C_{N,t} = \frac{N}{2}\langle\tilde\varphi_t, [N^3V(N\,\cdot)w_\ell(N\,\cdot) * |\tilde\varphi_t|^2]\tilde\varphi_t\rangle - \frac{1}{2}\langle\tilde\varphi_t, [N^3V(N\,\cdot) * |\tilde\varphi_t|^2]\tilde\varphi_t\rangle.$$

The properties of the other terms are described in the next proposition.

Proposition 5.5. *Under the same assumptions as in Theorem 4.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} |\langle \xi, e^{-B(\eta_t)}(\mathcal{L}^{(0)}_{N,t} - C_{N,t})e^{B(\eta_t)}\xi\rangle| &\le C\langle \xi, (\mathcal{N}+1)\xi\rangle, \\ |\langle \xi, [\mathcal{N}, e^{-B(\eta_t)}(\mathcal{L}^{(0)}_{N,t} - C_{N,t})e^{B(\eta_t)}]\xi\rangle| &\le C\langle \xi, (\mathcal{N}+1)\xi\rangle, \\ |\langle \xi, [a^*(g_1)a(g_2), e^{-B(\eta_t)}(\mathcal{L}^{(0)}_{N,t} - C_{N,t})e^{B(\eta_t)}]\xi\rangle| &\le C\|g_1\|\,\|g_2\|\langle \xi, (\mathcal{N}+1)\xi\rangle, \\ |\partial_t\langle \xi, e^{-B(\eta_t)}(\mathcal{L}^{(0)}_{N,t} - C_{N,t})e^{B(\eta_t)}\xi\rangle| &\le Ce^{c|t|}\langle \xi, (\mathcal{N}+1)\xi\rangle \end{aligned} \tag{5-31}$$

for all $t \in \mathbb{R}$, $g_1, g_2 \in L^2(\mathbb{R}^3)$, $\xi \in \mathcal{F}^{\le N}$.

In order to show Proposition 5.5, we need to conjugate the number of particles operator $\mathcal{N}$ with the generalized Bogoliubov transformation $e^{-B(\eta_t)}$. To this end, we make use of the following lemma, where, for later convenience, we consider conjugation of more general quadratic operators.

Lemma 5.6. *Let $r : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be a bounded linear operator. Consider the Fock-space operators*

$$R_1 = \int dx\, dy\, r(y; x) b^*_x b_y \quad \text{and} \quad R_2 = \int dx\, dy\, r(y; x) a^*_x a_y$$

mapping $\mathcal{F}^{\le N}$ in itself. Then we have the bounds

$$\begin{aligned} |\langle \xi_1, e^{-B(\eta_t)} R_i e^{B(\eta_t)}\xi_2\rangle| &\le C\|r\|_{\text{op}}\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \\ |\langle \xi_1, [\mathcal{N}, e^{-B(\eta_t)} R_i e^{B(\eta_t)}]\xi_2\rangle| &\le C\|r\|_{\text{op}}\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \\ |\langle \xi_1, [a^*(g_1)a(g_2), e^{-B(\eta_t)} R_i e^{B(\eta_t)}]\xi_2\rangle| &\le C\|r\|_{\text{op}}\|g_1\|\,\|g_2\|\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\| \end{aligned} \tag{5-32}$$

for $i = 1, 2$ and all $\xi_1, \xi_2 \in \mathcal{F}^{\le N}$. Furthermore, if $r = r_t$ is differentiable in t, we find

$$|\partial_t\langle \xi_1, e^{-B(\eta_t)} R_i e^{B(\eta_t)}\xi_2\rangle| \le Ce^{c|t|}(\|r\|_{\text{op}} + \|\dot r\|_{\text{op}})\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\| \tag{5-33}$$

for $i = 1, 2$ and all $\xi_1, \xi_2 \in \mathcal{F}^{\le N}$.

Proof. We consider first the operator R_1. By Lemma 3.3, we expand

$$e^{-B(\eta_t)} R_1 e^{B(\eta_t)} = \int dx\, e^{-B(\eta_t)} b^*(r_x) b_x e^{B(\eta_t)} = \sum_{k,n\ge 0} \frac{(-1)^{k+n}}{k!\,n!} \int dx\, \text{ad}^{(n)}_{B(\eta_t)}(b^*(r_x))\text{ad}^{(k)}_{B(\eta_t)}(b_x), \tag{5-34}$$

with the notation $r_x(y) = r(x; y)$. According to Lemma 3.2 the operator

$$\int dx\, \text{ad}^{(n)}_{B(\eta_t)}(b^*(r_x))\, \text{ad}^{(k)}_{B(\eta_t)}(b_x)$$

is given by the sum of $2^{n+k} n!\, k!$ terms having the form

$$\begin{aligned} \text{E} := \int dx\, N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1)}_{t,\diamondsuit} r)_x)^* \\ \times \Lambda^*_{i_1}\cdots\Lambda^*_1\Lambda'_1\cdots\Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'}), \end{aligned} \tag{5-35}$$

where $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0,\ j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 1$, and where each operator $\Lambda_i,\ \Lambda'_i$ is a factor $(N-\mathcal{N})/N$, a factor $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1}, \dots, \eta^{(q_p)}_{t,\underline{\natural}_p}) \tag{5-36}$$

for a $p \geq 1$ and powers $q_1, \dots, q_p \geq 1$. With Cauchy–Schwarz we find

$$\begin{aligned} |\langle \xi_1, \mathrm{E}\xi_2\rangle| \leq \int dx\, \|\Lambda_1 \cdots \Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1)}_{t,\diamondsuit} r)_x)\xi_1\| \\ \times \|\Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi_2\| \end{aligned} \tag{5-37}$$

for every $\xi_1, \xi_2 \in \mathcal{F}^{\leq N}$. With Lemma 5.2(ii), we find that

$$|\langle \xi_1, \mathrm{E}\xi_2\rangle| \leq C^{k+n}\|r\|_{\mathrm{op}}\|\eta_t\|^{n+k}\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \tag{5-38}$$

where we used the fact that

$$\int dx\, \|a(r_x)\xi_1\|^2 = \langle \xi_1, d\Gamma(r^2)\xi_1\rangle \leq \|r^2\|_{\mathrm{op}}\|\mathcal{N}^{1/2}\xi_1\|^2 \leq \|r\|^2_{\mathrm{op}}\|\mathcal{N}^{1/2}\xi_1\|^2.$$

From (5-34), we conclude that, if $\sup_t \|\eta_t\|$ is small enough,

$$|\langle \xi_1, e^{-B(\eta_t)}R_1 e^{B(\eta_t)}\xi_2\rangle| \leq C\|r\|_{\mathrm{op}}\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|. \tag{5-39}$$

This proves the first bound in (5-32), if $i=1$. The other two bounds in (5-32) and the bound in (5-33) for $i=1$ can be proven similarly. To be more precise, we first expand the operator $e^{-B(\eta_t)}R_1e^{B(\eta_t)}$ as in (5-34), where the (n,k)-th term can be written as the sum of $2^{n+k}k!\,n!$ terms of the form (5-35). Then we use Lemma 5.1 to express the commutator of (5-35) with $\mathcal{N}$ or with $a^*(g_1)a(g_2)$ or its time-derivative as a sum of at most $2(k+n+1)$ terms having again the form (5-35), with just one of the η_t-kernels appropriately replaced. Finally, we proceed as above to show that the matrix elements of such a term can be bounded as in (5-38). We omit further details.

Let us now consider the operator R_2. We start by writing

$$\begin{aligned} e^{-B(\eta_t)}R_2e^{B(\eta_t)} &= R_2 + \int_0^1 ds\, e^{-sB(\eta_t)}[R_2, B(\eta_t)]e^{sB(\eta_t)} \\ &= R_2 + \int_0^1 ds \int dx\, dy\, r(y;x)e^{-sB(\eta_t)}[a^*_x a_y, B(\eta_t)]e^{sB(\eta_t)} \\ &= R_2 + \int_0^1 ds \int dx\, e^{-sB(\eta_t)}[b((\eta_t r)_x)b_x + \text{h.c.}]e^{sB(\eta_t)}. \end{aligned}$$

Expanding as in Lemma 3.3 and then integrating over s, we find

$$e^{-B(\eta_t)}R_2e^{B(\eta_t)} = R_2 + \sum_{k,n\geq 0}\frac{(-1)^{k+n}}{k!\,n!\,(k+n+1)}\int dx\, [\mathrm{ad}^{(n)}_{B(\eta_t)}(b((\eta_t r)_x))\mathrm{ad}^{(k)}_{B(\eta_t)}(b_x) + \text{h.c.}]. \tag{5-40}$$

With Lemma 3.2, we can write the operator

$$\int dx\, \mathrm{ad}^{(n)}_{B(\eta_t)}(b((\eta_t r)_x))\mathrm{ad}^{(k)}_{B(\eta_t)}(b_x) \tag{5-41}$$

as a sum of $2^{n+k}k!\,n!$ contributions of the form

$$\begin{aligned} \mathrm{E} = \int dx\; \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x) \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'}), \end{aligned} \tag{5-42}$$

where each Λ_i and Λ'_i is $(N-\mathcal{N})/N$, $(N+1-\mathcal{N})/N$ or an operator of the form

$$N^{-p}\, \Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1}, \dots, \eta^{(q_p)}_{t,\underline{\natural}_p}). \tag{5-43}$$

From Lemma 5.3, we obtain that

$$\begin{aligned} |\langle \xi_1, \mathrm{E}\xi_2\rangle| &\le \|(\mathcal{N}+1)^{1/2}\xi_1\| \int dx\; \big\|(\mathcal{N}+1)^{-1/2}\Lambda_1 \cdots \Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; (\eta^{(\ell_1+1)}_{t,\diamondsuit} r)_x) \\ &\qquad \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{x,\diamondsuit'})\xi_2\big\| \\ &\le C^{n+k}\|r\|_{\mathrm{op}}\,\|\eta_t\|^{k+n+1}\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|. \end{aligned}$$

This implies that, if $\sup_t \|\eta_t\|$ is small enough,

$$|\langle \xi_1, e^{-B(\eta_t)} R_2 e^{B(\eta_t)}\xi_2\rangle| \le C\|r\|_{\mathrm{op}}\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|.$$

As in the analysis of R_1 above, also here one can show the other bounds in (5-32) for the commutators of $e^{-B(\eta_t)}R_1 e^{B(\eta_t)}$ with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and for its time-derivative. □

Next, we use Lemma 5.6 to show Proposition 5.5.

Proof of Proposition 5.5. To control $\mathcal{L}^{(0)}_{N,t}$ we start by noticing that, with Young's inequality,

$$\begin{aligned} |\langle \tilde{\varphi}_t, [N^3V(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle| &\le \int N^3V(N(x-y))|\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2\,dx\,dy \\ &\le C\|\tilde{\varphi}_t\|_4^4 \le C\|\tilde{\varphi}_t\|^4_{H^1} \le C \end{aligned} \tag{5-44}$$

and

$$|\partial_t\langle \tilde{\varphi}_t, [N^3V(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle| \le C\|\tilde{\varphi}_t\|_4^3\|\dot{\tilde{\varphi}}_t\|_4 \le C\|\tilde{\varphi}_t\|^3_{H^1}\|\tilde{\varphi}_t\|_{H^3} \le Ce^{c|t|} \tag{5-45}$$

for constants $C, c > 0$. Similarly, we also have

$$\begin{aligned} |\langle \tilde{\varphi}_t, [N^3V(N\,\cdot)w_\ell(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle| &\le C, \\ |\partial_t\langle \tilde{\varphi}_t, [N^3V(N\,\cdot)w_\ell(N\,\cdot)*|\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle| &\le Ce^{c|t|}. \end{aligned} \tag{5-46}$$

By (5-44), (5-45), (5-46), it is enough to show the four bounds in (5-31) with $\mathcal{L}^{(0)}_{N,t} - C_{N,t}$ replaced by $\mathcal{N}$ and by $\mathcal{N}^2/N$. If we replace $\mathcal{L}^{(0)}_{N,t} - C_{N,t}$ with $\mathcal{N}$, the bounds in (5-31) follow from Lemma 5.6. To prove that these bounds also hold for $\mathcal{N}^2/N$, we use again Lemma 5.6. Setting $\xi_2 = e^{-B(\eta_t)}(\mathcal{N}/N)e^{B(\eta_t)}\xi$, we have

$$|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\xi\rangle| = |\langle \xi, e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}\xi_2\rangle| \le C\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|.$$

Since, by Lemma 3.1,

$$\begin{aligned}\|(\mathcal{N}+1)^{1/2}\xi_2\|^2 &= N^{-2}\langle \xi, e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}(\mathcal{N}+1)e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}\xi\rangle \\ &\le N^{-2}\langle \xi, (\mathcal{N}+1)^3\xi\rangle \le C\langle \xi, (\mathcal{N}+1)\xi\rangle\end{aligned}$$

for all $\xi \in \mathcal{F}^{\le N}$, we have

$$|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}\xi\rangle| \le C\|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

Using Lemma 5.6 and the Leibniz rule, we also find

$$\begin{aligned}|\langle \xi, [\mathcal{N}, e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}]\xi\rangle| &\le C\|(\mathcal{N}+1)^{1/2}\xi\|^2, \\ |\langle \xi, [a^*(g_1)a(g_2), e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)}]\xi\rangle| &\le C\|g_1\|\,\|g_2\|\,\|(\mathcal{N}+1)^{1/2}\xi\|^2, \\ |\langle \xi, \partial_t(e^{-B(\eta_t)}(\mathcal{N}^2/N)e^{B(\eta_t)})\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2.\end{aligned}$$

□

5C. *Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(1)}_{N,t}e^{B(\eta_t)}$.* We recall that

$$\mathcal{L}^{(1)}_{N,t} = \sqrt{N}b(h_{N,t}) - \frac{\mathcal{N}+1}{\sqrt{N}}b(\tilde{h}_{N,t}) + \text{h.c.},$$

where we used the notation

$$\begin{aligned}h_{N,t} &= (N^3V(N\,\cdot\,)w_\ell(N\,\cdot\,) * |\tilde{\varphi}_t|^2)\tilde{\varphi}_t, \\ \tilde{h}_{N,t} &= (N^3V(N\,\cdot\,) * |\tilde{\varphi}_t|^2)\tilde{\varphi}_t.\end{aligned}$$

We write

$$e^{-B(\eta_t)}\mathcal{L}^{(1)}_{N,t}e^{B(\eta_t)} = \sqrt{N}[b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(\bar{h}_{N,t})) + \text{h.c.}] + \mathcal{E}^{(1)}_{N,t}. \tag{5-47}$$

In the next proposition we show that the operator $\mathcal{E}^{(1)}_{N,t}$, defined in (5-47), its commutator with $\mathcal{N}$ and its time-derivative can all be controlled by the number of particles operator $\mathcal{N}$ (while the first term on the right-hand side of (5-47) will cancel with contributions arising from conjugation of $\mathcal{L}^{(3)}_{N,t}$).

Proposition 5.7. *Under the same assumptions as in Theorem 4.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned}|\langle \xi, \mathcal{E}^{(1)}_{N,t}\xi\rangle| &\le C\langle \xi, (\mathcal{N}+1)\xi\rangle, \\ |\langle \xi, [\mathcal{N}, \mathcal{E}^{(1)}_{N,t}]\xi\rangle| &\le C\langle \xi, (\mathcal{N}+1)\xi\rangle, \\ |\langle \xi, [a^*(g_1)a(g_2), \mathcal{E}^{(1)}_{N,t}]\xi\rangle| &\le C\|g_1\|\,\|g_2\|\langle \xi, (\mathcal{N}+1)\xi\rangle, \\ |\partial_t\langle \xi, \mathcal{E}^{(1)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\langle \xi, (\mathcal{N}+1)\xi\rangle\end{aligned} \tag{5-48}$$

for all $\xi \in \mathcal{F}^{\le N}$.

Proof. We start with the observation that

$$\begin{aligned} \|h_{N,t}\|,\ \|\tilde{h}_{N,t}\| &\le C\|\tilde{\varphi}_t\|_{H^1}^3 \le C, \\ \|\partial_t h_{N,t}\|,\ \|\partial_t \tilde{h}_{N,t}\| &\le \|\tilde{\varphi}_t\|_{H^1}^2 \|\tilde{\varphi}_t\|_{H^3} \le Ce^{c|t|} \end{aligned} \tag{5-49}$$

uniformly in N and for all $t \in \mathbb{R}$. Recall that, by (5-47),

$$\begin{aligned} \mathcal{E}_{N,t}^{(1)} &= \big[e^{-B(\eta_t)}\mathcal{L}_{N,t}^{(1)}e^{B(\eta_t)} - \sqrt{N}\big(b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(h_{N,t})) + \text{h.c.}\big)\big] \\ &= \sqrt{N}\big[e^{-B(\eta_t)}b(h_{N,t})e^{B(\eta_t)} - \big(b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(h_{N,t}))\big)\big] + \text{h.c.} \\ &\qquad + N^{-1/2}e^{-B(\eta_t)}(\mathcal{N}+1)b(\tilde{h}_{N,t})e^{B(\eta_t)}. \end{aligned} \tag{5-50}$$

Set

$$D(g) = e^{-B(\eta_t)}b(g)e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t}(g)).$$

We observe that Proposition 5.7 follows if we prove that

$$\begin{aligned} |\langle \xi_1, D(g)\xi_2\rangle| &\le CN^{-1/2}\|g\|\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \\ |\langle \xi_1, [\mathcal{N}, D(g)]\xi_2\rangle| &\le CN^{-1/2}\|g\|\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \\ |\langle \xi_1, [a^*(g_1)a(g_2), D(g)]\xi_2\rangle| &\le CN^{-1/2}\|g\|\,\|g_1\|\,\|g_2\|\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \\ |\langle \xi_1, \partial_t D(g)\xi_2\rangle| &\le CN^{-1/2}(\|g\|+\|\dot{g}\|)\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\| \end{aligned} \tag{5-51}$$

for every, possibly time-dependent, $g \in L^2(\mathbb{R}^3)$. In fact, applying (5-51) with $g = h_{N,t}$, we obtain the desired bounds for the first line on the right-hand side of (5-50). To bound the expectation of the operator on the second line on the right-hand side of (5-50), on the other hand, we apply (5-51) with $g = \tilde{h}_{N,t}$, $\xi_1 = \xi$ and $\xi_2 = e^{-B(\eta_t)}(\mathcal{N}+1)e^{B(\eta_t)}\xi$. We find

$$\begin{aligned} &N^{-1/2}|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)b(\tilde{h}_{N,t})e^{B(\eta_t)}\xi\rangle| \\ &\quad = N^{-1/2}|\langle \xi_2, e^{-B(\eta_t)}b(\tilde{h}_{N,t})e^{B(\eta_t)}\xi\rangle| \\ &\quad \le N^{-1/2}|\langle \xi_2, [b(\cosh_{\eta_t}(\tilde{h}_{N,t})) + b^*(\sinh_{\eta_t}(\tilde{h}_{N,t}))]\xi\rangle| + CN^{-1}\|\tilde{h}_{N,t}\|\,\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\| \\ &\quad \le CN^{-1/2}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|\xi_2\| + CN^{-1}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\|, \end{aligned} \tag{5-52}$$

where we used Lemma 2.2, the fact that $\cosh_{\eta_t}$, $\sinh_{\eta_t}$ are bounded operators (uniformly in t and N), and (5-49). From Lemma 3.1, we obtain

$$\|\xi_2\|^2 = \langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)^2e^{B(\eta_t)}\xi\rangle \le C\langle \xi, (\mathcal{N}+1)^2\xi\rangle = C\|(\mathcal{N}+1)\xi\|^2$$

and, similarly,

$$\begin{aligned} \|(\mathcal{N}+1)^{1/2}\xi_2\|^2 &= \langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)e^{B(\eta_t)}(\mathcal{N}+1)e^{-B(\eta_t)}(\mathcal{N}+1)e^{B(\eta_t)}\xi\rangle \\ &\le C\langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)^3e^{B(\eta_t)}\xi\rangle \\ &\le C\langle \xi, (\mathcal{N}+1)^3\xi\rangle = C\|(\mathcal{N}+1)^{3/2}\xi\|^2. \end{aligned}$$

Inserting the last two bounds in the right-hand side of (5-52), we conclude that

$$N^{-1/2}|\langle \xi, e^{-B(\eta_t)}(\mathcal{N}+1)b(\tilde{h}_{N,t})e^{B(\eta_t)}\xi\rangle| \le C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

for all $\xi \in \mathcal{F}^{\le N}$. Similarly, we can control the commutators of the second line on the right-hand side of (5-50) with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and its time-derivative.

We still have to show (5-51). To this end, we use Lemma 3.3 to expand

$$e^{-B(\eta_t)}b(g)e^{B(\eta_t)} = \sum_{n\ge 0} \frac{(-1)^n}{n!}\mathrm{ad}^{(n)}_{B(\eta_t)}(b(g)). \tag{5-53}$$

According to Lemma 3.2, the nested commutator $\mathrm{ad}^{(n)}_{B(\eta_t)}(b(g))$ can be written as a sum of $2^n n!$ terms, having the form

$$\Lambda_1 \cdots \Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(s)}_{t,\natural_{k+1}}(g_\diamond)), \tag{5-54}$$

where each Λ_m is $(N-\mathcal{N})/N$, $(N-\mathcal{N}+1)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_p)}_{t,\natural'_p}). \tag{5-55}$$

Exactly one of these $2^n n!$ terms has the form

$$\begin{cases} \dfrac{(N-\mathcal{N})^r}{N^r}\dfrac{(N+1-\mathcal{N})^r}{N^r}b(\eta_t^{(2r)}(g)) & \text{if } n=2r \text{ is even},\\ -\dfrac{(N-\mathcal{N})^{r+1}}{N^{r+1}}\dfrac{(N+1-\mathcal{N})^r}{N^r}b^*(\eta_t^{(2r+1)}(\bar{g})) & \text{if } n=2r+1 \text{ is odd}. \end{cases} \tag{5-56}$$

All other terms are of the form (5-54), with either $k>0$ or with at least one factor Λ_i being of the form (5-55). Let us suppose that $n=2r$ is even. Then we write (5-56) as

$$\frac{(N-\mathcal{N})^r}{N^r}\frac{(N+1-\mathcal{N})^r}{N^r}b(\eta_t^{(2r)}(g)) = b(\eta_t^{(2r)}(g)) + \left[\frac{(N-\mathcal{N})^r}{N^r}\frac{(N+1-\mathcal{N})^r}{N^r}-1\right]b(\eta_t^{(2r)}(g)). \tag{5-57}$$

Inserting the term $b(\eta_t^{(2r)}(g))$ on the right-hand side of (5-53) and summing over all $r\in\mathbb{N}$, we reconstruct

$$\sum_{r\ge 0}\frac{1}{(2r)!}b(\eta_t^{(2r)}(g)) = b(\cosh_{\eta_t}(g)).$$

On the other hand, the contribution of the second term on the right-hand side of (5-57) has matrix elements bounded by

$$\begin{aligned} &\left|\left\langle \xi_1, \left[\frac{(N-\mathcal{N})^r}{N^r}\frac{(N+1-\mathcal{N})^r}{N^r}-1\right]b(\eta_t^{(2r)}(g))\xi_2\right\rangle\right| \\ &\qquad\le \left\|\left[\frac{(N-\mathcal{N})^r}{N^r}\frac{(N+1-\mathcal{N})^r}{N^r}-1\right]\xi_1\right\| \|b(\eta_t^{(2r)}(g))\xi_2\| \\ &\qquad\le 2rN^{-1/2}\|\eta_t\|^{2r}\|g\|\,\|(\mathcal{N}+1)^{1/2}\xi_1\|\,\|(\mathcal{N}+1)^{1/2}\xi_2\| \end{aligned} \tag{5-58}$$

since $1-(1-x)^r \le rx$ for all $0 \le x \le 1$. Similarly, the contribution (5-56) with $n = 2r+1$ odd can be shown to reconstruct the operator $b^*(\sinh_{\eta_t}(\bar{g}))$, up to an error that can be estimated as in (5-58).

As for the other terms of the form (5-54), excluding (5-56), we can bound their matrix elements using part (i) of Lemma 5.2. We obtain

$$\begin{aligned} &|\langle \xi_1, \Lambda_1 \cdots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}_{t,\natural_{k+1}}) \xi_2 \rangle| \\ &\qquad \le \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{-1/2} \, \Lambda_1 \cdots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_k)}_{t,\natural_k}; \eta^{(s)}_{t,\natural_{k+1}}(g_\diamond))\xi_2\| \\ &\qquad \le C^n \|\eta_t\|^n N^{-1/2} \|g\| \, \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{1/2}\xi_2\|. \end{aligned} \tag{5-59}$$

We conclude that

$$\begin{aligned} &|\langle \xi_1, \{e^{-B(\eta_t)} b(g) e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t}(\bar{g}))\}\xi_2\rangle| \\ &\qquad \le N^{-1/2}\|g\| \, \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{1/2}\xi_2\| \sum_{n\ge 2} nC^n \|\eta_t\|^n \\ &\qquad \le CN^{-1/2}\|g\| \, \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{1/2}\xi_2\| \end{aligned} \tag{5-60}$$

if the parameter $\ell > 0$ in the definition (4-16) of the kernel η_t is small enough.

Since, by Lemma 5.1(i), the commutator of every term of the form (5-54) with $\mathcal{N}$ is again a term of the same form, just multiplied with a constant $\kappa \in \{0, \pm 1, \pm 2\}$, we conclude that

$$\begin{aligned} &|\langle \xi_1, [\mathcal{N}, \{e^{-B(\eta_t)} b(g) e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t}(\bar{g}))\}]\xi_2\rangle| \\ &\qquad \le CN^{-1/2}\|g\| \, \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{1/2}\xi_2\|. \end{aligned} \tag{5-61}$$

Since, again by Lemma 5.1, parts (ii) and (iii), the commutator of every term of the form (5-54) with $a^*(g_1)a(g_2)$ can be written as a sum of at most $2n$ terms having again the form (5-54), just with one of the η_t-kernels or with the function $\eta^{(s)}_{t,\natural_{k+1}}(g_\diamond)$ appearing in the $\Pi^{(1)}$-operator replaced according to (5-4) and (5-9), we also find that

$$\begin{aligned} &|\langle \xi_1, [a^*(g_1)a(g_2), \{e^{-B(\eta_t)} b(g) e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t}(\bar{g}))\}]\xi_2\rangle| \\ &\qquad \le CN^{-1/2}\|g\| \, \|g_1\| \, \|g_2\| \, \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{1/2}\xi_2\|. \end{aligned} \tag{5-62}$$

Finally, since by Lemma 5.1(iv) the time-derivative of each term of the form (5-54) can be written as a sum of at most $n+1$ terms having again the form (5-54), but with one of the η_t-kernels or the function $\eta^{(s)}_{t,\natural_{k+1}}(g_\diamond)$ appearing in the $\Pi^{(1)}$-operator replaced by their time-derivative, we get (since $\|\dot{\eta}_t\| \le Ce^{c|t|}$)

$$\begin{aligned} &|\partial_t \langle \xi_1, [e^{-B(\eta_t)} b(g) e^{B(\eta_t)} - b(\cosh_{\eta_t}(g)) - b^*(\sinh_{\eta_t}(\bar{g}))]\xi_2\rangle| \\ &\qquad \le CN^{-1/2}e^{c|t|}(\|g\| + \|\dot{g}\|) \|(\mathcal{N}+1)^{1/2}\xi_1\| \, \|(\mathcal{N}+1)^{1/2}\xi_2\|, \end{aligned} \tag{5-63}$$

completing the proof. □

5D. ***Analysis of*** $e^{-B(\eta_t)}\mathcal{L}^{(2)}_{N,t}e^{B(\eta_t)}$. Recall that

$$\begin{aligned}\mathcal{L}^{(2)}_{N,t} = \mathcal{K} + \int dx\,dy\,N^3V(N(x-y))|\tilde{\varphi}_t(y)|^2\Big[b^*_xb_x - \frac{1}{N}a^*_xa_x\Big]\\ + \int dx\,dy\,N^3V(N(x-y))\tilde{\varphi}_t(x)\bar{\tilde{\varphi}}_t(y)\Big[b^*_xb_y - \frac{1}{N}a^*_xa_y\Big]\\ + \frac{1}{2}\int dx\,dy\,N^3V(N(x-y))[\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)b^*_xb^*_y + \text{h.c.}],\end{aligned} \tag{5-64}$$

with the notation

$$\mathcal{K} = \int dx\,\nabla_xa^*_x\nabla_xa_x$$

for the kinetic energy operator.

In the next two subsections we consider first the conjugation of the kinetic energy operator and then of the rest of $\mathcal{L}^{(2)}_{N,t}$ with $e^{B(\eta_t)}$.

5D1. *Analysis of* $e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)}$. We write

$$\begin{aligned}&e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)}\\ &= \mathcal{K} + \int|\nabla_xk_t(x;y)|^2\,dx\,dy + \int dx\,dy\,(\Delta w_\ell)(N(x-y))[\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)b^*_xb^*_y + \text{h.c.}] + \mathcal{E}^{(K)}_{N,t}.\end{aligned} \tag{5-65}$$

In the next proposition, we collect important properties of the error term $\mathcal{E}^{(K)}_{N,t}$ defined in (5-65).

Proposition 5.8. *Under the same assumptions as in Theorem 4.4, there exist constants* $C, c > 0$ *such that*

$$\begin{aligned}|\langle\xi, \mathcal{E}^{(K)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle\xi, [\mathcal{N}, \mathcal{E}^{(K)}_{N,t}]\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle\xi, [a^*(g_1)a(g_2), \mathcal{E}^{(K)}_{N,t}]\xi\rangle| &\le Ce^{c|t|}\|g_1\|_{H^1}\|g_2\|_{H^1}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\\ |\partial_t\langle\xi, \mathcal{E}^{(K)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\end{aligned} \tag{5-66}$$

where we used the notation $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, *with*

$$\mathcal{V}_N = \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))a^*_xa^*_ya_ya_x. \tag{5-67}$$

Proof. We write

$$e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} - \mathcal{K} = \int_0^1 e^{-sB(\eta_t)}[\mathcal{K}, B(\eta_t)]e^{sB(\eta_t)} = \int_0^1 ds\int dx\,e^{-sB(\eta_t)}[\nabla_xa^*_x\nabla_xa_x, B(\eta_t)]e^{sB(\eta_t)}.$$

From (3-14), we find

$$\begin{aligned}e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} - \mathcal{K} &= \int_0^1 ds\int dx\,[e^{-sB(\eta_t)}b(\nabla_x\eta_x)\nabla_xb_xe^{sB(\eta_t)} + \text{h.c.}]\\ &= \sum_{k,n\ge0}\frac{(-1)^{k+n}}{k!\,n!\,(k+n+1)}\int dx\,[\text{ad}^{(n)}_{B(\eta_t)}(b(\nabla_x\eta_x))\text{ad}^{(k)}_{B(\eta_t)}(\nabla_xb_x) + \text{h.c.}].\end{aligned}$$

From the sum on the right-hand side we extract the term with $k = n = 0$ and also the term with $n = 0$, $k = 1$. We obtain

$$\begin{aligned} e^{-B(\eta_t)}\mathcal{K}e^{B(\eta_t)} - \mathcal{K} &= \int dx\, [b(\nabla_x\eta_x)\nabla_x b_x + \text{h.c.}] \\ &\quad + \int dx\, b(\nabla_x\eta_x)b^*(\nabla_x\eta_x) - \frac{1}{N}\int dx\, b(\nabla_x\eta_x)\mathcal{N}b^*(\nabla_x\eta_x) \\ &\quad - \frac{1}{2N}\int dx\, dz\, dy [\eta_t(z,y)b(\nabla_x\eta_x)b_y^* a_z^* \nabla_x a_x + \text{h.c.}] \\ &\quad + \sum_{k,n}^{*} \frac{(-1)^{k+n}}{k!\,n!\,(k+n+1)} \int dx [\text{ad}_{B(\eta_t)}^{(n)}(b(\nabla_x\eta_x))\text{ad}_{B(\eta_t)}^{(k)}(\nabla_x b_x) + \text{h.c.}], \end{aligned} \tag{5-68}$$

where $\sum^{*}$ denotes the sum over all indices $k, n \geq 0$, excluding the two pairs $(k,n) = (0,0)$ and $(k,n) = (1,0)$. We discuss now the terms on the right-hand side of (5-68) separately.

The first term on the right-hand side of (5-68) can be decomposed as in (4-17), giving

$$\int dx\, b(\nabla_x\eta_x)\nabla_x b_x = \int dx\, b(\nabla_x k_x)\nabla_x b_x + \int dx\, b(\nabla_x\mu_x)\nabla_x b_x. \tag{5-69}$$

The second term on the right-hand side of (5-69) contributes to the error $\mathcal{E}_{N,t}^{(K)}$. Its expectation is bounded by

$$\begin{aligned} \left| \int dx\, \langle \xi, b(\nabla_x\mu_x)\nabla_x b_x \xi\rangle \right| &\leq \|(\mathcal{N}+1)^{1/2}\xi\| \int dx\, \|\nabla_x\mu_x\|\, \|\nabla_x b_x\xi\| \\ &\leq \|\nabla_x\mu\|\, \|(\mathcal{N}+1)^{1/2}\xi\|\, \|\mathcal{K}^{1/2}\xi\| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|\, \|\mathcal{K}^{1/2}\xi\|. \end{aligned}$$

The expectation of the commutators of this term with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and also its time-derivative can be bounded similarly, using the formula

$$[a^*(g_1)a(g_2), b(\nabla_x\mu_x)\nabla_x b_x] = \langle g_1, \nabla_x\mu_x\rangle b(g_2)\nabla_x b_x + b(\nabla_x\mu_x)\nabla g_1(x) b(g_2)$$

and the fact that $\|\partial_t\nabla_x\mu_t\| < Ce^{c|t|}$, uniformly in N.

As for the first term on the right-hand side of (5-69), we integrate by parts and we use the definition (4-15), to write

$$\begin{aligned} \int dx\, b(\nabla_x k_x)\nabla_x b_x &= \int dx\, dy\, N^3(\Delta w_\ell)(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\, b_x b_y \\ &\quad + 2\int dx\, dy\, N^2(\nabla w_\ell)(N(x-y))(\nabla\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y)\, b_x b_y \\ &\quad + \int dx\, dy\, N w_\ell(N(x-y))(\Delta\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y) b_x b_y. \end{aligned} \tag{5-70}$$

The first term on the right-hand side of (5-70) is exactly the (hermitian conjugate of the) contribution that we isolated on the second line of (5-65); it does not enter the error term $\mathcal{E}_{N,t}^{(K)}$. The second and third terms on the right-hand side of (5-70), on the other hand, are included in $\mathcal{E}_{N,t}^{(K)}$. The expectation of the third term is bounded by

$$\begin{aligned}\left|\int dx\,dy\,Nw_\ell(N(x-y))(\Delta\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y)\,\langle\xi, b_x b_y\xi\rangle\right| \\ &\le \int dx\,|\Delta\tilde{\varphi}_t(x)|\,\|b^*(Nw_\ell(N(x-\cdot))\tilde{\varphi}_t)\xi\|\,\|b_x\xi\| \\ &\le \sup_x \|Nw_\ell(N(x-\cdot))\tilde{\varphi}_t\|\,\|\Delta\tilde{\varphi}_t\|\,\|(\mathcal{N}+1)^{1/2}\xi\|^2 \le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2. \qquad (5\text{-}71)\end{aligned}$$

To bound the expectation of the second term on the right-hand side of (5-70), we integrate by parts. We find

$$\begin{aligned}\int dx\,dy\,N^2(\nabla w_\ell)(N(x-y))(\nabla\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y)\,\langle\xi, b_x b_y\xi\rangle \\ &= -\int dx\,dy\,Nw_\ell(N(x-y))(\Delta\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y)\langle\xi, b_x b_y\xi\rangle \\ &\quad -\int dx\,dy\,Nw_\ell(N(x-y))(\nabla\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y)\langle\xi, b_y\nabla_x b_x\xi\rangle.\end{aligned}$$

Proceeding as in (5-71), we conclude that

$$\begin{aligned}\left|\int dx\,dy\,N^2(\nabla w_\ell)(N(x-y))(\nabla\tilde{\varphi}_t)(x)\tilde{\varphi}_t(y)\,\langle\xi, b_x b_y\xi\rangle\right| \\ &\le \sup_x \|Nw_\ell(N(x-\cdot))\tilde{\varphi}_t\|\left[\|\Delta\tilde{\varphi}_t\|\,\|(\mathcal{N}+1)^{1/2}\xi\|^2 + \|\nabla\tilde{\varphi}_t\|\,\|(\mathcal{N}+1)^{1/2}\xi\|\,\|\mathcal{K}^{1/2}\xi\|\right] \\ &\le Ce^{c|t|}\left[\|(\mathcal{N}+1)^{1/2}\xi\|^2 + \|(\mathcal{N}+1)^{1/2}\xi\|\,\|\mathcal{K}^{1/2}\xi\|\right].\end{aligned}$$

Notice that the last estimate and the estimate (5-71) for the third term on the right-hand side of (5-70) continue to hold, if we replace the operator whose expectation we are bounding with its commutator with $\mathcal{N}$ or with $a^*(g_1)a(g_2)$ or with its time-derivative.

Now, let us consider the second term on the right-hand side of (5-68). We observe that

$$\begin{aligned}\int dx\,b(\nabla_x\eta_x)b^*(\nabla_x\eta_x) \\ &= \|\nabla_x\eta_x\|^2 - \frac{\mathcal{N}}{N}\|\nabla_x\eta_x\|^2 + \int dx\,dy\,dz\,\nabla_x\eta_t(x;z)\nabla_x\bar{\eta}_t(y;x)\left(b_z^*b_y - \frac{1}{N}a_z^*a_y\right). \qquad (5\text{-}72)\end{aligned}$$

Denoting by D the operator with the integral kernel

$$D(z;y) = \int dx\,\nabla_x\eta_t(z;x)\nabla_x\bar{\eta}_t(x;y), \qquad (5\text{-}73)$$

we have

$$\left|\int dx\,dy\,dz\,\nabla_x\eta_t(x;z)\nabla_x\bar{\eta}_t(y;x)\langle\xi, b_z^*b_y\xi\rangle\right| \le |\langle\xi, d\Gamma(D)\xi\rangle| \le \|D\|_2\,\|\mathcal{N}^{1/2}\xi\|^2. \qquad (5\text{-}74)$$

Since, by Lemma 4.3, $\|D\|_2 \le C$, we obtain

$$\left|\int dx\,dy\,dz\,\nabla_x\eta_t(x;z)\nabla_x\bar{\eta}_t(y;x)\langle\xi, b_z^*b_y\xi\rangle\right| \le C\|\mathcal{N}^{1/2}\xi\|^2$$

and similarly for the $a_z^* a_y$ term. As for the first term on the right-hand side of (5-72), we use the decomposition $\eta_t = k_t + \mu_t$. Since $\|\nabla_x \mu_t\|$ is finite, uniformly in N and in t, we find

$$\left| \int dx \, \|\nabla_x \eta_x\|^2 - \int dx \, dy \, |\nabla_x k_t(x;y)|^2 \right| \le C.$$

The second term on the right-hand side of (5-72) can be controlled using $N^{-1}\|\nabla_x \eta_x\|^2 \le C$. Furthermore, one can show that

$$\int dx \, \langle \xi, [\mathcal{N}, b(\nabla_x \eta_x) b^*(\nabla_x \eta_x)] \xi \rangle = 0,$$

$$\left| \int dx \, \langle \xi, [a^*(g_1) a(g_2), b(\nabla_x \eta_x) b^*(\nabla_x \eta_x)] \xi \rangle \right| \le C \|g_1\| \, \|g_2\| \, \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

and

$$\left| \partial_t \left[\int dx \, \langle \xi, [b(\nabla_x \eta_x) b^*(\nabla_x \eta_x)] \xi \rangle - \int dx \, dy |\nabla_x k_t(x;y)|^2 \right] \right| \le C e^{K|t|} \|(\mathcal{N}+1)^{1/2} \xi\|^2.$$

Here we used the formula

$$\left[a^*(g_1) a(g_2), \int dx \, b(\nabla_x \eta_x) b^*(\nabla_x \eta_x) \right] = \int dx \, \langle \nabla_x \eta_x, g_1 \rangle b(g_2) b^*(\nabla_x \eta_x) + \int dx \, \langle g_2, \nabla_x \eta_x \rangle b(\nabla_x \eta_x) b^*(g_1)$$

for the commutator with $a^*(g_1) a(g_2)$ and the bounds in Proposition 4.2 for $\partial_t \tilde{\varphi}_t$.

The third term on the right-hand side of (5-68) can be controlled similarly.

To control the fourth term on the right-hand side of (5-68) we proceed as follows. First of all, we commute the annihilation operator $b(\nabla_x \eta_x)$ to the right of the two creation operators $b_y^* a_z^*$. Using (2-7), we find

$$\begin{aligned} \frac{1}{2N} \int dx \, dy \, dz \, \eta_t(z;y) b(\nabla_x \eta_x) b_y^* a_z^* \nabla_x a_x &= \frac{1}{2N} \int dx \, dy \, dz \, \eta_t(z;y) b_y^* a_z^* a(\nabla_x \eta_x) \nabla_x b_x \\ &\quad + \frac{1}{N} \int dx \, dy \, dz \, \eta_t(z;y) \nabla_x \eta_t(x;y) \left(1 - \frac{\mathcal{N}}{N} - \frac{1}{2N}\right) a_z^* \nabla_x a_x \\ &\quad - \frac{1}{2N^2} \int dx \, dy \, dz \, \eta_t(z;y) a_y^* a(\nabla_x \eta_x) a_z^* \nabla_x a_x. \end{aligned} \tag{5-75}$$

To bound the expectation of the last term, we use the additional N^{-1} factor to compensate for $\|\nabla_x \eta_t\| \simeq N^{1/2}$. We find

$$\begin{aligned} &\left| \frac{1}{2N^2} \int dx \, dy \, dz \, \eta_t(z;y) \langle \xi, a_y^* a(\nabla_x \eta_x) a_z^* \nabla_x a_x \xi \rangle \right| \\ &\qquad \le \frac{1}{2N^2} \left[\int dx \, dy \, dz \, |\eta_t(y;z)|^2 \, \|\nabla_x a_x \xi\|^2 \right]^{1/2} \left[\int dx \, dy \, dz \, \|a_z a^*(\nabla_x \eta_x) a_y \xi\|^2 \right]^{1/2} \\ &\qquad \le \frac{\|\eta_t\| \, \|\nabla_x \eta_t\|}{2N^2} \|\mathcal{K}^{1/2} \xi\| \, \|(\mathcal{N}+1)^{3/2} \xi\| \\ &\qquad \le C N^{-1/2} \|\mathcal{K}^{1/2} \xi\| \, \|(\mathcal{N}+1)^{1/2} \xi\|. \end{aligned}$$

Similarly, the expectation of the second term on the right-hand side of (5-75) is bounded by

$$\begin{aligned}\left|\frac{1}{N}\int dx\,dy\,dz\,\eta_t(z;y)\nabla_x\eta_t(x;y)\left\langle\xi,\left(1-\frac{\mathcal{N}}{N}-\frac{1}{2N}\right)a_z^*\nabla_x a_x\xi\right\rangle\right| &\\ \le \frac{1}{N}\left[\int dx\,dy\,dz\,|\eta_t(z;y)|^2\,\|\nabla_x a_x\xi\|^2\right]^{1/2}&\left[\int dx\,dy\,dz\,|\nabla_x\eta_t(x;y)|^2\,\|a_z\xi\|^2\right]^{1/2}\\ \le \frac{\|\eta_t\|\,\|\nabla_x\eta_t\|}{N}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|\mathcal{K}^{1/2}\xi\|&\\ \le CN^{-1/2}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|\mathcal{K}^{1/2}\xi\|.&\end{aligned}$$

We are left with the first term on the right-hand side of (5-75). Here, we consider the decomposition

$$\begin{aligned}\frac{1}{2N}\int dx\,dy\,dz\,\eta_t(z;y)b_y^*a_z^*a(\nabla_x\eta_x)\nabla_x b_x &\\ =\frac{1}{2N}\int dx\,dy\,dz\,\eta_t(z;y)b_y^*a_z^*a(\nabla_x k_x)\nabla_x b_x&\\ +\frac{1}{2N}\int dx\,dy\,dz\,\eta_t(z;y)b_y^*a_z^*a(\nabla_x\mu_x)\nabla_x b_x &=: \mathrm{M}_1+\mathrm{M}_2. \qquad (5\text{-}76)\end{aligned}$$

Since $\nabla_x\mu_t\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$, with norm bounded uniformly in N and t, we easily find

$$|\langle\xi, M_2\xi\rangle|\le CN^{-1/2}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|\mathcal{K}^{1/2}\xi\|.$$

To control the term M_1, on the other hand, we integrate by parts. We obtain

$$\begin{aligned}\mathrm{M}_1&=\frac{1}{2N}\int dx\,dy\,dz\,dw\,\eta_t(z;y)(-\Delta_x k_t)(x;w)b_y^*a_z^*a_w b_x\\ &=\frac{N^2}{2}\int dx\,dy\,dz\,dw\,\eta_t(z;y)(\Delta w_\ell)(N(x-w))\tilde\varphi_t(x)\tilde\varphi_t(w)b_y^*a_z^*a_w b_x\\ &\quad+\frac{N}{2}\int dx\,dy\,dz\,dw\,\eta_t(z;y)(\nabla w_\ell)(N(x-w))\nabla\tilde\varphi_t(x)\tilde\varphi_t(w)b_y^*a_z^*a_w b_x\\ &\quad+\frac{1}{2}\int dx\,dy\,dz\,dw\,\eta_t(z;y)w_\ell(N(x-w))\Delta\tilde\varphi_t(x)\tilde\varphi_t(w)b_y^*a_z^*a_w b_x\\ &=\mathrm{M}_{11}+\mathrm{M}_{12}+\mathrm{M}_{13}. \qquad (5\text{-}77)\end{aligned}$$

Since $|(\nabla w_\ell)(Nx)|\le C/(N^2|x|^2)$, we have

$$\begin{aligned}&|\langle\xi,\mathrm{M}_{12}\xi\rangle|\\ &\quad\le CN^{-1}\int dx\,dy\,dz\,dw\,|\eta_t(z;y)|\frac{|\nabla\tilde\varphi_t(x)|\,|\tilde\varphi_t(w)|}{|x-w|^2}\|a_z b_y\xi\|\,\|a_w b_x\xi\|\\ &\quad\le CN^{-1}\left[\int dx\,dy\,dz\,dw\,\frac{|\nabla\tilde\varphi_t(x)|^2\,|\tilde\varphi_t(w)|^2}{|x-w|^2}\|a_z b_y\xi\|^2\right]^{1/2}\left[\int dx\,dy\,dz\,dw\,\frac{|\eta_t(y;z)|^2}{|x-w|^2}\|a_w b_x\xi\|^2\right]^{1/2}\\ &\quad\le CN^{-1}\|\eta_t\|\,\|(\mathcal{N}+1)\xi\|\,\|(\mathcal{N}+1)^{1/2}(\mathcal{K}+\mathcal{N})^{1/2}\xi\|\\ &\quad\le C\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{K}+\mathcal{N})^{1/2}\xi\|,\end{aligned}$$

where we used Hardy's inequality $|x|^{-2} \leq C(1-\Delta)$. The expectation of M_{13} can be bounded analogously. Let us focus now on the term M_{11}. Here we use the fact that $f_\ell = 1 - w_\ell$ solves the Neumann problem (4-1) to write

$$\begin{aligned}\mathrm{M}_{11} &= -\frac{N^2}{2}\int dx\,dy\,dz\,dw\,\eta_t(z;y)V(N(x-w))f_\ell(N(x-w))\tilde{\varphi}_t(x)\tilde{\varphi}_t(w)b_y^* a_z^* a_w b_x \\ &\quad + N^2\lambda_\ell\int dx\,dy\,dz\,dw\,\eta_t(z;y)f_\ell(N(x-w))\chi(|x-w|\leq\ell)\tilde{\varphi}_t(x)\tilde{\varphi}_t(w)b_y^* a_z^* a_w b_x \\ &=: \mathrm{M}_{111}+\mathrm{M}_{112}. \end{aligned} \tag{5-78}$$

Since, by Lemma 4.1, $\lambda_\ell \leq CN^{-3}$ and $0 \leq f_\ell \leq 1$, it is easy to check that

$$|\langle \xi, \mathrm{M}_{112}\xi\rangle| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

As for the first term on the right-hand side of (5-78), it can be estimated by

$$\begin{aligned}|\langle \xi, \mathrm{M}_{111}\xi\rangle| &\leq \int dx\,dy\,dz\,dw\,|\eta_t(z;y)|N^2V(N(x-w))|\tilde{\varphi}_t(w)|\,|\tilde{\varphi}_t(x)|\,\|a_z b_y\xi\|\,\|a_w b_x\xi\| \\ &\leq \left[\int dx\,dy\,dz\,dw\,|\eta_t(z;y)|^2 N^2 V(N(x-w))\|a_w b_x\xi\|^2\right]^{1/2} \\ &\qquad\times\left[\int dx\,dy\,dz\,dw\,N^2V(N(x-w))|\tilde{\varphi}_t(w)|^2|\tilde{\varphi}_t(x)|^2\|a_z b_y\xi\|^2\right]^{1/2} \\ &\leq CN^{-1/2}\|\eta_t\|\,\|\mathcal{V}_N^{1/2}\xi\|\,\|(\mathcal{N}+1)\xi\| \leq C\|\mathcal{V}_N^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\end{aligned}$$

where we used the fact that $0 \leq f_\ell \leq 1$ and the notation (5-67).

Summarizing, we have shown that the expectation of the fourth term on the right-hand side of (5-68) can be bounded by

$$\left|\frac{1}{2N}\int dx\,dy\,dz\,\eta_t(y;z)\langle \xi, b(\nabla_x\eta_x)b_y^* a_z^*\nabla_x a_x\xi\rangle\right| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{K}+\mathcal{N}+\mathcal{V}_N+1)^{1/2}\xi\|. \tag{5-79}$$

Also in this case, it is easy to check that the same estimate holds true for the expectations of the commutators of this term with $\mathcal{N}$ and with $a^*(g_1)a(g_2)$ and for the expectation of its time-derivative.

Finally, we have to deal with the last term on the right-hand side of (5-68). According to Lemma 3.2, the operator

$$\int dx\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b(\nabla_x\eta_x))\mathrm{ad}^{(k)}_{B(\eta_t)}(\nabla_x b_x)$$

is given by the sum of $2^{n+k}n!\,k!$ terms, all having the form

$$\begin{aligned}\mathrm{E} := \int dx\,\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\nabla_x\eta^{(\ell_1+1)}_{x,\diamondsuit}) \\ \times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\nabla_x\eta^{(\ell_2)}_{x,\diamondsuit'}),\end{aligned} \tag{5-80}$$

with $k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 1$, and where each operator Λ_i, Λ'_i is a factor $(N - \mathcal{N})/N$, a factor $(N + 1 - \mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p} \Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1}, \dots, \eta^{(q_p)}_{t,\underline{\natural}_p}), \tag{5-81}$$

with $p, q_1, \dots, q_p \geq 1$. Here we used the fact that $\eta^{(\ell_1)}_{\natural}(\nabla_x \eta_{x,\diamondsuit}) = \nabla_x \eta^{(\ell_1+1)}_{x,\diamondsuit'}$ for an appropriate choice of $\diamondsuit' \in \{\cdot, *\}^{\ell_1+1}$.

We study the expectation of a term of the form (5-80), distinguishing several cases, depending on the values of $\ell_1, \ell_2 \in \mathbb{N}$.

<u>Case 1</u>: $\ell_1 \geq 1$, $\ell_2 \geq 2$. In this case, $\nabla_x \eta^{(\ell_1+1)}_{t,\diamondsuit}$, $\nabla_x \eta^{(\ell_2)}_{t,\diamondsuit} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, with norm bounded uniformly in N and t. Hence, with Lemma 2.4, we can bound

$$|\langle \xi, \mathrm{E}\xi \rangle| \leq C^{k+n} \|\eta_t\|^{k+n-\ell_1-\ell_2} \|\nabla_x \eta^{(\ell_1+1)}_t\| \, \|\nabla_x \eta^{(\ell_2)}_t\| \, \|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

Now we observe that, for example,

$$\|\nabla_x \eta^{(\ell_2)}_t\| \leq \|\nabla_x \eta^{(2)}_t\| \, \|\eta^{(\ell_2-2)}_t\| \leq \|\nabla_x \eta^{(2)}_t\| \, \|\eta_t\|^{\ell_2-2} \leq C\|\eta_t\|^{\ell_2-2}.$$

Similarly, $\|\nabla_x \eta^{(\ell_1+1)}_t\| \leq C\|\eta_t\|^{\ell_1-1}$. Hence, in this case,

$$|\langle \xi, \mathrm{E}\xi \rangle| \leq C^{k+n} \|\eta_t\|^{k+n-3} \|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

<u>Case 2</u>: $\ell_1 \geq 1$, $\ell_2 = 1$. In this case we integrate by parts, writing

$$\begin{aligned}\langle \xi, \mathrm{E}\xi \rangle = \int dx \, \big\langle \xi, \Lambda_1 \cdots \Lambda_{i_1} N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; -\Delta_x \eta^{(\ell_1+1)}_{x,\diamondsuit}) \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta_{x,\diamondsuit'})\xi \big\rangle.\end{aligned}$$

Since, by Lemma 4.3, $\|\Delta_x \eta^{(2)}_t\| \leq Ce^{c|t|}$, we conclude by Lemma 2.4 that, in this case,

$$|\langle \xi, \mathrm{E}\xi \rangle| \leq C^{k+n} \|\eta_t\|^{k+n-1} \|\Delta_x \eta^{(2)}_t\| \, \|(\mathcal{N}+1)^{1/2}\xi\|^2 \leq C^{k+n} e^{c|t|} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

<u>Case 3</u>: $\ell_1 \geq 1$, $\ell_2 = 0$. In this case, the second $\Pi^{(1)}$-operator in (5-80) has the form

$$N^{-k_2} \Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \nabla_x \delta_x) = N^{-k_2} \int b^{\flat_0}_{x_1} \prod_{j=1}^{k_2-1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_{k_2}}_{y_{k_2}} \nabla_x a_x \prod_{j=1}^{k_2} \eta^{(m_j)}_{t,\natural'_j}(x_j; y_j) \, dx_j \, dy_j.$$

Here we used part (v) of Lemma 3.2 to conclude that the last field on the right, the one carrying the derivative, must be an annihilation operator (or possibly a b-operator). Repeatedly applying Lemma 2.1 on pairs of creation and annihilation operators, but leaving the last annihilation operator $\nabla_x a_x$ untouched, we find

$$\begin{aligned}|\langle \xi, \mathrm{E}\xi \rangle| &\leq C^{k+n} \|\eta_t\|^{k+n-\ell_1} \|(\mathcal{N}+1)^{1/2}\xi\| \int dx \, \|\nabla_x \eta^{(\ell_1+1)}_x\| \, \|\nabla_x a_x \xi\| \\ &\leq C^{k+n} \|\eta_t\|^{k+n-\ell_1} \|\nabla_x \eta^{(\ell_1+1)}_t\| \, \|(\mathcal{N}+1)^{1/2}\xi\| \, \|\mathcal{K}^{1/2}\xi\| \\ &\leq C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

Case 4: $\ell_1 = 0,\ \ell_2 \geq 2$. Here we proceed as in Case 2, integrating by parts and moving the derivative over x from $\nabla_x \eta_{x,\diamondsuit}$ (whose L^2 norm blows up) to $\nabla_x \eta^{(\ell_2)}_{x,\diamondsuit'}$ (using the fact that $\|\Delta_x \eta^{(2)}_t\| < \infty$).

Case 5: $\ell_1 = 0,\ \ell_2 = 1$. In this case, by part (v) of Lemma 3.2, the two $\Pi^{(1)}$-operators in (5-80) have the form

$$\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \dots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \nabla_x \eta^{(\ell_1+1)}_{x,\diamondsuit}) = \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} a(\nabla_x \eta_x) \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i)\, dx_i\, dy_i \tag{5-82}$$

and

$$\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \dots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \nabla_x \eta^{(\ell_2)}_{x,\diamondsuit'}) = \int b^{\flat'_0}_{x_1} \prod_{j=1}^{k_2} a^{\sharp'_j}_{y_j} a^{\flat'_j}_{x_{j+1}} a^{\sharp'_n}_{y_n} a^*(\nabla_x \eta_x) \prod_{i=1}^{k_2} \eta^{(m_i)}_{t,\natural_i}(x_i; y_i)\, dx_i\, dy_i. \tag{5-83}$$

Since $\|\nabla_x \eta_t\| \simeq N^{1/2}$ blows up as $N \to \infty$, to estimate (5-80) in this case we first have to commute the annihilation operator $a(\nabla_x \eta_{x,\diamondsuit})$ in (5-82) with the creation operator $a^*(\nabla_x \eta_{x,\diamondsuit'})$ in (5-83). We proceed much as we did to bound the second term on the right-hand side of (5-68) in the case $n = 0,\ k = 1$, starting in (5-72). Here, however, we first have to commute the annihilation operator $a(\nabla_x \eta_{x,\diamondsuit})$ through the Λ'_i-operators and through the creation operators in (5-83).

If $\Lambda'_i = (N - \mathcal{N})/N$ or $\Lambda'_i = (N + 1 - \mathcal{N})/N$, we just pull the annihilation operator $a(\nabla_x \eta_{x,\diamondsuit})$ through, using the fact that $a(\nabla_x \eta_{x,\diamondsuit})\mathcal{N} = (\mathcal{N} + 1)a(\nabla_x \eta_{x,\diamondsuit})$. On the other hand, to commute $a(\nabla_x \eta_{x,\diamondsuit})$ through the Λ'_i-operators having the form (5-81) and through the creation operators in (5-83) (excluding the very last one on the right), we use the canonical commutation relations (2-1). The important observation here is the fact that every creation operator appearing in (5-81) and in (5-83) is associated with an η_t-kernel; the commutator produces a new creation or annihilation operator, this time with a wave function whose L^2-norm remains bounded, uniformly in N. For example, we have

$$\left[a(\nabla_x \eta_x), \int a^*_{x_i} a_{y_i} \eta^{(m_i)}(x_i; y_i)\, dx_i\, dy_i \right] = a(\nabla_x \eta^{(m_i+1)}_x). \tag{5-84}$$

Since $m_i + 1 \geq 2$, we have $\|\nabla_x \eta^{(m_i+1)}\| \leq C$, uniformly in N. Similar formulas hold for commutators of $a(\nabla_x \eta_x)$ with a pair of not normally ordered creation and annihilation operators or with the product of two creation operators. In fact, not only the L^2-norm but even the H^1-norm of the wave function of the annihilation operator on the right-hand side of (5-84) is bounded, uniformly in N. This means that terms resulting from commutators like (5-84) can be bounded integrating by parts and moving the derivative in (5-83) to the argument of the annihilation operator in (5-84). We conclude that $\mathrm{E} = \mathrm{F}_1 + \mathrm{F}_2$, where

$$\begin{aligned}\mathrm{F}_1 = \int dx\, \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i)\, dx_i\, dy_i \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i)\, dx'_i\, dy'_i\, a(\nabla_x \eta_{x,\diamondsuit}) a^*(\nabla_x \eta_{x,\diamondsuit'}),\end{aligned}$$

while F_2, which contains the contribution of all commutators, is bounded by

$$|\langle \xi, \mathrm{F}_2 \xi \rangle| \leq n C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N} + 1)^{1/2} \xi\|^2.$$

To estimate F_1, we write it as $\mathrm{F}_1 = \mathrm{F}_{11} + \mathrm{F}_{12}$, with

$$\mathrm{F}_{11} = \|\nabla_x \eta_t\|^2\, \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i;\, y_i)\, dx_i\, dy_i \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i;\, y'_i)\, dx'_i\, dy'_i \quad (5\text{-}85)$$

and

$$\mathrm{F}_{12} = \int dx\, \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i;\, y_i)\, dx_i\, dy_i \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i;\, y'_i)\, dx'_i\, dy'_i\, a^*(\nabla_x \eta_{x,\diamondsuit'})\, a(\nabla_x \eta_{x,\diamondsuit}). \quad (5\text{-}86)$$

The contribution F_{11} can be estimated by

$$|F_{11}| \le C^{k+n} \|\eta_t\|^{k+n-1} \|\nabla_x \eta_t\|^2 N^{-\alpha} \|(\mathcal{N}+1)^{\alpha/2} \xi\|^2, \quad (5\text{-}87)$$

where $\alpha = k_1 + p_1 + \cdots + p_r + k_2 + p'_1 + \cdots + p'_{r'}$ if r of the operators $\Lambda_1, \dots, \Lambda_{i_1}$ and r' of the operators $\Lambda'_1, \dots, \Lambda'_{i_2}$ are $\Pi^{(2)}$-operators of the form (5-81), with orders $p_1, \dots, p_r > 0$ and, respectively, $p'_1, \dots, p'_{r'} > 0$. Now observe that, since $\ell_2 = 1$, we must have $k \ge 1$. Since we are excluding here the case $n = 0,\ k = 1$, we must either have $n \ge 1$ and $k = 1$, or $k \ge 2$. In both cases $k + n \ge 2$. According to Lemma 3.2, the total number of η_t-kernels in every term of the form (5-80) is equal to $k + n + 1 \ge 3$. This implies that there is at least one η_t-kernel, in addition to the two η_t-kernels which produced the commutator $\|\nabla_x \eta_t\|^2$ in (5-85). We conclude that, in (5-87), we have $\alpha \ge 1$, and therefore, on $\mathcal{F}^{\le N}$,

$$|\mathrm{F}_{11}| \le C^{k+n} \|\eta_t\|^{k+n-1} \|\nabla_x \eta_t\|^2 N^{-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2 \le C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2$$

since $\|\nabla_x \eta_t\|^2 \le CN$ by Lemma 4.3. To control F_{12} we notice that, with the operator D defined in (5-73),

$$0 \le \int dx\, a^*(\nabla_x \eta_{x,\diamondsuit'})\, a(\nabla_x \eta_{x,\diamondsuit}) = d\Gamma(D) \le \|D\|_2 \mathcal{N} \le C\mathcal{N}.$$

This easily implies that

$$|\langle \xi, \mathrm{F}_{12} \xi \rangle| \le C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2.$$

We conclude that, in this case,

$$|\langle \xi, \mathrm{E} \xi \rangle| \le n C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2} \xi\|^2.$$

Case 6: $\ell_1 = 0,\ \ell_2 = 0$. In this case, the term (5-80) has the form

$$\mathrm{E} = \int dx\, \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_i}_{y_i} a^{\flat_i}_{x_{i+1}} a^{\sharp_n}_{y_n} a(\nabla_x \eta_{x,\diamondsuit}) \prod_{i=1}^{k_1} \eta^{(j_i)}(x_i;\, y_i)\, dx_i\, dy_i \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_i}_{y'_i} a^{\flat'_i}_{x'_{i+1}} a^{\sharp'_n}_{y'_n} \nabla_x a_x \prod_{i=1}^{k_2} \eta^{(m_i)}(x'_i;\, y'_i)\, dx'_i\, dy'_i. \quad (5\text{-}88)$$

Notice that a term of this form (with $n=0$ and $k=1$) already appears in the fourth line of (5-68) and was studied starting in (5-75) (to be more precise, in this case the first $\Pi^{(1)}$-operator in (5-80) is of order zero (for $n=0$, there is no other choice), and therefore the operator $a(\nabla_x \eta_{x,\Diamond})$ appearing in (5-88) is replaced by $b(\nabla_x \eta_{x,\Diamond})$). We will bound (5-88) following the same strategy used in (5-75). First we have to commute the operator $a(\nabla_x \eta_{x,\Diamond})$ in (5-88) to the right, close to the $\nabla_x a_x$-operator. As already explained in Case 5, the annihilation and creation operators produced while commuting $a(\nabla_x \eta_{x,\Diamond})$ to the right will have wave function with H^1-norm bounded, uniformly in N. Integrating by parts over x, we obtain $\mathrm{E} = \mathrm{G}_1 + \mathrm{G}_2$, with

$$\begin{aligned}\mathrm{G}_1 = \int dx\, \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i)\, dx_i\, dy_i \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i)\, dx'_i\, dy'_i\, a(\nabla_x \eta_{x,\Diamond}) \nabla_x a_x\end{aligned}$$

and

$$|\langle \xi, \mathrm{G}_2 \xi\rangle| \le n C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

To bound G_1, we proceed exactly as we did starting in (5-76). Using the decomposition $\eta_t = \mu_t + k_t$ and the fact that $\nabla_x \mu_t$ has bounded L^2-norm, uniformly in N, we conclude that $\mathrm{G}_1 = \mathrm{G}_{11} + \mathrm{G}_{12}$, with

$$\begin{aligned}\mathrm{G}_{11} = \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \int b^{\flat_0}_{x_1} \prod_{i=1}^{k_1} a^{\sharp_j}_{y_j} a^{\flat_j}_{x_{j+1}} a^{\sharp_n}_{y_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x_i; y_i)\, dx_i\, dy_i \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \int b^{\flat'_0}_{x'_1} \prod_{i=1}^{k_1} a^{\sharp'_j}_{y'_j} a^{\flat'_j}_{x'_{j+1}} a^{\sharp'_n}_{y'_n} \prod_{i=1}^{k_1} \eta^{(j_i)}_{t,\natural_i}(x'_i; y'_i)\, dx'_i\, dy'_i \int dx\, (-\Delta_x k_t)(x;y)\, a_x a_y\end{aligned}$$

and

$$|\langle \xi, \mathrm{G}_{12} \xi\rangle| \le C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\|\, \|\mathcal{K}^{1/2}\xi\|.$$

By Cauchy–Schwarz, the term G_{11} is bounded by

$$|\langle \xi, \mathrm{G}_{11}\xi\rangle| \le C^{k+n} \|\eta_t\|^{k+n-1} N^{-\alpha} \|(\mathcal{N}+1)^{\alpha}\xi\| \int dx\, dy\, |\Delta_x k_t(x;y)|\, \|a_x a_y \xi\|, \tag{5-89}$$

where $\alpha = k_1 + p_1 + \cdots + p_r + k_2 + p'_1 + \cdots + p'_{r'}$ if r of the operators $\Lambda_1, \ldots, \Lambda_{i_1}$ and r' of the operators $\Lambda'_1, \ldots, \Lambda'_{i_2}$ are $\Pi^{(2)}$-operators of the form (5-81), with orders $p_1, \ldots, p_r > 0$ and, respectively, $p'_1, \ldots, p'_r > 0$. The important observation now is that, since we excluded the case $k = n = 0$, we have $k+n \ge 1$, and therefore every term of the form (5-80) must have at least two η_t-kernels in it. This implies that, in (5-89), $\alpha \ge 1$, and therefore that

$$|\mathrm{G}_{11}| \le C^{k+n} \|\eta_t\|^{k+n-1} N^{-1/2} \|(\mathcal{N}+1)^{1/2}\xi\| \int dx\, dy\, |\Delta_x k_t(x;y)|\, \|a_x a_y \xi\|.$$

Proceeding as we did from (5-77) to (5-79), we conclude that

$$|\mathrm{G}_{11}| \le C^{k+n} \|\eta_t\|^{k+n-1} \|(\mathcal{N}+1)^{1/2}\xi\|\, \|(\mathcal{H}_N + \mathcal{N}+1)^{1/2}\xi\|.$$

Summarizing, we proved that the last term on the right-hand side of (5-68) is a sum over all $(k,n)\neq(0,0),(1,0)$ of $2^{n+k}n!\,k!$ terms of the form (5-80), each of them having expectation bounded by

$$|\langle\xi,\mathrm{E}\xi\rangle|\le C^{k+n}e^{c|t|}\|\eta_t\|^{\max(0,k+n-3)}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|.$$

Similarly, one can show that

$$\begin{aligned}|\langle\xi,[\mathcal{N},\mathrm{E}]\xi\rangle|&\le C^{k+n}e^{c|t|}\|\eta_t\|^{\max(0,k+n-3)}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle\xi,[a^*(g_1)a(g_2),\mathrm{E}]\xi\rangle|&\le C^{k+n}e^{c|t|}\|\eta_t\|^{\max(0,k+n-3)}\|g_1\|_{H^1}\|g_2\|_{H^1}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle\xi,\partial_t[\mathrm{E}]\xi\rangle|&\le C^{k+n}e^{c|t|}\|\eta_t\|^{\max(0,k+n-3)}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|.\end{aligned}$$

Inserting in (5-68) we conclude that, if $\sup_{t\in\mathbb{R}}\|\eta_t\|$ is small enough, the operator $\mathcal{E}^{(K)}_{N,t}$ defined in (5-65) satisfies the bounds in (5-66). □

5D2. *Analysis of* $e^{-B(\eta_t)}(\mathcal{L}^{(2)}_{N,t}-\mathcal{K})e^{B(\eta_t)}$. Recall that

$$\begin{aligned}\mathcal{L}^{(2)}_{N,t}-\mathcal{K}&=\int dx\,(N^3V(N\cdot)*|\tilde\varphi_t|^2)(x)[b^*_xb_x-N^{-1}a^*_xa_x]\\&\quad+\int dx\,dy\,N^3V(N(x-y))\tilde\varphi_t(x)\bar{\tilde\varphi}_t(y)[b^*_xb_y-N^{-1}a^*_xa_y]\\&\quad+\frac12\int dx\,dy\,N^3V(N(x-y))[\tilde\varphi_t(x)\tilde\varphi_t(y)b^*_xb^*_y+\text{h.c.}].\end{aligned}\tag{5-90}$$

We define the error term $\mathcal{E}^{(2)}_{N,t}$ through the equation

$$\begin{aligned}e^{-B(\eta_t)}(\mathcal{L}^{(2)}_{N,t}-\mathcal{K})e^{B(\eta_t)}&=\mathrm{Re}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde\varphi}_t(x)\bar{\tilde\varphi}_t(y)k_t(y;x)\\&\quad+\frac12\int dx\,dy\,N^3V(N(x-y))[\tilde\varphi_t(x)\tilde\varphi_t(y)b^*_xb^*_y+\text{h.c.}]+\mathcal{E}^{(2)}_{N,t}.\end{aligned}\tag{5-91}$$

The properties of the error term $\mathcal{E}^{(2)}_{N,t}$ are described in the next proposition.

Proposition 5.9. *Under the same assumptions as in Theorem 4.4, there exist constants* $C,c>0$ *such that*

$$\begin{aligned}|\langle\xi,\mathcal{E}^{(2)}_{N,t}\xi\rangle|&\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle\xi,[\mathcal{N},\mathcal{E}^{(2)}_{N,t}]\xi\rangle|&\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle\xi,[a^*(g_1)a(g_2),\mathcal{E}^{(2)}_{N,t}]\xi\rangle|&\le Ce^{c|t|}\|g_1\|_{H^2}\|g_2\|_{H^2}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N^{1/2}+\mathcal{N}+1)^{1/2}\xi\|,\\ |\partial_t\langle\xi,\mathcal{E}^{(2)}_{N,t}\xi\rangle|&\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N^{1/2}+\mathcal{N}+1)^{1/2}\xi\|\end{aligned}\tag{5-92}$$

for all $\xi\in\mathcal{F}^{\le N}$.

Proof. The conjugation of the first two terms on the right-hand side of (5-90) can be controlled with Lemma 5.6, taking r to be the multiplication operator with the convolution $N^3V(N\cdot)*|\tilde\varphi_t|^2$ in the first case (so that $\|r\|_{\text{op}}=\|N^3V(N\cdot)*|\tilde\varphi_t|^2\|_\infty\le C\|\tilde\varphi_t\|^2_\infty\le Ce^{c|t|}$) and the operator with integral kernel $r(x;y)=N^3V(N(x-y))\tilde\varphi_t(x)\tilde\varphi_t(y)$ in the second case (then $\|r\|_{\text{op}}\le\sup_x\int|r(x;y)|\,dy\le Ce^{c|t|}$, uniformly in N). Hence, to show Proposition 5.9 it is enough to prove the bounds (5-92), with $\mathcal{E}^{(2)}_{N,t}$ replaced by

$$\begin{aligned}\widetilde{\mathcal{E}}^{(2)}_{N,t} = \frac{1}{2}\int dx\,dy\,N^3V(N(x-y))[\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)e^{-B(\eta_t)}b_xb_ye^{B(\eta_t)}+\text{h.c.}]\\ -\operatorname{Re}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)k_t(x;y)\\ -\frac{1}{2}\int dx\,dy\,N^3V(N(x-y))[\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)b_xb_y+\text{h.c.}].\end{aligned} \tag{5-93}$$

By Lemma 3.3, we can write

$$\begin{aligned}&\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)e^{-B(\eta_t)}b_xb_ye^{B(\eta_t)}\\ &\quad=\sum_{n,k\geq 0}\frac{(-1)^{k+n}}{k!\,n!}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\operatorname{ad}^{(n)}_{B(\eta_t)}(b_x)\operatorname{ad}^{(k)}_{B(\eta_t)}(b_y)\\ &\quad=\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)b_xb_y\\ &\qquad-\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)b_x[B(\eta_t),b_y]\\ &\qquad+\sum_{n,k}^{*}\frac{(-1)^{k+n}}{k!\,n!}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\operatorname{ad}^{(n)}_{B(\eta_t)}(b_x)\operatorname{ad}^{(k)}_{B(\eta_t)}(b_y),\end{aligned} \tag{5-94}$$

where we isolated the terms with $(n,k)=(0,0)$ and $(n,k)=(0,1)$ and the sum $\sum^{*}$ runs over all other pairs $(n,k)\in\mathbb{N}\times\mathbb{N}$. The first term on the right-hand side of (5-94) (the one associated with $(k,n)=(0,0)$) is subtracted in (5-93) and does not enter the error term $\widetilde{\mathcal{E}}^{(2)}_{N,t}$. The second term on the right-hand side of (5-94), on the other hand, is given by

$$\begin{aligned}\mathrm{P}&:=-\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)b_x[B(\eta_t),b_y]\\ &=\frac{N-1-\mathcal{N}}{N}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\,b_xb^*(\eta_y)\\ &\qquad-\frac{1}{N}\int dx\,dy\,dw\,dz\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\,\eta_t(z;w)\,b_xb^*_za^*_wa_y.\end{aligned}$$

Commuting in both terms the annihilation field b_x to the right, we find

$$\begin{aligned}\mathrm{P}&=\frac{N-1-\mathcal{N}}{N}\frac{N-\mathcal{N}}{N}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\,\eta_t(x;y)\\ &\quad+\frac{N-1-\mathcal{N}}{N}\int dx\,dy\,dz\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\left[b^*(\eta_y)b_x-\frac{1}{N}a^*(\eta_y)\,a_x\right]\\ &\quad-2\frac{N-\mathcal{N}}{N^2}\int dx\,dy\,dz\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\,a^*(\eta_y)\,a_x\\ &\quad-\frac{N-\mathcal{N}}{N^2}\int dx\,dy\,dz\,dw\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)\,\eta_t(z;w)\,a^*_wa^*_za_xa_y\\ &=:\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3+\mathrm{P}_4.\end{aligned} \tag{5-95}$$

Writing $\eta_t = k_t + \mu_t$, and using the pointwise bounds $|\mu_t(x;y)| \le C|\tilde{\varphi}_t(x)||\tilde{\varphi}_t(y)|$ and $|k_t(x;y)| \le CN|\tilde{\varphi}_t(x)||\tilde{\varphi}_t(y)|$ from Lemma 4.3, we obtain that

$$\left| \langle \xi, \mathrm{P}_1 \xi \rangle - \int dx\, dy\, N^3 V(N(x-y)) \bar{\tilde{\varphi}}_t(x) \bar{\tilde{\varphi}}_t(y) k_t(x;y) \right| \le C \|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

The expectation of the operator P_2, and analogously the expectation of the operator P_3, can be bounded by

$$\begin{aligned} &|\langle \xi, \mathrm{P}_2 \xi \rangle| \\ &\quad \le \|(\mathcal{N}+1)^{1/2}\xi\| \int dx\, dy\, N^3 V(N(x-y)) |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)| \, \|\eta_y\| \, \|b_x \xi\| \\ &\quad \le \|\tilde{\varphi}_t\|_\infty^2 \, \|(\mathcal{N}+1)^{1/2}\xi\| \left[\int dx\, dy\, N^3 V(N(x-y)) \|\eta_y\|^2 \right]^{1/2} \left[\int dx\, dy\, N^3 V(N(x-y)) \|b_x \xi\|^2 \right]^{1/2} \\ &\quad \le C e^{c|t|} \|\eta_t\| \, \|(\mathcal{N}+1)^{1/2}\xi\|^2. \end{aligned}$$

As for the last term on the right-hand side of (5-95), its expectation is estimated by

$$\begin{aligned} &|\langle \xi, \mathrm{P}_3 \xi \rangle| \\ &\le \|\eta_t\| \, \|(\mathcal{N}+1)\xi\| \int dx\, dy\, N^2 V(N(x-y)) |\tilde{\varphi}_t(x)| |\tilde{\varphi}_t(y)| \, \|a_x a_y \xi\| \\ &\le \|\eta_t\| \, \|(\mathcal{N}+1)\xi\| \left[\int dx\, dy\, N^2 V(N(x-y)) \|a_x a_y \xi\|^2 \right]^{1/2} \left[\int dx\, dy\, N^2 V(N(x-y)) |\tilde{\varphi}_t(x)|^2 |\tilde{\varphi}_t(y)|^2 \right]^{1/2} \\ &\le C \|\eta_t\| \, \|(\mathcal{N}+1)^{1/2}\xi\| \, \|\mathcal{V}_N^{1/2}\xi\|. \end{aligned}$$

We conclude that

$$\begin{aligned} &\left| \langle \xi, \mathrm{P}\xi \rangle - \int dx\, dy\, N^3 V(N(x-y)) \bar{\tilde{\varphi}}_t(x) \bar{\tilde{\varphi}}_t(y) k_t(x;y) \right| \\ &\qquad\qquad \le C e^{c|t|} \|\eta_t\| \, \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N + \mathcal{N} + 1)^{1/2}\xi\|. \end{aligned} \tag{5-96}$$

Let us now consider the terms in the sum on the last line of (5-94), where we excluded the pairs $(k,n) = (0,0)$ and $(k,n) = (0,1)$. By Lemma 3.2, the operator

$$\int dx\, dy\, N^3 V(N(x-y)) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \mathrm{ad}^{(n)}_{B(\eta_t)}(b_x) \mathrm{ad}^{(k)}_{B(\eta_t)}(b_y) \tag{5-97}$$

can be expressed as the sum of $2^{n+k} n!\, k!$ terms having the form

$$\begin{aligned} \mathrm{E} = \int dx\, dy\, N^3 V(N(x-y)) \tilde{\varphi}_t(x) \tilde{\varphi}_t(y) \Lambda_1 \cdots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1}, \ldots, \eta^{(j_{k_1})}_{t,\natural_{k_1}}; \eta^{(\ell_1)}_{x,\diamondsuit}) \\ \times \Lambda'_1 \cdots \Lambda'_{i_2} N^{-k_2} \Pi^{(2)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1}, \ldots, \eta^{(m_{k_2})}_{t,\natural'_{k_2}}; \eta^{(\ell_2)}_{y,\diamondsuit'}), \end{aligned} \tag{5-98}$$

where $k_1, k_2, i_1, i_2 \ge 0$, $j_1, \ldots, j_{k_1}, m_1, \ldots, m_{k_2} > 0$ and where each Λ_i, Λ'_i is a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\, \Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1}, \ldots, \eta^{(q_p)}_{t,\underline{\natural}_p}). \tag{5-99}$$

With Lemma 5.4, we obtain

$$\begin{aligned}
&|\langle \xi, \mathrm{E}\xi\rangle| \\
&\quad\le \|(\mathcal{N}+1)^{1/2}\xi\| \int dx\, dy\, N^3 V(N(x-y))|\tilde{\varphi}_t(x)|\,|\tilde{\varphi}_t(y)| \\
&\qquad\qquad \times \|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1} N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{x,\diamondsuit}) \\
&\qquad\qquad\qquad\qquad \times \Lambda'_1\cdots\Lambda'_{i_2} N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{y,\diamondsuit'})\xi\| \\
&\quad\le C^{k+n}\|\eta_t\|^{n+k-2}\|(\mathcal{N}+1)^{1/2}\xi\| \int dx\, dy\, N^3 V(N(x-y))|\tilde{\varphi}_t(x)|\,|\tilde{\varphi}_t(y)| \\
&\qquad\qquad \times \big\{ n\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|\,\|\eta_y\|\,\|a_x\xi\| \\
&\qquad\qquad\qquad\qquad + Ce^{c|t|}\|\eta_t\|\,\|(\mathcal{N}+1)^{1/2}\xi\| + N^{-1/2}\|\eta_t\|^2\,\|a_x a_y\xi\| \big\},
\end{aligned}$$

where (in the last term in the braces) we used the pointwise bound

$$N^{-1}|\eta_t(x;y)| \le Ce^{c|t|}$$

from Lemma 4.3. The contribution of the first three terms in the braces can be bounded by Cauchy–Schwarz, since $\|\tilde{\varphi}_t\|_\infty \le Ce^{c|t|}$. We find

$$|\langle \xi, \mathrm{E}\xi\rangle| \le C^{k+n} n e^{c|t|}\|\eta_t\|^{k+n-1}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|.$$

Since the expectation of (5-97) is the sum of $2^{n+k}k!\,n!$ such contributions, inserting in (5-94) and taking into account also (5-96), we conclude that

$$|\langle \xi, \widetilde{\mathcal{E}}^{(2)}_{N,t}\xi\rangle| \le Ce^{c|t|}\,\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough. As usual, we can prove similarly that the same bounds hold true for the expectation of the commutators of $\widetilde{\mathcal{E}}^{(2)}_{N,t}$ with the number of particles operator $\mathcal{N}$ and with $a^*(g_1)a(g_2)$, for arbitrary $g_1, g_2 \in H^2(\mathbb{R}^3)$ (this assumption allows us to extract $\|g_j\|_\infty \le C\|g_j\|_{H^2}$) and also for the time-derivative of $\widetilde{\mathcal{E}}^{(2)}_{N,t}$. □

5E. *Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)}$.* Recall from (5-3) that

$$\mathcal{L}^{(3)}_{N,t} = \int dx\, dy\, N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)[b^*_x a^*_y a_x + \text{h.c.}].$$

We conjugate $\mathcal{L}^{(3)}_{N,t}$ with the unitary operator $e^{B(\eta_t)}$. We define the error term $\mathcal{E}^{(3)}_{N,t}$ through the equation

$$e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)} = -\sqrt{N}\,[b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(\bar{h}_{N,t})) + \text{h.c.}] + \mathcal{E}^{(3)}_{N,t}, \tag{5-100}$$

where we recall, from (5-47) that,

$$h_{N,t} = (N^3V(N\,\cdot\,)w_\ell(N\,\cdot\,) * |\tilde{\varphi}_t|^2)\tilde{\varphi}_t.$$

In the next proposition we collect the important properties of the error term $\mathcal{E}^{(3)}_{N,t}$.

Proposition 5.10. *Under the same assumptions as in Theorem 4.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} |\langle \xi, \mathcal{E}^{(3)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|, \\ |\langle \xi, [\mathcal{N}, \mathcal{E}^{(3)}_{N,t}]\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|, \\ |\langle \xi, [a^*(g_1)a(g_2), \mathcal{E}^{(3)}_{N,t}]\xi\rangle| &\le Ce^{c|t|}\|g_1\|_{H^2}\|g_2\|_{H^2}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|, \\ |\partial_t\langle \xi, \mathcal{E}^{(3)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \end{aligned} \tag{5-101}$$

for all $\xi \in \mathcal{F}^{\le N}$.

Proof. We start by writing

$$\begin{aligned} e^{-B(\eta_t)}a^*_y a_x e^{B(\eta_t)} &= a^*_y a_x + \int_0^1 ds\, e^{-sB(\eta_t)}[a^*_y a_x, B(\eta_t)]e^{sB(\eta_t)} \\ &= a^*_y a_x + \int_0^1 e^{-sB(\eta_t)}[b^*_y b^*(\eta_x) + b(\eta_y)b_x]e^{sB(\eta_t)}. \end{aligned}$$

From Lemma 3.3, we conclude that

$$e^{-B(\eta_t)}a^*_y a_x e^{B(\eta_t)} = a^*_y a_x + \sum_{k,r\ge 0}\frac{(-1)^{k+r}}{k!\,r!\,(k+r+1)}[\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) + \mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)].$$

Inserting in the expression for $\mathcal{L}^{(3)}_{N,t}$, we conclude that

$$\begin{aligned} &e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)} \\ &\quad= \sum_{n\ge 0}\frac{(-1)^n}{n!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)a^*_y a_x \\ &\qquad + \sum_{n,k,r\ge 0}\frac{(-1)^{n+k+r}}{n!\,k!\,r!\,(k+r+1)}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x) \\ &\qquad\qquad \times [\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) + \mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)] + \text{h.c.} \end{aligned}$$

We divide the triple sum into several parts. We find

$$\begin{aligned} &e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)} \\ &\quad= \sum_{n\ge 0}\frac{(-1)^n}{n!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)a^*_y a_x \\ &\qquad + \sum_{n,r\ge 0}\frac{(-1)^{n+r}}{n!\,(r+1)!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &\qquad + \sum_{n,r\ge 0,k\ge 1}\frac{(-1)^{n+k+r}}{n!\,k!\,r!\,(k+r+1)}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &\qquad + \sum_{n,r\ge 0,k\ge 1}\frac{(-1)^{n+k+r}}{n!\,k!\,r!\,(k+r+1)}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x) \\ &\qquad + \text{h.c.} \end{aligned}$$

In the terms with $k=0$, we distinguish furthermore the case $n=1$ from $n\neq 1$. We find

$$\begin{aligned}
&e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)}\\
&=-\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)[B(\eta_t),b^*_x]a^*_y a_x\\
&\quad-\sum_{r\geq 0}\frac{(-1)^r}{(r+1)!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)[B(\eta_t),b^*_x]b^*_y\,\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\\
&\quad+\sum_{n\neq 1}\frac{(-1)^n}{n!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\,a^*_y a_x\\
&\quad+\sum_{n\neq 1,r\geq 0}\frac{(-1)^{n+r}}{n!\,(r+1)!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\\
&\quad+\sum_{n,r\geq 0,k\geq 1}\frac{(-1)^{n+k+r}}{n!k!r!\,(k+r+1)}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\\
&\quad+\sum_{n,r\geq 0,k\geq 1}\frac{(-1)^{n+k+r}}{n!k!r!\,(k+r+1)}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\\
&\quad+\text{h.c.}
\end{aligned}\tag{5-102}$$

We start by estimating the contribution of the last term on the right-hand side of (5-102). We are interested in the expectation

$$\begin{aligned}
&\left|\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\,\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\rangle\right|\\
&\qquad\leq\int dx\,dy\,N^{5/2}V(N(x-y))|\tilde{\varphi}_t(y)|\,\|\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\,\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\|
\end{aligned}$$

for $n,r\geq 0$ and $k\geq 1$. According to Lemma 3.3, the norm $\|\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^n n!$ terms of the form

$$\mathrm{P}_{1,x}=\|\Lambda_1\cdots\Lambda_i N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_k)}_{t,\natural_k};\eta^{(s)}_{x,\diamond})\xi\|$$

for $i,k,s\geq 0$, $j_1,\dots,j_k\geq 1$, where each Λ_i is a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\sharp',\flat'}(\eta^{(q_1)}_{t,\natural'_1},\dots,\eta^{(q_p)}_{t,\natural'_p}).\tag{5-103}$$

From Lemma 5.2, we find

$$\mathrm{P}_{1,x}\leq\begin{cases}C^n\|\eta_t\|^{n-1}\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } s\geq 1,\\ C^n\|\eta_t\|^n\,\|a_x\xi\| & \text{if } s=0\end{cases}\tag{5-104}$$

for all $x\in\mathbb{R}^3$. Similarly, the norm $\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{k+r}k!\,r!$ terms having the form

$$\mathrm{P}_{2,x,y}=\|\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1+1)}_{y,\diamond})\times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamond'})\xi\|,$$

which can be estimated (with Lemma 5.4) by

$$\mathrm{P}_{2,x,y} \leq \begin{cases} C^{k+r}\|\eta_t\|^{k+r-2}\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)\xi\| & \text{if } \ell_2 \geq 1, \\ C^{k+r}\|\eta_t\|^{k+r-1}\|\eta_y\|\,\|a_x(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_2 = 0 \end{cases}$$

for all $x, y \in \mathbb{R}^3$. Combining this estimate with (5-104) we find that

$$\begin{aligned}
&\left|\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\,\langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\rangle\right| \\
&\quad\leq n!\,k!\,r!\,C^{n+k+r}\|\eta_t\|^{n+k+r-3}\int dx\,dy N^{5/2}V(N(x-y))|\tilde{\varphi}_t(y)|\,\|\eta_y\| \\
&\qquad\qquad \times\big[\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|\,\|a_x\xi\|\big] \\
&\qquad\qquad\qquad \times\big[\|\eta_x\|\,\|(\mathcal{N}+1)\xi\| + \|\eta_t\|\,\|a_x(\mathcal{N}+1)^{1/2}\xi\|\big] \\
&\quad\leq n!\,k!\,r!\,C^{n+k+r}e^{c|t|}N^{-1/2}\|\eta_t\|^{n+k+r-3} \\
&\qquad\times\int dx\,\big[\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|\|a_x\xi\|\big]\big[\|\eta_x\|\,\|(\mathcal{N}+1)\xi\| + \|\eta_t\|\,\|a_x(\mathcal{N}+1)^{1/2}\xi\|\big],
\end{aligned}$$

where we first used the bounds $\|\tilde{\varphi}_t\|_\infty \leq Ce^{c|t|}$ from Proposition 4.2 and $\sup_y \|\eta_y\| \leq Ce^{c|t|}$ from Lemma 4.3, and then we integrated over y to obtain the $N^{-1/2}$ factor. Applying Cauchy–Schwarz in the x-integral, we conclude that

$$\begin{aligned}
&\left|\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\,\langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b(\eta_y))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_x)\xi\rangle\right| \\
&\qquad\leq n!\,k!\,r!\,C^{n+k+r}e^{c|t|}N^{-1/2}\|\eta_t\|^{n+k+r-1}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)\xi\| \\
&\qquad\leq n!\,k!\,r!\,C^{n+k+r}e^{c|t|}\|\eta_t\|^{n+k+r-1}\|(\mathcal{N}+1)^{1/2}\xi\|^2
\end{aligned} \tag{5-105}$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Let us now consider the fifth sum on the right-hand side of (5-102). The expectation of every term in this sum is bounded by

$$\begin{aligned}
&\left|\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\rangle\right| \\
&\qquad\leq \int dx\,dy\,N^{5/2}V(N(x-y))|\tilde{\varphi}_t(y)|\,\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\,\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|,
\end{aligned} \tag{5-106}$$

where we assume $k \geq 1,\ n, r \geq 0$. According to Lemma 3.2, $\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$ is bounded by the sum of $2^r r!$ terms of the form

$$\mathrm{Q}_{1,x} = \|\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1+1)}_x)\,\xi\|$$

for a $i_1, k_1, \ell_1 \geq 0$ and $j_1, \ldots, j_{k_1} \geq 1$. Each Λ_i is a factor $(N-\mathcal{N})/N$, a factor $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form (5-103). From Lemma 5.2, we have

$$\mathrm{Q}_{1,x} \leq C^r\|\eta_t\|^r\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\|$$

for all $x \in \mathbb{R}^3$. On the other hand, using again Lemma 3.2, we can bound the norm $\|\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\,\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ by the sum of $2^{n+k}k!\,n!$ terms having the form

$$\mathrm{Q}_{2,x,y} = \|\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_2}};\eta^{(\ell_1)}_{y,\diamondsuit})\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\ldots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\|,$$

where $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$ and $j_1,\ldots,j_{k_1}, m_1,\ldots,m_{k_2} \geq 1$ and where each Λ_i- and Λ'_i-operator is a factor $(N-\mathcal{N})/N$, a factor $(N-\mathcal{N}+1)/N$ or a $\Pi^{(2)}$-operator of the form (5-103). Using Lemma 5.4, we obtain (using the assumption $k \geq 1$ to apply (5-27) and using (5-28) with $\alpha = 1$)

$$\mathrm{Q}_{2,x,y} \leq C^{n+k}\|\eta_t\|^{n+k-2}\big\{\big[(n+1)\|\eta_x\|\,\|\eta_y\| + \|\eta_t\|N^{-1}|\eta_t(x;y)|\big]\|(\mathcal{N}+1)\xi\| \\ + \|\eta_y\|\,\|\eta_t\|\,\|a_x(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|^2\,\|a_x a_y\xi\|\big\}$$

for all $x, y \in \mathbb{R}^3$. With the bound $\sup_x \|\eta_x\|,\ \sup_{x,y} N^{-1}|\eta_t(x;y)| \leq Ce^{c|t|}$ from Lemma 4.3, we conclude that

$$\bigg|\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(r)}_{B(\eta_t)}(\eta_x)\xi\rangle\bigg| \\ \leq n!\,k!\,r!\,C^{n+k+r}e^{c|t|}\|\eta_t\|^{n+k+r}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\| \quad (5\text{-}107)$$

for all $\xi \in \mathcal{F}^{\leq N}$.

Let us now study the fourth term on the right-hand side of (5-102). As we did for the other terms, we bound the expectation

$$\bigg|\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)\langle\xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)b^*_y\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\rangle\bigg| \\ \leq \int dx\,dy\,N^{5/2}V(N(x-y))|\tilde{\varphi}_t(y)|\,\|b_y\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\,\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|, \quad (5\text{-}108)$$

where we assume that $n \neq 1,\ r \geq 0$. According to Lemma 3.2, $\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$ can be bounded by the sum of $2^r r!$ terms of the form

$$\mathrm{R}_{1,x} = \|\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1+1)}_{x,\diamondsuit})\xi\|$$

for $i_1, k_1, \ell_1 \geq 0$ and $j_1,\ldots,j_{k_1} \geq 1$. According to Lemma 5.2, such a term can always be estimated by

$$\mathrm{R}_{1,x} \leq C^r\|\eta_t\|^r\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| \quad (5\text{-}109)$$

for all $x \in \mathbb{R}^3$. On the other hand, the norm $\|b_y\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ can be bounded by the sum of $2^n n!$ contributions having the form

$$\mathrm{R}_{2,x,y} = \|b_y\Lambda_1\cdots\Lambda_{i_1}\Pi^{(k_1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{x,\diamondsuit})\xi\| \quad (5\text{-}110)$$

for $i_1, k_1, \ell_1 \geq 0$ and $j_1,\ldots,j_{k_1} \geq 1$. With Lemma 5.4, we find that

$$\mathrm{R}_{2,x,y} \leq C^n\|\eta_t\|^{n-2}\big\{\big[(1+n/N)\|\eta_x\|\,\|\eta_y\| + \|\eta_t\|N^{-1}|\eta_t(x;y)|\big]\|(\mathcal{N}+1)\xi\| \\ + \|\eta_t\|\,\|\eta_x\|\,\|a_y(\mathcal{N}+1)^{1/2}\xi\| + (n/N)\|\eta_t\|\,\|\eta_y\|\,\|a_x(\mathcal{N}+1)^{1/2}\xi\| + \|\eta_t\|^2\,\|a_x a_y\xi\|\big\}$$

for all $x, y \in \mathbb{R}^3$. With $\|\tilde{\varphi}_t\|_\infty \le Ce^{c|t|}$ and $\sup_{x,y} N^{-1}|\eta_t(x; y)| \le Ce^{c|t|}$ we conclude, similarly to (5-107), that

$$\left| \int dx\,dy\, N^{5/2} V(N(x-y))\tilde{\varphi}_t(y) \langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x) b^*_y \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\rangle \right| \\ \le (n+1)!\, r!\, C^{n+r} e^{c|t|} \|\eta_t\|^{r+n} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|. \quad (5\text{-}111)$$

The expectation of terms in the third sum on the right-hand side of (5-102) is bounded by

$$\left| \int dx\,dy\, N^{5/2} V(N(x-y))\tilde{\varphi}_t(y) \langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x) a^*_y a_x \xi\rangle \right| \\ \le \int dx\,dy\, N^{5/2} V(N(x-y)) |\tilde{\varphi}_t(y)| \, \|a_y \mathrm{ad}^{(n)}_{B(\eta_t)} \xi\| \, \|a_x \xi\|,$$

which is similar to the right-hand side of (5-108), the only difference being that instead of the norm $\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|$ we have $\|a_x\xi\|$ (and the fact that in the other norm, we have the field a_y instead of b_y; it is clear, however, that both fields can be treated similarly). Analogously to (5-111), we conclude that

$$\left| \int dx\,dy\, N^{5/2} V(N(x-y))\tilde{\varphi}_t(y) \langle \xi, \mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x) a^*_y a_x \xi\rangle \right| \\ \le (n+1)!\, C^{n} e^{c|t|} \|\eta_t\|^{n-1} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|. \quad (5\text{-}112)$$

Let us now switch to the second term on the right-hand side of (5-102) (the sum over $r \ge 0$). First of all, we compute the commutator

$$[B(\eta_t), b^*_x] = -b(\eta_x)\left(1 - \frac{\mathcal{N}}{N}\right) + \frac{1}{N}\int dz\,dw\, \bar{\eta}(z; w) a^*_x a_w b_z.$$

Hence the r-th term in the sum is proportional to

$$-\int dx\,dy\, N^{5/2} V(N(x-y))\tilde{\varphi}_t(y) \frac{(N-1-\mathcal{N})}{N} b(\eta_x) b^*_y \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ + \int dx\,dy\, N^{5/2} V(N(x-y))\tilde{\varphi}_t(y) N^{-1} \Pi^{(1)}_{(*,\cdot),*}(\eta_t, \delta_x)^* b^*_y \mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ =: \mathrm{S}_1 + \mathrm{S}_2. \quad (5\text{-}113)$$

The expectation of S_2 can be bounded as follows:

$$|\langle \xi, \mathrm{S}_2 \xi\rangle| \le \int dx\,dy\, N^{5/2} V(N(x-y)) |\tilde{\varphi}_t(y)| \, \|b_y N^{-1} \Pi^{(1)}_{(*,\cdot),*}(\eta_t, \delta_x)\xi\| \, \|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\|.$$

As in (5-109), we find

$$\|\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))\xi\| \le C^r r! \|\eta_t\|^r \|\eta_x\| \, \|(\mathcal{N}+1)^{1/2}\xi\|.$$

Since, on the other hand,

$$\|b_y N^{-1} \Pi^{(1)}_{(*,\cdot),*}(\eta_t, \delta_x)\xi\| \le C N^{-1} \|\eta_y\| \, \|a_x (\mathcal{N}+1)^{1/2}\xi\| + C\|\eta_t\| \, \|a_x a_y \xi\|,$$

we conclude that

$$|\langle \xi, \mathrm{S}_2 \xi\rangle| \le C^r e^{c|t|} \|\eta_t\|^{r+1} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

for all $\xi \in \mathcal{F}^{\le N}$. We are left with the operator S_1 defined in (5-113). Commuting $b(\eta_x)$ with b_y^* we write it as

$$\begin{aligned}\mathrm{S}_1 &= -\int dx\,dy\, N^{5/2} V(N(x-y))\eta_t(x;y)\tilde{\varphi}_t(y)\frac{(N-\mathcal{N})(N-\mathcal{N}-1)}{N^2}\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &\quad -\int dx\,dy\, N^{5/2} V(N(x-y))\tilde{\varphi}_t(y)\frac{(N-\mathcal{N}-1)}{N}\left[b_y^* b(\eta_x) - \frac{1}{N}a_y^* a(\eta_x)\right]\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &=: \mathrm{S}_{11}+\mathrm{S}_{12}.\end{aligned}$$

The expectation of S_{12} is estimated by

$$|\langle \xi, \mathrm{S}_{12}\xi\rangle| \le C^r e^{c|t|}\|\eta_t\|^{r+1}\|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

As for S_{11}, we decompose it as

$$\begin{aligned}\mathrm{S}_{11} &= -\int dx\,dy\, N^{5/2} V(N(x-y))k_t(x;y)\tilde{\varphi}_t(y)\frac{(N-\mathcal{N})(N-\mathcal{N}-1)}{N^2}\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)) \\ &\quad -\int dx\,dy\, N^{5/2} V(N(x-y))\mu_t(x;y)\tilde{\varphi}_t(y)\frac{(N-\mathcal{N})(N-\mathcal{N}-1)}{N^2}\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x)), \\ &=: \mathrm{S}_{111}+\mathrm{S}_{112}.\end{aligned}$$

Since $|\mu_t(x;y)| \le Ce^{c|t|}$ from Lemma 4.3, it is easy to estimate the expectation of the term S_{112} by

$$|\langle \xi, \mathrm{S}_{112}\xi\rangle| \le C^r e^{c|t|}\|\eta_t\|^{r+1}\|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

As for the term S_{111}, we use the fact that, by Lemma 3.2, the nested commutator $\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))$ is given by

$$\left(1-\frac{\mathcal{N}-1}{N}\right)^m \left(1-\frac{\mathcal{N}-2}{N}\right)^m b^*((\eta_t\bar{\eta}_t)^m\eta_x)$$

if $r=2m$ is even and by

$$-\left(1-\frac{\mathcal{N}+1}{N}\right)^{m+1}\left(1-\frac{\mathcal{N}}{N}\right)^m b((\eta_t\bar{\eta}_t)^{m+1}_x)$$

if $r=2m+1$ is odd, up to terms ($2^r r!-1$ of them) having the form

$$\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1+1)}_{x,\diamond}),$$

where either $k_1 \ge 1$ or at least one of the Λ-operators is a $\Pi^{(2)}$-operator of the form (5-103). We conclude that, if $r=2m$ is even,

$$\mathrm{S}_{111} = \sqrt{N}\int dx\,dy\, N^3 V(N(x-y))w_\ell(N(x-y))|\tilde{\varphi}_t(y)|^2\tilde{\varphi}_t(x)b^*((\eta_t\bar{\eta}_t)^m\eta_x)+\mathrm{S}_{1112}, \qquad \text{(5-114)}$$

while, if $r = 2m+1$ is odd,

$$\mathrm{S}_{111} = -\sqrt{N}\int dx\,dy\,N^3V(N(x-y))w_\ell(N(x-y))|\tilde{\varphi}_t(y)|^2\tilde{\varphi}_t(x)b^*((\eta_t\bar{\eta}_t)_x^{m+1}) + \mathrm{S}_{1112}, \tag{5-115}$$

where, in both cases, the expectation of the error term S_{1112} is bounded by

$$\begin{aligned}|\langle\xi, \mathrm{S}_{1112}\xi\rangle| &\le C^r\|\eta_t\|^r\int dx\,dy\,N^{3/2}V(N(x-y))\,|k_t(x;y)|\,\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)\xi\|\\ &\le C^r\|\eta_t\|^{r+1}\|(\mathcal{N}+1)^{1/2}\xi\|^2\end{aligned}$$

for all $\xi\in\mathcal{F}^{\le N}$. Here, once again, we used the fact that $N^{-1}|\eta_t(x;y)|\le C$. Summing over all $r\ge 0$, we conclude that

$$\begin{aligned}-\sum_{r\ge 0}\frac{(-1)^r}{(r+1)!}\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)[B(\eta_t), b_x^*]b_y^*\mathrm{ad}^{(r)}_{B(\eta_t)}(b^*(\eta_x))&\\ = -\sqrt{N}\,[b((\cosh_{\eta_t}-1)(h_{N,t})) + b^*(\sinh_{\eta_t}(h_{N,t}))] + \mathrm{S},&\end{aligned}$$

where

$$|\langle\xi, S\xi\rangle| \le e^{c|t|}\sum_{r\ge 0}(C\|\eta_t\|)^r\|(\mathcal{N}+1)^{1/2}\xi\|^2 \le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2 \tag{5-116}$$

for all $\xi\in\mathcal{F}^{\le N}$.

Finally, we consider the first term on the right-hand side of (5-102). This term can be handled much as we did with the second term (the sum over $r\ge 0$). We obtain that

$$-\int dx\,dy\,N^{5/2}V(N(x-y))\tilde{\varphi}_t(y)[B(\eta_t), b_x^*]a_y^*a_x = -\sqrt{N}b(h_{N,t}) + \widetilde{S},$$

where the expectation of $\widetilde{S}$ can be bounded as we did with the expectation of S in (5-116).

Recalling the definition of $\mathcal{E}^{(3)}_{N,t}$ in (5-100), it follows from (5-105), (5-107), (5-111), (5-112) and (5-116) that

$$|\langle\xi, \mathcal{E}^{(3)}_{N,t}\xi\rangle| \le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|.$$

The bounds in (5-101) for the expectations of $[\mathcal{N}, \mathcal{E}^{(3)}_{N,t}]$, $[a^*(g_1)a(g_2), \mathcal{E}^{(3)}_{N,t}]$ and of the time-derivative $\partial_t\mathcal{E}^{(3)}_{N,t}$ can be proven analogously. We omit the details. □

5F. *Analysis of $e^{-B(\eta_t)}\mathcal{L}^{(4)}_{N,t}e^{B(\eta_t)}$.* Recall from (5-3) that

$$\mathcal{L}^{(4)}_{N,t} = \mathcal{V}_N = \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))a_x^*a_y^*a_ya_x.$$

We conjugate $\mathcal{L}^{(4)}_{N,t}$ with the unitary operator $e^{B(\eta_t)}$. We define the error term $\mathcal{E}^{(4)}_{N,t}$ through the equation

$$\begin{aligned}e^{-B(\eta_t)}\mathcal{L}^{(4)}_{N,t}e^{B(\eta_t)} = \mathcal{V}_N &+ \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))|k_t(x;y)|^2\\ &+ \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))[k_t(x;y)b_x^*b_y^* + \mathrm{h.c.}] + \mathcal{E}^{(4)}_{N,t}.\end{aligned} \tag{5-117}$$

In the next proposition we collect some important properties of the operator $\mathcal{E}^{(4)}_{N,t}$.

Proposition 5.11. *Under the same assumptions as in Theorem 4.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} |\langle \xi, \mathcal{E}_{N,t}^{(4)} \xi \rangle| &\le C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\|, \\ |\langle \xi, [\mathcal{N}, \mathcal{E}_{N,t}^{(4)}] \xi \rangle| &\le C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\|, \\ |\langle \xi, [a^*(g_1)a(g_2), \mathcal{E}_{N,t}^{(4)}] \xi \rangle| &\le C e^{c|t|} \|g_1\|_{H^2} \|g_2\|_{H^2} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\|, \\ |\partial_t \langle \xi, \mathcal{E}_{N,t}^{(4)} \xi \rangle| &\le C e^{c|t|} \|(\mathcal{N}+1)^{1/2}\xi\| \, \|(\mathcal{V}_N + \mathcal{N}+1)^{1/2}\xi\| \end{aligned} \tag{5-118}$$

for all $\xi \in \mathcal{F}^{\le N}$.

Proof. We start by writing

$$e^{-B(\eta_t)} a_x^* a_y^* a_y a_x e^{B(\eta_t)} = a_x^* a_y^* a_y a_x + \int_0^1 ds \, e^{-sB(\eta_t)} [a_x^* a_y^* a_y a_x, B(\eta_t)] e^{sB(\eta_t)}.$$

A straightforward computation gives

$$e^{-B(\eta_t)} a_x^* a_y^* a_y a_x e^{B(\eta_t)} = a_x^* a_y^* a_y a_x + \int_0^1 ds \, e^{-sB(\eta_t)} [b_x^* b_y^* (a_x a^*(\eta_y) + a^*(\eta_x) a_y) + \text{h.c.}] e^{sB(\eta_t)}. \tag{5-119}$$

Now we observe that

$$\begin{aligned} &e^{-sB(\eta_t)} [a_x a^*(\eta_y) + a^*(\eta_x) a_y] e^{sB(\eta_t)} \\ &\quad = a_x a^*(\eta_y) + a^*(\eta_x) a_y + \int_0^s d\tau \, e^{-\tau B(\eta_t)} [a_x a^*(\eta_y) + a^*(\eta_x) a_y, B(\eta_t)] e^{\tau B(\eta_t)} \\ &\quad = \eta_t(x;y) + a^*(\eta_y) a_x + a^*(\eta_x) a_y + \int_0^s d\tau \, e^{-\tau B(\eta_t)} [2b^*(\eta_x) b^*(\eta_y) + b(\eta_y^{(2)}) b_x + b(\eta_x^{(2)}) b_y] e^{\tau B(\eta_t)}. \end{aligned}$$

Inserting in (5-119), expanding as in Lemma 3.3, and integrating over s, τ, we obtain

$$e^{-B(\eta_t)} \mathcal{L}_{N,t}^{(4)} e^{B(\eta_t)} = \mathcal{V}_N + \mathrm{W}_1 + \mathrm{W}_2 + \mathrm{W}_3 + \mathrm{W}_4, \tag{5-120}$$

where

$$\begin{aligned} \mathrm{W}_1 &= \frac{1}{2} \sum_{n,k \ge 0} \frac{(-1)^{n+k}}{n!\,k!\,(n+k+1)} \int dx\,dy\, N^2 V(N(x-y)) \eta_t(x;y) \, \mathrm{ad}_{B(\eta_t)}^{(n)}(b_x^*) \mathrm{ad}_{B(\eta_t)}^{(k)}(b_y^*), \\ \mathrm{W}_2 &= \sum_{n,k \ge 0} \frac{(-1)^{n+k}}{n!\,k!\,(n+k+1)} \int dx\,dy\, N^2 V(N(x-y)) \, \mathrm{ad}_{B(\eta_t)}^{(n)}(b_x^*) \mathrm{ad}_{B(\eta_t)}^{(k)}(b_y^*) a^*(\eta_x) a_y, \\ \mathrm{W}_3 &= \sum_{n,k,m,r \ge 0} \frac{(-1)^{n+k+m+r}}{n!\,k!\,m!\,r!\,(m+r+1)(n+k+m+r+2)} \\ &\qquad \times \int dx\,dy\, N^2 V(N(x-y)) \, \mathrm{ad}_{B(\eta_t)}^{(n)}(b_x^*) \mathrm{ad}_{B(\eta_t)}^{(k)}(b_y^*) \mathrm{ad}_{B(\eta_t)}^{(m)}(b(\eta_x^{(2)})) \mathrm{ad}_{B(\eta_t)}^{(r)}(b_y), \\ \mathrm{W}_4 &= \sum_{n,k,m,r \ge 0} \frac{(-1)^{n+k+m+r}}{n!\,k!\,m!\,r!\,(m+r+1)(m+r+n+k+2)} \\ &\qquad \times \int dx\,dy\, N^2 V(N(x-y)) \, \mathrm{ad}_{B(\eta_t)}^{(n)}(b_x^*) \mathrm{ad}_{B(\eta_t)}^{(k)}(b_y^*) \mathrm{ad}_{B(\eta_t)}^{(m)}(b^*(\eta_x)) \mathrm{ad}_{B(\eta_t)}^{(r)}(b^*(\eta_y)). \end{aligned}$$

Let us now estimate the expectation of W_2. By Cauchy–Schwarz, we have

$$\begin{aligned}\left|\int dx\,dy\,N^2V(N(x-y))\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)a^*(\eta_x)a_y\xi\rangle\right|\\ \leq\int dx\,dy\,N^2V(N(x-y))\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\,\|(\mathcal{N}+1)^{-1/2}a^*(\eta_x)a_y\xi\|.\end{aligned}$$

We bound

$$\|(\mathcal{N}+1)^{-1/2}a^*(\eta_x)a_y\xi\|\leq\|\eta_x\|\,\|a_y\xi\| \tag{5-121}$$

On the other hand, according to Lemma 3.3, $\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{n+k}n!\,k!$ contributions having the form

$$\begin{aligned}\mathrm{T}_{x,y}=\big\|(\mathcal{N}+1)^{1/2}\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{y,t,\diamondsuit})\\ \times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural'_1},\ldots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\big\|,\end{aligned} \tag{5-122}$$

with $i_1,i_2,k_1,k_2,\ell_1,\ell_2\geq 0$, $j_1,\ldots,j_{k_1},m_1,\ldots,m_{k_2}\geq 0$ and where each Λ_i- or Λ'_i-operator is a factor $(N-\mathcal{N})/N$, a factor $(N-\mathcal{N}+1)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1},\ldots,\eta^{(q_p)}_{t,\underline{\natural}_p}). \tag{5-123}$$

According to Lemma 5.4, we have

$$\begin{aligned}\mathrm{T}_{x,y}\leq(n+1)C^{k+n}\|\eta_t\|^{k+n-2}\big\{\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)^{3/2}\xi\|\\ +\|\eta_t\|\,\|\eta_x\|\,\|a_y(\mathcal{N}+1)\xi\|+\|\eta_t\|\,\|\eta_y\|\,\|a_x(\mathcal{N}+1)\xi\|\\ +\|\eta_t\|\,|\eta_t(x;y)|\,\|(\mathcal{N}+1)^{1/2}\xi\|+\|\eta_t\|^2\sqrt{N}\|a_xa_y\xi\|\big\}\end{aligned} \tag{5-124}$$

for all $x,y\in\mathbb{R}^3$. For $\xi\in\mathcal{F}^{\leq N}$, we obtain

$$\begin{aligned}&\left|\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)a^*(\eta_x)a_y\xi\rangle\right|\\ &\leq(n+1)!k!\,C^{n+k}\|\eta_t\|^{n+k-2}\int dx\,dy\,N^2V(N(x-y))\|\eta_x\|\,\|a_y\xi\|\\ &\qquad\times\big\{\big[N\|\eta_x\|\,\|\eta_y\|+\|\eta_t\|\,|\eta_t(x;y)|\big]\|(\mathcal{N}+1)^{1/2}\xi\|\\ &\qquad\qquad+N\|\eta_t\|\,\|\eta_y\|\,\|a_x\xi\|+N\|\eta_t\|\,\|\eta_x\|\,\|a_y\xi\|+N^{1/2}\|a_xa_y\xi\|\big\}\\ &\leq(n+1)!k!\,C^{n+k}\|\eta_t\|^{n+k}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|\end{aligned}$$

and therefore

$$|\langle\xi,\mathrm{W}_2\xi\rangle|\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

Now, let us consider the expectation of the term W_3. By Cauchy–Schwarz, we have

$$\left|\int dx\,dy\,N^2V(N(x-y))\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta^{(2)}_x))\mathrm{ad}^{(r)}(b_y)\xi\rangle\right| \\ \leq \int N^2V(N(x-y))\,\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\,\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta^{(2)}_x))\mathrm{ad}^{(r)}(b_y)\xi\|.$$

Expanding $\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta^{(2)}_x))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_y)$ as in Lemma 3.2 and using Lemma 5.4, we obtain

$$\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta^{(2)}_x))\mathrm{ad}^{(r)}_{B(\eta_t)}(b_y)\xi\| \\ \leq m!\,r!\,C^{m+r}\|\eta_t\|^{m+r}[\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)^{1/2}\xi\|+\|\eta_t\|\,\|\eta_x\|\,\|a_y\xi\|]. \quad (5\text{-}125)$$

As for the norm $\|(\mathcal{N}+1)^{1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$, we can estimate it as the sum of $2^{n+k}n!\,k!$ contributions of the form (5-122). Using (5-124) and integrating over x, y, we conclude

$$|\langle\xi,\mathrm{W}_3\xi\rangle|\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

Let us now switch to W_4. We proceed analogously as we did for W_3. The only difference is that, instead of (5-125), we need to bound

$$\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(m)}_{B(\eta_t)}(b(\eta_x))\mathrm{ad}^{(r)}_{B(\eta_t)}(b(\eta_y))\xi\|\leq m!\,r!\,C^{m+r}\|\eta_t\|^{m+r}\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)^{1/2}\xi\|.$$

We find

$$|\langle\xi,\mathrm{W}_4\xi\rangle|\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

Finally, we consider the term W_1 in (5-120). We extract from the sum over $n,k\geq 0$ the terms with $(n,k)=(0,0)$ and $(n,k)=(0,1)$. We obtain that

$$\mathrm{W}_1=\frac{1}{2}\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)b^*_xb^*_y \\ -\frac{1}{4}\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)[B(\eta_t),b^*_x]b^*_y+\widetilde{\mathrm{W}}_1, \quad (5\text{-}126)$$

with

$$\widetilde{\mathrm{W}}_1=\frac{1}{2}\sum^{*}_{n,k}\frac{(-1)^{n+k}}{n!\,k!\,(n+k+1)}\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y), \quad (5\text{-}127)$$

where $\sum^{*}$ excludes the terms $(n,k)=(0,0),(1,0)$. We bound the expectation of $\widetilde{W}_1$ by

$$\left|\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)\langle\xi,\mathrm{ad}^{(n)}_{B(\eta_t)}(b^*_x)\mathrm{ad}^{(k)}_{B(\eta_t)}(b^*_y)\xi\rangle\right| \\ \leq\int dx\,dy\,N^2V(N(x-y))|\eta_t(x;y)|\,\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b_y)\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|.$$

Following Lemma 3.3, we can bound the norm $\|(\mathcal{N}+1)^{-1/2}\text{ad}^{(k)}_{B(\eta_t)}(b_y)\text{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ by the sum of $2^{n+k}n!\,k!$ terms of the form

$$\widetilde{\mathrm{T}}_{x,y}=\big\|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\ldots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};\eta^{(\ell_1)}_{y,t,\diamondsuit}) \times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp,\flat}(\eta^{(m_1)}_{t,\natural_1},\ldots,\eta^{(m_{k_2})}_{t,\natural_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\big\|, \quad (5\text{-}128)$$

with $i_1, i_2, k_1, k_2, \ell_1, \ell_2 \geq 0$, $j_1,\ldots,j_{k_1}, m_1,\ldots,m_{k_2}\geq 0$ and where each Λ_i- or Λ'_i-operator is a factor $(N-\mathcal{N})/N$, a factor $(N-\mathcal{N}+1)/N$ or a $\Pi^{(2)}$-operator of the form (5-123). With Lemma 5.4 we find

$$\widetilde{\mathrm{T}}_{x,y}\leq(n+1)C^{k+n}\|\eta_t\|^{k+n-2}\big\{\|\eta_x\|\,\|\eta_y\|\,\|(\mathcal{N}+1)^{1/2}\xi\|+\|\eta_t\|\,\|\eta_x\|\,\|a_y\xi\|+\|\eta_t\|\eta_y\|\,\|a_x\xi\| +\|\eta_t\|N^{-1}|\eta_t(x;y)|\,\|(\mathcal{N}+1)^{1/2}\xi\|+\|\eta_t\|^2\,\|a_xa_y\xi\|\big\}$$

for all $x, y\in\mathbb{R}^3$. The important difference with respect to (5-124) is that here, when we consider the cases $\ell_1=\ell_2=0$ and $\ell_1=0$, $\ell_2=1$ we can apply (5-27) and (5-29), rather than (5-26) and (5-28), because the assumption $(n,k)\neq(0,0),(1,0)$ implies that $k+n\geq 2$ (the case $(n,k)=(0,1)$ is not compatible with $\ell_2=1$). Using $\sup_{x,y}N^{-1}|\eta_t(x;y)|\leq Ce^{c|t|}$ from Lemma 4.3, we conclude that

$$|\langle\xi,\widetilde{W}_1\xi\rangle|\leq Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|$$

if $\sup_t\|\eta_t\|$ is small enough.

As for the second term on the right-hand side of (5-126), we have

$$[B(\eta_t),b^*_x]=-b(\eta_x)\frac{N-\mathcal{N}}{N}+\frac{1}{N}\int dz\,dw\,a^*_xa_zb_w\,\eta_t(z;w).$$

Hence

$$\begin{aligned}
-\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)[B(\eta_t),b^*_x]b^*_y
&=\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)b(\eta_x)b^*_y\frac{N-\mathcal{N}+1}{N}\\
&\quad-N^{-1}\int dx\,dy\,dz\,dw\,N^2V(N(x-y))\eta_t(x;y)\eta_t(z;w)\,a^*_xa_zb_wb^*_y\\
&=\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)b^*_yb(\eta_x)\frac{N-\mathcal{N}+1}{N}\\
&\quad+\int dx\,dy\,N^2V(N(x-y))|\eta_t(x;y)|^2\frac{N-\mathcal{N}}{N}\frac{N-\mathcal{N}+1}{N}\\
&\quad-N^{-1}\int dx\,dy\,dz\,N^2V(N(x-y))\eta_t(x;y)\eta_t(x;z)\,a^*_ya_z\frac{N-\mathcal{N}+1}{N}\\
&\quad-N^{-1}\int dx\,dy\,dz\,dw\,N^2V(N(x-y))\eta_t(x;y)\eta_t(z;w)\,a^*_xa_zb_wb^*_y.
\end{aligned}$$

We conclude that

$$-\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)[B(\eta_t),b^*_x]b^*_y=\int dx\,dy\,N^2V(N(x-y))|k_t(x;y)|^2+\mathrm{W}_{12},$$

where
$$|\langle \xi, \mathrm{W}_{12}\xi\rangle| \le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{V}_N+\mathcal{N}+1)^{1/2}\xi\|.$$
Similarly, the first term on the right-hand side of (5-126) can be decomposed as
$$\int dx\,dy\,N^2V(N(x-y))\eta_t(x;y)b_x^*b_y^* = \int dx\,dy\,N^2V(N(x-y))k_t(x;y)b_x^*b_y^* + \mathrm{W}_{11},$$
where
$$\mathrm{W}_{11} = \int dx\,dy\,N^2V(N(x-y))\mu_t(x;y)b_x^*b_y^*$$
is such that
$$|\langle \xi, \mathrm{W}_{11}\xi\rangle| \le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|\,\|\mathcal{V}_N^{1/2}\xi\|$$
since $|\mu(x;y)| \le Ce^{c|t|}$ uniformly in N. □

5G. *Analysis of* $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$**.** This subsection is devoted to the study of the first term in the generator $\mathcal{G}_{N,t}$ in (5-1). The properties of $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$ are collected in the next proposition.

Proposition 5.12. *Under the same assumptions as in Theorem 4.4, there exist constants $C, c > 0$ such that*
$$\begin{aligned}
|\langle \xi, (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\xi\rangle| &\le C\|(\mathcal{N}+1)^{1/2}\xi\|^2,\\
|\langle \xi, [\mathcal{N}, (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}]\xi\rangle| &\le C\|(\mathcal{N}+1)^{1/2}\xi\|^2,\\
|\langle \xi, [a^*(g_1)a(g_2), (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}]\xi\rangle| &\le C\|g_1\|\,\|g_2\|\,\|(\mathcal{N}+1)^{1/2}\xi\|^2,\\
|\langle \xi, [\partial_t(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}]\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{N}+1)^{1/2}\xi\|^2
\end{aligned} \tag{5-129}$$
for all $\xi \in \mathcal{F}^{\le N}$.

Proof. As in Section 6.5 of [Benedikter et al. 2015], we expand $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$ as
$$\begin{aligned}
(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} &= -\int_0^1 ds\, e^{-sB(\eta_t)}[i\partial_t B(\eta_t)]e^{sB(\eta_t)}\\
&= \frac{i}{2}\sum_{k,n\ge 0}\frac{(-1)^{n+k}}{k!\,n!\,(n+k+1)}\int dx\,\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x) + \text{h.c.}
\end{aligned} \tag{5-130}$$
We bound the expectations
$$\begin{aligned}
&\left|\int dx\,\langle \xi, \mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\rangle\right|\\
&\qquad\le \|(\mathcal{N}+1)^{1/2}\xi\|\int dx\,\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|.
\end{aligned}$$
According to Lemma 3.3, the norm $\|(\mathcal{N}+1)^{-1/2}\mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\|$ is bounded by the sum of $2^{n+k}n!\,k!$ terms of the form
$$\begin{aligned}
\mathrm{Z}_x = \|(\mathcal{N}+1)^{-1/2}\Lambda_1\cdots\Lambda_{i_1}N^{-k_1}\Pi^{(1)}_{\sharp,\flat}(\eta^{(j_1)}_{t,\natural_1},\dots,\eta^{(j_{k_1})}_{t,\natural_{k_1}};(\eta^{(\ell_1)}_{t,\diamondsuit}\partial_t\eta_t)_x)\\
\times\Lambda'_1\cdots\Lambda'_{i_2}N^{-k_2}\Pi^{(1)}_{\sharp',\flat'}(\eta^{(m_1)}_{t,\natural'_1},\dots,\eta^{(m_{k_2})}_{t,\natural'_{k_2}};\eta^{(\ell_2)}_{x,\diamondsuit'})\xi\|,
\end{aligned} \tag{5-131}$$

with integers $i_1, k_1, \ell_1, i_2, k_2, \ell_2 \geq 0$, $j_1, \dots, j_{k_1}, m_1, \dots, m_{k_2} \geq 1$ and where each Λ_i, Λ'_i is a factor $(N-\mathcal{N})/N$ or $(N+1-\mathcal{N})/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-p}\Pi^{(2)}_{\underline{\sharp},\underline{\flat}}(\eta^{(q_1)}_{t,\underline{\natural}_1}, \dots, \eta^{(q_p)}_{t,\underline{\natural}_p}).$$

From Lemma 5.3, we conclude that

$$\mathrm{Z}_x \leq \begin{cases} C^{n+k}\|\eta_t\|^{n+k-1}\|(\partial_t\eta_t)_x\|\,\|\eta_x\|\,\|(\mathcal{N}+1)^{1/2}\xi\| & \text{if } \ell_2 > 0, \\ C^{n+k}\|\eta_t\|^{n+k}\|(\partial_t\eta_t)_x\|\,\|a_x\xi\| & \text{if } \ell_2 = 0 \end{cases}$$

for all $x \in \mathbb{R}^3$. With Cauchy–Schwarz, we obtain

$$\left|\int dx\, \langle \xi, \mathrm{ad}^{(k)}_{B(\eta_t)}(b((\partial_t\eta_t)_x))\mathrm{ad}^{(n)}_{B(\eta_t)}(b_x)\xi\rangle\right| \leq n!\,k!\,C^{n+k}\|\eta_t\|^{n+k}\|\partial_t\eta_t\|\,\|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

From (5-130), we conclude that, if $\sup_t \|\eta_t\|$ is sufficiently small,

$$|\langle \xi, (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}\xi\rangle| \leq C\|(\mathcal{N}+1)^{1/2}\xi\|^2.$$

The other bounds in (5-129) can be proven analogously, first expanding $(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)}$ as in (5-130), then using Lemmas 3.3 and 3.2 to write the nested commutators on the right-hand side of (5-130) as sums of factors like in (5-131), and then commuting each of these factors with $\mathcal{N}$, with $a^*(g_1)a(g_2)$, or taking its time-derivative; we omit the details. □

5H. *Proof of Theorem 4.4.* Recall from (5-1) that

$$\mathcal{G}_{N,t} = (i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} + \sum_{j=0}^{4} e^{-B(\eta_t)}\mathcal{L}^{(j)}_{N,t}e^{B(\eta_t)},$$

with $\mathcal{L}^{(j)}_{N,t}$ defined as in (5-3), for $j = 0, \dots, 4$. It follows from Propositions 5.5 and 5.7–5.12 that

$$\begin{aligned}
e^{-B(\eta_t)}\mathcal{L}^{(0)}_{N,t}e^{B(\eta_t)} &=_{N,t} + \widetilde{\mathcal{E}}^{(0)}_{N,t}, \\
e^{-B(\eta_t)}\mathcal{L}^{(1)}_{N,t}e^{B(\eta_t)} &= \sqrt{N}[b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(\bar{h}_{N,t})) + \text{h.c.}] + \widetilde{\mathcal{E}}^{(1)}_{N,t}, \\
e^{-B(\eta_t)}\mathcal{L}^{(2)}_{N,t}e^{B(\eta_t)} &= \mathcal{K} + \int |\nabla_x k_t(x;y)|^2\,dx\,dy \\
&\quad + \int dx\,dy\,(\Delta w_\ell)(N(x-y))[\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)b^*_x b^*_y + \text{h.c.}] \\
&\quad + \mathrm{Re}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)k_t(y;x) \\
&\quad + \frac{1}{2}\int dx\,dy\,N^3V(N(x-y))[\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)b^*_x b^*_y + \text{h.c.}] + \widetilde{\mathcal{E}}^{(2)}_{N,t}, \qquad (5\text{-}132)\\
e^{-B(\eta_t)}\mathcal{L}^{(3)}_{N,t}e^{B(\eta_t)} &= -\sqrt{N}\,[b(\cosh_{\eta_t}(h_{N,t})) + b^*(\sinh_{\eta_t}(\bar{h}_{N,t})) + \text{h.c.}] + \widetilde{\mathcal{E}}^{(3)}_{N,t}, \\
e^{-B(\eta_t)}\mathcal{L}^{(4)}_{N,t}e^{B(\eta_t)} &= \mathcal{V}_N + \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))|k_t(x;y)|^2 \\
&\quad + \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))[k_t(x;y)b^*_x b^*_y + \text{h.c.}] + \widetilde{\mathcal{E}}^{(4)}_{N,t}, \\
(i\partial_t e^{-B(\eta_t)})e^{B(\eta_t)} &= \widetilde{\mathcal{E}}^{(5)}_{N,t},
\end{aligned}$$

where the error terms $\widetilde{\mathcal{E}}^{(j)}_{N,t}$ are such that

$$\begin{aligned}|\langle \xi, \widetilde{\mathcal{E}}^{(j)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle \xi, [\mathcal{N}, \widetilde{\mathcal{E}}^{(j)}_{N,t}]\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\\ |\langle \xi, [a^*(g_1)a(g_2), \widetilde{\mathcal{E}}^{(j)}_{N,t}]\xi\rangle| &\le Ce^{c|t|}\|g_1\|_{H^2}\|g_2\|_{H^2}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|,\\ |\partial_t\langle \xi, \widetilde{\mathcal{E}}^{(j)}_{N,t}\xi\rangle| &\le Ce^{c|t|}\|(\mathcal{H}_N+\mathcal{N}+1)^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned} \tag{5-133}$$

for all $j=0,1,\dots,5$. With the scattering equation (4-2), we conclude that

$$\begin{aligned}\mathcal{G}_{N,t} &= C_{N,t}+\mathcal{H}_N+\widetilde{\mathcal{E}}_{N,t}+N\int dx\,dy\,\big[-\Delta+\tfrac{1}{2}N^2V(N(x-y))\big](1-w_\ell(N(x-y)))\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)b^*_xb^*_y+\text{h.c.}\\ &= C_{N,t}+\mathcal{H}_N+\mathrm{A}+\widetilde{\mathcal{E}}_{N,t},\end{aligned} \tag{5-134}$$

with

$$\mathrm{A}=N^3\lambda_\ell\int dx\,dy\ f_\ell(N(x-y))\chi(|x-y|\le\ell)[\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)b^*_xb^*_y+\text{h.c.}]$$

and where $\widetilde{\mathcal{E}}_{N,t}$ satisfies the same estimates (5-133) as all error terms $\widetilde{\mathcal{E}}^{(j)}_{N,t}$, $j=0,\dots,5$. Since $N^3\lambda_\ell\le C$ (see Lemma 4.1) and $f_\ell(N(x-y))\le 1$ we have, with Lemma 2.2,

$$|\langle \xi, \mathrm{A}\xi\rangle|\le C\|(\mathcal{N}+1)^{1/2}\xi\|^2$$

and similarly, $\pm[\mathcal{N},\mathrm{A}]$, $\pm[a^*(g_1)a(g_2),\mathrm{A}]$, $\pm\partial_t\mathrm{A}\le C(\mathcal{N}+1)$. Setting $\mathcal{E}_{N,t}=\mathrm{A}+\widetilde{\mathcal{E}}_{N,t}$, we conclude that

$$\mathcal{G}_{N,t}=C_{N,t}+\mathcal{H}_N+\mathcal{E}_{N,t},$$

where $\mathcal{E}_{N,t}$ satisfies again the same bounds (5-133) as $\widetilde{\mathcal{E}}_{N,t}$. This immediately implies that, in the sense of forms on $\mathcal{F}^{\le N}_{\perp\tilde{\varphi}_t}\times\mathcal{F}^{\le N}_{\perp\tilde{\varphi}_t}$,

$$\begin{aligned}\tfrac{1}{2}\mathcal{H}_N-Ce^{c|t|}(\mathcal{N}+1)\le \mathcal{G}_{N,t}-C_{N,t}&\le 2\mathcal{H}_N+Ce^{c|t|}(\mathcal{N}+1),\\ \pm i[\mathcal{G}_{N,t},\mathcal{N}]&\le \mathcal{H}_N+Ce^{c|t|}(\mathcal{N}+1),\\ \partial_t[\mathcal{G}_{N,t}-C_{N,t}]&\le \mathcal{H}_N+Ce^{c|t|}(\mathcal{N}+1).\end{aligned}$$

Moreover, since

$$\begin{aligned}[\mathcal{H}_N, a^*(g_1)a(g_2)] &= \int dx\,\nabla g_1(x)\nabla_x a^*_x a(g_2)-\int dx\,a^*(g_1)\nabla\bar{g}_2(x)\nabla_x a_x\\ &\quad+\int dx\,dy\,N^2V(N(x-y))g_1(y)a^*_xa^*_ya_xa(g_2)\\ &\quad-\int dx\,dy\,N^2V(N(x-y))\,\bar{g}_2(x)a^*(g_1)a^*_ya_ya_x,\end{aligned}$$

we obtain that

$$\begin{aligned}&|\langle\xi,[\mathcal{H}_N,a^*(g_1)a(g_2)]\xi\rangle|\\ &\quad\le\big[\|\nabla g_1\|\|g_2\|+\|g_1\|\|\nabla g_2\|\big]\|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}^{1/2}\xi\|+\big[\|g_2\|\|g_1\|_\infty+\|g_1\|\|g_2\|_\infty\big]\\ &\qquad\times\bigg[\int dx\,dy\,N^2V(N(x-y))\|a_xa_y\xi\|^2\bigg]^{1/2}\bigg[\int dx\,dy\,N^2V(N(x-y))\|a_y(\mathcal{N}+1)^{1/2}\xi\|^2\bigg]^{1/2}\\ &\quad\le\|g_1\|_{H^2}\|g_2\|_{H^2}\|\mathcal{H}_N^{1/2}\xi\|\,\|(\mathcal{N}+1)^{1/2}\xi\|\end{aligned}$$

for all $\xi \in \mathcal{F}^{\leq N}$. Combining with the bounds (5-133) for the error operator $\mathcal{E}_{N,t}$, and choosing $g_1 = \partial_t \tilde{\varphi}_t$ and $g_2 = \tilde{\varphi}_t$, we find that

$$\pm \operatorname{Re}[\mathcal{G}_{N,t}, a^*(\partial_t \tilde{\varphi}_t) a(\tilde{\varphi}_t)] \leq \mathcal{H}_N + C e^{K|t|}(\mathcal{N}+1).$$

This concludes the proof of Theorem 4.4.

6. Bounds on the growth of fluctuations

In this section, we are going to complete the proofs of Theorems 1.1 and 1.2. The main ingredient to reach this goal is a bound on the growth of the expectation of the number of particles operator with respect to the fluctuation dynamics $\mathcal{W}_{N,t}$, which we prove in the next proposition using the properties of the generator $\mathcal{G}_{N,t}$ established in Theorem 4.4.

Proposition 6.1. *Under the same assumptions as in Theorem 4.4, there exist constants $C, c > 0$ such that*

$$\begin{aligned} \langle \mathcal{W}_{N,t}\,\xi, \mathcal{N}\mathcal{W}_{N,t}\xi\rangle &\leq C\,\langle \xi, ((\mathcal{G}_{N,0} - C_{N,0}) + (\mathcal{N}+1))\xi\rangle \exp(c\exp(c|t|)), \\ \langle \mathcal{W}_{N,t}\,\xi, \mathcal{H}_N\mathcal{W}_{N,t}\xi\rangle &\leq C\,\langle \xi, ((\mathcal{G}_{N,0} - C_{N,0}) + (\mathcal{N}+1))\xi\rangle \exp(c\exp(c|t|)) \end{aligned} \tag{6-1}$$

for all $\xi \in \mathcal{F}^{\leq N}_{\perp\varphi}$. Here $\mathcal{H}_N$ is the Hamilton operator defined in (4-27).

Remark. From (4-26), we also have

$$\begin{aligned} \langle \mathcal{W}_{N,t}\,\xi, \mathcal{N}\mathcal{W}_{N,t}\xi\rangle &\leq C\,\langle \xi, (\mathcal{H}_N + \mathcal{N}+1)\xi\rangle \exp(c\exp(c|t|)), \\ \langle \mathcal{W}_{N,t}\,\xi, \mathcal{H}_N\mathcal{W}_{N,t}\xi\rangle &\leq C\,\langle \xi, (\mathcal{H}_N + \mathcal{N}+1)\xi\rangle \exp(c\exp(c|t|)). \end{aligned}$$

Proof. First of all, we observe that, from the first bound in (4-26),

$$\tfrac{1}{2}\mathcal{H}_N + \mathcal{N} \leq (\mathcal{G}_{N,t} - C_{N,t}) + C e^{K|t|}(\mathcal{N}+1). \tag{6-2}$$

Hence, it is enough to control the growth of the expectation of the operator on the right-hand side. We follow here the approach of [Lewin et al. 2015a]. We define $q_t = 1 - |\tilde{\varphi}_t\rangle\langle\tilde{\varphi}_t|$ as the orthogonal projection onto $L^2_{\perp\tilde{\varphi}_t}(\mathbb{R}^3)$. We define moreover $\Gamma_t : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}_{\perp\varphi_t}$ by imposing that $\Gamma_t|_{\mathcal{F}_j} = q_t^{\otimes j}$ for all $j = 1, \dots, N$ ($\mathcal{F}_j$ is the sector of $\mathcal{F}^{\leq N}$ with exactly j particles). We have, restricting our attention to $t \geq 0$ (the case $t < 0$ can be handled very similarly),

$$\langle \mathcal{W}_{N,t}\,\xi, [(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle = \langle \mathcal{W}_{N,t}\,\xi, [(\Gamma_t\mathcal{G}_{N,t}\Gamma_t - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle.$$

Hence, since $\mathcal{N}$ commutes with Γ_t,

$$\begin{aligned} &i\partial_t\langle \mathcal{W}_{N,t}\,\xi, [(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle \\ &\quad = \langle \mathcal{W}_{N,t}\,\xi, [\Gamma_t\mathcal{G}_{N,t}\Gamma_t, (\Gamma_t\mathcal{G}_{N,t}\Gamma_t - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle \\ &\qquad\qquad + \langle \mathcal{W}_{N,t}\,\xi, \partial_t[(\Gamma_t\mathcal{G}_{N,t}\Gamma_t - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle \\ &\quad = Ce^{Kt}\langle \mathcal{W}_{N,t}\,\xi, [\mathcal{G}_{N,t}, \mathcal{N}]\mathcal{W}_{N,t}\,\xi\rangle + \langle \mathcal{W}_{N,t}\,\xi, \partial_t[(\Gamma_t\mathcal{G}_{N,t}\Gamma_t - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle. \end{aligned} \tag{6-3}$$

We observe that

$$0 = \partial_t\|\tilde{\varphi}_t\|_2^2 = \langle \dot{\tilde{\varphi}}_t, \tilde{\varphi}_t\rangle + \langle \tilde{\varphi}_t, \dot{\tilde{\varphi}}_t\rangle.$$

This implies that

$$\dot{q}_t = -|\tilde{\varphi}_t\rangle\langle\dot{\tilde{\varphi}}_t| - |\dot{\tilde{\varphi}}_t\rangle\langle\tilde{\varphi}_t| = -|\tilde{\varphi}_t\rangle\langle q_t\dot{\tilde{\varphi}}_t| - |q_t\dot{\tilde{\varphi}}_t\rangle\langle\tilde{\varphi}_t|.$$

Therefore

$$\begin{aligned}\partial_t\Gamma_t^{(j)} &= -\sum_{i=1}^{j} q_t\otimes\cdots\otimes[|\tilde{\varphi}_t\rangle\langle q_t\dot{\tilde{\varphi}}_t|q_t + q_t|q_t\dot{\tilde{\varphi}}_t\rangle\langle\tilde{\varphi}_t|]\otimes\cdots\otimes q_t\\ &= -\sum_{i=1}^{j}[|\tilde{\varphi}_t\rangle\langle q_t\dot{\tilde{\varphi}}_t|_i\Gamma_t^{(j)} - \Gamma_t^{(j)}|q_t\dot{\tilde{\varphi}}_t\rangle\langle\tilde{\varphi}_t|_i].\end{aligned}$$

We conclude that

$$\partial_t\Gamma_t = -a^*(\tilde{\varphi}_t)a(q_t\dot{\tilde{\varphi}}_t)\Gamma_t - \Gamma_t a^*(q_t\dot{\tilde{\varphi}}_t)a(\tilde{\varphi}_t).$$

Thus

$$\begin{aligned}&\langle\mathcal{W}_{N,t}\,\xi,\,\partial_t[(\Gamma_t\mathcal{G}_{N,t}\Gamma_t - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle\\ &\quad= \langle\mathcal{W}_{N,t}\,\xi,\,[(\partial_t\Gamma_t)(\mathcal{G}_{N,t} - C_{N,t}) + (\mathcal{G}_{N,t} - C_{N,t})(\partial_t\Gamma_t)]\mathcal{W}_{N,t}\,\xi\rangle\\ &\qquad\qquad + \langle\mathcal{W}_{N,t}\,\xi,\,[\partial_t(\mathcal{G}_{N,t} - C_{N,t}) + CKe^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle\\ &\quad= 2\,\mathrm{Re}\langle\mathcal{W}_{N,t}\,\xi,\,[a^*(q_t\dot{\tilde{\varphi}}_t)a(\tilde{\varphi}_t),\mathcal{G}_{N,t}]\mathcal{W}_{N,t}\,\xi\rangle + \langle\mathcal{W}_{N,t}\,\xi,\,[\partial_t(\mathcal{G}_{N,t} - C_{N,t}) + CKe^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle,\end{aligned}$$

where we used the fact that $a(\tilde{\varphi}_t)\mathcal{W}_{N,t}\xi = 0$ for all $t\in\mathbb{R}$. Together with (6-3), we find

$$\begin{aligned}&i\partial_t\langle\mathcal{W}_{N,t}\,\xi,\,[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle\\ &\quad= Ce^{Kt}\langle\mathcal{W}_{N,t}\,\xi,\,[\mathcal{G}_{N,t},\mathcal{N}]\mathcal{W}_{N,t}\,\xi\rangle + \langle\mathcal{W}_{N,t}\,\xi,\,[\partial_t(\mathcal{G}_{N,t} - C_{N,t}) + CKe^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle\\ &\qquad\qquad + 2\,\mathrm{Re}\langle\mathcal{W}_{N,t}\,\xi,\,[a^*(q_t\dot{\tilde{\varphi}}_t)a(\tilde{\varphi}_t),\mathcal{G}_{N,t}]\mathcal{W}_{N,t}\,\xi\rangle.\end{aligned}$$

From Theorem 4.4, we obtain that

$$\begin{aligned}|\partial_t\langle\mathcal{W}_{N,t}\,\xi,\,[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\,\xi\rangle| &\le \widetilde{C}e^{K|t|}\langle\mathcal{W}_{N,t}\,\xi,\,[\mathcal{H}_N + Ce^{Kt}(\mathcal{N}+1)]\mathcal{W}_{N,t}\xi\rangle\\ &\le \widetilde{C}e^{K|t|}\langle\mathcal{W}_{N,t}\,\xi,\,[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{K|t|}(\mathcal{N}+1)]\mathcal{W}_{N,t}\xi\rangle.\end{aligned}$$

Applying Gronwall's inequality, we find a constant $c>0$ such that

$$\langle\mathcal{W}_{N,t}\,\xi,\,[(\mathcal{G}_{N,t} - C_{N,t}) + Ce^{K|t|}(\mathcal{N}+1)]\mathcal{W}_{N,t}\xi\rangle \le \langle\xi,\,[(\mathcal{G}_{N,0} - C_{N,0}) + C(\mathcal{N}+1)]\xi\rangle\ \exp(c\exp(c|t|)),$$

With (6-2), we conclude that

$$\begin{aligned}\langle\mathcal{W}_{N,t}\xi,\,\mathcal{N}\mathcal{W}_{N,t}\xi\rangle &\le C\langle\xi,\,[(\mathcal{G}_{N,0} - C_{N,0}) + (\mathcal{N}+1)]\xi\rangle\ \exp(c\exp(c|t|)),\\ \langle\mathcal{W}_{N,t}\xi,\,\mathcal{H}_N\mathcal{W}_{N,t}\xi\rangle &\le C\langle\xi,\,[(\mathcal{G}_{N,0} - C_{N,0}) + (\mathcal{N}+1)]\xi\rangle\ \exp(c\exp(c|t|))\end{aligned}$$

as claimed. □

To apply Proposition 6.1 to the proof of Theorems 1.1 and 1.2, we need to control the expectation on the right-hand side of (6-1) for vectors $\xi\in\mathcal{F}_{\perp\varphi}^{\le N}$ describing orthogonal excitations around the condensate wave function φ for initial N-particle wave functions ψ_N satisfying (1-10). To this end, we use the next lemma.

Lemma 6.2. *As in* (4-25), *let*

$$\begin{aligned}C_{N,t} = \frac{1}{2}\big\langle\tilde{\varphi}_t, \big([N^3V(N\,\cdot\,)(N-1-2Nf_\ell(N\,\cdot\,))] * |\tilde{\varphi}_t|^2\big)\tilde{\varphi}_t\big\rangle \\ + \int dx\,dy\,|\nabla_x k_t(x;y)|^2 + \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))|k_t(x;y)|^2 \\ + \operatorname{Re}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)k_t(x;y),\end{aligned}$$

where $\tilde{\varphi}_t$ is the solution of the modified Gross–Pitaevskii equation (4-8), *with initial data $\tilde{\varphi}_{t=0} = \varphi$ (we assumed in the construction of the fluctuation dynamics that $\varphi \in H^4(\mathbb{R}^3)$; in this lemma, we only need $\varphi \in H^1(\mathbb{R}^3)$). Then there is a constant $C > 0$, independent of N and t, such that*

$$|[C_{N,t} + N\langle i\partial_t\tilde{\varphi}_t, \tilde{\varphi}_t\rangle] - N\mathcal{E}_{\mathrm{GP}}(\varphi)| \le C,$$

with the translation-invariant Gross–Pitaevskii energy functional $\mathcal{E}_{\mathrm{GP}}$ defined in (1-15).

Proof. We have

$$N\langle i\partial_t\tilde{\varphi}_t, \tilde{\varphi}_t\rangle = N\langle\tilde{\varphi}_t, -\Delta\tilde{\varphi}_t\rangle + N\langle\tilde{\varphi}_t, (N^3V(N\,\cdot\,)f_\ell(N\,\cdot\,) * |\tilde{\varphi}_t|^2)\tilde{\varphi}_t\rangle.$$

Therefore

$$\begin{aligned}C_{N,t} + N\langle i\partial_t\tilde{\varphi}_t, \tilde{\varphi}_t\rangle = N\|\nabla\tilde{\varphi}_t\|^2 + \frac{(N-1)}{2}\langle\tilde{\varphi}_t, [N^3V(N\,\cdot\,) * |\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle \\ + \int dx\,dy\,|\nabla_x k_t(x;y)|^2 + \frac{1}{2}\int dx\,dy\,N^2V(N(x-y))|k_t(x;y)|^2 \\ + \operatorname{Re}\int dx\,dy\,N^3V(N(x-y))\bar{\tilde{\varphi}}_t(x)\bar{\tilde{\varphi}}_t(y)k_t(x;y).\end{aligned} \tag{6-4}$$

Obviously,

$$\frac{(N-1)}{2}\langle\tilde{\varphi}_t, [N^3V(N\,\cdot\,) * |\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle = \frac{N}{2}\langle\tilde{\varphi}_t, [N^3V(N\,\cdot\,) * |\tilde{\varphi}_t|^2]\tilde{\varphi}_t\rangle + \mathcal{O}(1), \tag{6-5}$$

where $\mathcal{O}(1)$ denotes a quantity with absolute value bounded by a constant, independent of N and of t. Furthermore

$$\begin{aligned}\frac{1}{2}\int dx\,dy\,N^2V(N(x-y))|k_t(x,y)|^2 \\ = \frac{N}{2}\int dx\,dy\,N^3V(N(x-y))w_\ell(N(x-y))^2|\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2.\end{aligned} \tag{6-6}$$

Finally, we consider the third term on the right-hand side of (6-4), the one with $\nabla_x k_t$. We recall that $k_t(x;y) = -Nw_\ell(N(x-y))\tilde{\varphi}_t(x)\tilde{\varphi}_t(y)$. Hence, we find

$$\begin{aligned}-\Delta_x k_t(x;y) = N^3(\Delta w_\ell)(N(x-y))\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) + Nw_\ell(N(x-y))\Delta\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) \\ + 2N^2(\nabla w_\ell)(N(x-y))\cdot\nabla\tilde{\varphi}_t(x)\tilde{\varphi}_t(y).\end{aligned} \tag{6-7}$$

Since, by (4-1), $\Delta w_\ell = -\Delta f_\ell = -\frac{1}{2} V f_\ell + \lambda_\ell f_\ell$ we have

$$\begin{aligned}
&\int dx\, dy\, \bar{k}_t(x;y)(-\Delta_x k_t)(x;y) \\
&\qquad = -\frac{N}{2}\int dx\, dy\, N^3 V(N(y-x))(w_\ell(N(x-y))-1)w_\ell(N(x-y))|\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2 \\
&\qquad\qquad - N^3\lambda_\ell \int dx\, dy\, f_\ell(N(x-y)) N w_\ell(N(x-y))\, |\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2 \\
&\qquad\qquad\quad + 2\int dx\, dy\, N w_\ell(N(y-x)) N^2 (\nabla w_\ell)(N(y-x)) \cdot \nabla\bar{\tilde{\varphi}}_t(x)\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2 \\
&\qquad\qquad\qquad - \int dx\, dy\, N^2 w_\ell^2(N(x-y))(\Delta\tilde{\varphi}_t)(x)\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2 \\
&\qquad = \frac{N}{2}\int dx\, dy\, N^3 V(N(y-x))(1-w_\ell(N(x-y)))w_\ell(N(x-y))|\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2 \\
&\qquad\qquad + 2\int dx\, dy\, N w_\ell(N(y-x)) N^2(\nabla w_\ell)(N(y-x)) \cdot \nabla\bar{\tilde{\varphi}}_t(x)\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2 + \mathcal{O}(1). \qquad (6\text{-}8)
\end{aligned}$$

In the last step, we used the bounds $N^3\lambda_\ell = \mathcal{O}(1)$, $N w_\ell(N(x-y)) \le C|x-y|^{-1}$ and $0 \le f_\ell(N(x-y)) \le 1$. Integrating by parts in the last term, we find

$$\begin{aligned}
&2\int dx\, dy\, N^2(\nabla w_\ell)(N(y-x))\cdot\nabla\bar{\tilde{\varphi}}_t(x) N w_\ell(N(y-x))\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2 \\
&= -\int dx\, dy\, \nabla_x(N^2 w_\ell(N(y-x))^2)\cdot\nabla\bar{\tilde{\varphi}}_t(x)\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2 \\
&= \int dx\, dy\, N^2 w_\ell(N(x-y))^2\Delta\bar{\tilde{\varphi}}_t(x)\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2 + \int dx\, dy\, N^2 w_\ell(N(x-y))^2\nabla\bar{\tilde{\varphi}}_t(x)\cdot\nabla\tilde{\varphi}_t(x)|\tilde{\varphi}_t(y)|^2.
\end{aligned}$$

With (6-8), this leads us (using again the bound $N w_\ell(N(x-y)) \le C|x-y|^{-1}$) to

$$\begin{aligned}
&\int dx\, dy\, \bar{k}_t(x;y)(-\Delta_x k_t)(x;y) \\
&\qquad\qquad = \frac{N}{2}\int dx\, dy\, N^3 V(N(y-x))(1-w_\ell(N(x-y)))w_\ell(N(x-y))|\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2 + \mathcal{O}(1).
\end{aligned}$$

Combining this bound with (6-5) and (6-6), we find

$$\begin{aligned}
&C_{N,t} + N\langle i\partial_t\tilde{\varphi}_t, \tilde{\varphi}_t\rangle \\
&\qquad = N\bigg[\int |\nabla\tilde{\varphi}_t(x)|^2\, dx + \frac{1}{2}\int dx\, dy\, N^3 V(N(x-y)) f_\ell(N(x-y))|\tilde{\varphi}_t(x)|^2|\tilde{\varphi}_t(y)|^2\bigg] + \mathcal{O}(1).
\end{aligned}$$

The expression in the brackets on the right-hand side is exactly the energy functional associated with the time-dependent modified Gross–Pitaevskii equation (4-8). By energy conservation, we conclude that

$$\begin{aligned}
&C_{N,t} + N\langle i\partial_t\tilde{\varphi}_t, \tilde{\varphi}_t\rangle \\
&\qquad = N\bigg[\int |\nabla\varphi(x)|^2\, dx + \frac{1}{2}\int dx\, dy\, N^3 V(N(x-y)) f_\ell(N(x-y))|\varphi(x)|^2|\varphi(y)|^2\bigg] + \mathcal{O}(1). \qquad (6\text{-}9)
\end{aligned}$$

Observe that, with (4-3),

$$\begin{aligned}\int dx\,dy\,N^3V(N(x-y))f_\ell(N(x-y))|\varphi(x)|^2|\varphi(y)|^2\\ &= \int dx\,dy V(y)f_\ell(y)|\varphi(x)|^2|\varphi(x+y/N)|^2\\ &= [8\pi a_0+\mathcal{O}(N^{-1})]\int|\varphi(x)|^4\,dx+\int dx\,dy\,V(y)f_\ell(y)|\varphi(x)|^2\,[|\varphi(x+y/N)|^2-|\varphi(x)|^2],\end{aligned} \tag{6-10}$$

where

$$\begin{aligned}\left|\int dx\,dy\,V(y)f(y)|\varphi(x)|^2[|\varphi(x+y/N)|^2-|\varphi(x)|^2]\right|\\ &\le N^{-1}\int_0^1 ds\int dx\,dy\,V(y)f(y)|\varphi(x)|^2|\nabla\varphi(x+sy/N)|\,|\varphi(x+y/N)|\,|y|\\ &\le CN^{-1}\end{aligned}$$

for a constant $C>0$ depending only on the H^1-norm of φ. Inserting the last bound and (6-10) in (6-9), we conclude that

$$C_{N,t}+N\langle i\partial_t\tilde{\varphi}_t,\tilde{\varphi}_t\rangle = N\mathcal{E}_{GP}(\varphi)+\mathcal{O}(1),$$

as claimed. □

With Proposition 6.1 and Lemma 6.2, we can now conclude the proof of our main theorems.

Proof of Theorems 1.1 and 1.2. We observe, first of all, that, by Proposition 4.2,

$$|\langle\varphi_t,\gamma_{N,t}^{(1)}\varphi_t\rangle-\langle\tilde{\varphi}_t,\gamma_{N,t}^{(1)}\tilde{\varphi}_t\rangle|\le 2\|\varphi_t-\tilde{\varphi}_t\|\le CN^{-1}\exp(c\exp(c|t|)). \tag{6-11}$$

Hence, it is enough to compute

$$\begin{aligned}\langle\tilde{\varphi}_t,\gamma_{N,t}^{(1)}\tilde{\varphi}_t\rangle &= \frac{1}{N}\langle e^{-iH_Nt}\psi_N,a^*(\tilde{\varphi}_t)a(\tilde{\varphi}_t)e^{-iH_Nt}\psi_N\rangle\\ &= \frac{1}{N}\langle U_{N,t}e^{-iH_Nt}\psi_N,(N-\mathcal{N})U_{N,t}e^{-iH_Nt}\psi_N\rangle\\ &= 1-\frac{1}{N}\langle U_{N,t}e^{-iH_Nt}\psi_N,\mathcal{N}U_{N,t}e^{-iH_Nt}\psi_N\rangle.\end{aligned}$$

We define $\xi=e^{-B(\eta_0)}U_{N,0}\psi_N\in\mathcal{F}_{\perp\varphi}^{\le N}$. Then we have $\psi_N=U_{N,0}^*e^{B(\eta_0)}\xi$ and therefore

$$1-\langle\tilde{\varphi}_t,\gamma_{N,t}^{(1)}\tilde{\varphi}_t\rangle=\frac{1}{N}\langle\mathcal{W}_{N,t}\xi,e^{-B(\eta_t)}\mathcal{N}e^{B(\eta_t)}\mathcal{W}_{N,t}\xi\rangle\le\frac{C}{N}\langle\mathcal{W}_{N,t}\xi,\mathcal{N}\mathcal{W}_{N,t}\xi\rangle,$$

where we applied Lemma 3.1. By Proposition 6.1, we conclude that

$$1-\langle\tilde{\varphi}_t,\gamma_{N,t}^{(1)}\tilde{\varphi}_t\rangle\le N^{-1}\exp(c\exp(c|t|))\,\langle\xi,[(\mathcal{G}_{N,0}-C_{N,0})+C(\mathcal{N}+1)]\xi\rangle. \tag{6-12}$$

In order to apply Proposition 6.1, we used here the assumption (valid in the proofs of both Theorem 1.1 and Theorem 1.2) that $\tilde{\varphi}_{t=0}=\varphi\in H^4(\mathbb{R}^3)$.

Recalling from (1-10) the definition $a_N = 1 - \langle \varphi, \gamma_N^{(1)} \varphi \rangle$, we bound, with the above definition of ξ,

$$\begin{aligned}\langle \xi, \mathcal{N}\xi \rangle &= \langle U_{N,0}\psi_N, e^{B(\eta_0)}\mathcal{N} e^{-B(\eta_0)} U_{N,0}\psi_N \rangle \\ &\le C\langle U_{N,0}\psi_N, \mathcal{N} U_{N,0}\psi_N \rangle \\ &= C\langle \psi_N, (N - a^*(\varphi)a(\varphi))\psi_N \rangle \\ &= CN(1 - \langle \varphi, \gamma_N^{(1)}\varphi \rangle) = CNa_N.\end{aligned}$$

We still have to bound the expectation of $(\mathcal{G}_{N,0} - C_{N,0})$ in the state ξ. We have

$$\mathcal{G}_{N,0} = i\partial_t e^{-B(\eta_t)}|_{t=0} e^{B(\eta_0)} + e^{-B(\eta_0)}[(i\partial_t U_{N,t})|_{t=0} U_{N,0}^* + U_{N,0} H_N U_{N,0}^*] e^{B(\eta_0)}.$$

With Proposition 5.12, we find

$$|\langle \xi, i\partial_t e^{-B(\eta_t)}|_{t=0} e^{B(\eta_0)} \xi \rangle| \le C\langle \xi, (\mathcal{N}+1)\xi \rangle \le CNa_N + C. \tag{6-13}$$

From (5-2), we obtain

$$\begin{aligned}&\langle e^{B(\eta_0)}\xi, (i\partial_t U_{N,t})|_{t=0}\, U_{N,0}^*\, e^{B(\eta_0)}\xi \rangle \\ &\quad = -\langle (i\partial_t \tilde{\varphi}_t)|_{t=0}, \varphi \rangle \langle U_{N,0}\psi_N, (N-\mathcal{N})U_{N,0}\psi_N \rangle - 2\,\mathrm{Re}\langle U_{N,0}\psi_N, \sqrt{N-\mathcal{N}}\, a(q_0 (i\partial_t \tilde{\varphi}_t)|_{t=0}) U_{N,0}\psi_N \rangle \\ &\quad = -N\langle (i\partial_t \tilde{\varphi}_t)|_{t=0}, \varphi \rangle + N\langle (i\partial_t \tilde{\varphi}_t)|_{t=0}, \varphi \rangle (1 - \langle \varphi, \gamma_N^{(1)}\varphi \rangle) - 2N\,\mathrm{Re}\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle.\end{aligned}$$

Combining this identity with the bound (6-13) and with the observation that, by the definition of ξ,

$$\langle \xi, e^{-B(\eta_0)} U_{N,0} H_N U_{N,0}^* e^{B(\eta_0)} \xi \rangle = \langle \psi_N, H_N \psi_N \rangle,$$

we conclude that

$$\begin{aligned}&\langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi \rangle \\ &\qquad \le [\langle \psi_N, H_N\psi_N \rangle - (C_{N,0} + N\langle (i\partial_t \tilde{\varphi}_t)|_{t=0}, \varphi \rangle)] - 2N\,\mathrm{Re}\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle + CNa_N + C.\end{aligned}$$

Hence, with Lemma 6.2, we get

$$\langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi \rangle \le [\langle \psi_N, H_N\psi_N \rangle - N\mathcal{E}_{\mathrm{GP}}(\varphi)] - 2N\,\mathrm{Re}\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle + CNa_N + C, \tag{6-14}$$

where $\mathcal{E}_{\mathrm{GP}}$ denotes the translation-invariant Gross–Pitaevskii functional defined in (1-15).

To bound the second term on the right-hand side of the last equation, we proceed differently depending on whether we want to show Theorem 1.1 or Theorem 1.2. To prove Theorem 1.2, we notice that

$$\begin{aligned}\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle &= \langle \varphi, \gamma_N^{(1)} (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle - \langle \varphi, \gamma_N^{(1)}\varphi \rangle \langle \varphi, (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle \\ &= \langle \varphi, (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle (1 - \langle \varphi, \gamma_N^{(1)}\varphi \rangle) + \langle \varphi, (\gamma^{(1)} - |\varphi\rangle\langle\varphi|)(i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle.\end{aligned}$$

With $\tilde{a}_N = \mathrm{tr}\,|\gamma_N^{(1)} - |\varphi\rangle\langle\varphi||$, we obtain that

$$|\langle \varphi, \gamma_N^{(1)} q_0 (i\partial_t \tilde{\varphi}_t)|_{t=0} \rangle| \le C(a_N + \tilde{a}_N).$$

Since $a_N \le \tilde{a}_N$, we conclude from (6-14) that

$$\langle \xi, (\mathcal{G}_{N,0} - C_{N,0})\xi \rangle \le C[N\tilde{a}_N + N\tilde{b}_N + 1].$$

Inserting in (6-12) and using (6-11), we arrive at

$$1-\langle\varphi_t,\gamma_N^{(1)}\varphi_t\rangle\le C[\tilde{a}_N+\tilde{b}_N+N^{-1}]\exp(c\exp(c|t|)).$$

This concludes the proof of Theorem 1.2.

To show Theorem 1.1, we use instead the fact that

$$i\partial_t\tilde{\varphi}_t|_{t=0}=-\Delta\varphi+(N^3V(N\,\cdot\,)f_\ell(N\,\cdot\,)*|\varphi|^2)\varphi.$$

Since here we assume that the initial data $\varphi=\phi_{\text{GP}}$ is the minimizer of the Gross–Pitaevskii energy functional (1-6), it must satisfy the Euler–Lagrange equation

$$-\Delta\varphi+V_{\text{ext}}\varphi+8\pi a_0|\varphi|^2\varphi=\mu\varphi$$

for some $\mu\in\mathbb{R}$. We find

$$i\partial_t\tilde{\varphi}_t|_{t=0}=\mu\varphi-V_{\text{ext}}\varphi+[(N^3V(N\,\cdot\,)f_\ell(N\,\cdot\,)*|\varphi|^2)-8\pi a_0|\varphi|^2]\varphi.$$

Using (4-3), the fact that the minimizer φ of (1-6) is continuously differentiable and vanishes at infinity, see [Lieb et al. 2000, Theorem 2.1], we obtain

$$\|[(N^3V(N\,\cdot\,)f_\ell(N\,\cdot\,)*|\varphi|^2)-8\pi a_0|\varphi|^2]\varphi\|_2\le CN^{-1}$$

and therefore

$$-2N\,\text{Re}\langle\varphi,\gamma_N^{(1)}q_0(i\partial_t\tilde{\varphi}_t)|_{t=0}\rangle\le 2N\,\text{Re}\langle\varphi,\gamma_N^{(1)}q_0(V_{\text{ext}}+\kappa)\varphi\rangle+C$$

for any constant $\kappa\in\mathbb{R}$. Choosing $\kappa\ge 0$ so that $V_{\text{ext}}+\kappa\ge 0$ (from the assumptions, V_{ext} is bounded below), we find

$$\begin{aligned}-2N\,\text{Re}\langle\varphi,\gamma_N^{(1)}q_0(i\partial_t\tilde{\varphi}_t)|_{t=0}\rangle&\le 2N\,\text{Re}\langle\varphi,\gamma_N^{(1)}(V_{\text{ext}}+\kappa)\varphi\rangle-2N\langle\varphi,\gamma_N^{(1)}\varphi\rangle\langle\varphi,(V_{\text{ext}}+\kappa)\varphi\rangle+C\\&\le 2N\,\text{Re}\langle\varphi,\gamma_N^{(1)}(V_{\text{ext}}+\kappa)\varphi\rangle-2N\langle\varphi,(V_{\text{ext}}+\kappa)\varphi\rangle+C(Na_N+1).\end{aligned}$$

With Cauchy–Schwarz and since $0\le\gamma_N^{(1)}\le 1$ implies that $(\gamma_N^{(1)})^2\le\gamma_N^{(1)}$, we get

$$\begin{aligned}-2N\,\text{Re}\langle\varphi,\gamma_N^{(1)}q_0(i\partial_t\tilde{\varphi}_t)|_{t=0}\rangle&\le N\langle\varphi,\gamma_N^{(1)}(V_{\text{ext}}+\kappa)\gamma_N^{(1)}\varphi\rangle-N\langle\varphi,(V_{\text{ext}}+\kappa)\varphi\rangle+C(Na_N+1)\\&\le N\,\text{tr}\;\gamma_N^{(1)}V_{\text{ext}}-N\langle\varphi,V_{\text{ext}}\varphi\rangle+C(Na_N+1).\end{aligned}$$

Inserting back in (6-14) we conclude that, under the assumptions of Theorem 1.1,

$$\langle\xi,(\mathcal{G}_{N,0}-C_{N,0})\xi\rangle\le[\langle\psi_N,H_N^{\text{trap}}\psi_N\rangle-N\mathcal{E}_{\text{GP}}^{\text{trap}}(\varphi)]+CNa_N+C\le C[Na_N+Nb_N+1].$$

With (6-12) and (6-11), we find now

$$1-\langle\varphi_t,\gamma_{N,t}^{(1)}\varphi_t\rangle\le C[a_N+b_N+N^{-1}]\exp(c\exp(c|t|)).$$

This concludes the proof of Theorem 1.1. □

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DIMENSIONAL CROSSOVER WITH A CONTINUUM OF CRITICAL EXPONENTS FOR NLS ON DOUBLY PERIODIC METRIC GRAPHS

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We investigate the existence of ground states for the focusing nonlinear Schrödinger equation on a prototypical doubly periodic metric graph. When the nonlinearity power is below 4, ground states exist for every value of the mass, while, for every nonlinearity power between 4 (included) and 6 (excluded), a mark of L^2-criticality arises, as ground states exist if and only if the mass exceeds a threshold value that depends on the power. This phenomenon can be interpreted as a continuous transition from a two-dimensional regime, for which the only critical power is 4, to a one-dimensional behavior, in which criticality corresponds to the power 6. We show that such a dimensional crossover is rooted in the coexistence of one-dimensional and two-dimensional Sobolev inequalities, leading to a new family of Gagliardo–Nirenberg inequalities that account for this continuum of critical exponents.

1. Introduction

Since the first appearance of branched structures in the modeling of organic molecules [Ruedenberg and Scherr 1953], through the development of the mathematical theory of quantum graphs [Berkolaiko and Kuchment 2013; Post 2012], networks (or metric graphs) have provided a general and flexible tool to describe dynamics in complex structures like systems of quantum wires, Josephson junctions, propagation of signals through waveguides, and some related technologies. Pioneering studies about nonlinear systems on metric graphs appeared in [Ali Mehmeti 1994; Ali Mehmeti et al. 2001], but more recently the research on such topics has grown rapidly, and several results have been achieved on propagation of solitary waves [Adami et al. 2011; Caudrelier 2015; Sobirov et al. 2010] and on stationary states [Sabirov et al. 2013; Cacciapuoti et al. 2015; Noja 2014; Noja et al. 2015; Pelinovsky and Schneider 2017; Gnutzmann and Waltner 2016].

In a series of recent works [Adami et al. 2015a; 2015b; 2016] we investigated the problem of existence of ground states for the energy functional associated to the focusing, L^2-subcritical and critical nonlinear Schrödinger (NLS) equation

$$i\,\partial_t u(t) = -u''(t) - |u(t)|^{p-2}u(t) \tag{1}$$

on finite noncompact metric graphs, i.e., branched structures with a finite number of vertices and edges, and at least one infinite edge (i.e., a half-line).

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Keywords: metric graphs, Sobolev inequalities, threshold phenomena, nonlinear Schrödinger equation.

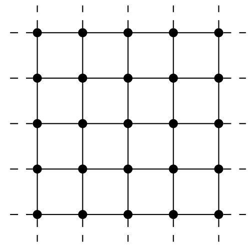

Figure 1. The grid $\mathcal{G}$.

Specifically, by *ground state* on a metric graph $\mathcal{G}$ we mean every global minimizer of the energy functional

$$E_p(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx \tag{2}$$

in the class of $H^1(\mathcal{G})$ functions with fixed L^2-norm (or mass) $\mu > 0$. The constraint is dynamically meaningful as the mass, as well as the energy, is conserved by the NLS flow, and the problem of the existence of ground states is particularly relevant in the physics of Bose–Einstein condensates; see, e.g., [Adami et al. 2015a; 2015b; 2016; 2017b, Section 1].

In this paper we extend the analysis of the existence of ground states to a prototypical *doubly periodic* metric graph $\mathcal{G}$, particularly relevant in the applications, for which the techniques developed in previous works (where noncompactness was due to one or more unbounded edges) do not apply: a two-dimensional infinite grid isometrically embedded in $\mathbb{R}^2$, with vertices on the lattice $\mathbb{Z}^2$ and edges of unit length (see Figure 1).

Schrödinger equations on periodic metric graphs have received considerable attention in the last few years. Linear problems have been extensively studied, and a fairly complete spectral analysis is now available for different types of coupling conditions. We refer for instance to the early papers [Exner 1996; Exner and Gawlista 1996] treating rectangular lattices, as well as to Chapter 4 in [Berkolaiko and Kuchment 2013] for a more up-to-date overview of several results in a general periodic setting. Concerning the square grid we focus on, we specifically quote [Exner and Turek 2010] for some results strictly rooted in the two-dimensional nature of the domain.

More recently, nonlinear problems have been addressed too. For instance, [Pelinovsky and Schneider 2017] considers a specific example of a structure periodic along a single direction, the so-called *necklace* graph, via bifurcation techniques. From a variational point of view, the first investigation for very general periodic graphs can be found in [Pankov 2018], where the approach is based on the Nehari method. We notice that, for this reason, in that paper the problem of the existence of ground states with prescribed mass cannot be dealt with.

Let us now discuss our results. We first note, roughly speaking, that macroscopically the grid $\mathcal{G}$ has dimension 2, while microscopically it is of dimension 1. This peculiarity is absent in graphs with a finite number of half-lines, where the two-dimensional scale is lacking, as well as in other two-dimensional structures like $\mathbb{Z}^2$, where edges are missing and there is of course no microscopic one-dimensional structure

[Weinstein 1999]. The presence of two scales in $\mathcal{G}$ results in a transition from a one-dimensional to a two-dimensional behavior, which emerges in functional inequalities and influences the existence of ground states. We shall refer to this phenomenon as *dimensional crossover*.

Before commenting further on this point, it is convenient to state our main results in a precise form. We define, for $\mu > 0$, the mass-constrained set

$$H^1_\mu(\mathcal{G}) = \left\{u \in H^1(\mathcal{G}) : \textstyle\int_{\mathcal{G}} |u|^2\,dx = \mu\right\} \tag{3}$$

and the corresponding "ground-state energy level"

$$\mathcal{E}_p(\mu) = \inf_{u \in H^1_\mu(\mathcal{G})} E_p(u), \tag{4}$$

considered as a function $\mathcal{E}_p : (0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ of the mass μ. By a "ground states of mass μ" we mean a function $u \in H^1_\mu(\mathcal{G})$ such that

$$E_p(u) = \mathcal{E}_p(\mu).$$

When $p \in (2, 4)$, ground states exist for every prescribed mass.

Theorem 1.1 (subcritical case). *Assume $2 < p < 4$. Then for every $\mu > 0$ there exists a ground state of mass μ, and $\mathcal{E}_p(\mu) < 0$.*

The picture changes as the exponent of the nonlinearity increases.

Theorem 1.2 (dimensional crossover). *For every $p \in [4, 6]$ there exists a critical mass $\mu_p > 0$ such that*:

(i) *If $p \in (4, 6)$ then ground states of mass μ exist if and only if $\mu \geq \mu_p$, and*

$$\begin{aligned} &\mathcal{E}_p(\mu) = 0 \quad \text{if } \mu \leq \mu_p, \\ &\mathcal{E}_p(\mu) < 0 \quad \text{if } \mu > \mu_p. \end{aligned} \tag{5}$$

(ii) *If $p = 4$ then ground states of mass μ exist if $\mu > \mu_4$, whereas they do not exist if $\mu < \mu_4$. Moreover* (5) *is valid also when $p = 4$.*

(iii) *If $p = 6$ then there are no ground states, regardless of the value of μ, and*

$$\begin{aligned} &\mathcal{E}_6(\mu) = 0 \qquad \text{if } \mu \leq \mu_6, \\ &\mathcal{E}_6(\mu) = -\infty \quad \text{if } \mu > \mu_6. \end{aligned} \tag{6}$$

We point out that, when $p = 4$, the existence of ground states of mass $\mu = \mu_4$ is still an open problem. For the sake of completeness, we also mention that when $p > 6$ one has $\mathcal{E}_p(\mu) \equiv -\infty$ for every μ, as one can easily see by a scaling argument.

In order to interpret Theorems 1.1 and 1.2, let us recall that in $\mathbb{R}^d$, for the minimization of the NLS energy under a mass constraint, there exists a *critical exponent* p^*_d such that

(1) if $p < p^*_d$, for every mass $\mu > 0$ the ground-state energy level is finite and negative, and is attained by a ground state;

(2) if $p > p^*_d$, for every mass $\mu > 0$ the ground-state energy level equals $-\infty$.

It is well known [Cazenave 2003] that $p_d^* = 4/d + 2$ for the NLS in $\mathbb{R}^d$, yielding $p_1^* = 6$ for $\mathbb{R}$ and $p_2^* = 4$ for $\mathbb{R}^2$. Furthermore, it has been proved in [Adami et al. 2015b; 2016] that for finite noncompact graphs (i.e., graphs with finitely many edges, at least one of them being unbounded) the critical exponent is 6, exactly as for $\mathbb{R}$. Thus the exponents considered in Theorem 1.1 are subcritical both in dimension 1 and 2, which reflects into the typical subcritical flavor of the result.

In fact, the main novelty of the paper emerges in Theorem 1.2 and lies in the "splitting" of the critical exponent p_d^* induced by the twofold nature (one-/two-dimensional) of the grid. Indeed, on the grid $\mathcal{G}$:

(1) $p = 4$ is the supremum of those exponents p such that $\mathcal{E}_p(\mu)$ is finite and negative (and attained by a ground state) for every $\mu > 0$.

(2) $p = 6$ is the infimum of those exponents p such that $\mathcal{E}_p(\mu) = -\infty$ for every $\mu > 0$.

Besides, let us stress another remarkable aspect of the dimensional crossover. In $\mathbb{R}^d$, as well as on noncompact finite graphs, the critical exponent is characterized by the existence of a *critical mass* in the following sense: for smaller masses every function has positive energy, while for larger masses there are functions with negative energy (as already mentioned, on a noncompact finite graph such a critical mass arises only when $p = 6$).

On the contrary, on the grid $\mathcal{G}$ a similar notion of critical mass (the number μ_p in Theorem 1.2) arises for every $p \in [4, 6]$, so that, in this respect, *every exponent within this range is, in fact, critical* (see Remark 2.5). Beyond this critical mass, however, the energy is still bounded from below and a ground state exists, as if the problem had kept track of the subcriticality of the exponent $p < 6$ at the microscopic scale.

From the point of view of functional analysis, the dimensional crossover is due to the simultaneous validity, for every function $u \in W^{1,1}(\mathcal{G})$, of the two inequalities

$$\|u\|_{L^\infty(\mathcal{G})} \le \|u'\|_{L^1(\mathcal{G})}, \quad \|u\|_{L^2(\mathcal{G})} \le \|u'\|_{L^1(\mathcal{G})}. \tag{7}$$

Of these, the former is typical of dimension 1, modeled on the well-known inequality

$$\|v\|_{L^\infty(\mathbb{R})} \le \tfrac{1}{2}\|v'\|_{L^1(\mathbb{R})} \quad \text{for all } v \in W^{1,1}(\mathbb{R}), \tag{8}$$

while the latter is the formal analogue of the Sobolev inequality in $\mathbb{R}^2$

$$\|v\|_{L^2(\mathbb{R}^2)} \le C\|\nabla v\|_{L^1(\mathbb{R}^2)} \quad \text{for all } v \in W^{1,1}(\mathbb{R}^2),$$

and is typical of dimension 2. As discussed in Section 2, either inequality in (7) yields a particular version of the Gagliardo–Nirenberg inequality in $H^1(\mathcal{G})$ ((12) and (18) respectively). By interpolation, one obtains the *critical* Gagliardo–Nirenberg inequalities

$$\int_{\mathcal{G}} |u|^p\,dx \le K_p \left(\int_{\mathcal{G}} |u|^2\,dx\right)^{(p-2)/2} \int_{\mathcal{G}} |u'|^2\,dx \quad \text{for all } u \in H^1(\mathcal{G}), \tag{9}$$

which, being valid for *every* exponent $p \in [4, 6]$, give rise to a continuum of critical exponents (see also Remark 2.5). Indeed, using (9), the NLS energy in (2) can be estimated from below as

$$E_p(u) \geq \frac{1}{2}\left(1 - \frac{2K_p}{p}\mu^{(p-2)/2}\right)\int_{\mathcal{G}} |u'|^2\,dx,$$

which shows that $E_p(u) \geq 0$ for every $u \in H^1_\mu(\mathcal{G})$, as soon as

$$\mu \leq \left(\frac{p}{2K_p}\right)^{2/(p-2)} =: \mu_p.$$

The number in the right-hand side of this inequality is thus the critical mass μ_p of Theorem 1.2.

Finally we would like to point out that we have chosen the grid $\mathcal{G}$ to illustrate our results because it is the simplest doubly periodic metric graph, on which computations and proofs are particularly transparent. It should be clear however that many other doubly periodic graphs can be treated with the methods developed in the present work. Among these, we explicitly mention the hexagonal grid, a model for *graphene*.

At the core of the results stands the double periodicity of the graph, which is responsible for the occurrence of phenomena such as the dimensional crossover. To exploit the double periodicity on a concrete given graph one must of course alter some parts of the proofs presented in this paper (e.g., the proof of Theorem 2.2) to adapt them to the particular features of the graph under study. We plan to illustrate this with the detailed study of some other particular graphs, significantly relevant for the applications, in forthcoming papers.

2. Inequalities

In this section we establish some fundamental inequalities for functions on the grid.

For notational purposes, it is convenient to describe the grid $\mathcal{G}$ as isometrically embedded in $\mathbb{R}^2$, with the lattice $\mathbb{Z}^2$ as set of vertices, and an edge of length 1 joining every pair of adjacent vertices. In this way, it is natural to interpret $\mathcal{G}$ as the union of horizontal lines $\{H_j\}$ and vertical lines $\{V_k\}$, which cross at every vertex $(k, j) \in \mathbb{Z}^2$.

As on any metric graph, to deal with the energy functional (2), the natural functional framework is given by the standard spaces $L^p(\mathcal{G})$ and $H^1(\mathcal{G})$. With the notation for $\mathcal{G}$ introduced above, for the L^p norms we have

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{j\in\mathbb{Z}} \|u\|^p_{L^p(H_j)} + \sum_{k\in\mathbb{Z}} \|u\|^p_{L^p(V_k)} = \sum_{j\in\mathbb{Z}} \int_{H_j} |u(x)|^p\,dx + \sum_{k\in\mathbb{Z}} \int_{V_k} |u(x)|^p\,dx < \infty \tag{10}$$

and

$$\|u\|_{L^\infty(\mathcal{G})} = \sup_{j,k}\{\|u\|_{L^\infty(H_j)}, \|u\|_{L^\infty(V_k)}\}, \tag{11}$$

while

$$\|u\|^2_{H^1(\mathcal{G})} = \|u\|^2_{L^2(\mathcal{G})} + \|u'\|^2_{L^2(\mathcal{G})}.$$

Here, as usual, $H^1(\mathcal{G})$ denotes the space of functions on $\mathcal{G}$ whose restriction to every horizontal and vertical line belongs to $H^1(\mathbb{R})$, and that, in addition, are continuous at every vertex of $\mathcal{G}$. In Theorem 2.2 we shall also need the space $W^{1,1}(\mathcal{G})$, similarly defined as the space of functions on $\mathcal{G}$ whose restriction to every horizontal and vertical line belongs to $W^{1,1}(\mathbb{R})$ and that, in addition, are continuous at every vertex.

Remark. In the following, symbols like $\|u\|_p$ stand for $\|u\|_{L^p(\mathcal{G})}$. When the domain of integration is different from $\mathcal{G}$, it will always be indicated in the norm.

First we recall the standard Gagliardo–Nirenberg inequality, which (up to a multiplicative constant $C > 1$ on the right-hand side) is valid on any noncompact metric graph; a proof in the general framework can be found in [Adami et al. 2016]. Here, for the sake of completeness, we shall give a short proof tailored to the grid $\mathcal{G}$ which, by the way, yields a slightly sharper estimate.

Theorem 2.1 (one-dimensional Gagliardo–Nirenberg inequality). *For every $p \in [2, \infty)$ one has*

$$\|u\|_p \le \|u\|_2^{1/2+1/p} \|u'\|_2^{1/2-1/p} \quad \textit{for all } u \in H^1(\mathcal{G}) \tag{12}$$

and, *moreover*,

$$\|u\|_\infty \le \|u\|_2^{1/2} \|u'\|_2^{1/2} \quad \textit{for all } u \in H^1(\mathcal{G}). \tag{13}$$

Proof. Since $\|u\|_p \le \|u\|_\infty^{1-2/p} \|u\|_2^{2/p}$, it suffices to prove (13). On the other hand, given $u \in H^1(\mathcal{G})$, we have $u^2 \in W^{1,1}(H_j)$ for every horizontal line H_j of $\mathcal{G}$. Then, applying (8) with $v = u^2$ on H_j yields

$$\|u\|^2_{L^\infty(H_j)} \le \int_{H_j} |u(x)u'(x)|\, dx \le \|u\|_{L^2(H_j)} \|u'\|_{L^2(H_j)} \le \|u\|_{L^2(\mathcal{G})} \|u'\|_{L^2(\mathcal{G})}.$$

Since clearly this inequality remains true if we replace H_j with any vertical line V_k, (13) follows immediately from (11). □

As already mentioned, inequalities like (12) and (13) hold for every noncompact graph. On the contrary, the next inequality, and its consequences below, rely on the two-dimensional web structure of the grid $\mathcal{G}$.

Theorem 2.2 (two-dimensional Sobolev inequality). *For every $u \in W^{1,1}(\mathcal{G})$*,

$$\|u\|_2 \le \tfrac{1}{2} \|u'\|_1. \tag{14}$$

Proof. Given $u \in W^{1,1}(\mathcal{G})$, we have

$$\|u\|_2^2 = \sum_{j \in \mathbb{Z}} \int_{H_j} |u(x)|^2\, dx + \sum_{k \in \mathbb{Z}} \int_{V_k} |u(y)|^2\, dy. \tag{15}$$

First observe that, for each k, using (8) we obtain

$$\int_{V_k} |u(y)|^2\, dy \le \|u\|_{L^\infty(V_k)} \int_{V_k} |u(y)|\, dy \le \tfrac{1}{2} \|u'\|_{L^1(V_k)} \int_{V_k} |u(y)|\, dy. \tag{16}$$

Then, for each $j \in \mathbb{Z}$, consider the horizontal lines H_j and H_{j+1}, and denote by P_j the path in $\mathcal{G}$ obtained by joining together the half-line of H_j to the left of V_k, the vertical segment of V_k between H_j and H_{j+1} (which we denote by I_j), and the half-line of H_{j+1} to the right of V_k (see Figure 2).

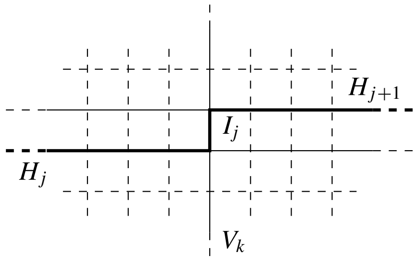

Figure 2. The path P_j (thick in the picture).

Since in particular $u \in W^{1,1}(P_j)$, and the metric graph P_j is isometric to $\mathbb{R}$, we find from (8)

$$|u(y)| \le \frac{1}{2}\int_{P_j} |u'(x)|\,dx \quad \text{for all } y \in I_j$$

and, since I_j has length 1, integrating this inequality over I_j yields

$$\int_{I_j} |u(y)|\,dy \le \frac{1}{2}\int_{P_j} |u'(x)|\,dx \quad \text{for all } j \in \mathbb{Z}. \tag{17}$$

Now observe that

$$V_k = \bigcup_{j\in\mathbb{Z}} I_j, \qquad \bigcup_{j\in\mathbb{Z}} P_j = V_k \cup \bigcup_{j\in\mathbb{Z}} H_j,$$

and moreover, up to a negligible set, the paths $\{P_j\}$ $(j \in \mathbb{Z})$ are mutually disjoint: therefore, summing (17) over $j \in \mathbb{Z}$ yields

$$\int_{V_k} |u(y)|\,dy \le \frac{1}{2}\Big(\int_{V_k} |u'(y)|\,dy + \sum_j \int_{H_j} |u'(x)|\,dx\Big) = \frac{1}{2}\Big(v_k + \sum_j h_j\Big)$$

having set, for brevity, $v_k = \int_{V_k} |u'(y)|\,dy$ and $h_j = \int_{H_j} |u'(x)|\,dx$. Combining with (16), and summing over k, one obtains

$$\sum_k \int_{V_k} |u(y)|^2\,dy \le \frac{1}{4}\sum_k v_k\Big(v_k + \sum_j h_j\Big).$$

Of course, by the symmetry of $\mathcal{G}$, we also have

$$\sum_j \int_{H_j} |u(x)|^2\,dx \le \frac{1}{4}\sum_j h_j\Big(h_j + \sum_k v_k\Big),$$

and summing the last two inequalities we find

$$\|u\|^2_{L^2(\mathcal{G})} \le \frac{1}{4}\Big(\sum_k (h_k^2 + v_k^2) + 2\sum_{j,k} h_j v_k\Big) \le \frac{1}{4}\Big(\sum_k h_k + v_k\Big)^2 = \frac{1}{4}\|u'\|^2_{L^1(\mathcal{G})}. \qquad \square$$

Theorem 2.3 (two-dimensional Gagliardo–Nirenberg inequality). *For every $p \in [2, \infty)$ one has*

$$\|u\|_p \le C\|u\|_2^{2/p}\|u'\|_2^{1-2/p} \quad \textit{for all } u \in H^1(\mathcal{G}), \tag{18}$$

where C is an absolute constant.

Proof. Given $p \in [2, \infty)$, we have

$$\|u\|_p \le \|u\|_2^{1-\theta} \|u\|_{p+2}^{\theta}, \tag{19}$$

where

$$\frac{1-\theta}{2} + \frac{\theta}{p+2} = \frac{1}{p}, \quad \text{i.e.,} \quad \theta = 1 - \frac{4}{p^2}. \tag{20}$$

Now observe that $u \in L^\infty(\mathcal{G})$ by (13), and hence $u^{1+p/2}$ belongs to $W^{1,1}(\mathcal{G})$ since $p \ge 2$. Therefore, we can replace u with $u^{1+p/2}$ in (14), thus obtaining

$$\|u\|_{p+2}^{1+p/2} \le \frac{p+2}{4} \int_{\mathcal{G}} |u(x)|^{p/2} |u'(x)|\, dx \le \frac{p+2}{4} \|u\|_p^{p/2} \|u'\|_2.$$

Raising to the power $2/(p+2)$ we find

$$\|u\|_{p+2} \le C \|u\|_p^{p/(p+2)} \|u'\|_2^{2/(p+2)}, \quad C = \sup_{p \ge 2} \left(\frac{p+2}{4} \right)^{2/(p+2)}; \tag{21}$$

one may take, e.g., $C = \frac{3}{2}$. Plugging this inequality into (19) gives

$$\|u\|_p \le \|u\|_2^{1-\theta} C^\theta \|u\|_p^{\theta p/(p+2)} \|u'\|_2^{2\theta/(p+2)}$$

and (18) follows using (20), after elementary computations. □

Corollary 2.4 (interdimensional Gagliardo–Nirenberg inequality). *There exists a universal constant $C > 0$ such that, for every $p \in [2, \infty)$,*

$$\|u\|_p \le C \|u\|_2^{1-\alpha} \|u'\|_2^{\alpha} \quad \textit{for all } \alpha \in \left[\frac{p-2}{2p}, \frac{p-2}{p} \right], \textit{ for all } u \in H^1(\mathcal{G}). \tag{22}$$

In particular, for every $p \in [4, 6]$ there exists a constant K_p, depending only on p, such that

$$\|u\|_p^p \le K_p \|u\|_2^{p-2} \|u'\|_2^2 \quad \textit{for all } u \in H^1(\mathcal{G}). \tag{23}$$

Proof. Observe that (22) reduces to (12) (where $C = 1$) when $\alpha = (p-2)/(2p)$, while it reduces to (18), where $C \le \frac{3}{2}$ by (21), when $\alpha = (p-2)/p$. Then (22) is established also for every intermediate value of α, since the right-hand side is a convex function of α, with a constant C independent of p and α.

Finally, when $p \in [4, 6]$, (23) is obtained letting $\alpha = 2/p$ in (22) (the condition $p \in [4, 6]$ guarantees that this choice of α is admissible). The constant K_p in (23) is the best possible (i.e., the smallest); of course $K_p \le C^p$ for every $p \in [4, 6]$, where C is the constant appearing in (22). □

Remark 2.5. In $\mathbb{R}^d$, when dealing with the NLS energy

$$\frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{p} \|u\|_{L^p(\mathbb{R}^d)}^p$$

in the presence of an L^2 mass constraint, the relevant version of the Gagliardo–Nirenberg (G-N) inequality is

$$\|u\|_{L^p(\mathbb{R}^d)} \le C \|u\|_{L^2(\mathbb{R}^d)}^{1-\alpha} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\alpha}, \quad \alpha = \frac{d(p-2)}{2p}, \tag{24}$$

valid as soon as $\alpha \in [0, 1)$; see [Leoni 2009]. When $p = 2+4/d$, this inequality becomes *critical* for the NLS energy because $\alpha = 2/p$ (i.e., the exponents in the inequality become as in (23)), and a critical mass μ_p comes into play. Now, while in (24) this *critical exponent* $p = 2+4/d$ is uniquely determined by the ambient space $\mathbb{R}^d$, on the grid $\mathcal{G}$ *every* $p \in [4, 6]$ is critical for the NLS energy, since one can let $\alpha = 2/p$ in (22) (and obtain (23)) not just for one particular p, but for every $p \in [4, 6]$.

Formally, solving for d in (24), for fixed α we can interpret (22) as a G-N inequality in dimension $d = 2\alpha p/(p-2)$: we call (22) *interdimensional* since d ranges over $[1, 2]$ as α varies (this is in contrast with (24), where the exponent α is uniquely determined by p and the space dimension d). With this interpretation, (23) (which is just (22) with $\alpha = 2/p$) can be seen as a critical G-N inequality in dimension $d = 4/(p-2)$ so that, formally, every $p \in [4, 6]$ can be seen as the critical exponent $p = 2+4/d$, in a fractal scaling dimension $d \in [1, 2]$.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Remark 3.1. Note that, for every $\mu > 0$ and $p < 6$, the one-dimensional Gagliardo–Nirenberg inequality (12) ensures that $\mathcal{E}_p(\mu)$ is finite and E_p is coercive on $H^1_\mu(\mathcal{G})$ [Adami et al. 2016].

Recalling (3) and (4), we first prove a dichotomy lemma for minimizing sequences, which is useful in proving the existence of ground states.

Lemma 3.2 (dichotomy). *Given $\mu > 0$ and $p \in (2, 6)$, let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence for E_p, i.e.,*

$$\lim_{n\to\infty} E_p(u_n) = \mathcal{E}_p(\mu),$$

and assume that $u_n \rightharpoonup u$ weakly in $H^1(\mathcal{G})$ and pointwise a.e. on $\mathcal{G}$. If

$$m := \mu - \|u\|_2^2 \in [0, \mu] \tag{25}$$

denotes the loss of mass in the limit, then either $m = 0$ or $m = \mu$.

Proof. We assume that $0 < m < \mu$ and seek a contradiction. According to the Brezis–Lieb lemma [1983], we can write

$$E_p(u_n) = E_p(u_n - u) + E_p(u) + o(1) \quad \text{as } n \to \infty, \tag{26}$$

and, since $u_n \rightharpoonup u$ in $L^2(\mathcal{G})$,

$$\|u_n - u\|_2^2 = \|u_n\|_2^2 + \|u\|_2^2 - 2\langle u_n, u\rangle_2 \to \mu - \|u\|_2^2 = m \tag{27}$$

as $n \to \infty$. Now, for n large enough,

$$\begin{aligned}\mathcal{E}_p(\mu) &\le E_p\left(\frac{\sqrt{\mu}}{\|u_n - u\|_2}(u_n - u)\right)\\ &= \frac{1}{2}\frac{\mu}{\|u_n - u\|_2^2}\|u_n' - u'\|_2^2 - \frac{1}{p}\frac{\mu^{p/2}}{\|u_n - u\|_2^p}\|u_n - u\|_p^p < \frac{\mu}{\|u_n - u\|_2^2}E_p(u_n - u),\end{aligned}$$

since $\|u_n - u\|_p \neq 0$ and $\|u_n - u\|_2^2 < \mu$. Thus,

$$E_p(u_n - u) > \frac{\|u_n - u\|_2^2}{\mu}\, \mathcal{E}_p(\mu),$$

and by (27)

$$\liminf_n E_p(u_n - u) \geq \frac{m}{\mu}\, \mathcal{E}_p(\mu).$$

Thus, taking the liminf in (26) we find

$$\mathcal{E}_p(\mu) \geq \frac{m}{\mu}\, \mathcal{E}_p(\mu) + E_p(u). \tag{28}$$

Similarly, since $u \not\equiv 0$ we also have

$$\mathcal{E}_p(\mu) \leq E_p\left(\frac{\sqrt{\mu}}{\sqrt{\mu - m}}\, u\right) \leq \frac{1}{2}\frac{\mu}{\mu - m}\|u'\|_2^2 - \frac{1}{p}\left(\frac{\mu}{\mu - m}\right)^{p/2}\|u\|_p^p < \frac{\mu}{\mu - m} E_p(u) \tag{29}$$

and, as $\mathcal{E}_p(\mu) > -\infty$ by Remark 3.1, from (28) we finally obtain

$$\mathcal{E}_p(\mu) > \frac{m}{\mu}\, \mathcal{E}_p(\mu) + \frac{\mu - m}{\mu}\, \mathcal{E}_p(\mu) = \mathcal{E}_p(\mu),$$

a contradiction. □

Proposition 3.3. *Assume $p < 6$ and $\mathcal{E}_p(\mu)$ is strictly negative. Then there exists $u \in H_\mu^1(\mathcal{G})$ such that*

$$E_p(u) = \mathcal{E}_p(\mu).$$

Proof. Let $\{u_n\} \subset H_\mu^1(\mathcal{G})$ be a minimizing sequence for E_p. Since $p < 6$, Remark 3.1 yields that $\mathcal{E}_p(\mu) > -\infty$ and u_n is bounded in $H^1(\mathcal{G})$, and by translating each u_n (exploiting the periodicity of $\mathcal{G}$) we can also assume that u_n attains its L^∞-norm on a compact set $\mathcal{K} \subset \mathcal{G}$ independent of n. Therefore, up to subsequences, u_n converges weakly in $H^1(\mathcal{G})$, and strongly in $L^\infty_{\text{loc}}(\mathcal{G})$, to some function $u \in H^1(\mathcal{G})$. Setting $m := \mu - \|u\|_2^2$, from Lemma 3.2 one sees that either $m = 0$ or $m = \mu$. If $m = \mu$ then $u \equiv 0$, but in this case $u_n \to 0$ in $L^\infty(\mathcal{G})$, since in particular, $u_n \to u \equiv 0$ uniformly on $\mathcal{K}$. Therefore we would have

$$E_p(u_n) \geq -\frac{1}{p}\|u_n\|_\infty^{p-2} \int_{\mathcal{G}} |u_n|^2\, dx = -\frac{\mu}{p}\|u_n\|_\infty^{p-2} \to 0,$$

contradicting the fact that $\mathcal{E}_p(\mu) < 0$.

Thus it must be that $m = 0$, so that $u_n \to u$ strongly in $L^2(\mathcal{G})$ and therefore $u \in H_\mu^1(\mathcal{G})$. Moreover, since u_n is bounded in $L^\infty(\mathcal{G})$, $u_n \to u$ strongly also in $L^p(\mathcal{G})$. Then

$$\mathcal{E}_p(\mu) \leq E_p(u) \leq \liminf_n E_p(u_n) = \mathcal{E}_p(\mu)$$

by weak lower semicontinuity, and the proof is complete. □

Remark 3.4. It is interesting to compare Proposition 3.3 with Theorem 3.3 in [Adami et al. 2016]. According to that result, in a finite noncompact graph the energy threshold under which the existence of a ground state of a given mass is guaranteed equals the energy of the soliton on $\mathbb{R}$ with the same mass. On the contrary, on the grid $\mathcal{G}$ the absence of half-lines and the periodicity pushes the energy threshold up to zero. This makes some proofs easier, since finding a function with negative energy is far easier than finding a function whose energy lies below a particular negative number. In fact, this task is immediately accomplished when $p < 4$, as we now show.

Proof of Theorem 1.1. In view of Proposition 3.3, it suffices to construct a function in $H^1_\mu(\mathcal{G})$ with negative energy. Given $\mu > 0$, for $\varepsilon > 0$ let

$$\kappa_\varepsilon = \left(\frac{\varepsilon\mu}{2} \, \frac{1 - e^{-2\varepsilon}}{1 + e^{-2\varepsilon}} \right)^{1/2} \tag{30}$$

and consider the function of two variables

$$\varphi_\varepsilon(x, y) = \kappa_\varepsilon e^{-\varepsilon(|x|+|y|)}, \qquad (x, y) \in \mathbb{R}^2.$$

Now, as described in Section 2, we can consider $\mathcal{G}$ isometrically embedded in $\mathbb{R}^2$, with its vertices on the lattice $\mathbb{Z}^2$, and we can define $u_\varepsilon : \mathcal{G} \to \mathbb{R}$ as the restriction of φ_ε to the grid $\mathcal{G}$. Observe that, on every horizontal line H_j of $\mathcal{G}$, u takes the form $\kappa_\varepsilon e^{-\varepsilon(|x|+|j|)}$, and a similar expression holds on vertical lines. Since for every $\lambda > 0$

$$\int_{\mathbb{R}} e^{-\lambda\varepsilon|x|} \, dx = \frac{2}{\lambda\varepsilon} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} e^{-\lambda\varepsilon|j|} = \frac{1 + e^{-\lambda\varepsilon}}{1 - e^{-\lambda\varepsilon}},$$

recalling (30) we obtain

$$\int_{\mathcal{G}} |u_\varepsilon|^2 \, dx = 2 \sum_{j\in\mathbb{Z}} \int_{H_j} |u_\varepsilon|^2 \, dx = 2\kappa_\varepsilon^2 \sum_{j\in\mathbb{Z}} e^{-2\varepsilon|j|} \int_{\mathbb{R}} e^{-2\varepsilon|x|} \, dx = \mu$$

and, since $|u'_\varepsilon(x)| = \varepsilon|u_\varepsilon(x)|$,

$$\int_{\mathcal{G}} |u'_\varepsilon|^2 \, dx = \varepsilon^2\mu.$$

This shows in particular that $u_\varepsilon \in H^1_\mu(\mathcal{G})$. Similarly, observing that $\kappa_\varepsilon \sim \varepsilon\sqrt{\mu/2}$ as $\varepsilon \to 0$, we obtain the expansion

$$\int_{\mathcal{G}} |u_\varepsilon|^p \, dx = 2 \sum_{j\in\mathbb{Z}} \int_{H_j} |u_\varepsilon|^p \, dx = 2\kappa_\varepsilon^p \frac{2}{\varepsilon p} \, \frac{1 + e^{-\varepsilon p}}{1 - e^{-\varepsilon p}} \sim C\mu^{p/2}\varepsilon^{p-2} \quad \text{as } \varepsilon \to 0,$$

where C depends only on p. Therefore, as $\varepsilon \to 0$,

$$E_p(u_\varepsilon) \sim \frac{1}{2}\varepsilon^2\mu - \frac{1}{p} C\mu^{p/2}\varepsilon^{p-2}, \tag{31}$$

so that $E_p(u_\varepsilon) < 0$ (for ε small enough) when $p < 4$. This proves that, when $p < 4$, $\mathcal{E}_p(\mu) < 0$ for every $\mu > 0$. Moreover, since in particular $p < 6$, Remark 3.1 guarantees that $\mathcal{E}_p(\mu)$ is finite. The result then follows from Proposition 3.3. □

4. Proof of Theorem 1.2

In the following we assume that the constants K_p in the Gagliardo–Nirenberg inequality (23) are the smallest possible. In other words, for $p \in [4, 6]$ we let

$$K_p = \sup_{\substack{u\in H^1(\mathcal{G}) \\ u\not\equiv 0}} Q_p(u), \qquad \text{where } Q_p(u) = \frac{\|u\|_p^p}{\|u\|_2^{p-2}\|u'\|_2^2}. \tag{32}$$

The critical masses μ_p mentioned in Theorem 1.2 are defined in terms of the constants K_p as follows.

Definition 4.1. For every $p \in [4, 6]$ we define the *critical mass* μ_p as the positive number

$$\mu_p = \left(\frac{p}{2K_p}\right)^{2/(p-2)}. \tag{33}$$

This definition is natural due to the identity

$$E_p(u) = \frac{1}{2}\|u'\|_2^2\Big(1 - \frac{2}{p}Q_p(u)\mu^{(p-2)/2}\Big) \quad \text{for all } u \in H^1_\mu(\mathcal{G}), \tag{34}$$

which, using $Q_p(u) \le K_p$ and (33), leads to the lower bound

$$E_p(u) \ge \frac{1}{2}\|u'\|_2^2\left(1 - \left(\frac{\mu}{\mu_p}\right)^{(p-2)/2}\right) \quad \text{for all } u \in H^1_\mu(\mathcal{G}), \tag{35}$$

which will be widely used in the sequel.

Remark 4.2. On the real line $\mathbb{R}$, when $p = 6$ the ground-state level

$$\mathcal{E}_6^{\mathbb{R}}(\mu) = \inf\left\{\tfrac{1}{2}\|w'\|^2_{L^2(\mathbb{R})} - \tfrac{1}{6}\|w\|^6_{L^6(\mathbb{R})} \;\middle|\; w \in H^1_\mu(\mathbb{R})\right\}, \quad \mu > 0, \tag{36}$$

is attained by a ground state if and only if $\mu = \mu_6^{\mathbb{R}}$, where the number

$$\mu_6^{\mathbb{R}} = \frac{\pi\sqrt{3}}{2} \tag{37}$$

is the critical mass of the real line; see [Adami et al. 2017a]. Up to sign and translations, the ground states (of mass $\mu_6^{\mathbb{R}}$) are the soliton $\varphi(x) = \operatorname{sech}(2x/\sqrt{3})^{1/2}$ together with all its mass-preserving rescalings $\varphi_\lambda(x) = \sqrt{\lambda}\varphi(\lambda x)$ $(\lambda > 0)$. There holds

$$\begin{aligned} \mathcal{E}_6^{\mathbb{R}}(\mu) &= 0 && \text{if } \mu \le \mu_6^{\mathbb{R}}, \\ \mathcal{E}_6^{\mathbb{R}}(\mu) &= -\infty && \text{if } \mu < \mu_6^{\mathbb{R}} \end{aligned} \tag{38}$$

so that in particular ground states have zero energy. Another related quantity is the optimal constant in the Gagliardo–Nirenberg inequality on $\mathbb{R}$, i.e., the number

$$K_6^{\mathbb{R}} = \sup_{\substack{w \in H^1(\mathbb{R}) \\ w \not\equiv 0}} \frac{\|w\|^6_{L^6(\mathbb{R})}}{\|w\|^4_{L^2(\mathbb{R})}\|w'\|^2_{L^2(\mathbb{R})}} = \frac{4}{\pi^2} \tag{39}$$

(note that $\mu_6^{\mathbb{R}} = (3/K_6^{\mathbb{R}})^{1/2}$, which is formally consistent with (33) when $p = 6$).

The following proposition gives a complete picture of the problem on the grid $\mathcal{G}$ when $p = 6$ and, moreover, provides the exact values of μ_6 and K_6.

Proposition 4.3. *There hold $\mu_6 = \mu_6^{\mathbb{R}} = \pi\sqrt{3}/2$ and $K_6 = K_6^{\mathbb{R}} = 4/\pi^2$. Moreover there holds $\mathcal{E}_6(\mu) = \mathcal{E}_6^{\mathbb{R}}(\mu)$ for every $\mu > 0$, but the infimum*

$$\mathcal{E}_6(\mu) = \inf\left\{\tfrac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \tfrac{1}{6}\|u\|^6_{L^6(\mathcal{G})} \;\middle|\; u \in H^1_\mu(\mathcal{G})\right\}, \quad \mu > 0, \tag{40}$$

is never attained.

Proof. By a density argument, the infimum in (36) can be restricted to functions $w \in H^1_\mu(\mathbb{R})$ having compact support. In fact, by a mass-preserving transformation $w(x) \mapsto w(x/\varepsilon^2)/\varepsilon$, one can restrict to functions supported in the interval $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$. Then, by interpreting this interval as one of the edges of the grid $\mathcal{G}$, any function $w \in H^1_\mu(\mathbb{R})$ supported in I can be embedded in $H^1_\mu(\mathcal{G})$ by setting $w \equiv 0$ on $\mathcal{G} \setminus I$, thus providing an admissible function in (40). This proves that $\mathcal{E}_6(\mu) \le \mathcal{E}_6^{\mathbb{R}}(\mu)$ for every $\mu > 0$. Similarly, starting from the supremum in (39), by the same argument one proves that $K_6 \ge K_6^{\mathbb{R}}$.

To prove the opposite inequalities we argue as follows. Given a nonnegative function $u \in H^1(\mathcal{G})$ ($u \not\equiv 0$), let $x_0 \in \mathcal{G}$ be a point where u achieves its absolute maximum $\|u\|_\infty$, and let P be any path in $\mathcal{G}$ such that $x_0 \in P$ and P is isometric to the real line $\mathbb{R}$ (a natural choice for P is the horizontal/vertical line of $\mathcal{G}$ that contains x_0). Since $u(x_0) = \|u\|_\infty$ and $u(x) \to 0$ as $x \to \pm\infty$ along P (in both directions away from x_0), the continuity of u guarantees that $N(t) \ge 2$ for every $t \in (0, \|u\|_\infty)$, where

$$N(t) = \#\{x \in \mathcal{G} \mid u(x) = t\} \tag{41}$$

counts the number of preimages in $\mathcal{G}$. Then, if $\hat{u} \in H^1(\mathbb{R})$ denotes the symmetric rearrangement of u on $\mathbb{R}$, applying Proposition 3.1 of [Adami et al. 2015b] we obtain

$$\|(\hat{u})'\|_{L^2(\mathbb{R})} \le \|u'\|_{L^2(\mathcal{G})}, \qquad \|\hat{u}\|_{L^r(\mathbb{R})} = \|u\|_{L^r(\mathcal{G})} \quad \text{for all } r \tag{42}$$

so that, by the definition of $K_6^{\mathbb{R}}$ in (39), we can estimate

$$\|u\|^6_{L^6(\mathcal{G})} = \|\hat{u}\|^6_{L^6(\mathbb{R})} \le K_6^{\mathbb{R}} \|\hat{u}\|^4_{L^2(\mathbb{R})} \|(\hat{u})'\|^2_{L^2(\mathbb{R})} \le K_6^{\mathbb{R}} \|u\|^4_{L^2(\mathcal{G})} \|u'\|^2_{L^2(\mathcal{G})}.$$

Therefore, $K_6 \le K_6^{\mathbb{R}}$ by (32). Similarly, for the NLS energy we have

$$\tfrac{1}{2}\|(\hat{u})'\|^2_{L^2(\mathbb{R})} - \tfrac{1}{6}\|\hat{u}\|^6_{L^6(\mathbb{R})} \le \tfrac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \tfrac{1}{6}\|u\|^6_{L^6(\mathcal{G})} \tag{43}$$

and, since $\hat{u} \in H^1_\mu(\mathbb{R})$ whenever $u \in H^1_\mu(\mathcal{G})$, this proves that $\mathcal{E}_6^{\mathbb{R}}(\mu) \le \mathcal{E}_6(\mu)$ for every $\mu > 0$.

Now assume that, for some μ, a function $u \in H^1_\mu(\mathcal{G})$ achieves the infimum $\mathcal{E}_6(\mu)$ in (40). Then, since $\mathcal{E}_6^{\mathbb{R}}(\mu) = \mathcal{E}_6(\mu)$, (43) shows that, necessarily (i) $\hat{u}$ achieves the infimum $\mathcal{E}_6^{\mathbb{R}}(\mu)$ in (36); (ii) equality must occur in (43), i.e., in (42). Now, condition (i) gives that $\hat{u}$ is a soliton on $\mathbb{R}$ (necessarily of mass $\mu_6^{\mathbb{R}}$), while (ii) implies, see Proposition 3.1 of [Adami et al. 2015b], that $N(t) = 2$ in (41), i.e., that $u^{-1}(t)$ has exactly two elements for almost every $t \in (0, \|u\|_\infty)$; then, since every vertex of $\mathcal{G}$ has degree 4, u must vanish at every vertex and is necessarily supported in a single edge of $\mathcal{G}$. So $\hat{u}$ has compact support too, which is incompatible with $\hat{u}$ being a soliton. This contradiction shows the infimum in (40) is not achieved.

Finally, (33) with $p = 6$ yields $\mu_6 = \sqrt{3/K_6} = \pi\sqrt{3}/2$; hence $\mu_6 = \mu_6^{\mathbb{R}}$ by (37). □

Proof of Theorem 1.2. The case where $p = 6$ has already been proved through Proposition 4.3. The rest of the proof is divided into three parts.

Computation of $\mathcal{E}_p(\mu)$ when $p \in [4, 6)$. First observe that, in the proof of Theorem 1.1, no restriction on p was used to construct u_ε and obtain (31), which is therefore valid also when $p \ge 4$. As a consequence, in this case, letting $\varepsilon \to 0$ in (31) we obtain

$$\mathcal{E}_p(\mu) \le \liminf_{\varepsilon\to 0} E_p(u_\varepsilon) \le 0 \quad \text{for all } p \ge 4, \text{ for all } \mu > 0. \tag{44}$$

Moreover, (35) shows that $\mathcal{E}_p(\mu) \ge 0$ when $\mu \le \mu_p$. This, combined with (44), proves the first part of (5), also when $p = 4$.

Now fix a mass $\mu > \mu_p$ and a number $\varepsilon > 0$. Since the quotient $Q_p(u)$ in (32) is unaltered if u is replaced with λu, there exists $u \in H^1_\mu(\mathcal{G})$ such that

$$Q_p(u) = \frac{\|u\|_p^p}{\mu^{(p-2)/2}\|u'\|_2^2} \ge K_p - \varepsilon. \tag{45}$$

Plugging this into (34), and then using (33), we can estimate

$$E_p(u) \le \frac{1}{2}\|u'\|_2^2\left(1 - \frac{2}{p}(K_p - \varepsilon)\mu^{(p-2)/2}\right) = \frac{1}{2}\|u'\|_2^2\left(1 - \left(\frac{\mu}{\mu_p}\right)^{(p-2)/2} + \frac{2\varepsilon}{p}\mu^{(p-2)/2}\right).$$

Since $\mu > \mu_p$, this quantity is strictly negative if ε is small enough. Thus, for $\mu > \mu_p$, $\mathcal{E}_p(\mu) < 0$. Moreover, when $p < 6$, $\mathcal{E}_p(\mu) > -\infty$ by Remark 3.1. This proves the second part of (5), also when $p = 4$.

Ground states when $p \in [4, 6)$ and $\mu \ne \mu_p$. When $\mu > \mu_p$, (5) (valid also when $p = 4$) shows that $\mathcal{E}_p(\mu)$ is finite and negative; hence a ground state exists by Proposition 3.3. When $\mu < \mu_p$, $\mathcal{E}_p(\mu) = 0$ by (5), but (35) reveals that $E_p(u) > 0$ for every $u \in H^1_\mu(\mathcal{G})$. Therefore, no ground state exists in this case.

Ground states when $p \in (4, 6)$ and $\mu = \mu_p$. Since by (5) $\mathcal{E}_p(\mu_p) = 0$, we can no longer rely on Proposition 3.3, and another argument is needed to show that $\mathcal{E}_p(\mu_p)$ is in fact achieved.

Arguing as for (45), let $u_n \in H^1_{\mu_p}(\mathcal{G})$ be a sequence of functions such that

$$\lim_n Q_p(u_n) = \lim_n \frac{\|u_n\|_p^p}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} = K_p. \tag{46}$$

We shall bound $Q_p(u_n)$ in two different ways. First, from the Gagliardo–Nirenberg inequality (12) we obtain

$$Q_p(u_n) \le \frac{\|u_n\|_2^{p/2+1}\|u_n'\|_2^{p/2-1}}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} = \frac{\mu_p^{(6-p)/4}}{\|u_n'\|_2^{(6-p)/2}}.$$

Secondly, interpolating and then using (23) with $p = 4$, we obtain

$$Q_p(u_n) \le \frac{\|u_n\|_\infty^{p-4}\|u_n\|_4^4}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} \le \|u_n\|_\infty^{p-4}\frac{K_4\|u_n\|_2^2\|u_n'\|_2^2}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} = \|u_n\|_\infty^{p-4}\frac{K_4}{\mu_p^{(p-4)/2}}.$$

Recalling (46), from these two bounds we infer that $\|u_n'\|_2 \le C$ (compactness) and $\|u_n\|_\infty \ge C^{-1}$ (nondegeneracy) for some constant $C > 0$ independent of n. Thus $\{u_n\}$ is bounded in $H^1(\mathcal{G})$ and, up to translations, we can also assume that each u_n achieves its L^∞ norm on some compact set $\mathcal{K} \subset \mathcal{G}$ independent of n. Then, up to subsequences, $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ for some $u \in H^1(\mathcal{G})$, and $u_n \to u$ in $L^\infty_{\text{loc}}(\mathcal{G})$; in particular, $u_n \to u$ uniformly on $\mathcal{K}$ and, since $\|u_n\|_{L^\infty(\mathcal{K})} > C^{-1}$, u is not identically zero.

Finally, writing (34) with $u = u_n$ and $\mu = \mu_p$, since $\|u_n'\|_2 \le C$ we find

$$|E_p(u_n)| \le \frac{C^2}{2}\left|1 - \frac{2}{p} Q_p(u_n)\mu_p^{(p-2)/2}\right| = \frac{C^2}{2}\left|1 - \frac{Q_p(u_n)}{K_p}\right|,$$

having used (33). Therefore, $E_p(u_n) \to 0$ by (46) and, since $\mathcal{E}_p(\mu_p) = 0$, u_n is a minimizing sequence for E_p, so that Lemma 3.2 applies: since we already know that u is not identically zero, we obtain that $\|u\|_2^2 = \mu_p$, i.e., $u \in H^1_{\mu_p}(\mathcal{G})$. But then u is the required minimizer: indeed, $u_n \to u$ strongly in $L^2(\mathcal{G})$ hence also in $L^p(\mathcal{G})$, and by weak lower semicontinuity we obtain

$$\mathcal{E}_p(\mu_p) \le E_p(u) \le \liminf_n E_p(u_n) = \mathcal{E}_p(\mu_p). \qquad \square$$

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ALEXANDROV'S THEOREM REVISITED

MATIAS GONZALO DELGADINO AND FRANCESCO MAGGI

We show that *among sets of finite perimeter* balls are the only volume-constrained critical points of the perimeter functional.

1. Introduction

1.1. ***Sets of finite perimeter and the isoperimetric problem.*** The Euclidean isoperimetric theorem is probably the most basic result in the calculus of variations. There are many different proofs of the isoperimetry of balls in different classes of competitors, thus motivating the question: which is the natural competition class in which the isoperimetric theorem can be formulated? From the perspective of the modern calculus of variations, the answer is found by looking at the relaxation of the perimeter functional. Following the seminal work of De Giorgi [1954; 1955] we consider as particularly natural his formulation of the Euclidean isoperimetric problem in the class of sets of finite perimeter. The characterization of Euclidean balls as the only isoperimetric sets among sets of finite perimeter was achieved in [De Giorgi 1958]. By using the compactness properties of sets of finite perimeter, De Giorgi shows the existence of global minimizers (isoperimetric sets). Next, he shows that distributional perimeter is decreased under Steiner symmetrization, thus deducing that Steiner symmetrization applied to an isoperimetric set leads to an equality case in the Steiner perimeter inequality. He finally derives some necessary conditions for being an equality case in the Steiner perimeter inequality, in order to deduce the sphericity of isoperimetric sets.

Despite the intimate connection between sets of finite perimeter and the isoperimetric problem, a characterization of balls as the only *critical points* in the isoperimetric problem *among sets of finite perimeter* is currently missing. The main result of this paper is showing the validity of this characterization.

The problem is already subtle in the case of local minimizers. By a local minimizer we mean a set of finite perimeter which minimizes perimeter among variations compactly supported in a fixed neighborhood of its own boundary. In particular, local minimality does not allow for perimeter comparison with sets obtained by symmetrization, thus ruling out the use of De Giorgi's original argument. In Euclidean spaces of dimension less than or equal to 7 the problem can be settled by the means of the regularity theory for local perimeter minimizers. In fact, in these dimensions any local minimizer is a bounded smooth set with constant mean curvature. One can then combine the strong maximum principle with the geometric construction known as the moving planes method (Alexandrov's theorem [1962]) to deduce the sphericity of the boundary. But this strategy fails in dimension 8 or larger, as boundaries of local

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perimeter minimizers could have, in principle, singular points, where local graphicality fails [Simons 1968]. Actually, it has been recently shown that local volume-constrained perimeter minimizers in nonconvex perturbations of the unit ball may indeed have singularities [Sternberg and Zumbrun 2018].

The problem is open in every dimension for critical points, that is, sets of finite perimeter and finite volume such that the first variation of perimeter under volume-fixing flows vanishes. These sets have constant mean curvature in a very natural (distributional) sense. However, at variance with the case of local minimizers, there seems to be no obvious way, even in low dimensions, to exploit regularity theorems and the moving planes method to conclude their sphericity.

Here we approach this problem by combining regularity theorems and maximum principles with various geometric constructions inspired by the proof of Alexandrov's theorem in [Montiel and Ros 1991]. We thus extend De Giorgi's isoperimetric theorem from the case of global minimizers to that of critical points in the isoperimetric problem.

Theorem 1. *Among sets of finite perimeter and finite volume, finite unions of balls with equal radii are the unique critical points of the Euclidean isoperimetric problem.*

Remark. Theorem 1 is stated in terms of finite unions of balls. By assuming indecomposability (the measure-theoretic analogue of connectedness) of our critical points, we can change "finite unions of balls" to "a single ball". However, it seems natural to consider finite unions of mutually tangent balls as genuinely distinct critical points of the perimeter functional. Indeed, as proved in [Ciraolo and Maggi 2017; Delgadino et al. 2018] (and as it has been known for a much longer time in the case of parametrized surfaces [Brezis and Coron 1984; Struwe 1984]), finite unions of mutually tangent balls are the unique limits of sequences of bounded connected smooth sets with bounded perimeters and scalar mean curvatures which converge to a constant. In short, finite unions of mutually tangent balls are the limit points of Palais–Smale sequences for the isoperimetric problem among connected open sets with smooth boundary.

Remark. Wente's torus [1986] provides an example of an integer rectifiable varifold with multiplicity 1 in $\mathbb{R}^3$ which has constant distributional mean curvature and is not a sphere. Clearly, Wente's torus is not the boundary of a set of finite perimeter. From this point of view, Theorem 1 seems to identify the most general family of surfaces such that constant distributional mean curvature implies sphericity.

While uniqueness and symmetry results for global minimizers can be obtained by a wealth of methods (symmetrization, mass transportation, etc.), the methods employed in the case of critical points/solutions to geometric PDEs, that we are aware of, require a sufficient degree of smoothness (e.g., the classical Alexandrov theorem [1962]). Addressing this kind of issue without assuming smoothness seems a novel aspect of Theorem 1. This point could be particularly useful in proving convergence of geometric flows to unions of balls. Indeed, without strong assumptions like convexity or star-shapedness, global-in-time existence results for geometric flows hold only in a weak (either distributional or viscous) sense. Corollary 2 below should be useful in this context. To better illustrate this point, and to state the corollary itself, we introduce some terminology. In Theorem 1 we consider Borel sets Ω in $\mathbb{R}^{n+1}$ with the following properties:

(i) *Finite perimeter*: There exists a Borel set $\partial^*\Omega$ which is covered, up to an $\mathcal{H}^n$-negligible set, by countably many graphs of C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, and a Borel vector field $\nu_\Omega : \partial^*\Omega \to \mathbb{S}^n$ such that a generalized version of the divergence theorem holds:

$$\int_\Omega \operatorname{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{1-1}$$

Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$.

(ii) *Constant distributional mean curvature*: There exists $\lambda \in \mathbb{R}$ such that

$$\int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^n = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{1-2}$$

Here $\operatorname{div}^{\partial^*\Omega} X = \operatorname{div} X - \nu_\Omega \cdot (\nabla X)[\nu_\Omega]$ is the tangential divergence of X along $\partial^*\Omega$. Condition (1-2) is equivalent to asking that Ω be a *critical point in the Euclidean isoperimetric problem*, that is,

$$\frac{d}{dt}\Big|_{t=0} P(f_t(\Omega)) = 0 \tag{1-3}$$

whenever $\{f_t\}_{|t|<1}$ is a volume-preserving variation of Ω. Namely, each f_t is a diffeomorphism with $f_t = \mathrm{Id}$ outside of a compact set, $f_0 \equiv \mathrm{Id}$, and $|f_t(\Omega)| = |\Omega|$ for every $|t| < 1$, where $|\Omega|$ denotes the Lebesgue measure, or volume, of Ω. When Ω is an open bounded set with C^2-boundary, as in Alexandrov's theorem, one simply has $\partial^*\Omega = \partial\Omega$ and (1-3) is equivalent to asking that $\partial\Omega$ have constant mean curvature.

With this terminology in place, we can state the following corollary of Theorem 1.

Corollary 2. *If* $\{\Omega_j\}_{j\in\mathbb{N}}$ *and* Ω *are sets of finite perimeter in* $\mathbb{R}^{n+1}$ *such that*

$$\lim_{j\to\infty} |\Omega_j \Delta \Omega| = 0, \qquad \lim_{j\to\infty} P(\Omega_j) = P(\Omega), \tag{1-4}$$

and if the distributional mean curvatures of the Ω_j *converge to a constant* $\lambda \in \mathbb{R}$, *i.e.*,

$$\lim_{j\to\infty} \int_{\partial^*\Omega_j} (\operatorname{div}^{\partial^*\Omega_j} X - \lambda X \cdot \nu_{\Omega_j}) \, d\mathcal{H}^n = 0 \quad \textit{for all } X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), \tag{1-5}$$

then $\lambda = nP(\Omega)/(n+1)|\Omega|$ *and* Ω *is a finite union of balls of radius* n/λ.

Remark. Notice that (1-5) holds whenever each Ω_j has distributional mean curvature $H_{\Omega_j} \in L^p(\mathcal{H}^n \llcorner \partial^*\Omega_j)$ for some $p \geq 1$ (see (2-7) and (2-16) below) and

$$\lim_{j\to\infty} \int_{\partial^*\Omega_j} |H_{\Omega_j} - \lambda|^p \, d\mathcal{H}^n = 0. \tag{1-6}$$

Remark. Global-in-time weak solutions of the volume-preserving mean curvature flow have been constructed in [Mugnai et al. 2016] following the method proposed by Almgren, Taylor and Wang [Almgren et al. 1993] and Luckhaus and Sturzenhecker [1995]. Considering [Mugnai et al. 2016, Theorem 2.3.2] and (1-5), it seems reasonable to conjecture that, for a large class of initial data and along time subsequences $t_j \to \infty$, the evolution $\{\Omega(t) : t \geq 0\}$ should converge to finite union of balls. This is indeed the case, with a single ball as the limit for $t \to \infty$, when the initial data is uniformly smooth and convex, as proved in a classical theorem of [Huisken 1987]. As geometric evolutions unavoidably produce singularities, Theorem 1 should turn out to be a fundamental ingredient in attacking such questions.

1.2. ***The Montiel–Ros argument.*** Our starting point is the beautiful proof of Alexandrov's theorem in [Montiel and Ros 1991], which we now recall. Assume that Ω is a bounded open set with smooth boundary and positive mean curvature H_Ω with respect to its outer unit normal ν_Ω. Denote by $\{\kappa_i\}_{i=1}^n$ the principal curvatures of $\partial\Omega$, indexed in increasing order so that $\kappa_n \ge H_\Omega/n > 0$, set $u(y) = \mathrm{dist}(y, \partial\Omega)$ for each $y \in \Omega$, and define

$$Z = \left\{(x,t) \in \partial\Omega \times \mathbb{R} : 0 < t \le \frac{1}{\kappa_n(x)}\right\}, \quad \zeta(x,t) = x - t\nu_\Omega(x), \quad (x,t) \in Z. \tag{1-7}$$

Let us denote by $B_\rho(x)$ the Euclidean ball in $\mathbb{R}^{n+1}$ with center at x and radius ρ. If $y \in \Omega$, then $B_{u(y)}(y)$ touches Ω from inside at a point $x \in \partial\Omega$, where $\kappa_n(x) \le 1/u(y)$, i.e., $u(y) \le 1/\kappa_n(x)$. In particular,

$$\Omega \subset \zeta(Z) \tag{1-8}$$

and by the area formula, with $J^Z\zeta$ denoting the tangential Jacobian of ζ along Z,

$$|\Omega| \le |\zeta(Z)| \le \int_{\zeta(Z)} \mathcal{H}^0(\zeta^{-1}(y))\,dy = \int_Z J^Z\zeta\, d\mathcal{H}^{n+1} = \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \prod_{i=1}^n (1 - t\kappa_i(x))\,dt.$$

By the arithmetic-geometric mean inequality and by $\kappa_n \ge H_\Omega/n$,

$$\begin{aligned} |\Omega| &\le \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \left(\frac{1}{n}\sum_{i=1}^n (1 - t\kappa_i(x))\right)^n dt \\ &\le \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{n/H_\Omega(x)} \left(1 - t\frac{H_\Omega(x)}{n}\right)^n dt = \frac{n}{n+1}\int_{\partial\Omega} \frac{d\mathcal{H}^n}{H_\Omega}, \end{aligned} \tag{1-9}$$

so that we have proved the *Heintze–Karcher inequality*

$$|\Omega| \le \frac{n}{n+1}\int_{\partial\Omega} \frac{d\mathcal{H}^n}{H_\Omega}. \tag{1-10}$$

If H_Ω is constantly equal to some $\lambda \in \mathbb{R}$, then, by combining the divergence theorems (1-1) and (1-2) (see (2-24) below), we find $\lambda = n\mathcal{H}^n(\partial\Omega)/(n+1)|\Omega|$. Hence equality holds throughout the argument, $\partial\Omega$ is umbilical, and thus is a sphere. In this way the Montiel–Ros argument provides a very effective proof of Alexandrov's theorem.

1.3. ***The Montiel–Ros argument revisited.*** As the Montiel–Ros argument heavily relies on the smoothness of $\partial\Omega$, it does not seem obvious how to adapt it to the case when Ω is a set with finite volume, finite perimeter and constant distributional mean curvature.

From the point of view of regularity of $\partial\Omega$, the starting point is given by the regularity theory of [Allard 1972]; see [Simon 1983; De Lellis 2008]. Up to modifying Ω on a set of volume zero, we can assume that Ω is open and that its topological boundary $\partial\Omega$ can be split into a closed subset Σ with $\mathcal{H}^n(\Sigma) = 0$, and a relatively open subset $\partial^*\Omega = \partial\Omega \setminus \Sigma$ which is locally an analytic constant mean curvature hypersurface, characterized by the property that for every $x \in \partial\Omega$

$$x \in \partial^*\Omega \quad \text{if and only if} \quad \lim_{\rho\to 0^+} \frac{\mathcal{H}^n(B_\rho(x)\cap\partial\Omega)}{\rho^n} = \omega_n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. It is thus natural to redefine Z by replacing $\partial\Omega$ with $\partial^*\Omega$ in (1-7); i.e.,

$$Z = \left\{(x,t) \in \partial^*\Omega \times \mathbb{R} : 0 < t \le \frac{1}{\kappa_n(x)}\right\}, \tag{1-11}$$

where it is still true that the largest principal curvature κ_n is positive along $\partial^*\Omega$.

Given this choice of Z, in order to obtain (1-8) we would need to show that, for every $y \in \Omega$, $B_{u(y)}(y)$ touches $\partial\Omega$ at a point $x \in \partial^*\Omega$. This is not obvious as we just know that $\Sigma = \partial\Omega \setminus \partial^*\Omega$ is $\mathcal{H}^n$-negligible. Actually, this is false for an arbitrary point $y \in \Omega$: this is the case when Ω is a union of two mutually tangent balls, x is a tangency point between two balls, and y is any point between x and the center of one of the balls. A cheap argument (see Lemma 3) shows that at each touching point x, $\partial\Omega$ blows up a hyperplane *with integer multiplicity possibly larger than* 1. So, near a touching point x, $\partial\Omega$ consists of finitely many sheets that are mutually tangent at x. The union of these sheets has constant mean curvature in the distributional sense defined by (1-2), although it is not immediate to extract information on the mean curvature of each separate sheet. A deep result of [Schätzle 2004] implies that the lower and upper sheets (with respect to any given direction) satisfy a measure-theoretic version of the strong maximum principle. This is crucial information, which is delicate to exploit, but fundamental to our argument.

We now describe our argument by referring to the main steps of the proof of Theorem 1, which is contained in detail in Section 3. We start by identifying a large subset $\Omega^\star$ of good points of Ω, meaning that

$$|\Omega^\star \setminus \zeta(Z)| = 0, \quad |\Omega \setminus \Omega^\star| = 0. \tag{1-12}$$

In other words, the projection of almost every point in $\Omega^\star$ onto $\partial\Omega$ is contained in $\partial^*\Omega$, and $\Omega^\star$ is equivalent to Ω. The definition of $\Omega^\star$ is as follows. First, for every $s > 0$, we set

$$\Omega_s = \{y \in \Omega : u(y) > s\}, \quad \partial\Omega_s = \{y \in \Omega : u(y) = s\}. \tag{1-13}$$

Clearly Ω_s satisfies an exterior ball condition of radius s at each point of $\partial\Omega_s$, but otherwise Ω_s is just a set of finite perimeter (for a.e. $s > 0$). We can also obtain an interior ball condition, restricting ourselves to the following subset. Setting $t > s > 0$, we define

$$\Gamma_s^t = \left\{y \in \partial\Omega_s : y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z \quad \text{for some } z \in \partial\Omega_t,\ x \in \partial\Omega\right\}. \tag{1-14}$$

Notice that Γ_s^t is just a compact subset of $\partial\Omega_s$, which could be very porous inside $\partial\Omega_s$. Some technical effort (see Step 1) is put into showing that Γ_s^t can be covered by countably many $C^{1,1}$-images of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, and that ∇u is tangentially differentiable along Γ_s^t (with bounds on the tangential derivatives corresponding to the exterior/interior ball conditions). Once these technical aspects are settled, we are allowed to use $\mathrm{Id} - r\nabla u$ to change variables between Γ_s^t and Γ_{s-r}^t and we can prove that $|\Omega \setminus \Omega^\star| = 0$, where $\Omega^\star$ is defined by

$$\Gamma_s^+ = \bigcup_{t>s} \Gamma_s^t, \quad \Omega^\star = \bigcup_{s>0} \Gamma_s^+. \tag{1-15}$$

This is done in Step 2 of the proof.

Showing that $|\Omega^\star \setminus \zeta(Z)| = 0$, see Steps 3 and 4, is considerably more delicate. We have to exclude that the points in a given Γ_s^t that are projected into the singular set $\Sigma = \partial\Omega \setminus \partial^*\Omega$ have positive $\mathcal{H}^n$-measure; in other words, we want

$$\mathcal{H}^n((\mathrm{Id} - s\nabla u)^{-1}(\Sigma) \cap \Gamma_s^t) = 0.$$

This may seem obvious, as $\mathrm{Id} - s\nabla u$ is almost injective on Γ_s^t (see (3-43)) and it is Lipschitz on each piece of a countable decomposition of Γ_s^t (see (3-16)), while at the same time $\mathcal{H}^n(\Sigma) = 0$. However we cannot derive a straightforward contradiction from the area formula, as the tangential Jacobian of $\mathrm{Id} - s\nabla u$ along Γ_s^t may be zero $\mathcal{H}^n$-a.e. In fact, this is the information that we obtain from the area formula; namely, the least principal curvature of Γ_s^t is equal to $-1/s$ along points in $(\mathrm{Id} - s\nabla u)^{-1}(\Sigma) \cap \Gamma_s^t$. Heuristically, this curvature for Γ_s^t can only be obtained when $\partial\Omega$ has a inward corner, which is ruled out by absolute continuity of the mean curvature. Following this guiding example, we change variable to show that the least principal curvature of Γ_{s-r}^t at corresponding points is thus as negative as we wish. This indicates that $\partial\Omega_{s-r}$ has negative mean curvature on a set of positive $\mathcal{H}^n$-measure for any r close enough to s. By the almost-everywhere second-order differentiability of u, swiping r over an interval we can find a paraboloid with negative mean curvature, locally contained inside $\partial\Omega_{s-r}$. By translating this object until it touches $\partial\Omega$ (at Σ) we can apply Schätzle's maximum principle and derive a contradiction.

As pointed out to us by a referee, our argument up to this point shares some similarities with the strategy adopted by Almgren [1986] in proving the isoperimetric inequality in higher codimension. Almgren's goal in that paper is showing that an upper bound on the length of the mean curvature vector implies a lower bound on the area, which is saturated by spheres. His arguments are also based on a viscosity approach, where sliding constructions and the maximum principle are combined to infer regularity properties. The referee's insight is that Almgren's argument could be adapted to our setting by updating some technical aspects along the lines of the recent work [Santilli 2017], or, better said, of a possible generalization of that paper to the bounded mean curvature case. This approach could provide a proof of (1-12) independent of Schätzle's maximum principle.

Having proved (1-12), we are ready to argue as Montiel and Ros. We thus find, from the equality case in their argument, that

$$|\zeta(Z) \setminus \Omega| = 0, \tag{1-16}$$

$$\mathcal{H}^0(\zeta^{-1}(y)) = 1, \quad \text{for a.e. } y \in \Omega, \tag{1-17}$$

$$\kappa_i(x) = \frac{H_\Omega}{n} \quad \text{for every } x \in \partial^*\Omega,\ i = 1, \dots, n. \tag{1-18}$$

Condition (1-18) implies that $\partial^*\Omega$ is umbilical, in addition to having constant mean curvature. In particular, $\partial^*\Omega$ consists of at most countably many open pieces of spheres with same curvature. Should these pieces be finitely many, one could conclude from the distributional constant mean curvature condition, in a rather direct way, that each piece is equal to a complete sphere. But as the number of the pieces could indeed be infinite, the pieces may have smaller and smaller areas and combine themselves in particular ways to achieve constant distributional mean curvature, creating at the same time a large singular set $\partial\Omega \setminus \partial^*\Omega$. To rule out this possibility, we exploit the information contained in (1-16) and (1-17) through a geometric argument. In this last step, we make once again use of Schätzle's strong maximum principle; see in particular (3-56).

We conclude with two remarks. First, as a by-product of this analysis, we obtain a *Heintze–Karcher inequality for sets of finite perimeter* which are mean convex in a viscous sense; see Theorem 8 below. This result is actually not needed to prove Theorem 1, but it is included as it may be considered of independent interest. Second, as recently shown by Brendle [2013], the Montiel–Ros approach to Alexandrov's theorem is quite flexible, as it allows one to show that constant mean curvature implies umbilicality in many warped product manifolds of physical and geometric interest. The methods of this paper should be naturally adaptable to these more general contexts. In this direction, in a preliminary version of this manuscript [Delgadino and Maggi 2017, Section 5], we prove that Wulff shapes are the only volume-constrained *local minimizers* of smooth uniformly elliptic surface tension energies. Of course the assumption of local minimality is considerably stronger than criticality.

1.4. ***Organization of the paper.*** The paper is organized as follows. In Section 2 we gather some background material from geometric measure theory. In Section 3 we prove Theorem 1 and Corollary 2. The generalized Heintze–Karcher inequality for sets of finite perimeter is stated and proved in Section 4.

2. Background material from geometric measure theory

In this section we review some preliminaries from the theory of rectifiable sets (Section 2.1), rectifiable varifolds (Section 2.2) and sets of finite perimeter (Section 2.3). We refer to [Simon 1983; Ambrosio et al. 2000; Maggi 2012; Evans and Gariepy 1992] for detailed accounts. Finally, in Section 2.4, we discuss some basic properties of volume-constrained critical points of the perimeter functional.

2.1. ***Rectifiable sets.*** Denote by $\mathcal{H}^n$ the Hausdorff measure on $\mathbb{R}^{n+1}$. A Borel set $M \subset \mathbb{R}^{n+1}$ is a *locally $\mathcal{H}^n$-rectifiable set* if M can be covered, up to a $\mathcal{H}^n$-negligible set, by countably many Lipschitz images of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, and if $\mathcal{H}^n \llcorner M$ is locally finite on $\mathbb{R}^{n+1}$. We say that M is *$\mathcal{H}^n$-rectifiable* if in addition $\mathcal{H}^n(M) < \infty$, and that M is *normalized* if $M = \operatorname{spt} \mathcal{H}^n \llcorner M$, i.e.,

$$x \in M \qquad \text{if and only if} \qquad \mathcal{H}^n(B_\rho(x) \cap M) > 0 \quad \text{for all } \rho > 0.$$

Basic properties of rectifiable sets needed in the sequel are:

(i) For $\mathcal{H}^n$-a.e. $x \in M$ there exists $T_x M \in G(n, n+1)$ (the space of n-dimensional planes in $\mathbb{R}^{n+1}$) such that

$$\lim_{\rho \to 0^+} \int_{(M-x)/\rho} \varphi \, d\mathcal{H}^n = \int_{T_x M} \varphi \, d\mathcal{H}^n \quad \text{for all } \varphi \in C_c^0(\mathbb{R}^{n+1}); \tag{2-1}$$

see [Maggi 2012, Theorem 10.2]. The plane $T_x M$ is called the *approximate tangent plane to M at x*.

(ii) If M_1 and M_2 are locally $\mathcal{H}^n$-rectifiable sets, then

$$T_x M_1 = T_x M_2 \quad \mathcal{H}^n\text{-a.e. on } M_1 \cap M_2; \tag{2-2}$$

see [Maggi 2012, Proposition 10.5].

(iii) Lipschitz functions are differentiable along approximate tangent planes; that is, if $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lipschitz function, then, for $\mathcal{H}^n$-a.e. $x \in M$ such that $T_x M$ exists, the restriction of f to $x + T_x M$ is differentiable at x, and the limit

$$(\nabla^M f)_x[\tau] = \lim_{h \to 0^+} \frac{f(x + h\tau) - f(x)}{h} \quad \text{for all } \tau \in T_x M$$

defines the *tangential gradient* $\nabla^M f(x) = (\nabla^M f)_x$ *of* f *along* M *at* x; see [Maggi 2012, Theorem 11.4].

(iv) The tangential gradient just depends on the restriction of f to M. In other words, if $f : M \to \mathbb{R}^{n+1}$ is a Lipschitz function, and $F, G : \mathbb{R}^{n+1} \to \mathbb{R}$ are Lipschitz functions such that $F = G = f$ on M, then

$$\nabla^M F = \nabla^M G \quad \mathcal{H}^n\text{-a.e. on } M. \tag{2-3}$$

(v) Finally, given a Lipschitz function $f : M \to \mathbb{R}^{n+1}$, the *tangential Jacobian of* f *along* M is defined at $\mathcal{H}^n$-a.e. $x \in M$ by

$$J^M f(x) = \sqrt{\det(\nabla^M f(x)^* \nabla^M f(x))} = \left| \bigwedge_{i=1}^n (\nabla^M f)_x[\tau_i(x)] \right|$$

provided $\{\tau_i(x)\}_{i=1}^n$ is an orthonormal basis of $T_x M$, and the area formula

$$\int_{f(M)} \mathcal{H}^0(f^{-1}(y))\, d\mathcal{H}^n_y = \int_M J^M f(x)\, d\mathcal{H}^n_x \tag{2-4}$$

holds [Maggi 2012, Theorem 11.6].

For the lack of precise reference we justify property (iv). If $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a Lipschitz map and $E \subset \mathbb{R}^n$ is a Borel set, then by [Maggi 2012, Lemmas 10.4 and 11.5] we have $T_x M = (\nabla \psi)_{\psi^{-1}(x)}[\mathbb{R}^n]$ for $\mathcal{H}^n$-a.e. $x \in M \cap \psi(E)$, with

$$(\nabla^M F)_x[\tau] = \nabla(F \circ \psi)_{\psi^{-1}(x)}[(\nabla \psi)_x^{-1}[\tau]] \quad \text{for all } \tau \in T_x M. \tag{2-5}$$

Since $F = G$ on M implies $\nabla(F \circ \psi) = \nabla(G \circ \psi)$ $\mathcal{H}^n$-a.e. on $E \cap \psi^{-1}(M)$ [Maggi 2012, Lemma 7.6] we deduce (2-3) from (2-5).

2.2. ***Integer rectifiable varifolds.*** If M is a C^2-hypersurface without boundary in $\mathbb{R}^{n+1}$, then the mean curvature vector $\boldsymbol{H}_M \in C^0(M; \mathbb{R}^{n+1})$ of M is such that

$$\int_M \operatorname{div}^M X\, d\mathcal{H}^n = \int_M \boldsymbol{H}_M \cdot X\, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), \tag{2-6}$$

with $\boldsymbol{H}_M(x) \cdot \tau = 0$ for every $\tau \in T_x M$. This basic fact motivates the following definitions.

Let M be a locally $\mathcal{H}^n$-rectifiable set, and consider a Borel measurable function $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^n \llcorner M; \mathbb{N})$. The *integer rectifiable varifold* $\mathrm{var}(M,\theta)$ defined by M and θ is the Radon measure on $\mathbb{R}^{n+1} \times G(n,n+1)$ defined as

$$\int_{\mathbb{R}^{n+1}\times G(n,n+1)} \Phi \, d\,\mathrm{var}(M,\theta) = \int_M \Phi(x, T_x M)\theta(x)\, d\mathcal{H}^n_x$$

for every bounded, compactly supported Borel function Φ on $\mathbb{R}^{n+1} \times G(n,n+1)$. To each $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ we associate the test function

$$\Phi_X(x,T) = (\mathrm{div}^T X)(x), \quad (x,T) \in \mathbb{R}^{n+1} \times G(n,n+1),$$

where $\mathrm{div}^T X$ is the divergence of X with respect to T. Motivated by (2-6), we say that $\mathrm{var}(M,\theta)$ has *distributional mean curvature vector* $\boldsymbol{H}_M \in L^1_{\mathrm{loc}}(\theta\mathcal{H}^n \llcorner M; \mathbb{R}^{n+1})$ if

$$\int_M \mathrm{div}^M X\theta \, d\mathcal{H}^n = \int_M \boldsymbol{H}_M \cdot X\theta\, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1}). \tag{2-7}$$

(The dependency of $\boldsymbol{H}_M$ from θ is omitted.) When $|\boldsymbol{H}_M|$ is constant ($\mathcal{H}^n$-a.e. on M) we say that $\mathrm{var}(M,\theta)$ has *constant distributional mean curvature on* $\mathbb{R}^{n+1}$; when $\boldsymbol{H}_M = 0$ we say that $\mathrm{var}(M,\theta)$ is *stationary on* $\mathbb{R}^{n+1}$. For example, if M is a union of finitely many *possibly intersecting* spheres with same radius, then M has constant distributional mean curvature in $\mathbb{R}^{n+1}$. Similarly, a finite union of hyperplanes is stationary in $\mathbb{R}^{n+1}$.

In the proof of Theorem 1 we will exploit two forms of the maximum principle for integer rectifiable varifolds. The first one is a simple fact, well-known to experts, whose proof is included for the sake of clarity.

Lemma 3. *Let M be a normalized locally $\mathcal{H}^n$-rectifiable set such that* $\mathrm{var}(M,\theta)$ *is stationary on $\mathbb{R}^{n+1}$. If M is a cone* (*that is, $M = tM$ for every $t > 0$*), *and M is contained in a closed half-space H with $0 \in \partial H$, then $M = \partial H$ and θ is constant. In particular, M cannot be contained in the convex intersection of two distinct, nonopposite half-spaces containing the origin.*

Proof. Let $H = \{z \in \mathbb{R}^{n+1} : z \cdot \nu < 0\}$, where $\nu \in \mathbb{S}^n$. Given $\varphi \in C^\infty_c([0,\infty))$ with $0 \le \varphi \le 1$, $\varphi(r) = 1$ on $[0,\varepsilon)$ for some $\varepsilon > 0$, and $\varphi'(r) < 0$ on $\{0 < \varphi < 1\}$, let us set $X(x) = \varphi(|x|)\nu$ for $x \in \mathbb{R}^{n+1}$. Then $X \in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ and $\nabla X = \varphi'(|x|)\nu \otimes \hat{x}$, where $\hat{x} = x/|x|$ if $x \ne 0$. Let $\nu_M : M \to \mathbb{S}^n$ be a Borel vector field such that $T_x M = \nu_M(x)^\perp$ for $\mathcal{H}^n$-a.e. $x \in M$. Since M is a cone, we have $\hat{x} \cdot \nu_M(x) = 0$ for $\mathcal{H}^n$-a.e. $x \in M$, and hence

$$\mathrm{div}^M X = \mathrm{div}\, X - \nu_M \cdot \nabla X[\nu_M] = \varphi'(|x|)(\nu\cdot\hat{x} - (\nu_M\cdot\nu)(\nu_M\cdot\hat{x})) = \varphi'(|x|)(\nu\cdot\hat{x}),$$

and thus, by the stationarity of M,

$$0 = \int_M \mathrm{div}^M X\theta\, d\mathcal{H}^n = \int_M \varphi'(|x|)(\nu\cdot\hat{x})\theta(x)\, d\mathcal{H}^n(x).$$

Since $M \subset H$ implies $\hat{x}\cdot\nu \le 0$ for every $x \in M$, $x \ne 0$, thanks to the arbitrariness of φ we find $\nu \cdot \hat{x} = 0$ for $\mathcal{H}^n$-a.e. $x \in M$. The lemma is proved. $\square$

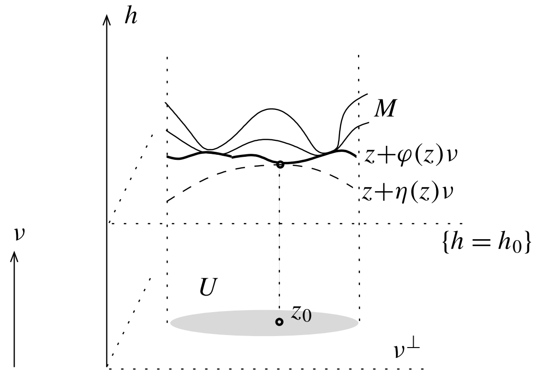

Figure 1. The strong maximum principle for integer varifolds. The rectifiable set M may consist of multiple sheets which, combined with the multiplicity function θ, have distributional mean curvature $\boldsymbol{H}_M$ in some L^p. The sheets may overlap in complicated ways along sets of positive area, so there is a nontrivial relation between the mean curvature vector $\boldsymbol{H}_M$ of the whole configuration and that of a single sheet. The function φ describes the lower sheet of M *above height* h_0 *with respect to the direction* ν and projecting over an open set $U \subset \nu^\perp$. This lower sheet is shown to satisfy a strong maximum principle. Notice that the role of h_0 is that of localizing the part of the varifold we are looking at. For example, in this picture, M could have many more points of the form $z + h\nu$ with $h < h_0$ and $z \in U$, but these points will not contribute to the definition of φ.

The second tool we shall use is a much deeper result, namely, Schätzle's strong maximum principle [2004] for integer rectifiable varifolds with sufficiently summable distributional mean curvature. The statement we adopt here is a slightly simplified version, still sufficient for our purposes, of [Schätzle 2004, Theorem 6.2].

Theorem 4. *Let M be a normalized locally $\mathcal{H}^n$-rectifiable set with distributional mean curvature vector $\boldsymbol{H}_M \in L^p(\theta\mathcal{H}^n \llcorner M; \mathbb{R}^{n+1})$ for some $p > \max\{2, n\}$.*

Pick $\nu \in \mathbb{S}^n$, $h_0 \in \mathbb{R}$, and consider a connected open set $U \subset \nu^\perp$ such that

$$\varphi(z) = \inf\{h > h_0 : z + h\nu \in M\}, \qquad z \in U, \tag{2-8}$$

satisfies $\varphi(z) \in (h_0, \infty)$ for every $z \in U$.

If $\eta \in W^{2,p}(U; (h_0, \infty))$ is such that $\eta \le \varphi$ on U and $\eta(z_0) = \varphi(z_0)$ for some $z_0 \in U$, then it cannot be that

$$-\operatorname{div}\Bigl(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\Bigr)(z) \le \boldsymbol{H}_M(z+\varphi(z)\nu)\cdot\frac{-\nabla\varphi(z)+\nu}{\sqrt{1+|\nabla\varphi(z)|^2}} \tag{2-9}$$

for $\mathcal{H}^n$-a.e. $z \in U$, unless $\eta = \varphi$ on U.

The signs in (2-9) and the geometric intuition behind Theorem 4 are illustrated in Figure 1. The left-hand side is the mean curvature of the subgraph of η with respect to its outer unit normal $(-\nabla\eta+\nu)/\sqrt{1+|\nabla\eta|^2}$, and, similarly, the right-hand side is the mean curvature of the subgraph of φ with respect to its outer unit normal. So, if η touches φ from below at z_0, it cannot be that the subgraph of η is in average bent upwards at least as much as the subgraph of η, unless $\eta = \varphi$. The considerable difficulty of the theorem lies in the fact that $\boldsymbol{H}_M$ does not come into play as the mean curvature of the graph of φ, but rather as the mean curvature of a more complex structure (the integer rectifiable varifold $\operatorname{var}(M, \theta)$), of which φ only represents a sort of lower envelope localized in the cylinder $\{z + t\nu : z \in U, t > h_0\}$.

2.3. ***Sets of finite perimeter.*** A Borel set $\Omega \subset \mathbb{R}^{n+1}$ has *locally finite perimeter* if there exists an $\mathbb{R}^{n+1}$-valued Radon measure μ_Ω on $\mathbb{R}^{n+1}$ such that

$$\int_\Omega \operatorname{div} X = \int_{\mathbb{R}^{n+1}} X \cdot d\mu_\Omega \quad \text{for all } X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-10}$$

The *perimeter of* Ω *relative to an open set* A is defined as $P(\Omega; A) = |\mu_\Omega|(A)$, where $|\mu_\Omega|$ is the total variation of μ_Ω, and Ω has *finite perimeter* if $P(\Omega) = P(\Omega; \mathbb{R}^{n+1}) < \infty$. In this case, either Ω or its complement has finite volume. By exploiting (2-10), the support of μ_Ω is seen to satisfy

$$\operatorname{spt} \mu_\Omega = \{x \in \mathbb{R}^{n+1} : 0 < |B_\rho(x) \cap \Omega| < \omega_n \rho^n \text{ for all } \rho > 0\} \subset \partial\Omega; \tag{2-11}$$

see [Maggi 2012, Proposition 12.19]. Notice that $\operatorname{spt} \mu_\Omega$ is invariant by zero-volume modifications of Ω, while of course $\partial\Omega$ is not. The *reduced boundary* of a set of locally finite perimeter Ω is defined as the set of points such that

$$\nu_\Omega(x) = \lim_{\rho \to 0^+} \frac{\mu_\Omega(B_\rho(x))}{|\mu_\Omega|(B_\rho(x))} \quad \text{exists and belongs to } \mathbb{S}^n. \tag{2-12}$$

The Borel vector field $\nu_\Omega : \partial^*\Omega \to \mathbb{S}^n$ is called the *measure-theoretic outer unit normal to* Ω, and we always have

$$\overline{\partial^*\Omega} = \operatorname{spt} \mu_\Omega. \tag{2-13}$$

Moreover by [Maggi 2012, Theorem 15.9], the reduced boundary is locally $\mathcal{H}^n$-rectifiable, with

$$\mu_\Omega = \nu_\Omega \mathcal{H}^n \llcorner \partial^*\Omega, \qquad P(\Omega; A) = \mathcal{H}^n(A \cap \partial^*\Omega)$$

for every open set $A \subset \mathbb{R}^{n+1}$, and thus (2-10) takes the form

$$\int_\Omega \operatorname{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-14}$$

In addition, for every $x \in \partial^*\Omega$, $\nu_\Omega(x)^\perp = T_x(\partial^*\Omega)$ is the approximate tangent plane to $\partial^*\Omega$ at x and in particular we have

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial^*\Omega)}{\rho^n} = \omega_n \quad \text{for all } x \in \partial^*\Omega. \tag{2-15}$$

To every set Ω of locally finite perimeter we can always associate in a natural way an integer rectifiable varifold $\operatorname{var}(\partial^*\Omega, 1)$. If $\operatorname{var}(\partial^*\Omega, 1)$ admits a distributional mean curvature vector $\boldsymbol{H}_{\partial^*\Omega}$, then the *distributional mean curvature of* Ω is defined by setting

$$H_\Omega = \boldsymbol{H}_{\partial^*\Omega} \cdot \nu_\Omega. \tag{2-16}$$

The subscript Ω on H_Ω is a reminder that we have used the outer orientation of Ω to specify the scalar curvature. With this notation, $H_{B_r} = n/r$ for every $r > 0$.

2.4. ***Basic properties of critical points.*** Here we prove some properties of critical points in the isoperimetric problem which descend from generally known facts about integer varifolds and sets of finite perimeter. A set of finite perimeter and finite volume Ω is a *critical point for the isoperimetric problem* if

$$\frac{d}{dt}\Big|_{t=0} P(f_t(\Omega)) = 0 \tag{2-17}$$

whenever $\{f_t\}_{|t|<1}$ is a one-parameter family of diffeomorphisms with $f_0 = \mathrm{Id}$, $|f_t(\Omega)| = |\Omega|$ and $\mathrm{spt}(f_t - \mathrm{Id}) \Subset \mathbb{R}^{n+1}$ for every $|t| < 1$. By [Maggi 2012, Theorem 17.20], (2-17) is equivalent to the existence of a constant $\lambda \in \mathbb{R}$ such that

$$\int_{\partial^*\Omega} \mathrm{div}^{\partial^*\Omega} X \, d\mathcal{H}^n = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-18}$$

Lemma 5. *If $\Omega \subset \mathbb{R}^{n+1}$ is a critical point for the isoperimetric problem, then Ω is (equivalent modulo sets of volume zero to) a bounded open set such that $\partial\Omega = \mathrm{spt}\,\mu_\Omega$ and $\mathcal{H}^n(\partial\Omega \setminus \partial^*\Omega) = 0$. Moreover, the constant λ in* (2-18) *is equal to*

$$H^0_\Omega = \frac{n P(\Omega)}{(n+1)|\Omega|}; \tag{2-19}$$

that is, $H_\Omega \equiv H^0_\Omega$. Finally,

$$\partial^*\Omega = \left\{ x \in \partial\Omega : \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial\Omega)}{\rho^n} = \omega_n \right\}$$

is locally an analytic hypersurface with constant mean curvature, relatively open in $\partial\Omega$.

Proof. By [Simon 1983, Theorem 17.6], condition (2-18) implies that for every $x \in \mathbb{R}^{n+1}$,

$$e^{|\lambda|\rho} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial^*\Omega)}{\rho^n} \quad \text{is increasing on } \rho > 0, \tag{2-20}$$

which combined with (2-15) and (2-13) gives

$$\mathcal{H}^n(B_\rho(x) \cap \partial^*\Omega) \ge \omega_n e^{-|\lambda|} \rho^n \quad \text{for all } \rho \in (0,1),\ x \in \mathrm{spt}\,\mu_\Omega. \tag{2-21}$$

A first consequence of the lower bound (2-21) is that

$$\mathcal{H}^n(\mathrm{spt}\,\mu_\Omega \setminus \partial^*\Omega) = 0; \tag{2-22}$$

see, e.g., [Maggi 2012, Exercise 17.19]. Moreover, by combining (2-21) with $P(\Omega) < \infty$ and a covering argument, we see that $\mathrm{spt}\,\mu_\Omega$ is bounded.

Let us now consider the open set Ω_1 of those $x \in \mathbb{R}^{n+1}$ such that $|\Omega \cap B_\rho(x)| = |B_\rho(x)|$ for every ρ small enough, and the open set Ω_0 of those $x \in \mathbb{R}^{n+1}$ such that $|\Omega \cap B_\rho(x)| = 0$ for every ρ small enough, so that

$$\mathrm{spt}\,\mu_\Omega = \mathbb{R}^{n+1} \setminus (\Omega_0 \cup \Omega_1), \tag{2-23}$$

thanks to (2-11). If $\Omega^{(1)}$ denotes the set of points of density 1 of Ω, then $\Omega_1 \subset \Omega^{(1)}$, while

$$|\Omega^{(1)} \setminus \Omega_1| = |\Omega^{(1)} \cap \Omega_0| + |\Omega^{(1)} \cap \operatorname{spt} \mu_\Omega| = |\Omega^{(1)} \cap \operatorname{spt} \mu_\Omega| = 0$$

as $\mathcal{H}^n(\operatorname{spt} \mu_\Omega) < \infty$ thanks to (2-22). Thus $|\Omega^{(1)} \Delta \Omega_1| = 0$, and then $|\Omega \Delta \Omega_1| = 0$ by the Lebesgue density theorem. Since Ω_0 and Ω_1 are disjoint open sets, (2-23) implies $\partial\Omega_1 \subset \operatorname{spt} \mu_\Omega$. At the same time, $|\Omega \Delta \Omega_1| = 0$ and the inclusion in (2-11) imply $\operatorname{spt} \mu_\Omega \subset \partial\Omega_1$. Hence $\operatorname{spt} \mu_\Omega = \partial\Omega_1$, and since $\operatorname{spt} \mu_\Omega = \partial\Omega_1$ is bounded and $|\Omega_1| < \infty$, we have that Ω_1 is bounded. The first part of the statement is proved.

We show that λ in (2-18) satisfies $\lambda = H^0_\Omega$ with H^0_Ω defined in (2-19). Since Ω is bounded we can test both (2-14) and (2-18) with $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, where $X(x) = x$ for x in a neighborhood of Ω. Hence,

$$\begin{aligned}(n+1)|\Omega| &= \int_\Omega \operatorname{div}(x)\, dx = \int_\Omega \operatorname{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n = \frac{1}{\lambda} \int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^n \\ &= \frac{1}{\lambda} \int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega}(x)\, d\mathcal{H}^n_x = \frac{n P(\Omega)}{\lambda}, \end{aligned} \tag{2-24}$$

and thus $\lambda = H^0_\Omega$.

Finally, by applying Allard's regularity theorem (see [Simon 1983, Theorem 24.2] or [De Lellis 2008]) to $\operatorname{var}(\partial\Omega, 1)$, we see that $\partial\Omega$ is an analytic constant mean curvature hypersurface in a neighborhood of every $x \in \partial\Omega$ such that

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial\Omega)}{\rho^n} = \omega_n. \tag{2-25}$$

In particular, if $x \in \partial\Omega$ satisfies (2-25) then there exists $\rho > 0$ such that $B_\rho(x) \cap \Omega$ is the epigraph of an analytic function, and thus $x \in \partial^*\Omega$. Vice versa, (2-25) holds everywhere on $\partial^*\Omega$ thanks to (2-15). □

We also notice a simple consequence of Lemma 3.

Lemma 6. *If $\Omega \subset \mathbb{R}^{n+1}$ is a critical point for the isoperimetric problem, $x \in \partial\Omega$, and $y_1, y_2 \in \Omega$ are such that $|y_i - x| = \operatorname{dist}(y_i, \partial\Omega)$ and $|x - y_1| = |x - y_2|$, then $x - y_1 = y_2 - x$.*

Proof. Since $\operatorname{var}(\partial\Omega, 1)$ is an integer varifold of constant distributional mean curvature, it admits at least one blow-up limit in the weak convergence of varifolds at x, and each such limit varifold is stationary and supported on a cone M; see [Simon 1983, Chapter 46]. By construction, M is contained in the half-spaces $\{z \cdot \nu_i \le 0\}$ defined by $\nu_i = (x - y_i)/|x - y_i|,\ i = 1, 2$. If $y_1 \neq y_2$, then $\nu_1 \neq \nu_2$, and Lemma 3 implies that $\nu_1 = -\nu_2$. □

3. Critical points of the isoperimetric problem

Referring to the Introduction for the general strategy, we now present the proof of Theorem 1. At the end of the section we also prove Corollary 2.

Proof of Theorem 1. Let Ω be a set with finite perimeter and finite volume which is a critical point for the isoperimetric problem. The conclusion of Lemma 5 is the starting point of our analysis, aimed at showing that Ω is a finite union of disjoint balls of radius n/H^0_Ω. We rescale Ω so that $H^0_\Omega = n$.

Properties of the distance function: We set $u(y) = \mathrm{dist}(y, \partial\Omega)$ for $y \in \mathbb{R}^{n+1}$ so that

$$N(y) = \nabla u(y) \in \mathbb{S}^n \quad \text{exists for a.e. } y \in \Omega, \tag{3-1}$$

thanks to Rademacher's theorem. For $s > 0$ we set

$$\Omega_s = \{y \in \Omega : u(y) > s\}, \qquad \partial\Omega_s = \{y \in \Omega : u(y) = s\},$$

and recall that, by the coarea formula [Maggi 2012, Theorems 13.1 and 18.1], Ω_s is a set of finite perimeter for a.e. $s > 0$, and for every Borel set $E \subset \mathbb{R}^{n+1}$,

$$|E| = \int_0^\infty \mathcal{H}^n(E \cap \partial^*\Omega_s)\, ds = \int_0^\infty \mathcal{H}^n(E \cap \partial\Omega_s)\, ds. \tag{3-2}$$

In particular,

$$\mathcal{H}^n(\partial\Omega_s \setminus \partial^*\Omega_s) = 0 \quad \text{for a.e. } s > 0. \tag{3-3}$$

We recall that for a.e. $y \in \Omega$, u admits a second-order Taylor expansion at y. Indeed, given $A \subset \Omega$ and $y \in \Omega$, denote by $\overline{\Theta}(u, A)(y)$ the infimum of the constants $c > 0$ such that for $a \in \mathbb{R}$ and $b \in \mathbb{R}^{n+1}$ we have

$$a + b \cdot z + c\frac{|z|^2}{2} \geq u(z) \quad \text{for all } z \in A,$$

with equality at y. For any $y \in \Omega$ we can pick $x \in \partial\Omega$ such that $|x - y| = u(y)$,

$$u(z) = \mathrm{dist}(z, \partial\Omega) \leq \mathrm{dist}(z, \{x\}) = |z - x| \quad \text{for all } z \in \Omega, \tag{3-4}$$

that is, $z \mapsto |z - x|$ touches u from above at y over Ω. At the same time we can construct a second-order polynomial that touches $z \mapsto |z - x|$ from above at y over $\mathbb{R}^{n+1}$. Indeed, it holds

$$|z - x| \leq |y - x| + \frac{y - x}{|y - x|} \cdot (z - y) + \frac{|z - y|^2}{2|y - x|} \quad \text{for all } z \in \mathbb{R}^{n+1}. \tag{3-5}$$

To check this set $y = x + tv$ for $t > 0$ and $|v| = 1$, and set $w = z - y$, so that (3-5) becomes

$$|tv + w| \leq t + v \cdot w + \frac{|w|^2}{2t} \quad \text{for all } w \in \mathbb{R}^{n+1}.$$

Taking squares this is equivalent to

$$\begin{aligned} t^2 + 2tv \cdot w + |w|^2 &\leq t^2 + 2tv \cdot w + |w|^2 + (v \cdot w)^2 + \frac{(v \cdot w)|w|^2}{t} + \frac{|w|^4}{4t^2} \\ &= t^2 + 2tv \cdot w + |w|^2 + \left(v \cdot w + \frac{|w|^2}{2t}\right)^2, \end{aligned}$$

which clearly holds for every $w \in \mathbb{R}^{n+1}$. Thanks to (3-5) there exists $a, b \in \mathbb{R}$ such that

$$|z - x| \leq a + b \cdot z + \frac{|z|^2}{2|y - x|} \quad \text{for all } z \in \mathbb{R}^{n+1},$$

with equality if $z = y$, so that, by the definition of $\overline{\Theta}$ and by (3-4)

$$\overline{\Theta}(u, \Omega)(y) \le \frac{1}{u(y)} \quad \text{for all } y \in \Omega. \tag{3-6}$$

Arguing as in [Caffarelli and Cabré 1995, Proposition 1.6], we see that u is twice differentiable a.e. in Ω.

Preliminary properties of the sets Γ_s^t: For every $t > s > 0$, we consider the compact set

$$\Gamma_s^t = \left\{ y \in \partial\Omega_s : y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z \text{ for some } z \in \partial\Omega_t,\ x \in \partial\Omega \right\}. \tag{3-7}$$

By definition, if $y \in \Gamma_s^t$, then there exist $x \in \partial\Omega$ and $z \in \partial\Omega_t$ such that

$$B_{t-s}(z) \subset \Omega_s \subset \mathbb{R}^{n+1} \setminus B_s(x), \quad \{y\} = \partial B_{t-s}(z) \cap \partial B_s(x). \tag{3-8}$$

In particular x and z are uniquely determined by the uniqueness of limits in L^1_{loc}. Indeed, when $\rho \to 0^+$,

$$\frac{\Omega_s - y}{\rho} \to [x - z]^- \quad \text{as characteristic functions in } L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}), \tag{3-9}$$

where $[v]^-$ denotes the negative half-space defined by, $v \ne 0$,

$$[v]^- = \{w \in \mathbb{R}^{n+1} : w \cdot v < 0\}.$$

Notice also that $\mathrm{Lip}(u; \mathbb{R}^{n+1}) \le 1$ and the inclusion $B_{s+\varepsilon}(y - \varepsilon(x - z)/|x - z|) \subset \Omega$ (which holds for $\varepsilon > 0$ small since $t > s$) imply that y has a unique projection onto $\partial\Omega$. This shows that u is differentiable at $y \in \Gamma_s^t$ with

$$N(y) = -\frac{x - z}{|x - z|} \quad \text{for all } y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z \in \Gamma_s^t. \tag{3-10}$$

In turn, (3-10) gives

$$y + rN(y) \in \partial\Omega_{s-r} \quad \text{for all } r \in [-s, t - s],\ y \in \Gamma_s^t. \tag{3-11}$$

By (3-11), if $y, y' \in \Gamma_s^t$ then

$$s^2 \le |y - sN(y) - y'|^2 = s^2 - 2sN(y) \cdot (y - y') + |y - y'|^2,$$
$$(t - s)^2 \le |y + (t - s)N(y) - y'|^2 = (t - s)^2 + 2(t - s)N(y) \cdot (y - y') + |y - y'|^2;$$

that is

$$|N(y) \cdot (y - y')| \le \max\left\{\frac{1}{s}, \frac{1}{t - s}\right\} \frac{|y - y'|^2}{2} \quad \text{for all } y, y' \in \Gamma_s^t. \tag{3-12}$$

Using (3-10) we easily see that N is continuous on Γ_s^t so that $(u, N) \in C^0(\Gamma_s^t; \mathbb{R} \times \mathbb{R}^{n+1})$ and satisfies (3-12). By Whitney's extension theorem, there exists $\phi \in C^1(\mathbb{R}^{n+1})$ such that $(\phi, \nabla\phi) = (u, N)$ on Γ_s^t. In particular, this implies the $\mathcal{H}^n$-rectifiability of Γ_s^t.

Decomposition of Ω *and covering by* $\zeta(Z)$: We define

$$\Gamma_s^+ = \bigcup_{t > s} \Gamma_s^t, \quad \Omega^\star = \bigcup_{s > 0} \Gamma_s^+ \subset \Omega, \quad Z = \left\{ (x, t) \in \partial^*\Omega \times \mathbb{R} : 0 < t \le \frac{1}{\kappa_n(x)} \right\}, \tag{3-13}$$

and set $\zeta(x,t) = x - t\nu_\Omega(x)$. We claim that

$$|\Omega \setminus \Omega^\star| = 0, \quad |\Omega^\star \setminus \zeta(Z)| = 0. \tag{3-14}$$

We divide the proof of (3-14) into four steps.

Step 1: We prove that N is tangentially differentiable along Γ_s^t at $\mathcal{H}^n$-a.e. $y \in \Gamma_s^t$, with

$$\begin{cases} \nabla^{\Gamma_s^t} N(y) = -\displaystyle\sum_{i=1}^n (\kappa_s^t)_i(y)\tau_i(y) \otimes \tau_i(y), \\ -\dfrac{1}{s} \le (\kappa_s^t)_i(y) \le (\kappa_s^t)_{i+1}(y) \le \dfrac{1}{t-s}, \end{cases} \tag{3-15}$$

where $\{\tau_i(y)\}_{i=1}^n$ is an orthonormal basis of $T_y\Gamma_s^t$. To this end, we first prove that Γ_s^t can be covered by compact sets $\{\mathcal{U}_j\}_{j\in\mathbb{N}}$ in such a way that the restriction of N to $\mathcal{U}_j$ is a Lipschitz map, that is,

$$|N(y_1) - N(y_2)| \le C_j |y_1 - y_2| \quad \text{for all } y_1, y_2 \in \mathcal{U}_j. \tag{3-16}$$

(In passing we notice that (3-16) implies the $C^{1,1}$-rectifiability of Γ_s^+, that is to say, the possibility of covering Γ_s^+ by graphs of $C^{1,1}$ functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$.)

We start by defining the sets $\mathcal{U}_j$. Let us denote by

$$\boldsymbol{C}(N,\rho) = \{z + hN : z \in N^\perp,\ |z| < \rho,\ |h| < \rho\}$$

the open cylinder centered at the origin with axis along $N \in \mathbb{S}^n$, radius $\rho > 0$, and height 2ρ. Notice that, by the interior/exterior ball condition, Γ_s^t admits an approximate tangent plane at $\mathcal{H}^n$-a.e. of its points, and this plane is then necessarily equal to $N(y)^\perp$; that is,

$$T_y\Gamma_s^t = N(y)^\perp \quad \text{for } \mathcal{H}^n\text{-a.e. } y \in \Gamma_s^t.$$

In particular (2-1) implies

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^n\big(\Gamma_s^t \cap (y + \boldsymbol{C}(N(y),\rho))\big)}{\rho^n} = \omega_n \quad \text{for } \mathcal{H}^n\text{-a.e. } y \in \Gamma_s^t.$$

By Egoroff's theorem, we can find compact sets $\mathcal{U}_j$ covering Γ_s^t such that

$$\mu_j^*(\rho) = \sup_{y\in\mathcal{U}_j} \left| 1 - \frac{\mathcal{H}^n\big(\Gamma_s^t \cap (y + \boldsymbol{C}(N(y),\rho))\big)}{\omega_n \rho^n} \right| \to 0 \quad \text{as } \rho \to 0^+. \tag{3-17}$$

Consider the function ϕ constructed in proving the $\mathcal{H}^n$-rectifiability of Γ_s^t. Since $\nabla\phi(y) = N(y) \ne 0$ at each $y \in \Gamma_s^t$, we can apply the implicit function theorem at y and find that Γ_s^t is a C^1-graph over a disk of radius ρ_y in a neighborhood of y. We can thus pick any sequence $\rho_j \to 0^+$, and up to further subdivision of $\mathcal{U}_j$ and relabeling the resulting pieces, we can assume that each $\mathcal{U}_j$ has the following property: for each $y \in \mathcal{U}_j$ there exists

$$\psi_j \in C^1(N(y)^\perp), \quad \psi_j(0) = 0, \quad \nabla\psi_j(0) = 0, \quad \|\nabla\psi_j\|_{C^0(N(y)^\perp)} \le 1 \tag{3-18}$$

such that, if

$$\mathcal{U}_j' = \text{projection of } \mathcal{U}_j \text{ on } N(y)^\perp \cap \{|z| < \rho_j\}, \tag{3-19}$$

then

$$\mathcal{U}_j \cap (y + \boldsymbol{C}(N(y), \rho_j)) = \Gamma_s^t \cap (y + \boldsymbol{C}(N(y), \rho_j)) = y + \{z + \psi_j(z)N(y) : z \in \mathcal{U}_j'\}. \tag{3-20}$$

(Notice that both ψ_j and $\mathcal{U}_j'$ depend on the point $y \in \mathcal{U}_j$ at which we are considering the "graphicality" property of $\mathcal{U}_j$, but that this dependency is not stressed to simplify the notation.) If we set

$$\mu_j(\rho) = \max\Big\{\mu_j^*(\rho), \max_{|z|\le\rho} |\nabla\psi_j(z)|\Big\}, \quad \rho \in (0, \rho_j], \tag{3-21}$$

then $\mu_j(\rho) \to 0$ as $\rho \to 0^+$ by (3-17) and continuity of $\nabla\psi_j$. This completes the definition of the sets $\mathcal{U}_j$.

We now prove (3-16). Fix $y_1, y_2 \in \mathcal{U}_j$. Let ρ_j and ψ_j be the functions associated to $\mathcal{U}_j$ and $y_2 \in \mathcal{U}_j$ as we have just described. For $r_j < \rho_j/3$ to be chosen, we can directly assume that

$$y_1 \in y_2 + \boldsymbol{C}(N(y_2), r_j) \tag{3-22}$$

for otherwise $|y_1 - y_2| \ge c(n) r_j$ and, trivially, $|N(y_1) - N(y_2)| \le 2 \le C_j |y_1 - y_2|$. Next we assume, as we can do without loss of generality up to a rigid motion, that

$$y_2 = (0, 0) \in \mathbb{R}^n \times \mathbb{R}, \quad N(y_2) = (0, 1) \in \mathbb{R}^n \times \mathbb{R}, \quad N(y_2)^\perp = \mathbb{R}^n.$$

In this way (3-20) takes the form

$$\{(z, h) \in \Gamma_s^t : |z| < \rho_j, |h| < \rho_j\} = \{(z, \psi_j(z)) : z \in \mathcal{U}_j'\}, \tag{3-23}$$

with

$$\psi_j \in C^1(\mathbb{R}^n), \quad \psi_j(0) = 0, \quad \nabla\psi_j(0) = 0, \quad \|\nabla\psi_j\|_{C^0(\mathbb{R}^n)} \le 1. \tag{3-24}$$

By (3-22), $y_1 = (z_1, \psi_j(z_1))$ for some $z_1 \in \mathcal{U}_j'$ with $|z_1| < r_j$. By continuity of N along Γ_s^t and since $N(0) = (0, 1)$, we find

$$N(y_1) = \frac{(-\nabla\psi_j(z_1), 1)}{\sqrt{1 + |\nabla\psi_j(z_1)|^2}}.$$

In particular,

$$\frac{|N(y_1) - N(y_2)|^2}{2} = 1 - \frac{1}{\sqrt{1 + |\nabla\psi_j(z_1)|^2}} \le \frac{|\nabla\psi_j(z_1)|^2}{2},$$

while at the same time $|y_1 - y_2|^2 = |z_1|^2 + \psi_j(z_1)^2 \ge |z_1|^2$. We are thus left to show

$$|\nabla\psi_j(z_1)| \le C_j |z_1|. \tag{3-25}$$

To this end we would like to exploit (3-12) with $y = y_1$ and $y' = y_0$ where $y_0 = (z_0, h_0)$ is defined, in terms of a suitable $e_0 \in \mathbb{S}^n$ (see (3-30) below), as

$$z_0 = z_1 - |z_1| e_0, \quad h_0 = \psi_j(z_0). \tag{3-26}$$

Since Γ_s^t may be very "porous", that is, its projection over $\{|z| < \rho_j\}$ could have lots of holes, it is not generally true that $y_0 \in \Gamma_s^t$ and thus that $y' = y_0$ is an admissible choice in (3-12). But when this is the case, by (3-12)

$$C|y_1-y_0|^2 \geq N(y_1)\cdot(y_1-y_0) = |z_1|\frac{\nabla\psi_j(z_1)\cdot(-e_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}} + \frac{\psi_j(z_1)-\psi_j(z_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}. \tag{3-27}$$

Now, in order to exploit (3-27), we notice that

$$|\psi_j(z)| \leq C|z|^2 \quad \text{for all } |z| < \rho_j \text{ such that } (z, \psi_j(z)) \in \Gamma_s^t, \tag{3-28}$$

which is an immediate consequence of the fact that, around $(0,0) = (0, \psi_j(0))$, Γ_s^t is trapped between two tangent balls (notice that we do not know this about the graph of ψ_j, and so we can apply (3-28) only to the points of this graph that lie in Γ_s^t). Since $|z_0| \leq 2|z_1| < 2r_j < \rho_j$, still *assuming* that $y_0 = (z_0, h_0) \in \Gamma_s^t$, by (3-28) we find that

$$|y_1-y_0|^2 = |z_1|^2 + (\psi_j(z_1)-\psi_j(z_0))^2 \leq C|z_1|^2,$$
$$\left|\frac{\psi_j(z_1)-\psi_j(z_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}\right| \leq |\psi_j(z_1)| + |\psi_j(z_0)| \leq C|z_1|^2,$$

and thus (3-27) takes the form

$$C|z_1|^2 \geq |z_1|\frac{\nabla\psi_j(z_1)\cdot(-e_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}. \tag{3-29}$$

Our choice of e_0 is thus clear; we want

$$e_0 = -\frac{\nabla\psi_j(z_1)}{|\nabla\psi_j(z_1)|} \tag{3-30}$$

to have a chance of proving (3-25).

We are now ready to prove (3-25). Set $y_0 = (z_0, h_0)$ for e_0 as in (3-30) and z_0 and h_0 as in (3-26). If $z_0 \in \mathcal{U}_j'$, and thus $y_0 \in \Gamma_s^t$, then, as explained, we are done. Otherwise, let ε_0 be the largest $\varepsilon > 0$ such that

$$\{|z-z_0| < \varepsilon\} \cap \mathcal{U}_j' = \varnothing.$$

Since $z_1 \in \mathcal{U}_j'$ and $|z_0 - z_1| = |z_1|$, we have $\varepsilon_0 \leq |z_1|$. In particular, since $|z_0| \leq 2|z_1|$, the ball $\{|z-z_0| < \varepsilon_0\}$ is contained in $\{|z| < 3|z_1|\} \subset \{|z| < \rho_j\}$ thanks to $3r_j < \rho_j$. By the definition of ε_0, there exists $z_* \in \mathcal{U}_j'$ with $|z_* - z_0| = \varepsilon_0$ and

$$\begin{aligned}\omega_n|z_0-z_*|^n &= \mathcal{H}^n(\{|z-z_0| < \varepsilon_0\}) \\ &\leq \mathcal{H}^n(\{|z| < 3|z_1|\} \setminus \mathcal{U}_j') = \omega_n(3|z_1|)^n - \mathcal{H}^n(\mathcal{U}_j' \cap \{|z| < 3|z_1|\}).\end{aligned} \tag{3-31}$$

On the one hand, since $\mathcal{U}_j$ is the graph of the Lipschitz function ψ_j over $\mathcal{U}_j'$,

$$\begin{aligned}\mathcal{H}^n(\mathcal{U}_j' \cap \{|z| < 3|z_1|\}) &\leq \int_{\mathcal{U}_j'\cap\{|z|<3|z_1|\}} \sqrt{1+|\nabla\psi_j|^2} = \mathcal{H}^n(\mathcal{U}_j \cap \boldsymbol{C}(N(y_2), 3|z_1|)) \\ &= \mathcal{H}^n(\Gamma_s^t \cap \boldsymbol{C}(N(y_2), 3|z_1|)) \\ &\leq \omega_n(3|z_1|)^n(1+\mu_j(3|z_1|))\end{aligned}$$

thanks to (3-21); on the other hand, again by the definition (3-21) of μ_j,

$$\mathcal{H}^n(\mathcal{U}_j'\cap\{|z|<3|z_1|\})=\int_{\mathcal{U}_j'\cap\{|z|<3|z_1|\}}\frac{\sqrt{1+|\nabla\psi_j|^2}}{\sqrt{1+|\nabla\psi_j|^2}}$$
$$\geq\frac{\mathcal{H}^n(\Gamma_s^t\cap\boldsymbol{C}(N(y_2),3|z_1|))}{\sqrt{1+\mu_h(3|z_1|)^2}}\geq\frac{1-\mu_j(3|z_1|)}{\sqrt{1+\mu_h(3|z_1|)^2}}\omega_n(3|z_1|)^n.$$

Combining the last two estimates into (3-31) we find

$$\omega_n|z_0-z_*|^n\leq C\mu_j(3|z_1|)\omega_n(3|z_1|)^n;$$

that is,

$$|z_0-z_*|\leq C\mu_j(3|z_1|)^{1/n}|z_1|. \tag{3-32}$$

In other words, after scaling out $|z_1|$, the best point we can use, z_*, is as close as we want to the point we would like to use, z_0. We conclude the argument setting $y_*=(z_*,\psi_j(z_*))$. Since $z_*\in\mathcal{U}_j'$, we have $y_*\in\Gamma_s^t$. We can apply (3-12) with $y=y_1=(z_1,\psi_j(z_1))$ and $y'=y_*$ to find

$$\begin{aligned}C|y_1-y_*|^2&\geq N(y_1)\cdot(y_1-y_*)\\&\geq\frac{(-\nabla\psi_j(z_1))\cdot(z_1-z_*)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}+\frac{\psi_j(z_1)-\psi_j(z_*)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}\\&\geq\frac{(-\nabla\psi_j(z_1))\cdot(z_1-z_*)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}-C(|z_1|^2+|z_*|^2)\\&\geq|\nabla\psi_j(z_1)|(1-C\mu_j(3|z_1|)^{1/n})\frac{|z_1|}{C}-C(|z_1|^2+|z_*|^2),\end{aligned} \tag{3-33}$$

where we have first applied (3-28) to z_1 and z_*, and then have decomposed z_1-z_* as the sum of $z_1-z_0=e_0|z_1|$ and of z_0-z_*, have recalled the definition of e_0, and have used (3-32). Similarly,

$$\begin{aligned}|y_1-y_*|&\leq|z_1-z_*|+|\psi_j(z_1)-\psi_j(z_*)|\\&\leq|z_1-z_0|+|z_0-z_*|+C(|z_1|^2+|z_*|^2)\leq C|z_1|,\end{aligned}$$

and thus (3-33) implies (3-25). This concludes the proof of (3-16). We now prove (3-15).

As noticed in Section 2.2, since N is a Lipschitz function on each $\mathcal{U}_j$, and since the $\mathcal{U}_j$ are covering Γ_s^t, we deduce that N is tangentially differentiable along Γ_s^t, and that its tangential gradient along Γ_s^t can be computed by looking at any Lipschitz extension of N to $\mathbb{R}^{n+1}$. Moreover, by (2-2), it is enough to work with $\mathcal{U}_j$ in place of Γ_s^t.

To construct a convenient extension of N we go back to the proof of the $\mathcal{H}^n$-rectifiability of Γ_s^t, and this time we construct $\phi\in C^{1,1}(\mathbb{R}^{n+1})$ such that $(u,N)=(\phi,\nabla\phi)$ on $\mathcal{U}_j$ by taking (3-12) and (3-16) into account. Then we can go back to the construction of the sets $\mathcal{U}_j$, and apply the $C^{1,1}$-implicit function theorem to deduce that for each $y\in\mathcal{U}_j$ there exists

$$\psi_j\in C^{1,1}(N(y)^\perp),$$

satisfying (3-18) and (3-20). In particular, we can consider the Lipschitz extension N_* of N from $\mathcal{U}_j \cap (y + \boldsymbol{C}(N(y), \rho_j))$ to $y + \boldsymbol{C}(N(y), \rho_j)$ given by

$$N_*(y + z + hN(y)) = \frac{-\nabla\psi_j(z) + N(y)}{\sqrt{1 + |\nabla\psi_j(z)|^2}} \quad \text{for all } z \in N(y)^\perp,\ |z| < \rho_j,\ |h| < \rho_j.$$

Setting $\Psi_j(z) = y + z + \psi_j(z)N(y)$ for $|z| < \rho_j$, by (2-5) we have that for $\mathcal{H}^n$-a.e. $y' \in \mathcal{U}_j$,

$$(\nabla^{\mathcal{U}_j} N)_{y'}[\tau] = \nabla(N_* \circ \Psi_j)_{\Psi_j^{-1}(y')}[e],$$

where $\tau \in T_{y'}\mathcal{U}_j$ and $e = (\nabla\Psi_j)_{\Psi_j^{-1}(y')}[\tau] \in \mathbb{R}^n$. When $\psi_j \in C^2(N(y)^\perp)$, a classical computation shows that

$$\nabla(N_* \circ \Psi_j)_z[e] = A_j(\Psi_j(z))[\tau],$$

where A_j denotes the second fundamental form to the graph of ψ_j, which is symmetric thanks to the commutativity property of the second derivatives of ψ_j, and where the eigenvalues of A_j are bounded from below by $-1/s$ and from above by $1/(t-s)$ thanks to $\mathcal{U}_j \subset \Gamma_s^t$. In our case the same computations hold for a.e. $|z| < \rho_j$ by the chain rule for Lipschitz functions, where the symmetry of A_j is guaranteed by the fact that $\nabla^2\psi_j$ is both a distributional gradient and an a.e. classical differential of $\nabla\psi_j$. Finally, the a.e.-pointwise estimates on the eigenvalues are deduced a.e. on $\mathcal{U}_j'$ thanks to the fact that $\nabla^2\psi_j$ is an a.e. classical differential. This proves (3-15).

Step 2: We claim that for every $t > s > 0$ we have

$$\mathcal{H}^n(\partial\Omega_t) \le (t/s)^n \mathcal{H}^n(\Gamma_s^t), \tag{3-34}$$

and then use (3-34) to prove

$$|\Omega \Delta \Omega^\star| = 0. \tag{3-35}$$

Indeed, for $r \in [-s, t-s]$ let us consider the map

$$f_r : \Gamma_s^t \to \partial\Omega_{s+r}, \quad f_r(y) = y + rN(y), \quad y \in \Gamma_s^t. \tag{3-36}$$

The fact that $f_r(y) \in \partial\Omega_{s+r}$ is immediate as every $y \in \Gamma_s^t$ has the form $y = (1-(s/t))x + (s/t)z$ for $x \in \partial\Omega$, $z \in \partial\Omega_t$. Notice that, again by the definition of Γ_s^t, the map f_{t-s} is surjective; that is, $\partial\Omega_t = f_{t-s}(\Gamma_s^t)$. Thus

$$\mathcal{H}^n(\partial\Omega_t) = \mathcal{H}^n(f_{t-s}(\Gamma_s^t)) \le \int_{f_{t-s}(\Gamma_s^t)} \mathcal{H}^0(f_{t-s}^{-1}(z))\, d\mathcal{H}^n_z = \int_{\Gamma_s^t} J^{\Gamma_s^t} f_{t-s}\, d\mathcal{H}^n,$$

where by (3-15), and in particular by the lower bound on $(\kappa_s^t)_i$,

$$J^{\Gamma_s^t} f_{t-s} = \prod_{i=1}^n (1 - (t-s)(\kappa_s^t)_i) \le \Big(1 + \frac{t-s}{s}\Big)^n \quad \mathcal{H}^n\text{-a.e. on } \Gamma_s^t.$$

This proves (3-34). To prove (3-35), we first apply the coarea formula (3-2) to find

$$|\Omega \Delta \Omega^\star| = \int_0^\infty \mathcal{H}^n((\Omega \Delta \Omega^\star) \cap \partial\Omega_s)\, ds = \int_0^\infty \mathcal{H}^n(\partial\Omega_s \setminus \Gamma_s^+)\, ds, \tag{3-37}$$

where $\Gamma_s^+ \subset \partial\Omega_s$. Again by the coarea formula, for a.e. $s > 0$,

$$\mathcal{H}^n(\partial\Omega_s) = \lim_{\varepsilon\to 0} \frac{|\Omega_s| - |\Omega_{s+\varepsilon}|}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^\varepsilon \mathcal{H}^n(\partial\Omega_{s+r})\,dr.$$

where by (3-34)

$$\frac{1}{\varepsilon}\int_0^\varepsilon \mathcal{H}^n(\partial\Omega_{s+r})\,dr \le \frac{1}{\varepsilon}\int_0^\varepsilon \Big(1+\frac{r}{s}\Big)^n \mathcal{H}^n(\Gamma_s^{s+r})\,dr \le \Big(1+\frac{\varepsilon}{s}\Big)^n \mathcal{H}^n(\Gamma_s^+).$$

Since $\Gamma_s^+ \subset \partial\Omega_s$, this proves

$$\mathcal{H}^n(\Gamma_s^+) = \mathcal{H}^n(\partial\Omega_s) \quad \text{for a.e. } s > 0, \tag{3-38}$$

which, combined with (3-37) gives in turn (3-35).

Step 3: For $r \in (0, s)$, let us consider the map

$$g_r : \Gamma_s^+ \to \Gamma_{s-r}^+, \quad g_r(y) = y - rN(y), \quad y \in \Gamma_s^+,$$

which is (clearly) a bijection between Γ_s^t and Γ_{s-r}^t for each $t > 0$. We claim that if y is a point of tangential differentiability of N along Γ_s^t, then $g_r(y)$ is a point of tangential differentiability of N along Γ_{s-r}^t, and

$$(\kappa_{s-r}^t)_i(g_r(y)) = \frac{(\kappa_s^t)_i(y)}{1 + r(\kappa_s^t)_i(y)} \quad \text{for all } i = 1, \dots, n. \tag{3-39}$$

Indeed, it is easily seen that

$$N(y) = N(g_r(y)) = N(y - rN(y)) \quad \text{for all } y \in \Gamma_s^t, \tag{3-40}$$

so that if y is a point of tangential differentiability of N along Γ_s^t and $\tau \in T_y\Gamma_s^t$, then $\tau \in T_{g_r(y)}\Gamma_s^t$ and

$$(\nabla^{\Gamma_s^t} N)_y[\tau] = (\nabla^{\Gamma_{s-r}^t} N)_{g_r(y)}[\tau - r(\nabla^{\Gamma_s^t} N)_y[\tau]].$$

Plugging in $\tau = \tau_i(y)$ as in (3-15) we find

$$-(\kappa_s^t)_i(y)\tau_i(y) = (1 + r(\kappa_s^t)_i(y))(\nabla^{\Gamma_{s-r}^t} N)_{g_r(y)}[\tau_i(y)];$$

that is,

$$-\tau_i(y)\cdot(\nabla^{\Gamma_{s-r}^t} N)_{g_r(y)}[\tau_i(y)] = \frac{(\kappa_s^t)_i(y)}{1 + r(\kappa_s^t)_i(y)}.$$

Thus $\{\tau_i(y)\}_{i=1}^n$ is an orthonormal basis for $T_{g_r(y)}\Gamma_{s-r}^t = T_y\Gamma_s^t$ made up of eigenvalues of $\nabla^{\Gamma_{s-r}^t} N(g_r(y))$, and the last formula is just (3-39).

Step 4: We prove that

$$|\Omega^\star \setminus \zeta(Z)| = 0. \tag{3-41}$$

By the coarea formula (3-2) and by (3-38)

$$\begin{aligned}|\Omega^\star \setminus \zeta(Z)| &= \int_0^\infty \mathcal{H}^n((\Omega^\star \setminus \zeta(Z)) \cap \partial\Omega_s)\,ds = \int_0^\infty \mathcal{H}^n((\Omega^\star \setminus \zeta(Z)) \cap \Gamma_s^+)\,ds \\ &= \int_0^\infty \mathcal{H}^n(\Gamma_s^+ \setminus \zeta(Z))\,ds.\end{aligned}$$

Since $x \in \partial^*\Omega$ and $y \in \Gamma_s^+$ are such that $y = x - s\nu_\Omega(x)$ if and only if $x = y - sN(y) = g_s(y)$, with g_s as in Step 3, we have

$$\zeta(Z) \cap \Gamma_s^+ = g_s^{-1}(\partial^*\Omega) \quad \text{for all } s > 0.$$

Taking into account that $\partial\Omega \setminus \partial^*\Omega = \Sigma$ (recall Lemma 5) and that $g_s^{-1}(\partial\Omega) \subset \Gamma_s^+$, in order to prove (3-41) we are left to show that for a.e. $s > 0$

$$\mathcal{H}^n(g_s^{-1}(\Sigma)) = 0. \tag{3-42}$$

In other words, the points in Γ_s^+ that, projected over $\partial\Omega$, end up on the singular set, have negligible $\mathcal{H}^n$-measure. We are actually going to show that (3-42) holds for every $s > 0$ such that $\mathcal{H}^n(\Gamma_s^+) = \mathcal{H}^n(\partial\Omega_s)$. We shall argue by contradiction, assuming that $\mathcal{H}^n(\Gamma_s^+) = \mathcal{H}^n(\partial\Omega_s)$ and

$$\mathcal{H}^n(g_s^{-1}(\Sigma)) > 0.$$

In particular, there exists $t > s$, such that $\mathcal{H}^n(\Gamma_s^t \cap g_s^{-1}(\Sigma)) > 0$.

As a preliminary step to derive a contradiction we first notice that

$$\mathcal{H}^0(g_s^{-1}(x)) \le 2 \quad \text{for all } x \in \partial\Omega. \tag{3-43}$$

Otherwise, $g_s^{-1}(x)$ would contain at least two points y_1, y_2 such that $(x - y_1)/|x - y_1|$ and $(x - y_2)/|x - y_2|$ are not antipodal. Any blow-up of $\mathrm{var}(\partial\Omega, x)$ would then be a stationary varifold contained in the intersection of two nonopposite half-spaces, a contradiction to Lemma 3. By (3-43) and by $\mathcal{H}^n(\Sigma) = 0$ (recall (3-3)) we find that

$$0 = 2\mathcal{H}^n(\Sigma) \ge \int_\Sigma \mathcal{H}^0(g_s^{-1}(x))\, d\mathcal{H}^n = \int_{g_s^{-1}(\Sigma)} J^{\Gamma_s^t} g_s \, d\mathcal{H}^n,$$

where

$$J^{\Gamma_s^t} g_s = \prod_{i=1}^n (1 + s(\kappa_s^t)_i) \ge 0 \quad \text{on } \Gamma_s^t$$

thanks to $-1/s \le (\kappa_s^t)_i$; see (3-15). Having assumed $\mathcal{H}^n(g_s^{-1}(\Sigma)) > 0$, and since $\{(\kappa_s^t)_i\}_i$ are ordered increasingly on i, we deduce in particular that

$$\mathcal{H}^n\Big(\Big\{y \in \Gamma_s^t : (\kappa_s^t)_1(y) = -\frac{1}{s}\Big\}\Big) \ge \mathcal{H}^n(\Gamma_s^t \cap g_s^{-1}(\Sigma)) > 0. \tag{3-44}$$

By (3-39) we see that

$$\Big\{\tilde{y} \in \Gamma_{s-r}^t : (\kappa_{s-r}^t)_1(\tilde{y}) = -\frac{1}{s-r}\Big\} = g_r\Big(\Big\{y \in \Gamma_s^t : (\kappa_s^t)_1(y) = -\frac{1}{s}\Big\}\Big).$$

Since $g_r : \Gamma_s^t \to \Gamma_{s-r}^t$ is injective, by the area formula

$$\mathcal{H}^n\Big(\Big\{\tilde{y} \in \Gamma_{s-r}^t : (\kappa_{s-r}^t)_1(\tilde{y}) = -\frac{1}{s-r}\Big\}\Big) = \int_{\{y \in \Gamma_s^t : (\kappa_s^t)_1(y) = -1/s\}} J^{\Gamma_s^t} g_r \, d\mathcal{H}^n.$$

Using again that $(\kappa_s^t)_i \ge -1/s$ on Γ_s^t, we have

$$J^{\Gamma_s^t} g_r = \prod_{i=1}^n (1 + r(\kappa_s^t)_i) \ge \Big(1 - \frac{r}{s}\Big)^n > 0 \quad \text{for all } r \in (0, s),$$

so that (3-44) implies that for every $r \in (0, s)$

$$\mathcal{H}^n(\Lambda_{s-r}^t) > 0 \quad \text{for } \Lambda_{s-r}^t = \Big\{\tilde{y} \in \Gamma_{s-r}^t : (\kappa_{s-r}^t)_1(\tilde{y}) = -\frac{1}{s-r}\Big\}. \tag{3-45}$$

By using (3-39) and the fact that $a \mapsto a/(1+ra)$ is increasing on $a \ge 0$, we see that for every $\tilde{y} \in \Lambda_{s-r}^t$, $\tilde{y} = g_r(y)$, we have

$$\sum_{i=1}^n (\kappa_{s-r}^t)_i(\tilde{y}) = -\frac{1}{s-r} + \sum_{i=2}^n \frac{(\kappa_s^t)_i(y)}{1 + r(\kappa_s^t)_i(y)} \le -\frac{1}{s-r} + (n-1)\frac{1/(t-s)}{1 + (r/(t-s))} \le 0, \tag{3-46}$$

provided $r \in (r_0, s)$ for $r_0 = r_0(s, t)$ suitably close to s, depending on s and t. Here the choice of 0 on the right-hand side of (3-46) is arbitrary. Any constant strictly less than n would suffice for the rest of the argument.

Now consider the set

$$\Lambda = \bigcup_{r_0 < r < s} \Lambda_{s-r}^t$$

so that by the coarea formula and (3-45)

$$|\Lambda| = \int_{r_0}^s \mathcal{H}^n(\Lambda \cap \partial\Omega_{s-r})\, dr = \int_{r_0}^s \mathcal{H}^n(\Lambda_{s-r}^t)\, dr > 0.$$

By the a.e. second-order differentiability of u, there exists $y_0 \in \Lambda$ such that u admits a second-order Taylor expansion at y_0. Moreover there exists $r \in (r_0, s)$ such that $y_0 \in \Lambda_{s-r}^t \subset \Gamma_{s-r}^t$, so that $\nabla^2 u(y_0)[N(y_0)] = 0$ by (3-40), and thus

$$\nabla^2 u(y_0) = \nabla^{\Gamma_{s-r}^t} N(y_0) = -\sum_{i=1}^n (\kappa_{s-r}^t)_i(y_0)\tau_i(y_0) \otimes \tau_i(y_0), \tag{3-47}$$

thanks to (3-15). Moreover, by (3-46), we definitely have

$$\sum_{i=1}^n (\kappa_{s-r}^t)_i(y_0) \le 0. \tag{3-48}$$

Let us now set $\nu = -N(y_0)$ and

$$\boldsymbol{D}_\rho = \{z \in \nu^\perp : |z| < \rho\}, \quad \boldsymbol{C}_\rho = \{z + h\nu : z \in \boldsymbol{D}_\rho, |h| < \rho\}, \quad \rho > 0.$$

For every $\varepsilon > 0$, the second-order differentiability of u at y_0, (3-48) and (3-47) imply the existence of $\rho > 0$ and of a second-order polynomial $\eta : \nu^\perp \equiv \mathbb{R}^n \to \mathbb{R}$ such that $\eta(0) = 0$, $\nabla\eta(0) = 0$,

$$-\operatorname{div}\Big(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\Big)(z) \le -\operatorname{div}\Big(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\Big)(0) + \varepsilon \le \sum_{i=1}^n (\kappa_{s-r}^t)_i(y_0) + 2\varepsilon \le 2\varepsilon \tag{3-49}$$

for every $z \in \boldsymbol{D}_\rho$ and

$$y_0 + \{z + h\nu : z \in \boldsymbol{D}_\rho, -\rho < h < \eta(z)\} \subset (y_0 + \boldsymbol{C}_\rho) \cap \Omega_{s-r}. \tag{3-50}$$

If we translate Ω by $(s-r)N(y_0)$, then

$$\Omega_{s-r} \subset (\Omega + (s-r)N(y_0)) \quad \text{with } y_0 \in \partial\Omega_{s-r} \cap \partial(\Omega + (s-r)N(y_0)).$$

We are now in the position to apply Theorem 4 with

$$M = \partial(\Omega + (s-r)N(y_0) - y_0),$$

$\nu = -N(y_0),\ U = \boldsymbol{D}_\rho,\ z_0 = 0,\ h_0 = \nu \cdot y_0 - \rho$ and η as in (3-49). Indeed by (3-50) we have that if we set

$$\varphi(z) = \inf\{h \in (h_0, \infty) : z + h\nu \in M\}, \quad z \in \boldsymbol{D}_\rho,$$

then $\infty > \varphi \geq \eta > h_0$ on $\boldsymbol{D}_\rho$, as well as $\varphi(0) = \eta(0) = 0$. However, by (3-49),

$$2\varepsilon \geq -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)(z) \quad \text{for all } z \in \boldsymbol{D}_\rho,$$

while by the constant mean curvature condition $n = H^0_\Omega = \boldsymbol{H}_{\partial\Omega} \cdot \nu_\Omega$ on $\partial^*\Omega$ we have

$$n = \boldsymbol{H}_M(z + \varphi(z)\nu) \cdot \frac{-\nabla\varphi(z) + \nu}{\sqrt{1+|\nabla\varphi(z)|^2}} \quad \text{for a.e. } z \in \boldsymbol{D}_\rho.$$

This is a contradiction to Theorem 4; hence we obtain (3-41).

Conclusion of the proof: Having proved (3-41), we can now apply the Montiel–Ros argument. By (3-35) and (3-41),

$$|\Omega| = |\Omega^\star| \leq |\zeta(Z)| \leq \int_Z \mathcal{H}^0(\zeta^{-1}(y))\,dy = \int_{\partial^*\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \prod_{i=1}^n (1 - t\kappa_i(x))\,dt,$$

where $Z = \{(x,t) \in \partial^*\Omega \times \mathbb{R} : 0 < t \leq 1/\kappa_n(x)\}$ and $\zeta(x,t) = x - t\nu_\Omega(x)$. Here we have used the fact that Z is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^{n+1} \times \mathbb{R}$ with

$$\mathcal{H}^{n+1} \llcorner ((\partial^*\Omega) \times \mathbb{R}) = (\mathcal{H}^n \llcorner \partial^*\Omega) \times \mathcal{H}^1, \tag{3-51}$$

see [Maggi 2012, Exercise 18.10], and that $J^Z\zeta = \prod_{i=1}^n (1 - t\kappa_i)$. By the arithmetic-geometric mean inequality and by $\kappa_n \geq H^0_\Omega/n$, arguing as in (1-9) we thus find

$$\begin{aligned}
\int_{\partial^*\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \prod_{i=1}^n (1 - t\kappa_i(x))\,dt &\leq \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \left(\frac{1}{n}\sum_{i=1}^n (1 - t\kappa_i(x))\right)^n dt \\
&\leq \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{n/H^0_\Omega} (1 - tH^0_\Omega)^n\,dt \\
&= \frac{n}{n+1}\int_{\partial\Omega} \frac{d\mathcal{H}^n}{H^0_\Omega} = |\Omega|,
\end{aligned}$$

so that equalities hold everywhere and

$$|\zeta(Z)\setminus\Omega| = 0, \tag{3-52}$$

$$\mathcal{H}^0(\zeta^{-1}(y)) = 1 \quad \text{for a.e. } y\in\Omega, \tag{3-53}$$

$$\kappa_i(x) = \frac{H^0_\Omega}{n} \quad \text{for every } x\in\partial^*\Omega,\ i = 1,\dots,n. \tag{3-54}$$

Recall that we have rescaled Ω so that $H^0_\Omega = n$. By (3-54), since $\partial^*\Omega$ is relatively open in $\partial\Omega$, we can find a family $\{S_i\}_{i\in I}$, $I\subset\mathbb{N}$, of mutually disjoint subsets of $\partial^*\Omega$ with $S_i\subset\partial B_1(x_i)$ for points $x_i\in\mathbb{R}^{n+1}$ such that

$$\partial^*\Omega = \bigcup_{i\in I} S_i, \quad S_i \text{ is relatively open in } \partial\Omega, \quad S_i \text{ is connected.} \tag{3-55}$$

Because $S_i\subset\partial\Omega$, we know that $u(x_i)\le 1$.

We claim that $u(x_i) = 1$ for every $i\in I$. Indeed if $\delta>0$ and $i\in I$ are such that $u(x_i) = 1-4\delta$, then $B_\delta(x_i)\cap A_i\subset\Omega$, where $A_i = \zeta(S_i\times(0,1))$ is an open subset of Ω. For any $y\in B_\delta(x_i)\cap A_i$, the triangle inequality implies $u(y) < 1-3\delta$, while clearly $d(y,S_i)\ge d(y,\partial B_1(x_i))\ge 1-\delta$. In particular, if $x\in\partial\Omega$ is such that $|x-y| = u(y)$, then $x\notin S_i$. Since (3-35) and (3-41) imply that for a.e. $y\in\Omega$ there exists $x\in\partial^*\Omega$ such that $|x-y| = u(y)$, we conclude from (3-55) that for a.e. $y\in B_\delta(x_i)\cap A_i$ there exist $j\ne i$ and $x\in S_j$ such that $|x-y| = u(y)$; in particular, $B_\delta(x_i)\cap A_i\cap A_j$ is nonempty, and since it is an open set, we have

$$0 < |B_\delta(x_i)\cap A_i\cap A_j|, \quad \text{where, if } i\ne j, \quad A_i\cap A_j\subset\{y\in\Omega:\mathcal{H}^0(\zeta^{-1}(y))\ge 2\}.$$

This is a contradiction to (3-53). Thus $u(x_i) = 1$ for every $i\in I$.

Now let T_i denote the closure of S_i in $\partial B_1(x_i)$. Since $u(x_i) = 1$ for every $i\in I$, we can apply Theorem 4 to $M = \partial\Omega$ at each $x\in T_i$ to find $\rho_x > 0$ such that

$$\partial\Omega\cap B_{\rho_x}(x) = \partial B_1(x_i)\cap B_{\rho_x}(x). \tag{3-56}$$

This in turn proves that $T_i = \partial B_1(x_i)$, and thus that $\partial B_1(x_i)\subset\partial\Omega$ for every $i\in I$.

Since $\mathcal{H}^n(\partial B_1(x)\cap\partial B_1(y)) = 0$ unless $x = y$, $P(\Omega) < \infty$ implies that I is finite. Since $\partial^*\Omega$ is covered by the S_i, Ω is the finite union of the balls $B_1(x_i)$, and owing to $\partial B_1(x_i)\subset\partial\Omega$, these balls must be disjoint (their closures can of course intersect). This completes the proof of Theorem 1. □

Proof of Corollary 2. Condition (1-4) implies that the vector-valued Radon measures

$$\mu_{\Omega_j} = \nu_{\Omega_j}\mathcal{H}^n \llcorner \partial^*\Omega_j$$

converge in weak-star sense to μ_Ω with $|\mu_{\Omega_j}| \overset{*}{\rightharpoonup} |\mu_\Omega|$ on $\mathbb{R}^{n+1}$. By Reshetnyak's continuity theorem [Ambrosio et al. 2000, Theorem 2.39]

$$\lim_{j\to\infty}\int_{\mathbb{R}^{n+1}}\Phi\left(x,\frac{d\mu_{\Omega_j}}{d|\mu_{\Omega_j}|}(x)\right)d|\mu_{\Omega_j}| = \int_{\mathbb{R}^{n+1}}\Phi\left(x,\frac{d\mu_\Omega}{d|\mu_\Omega|}(x)\right)d|\mu_\Omega|$$

whenever $\Phi \in C^0_c(\mathbb{R}^{n+1}\times\mathbb{S}^n)$. Given $X\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$,

$$\Phi(x,\nu)=\operatorname{div}X(x)-\nu\cdot\nabla X(x)[\nu],\quad (x,\nu)\in\mathbb{R}^{n+1}\times\mathbb{S}^n,$$

belongs to $C^0_c(\mathbb{R}^{n+1}\times\mathbb{S}^n)$ and thus we find

$$\lim_{j\to\infty}\int_{\partial^*\Omega_j}\operatorname{div}^{\partial^*\Omega_j}X\,d\mathcal{H}^n=\int_{\partial^*\Omega}\operatorname{div}^{\partial^*\Omega}X\,d\mathcal{H}^n.$$

By (1-5) and by $\mu_{\Omega_j}\overset{*}{\rightharpoonup}\mu_\Omega$

$$\lim_{j\to\infty}\int_{\partial^*\Omega_j}\operatorname{div}^{\partial^*\Omega_j}X\,d\mathcal{H}^n=\lambda\lim_{j\to\infty}\int_{\partial^*\Omega_j}X\cdot\nu_{\Omega_j}\,d\mathcal{H}^n=\lambda\int_{\partial^*\Omega}X\cdot\nu_\Omega\,d\mathcal{H}^n.$$

We have thus proved that Ω is a set of finite perimeter, finite volume and constant distributional mean curvature. We conclude by Theorem 1. □

4. The Heintze–Karcher inequality for sets of finite perimeter

The proof of Theorem 1 also shows that the Heintze–Karcher inequality can be generalized to sets of finite perimeter. In this section we explain how this is done. As usual, set $u(y)=\operatorname{dist}(y,\partial\Omega)$ for $y\in\Omega$.

Lemma 7. *If Ω is an open set with finite perimeter and finite volume in $\mathbb{R}^{n+1}$, then $\Omega_s=\{y\in\Omega: u(y)>s\}$ is an open set of finite perimeter with $\mathcal{H}^n(\partial\Omega_s\setminus\Gamma_s^+)=0$ for a.e. $s>0$, where $\Gamma_s^+=\bigcup_{t>0}\Gamma_s^t$ and Γ_s^t is defined as in* (1-14)*. Moreover*:

(i) *For every $s>0$, Γ_s^+ can be covered by countably many graphs of $C^{1,1}$-functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$.*

(ii) *For every $s>0$, the principal curvatures $(\kappa_s)_i$ of Γ_s^+ are defined $\mathcal{H}^n$-a.e. on Γ_s^+ by setting*

$$(\kappa_s)_i=(\kappa_s^t)_i\quad\text{on } \Gamma_s^t \text{ for each } t>s,$$

for $(\kappa_s^t)_i$ as in (3-15)*. Correspondingly, $\mathcal{H}^n$-a.e. on Γ_s^+ we can define*

$$H_{\Omega_s}=\sum_{i=1}^n(\kappa_s)_i,\quad |A_{\Omega_s}|^2=\sum_{i=1}^n(\kappa_s)_i^2$$

as natural generalizations of the mean curvature and of the length of the second fundamental form of $\partial\Omega_s$ with respect to ν_{Ω_s} at points in $\Gamma_s^+\subset\partial\Omega_s$.

(iii) *For every $r<s<t$, the map $g_r:\Gamma_s^t\to\Gamma_{s-r}^t$, defined by $g(y)=y-r\nabla u(y)$ for $y\in\Gamma_s^t$, is a Lipschitz bijection from Γ_s^t to Γ_{s-r}^t, with*

$$J^{\Gamma_s^t}g_r(y)=\prod_{i=1}^n(1+r(\kappa_s)_i(y)),\quad (\kappa_{s-r})_i(g_r(y))=\frac{(\kappa_s)_i(y)}{1+r(\kappa_s)_i(y)}\tag{4-1}$$

for $\mathcal{H}^n$-a.e. $y\in\Gamma_s^t$.

Proof. All these conclusions are contained in Steps 1, 2 and 3 of the proof of Theorem 1, where at no stage the constant distributional mean curvature condition, or the regularity of $\partial^*\Omega$ implied by it, have been used. □

As a consequence of Lemma 7, we see that for every $x \in g_s(\Gamma_s^+) \subset \partial\Omega$, the limit

$$\kappa_i(x) = \lim_{r \to s^-} (\kappa_{s-r})_i(x) \in [-\infty, \infty) \tag{4-2}$$

exists by monotonicity; see (4-1). We thus give the following definitions: given an open set of finite perimeter and finite volume $\Omega \subset \mathbb{R}^{n+1}$ we define the *viscosity boundary of* Ω as

$$\partial^v \Omega = \bigcup_{s>0} g_s(\Gamma_s^+)$$

and the *viscosity mean curvature of* Ω by

$$H_\Omega^v(x) = \sum_{i=1}^n \kappa_i(x) \quad \text{for all } x \in \partial^v \Omega. \tag{4-3}$$

Notice that $\partial^v\Omega$ is covered by countably many $\mathcal{H}^n$-rectifiable sets, although it may contain points of $\operatorname{spt} \mu_\Omega$ that are outside the reduced boundary, or that have density 1 for Ω. It is not obvious if, at this level of generality, $\partial^v\Omega$ is $\mathcal{H}^n$-finite. In any case, our only reason for introducing these concepts is to formulate the following definition: a set of finite perimeter and finite volume Ω is *mean convex in the viscosity sense* if H_Ω^v defined in (4-3) is positive along $\partial^v\Omega$. It is easy to see that if $\partial\Omega$ is C^2, then $\partial^v\Omega = \partial\Omega$ and $H_\Omega^v(x) = H_\Omega(x)$ for any $x \in \partial\Omega$. Hence, the viscosity notion generalizes the mean convexity in the classical sense.

This said, following Brendle's point of view [2013] on the Montiel–Ros argument, we have the following generalized form of the Heintze–Karcher inequality; see (4-4) below.

Theorem 8 (Heintze–Karcher inequality for sets of finite perimeter). *If $\Omega \subset \mathbb{R}^{n+1}$ is an open set of finite perimeter and finite volume which is mean convex in the viscosity sense, then for every $s > 0$*

$$|\Omega_s| \le \frac{n}{n+1} \int_{\Gamma_s^+} \frac{d\mathcal{H}^n}{H_{\Omega_s}}. \tag{4-4}$$

Moreover, the limit of the right-hand side of (4-4) *as $s \to 0^+$ always exists in* $(0, \infty]$.

Proof. The mean convexity assumption on Ω and the monotonicity property behind the definition (4-2) of κ_i imply that $\sum_{i=1}^n (\kappa_s)_i > 0$ on Γ_s^+. We define for every $s > 0$

$$Q(s) = \int_{\Gamma_s^+} \frac{d\mathcal{H}^n}{H_{\Omega_s}} > 0.$$

Moreover, for every $t > 0$ we define $Q^t : (0, t) \to (0, \infty)$ by setting

$$Q^t(s) = \int_{\Gamma_s^t} \frac{d\mathcal{H}^n}{H_{\Omega_s}}, \quad s \in (0, t).$$

Notice that

$$Q(s) \ge Q^t(s) \ge Q^{t+\varepsilon}(s) \quad \text{for all } t > s,\ \varepsilon > 0, \tag{4-5}$$

and recall that $\mathcal{H}^n(\Gamma_s^t)$ converges monotonically to $\mathcal{H}^n(\Gamma_s^+)$ as $t \to s^+$, so that

$$Q(s) = \lim_{t \to s^+} Q^t(s) = \sup_{t > s} Q^t(s) \quad \text{for every } s > 0. \tag{4-6}$$

For $r \in (0, s)$ by Lemma 7(iii) we have

$$Q^t(s-r) - Q^t(s) = \int_{\Gamma_s^t} \left(\frac{\prod_{i=1}^n (1 + r(\kappa_s)_i)}{\sum_{i=1}^n (\kappa_s)_i / (1 + r(\kappa_s)_i)} - \frac{1}{H_{\Omega_s}} \right) d\mathcal{H}^n$$
$$= \int_{\Gamma_s^t} \left(\frac{1 + r H_{\Omega_s} + O_t(r^2)}{H_{\Omega_s} - r|A_{\Omega_s}|^2 + O_t(r^2)} - \frac{1}{H_{\Omega_s}} \right) d\mathcal{H}^n,$$

where $O_t(r^2)/r \to 0$ uniformly on Γ_s^t as $r \to 0$. We thus find that Q^t is differentiable on $(0, t)$ with

$$(Q^t)'(s) = -\int_{\Gamma_s^t} 1 + \frac{|A_{\Omega_s}|^2}{H_{\Omega_s}^2} \, d\mathcal{H}^n \quad \text{for all } s \in (0, t).$$

By the Cauchy–Schwarz inequality, $H_{\Omega_s}^2 \le n|A_{\Omega_s}|^2$. Hence,

$$(Q^t)'(s) \le -\frac{n+1}{n} \mathcal{H}^n(\Gamma_s^t) \quad \text{for all } s \in (0, t). \tag{4-7}$$

If $0 < s_1 < s_2$, then by (4-6), (4-5) and (4-7) respectively, we have

$$\begin{aligned} Q(s_1) - Q(s_2) &= \lim_{\varepsilon \to 0^+} Q^{s_1+\varepsilon}(s_1) - Q^{s_2+\varepsilon}(s_2) \\ &\ge \lim_{\varepsilon \to 0^+} Q^{s_2+\varepsilon}(s_1) - Q^{s_2+\varepsilon}(s_2) = Q^{s_2}(s_1) - Q^{s_2}(s_2) \\ &\ge \frac{n+1}{n} \int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^{s_2}) \, ds, \end{aligned} \tag{4-8}$$

and, in particular, Q is decreasing on $(0, \infty)$. Again by Lemma 7(iii)

$$\mathcal{H}^n(\Gamma_{s-r}^t) = \int_{\Gamma_s^t} \prod_{i=1}^n (1 + r(\kappa_i)_s) \, d\mathcal{H}^n,$$

where $1 + r(\kappa_i)_s \to 1$ uniformly on Γ_s^t as $r \to 0$ thanks to $1/(t-s) \ge (\kappa_s)_i \ge -1/s$ for every $i = 1, \dots, n$. Thus $\mathcal{H}^n(\Gamma_s^t)$ is continuous on $s \in (0, t)$, and

$$\int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^{s_2}) \, ds = (s_2 - s_1) \mathcal{H}^n(\Gamma_{s*}^{s_2})$$

for a suitable $s_* \in (s_1, s_2)$. But (3-34) implies

$$\liminf_{s \to (s_2)^-} \mathcal{H}^n(\Gamma_s^{s_2}) \ge \mathcal{H}^n(\partial\Omega_{s_2})$$

so that, in conclusion,

$$\liminf_{s_1 \to (s_2)^-} \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^{s_2}) \, ds \ge \mathcal{H}^n(\partial\Omega_{s_2}) \quad \text{for all } s_2 > 0.$$

Coming back to (4-8), and noticing that $Q'(s)$ exists for a.e. $s > 0$ by monotonicity, we conclude that

$$-Q'(s) \ge \frac{n+1}{n} \mathcal{H}^n(\partial\Omega_s) \quad \text{for a.e. } s > 0.$$

We integrate this inequality over (s, ∞) to complete the proof of (4-4). □

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