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LONG TIME BEHAVIOR OF THE MASTER EQUATION IN MEAN FIELD GAME THEORY

PIERRE CARDALIAGUET AND ALESSIO PORRETTA

Mean field game (MFG) systems describe equilibrium configurations in games with infinitely many interacting controllers. We are interested in the behavior of this system as the horizon becomes large, or as the discount factor tends to 0. We show that, in these two cases, the asymptotic behavior of the mean field game system is strongly related to the long time behavior of the so-called master equation and to the vanishing discount limit of the discounted master equation, respectively. Both equations are nonlinear transport equations in the space of measures. We prove the existence of a solution to an ergodic master equation, towards which the time-dependent master equation converges as the horizon becomes large, and towards which the discounted master equation converges as the discount factor tends to 0. The whole analysis is based on new estimates for the exponential rates of convergence of the time-dependent and the discounted MFG systems, respectively.

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Given a terminal time T and an initial measure m_0, we consider the solution to the mean field game (MFG) system

$$\begin{cases} -\partial_t u^T - \Delta u^T + H(x, Du^T) = F(x, m^T) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m^T - \Delta m^T - \operatorname{div}(m^T H_p(x, Du^T)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m^T(0,\cdot) = m_0, \quad u^T(T,\cdot) = G(\cdot, m^T(T)) & \text{in } \mathbb{T}^d, \end{cases} \tag{1}$$

where $\mathbb{T}^d$ is the d-dimensional flat torus $\mathbb{R}^d/\mathbb{Z}^d$, F, G are functions defined on $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ (the space of probability measures on $\mathbb{T}^d$) and H is a function, defined on $\mathbb{T}^d \times \mathbb{R}^d$, which is convex in the second variable.

Let us recall that this system appears in mean field games theory, introduced by Lasry and Lions [2006a; 2006b; 2007] and by Huang, Caines and Malhamé [Huang et al. 2006]. Mean field games are dynamic

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games with infinitely many players. The first equation in (1) can be interpreted as the value function of a small player whose cost depends on the density $m(t)$ of the players, while the second equation describes the evolution in time of the density of the players. Note that the first equation is backward in time (and with a terminal condition) while the second one is forward, with the initial condition $m(0) = m_0$, m_0 being the initial repartition of the players.

The study of the long time average of the MFG system was initiated in [Lions 2010] and then discussed in several different contexts [Cardaliaguet et al. 2012; 2013; Cardaliaguet 2013; Cardaliaguet and Graber 2015; Gomes et al. 2010].

In [Cardaliaguet et al. 2013] the long time average of u^T is investigated when $H(x, p) = \frac{1}{2}|p|^2$ and $F(x, m)$, $G(x, m)$ satisfy suitable smoothing conditions with respect to the measure m. Then it is proved that there exists a constant $\bar{\lambda} \in \mathbb{R}$ such that the scaled function $(s, x) \to u^T(Ts, x)/T$ locally uniformly converges to the map $(s, x) \to -\bar{\lambda}s$ as $T \to \infty$ on $(0, 1) \times \mathbb{T}^d$, while the rescaled measure $(s, x) \to m^T(sT, x)$ converges to a time-invariant measure $\bar{m}$ in $L^1((0, 1) \times \mathbb{T}^d)$. The constant $\bar{\lambda}$ and the measure $\bar{m}$ are characterized as solutions of the ergodic MFG system; namely, there exists a unique triple $(\bar{\lambda}, \bar{u}, \bar{m})$ which solves

$$\begin{cases} \bar{\lambda} - \Delta\bar{u} + H(x, D\bar{u}) = F(x, \bar{m}) & \text{in } \mathbb{T}^d, \\ -\Delta\bar{m} - \operatorname{div}(\bar{m}H_p(x, D\bar{u})) = 0 & \text{in } \mathbb{T}^d, \\ \bar{m} \geq 0, \ \int_{\mathbb{T}^d} \bar{m} = 1, \quad \int_{\mathbb{T}^d} \bar{u} = 0 & \text{in } \mathbb{T}^d, \end{cases} \tag{2}$$

and $Du^T(sT, x)$ actually converges to $D\bar{u}(x)$. The result holds under a monotonicity condition on F and G:

$$\int_{\mathbb{T}^d} (F(x, m) - F(x, m'))(m - m')\,dx \geq 0, \qquad \int_{\mathbb{T}^d} (G(x, m) - G(x, m'))(m - m')\,dx \geq 0$$

for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$. Moreover it is proved in [Cardaliaguet et al. 2013] that the convergence holds with an exponential rate. Precisely, under some additional conditions on the smoothing properties of the coupling terms F and G, one has

$$\|m^T(t) - \bar{m}\|_{C^{2+\alpha}} + \|Du^T(t) - D\bar{u}\|_{C^{2+\alpha}} \leq C(e^{-\omega t} + e^{-\omega(T-t)})$$

for some constants $C, \omega > 0$ and $\alpha \in (0, 1)$.

This paper is devoted to the long time behavior of u^T, i.e., the convergence, as $T \to \infty$, of the map $(t, x) \to u^T(t, x) - \bar{\lambda}(T - t)$. This question is inspired by results of Fathi [1997a; 1997b], Roquejoffre [1998], Namah and Roquejoffre [1999] and Barles and Souganidis [2000] for Hamilton–Jacobi equations. In that framework, it is known that if u solves the (forward) Hamilton–Jacobi equation

$$\partial_t u - \Delta u + H(x, Du) = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^d,$$

with associated ergodic constant $\bar{\lambda}$, then $u(t, x) - \bar{\lambda}t$ converges, as $t \to +\infty$, to a solution $\bar{u}$ of the associated ergodic problem. One may wonder what remains of this result for the MFG system.

The convergence of the difference $u^T(t, \cdot) - \bar{\lambda}(T - t)$, as $T \to \infty$, has been an open (and puzzling) question since [Cardaliaguet et al. 2013]. We prove in this paper that the limit of $u^T(t, \cdot) - \bar{\lambda}(T - t)$

indeed exists, although it cannot be described just in terms of the $\bar{u}$-component of the MFG ergodic system (2). In order to describe this long-time behavior, we have to keep track of the initial measure m_0. To do so, we rely on the master equation, which is the following (backward) transport equation in the space of measures:

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - F(x,m) \\ \qquad - \int_{\mathbb{T}^d} \operatorname{div}(D_m U(t,x,m,y))\, dm(y) \\ \qquad\qquad + \int_{\mathbb{T}^d} D_m U(t,x,m,y)\cdot H_p(y, D_x U(t,y,m))\, dm(y) = 0 & \text{in } (-\infty,0)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d), \\ U(0,x,m) = G(x,m) & \text{in } \mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d). \end{cases} \tag{3}$$

In the above equation, the unknown $U = U(t,x,m)$ depends on time, space and the measure on the space; moreover, the notation $D_m U$ denotes a suitable derivative with respect to probability measures, which will be described in Section 1A. Note that, in contrast with the MFG system, the master equation is a classical evolution equation, so its long time behavior may be described in a usual form. We recall, see [Lions 2010; Gangbo and Święch 2015; Chassagneux et al. 2014; Cardaliaguet et al. 2019], that the master equation is well-posed under the monotonicity condition on F and G and that the MFG system (1) plays the role of characteristics for this equation. Namely, if (u^T, m^T) solves (1), then

$$U(-T, x, m_0) = u^T(0,x) \quad \text{for all } x\in\mathbb{T}^d.$$

Our main result (Theorem 5.1) states that $U(t,\cdot,\cdot)+\bar{\lambda}t$ has a limit $\chi = \chi(x,m)$ as $t\to-\infty$. This limit solves (in a weak sense) the ergodic master equation

$$\bar{\lambda} - \Delta_x\chi(x,m) + H(x, D_x\chi(x,m)) - \int_{\mathbb{T}^d} \operatorname{div}(D_m\chi(x,m,y))\, dm(y) \\ + \int_{\mathbb{T}^d} D_m\chi(x,m,y)\cdot H_p(y, D_x\chi(y,m))\, dm(y) = F(x,m) \quad \text{in } T^d\times\mathcal{P}(\mathbb{T}^d). \tag{4}$$

As a consequence, the limit $u^T(0,\cdot)-\bar{\lambda}T$ exists as $T\to\infty$ and is equal to $\chi(\cdot, m_0)$. Note that, in general, $u^T(0,\cdot)-\bar{\lambda}T$ does not converge to $\bar{u}$, since it is not always true that $\chi(\cdot,m_0)=\bar{u}$ (even up to an additive constant); this is however the case if $m_0=\bar{m}$.

We are also interested in the infinite-horizon MFG system

$$\begin{cases} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = F(x, m^\delta(t)) & \text{in } (0,+\infty)\times\mathbb{T}^d, \\ \partial_t m^\delta - \Delta m^\delta - \operatorname{div}(m^\delta H_p(x, Du^\delta)) = 0 & \text{in } (0,+\infty)\times\mathbb{T}^d, \\ m^\delta(0,\cdot) = m_0 \quad \text{in } (0,+\infty)\times\mathbb{T}^d, \qquad u^\delta \text{ bounded.} \end{cases} \tag{5}$$

In the first-order stationary Hamilton–Jacobi (HJ) setting, where the equation reads

$$\delta u^\delta + H(x, Du^\delta) = 0 \quad \text{in } \mathbb{T}^d,$$

Gomes [2008] and Davini, Fathi, Iturriaga and Zavidovique [2016] have proved the convergence of $u^\delta - \delta^{-1}\bar{\lambda}$ as δ tends to 0 and characterized the limit. The result has been generalized to the second-order

HJ setting by Mitake and Tran [2017]; see also [Le et al. 2017; Ishii et al. 2017]. In the viscous case, the result is that, if u^δ solves the infinite-horizon problem

$$\delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = 0 \quad \text{in } \mathbb{T}^d,$$

then $u^\delta - \delta^{-1}\bar\lambda$ converges as $\delta \to 0$ to the unique solution $\bar u$ of the ergodic cell problem

$$-\Delta \bar u + H(x, D\bar u) = 0 \quad \text{in } \mathbb{T}^d$$

such that $\int_{\mathbb{T}^d} \bar u \bar m = 0$, where $\bar m$ solves

$$-\Delta \bar m - \operatorname{div}(\bar m H_p(x, D\bar u)) = 0 \quad \text{in } \mathbb{T}^d, \quad \bar m \ge 0, \ \int_{\mathbb{T}^d} \bar m = 1.$$

Here again, one may wonder if such a result remains true for the infinite-horizon MFG system (5) (which, in contrast with the Hamilton–Jacobi case, is time-dependent). As for the time-evolution MFG problem, we rely on a master equation. Following [Cardaliaguet et al. 2019], this infinite-horizon master equation takes the form[1]

$$\delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - \int_{\mathbb{T}^d} \operatorname{div}_y (D_m U^\delta(x, m, y))\, dm(y) \\ + \int_{\mathbb{T}^d} D_m U^\delta(x, m, y) \cdot H_p(y, D_x U^\delta(y, m))\, dm(y) = F(x, m) \quad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \tag{6}$$

Our second main result (Theorem 6.1) is that $U^\delta - \delta^{-1}\bar\lambda$ converges to the unique solution χ of the master ergodic problem (4) satisfying $\chi(x, \bar m) = \bar u$, where $\bar u$ is the unique solution of the ergodic MFG system (2) for which the following (new) linearized ergodic MFG system has a solution $(\bar v, \bar\mu)$:

$$\begin{cases} \bar u - \Delta \bar v + H_p(x, D\bar u) \cdot D\bar v = \dfrac{\delta F}{\delta m}(x, \bar m)(\bar\mu) & \text{in } \mathbb{T}^d, \\ -\Delta \bar\mu - \operatorname{div}(\bar\mu H_p(x, D\bar u)) - \operatorname{div}(\bar m H_{pp}(x, D\bar u) D\bar v) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \bar\mu = \int_{\mathbb{T}^d} \bar v = 0, \end{cases}$$

(the definition of the derivative $\delta F/\delta m$ is explained in Section 1). This implies the convergence of $u^\delta(0, \cdot) - \delta^{-1}\bar\lambda$ to $\chi(\cdot, m_0)$ as δ tends to 0. Note that if $F \equiv 0$, i.e., in the Hamilton–Jacobi case, one recovers the condition $\int_{\mathbb{T}^d} \bar u \bar m = 0$ by integrating the $\bar v$-equation against the measure $\bar m$. The MFG setting is more subtle since it keeps track of the coupling between the equations.

Let us now say a few words about the method of proofs. As in the Hamilton–Jacobi setting, the argument relies on compactness arguments and, therefore, on the regularity (Lipschitz estimates) for the solution U of the master equation (3) and for the solution U^δ of the infinite-horizon master equation (6). The main difficulty comes from the fact that these equations *do not satisfy a comparison principle* (in contrast to the HJ equation). Moreover, as can be seen plainly from (3) and (6), the equations do not provide easy bounds on the derivatives with respect to m of U and U^δ.

[1] See in particular the comments in the introduction of [Cardaliaguet et al. 2019], which explain that the approach of that work also applies to get the existence and uniqueness of solutions to this equation.

The key Lipschitz estimates come from the fact that the characteristics (1) and (5) of these master equations stabilize exponentially fast in time to the solution of the ergodic MFG system (2) and, respectively, to the solution of the time-invariant infinite-horizon problem

$$\begin{cases} \delta \bar{u}^\delta - \Delta \bar{u}^\delta + H(x, D\bar{u}^\delta) = F(x, \bar{m}^\delta) & \text{in } \mathbb{T}^d, \\ -\Delta \bar{m}^\delta - \operatorname{div}(\bar{m}^\delta H_p(x, D\bar{u}^\delta)) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \bar{m}^\delta = 1; \end{cases} \tag{7}$$

see Theorems 2.6 and 3.7 respectively. These exponential convergence rates were only known for system (1) when $H(x, p) = |p|^2$, see [Cardaliaguet et al. 2013], where the argument relied on some commutation properties which do not hold for general Hamiltonians. To prove the exponential convergence in our setting, we use a technique developed by one of us with E. Zuazua [Porretta and Zuazua 2013] to establish the so-called turnpike property for optimal control problems. The exponential rate for the infinite-horizon MFG system is new, but uses similar ideas.

The starting point of this analysis consists in studying the linearized MFG systems. For simplicity, let us explain this idea for the time-dependent problem, i.e., for U. In this framework, the MFG linearized system reads

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, Du) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m)(\mu(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, Du)) - \operatorname{div}(m H_{pp}(x, Du) Dv) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0, \qquad v(T, x) = \dfrac{\delta G}{\delta m}(x, m)(\mu(T)) \quad \text{in } \mathbb{T}^d, \end{cases}$$

where (u, m) is the solution of (1) and μ_0 is given. When $(u, m) = (\bar{u}, \bar{m})$, the analysis of the above system (the exponential decay of the solutions) provides an exponential convergence of the solution of the MFG system to $(\bar{u}, \bar{m})$ — at least, this holds true for the m-component. A very interesting point is that this linearized system turns out to be also strongly related to the derivative of U with respect to m: indeed, as explained in [Cardaliaguet et al. 2019], we have

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(0, x, m_0, y) \mu_0(y)\, dy = v(0, x) \quad \text{for all } x \in \mathbb{T}^d.$$

Thus controlling v allows us to control the variations of U with respect to m. Once the Lipschitz estimates for U and for U^δ are obtained, the construction of a corrector χ (solution of the ergodic master equation (4)) follows in a standard way; see Theorem 4.2.

However, the convergence of the solution of the time-dependent master equation (3) requires new ideas since, in contrast with the Hamilton–Jacobi setting, see [Fathi 2008; Barles and Souganidis 2000], there is no obvious quantity which is monotone in time; the reason is that the master equation does not satisfy a comparison principle. To overcome this issue, we rely again on the exponential convergence rate from which we derive a suitable convergence of the solution of the master equation when evaluated at $\bar{m}$ as time tends to $-\infty$ (see Proposition 2.7). Then we obtain the convergence of the map U by a compactness argument and using again the convergence of the characteristics.

The convergence of U^δ is more subtle: the key point is that two solutions of the ergodic master equation differ only by a constant. Thus we only have to show that $U^\delta(\cdot, m) - \delta^{-1}\bar\lambda$ has a limit for some m. The good choice turns out to be $m = \bar m^\delta$, where $(\bar u^\delta, \bar m^\delta)$ solves (7); indeed, we have then $U^\delta(\cdot, \bar m^\delta) = \bar u^\delta$ and we expect $(\bar u^\delta, \bar m^\delta)$ to be close to $(\bar u, \bar m)$ in some sense, where $(\bar u, \bar m)$ satisfies (2). Actually a formal expansion yields $(\bar u^\delta, \bar m^\delta) = (\delta^{-1}\bar\lambda + \bar u + \bar\theta + \delta\bar v, \bar m + \delta\bar\mu)$, where $(\bar\theta, \bar v, \bar\mu)$ solves

$$\begin{cases} \bar u + \bar\theta - \Delta\bar v + H_p(x, D\bar u)\cdot D\bar v = \dfrac{\delta F}{\delta m}(x, \bar m)(\bar\mu) & \text{in } \mathbb{T}^d, \\ -\Delta\bar\mu - \operatorname{div}(\bar\mu H_p(x, D\bar u)) - \operatorname{div}(\bar m H_{pp}(x, D\bar u)D\bar v) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d}\bar\mu = \int_{\mathbb{T}^d}\bar v = 0. \end{cases}$$

The rigorous justification is given in Proposition 6.5.

The paper is organized in the following way. In Section 1 we recall the notion of derivative in the space of measures and state our main assumptions. We also recall some decay and regularity estimates which hold separately for the two equations of the system and we provide the basic fundamental estimates for (1) which are independent of the horizon T. Section 2 is devoted to the exponential convergence rate, as $T \to \infty$, of solutions of (1) towards the pair $(\bar u, \bar m)$, a solution of (2). For this purpose, first we develop decay estimates in L^2 for the linearized system, and then we export the estimates (in stronger norms) to $(u^T - \bar u, m^T - \bar m)$ by using a fixed-point argument. A similar strategy is used in Section 3 for the infinite-horizon discounted problem (5); in this case we prove the exponential convergence as $t \to \infty$ towards the stationary pair $(\bar u^\delta, \bar m^\delta)$, a solution of (7). In both Sections 2 and 3, the analysis of the linearized systems is a crucial step, and this will also play a key role in the study of the master equations, both the time-dependent (3) and the stationary one (6), respectively. This is the content of Sections 4–6. More precisely, in Section 4 we prove the existence of a solution to the ergodic master equation, obtained as the limit, when $\delta \to 0$, of a subsequence of solutions of (6). The long-time behavior of the time-dependent master equation (3) is addressed in Section 5. Finally, the limit of the whole sequence of solutions of (6) is proved in Section 6.

1. Notation, assumptions and preliminary estimates

1A. ***Notation and assumptions.*** Throughout the paper we work on the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$; this means that all equations are $\mathbb{Z}^d$-periodic in space. This assumption is standard in the framework of the long time behavior. We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed with the Monge–Kantorovich distance $\boldsymbol{d}_1$

$$\boldsymbol{d}_1(m, m') = \sup_\phi \int_{\mathbb{T}^d} \phi\, d(m - m') \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d),$$

where the supremum is taken over all 1-Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$.

For $\alpha \in [0, 1]$, we denote by $C^\alpha([0, T], \mathcal{P}(\mathbb{T}^d))$ the set of maps $m : [0, T] \to \mathcal{P}(\mathbb{T}^d)$ which are α-Hölder continuous if $\alpha \in (0, 1)$ and continuous if $\alpha = 0$.

Next we recall the notion of derivative of a map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ as introduced in [Cardaliaguet et al. 2019]. We say that U is C^1 if there exists a continuous map $\delta U/\delta m : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-t)m + tm', y)\, d(m'-m)(y)\, dt \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d).$$

We observe that if U can be extended to $L^2(\mathbb{T}^d)$ then $y \mapsto (\delta U/\delta m)(m, y)$ is nothing but the representation in L^2 of the Gâteaux derivative of U computed at m. The fact that U is defined on probability measures, i.e., with the constraint of mass 1, lets $(\delta U/\delta m)(m, y)$ be defined up to a constant. We normalize the derivative by the condition

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m, y)\, dm(y) = 0 \quad \text{for all } m \in \mathcal{P}(\mathbb{T}^d). \tag{8}$$

We write interchangeably $(\delta U/\delta m)(m)(\mu)$ and $\int_{\mathbb{T}^d} (\delta U/\delta m)(m, y)\, d\mu(y)$ for a signed measure μ with finite mass.

When the map $\delta U/\delta m = (\delta U/\delta m)(m, y)$ is differentiable with respect to the last variable, we denote by $D_m U(m, y)$ its gradient:

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$

Let us recall [Cardaliaguet et al. 2019] that $D_m U$ can be used to estimate the Lipschitz regularity of U in the m-variable:

$$|U(m) - U(m')| \le \boldsymbol{d}_1(m, m') \Big[\sup_{m'' \in \mathcal{P}(\mathbb{T}^d),\, y \in \mathbb{T}^d} |D_m U(m'', y)| \Big] \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d).$$

For $p = 1, 2, \infty$, we denote by $\|\cdot\|_{L^p}$ the L^p norm of a map on $\mathbb{T}^d$ (we often use the notation $\|\cdot\|_\infty$ for $\|\cdot\|_{L^\infty}$). For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, we denote by $\|\cdot\|_{C^k}$ and $\|\cdot\|_{C^{k+\alpha}}$ the standard norm on the set of maps defined on $\mathbb{T}^d$ and which are, respectively, of class C^k and of class C^k with a k-th derivative which is α-Hölder continuous. By $\|\cdot\|_{(C^{k+\alpha})'}$ we mean the norm in the dual space:

$$\|\phi\|_{(C^{k+\alpha})'} := \sup \left\{ \int_{\mathbb{T}^d} \phi\psi,\ \|\psi\|_{C^{k+\alpha}} \le 1 \right\}.$$

For a map ϕ depending of two spatial variables, we denote by $\|\phi(\cdot,\cdot)\|_{k+\alpha, k'+\alpha}$ the supremum of the α-Hölder norm of the partial derivatives of order $l \le k$ and $l' \le k'$ respectively of the map ϕ.

Finally, if $\phi = \phi(x)$, we systematically denote by $\langle \phi \rangle := \int_{\mathbb{T}^d} \phi(x)\, dx$ the average of ϕ.

If $u : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ is a sufficiently smooth map, we denote by $Du(t, x)$ and $\Delta u(t, x)$ its spatial gradient and spatial Laplacian and by $\partial_t u(t, x)$ its partial derivative with respect to the time variable. We will also use the classical parabolic Hölder spaces: for $\alpha \in (0, 1)$, we denote by $C^{\alpha/2,\alpha}$ the set of maps which are α-Hölder in space and $\alpha/2$-Hölder in time and by $C^{1+\alpha/2, 2+\alpha}$ the set of maps u such that $\partial_t u$ and $D^2 u$ are in $C^{\alpha/2,\alpha}$.

Assumptions. The following assumptions are in force throughout the paper.

(H) The Hamiltonian $H = H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is of class C^2 and the function $p \mapsto D^2_{pp} H(x, p)$ is Lipschitz continuous, uniformly with respect to x, and satisfies the growth condition

$$C^{-1} I_d \le D^2_{pp} H(x, p) \le C I_d \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{9}$$

Moreover we suppose that there exist $\theta \in (0, 1)$ and $C > 0$ such that

$$|D_{xx} H(x, p)| \le C(1 + |p|)^{1+\theta}, \quad |D_{xp} H(x, p)| \le C(1 + |p|)^{\theta} \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{10}$$

This latter assumption is a little awkward, since it requires the quadratic part of H to be independent of the space variable, but we actually need it in order to ensure uniform Lipschitz regularity of a solution u^T of (1) and of a solution u^δ of (5) independently of T and δ: see Lemmas 1.5 and 3.6. If the same bounds were available with different arguments, then we could get rid of this condition, since in the rest of the paper we do not use it at all.

(FG) The coupling functions $F, G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are assumed to be of class C^1 and their first derivatives satisfy the following Lipschitz conditions:

(FGa) F, G are twice differentiable in the x-variable and $F_{xx}(x, m)$, $G_{xx}(x, m)$ are bounded uniformly in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.

(FGb) $(\delta F/\delta m)(x, m, y)$, $(\delta G/\delta m)(x, m, y)$ are differentiable with respect to (x, y) and Lipschitz continuous in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d$ (i.e., globally Lipschitz in the three variables).

Even if this will not be strictly needed, an extra regularity condition is assumed in order to get to smooth solutions of the master equation as stated in [Cardaliaguet et al. 2019]. Namely we assume that:

(FGc) For any $\alpha \in (0, 1)$, $F(\cdot, m)$ and $(\delta F/\delta m)(\cdot, m, \cdot)$ are of class $C^{2+\alpha}$ in all space variables, uniformly in m, and $\delta F/\delta m$ is Lipschitz continuous in m with respect to $C^{2+\alpha}$ in space. The same holds for G in norm $C^{3+\alpha}$.

(FGd) The maps F and G are assumed to be monotone: for any $m \in \mathcal{P}(\mathbb{T}^d)$ and for any centered Radon measure μ,

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\delta F}{\delta m}(x, m, y)\mu(x)\mu(y)\, dx\, dy \ge 0, \quad \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\delta G}{\delta m}(x, m, y)\mu(x)\mu(y)\, dx\, dy \ge 0. \tag{11}$$

Let us comment upon our assumptions.

The regularity of H as well as the uniform convexity with respect to the second variable are standard in MFG theory. Here these assumptions are all the more important because we make systematic use of the duality inequality, see [Lasry and Lions 2007], which provides uniqueness and quantified stability for the MFG system under this strong convexity assumption.

The regularity assumption on $\delta F/\delta m$ (and on $\delta G/\delta m$) allows for instance inequalities of the form

$$\left\| \frac{\delta F}{\delta m}(\cdot, m)(\mu) \right\|_{C^2} \le C \|\mu\|_{(C^2)'}$$

for any $m \in \mathcal{P}(\mathbb{T}^d)$ and any distribution μ on $\mathbb{T}^d$.

The monotonicity assumption (11) implies (and, under our regularity assumptions, is equivalent to) the more standard one

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))\, d(m-m')(x) \ge 0, \qquad \int_{\mathbb{T}^d} (G(x,m) - G(x,m'))\, d(m-m')(x) \ge 0$$

for any measures $m, m' \in \mathcal{P}(\mathbb{T}^d)$. This condition ensures the well-posedness of the MFG system (1) for large time intervals and the well-posedness of the ergodic MFG system (2). Without this assumption, these MFG systems may have several solutions and the long time average (and a fortiori the long time behavior) of the MFG system (1) is not known.

Let us illustrate our assumptions by examples. The Hamiltonian functions we have in mind are for instance of the form

$$H(x,p) = \tfrac{1}{2}|p|^2 + V(x)\cdot p + g(x),$$

where $V : \mathbb{T}^d \to \mathbb{R}^d$ is a smooth vector field and $g : \mathbb{T}^d \to \mathbb{R}$ is a smooth map. Typical examples of coupling maps F and G satisfying our conditions take the form

$$\Phi(x,m) = [\phi(\,\cdot\,, (\rho \star m)(\,\cdot\,)) \star \rho](x),$$

where $\star$ denotes the usual convolution product in $\mathbb{R}^d$, $\phi : \mathbb{R}^2 \to \mathbb{R}$ is smooth and nondecreasing with respect to the second variable and $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth, even function with compact support; see for instance Example 2.3.1. in [Cardaliaguet et al. 2019].

Let us stress that, in the following, we will denote generically by C possibly different constants appearing in the estimates which depend on the data F, G and H through the above assumptions. In particular, those constants will depend on the sup-norm of F_{xx}, G_{xx} (which are bounded uniformly with respect to x and m from (FGa)), the Lipschitz constants of $\delta F/\delta m$, $\delta G/\delta m$ and the conditions (9)–(10), respectively. Actually, those constants will also depend on the unique solution $\bar\lambda, \bar u, \bar m$ of (2), but this triple is also meant as (uniquely) depending on the data F, G and H, so we will not mention this kind of dependence otherwise.

1B. *Preliminary estimates.* We will use throughout the text the following estimates on linear equations which are independent of the time horizon. The first one is about linear equations in divergence form; see [Cardaliaguet et al. 2013, Lemmas 7.1 and 7.6].

Lemma 1.1. *Let V be a bounded vector field on $(0,T)\times\mathbb{T}^d$, let $B \in L^2((0,T)\times\mathbb{T}^d)$ and let μ be the solution to*

$$\begin{cases} \partial_t \mu - \Delta\mu + \operatorname{div}(\mu V) = \operatorname{div}(B) & \text{in } (0,T)\times\mathbb{T}^d, \\ \mu(0) = \mu_0 & \text{in } \mathbb{T}^d, \end{cases} \tag{12}$$

with $\int_{\mathbb{T}^d} \mu_0 = 0$.

There exist constants $\omega > 0$ and $C > 0$, depending only on $\|V\|_\infty$, such that

$$\|\mu(t)\|_{L^2} \le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^t \|B(s)\|_{L^2}^2\, ds\right]^{1/2}.$$

If $B \equiv 0$, we also have, for any $\tau > 0$,

$$\|\mu(t)\|_\infty \le C_\tau e^{-\omega t}\|\mu_0\|_{L^1} \quad \text{for all } t \ge \tau,$$

where the constant C_τ depends on τ and $\|V\|_\infty$ only.

The second lemma is about a viscous transport equation; see [Cardaliaguet et al. 2013, Lemmas 7.4 and 7.5].

Lemma 1.2. *Let V be a bounded vector field, $A \in L^2((0,T)\times\mathbb{T}^d)$ and v be the solution to the backward equation*

$$-\partial_t v - \Delta v + V\cdot Dv = A \quad \text{in } (0,T)\times\mathbb{T}^d. \tag{13}$$

There exist constants $\omega > 0$ and $C > 0$, depending only on $\|V\|_\infty$, such that

$$\|v(t)-\langle v(t)\rangle\|_{L^2} \le Ce^{-\omega(T-t)}\|v(T)-\langle v(T)\rangle\|_{L^2} + C\int_t^T e^{-\omega(s-t)}\|A(s)\|_{L^2}\,ds$$

and, if $A \in L^\infty((0,T)\times\mathbb{T}^d)$,

$$\|v(t)-\langle v(t)\rangle\|_{L^\infty} \le Ce^{-\omega(T-t)}\|v(T)-\langle v(T)\rangle\|_{L^\infty} + C\int_t^T e^{-\omega(s-t)}\|A(s)\|_{L^\infty}\,ds,$$

where $\langle\phi\rangle = \int_{\mathbb{T}^d}\phi$ for any map ϕ. Moreover, for any $0 \le t < t_0 \le T$,

$$(t_0-t)\|Dv(t)\|_{L^2} \le C(t_0-t+1)\big(\|v(t_0)-\langle v(t_0)\rangle\|_{L^2} + \|A\|_{L^2((t,t_0)\times\mathbb{T}^d)} + \|v-\langle v\rangle\|_{L^2((t,t_0)\times\mathbb{T}^d)}\big).$$

We note for later use a simple consequence of Lemma 1.1:

Corollary 1.3. *Let V and B be (time-independent) vector fields. Then any L^2 solution of*

$$-\Delta\mu + \operatorname{div}(\mu V) = \operatorname{div}(B) \quad \text{in } \mathbb{T}^d,$$

with $\int_{\mathbb{T}^d}\mu = 0$, satisfies

$$\|\mu\|_{H^1} \le C\|B\|_{L^2},$$

where C depends only on $\|V\|_\infty$.

Proof. It is enough to apply Lemma 1.1:

$$\|\mu\|_{L^2} \le Ce^{-\omega t}\|\mu\|_{L^2} + C\|B\|_{L^2}t^{1/2}.$$

Choosing t large enough, this gives

$$\|\mu\|_{L^2} \le C\|B\|_{L^2}.$$

Then, multiplying the equation by μ, the standard energy estimate gives

$$\|D\mu\|_{L^2} \le [\|V\|_\infty\|\mu\|_{L^2} + \|B\|_{L^2}],$$

which gives the result. □

We conclude this section with a further bound for the solutions of the Fokker–Planck equation.

Lemma 1.4. *Let V be a bounded vector field on $(0,T)\times\mathbb{T}^d$ with bounded space derivatives and μ be a weak solution to* (12) *with $B\equiv 0$. Then, for any $\tau>0$,*

$$\|\mu(t)\|_\infty \le C_\tau e^{-\omega t}\|\mu_0\|_{(C^{2+\alpha})'} \quad \textit{for all } t\ge\tau,$$

where ω is given by Lemma 1.1, $\alpha\in(0,1)$ and $C_\tau>0$ depends on $\|V\|_{L^\infty}$, $\|DV\|_{L^\infty}$ and τ.

Proof. Let $\tau>0$ and v be the solution to the transport equation

$$\begin{cases} -\partial_t v-\Delta v+V\cdot Dv=0 & \text{in } (0,\tau)\times\mathbb{T}^d,\\ v(\tau,x)=v_\tau(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{14}$$

where v_τ is in $C^\infty(\mathbb{T}^d)$. One easily checks that

$$\sup_t \|v(t)\|_{L^2}+\|Dv\|_{L^2((0,\tau)\times\mathbb{T}^d)}\le C\|v_\tau\|_{L^2},$$

where C depends on $\|V\|_\infty$ and τ only. Standard parabolic regularity [Ladyženskaja et al. 1968, Theorem III.11.1] then implies

$$\|Dv\|_{C^{\alpha/2,\alpha}([0,\tau/2]\times\mathbb{T}^d)}\le C\|v_\tau\|_{L^2}$$

for some α and C depending on $\|V\|_\infty$ and τ only. For any $i\in\{1,\dots,d\}$, the derivative v_{x_i} solves

$$-\partial_t v_{x_i}-\Delta v_{x_i}+V\cdot Dv_{x_i}+V_{x_i}\cdot Dv=0 \quad \text{in } (0,\tau/2)\times\mathbb{T}^d.$$

By parabolic regularity [Ladyženskaja et al. 1968, Theorem III.11.1], we infer that

$$\|D^2v\|_{C^{\alpha/2,\alpha}([0,\tau/4]\times\mathbb{T}^d)}\le C\|Dv\|_{L^\infty((0,\tau/2)\times\mathbb{T}^d)}\le C\|v_\tau\|_{L^2}$$

for some α and C depending on $\|V\|_\infty$, $\|DV\|_\infty$ and τ only. We have, since (14) is the dual equation of (13),

$$\int_{\mathbb{T}^d} v_\tau\mu(\tau)=\int_{\mathbb{T}^d} v(0)\,d\mu_0(x).$$

So taking the supremum over v_τ such that $\|v_\tau\|_{L^2}\le 1$, we infer that

$$\|\mu(\tau)\|_{L^2}\le C_\tau\|\mu_0\|_{(C^{2+\alpha})'} \quad \text{for all } \tau>0.$$

We can then derive the conclusion by Lemma 1.1. □

1C. *Regularity of the MFG system.*

The aim of this section is to provide additional basic estimates on the solution to the MFG system

$$\begin{cases} -\partial_t u+\bar\lambda-\Delta u+H(x,Du)=F(x,m) & \text{in } (0,T)\times\mathbb{T}^d,\\ \partial_t m-\Delta m-\operatorname{div}(mH_p(x,Du))=0 & \text{in } (0,T)\times\mathbb{T}^d,\\ m(0,\cdot)=m_0,\quad u(T,\cdot)=g & \text{in } \mathbb{T}^d, \end{cases} \tag{15}$$

where $m_0\in\mathcal{P}(\mathbb{T}^d)$. Let us recall that $\bar\lambda\in\mathbb{R}$ is the unique ergodic constant and $(\bar u,\bar m)$ the unique solution to the ergodic MFG system (2).

The following estimates have been mostly well known since [Cardaliaguet et al. 2013], but we collect them for the sake of completeness. The whole point is to get estimates which are independent of the time horizon or of the discount rate. For this purpose we rely on conditions (9)–(10), as well as on the smoothing assumption (FGa) for the couplings.

Lemma 1.5. *For any $M>0$, there exists a constant $C>0$ such that for any horizon $T>0$, if (u,m) is a solution to the MFG system* (15) *and $\|g\|_{C^2(\mathbb{T}^d)}\le M$, then*

$$\|Du\|_\infty \le C.$$

Proof. As in Lemma 3.2 in [Cardaliaguet et al. 2013], the proof relies on the uniform semiconcavity of the solution. Let us recall that, for any smooth map $\phi\in C^\infty(\mathbb{T}^d)$, we have

$$\|D\phi\|_\infty \le d^{1/2} \sup_{x\in\mathbb{T}^d,\ |z|\le 1} (D^2\phi(x)z\cdot z)_+. \tag{16}$$

Let ξ with $|\xi|\le 1$ be a direction for which $C_0 := \sup_{(t,x)} D^2u(t,x)\xi\cdot\xi$ is maximal (and thus nonnegative). We set $w(t,x)=D^2u(t,x)\xi\cdot\xi = u_{\xi\xi}(t,x)$. Then w solves

$$-\partial_t w-\Delta w+H_{\xi\xi}(x,Du)+2H_{\xi p}(x,Du)\cdot Du_\xi+H_{pp}(x,Du)Du_\xi\cdot Du_\xi+H_p(x,Du)\cdot Dw=F_{\xi\xi}(x,m(t)).$$

If the maximum of w is reached at T, then

$$C_0 \le \max_{x\in\mathbb{T}^d} D^2g(x)\xi\cdot\xi \le M.$$

Otherwise, one has at the maximum point (t,x) of w:

$$H_{\xi\xi}(x,Du)+2H_{\xi p}(x,Du)\cdot Du_\xi + H_{pp}(x,Du)Du_\xi\cdot Du_\xi \le F_{\xi\xi}(x,m(t)),$$

where by our standing assumptions on H we have

$$H_{\xi\xi}(x,Du)\ge -C(1+|Du|)^{1+\theta},$$
$$H_{pp}(x,Du)Du_\xi\cdot Du_\xi + 2H_{\xi p}(x,Du)\cdot Du_\xi \ge C^{-1}|Du_\xi|^2 - C(1+|Du|)^{2\theta}.$$

Since (16) implies $\|Du\|_\infty\le d^{1/2}C_0$, we deduce that

$$-C(1+C_0)^{1+\theta}-C(1+C_0)^{2\theta}+C^{-1}|Du_\xi|^2\le C$$

and since $|Du_\xi|\ge C_0$ at the maximum point of $w(t,x)$, because $\theta<1$ we conclude that C_0 is bounded. By (16), we infer the Lipschitz estimate for u. □

Remark 1.6. Thanks to Lemma 1.5, the drift $H_p(x,Du)$ in the Fokker–Planck equation is uniformly bounded. As a consequence, as it is well known (see, e.g., in [Cardaliaguet 2010, Lemma 3.4]), the solution m satisfies the following Hölder continuity estimate in time:

$$\boldsymbol{d}_1(m(t),m(s))\le C|t-s|^{1/2} \quad \text{for all } t,s\in(0,T) \text{ such that } |t-s|\le 1, \tag{17}$$

for some constant C independent of T.

Next result exploits the stability of the system which stems from the monotonicity of F and the convexity of H; see [Lasry and Lions 2007]. In particular, whenever H is uniformly convex, as is assumed in (9), the following estimate holds for any pair of solutions (u_1, m_1) and (u_2, m_2) of the system (15):

$$C^{-1}\int_{\mathbb{T}^d}(m_1+m_2)|D(u_1-u_2)|^2 \le -\frac{d}{dt}\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2). \tag{18}$$

Lemma 1.7. *For any $\varepsilon > 0$ and $M > 0$, there exist times $\widehat{T} > \tau > 0$ (depending only on ε, M and the data of the problem) such that, if $T \ge \widehat{T}$ and if (u, m) is a solution to the MFG system (15) and $\|g\|_{C^2(\mathbb{T}^d)} \le M$, we have, for some $\alpha \in (0, 1)$,*

$$\|m(t)-\bar{m}\|_{C^\alpha} + \|Du(t)-D\bar{u}\|_{C^\alpha} \le \varepsilon \quad \textit{for all } t \in [\tau, T-\tau].$$

Proof. We follow closely the argument of Lemma 3.5 of [Cardaliaguet et al. 2013] (in the case $H = |p|^2$) and, for this reason, we only sketch the proof. By Lemma 1.5, u is uniformly Lipschitz continuous in space, with a Lipschitz constant depending only on the regularity of H, F and on $\|Dg\|_\infty + \|D^2 g\|_\infty$. So, by Lemma 1.1, we have

$$\sup_{t\ge 1}\|m(t)\|_\infty \le C,$$

where C depends only on $\|H_p(\,\cdot\,, Du(\,\cdot\,))\|_\infty$, and thus only on the data. Applying (18) to (u, m) and $(\bar{u}, \bar{m})$, and using $\bar{m} > 0$ in $\mathbb{T}^d$, we have

$$C^{-1}\int_{t_1}^{t_2}\|D(u(t)-\bar{u})\|_{L^2}^2\,dt \le -\left[\int_{\mathbb{T}^d}(u(t)-\bar{u})(m(t)-\bar{m})\right]_{t_1}^{t_2}. \tag{19}$$

Thus

$$\int_0^T \|D(u(t)-\bar{u})\|_{L^2}^2\,dt \le C,$$

because u is uniformly Lipschitz continuous in space and $m(t)$ and $\bar{m}$ are probability measures. In particular, if $T \ge 3\varepsilon^{-1}$, there exist times $t_1 \in [1, \varepsilon^{-1}],\ t_2 \in [T-\varepsilon^{-1}, T]$ such that

$$\|D(u(t_i)-\bar{u})\|_{L^2} \le C\varepsilon^{1/2} \quad \text{for } i = 1, 2. \tag{20}$$

Coming back to (19), we infer by Poincaré's inequality that

$$\begin{aligned} C^{-1}\int_{1/\varepsilon}^{T-1/\varepsilon}\|D(u(t)-\bar{u})\|_{L^2}^2\,dt &\le C^{-1}\int_{t_1}^{t_2}\|D(u(t)-\bar{u})\|_{L^2}^2\,dt \\ &\le \|D(u(t_1)-\bar{u})\|_{L^2}\|m(t_1)-\bar{m}\|_{L^2} + \|D(u(t_2)-\bar{u})\|_{L^2}\|m(t_2)-\bar{m}\|_{L^2} \\ &\le C\varepsilon^{1/2}. \end{aligned}$$

As $\mu := m - \bar{m}$ satisfies

$$\partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, Du)) = -\operatorname{div}(\bar{m}(H_p(x, D\bar{u}) - H_p(x, Du))), \tag{21}$$

and still using the fact that Du is bounded, we have from Lemma 1.1 that, for any $t \in [1/\varepsilon, T - 1/\varepsilon]$,

$$\begin{aligned}\|m(t) - \bar{m}\|_{L^2} &\leq Ce^{-\omega(t-1/\varepsilon)}\|m(1/\varepsilon) - \bar{m}\|_{L^2} + C\left[\int_{1/\varepsilon}^{T-1/\varepsilon} \|D(u(t) - \bar{u})\|_{L^2}^2 \, dt\right]^{1/2} \\ &\leq C(e^{-\omega(t-1/\varepsilon)} + \varepsilon^{1/4}).\end{aligned}$$

So we can choose τ large enough (depending only on ε, on the data and on M) such that the right-hand side is less than $C\varepsilon^{1/4}$ if $t \in [\tau - 1, T - 1/\varepsilon]$.

Let us now upgrade this inequality into an L^∞ estimate for the interval $[\tau, T - 1/\varepsilon]$. For this, we recall from (21) that μ solves a parabolic equation of the type

$$\partial_t \mu - \Delta\mu - \operatorname{div}(\mu b + B) = 0,$$

where b is bounded in L^∞ and B is bounded in L^p for any $p \geq 2$ since

$$\int_{1/\varepsilon}^{T-1/\varepsilon} \|B(t)\|_{L^p}^p \leq C\int_{1/\varepsilon}^{T-1/\varepsilon}\int_{\mathbb{T}^d} |D(u(t) - \bar{u})|^p \leq C\int_{1/\varepsilon}^{T-1/\varepsilon}\int_{\mathbb{T}^d} |D(u(t) - \bar{u})|^2 \leq C\varepsilon^{1/2},$$

where we used the global bound for $Du(t)$. Since we already know that $\|\mu(t)\|_{L^2} \leq C\varepsilon^{1/4}$, by choosing p sufficiently large we deduce (see, e.g., [Ladyženskaja et al. 1968, Theorem III.8.1, p. 196]) that μ is bounded in $C^{\alpha/2,\alpha}$ for some $\alpha \in (0, 1)$ and

$$\|\mu(t)\|_{C^\alpha} \leq C\Big(\sup_{s\in(\tau-1,T-1/\varepsilon)} \|\mu(s)\|_{L^2} + \|B\|_{L^p((1/\varepsilon,T-1/\varepsilon)\times\mathbb{T}^d)}\Big) \leq C(\varepsilon^{1/4} + \varepsilon^{1/(2p)})$$

for any $t \in [\tau, T - 1/\varepsilon]$. This concludes the bound for $\|m(t) - \bar{m}\|_{C^\alpha}$. In order to prove the estimate for u, let us note that $v = u - \bar{u}$ satisfies

$$-\partial_t v - \Delta v + V \cdot Dv = F(x, m(t)) - F(x, \bar{m}),$$

where V is the bounded vector field

$$V(t, x) = \int_0^1 H_p(x, \lambda Du(t, x) + (1 - \lambda)D\bar{u}(x)) \, d\lambda.$$

By Lemma 1.2 we have, for $t \in [1/\varepsilon, T - 1/\varepsilon]$,

$$\begin{aligned}&\|v(t) - \langle v(t)\rangle\|_\infty \\ &\quad\leq \|v(T - 1/\varepsilon) - \langle v(T - 1/\varepsilon)\rangle\|_\infty e^{-\omega(T-1/\varepsilon-t)} + C\int_t^{T-1/\varepsilon} e^{-\omega(s-t)}\|F(x, m(t)) - F(x, \bar{m})\|_\infty \, ds \\ &\quad\leq C(e^{-\omega(T-1/\varepsilon-t)} + \varepsilon^{1/(2p)}).\end{aligned}$$

Choosing $\tau > 1/\varepsilon$ large enough then implies

$$\|v(t) - \langle v(t)\rangle\|_\infty \leq C\varepsilon^{1/(2p)} \quad \text{for all } t \in [\tau, T - \tau].$$

Finally, we can replace the left-hand side by $\|Dv(t)\|_{C^\alpha}$ by using again Lemma 1.2. Indeed, whenever v satisfies

$$-\partial_t v - \Delta v + V \cdot Dv = A$$

with V, A bounded, we estimate, for any interval $[t, t+1]$,

$$\begin{aligned}\|Dv(t)\|_{C^\alpha} &\le C \sup_{s\in(t,t+1/2)} [\|v(s)-\langle v(s)\rangle\|_\infty + \|A(s)\|_\infty + \|Dv(s)\|_{L^2}] \\ &\le C \sup_{s\in(t,t+1)} [\|v(s)-\langle v(s)\rangle\|_\infty + \|A(s)\|_\infty].\end{aligned}$$

Since $A = F(x, m(t)) - F(x, \bar{m})$, the previous estimates give the conclusion. □

2. Exponential rate of convergence for the finite-horizon MFG system

In this section we provide several convergence results with an exponential rate of convergence for finite-horizon MFG systems. The results of this section extend to general Hamiltonians the main results of [Cardaliaguet et al. 2013] (though requiring slightly stronger assumptions on the coupling F). Although the results are interesting themselves, they are nevertheless motivated by the rest of the paper, in which they play a central role.

The method of proof for these exponential rates differs completely from [Cardaliaguet et al. 2013], where it relied on an algebraic structure of the linearized system. We start with the linearized systems and first get a crude estimate on the solution. Using the monotonicity assumption, the duality method shows that a suitable quantity is monotone in time and bounded (thanks to the rough estimate). A compactness argument, borrowed from [Porretta and Zuazua 2013], then shows that the limit of this quantity must vanish. We then use the linearity property of the system to get an exponential rate of convergence. The nonlinear equations are treated as perturbations of the linear ones. Note that the key argument is inspired by [Porretta and Zuazua 2013], where the long time behavior of optimality systems is analyzed by using the stabilizing properties of the Riccati feedback operator. However, in contrast with that paper, our system does not come from an optimal control problem in general, which makes a substantial difference.

2A. ***Estimates for the linearized system.*** We now study the linearized MFG system around the stationary ergodic solution $(\bar{u}, \bar{m})$: namely, given $\mu_0, v_T : \mathbb{T}^d \to \mathbb{R}$ smooth with $\int_{\mathbb{T}^d} \mu_0 = 0$, we consider a solution (v, μ) to

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, D\bar{u}) \cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ \mu(0,\cdot) = \mu_0, \quad v(T,x) = \dfrac{\delta G}{\delta m}(x,\bar{m})(\mu(T)) + v_T(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{22}$$

Thanks to the assumptions made upon $\delta F/\delta m$ and $\delta G/\delta m$, and to the smoothness of $(\bar{u}, \bar{m})$, problem (22) can be considered in a standard framework of weak solutions with finite energy, i.e., $v, m \in L^2((0,T); H^1(\mathbb{T}^d))$. Solutions will eventually be more regular, but we are not considering this issue here; our main purpose, which is the following result, is to show the L^2 decay estimates for μ and Dv, assuming the same regularity on the initial-terminal conditions.

Proposition 2.1. *There exist $C_0 > 0$, $\lambda > 0$ such that, if (v, μ) is a solution to the MFG linearized system* (22) *with $\int_{\mathbb{T}^d} \mu_0 = 0$, then we have*

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C_0(e^{-\lambda t} + e^{-\lambda(T-t)})(\|\mu_0\|_{L^2} + \|Dv_T\|_{L^2}) \quad \textit{for all } t \in [0, T].$$

Let us start the proof with a lemma which explains that the solution is uniformly bounded, with a bound depending on $\|\mu_0\|_{L^2}$ only.

Lemma 2.2. *There is a constant $C_0 > 0$, depending only on the data H, F and G, but not on T, such that, if (v, μ) is a solution of the linearized problem* (22), *then*

$$\int_0^T \|Dv\|_{L^2}^2 + \sup_{t\in[0,T]} (\|\mu(t)\|_{L^2}^2 + \|Dv(t)\|_{L^2}^2) \le C_0(\|\mu_0\|_{L^2}^2 + \|Dv_T\|_{L^2}^2). \tag{23}$$

Proof. Note that $\int_{\mathbb{T}^d} \mu(t) = 0$ for any t. Multiplying the equation for v by μ and the equation for μ by v, integrating in time and space and adding the resulting relations, we have, for any $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{T}^d\times\mathbb{T}^d} \frac{\delta F}{\delta m}(x, \bar m, y)\mu(t, y)\mu(t, x)\, dy\, dx\, dt \\ + \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \bar m H_{pp}(x, D\bar u(x))Dv(t, x)\cdot Dv(t, x)\ \, dx\, dt = -\left[\int_{\mathbb{T}^d} v\mu\right]_{t_1}^{t_2},\end{aligned} \tag{24}$$

so, by the monotonicity of F and G, see assumption (11),

$$\begin{aligned}C^{-1}\int_0^T \|Dv(t)\|_{L^2}^2\, dt &\le \int_{\mathbb{T}^d} (v(0) - \langle v(0)\rangle)\mu_0 - \int_{\mathbb{T}^d} (v_T - \langle v_T\rangle)\mu(T) \\ &\le C(\|Dv(0)\|_{L^2}\|\mu_0\|_{L^2} + \|Dv_T\|_{L^2}\|\mu(T)\|_{L^2}),\end{aligned} \tag{25}$$

thanks to Poincaré's inequality. Using Lemma 1.1, we have

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^t \|\bar m H_{pp}(\,\cdot\,, D\bar u)Dv\|_{L^2}^2\right]^{1/2} \le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^T \|Dv\|_{L^2}^2\right]^{1/2} \\ &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + C(\|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}^{1/2}\|\mu(T)\|_{L^2}^{1/2}).\end{aligned}$$

For $t = T$, we get, after simplification,

$$\|\mu(T)\|_{L^2} \le C(\|\mu_0\|_{L^2} + \|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}),$$

from which we deduce that

$$\sup_{t\in[0,T]} \|\mu(t)\|_{L^2} \le C(\|\mu_0\|_{L^2} + \|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}). \tag{26}$$

Note that the derivative v_{x_i} of v satisfies

$$\begin{cases} -\partial_t v_{x_i} - \Delta v_{x_i} + H_p\cdot Dv_{x_i} + D_{x_i}[H_p]\cdot Dv = D_{x_i}\dfrac{\delta F}{\delta m}(x, \bar m)(\mu(t)) & \text{in } (0, T)\times\mathbb{T}^d, \\ v_{x_i}(T, x) = D_{x_i}\dfrac{\delta G}{\delta m}(x, \bar m)(\mu(T)) + D_{x_i}v_T(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{27}$$

where, to simplify the notation, we have set $H_p = H_p(x, D\bar{u})$, etc. Then Lemma 1.2 gives, in view of our assumptions on $\delta F/\delta m$ and $\delta G/\delta m$,

$$\begin{aligned}
\|v_{x_i}(t)\|_{L^2} &\leq Ce^{-\omega(T-t)}\Big(\|Dv_T\|_{L^2} + \Big\|D_{x_i}\frac{\delta G}{\delta m}(\cdot, \bar{m})(\mu(T))\Big\|_{L^2}\Big) \\
&\qquad + C\int_t^T e^{-\omega(s-t)}\Big(\|D_{x_i}[H_p]\cdot Dv\|_{L^2} + \Big\|D_{x_i}\frac{\delta F}{\delta m}(\cdot, \bar{m})(\mu(t))\Big\|_{L^2}\Big)\, ds \\
&\leq Ce^{-\omega(T-t)}(\|Dv_T\|_{L^2} + \|\mu(T)\|_{L^2}) + C\int_t^T e^{-\omega(s-t)}(\|Dv\|_{L^2} + \|\mu(t)\|_{L^2})\, ds \\
&\leq Ce^{-\omega(T-t)}\|Dv_T\|_{L^2} + C\Big(\int_t^T \|Dv\|_{L^2}^2\Big)^{1/2} + C\sup_{s\geq t}\|\mu(s)\|_{L^2}.
\end{aligned} \tag{28}$$

Combining this with (25) and with the estimate for μ in (26), we find, for any $t \in [0, T]$,

$$\|Dv(t)\|_{L^2} \leq C(\|\mu_0\|_{L^2} + \|Dv(0)\|_{L^2}^{1/2}\|\mu_0\|_{L^2}^{1/2} + \|Dv_T\|_{L^2}).$$

In particular, for $t = 0$, we get, after simplification,

$$\|Dv(0)\|_{L^2} \leq C(\|\mu_0\|_{L^2} + \|Dv_T\|_{L^2}),$$

which jointly with (25) and (26) gives the desired statement. □

Remark 2.3. The above lemma also provides an argument for proving the existence of a solution (v, μ) to (22). Indeed, the a priori estimate (23) allows for a standard application of Schaefer's fixed-point theorem by freezing μ in the right-hand side as well as in the final value of the equation of v.

Proof of Proposition 2.1. For $\tau \geq 0$, let us set

$$\rho(\tau) = \sup_{(T,t,\mu_0,v_T)\in S(\tau)}\Big|\int_{\mathbb{T}^d}\mu(t)v(t)\Big|,$$

where the supremum is taken over the set $S(\tau)$ defined as

$$S(\tau) := \{(T, t, \mu_0, v_T) : T \geq 2\tau,\ t \in [\tau, T-\tau],\ \|\mu_0\|_{L^2} \leq 1 \text{ and } \|Dv_T\|_{L^2} \leq 1\},$$

the pair (v, μ) being a solution to (22). According to Lemma 2.2, $\rho(\tau)$ is bounded for any τ, since, using that μ has zero average, one has for any t

$$\Big|\int_{\mathbb{T}^d}\mu(t)v(t)\Big| \leq C\|\mu(t)\|_{L^2}\|Dv(t)\|_{L^2}$$

by Poincaré's inequality. By definition, the map ρ is nonincreasing, since $S(\tau) \subseteq S(\tau')$ if $\tau > \tau'$. Let us denote by ρ_∞ the limit of $\rho(\tau)$ as $\tau \to +\infty$. The key step consists in proving that $\rho_\infty = 0$.

Let $\tau_n \to +\infty$, $T_n \geq 2\tau_n$, $t_n \in [\tau_n, T_n - \tau_n]$, μ_0^n with $\|\mu_0^n\|_{L^2} \leq 1$ and v_T^n with $\|Dv_T^n\|_{L^2} \leq 1$ be such that

$$\Big|\int_{\mathbb{T}^d}\mu^n(t_n)v^n(t_n)\Big| \geq \rho_\infty - \frac{1}{n}.$$

We set

$$\tilde{\mu}^n(t,x) = \mu^n(t_n+t,x), \quad \tilde{v}^n(t,x) = v^n(t_n+t,x) - \langle v^n(t_n)\rangle \quad \text{for all } t \in [-t_n, T_n - t_n],\ x \in \mathbb{T}^d.$$

By the estimates of Lemma 2.2, the $(\tilde{v}^n, \tilde{\mu}^n)$ are locally bounded in L^2. By parabolic regularity (from [Ladyženskaja et al. 1968, Theorem III.8.1, p. 196] combined with Theorem III.10.1, p. 204, and Theorem III.11.1, p. 211, of the same work), the $\tilde{v}^n$ and $D\tilde{v}^n$ are locally bounded in $C^{\alpha/2,\alpha}$, while the $\tilde{\mu}^n$ are bounded in $C^{\alpha/2,\alpha}$ for some $\alpha \in (0,1)$. So the pair $(\tilde{v}^n, \tilde{\mu}^n)$ locally uniformly converges to some (v,μ) which satisfies the linearized MFG system on $\mathbb{R}\times\mathbb{T}^d$. Moreover, we have

$$\left|\int_{\mathbb{T}^d} \mu(0)v(0)\right| = \lim_n \left|\int_{\mathbb{T}^d} \mu^n(t_n)v^n(t_n)\right| = \rho_\infty.$$

On the other hand, for any $t \in \mathbb{R}$ and for n large enough, we have that $t_n + t \in [\tau_n - |t|, T_n - (\tau_n - |t|)]$, so

$$\left|\int_{\mathbb{T}^d} \mu(t)v(t)\right| = \lim_n \left|\int_{\mathbb{T}^d} \mu^n(t_n+t)v^n(t_n+t)\right| \le \lim_n \rho(\tau_n - |t|) = \rho_\infty.$$

The duality equality (24) implies that, for any $t_1 \le t_2$, we have

$$C^{-1}\int_{t_1}^{t_2} \|Dv\|_{L^2}^2 \le -\left[\int_{\mathbb{T}^d} \mu v\right]_{t_1}^{t_2}. \tag{29}$$

Therefore the map $t \to \int_{\mathbb{T}^d} \mu(t)v(t)$ is nonincreasing, with a derivative bounded above by $-\|Dv(0)\|_{L^2}^2$ at $t=0$, while the map $t \to \left|\int_{\mathbb{T}^d} \mu(t)v(t)\right|$ has a maximum ρ_∞ at $t=0$; this implies $Dv(0)=0$. As $\int_{\mathbb{T}^d} v(0) = 0$, we can infer that

$$\rho_\infty = \left|\int_{\mathbb{T}^d} \mu(0)v(0)\right| = 0.$$

We now prove that $\rho(t)$ converges to 0 with an exponential rate. Let $T>0$ and (v,μ) be a solution of the MFG linearized system with $\|\mu(0)\|_{L^2} \le 1$ and $\|Dv_T\|_{L^2} \le 1$. Using Lemma 1.1 and (29), we have, for $\tau \ge 0$ and $t \in [\tau, T-\tau]$

$$\|\mu(t)\|_{L^2} \le Ce^{-\omega(t-\tau/2)}\|\mu(\tau/2)\|_{L^2} + C\left(-\left[\int_{\mathbb{T}^d} \mu v\right]_{\tau/2}^{t}\right)^{1/2} \le Ce^{-\omega\tau/2} + C[2\rho(\tau/2)]^{1/2},$$

because μ is uniformly bounded in L^2 (Lemma 2.2). Thus

$$\sup_{t\in[\tau,T-\tau]} \|\mu(t)\|_{L^2} \le C(e^{-\omega\tau/2} + (\rho(\tau/2))^{1/2}). \tag{30}$$

Coming back to (28), we have, for all $t \in [2\tau, T-2\tau]$,

$$\begin{aligned}
\|Dv(t)\|_{L^2} &\le Ce^{-\omega(T-\tau-t)}\|Dv(T-\tau)\|_{L^2} + C\left(\int_t^{T-\tau} \|Dv\|_{L^2}^2\right)^{1/2} + C\sup_{s\in[t,T-\tau]} \|\mu(s)\|_{L^2} \\
&\le Ce^{-\omega\tau} + C\left(-\left[\int_{\mathbb{T}^d} \mu(s)v(s)\right]_t^{T-\tau}\right)^{1/2} + C\sup_{s\in[t,T-\tau]} \|\mu(s)\|_{L^2} \\
&\le Ce^{-\omega\tau} + C\rho^{1/2}(\tau) + C(e^{-\omega\tau/2} + (\rho(\tau/2))^{1/2}),
\end{aligned} \tag{31}$$

because Dv is uniformly bounded in L^2 (Lemma 2.2). In view of (30) and (31), we can fix $\tau > 0$ large enough so that, for any $T \geq 4\tau$ and any (v, μ) as above, one has

$$\sup_{t\in[2\tau,T-2\tau]} (\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2}) \leq \varepsilon,$$

where $\varepsilon \in \left(0, \frac{1}{4}\right)$ is to be chosen below. Notice that, by the definition of ρ and by Poincaré's inequality, this also implies $\rho(2\tau) \leq C\varepsilon \leq \frac{1}{4}$ for a suitable choice of ε. Now we can iterate the previous estimate. Indeed, for $T \geq 4\tau$, the restriction to $[2\tau, T-2\tau]$ of (v, μ) is a solution of the linearized MFG system (22) on $[2\tau, T-2\tau]$ with boundary conditions $\|\mu(2\tau)\|_{L^2} \leq \frac{1}{2}$ and $\|Dv(T-2\tau)\|_{L^2} \leq \frac{1}{2}$. As the problem is invariant by time translation, we deduce that

$$\sup_{t\in[4\tau,T-4\tau]} (\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2}) \leq \tfrac{1}{4},$$

(and similarly $\rho(4\tau) \leq 1/4^2$). By a standard iteration, this shows that there exists λ such that

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C(e^{-\lambda t} + e^{-\lambda(T-t)}) \quad \text{for all } t \in [0, T]. \qquad \square$$

Proposition 2.4. *Let λ be as in Proposition 2.1. There exists C_1 such that, if $B = B(t, x)$ satisfies*

$$\|B(t)\|_{L^2} \leq e^{-\lambda t} + e^{-\lambda(T-t)}, \tag{32}$$

and if (v, μ) is a solution to the MFG linearized system

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, D\bar{u}) \cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv) = \operatorname{div}(B) & \text{in } (0, T) \times \mathbb{T}^d \\ \mu(0, \cdot) = 0, \quad v(T, x) = 0 & \text{in } \mathbb{T}^d, \end{cases} \tag{33}$$

then

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C_1((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}) \quad \text{for all } t \in [0, T].$$

Proof. Let us first prove that (v, μ) is bounded. Multiplying the equation for v by μ and the equation for μ by v, integrating in time and space and adding the resulting relations gives, for any $0 \leq t_1 \leq t_2 \leq T$,

$$C^{-1} \int_{t_1}^{t_2} \|Dv\|_{L^2}^2 \, dt \leq -\left[\int_{\mathbb{T}^d} v\mu\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{T}^d} B \cdot Dv.$$

Thus, by Young's inequality,

$$C^{-1} \int_{t_1}^{t_2} \|Dv\|_{L^2}^2 \, dt \leq -\left[\int_{\mathbb{T}^d} v\mu\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \|B\|_{L^2}^2 \, ds.$$

Using the homogeneous boundary conditions at $t = 0$, $t = T$, we obtain the bound

$$\int_0^T \|Dv\|_{L^2}^2 \, dt \leq C \int_0^T \|B\|_{L^2}^2 \, ds.$$

This implies, with the same arguments as in Lemma 2.2,

$$\sup_{t\in[0,T]} \|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C\left[\int_0^T \|B\|_{L^2}^2\right]^{1/2} \leq C,$$

where the last inequality comes from (32).

For $\tau \geq 0$, we set

$$\rho(\tau) = \sup_{T,t,B} (\|\mu(t)\|_2 + \|Dv(t)\|_{L^2}), \tag{34}$$

where the supremum is taken over any $T \geq 2\tau$, $t \in [\tau, T-\tau]$ and any B satisfying (32), the pair (v, μ) being a solution to (33). In view of the previous discussion, $\rho(\tau)$ is bounded for any τ.

The restriction $(\tilde{v}, \tilde{\mu})$ of (v, μ) to $[\tau, T-\tau]$ can be written as

$$(\tilde{v}, \tilde{\mu}) = (\tilde{v}_1, \tilde{\mu}_1) + (\tilde{v}_2, \tilde{\mu}_2),$$

where $(\tilde{v}_1, \tilde{\mu}_1)$ solves the homogeneous MFG linearized system (22) with boundary conditions $\tilde{v}_1(T-\tau) = v(T-\tau)$ and $\tilde{\mu}_1(\tau) = \mu(\tau)$, while $(\tilde{v}_2, \tilde{\mu}_2)$ solves the linearized MFG system (33) on the time interval $[\tau, T-\tau]$ with homogeneous boundary conditions.

From Proposition 2.1, we have, for any $t \in [\tau, T-\tau]$,

$$\begin{aligned}\|\tilde{\mu}_1(t)\|_{L^2} + \|D\tilde{v}_1(t)\|_{L^2} &\leq C_0(e^{-\lambda(t-\tau)} + e^{-\lambda(T-\tau-t)})(\|\mu(\tau)\|_{L^2} + \|Dv(T-\tau)\|_{L^2}) \\ &\leq C(e^{-\lambda(t-\tau)} + e^{-\lambda(T-\tau-t)}).\end{aligned}$$

Note that the restriction of B to $[\tau, T-\tau]$ satisfies

$$\|B(t)\|_{L^2} \leq e^{-\lambda\tau}[e^{-\lambda(t-\tau)} + e^{-\lambda(T-\tau-t)}].$$

So by the linearity and the invariance in time of the equation, we get

$$\|\tilde{\mu}_2(t)\|_2 + \|D\tilde{v}_2(t)\|_{L^2} \leq e^{-\lambda\tau}\rho(t-\tau) \quad \text{for all } t \in [\tau, T-\tau].$$

Putting together the estimates of $(\tilde{v}_1, \tilde{\mu}_1)$ and $(\tilde{v}_2, \tilde{\mu}_2)$, we obtain, for any $t \geq \tau$,

$$\begin{aligned}\sup_{s\in[t+\tau, T-\tau-t]} (\|\mu(s)\|_{L^2} + \|Dv(s)\|_{L^2}) &\leq \sup_{s\in[t+\tau, T-\tau-t]} C(e^{-\lambda(s-\tau)} + e^{-\lambda(T-\tau-s)}) + e^{-\lambda\tau}\rho(s-\tau) \\ &\leq Ce^{-\lambda t} + e^{-\lambda\tau}\rho(t).\end{aligned}$$

Taking the supremum over (v, μ) and multiplying by $e^{\lambda(t+\tau)}$ gives

$$e^{\lambda(t+\tau)}\rho(t+\tau) \leq Ce^{\lambda\tau} + e^{\lambda t}\rho(t),$$

from which we infer that

$$\rho(t) \leq C(1+t)e^{-\lambda t}.$$

By the definition of ρ in (34), this implies the conclusion when choosing $\tau = t$ if $t \in [0, T/2]$ and $\tau = T - t$ otherwise. □

Collecting the above propositions we finally obtain:

Theorem 2.5. *Let λ be as in Proposition 2.1. There exists $C_0 > 0$ such that, if $A = A(t, x)$ and $B = B(t, x)$ satisfy*

$$\|A(t)\|_{L^2} + \|B(t)\|_{L^2} \leq M(e^{-\lambda t} + e^{-\lambda(T-t)}), \tag{35}$$

and if (v, μ) *is a solution to the MFG linearized system*

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, D\bar{u}) \cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) + A(t, x) & \textit{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv) = \operatorname{div}(B) & \textit{in } (0, T) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0, \quad v(T, x) = \dfrac{\delta G}{\delta m}(x, \bar{m})(\mu(T)) + v_T(x) & \textit{in } \mathbb{T}^d, \end{cases} \tag{36}$$

with $\int_{\mathbb{T}^d} \mu_0 = 0$, *we have*

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C_0((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)})(\|Dv_T\|_{L^2} + \|\mu_0\|_{L^2} + M)$$

for any $t \in [0, T]$.

Proof. Let $\tilde{v}$ be the solution to

$$\begin{cases} -\partial_t \tilde{v} - \Delta \tilde{v} + H_p(x, D\bar{u}) \cdot D\tilde{v} = A(t, x) & \text{in } (0, T) \times \mathbb{T}^d, \\ \tilde{v}(T, x) = 0 & \text{in } \mathbb{T}^d. \end{cases}$$

Note for later use that, assuming $\lambda < \omega$, we have

$$\|D\tilde{v}(t)\|_{L^2} \le CM(e^{-\lambda t} + e^{-\lambda(T-t)}). \tag{37}$$

Indeed, using Lemma 1.2, we have

$$\|\tilde{v}(t) - \langle \tilde{v}(t) \rangle\|_{L^2} \le C \int_t^T e^{-\omega(s-t)} \|A(s)\|_{L^2}\, ds \le CM(e^{-\lambda t} + e^{-\lambda(T-t)}).$$

Then the regularizing property of the equation leads to (37).

The pair $(v_1, \mu_1) := (v - \tilde{v}, \mu)$ solves

$$\begin{cases} -\partial_t v - \Delta v_1 + H_p(x, D\bar{u}) \cdot Dv_1 = \dfrac{\delta F}{\delta m}(x, \bar{m})(\mu(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu_1 - \Delta \mu_1 - \operatorname{div}(\mu_1 H_p(x, D\bar{u})) - \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv_1) = \operatorname{div}(B + \bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}) & \text{in } (0, T) \times \mathbb{T}^d, \\ \mu_1(0, \cdot) = \mu_0, \quad v_1(T, x) = \dfrac{\delta G}{\delta m}(x, \bar{m})(\mu_1(T)) + v_T(x) & \text{in } \mathbb{T}^d, \end{cases}$$

where, by (35) and (37),

$$\|B(t) + \bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}(t)\|_{L^2} \le CM(e^{-\lambda t} + e^{-\lambda(T-t)}).$$

Using Propositions 2.1 and 2.4, we get

$$\|\mu_1(t)\|_{L^2} + \|Dv_1(t)\|_{L^2} \le C((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)})(\|Dv_T\|_{L^2} + \|\mu_0\|_{L^2} + M)$$

for any $t \in [0, T]$. Recalling the definition of (v_1, μ_1) and using again inequality (37) gives the result. □

2B. ***Estimates for the nonlinear system.*** Now we consider the nonlinear MFG systems. For the finite-horizon problem, we have:

Theorem 2.6. *There exists $\gamma > 0$ and $C > 0$ such that, if (u, m) is a solution of the MFG system with initial condition $m_0 \in \mathcal{P}(\mathbb{T}^d)$*

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = m_0, \quad u(T,x) = G(x, m(T)) & \text{in } \mathbb{T}^d, \end{cases} \tag{38}$$

then, for some $\alpha \in (0,1)$,

$$\|Du(t) - D\bar u\|_{C^{1+\alpha}} \le C(e^{-\gamma t} + e^{-\gamma(T-t)}) \quad \text{for all } t \in [0,T],$$
$$\|m(t) - \bar m\|_{C^{\alpha}} \le C(e^{-\gamma t} + e^{-\gamma(T-t)}) \quad \text{for all } t \in [1,T].$$

In particular,

$$\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} |u(t,x) - \bar u(x) - \bar\lambda(T-t)| \le C.$$

Proof. We use a fixed-point argument. Let us start with the proof for initial and terminal conditions which are sufficiently close to $\bar m$ and $\bar u$ respectively. Let $\widehat K > 0$ be small enough and $\gamma \in (\lambda/2, \lambda)$, where λ is given by Proposition 2.1. Let E be the set of continuous maps (v, μ) on $[0,T]\times\mathbb{T}^d$ such that Dv is also continuous and

$$\|Dv(t)\|_{L^\infty} + \|\mu(t)\|_{L^\infty} \le \widehat K(e^{-\gamma t} + e^{-\gamma(T-t)}).$$

We suppose that $\widehat K$ is such that

$$\bar m(x) > \widehat K \quad \text{for all } x \in \mathbb{T}^d.$$

We also assume that the initial condition m_0 and the terminal condition u_T are close to $\bar m$ and $\bar u$ (plus a constant) respectively, namely that $\mu_0 := m_0 - \bar m$ and $v_T := u_T - \bar u$ satisfy

$$\|\mu_0\|_{L^\infty} + \|Dv_T\|_\infty \le \widehat K^2. \tag{39}$$

We may suppose further that μ_0 and Dv_T belong to $C^\alpha(\mathbb{T}^d)$ for some $\alpha \in (0,1)$.

For $(v,\mu) \in E$, we consider the solution $(\tilde v, \tilde \mu)$ to the linearized system

$$\begin{cases} -\partial_t \tilde v - \Delta \tilde v + H_p(x, D\bar u)\cdot D\tilde v = \dfrac{\delta F}{\delta m}(x, \bar m)(\tilde\mu(t)) + A(t,x) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t \tilde\mu - \Delta\tilde\mu - \operatorname{div}(\tilde\mu H_p(x, D\bar u)) - \operatorname{div}(\bar m H_{pp}(x, D\bar u) D\tilde v) = \operatorname{div}(B) & \text{in } (0,T)\times\mathbb{T}^d, \\ \tilde\mu(0,\cdot) = \mu_0, \quad \tilde v(T,x) = v_T(x) & \text{in } \mathbb{T}^d, \end{cases}$$

with

$$A(t,x) = -H(x, D(\bar u + v)) + H(x, D\bar u) + H_p(x, D\bar u)\cdot Dv + F(x, \bar m + \mu) - F(x, \bar m) - \frac{\delta F}{\delta m}(x, \bar m)(\mu)$$

and

$$B(t,x) = (\bar m + \mu) H_p(x, D(\bar u + v)) - \bar m H_p(x, D\bar u) - \mu H_p(x, D\bar u) - \bar m H_{pp}(x, D\bar u) Dv.$$

We note that $\bar m + \mu \ge 0$ on $[0,T]\times\mathbb{T}^d$ and

$$\|A(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} \le C\widehat K^2(e^{-2\gamma t} + e^{-2\gamma(T-t)}).$$

Here we used that $m \mapsto (\delta F/\delta m)(x, m, y)$ is Lipschitz (uniformly with respect to (x, y)), and H_{pp} is Lipschitz as well.

From Theorem 2.5 we have, as $\gamma \in (\lambda/2, \lambda)$,

$$\|\tilde{\mu}(t)\|_{L^2} + \|D\tilde{v}(t)\|_{L^2} \leq C\widehat{K}^2((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}).$$

We upgrade the previous estimates to L^∞ norms with our usual arguments: from Lemma 1.2 we have

$$\begin{aligned}\|\tilde{v}(t) - \langle \tilde{v}(t)\rangle\|_{L^\infty} &\\ &\leq Ce^{-\omega(T-t)}\|v(T) - \langle v(T)\rangle\|_{L^\infty} + C\int_t^T e^{-\omega(s-t)}\Big(\Big\|\frac{\delta F}{\delta m}(x, \bar{m})(\tilde{\mu}(s))\Big\|_{L^\infty} + \|A(s)\|_{L^\infty}\Big)\, ds \\ &\leq C\widehat{K}^2((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}).\end{aligned}$$

Then, in any interval $[t, t+1]$, we have, by using the uniform parabolicity of the equation,

$$\|D\tilde{v}(t)\|_\infty \leq C \sup_{s\in(t,t+1)}\Big[\|\tilde{v}(s) - \langle \tilde{v}(s)\rangle\|_\infty + \Big\|\frac{\delta F}{\delta m}(x, \bar{m})(\tilde{\mu}(s))\Big\|_{L^\infty} + \|A(s)\|_\infty\Big],$$

and this concludes the estimate for $\|D\tilde{v}(t)\|_{L^\infty}$. Now, using the bound for $D\tilde{v}$ and B, we have

$$\|\tilde{\mu}(t)\|_\infty \leq C \sup_{s\in(t-1,t)} [\|\tilde{\mu}(s)\|_{L^2} + \|D\tilde{v}(s)\|_{L^\infty} + \|B(s)\|_\infty]$$

and we conclude the estimate for $\|\tilde{\mu}(t)\|_\infty$. Notice that the above bounds hold up to $t = 0$ and $t = T$ by using the condition (39) assumed on μ_0 and v_T. Eventually, we obtain that

$$\|\tilde{\mu}(t)\|_{L^\infty} + \|D\tilde{v}(t)\|_{L^\infty} \leq C\widehat{K}^2((1+t)e^{-\lambda t} + (1+T)e^{-\lambda(T-t)}).$$

Since $\gamma < \lambda$, for $\widehat{K}$ small enough we infer that

$$\|\tilde{\mu}(t)\|_{L^\infty} + \|D\tilde{v}(t)\|_{L^\infty} \leq \widehat{K}(e^{-\gamma t} + e^{-\gamma(T-t)})$$

and $(\tilde{v}, \tilde{\mu})$ belongs to E. In addition, $\tilde{\mu}$ and $\tilde{v} - \langle \tilde{v}\rangle$ solve linear parabolic equations with bounded coefficients, so classical parabolic estimates [Ladyženskaja et al. 1968, Theorems III.8.1, III.10.1 and III.11.1, p. 196] imply that $\tilde{\mu}$ and $D\tilde{v}$ are locally bounded in $C^{\alpha/2,\alpha}$ for some $\alpha \in (0, 1)$, with bounds that only depend on the L^∞ norm of the coefficients. In particular, the map $(v, \mu) \to (\tilde{v}, \tilde{\mu})$ is compact and it has a fixed point (v, μ). Then $(u, m) := (\bar{u}, \bar{m}) + (v, \mu)$ is a solution to (38) with terminal condition u_T and which satisfies the decay

$$\|Du(t) - D\bar{u}(t)\|_{C^\alpha} + \|m(t) - \bar{m}\|_{C^\alpha} \leq \widehat{K}(e^{-\gamma t} + e^{-\gamma(T-t)}).$$

We now remove the smallness and regularity assumptions on the initial condition m_0 and the terminal condition u_T. Let (u, m) be the solution to (38). From Lemma 1.7 there exists $0 < \tau < \widehat{T}$ such that, if $T \geq \widehat{T}$, then the solution to (38) satisfies, again for some $\alpha \in (0, 1)$,

$$\|m(t) - \bar{m}\|_{C^\alpha} + \|Du(t) - D\bar{u}\|_{C^\alpha} \leq \widehat{K}^2 \quad \text{for all } t \in [\tau, T-\tau]. \tag{40}$$

From the first step we conclude that

$$\|m(t)-\bar m\|_{C^\alpha}+\|Du(t)-D\bar u\|_{C^\alpha}\le \widehat{K}(e^{-\gamma(t-\tau)}+e^{-\gamma(T-\tau-t)}) \quad \text{for all } t\in[\tau,T-\tau].$$

Using Lemma 1.1 and changing the constant if necessary, we can extend this inequality for m to the time interval $[1,T]$. Moreover, $Du(t)-D\bar u$ also satisfies a parabolic equation with uniformly bounded coefficients. Thus it is bounded in $C^{1+\alpha/2,1+\alpha}$ (for some possibly different α, depending on the data only) and we can improve the above inequality for u into

$$\|Du(t)-D\bar u\|_{C^{1+\alpha}}\le C(e^{-\gamma t}+e^{-\gamma(T-t)}) \quad \text{for all } t\in[0,T].$$

We finally prove the last bound on $v:=u-\bar u-\bar\lambda(T-t)$. Note that v satisfies

$$-\partial_t v-\Delta v=A(t,x),$$

where

$$A(t,x)=-(H(x,Du)-H(x,D\bar u))+F(x,m(t))-F(x,\bar m),$$

so

$$\|A(t)\|_{L^\infty}\le C(e^{-\gamma t}+e^{-\gamma(T-t)}) \quad \text{for all } t\in[0,T].$$

Thus, by a standard heat estimate,

$$\|v(t)\|_{L^\infty}\le Ce^{-\omega(T-t)}\|v(T)\|_{L^\infty}+C\int_t^T e^{-\omega(s-t)}\|A(s)\|_{L^\infty}\,ds\le C. \qquad \square$$

Let us stress that the above proof provides an explicit smallness estimate on $D(u-\bar u)$ and $m-\bar m$ for initial-terminal data which are correspondingly small. This allows us to derive the convergence of $u^T(0,x)$ as the time horizon tends to infinity, for the special case with initial measure $m_0=\bar m$. This result is a first key argument in the analysis of the long time behavior of the general MFG system and of the master equation (Theorem 5.1 and Corollary 5.2).

Proposition 2.7. *For any $T>0$, let (u^T,m^T) be a solution to*

$$\begin{cases} -\partial_t u^T-\Delta u^T+H(x,Du^T)=F(x,m^T(t)) & \text{in } (0,T)\times\mathbb{T}^d,\\ \partial_t m^T-\Delta m^T-\operatorname{div}(m^T H_p(x,Du^T))=0 & \text{in } (0,T)\times\mathbb{T}^d,\\ m^T(0,\cdot)=\bar m, \quad u^T(T,x)=G(x,m(T)) & \text{in } \mathbb{T}^d. \end{cases} \tag{41}$$

Then there exists a constant $\bar c$ such that

$$\lim_{T\to+\infty} u^T(0,x)-\bar\lambda T=\bar u(x)+\bar c,$$

where the limit is uniform in $x\in\mathbb{T}^d$.

Proof. The proof consists in showing that the quantity $u^T(0,x)-\bar\lambda T-\bar u(x)$ is Cauchy in T in the uniform topology and converges to a constant. In a first step, we show that there exists $\tau>0$ large enough such that $u^T(T-\tau)$ and $u^{T'}(T'-\tau)$ are close in L^∞ for $T,T'\ge 2\tau$. Then we use Theorem 2.6 (and its proof) to extend this proximity up to time $t=0$.

Let us fix $\varepsilon > 0$ small. Theorem 2.6 states that

$$\|Du^T(t) - D\bar{u}\|_{C^{1+\alpha}} + \|m^T(t) - \bar{m}\|_{L^\infty} \le C(e^{-\gamma t} + e^{-\gamma(T-t)}) \quad \text{for all } t \in [1, T], \tag{42}$$

for some constant C independent of T. Fix τ large enough and let $T, T' \ge 2\tau$. If we consider $(\hat{u}^T, \hat{m}^T)(t, x) := (\hat{u}^T, \hat{m}^T)(t + T, x)$ and $(\hat{u}^{T'}, \hat{m}^{T'})(t, x) := (\hat{u}^{T'}, \hat{m}^{T'})(t + T', x)$, which are both solutions of the MFG system in $(-\tau, 0)$, the energy inequality gives

$$\begin{aligned} C^{-1}\int_{-\tau}^{0}\int_{\mathbb{T}^d}(\hat{m}^T + \hat{m}^{T'})|D(\hat{u}^T - \hat{u}^{T'})|^2 &\le -\left[\int_{\mathbb{T}^d}(\hat{u}^T(t) - \hat{u}^{T'}(t))(\hat{m}^T(t) - \hat{m}^{T'}(t))\right]_{-\tau}^{0} \\ &\le \int_{\mathbb{T}^d}(\hat{u}^T(-\tau) - \hat{u}^{T'}(-\tau))(\hat{m}^T(-\tau) - \hat{m}^{T'}(-\tau)), \end{aligned}$$

where we used that $(\hat{u}^T - \hat{u}^{T'})(0) = G(x, \hat{m}^T(0)) - G(x, \hat{m}^{T'}(0))$ and the monotonicity of G. Using (42) and the fact that $T, T' \ge 2\tau$ we deduce that

$$\int_{-\tau}^{0}\int_{\mathbb{T}^d}(\hat{m}^T + \hat{m}^{T'})|D(\hat{u}^T - \hat{u}^{T'})|^2 \le Ce^{-2\gamma\tau},$$

where C is independent of T, T'. Now we apply Lemma 1.1 and (42) to $\hat{m}^T - \hat{m}^{T'}$ in the interval $(-\tau, 0)$ and we get

$$\|\hat{m}^T(t) - \hat{m}^{T'}(t)\|_{L^2} \le C\|\hat{m}^T(-\tau) - \hat{m}^{T'}(-\tau)\|_{L^2} + C\left(\int_{-\tau}^{0}\int_{\mathbb{T}^d}(\hat{m}^{T'})^2|D(\hat{u}^T - \hat{u}^{T'})|^2\,dt\right)^{1/2} \le Ce^{-\gamma\tau}.$$

In particular, by the assumptions on F, G, there exists $C > 0$ such that

$$\sup_{t\in(-\tau,0)} \|F(x, \hat{m}^T(t)) - F(x, \hat{m}^{T'}(t))\|_{L^\infty} + \|G(x, \hat{m}^T(0)) - G(x, \hat{m}^{T'}(0))\|_{L^\infty} \le Ce^{-\gamma\tau}.$$

By the comparison principle between $\hat{u}^T$ and $\hat{u}^{T'}$ in $(-\tau, 0)$, we conclude that

$$\|\hat{u}^T(-\tau) - \hat{u}^{T'}(-\tau)\|_\infty \le C(1 + \tau)e^{-\gamma\tau}.$$

Hence we can choose τ sufficiently large such that

$$\|u^T(T - \tau) - u^{T'}(T' - \tau)\|_\infty \le \varepsilon \tag{43}$$

for any T, T' large enough.

Now we extend the proximity of u^T and $u^{T'}$ up to time $t = 0$. Recalling that, by (42),

$$\|Du^T(T - \tau) - D\bar{u}\|_\infty \le \varepsilon$$

for any T large enough, there exists $\bar{c}_0(T)$ such that

$$\|u^T(T - \tau) - \bar{u} - \bar{c}_0(T)\|_\infty \le C\varepsilon. \tag{44}$$

Note that (43) implies that $(\bar{c}_0(T))$ is Cauchy as $T \to +\infty$ and thus converges to a limit $\bar{c}$. Let $\gamma > 0$ be defined in the first step of the proof of Theorem 2.6; since (u^T, m^T) satisfy (39) with $\widehat{K} = \varepsilon^{1/2}$, we can choose ε small enough so that the fixed-point argument of Theorem 2.6 applies. Then, the restriction of (u^T, m^T) to $[0, T-\tau]$ satisfies

$$\|Du^T(t) - D\bar{u}\|_{L^\infty} + \|m^T(t) - \bar{m}\|_\infty \le \varepsilon^{1/2}(e^{-\gamma t} + e^{-\gamma(T-\tau-t)}) \quad \text{for all } t \in [0, T-\tau]. \tag{45}$$

Integrating in space the equation satisfied by $u^T - \bar{u}$, we get

$$\left| \int_{\mathbb{T}^d} (u^T(0) - u^T(T-\tau)) - \bar{\lambda}(T-\tau) \right| \le \int_0^{T-\tau} \int_{\mathbb{T}^d} |H(x, Du^T) - H(x, D\bar{u})| + |F(x, m^T(t)) - F(x, \bar{m})| \, dx \, dt \le C\varepsilon^{1/2}.$$

Using (45) (at time $t = 0$ and at time $t = T - \tau$) and Poincaré's inequality, we infer therefore that

$$\|u^T(0) - u^T(T-\tau) - \bar{\lambda}(T-\tau)\|_\infty \le C\varepsilon^{1/2}. \tag{46}$$

Combining (43), (44) and (46), we conclude that, for any T, T' large enough,

$$\|u^T(0) - \bar{u} - \bar{c}_0(T) - \bar{\lambda}(T-\tau)\|_\infty \le C\varepsilon^{1/2}.$$

From this we can deduce that $(u^T(0, x) - \bar{\lambda}T)$ converges uniformly to $\bar{u}(x) + \bar{c}$ as T tends to ∞. □

We also deduce from Theorem 2.6 crucial estimates for the linearized system around *any* solution (u, m) of (38).

Corollary 2.8. *There exists $\gamma > 0$ and $C > 0$ such that, if (u, m) is a solution of the MFG system* (38), *and if (v, μ) is a solution to the linearized MFG system*

$$\begin{cases} -\partial_t v - \Delta v + H_p(x, Du) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m)(\mu) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, Du)) - \operatorname{div}(m H_{pp}(x, Du) Dv) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0, \quad v(T, \cdot) = \dfrac{\delta G}{\delta m}(x, m(T))(\mu(T)) & \text{in } \mathbb{T}^d, \end{cases}$$

with $\int_{\mathbb{T}^d} \mu_0 = 0$, we have

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C(e^{-\gamma t} + e^{-\gamma(T-t)})\|\mu_0\|_{L^2} \tag{47}$$

and, for some $\alpha \in (0, 1)$ depending only on the data,

$$\sup_{t \in [0,T]} \|v\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}. \tag{48}$$

Proof. We first need a priori estimates on (v, μ). To this end we assume that $\mu_0 \in L^2(\mathbb{T}^d)$, and we proceed exactly as in Lemma 2.2 obtaining

$$\int_0^T \int_{\mathbb{T}^d} m|Dv|^2 + \sup_{t \in [0,T]} (\|\mu(t)\|_{L^2}^2 + \|Dv(t)\|_{L^2}^2) \le C_0\|\mu_0\|_{L^2}^2. \tag{49}$$

Next we note that (v, μ) is the solution to (36) with

$$A = -(H_p(x, Du) - H_p(x, D\bar{u})) \cdot Dv + \frac{\delta F}{\delta m}(x, m(t))(\mu(t)) - \frac{\delta F}{\delta m}(x, \bar{m})(\mu(t)),$$
$$B = \mu(H_p(x, Du) - H_p(x, D\bar{u})) + (mH_{pp}(x, Du) - \bar{m}H_{pp}(x, D\bar{u}))Dv$$

and

$$v_T(x) = \frac{\delta G}{\delta m}(x, m(T))(\mu(T)) - \frac{\delta G}{\delta m}(x, \bar{m})(\mu(T)).$$

Note that

$$\|A(t)\|_{L^2} \le C\|Du - D\bar{u}\|_\infty \|Dv\|_{L^2} + C\boldsymbol{d}_1(m(t), \bar{m})\|\mu(t)\|_{L^2},$$

while

$$\|B(t)\|_{L^2} \le C\|Du - D\bar{u}\|_\infty \|\mu(t)\|_{L^2} + C(\boldsymbol{d}_1(m(t), \bar{m}) + \|Du(t) - D\bar{u}\|_\infty)\|Dv(t)\|_{L^2}$$

and

$$\|v_T\| \le C\boldsymbol{d}_1(m(T), \bar{m})\|\mu(T)\|_{L^2}.$$

Here we used once more that $m \mapsto (\delta F/\delta m)(x, m, y)$, $m \mapsto (\delta G/\delta m)(x, m, y)$ and $p \mapsto H_{pp}(x, p)$ are Lipschitz.

Using Theorem 2.6 and (49), we deduce

$$\|A(t)\|_{L^2} + \|B(t)\|_{L^2} \le C\|\mu_0\|_{L^2}(e^{-\gamma t} + e^{-\gamma(T-t)}).$$

Then Theorem 2.5 (used with $\lambda = \gamma$) and the bound (49) imply

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C((1+t)e^{-\gamma t} + (1+T)e^{-\gamma(T-t)})\|\mu_0\|_{L^2}.$$

So we deduce (47), possibly for a smaller value of γ.

Now we upgrade the above estimate by using weaker norms for μ_0 and stronger norms for v. For this, we use Lemma 2.9 below, which states that

$$\|\mu(1)\|_{L^2} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Applying our previous estimate (47) to the time interval $[1, T]$, we find that, for any $t \ge 1$,

$$\begin{aligned}\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} &\le C(e^{-\gamma(t-1)} + e^{-\gamma(T-(t-1))})\|\mu(1)\|_{L^2} \\ &\le C(e^{-\gamma t} + e^{-\gamma(T-t)})\|\mu_0\|_{(C^{2+\alpha})'}.\end{aligned}$$

Lemma 2.9 also states that

$$\sup_{t\in[0,T]} \|v(t) - \langle v(t)\rangle\|_{C^{2+\alpha}} + \sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} \le C\|\mu_0\|_{(C^{2+\alpha})'}, \tag{50}$$

so we also have

$$\sup_{t\in[0,1]} \|Dv(t)\|_{L^2} + \sup_{t\in[0,1]} \|\mu(t)\|_{(C^{2+\alpha})'} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Integrating in space the equation for v and using the above bounds on Dv and μ then implies

$$|\langle v(t)\rangle| \le C\|\mu_0\|_{(C^{2+\alpha})'} \quad \text{for all } t \in [0, T].$$

We can then deduce (48) from (50) and the above inequality. □

Lemma 2.9. *Under the assumptions of Corollary 2.8, there exists a constant* $C > 0$ (*independent of* T, m_0 *and* μ_0) *such that*

$$\sup_{t\in[0,T]} \|v(t) - \langle v(t)\rangle\|_{C^{2+\alpha}} + \sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} + \|\mu(1)\|_{L^2} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Proof. The estimate (49) gives

$$c\int_0^t\int_{\mathbb{T}^d} m|Dv|^2 \le \int_0^T\int_{\mathbb{T}^d} mH_{pp}(x, Du)Dv\cdot Dv \le \int_{\mathbb{T}^d} v(0)\mu_0, \tag{51}$$

where we used that $(\delta G/\delta m)(x, m(T))$ is a nonnegative operator. By duality, we also have

$$\int_{\mathbb{T}^d} \mu(t)\xi = -\int_0^t\int_{\mathbb{T}^d} mH_{pp}(Du)Dv\cdot D\psi + \int_{\mathbb{T}^d} \psi(0)\mu_0,$$

where ψ solves (for some smooth terminal condition ξ at time t)

$$\begin{cases} -\partial_t\psi - \Delta\psi + H_p(x, Du)\cdot D\psi = 0 & \text{in } (0,t)\times\mathbb{T}^d,\\ \psi(t,\cdot) = \xi & \text{in } \mathbb{T}^d. \end{cases}$$

Since, by Lemma 1.2, $\|\psi(s) - \langle\psi(s)\rangle\|_{L^2} \le ce^{-\omega(t-s)}\|\xi\|_{L^2}$, we have by standard estimates

$$\int_0^t\int_{\mathbb{T}^d} |D\psi|^2 \le \|\xi\|_2^2 + C\int_0^t\int_{\mathbb{T}^d} |\psi - \langle\psi\rangle|^2 \le C\|\xi\|_{L^2}^2.$$

Therefore,

$$\int_{\mathbb{T}^d} \mu(t)\xi \le C\left(\int_0^t\int_{\mathbb{T}^d} m|Dv|^2\right)^{1/2} \|\xi\|_{L^2} + \|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}}\|\mu_0\|_{(C^{2+\alpha})'}.$$

From (51) we deduce

$$\int_{\mathbb{T}^d} \mu(t)\xi \le C\big(\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}\|\mu_0\|_{(C^{2+\alpha})'}\big)^{1/2}\|\xi\|_{L^2} + \|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}}\|\mu_0\|_{(C^{2+\alpha})'}. \tag{52}$$

To estimate last term, we note that, if $t \le 1$, we have by Schauder estimates that

$$\|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}} \le C\|\xi\|_{C^{2+\alpha}},$$

while, if $t \ge 1$, we have, by Schauder interior estimates

$$\|\psi(0) - \langle\psi(0)\rangle\|_{C^{2+\alpha}} \le C\|\psi(1) - \langle\psi(1)\rangle\|_{L^2} \le C\|\xi\|_{L^2} \le C\|\xi\|_{C^{2+\alpha}}. \tag{53}$$

Coming back to (52) and taking the supremum over the ξ with $\|\xi\|_{C^{2+\alpha}} \le 1$, this implies

$$\sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} \le C\big(\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}^{1/2}\|\mu_0\|_{(C^{2+\alpha})'}^{1/2} + \|\mu_0\|_{(C^{2+\alpha})'}\big). \tag{54}$$

Similarly, from (52) and (53) we also estimate

$$\|\mu(1)\|_{L^2} \le C\big(\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}^{1/2}\|\mu_0\|_{(C^{2+\alpha})'}^{1/2} + \|\mu_0\|_{(C^{2+\alpha})'}\big). \tag{55}$$

We now have to estimate $v(0) - \langle v(0)\rangle$. First we have, by Lemma 1.2, that for any $t \in [0, T]$

$$\begin{aligned}\|v(t) - \langle v(t)\rangle\|_\infty &\le e^{-\omega(T-t)} \Big\| \frac{\delta G}{\delta m}(x, m(T))\mu(T)\Big\|_\infty + \int_t^T e^{-\omega(s-t)} \Big\|\frac{\delta F}{\delta m}(\,\cdot\,, m(s))(\mu(s))\Big\|_\infty ds \\ &\le C \sup_{[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'},\end{aligned} \tag{56}$$

where we used that $\delta F/\delta m, \delta G/\delta m$ are $C^{2+\alpha}$ with respect to y. We also estimate Dv in L^2 in terms of the same quantity due to Lemma 1.2. Next, the regularizing property of the equation for $v - \langle v\rangle$ [Ladyženskaja et al. 1968, Theorem IV.9.1] implies that, for any $t \in \big[0, T - \frac{1}{2}\big]$ and any $\beta \in (0, 1)$,

$$\begin{aligned}\|v(t) - \langle v(t)\rangle\|_{C^{1+\beta}} &\le \big\|v\big(t + \tfrac{1}{2}\big) - \big\langle v\big(t + \tfrac{1}{2}\big)\big\rangle\big\|_\infty + C \sup_{s\in[t,t+1/2]} \|\mu(s)\|_{(C^{2+\alpha})'} \\ &\le C \sup_{[0,T]} \|\mu(s)\|_{(C^{2+\alpha})'},\end{aligned}$$

(where the constant depends on β). Then considering the equation for v_{x_i} (for $i \in \{1, \dots, d\}$) and using the uniform C^2 regularity of u as well as the C^2 regularity of $D_x(\delta F/\delta m)$ in the y-variable as in (56), we obtain in the same way, for any $t \in [0, T - 1]$

$$\begin{aligned}\|v_{x_i}(t)\|_{C^{1+\beta}} &\le \big\|v_{x_i}\big(t + \tfrac{1}{2}\big)\big\|_\infty + C\Big(\sup_{s\in[t,t+1/2]} \|Dv(s)\|_\infty + \sup_{s\in[t,t+1/2]} \|\mu(s)\|_{(C^{2+\alpha})'}\Big) \\ &\le C \sup_{[0,T]} \|\mu(s)\|_{(C^{2+\alpha})'}.\end{aligned}$$

Choosing $\beta = \alpha$, we have proved therefore that

$$\sup_{s\in[0,T-1]} \|v(s) - \langle v(s)\rangle\|_{C^{2+\alpha}} \le C \sup_{s\in[0,T]} \|\mu(s)\|_{(C^{2+\alpha})'}.$$

Using this inequality for $\|v(0) - \langle v(0)\rangle\|_{C^{2+\alpha}}$ in (54) then gives

$$\sup_{t\in[0,T]} \|\mu(t)\|_{(C^{2+\alpha})'} \le C\|\mu_0\|_{(C^{2+\alpha})'},$$

which in turn implies

$$\sup_{t\in[0,T-1]} \|v(s) - \langle v(s)\rangle\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Note that we can extend this inequality to the time interval $[T-1, T]$ by using the regularity of the equation satisfied by v on this interval, the regularity of the terminal condition and the bound on $\|\mu(t)\|_{(C^{2+\alpha})'}$.

In the same way, from (55) we obtain

$$\|\mu(1)\|_{L^2} \le C\|\mu_0\|_{(C^{2+\alpha})'}. \qquad \square$$

Remark 2.10. In order to estimate v in the C^2 norm, we have used in Lemma 2.9 the regularity condition (FGc) on the couplings. However, by only using condition (FGb), we could similarly obtain a milder estimate as

$$\sup_{t\in[0,T]} \|v(t) - \langle v(t)\rangle\|_{C^1} + \sup_{t\in[0,T]} \|\mu(t)\|_{(C^1)'} \le C\|\mu_0\|_{(C^1)'}. \tag{57}$$

Indeed, an estimate similar to (56) would hold in terms of $\|\mu(t)\|_{(C^1)'}$ by using condition (FGb) since

$$\|v(t)-\langle v(t)\rangle\|_\infty \le e^{-\omega(T-t)}\Big\|\frac{\delta G}{\delta m}(x,m(T))\mu(T)\Big\|_\infty + \int_t^T e^{-\omega(s-t)}\Big\|\frac{\delta F}{\delta m}(\,\cdot\,,m(s))(\mu(s))\Big\|_\infty\,ds$$
$$\le C\sup_{[0,T]}\|\mu(t)\|_{(C^1)'},$$

where we only used that $\delta F/\delta m$, $\delta G/\delta m$ are C^1 and globally Lipschitz with respect to y. Under the same condition the estimate for Dv in L^∞ would follow. Eventually, with the same strategy as in the above proof, by using C^1 rather than $C^{2+\alpha}$ and using estimates on v, we would get (57).

3. Exponential rate of convergence for the infinite-horizon MFG system

We now study the infinite-horizon discounted problem and show an exponential convergence towards a stationary solution. The existence of this solution is new, as well as the convergence rate towards this solution. The method of proof is close to the one employed in the previous section for the finite horizon.

3A. *The stationary solution of the infinite-horizon problem.*

Proposition 3.1. *There exists $\delta_0>0$ such that, if $\delta\in(0,\delta_0)$, there is a unique solution $(\bar u^\delta,\bar m^\delta)$ to the problem* (7). *Moreover, for any $\delta\in(0,\delta_0)$,*

$$\|D\bar u^\delta\|_\infty+\delta\|\bar u^\delta\|_\infty+\|\bar m^\delta\|_\infty\le C \qquad\text{and}\qquad \bar m^\delta(x)\ge C^{-1}\quad\text{for all } x\in\mathbb{T}^d,$$

for some constant $C>0$.

Proof. The existence of a solution can be achieved by a standard fixed-point argument, so we omit it. In the same way, the regularity of $\bar u^\delta$ and $\bar m^\delta$ is standard. The strong maximum principle implies that m^δ is bounded below by a constant independent of δ. For proving the uniqueness, we argue as usual by duality, see [Lasry and Lions 2007]: Let (u_1,m_1) and (u_2,m_2) be two solutions. We multiply the equation for u_1-u_2 by m_1-m_2 and the equation for m_1-m_2 by u_1-u_2, we integrate in time and space and add the resulting quantities to obtain, by Poincaré's inequality,

$$C^{-1}\|D(u_1-u_2)\|_{L^2}^2\le\delta\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)\le C\delta\|D(u_1-u_2)\|_{L^2}\|m_1-m_2\|_{L^2}.$$

Thus

$$\|D(u_1-u_2)\|_{L^2}\le C\delta\|m_1-m_2\|_{L^2}. \tag{58}$$

On another hand, by Corollary 1.3, we have

$$\|m_1-m_2\|_{L^2}\le C\|H_p(\,\cdot\,,Du_1)-H_p(\,\cdot\,,Du_2)\|_{L^2}\le C\|D(u_1-u_2)\|_{L^2} \tag{59}$$

For δ small enough, we deduce from (58)–(59) that $m_1=m_2$ and $Du_1=Du_2$, whence $u_1=u_2$. □

We now note that the solution $(\bar u^\delta,\bar m^\delta)$ is close to $(\bar u,\bar m)$, where $(\bar\lambda,\bar u,\bar m)$ is the solution of the ergodic problem (2):

Proposition 3.2. *We have*

$$\|\delta\bar{u}^\delta - \bar{\lambda}\|_\infty + \|D(\bar{u}^\delta - \bar{u})\|_{L^2} + \|\bar{m}^\delta - \bar{m}\|_{L^2} \le C\delta^{1/2}.$$

Proof. We use again the duality argument (consisting in multiplying the equation for $u^\delta - \bar{u}$ by $m^\delta - \bar{m}$ and the equation for $m^\delta - \bar{m}$ by $u^\delta - \bar{u}$, integrating in space and adding the resulting quantities) to get

$$C^{-1}\|D(\bar{u}^\delta - \bar{u})\|_{L^2}^2 \le \int_{\mathbb{T}^d} (\delta\bar{u}^\delta - \bar{\lambda})(\bar{m}^\delta - \bar{m}) \le C\delta\|D\bar{u}^\delta\|_\infty \le C\delta.$$

Thus

$$\|D(\bar{u}^\delta - \bar{u})\|_{L^2} \le C\delta^{1/2}.$$

By Corollary 1.3, we have

$$\|\bar{m}^\delta - \bar{m}\|_{L^2} \le C\|D(\bar{u}^\delta - \bar{u})\|_{L^2} \le C\delta^{1/2}.$$

The estimate between $\delta\bar{u}^\delta$ and $\bar{\lambda}$ then comes from the comparison principle. □

3B. *Exponential rate for the linearized system.* Let $(\bar{u}^\delta, \bar{m}^\delta)$ be the solution to (7). We consider the linearized discounted problem around this solution

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, D\bar{u}^\delta)\cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu(t)) & \text{in } (0,+\infty)\times\mathbb{T}^d, \\ \partial_t\mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u}^\delta)) - \operatorname{div}(\bar{m}^\delta H_{pp}(x, D\bar{u}^\delta)Dv) = 0 & \text{in } (0,+\infty)\times\mathbb{T}^d, \\ \mu(0,\cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded}, \end{cases} \tag{60}$$

with $\int_{\mathbb{T}^d}\mu_0 = 0$. As in Section 2A, the existence of a solution to (60) can be proved for $\mu_0 \in L^2(\mathbb{T}^d)$ by using fixed-point arguments and relying on the conditions enjoyed by $\delta F/\delta m$ and the smoothness of $(\bar{u}^\delta, \bar{m}^\delta)$. In particular, one can first solve the system in a finite horizon $t \in (0, n)$ with terminal condition $v(n) = 0$, and then obtain a solution to (60) by letting $n \to \infty$. Since $\delta > 0$, here $\|\delta F/\delta m\|_\infty \delta^{-1}$ is a uniform bound with respect to n and leads to a bounded v in (60).

In the rest of this section, we are going to show that v actually enjoys a bound which is uniform in δ and that μ, Dv decay exponentially in L^2 as $t \to \infty$, uniformly with respect to δ.

Lemma 3.3. *Let* (v, μ) *be a solution to* (60). *Then we have*

$$\int_{\mathbb{T}^d} \mu(t)v(t) \ge 0 \quad \textit{for all } t \ge 0$$

and there exists a constant $C_0 > 0$, *independent of* μ_0 *and* δ, *such that*, *for any* $t \ge 0$,

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C_0\|\mu_0\|_{L^2} e^{\delta t/2}.$$

Proof. We consider the duality between $e^{-\delta t}v$ and μ (i.e, we multiply the equation for $e^{-\delta t}v$ by μ and the equation for μ by $e^{-\delta t}v$, we integrate in time and space and we add the resulting quantities); using properties of $(\bar{u}_\delta, \bar{m}_\delta)$ from Proposition 3.1 we get

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta t}\|Dv(t)\|_{L^2}^2\,dt \le -\left[e^{-\delta t}\int_{\mathbb{T}^d} v(t)\mu(t)\right]_{t_1}^{t_2}. \tag{61}$$

Next we claim that

$$C^{-1}\int_0^\infty e^{-\delta t}\|Dv(t)\|_{L^2}^2\,dt \le \int_{\mathbb{T}^d}\mu_0 v(0) \le C\|\mu_0\|_{L^2}\|v(0)-\langle v(0)\rangle\|_{L^2}. \tag{62}$$

This inequality is obvious from (61) if we know that the limit $e^{-\delta t}\int_{\mathbb{T}^d} v(t)\mu(t)$ vanishes as $t\to+\infty$. For this we need a first rough bound on μ. By Lemma 1.1 we have

$$\|\mu(t)\|_{L^2} \le Ce^{-\omega t}\|\mu_0\|_{L^2} + C\left[\int_0^t \|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}.$$

By (61), we get

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\left[\int_0^t e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\|v\|_\infty^{1/2}[\|\mu_0\|_{L^2}^{1/2} + e^{-\delta t/2}\|\mu(t)\|_{L^2}^{1/2}],\end{aligned}$$

so

$$\|\mu(t)\|_{L^2} \le C_\delta e^{\delta t/2},$$

where C_δ depends on μ_0 and δ. This inequality then implies

$$\lim_{t\to+\infty} e^{-\delta t}\int_{\mathbb{T}^d}\mu(t)v(t) = 0$$

and (62) holds. Note that (61) implies that the map $t\to e^{-\delta t}\int_{\mathbb{T}^d}\mu(t)v(t)$ is nonincreasing, and we just proved that it has limit 0 as $t\to+\infty$. Thus it is nonnegative.

In light of (62) we revisit the estimate of μ. We have

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\left[\int_0^t e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{-\omega t}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}.\end{aligned}$$

We plug this inequality into the usual estimate for v (Lemma 1.2): for any $0\le t\le t_1$,

$$\begin{aligned}&\|v(t)-\langle v(t)\rangle\|_{L^2}\\ &\quad\le Ce^{-\omega(t_1-t)}\|v(t_1)-\langle v(t_1)\rangle\|_{L^2} + C\int_t^{t_1} e^{-\omega(s-t)}\|\mu(s)\|_{L^2}\,ds\\ &\quad\le Ce^{-\omega(t_1-t)}\|v(t_1)-\langle v(t_1)\rangle\|_{L^2} + C\int_t^{t_1} e^{-\omega(s-t)}\big(e^{-\omega s}\|\mu_0\|_{L^2} + Ce^{\delta s/2}\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}\big)\,ds\\ &\quad\le Ce^{-\omega(t_1-t)}\|v(t_1)-\langle v(t_1)\rangle\|_{L^2} + C\|\mu_0\|_{L^2}e^{-\omega t} + C\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}e^{\delta t/2}.\end{aligned}$$

Letting $t_1\to+\infty$ gives

$$\|v(t)-\langle v(t)\rangle\|_{L^2} \le C\|\mu_0\|_{L^2}e^{-\omega t} + C\|\mu_0\|_{L^2}^{1/2}\|v(0)-\langle v(0)\rangle\|_{L^2}^{1/2}e^{\delta t/2}.$$

Choosing $t=0$ and rearranging we find

$$\|v(0)-\langle v(0)\rangle\|_{L^2} \le C\|\mu_0\|_{L^2}.$$

So we have for any $t\ge 0$

$$\|\mu(t)\|_{L^2} + \|v(t)-\langle v(t)\rangle\|_{L^2} \le C\|\mu_0\|_{L^2}e^{\delta t/2}.$$

We can then conclude by Lemma 1.2. □

Proposition 3.4. *Let* $(\bar{u}^\delta, \bar{m}^\delta)$ *be the solution to* (7). *There exist* $\delta_0, C_0, \lambda > 0$ *such that, if* (v, μ) *is the solution to* (60) *associated with* $(\bar{u}^\delta, \bar{m}^\delta)$ *and* $\int_{\mathbb{T}^d} \mu_0 = 0$, *and if* $\delta \in (0, \delta_0)$, *then*

$$\|Dv(t)\|_{L^2} + \|\mu(t)\|_{L^2} \le C_0 \|\mu_0\|_{L^2} e^{-\lambda t} \quad \textit{for all } t \ge 0.$$

In particular,

$$\|v\|_{L^\infty} \le C.$$

Proof. Let us set

$$\rho^\delta(t) := \sup_{\mu_0} e^{-\delta t} \int_{\mathbb{T}^d} \mu(t) v(t),$$

where the supremum is taken over $\|\mu_0\|_{L^2} \le 1$ and where (v, μ) is the solution to (60) with initial condition $\mu(0) = \mu_0$. In view of the inequality (61), the map ρ^δ is nonincreasing. Moreover, Lemma 3.3 states that $\rho^\delta(t)$ is bounded independently of δ and nonnegative. Then we set

$$\rho(t) = \limsup_{\delta \to 0} \rho^\delta(t).$$

Note that ρ is also nonincreasing, nonnegative and bounded. We denote by ρ_∞ its limit as $t \to +\infty$. We claim that $\rho_\infty = 0$.

Indeed, let $t_n \to +\infty$, $\delta_n \to 0$, and μ_0^n with $\|\mu_0^n\|_{L^2} \le 1$ be such that

$$e^{-\delta_n t_n} \int_{\mathbb{T}^d} \mu^n(t_n) v^n(t_n) \ge \rho_\infty - \frac{1}{n}.$$

We let, for $s \in [-t_n, +\infty)$,

$$\tilde{v}^n(s) = e^{-\delta_n t_n/2}(v^n(t_n + s) - \langle v^n(t_n)\rangle), \quad \tilde{\mu}^n(s) = e^{-\delta_n t_n/2} \mu^n(t_n + s).$$

From Lemma 3.3 we know that $\tilde{v}^n$, $D\tilde{v}^n$ and $\tilde{\mu}^n$ are locally bounded in L^2. As the pair $(\tilde{v}^n, \tilde{\mu}^n)$ satisfies an equation of the form (60), standard regularity estimates for parabolic equations with bounded coefficients [Ladyženskaja et al. 1968, Theorem III.10.1] imply that $\tilde{v}^n$, $D\tilde{v}^n$ and $\tilde{\mu}^n$ are locally bounded in $C^{\beta/2,\beta}$ for some $\beta \in (0, 1)$. Therefore, up to a subsequence, denoted in the same way, $(\tilde{v}^n)$ converges to $\tilde{v}$ and $(\tilde{\mu}^n)$ converges to $\tilde{\mu}$ locally uniformly, where by linearity $(\tilde{v}, \tilde{\mu})$ solves

$$\begin{cases} -\partial_t \tilde{v} - \Delta \tilde{v} + H_p(x, D\bar{u}^\delta) \cdot D\tilde{v} = \dfrac{\delta F}{\delta m}(x, \bar{m}^\delta)(\tilde{\mu}(t)) & \text{in } (-\infty, 0) \times \mathbb{T}^d, \\ \partial_t \tilde{\mu} - \Delta \tilde{\mu} - \operatorname{div}(\tilde{\mu} H_p(x, D\bar{u}^\delta)) - \operatorname{div}(\bar{m}^\delta H_{pp}(x, D\bar{u}^\delta) D\tilde{v}) = 0 & \text{in } (-\infty, 0) \times \mathbb{T}^d. \end{cases}$$

For any $s \le 0$ and any $\tau \ge 0$, we have, for n large enough,

$$\int_{\mathbb{T}^d} \tilde{\mu}^n(s) \tilde{v}^n(s) = e^{-\delta_n t_n} \int_{\mathbb{T}^d} \mu^n(t_n + s) v^n(t_n + s) \le e^{\delta_n s} \rho^{\delta_n}(t_n + s) \le e^{\delta_n s} \rho^{\delta_n}(\tau),$$

so

$$\int_{\mathbb{T}^d} \tilde{\mu}(s) \tilde{v}(s) \le \rho(\tau).$$

Letting $\tau \to +\infty$, we find therefore

$$\int_{\mathbb{T}^d} \tilde\mu(s)\tilde v(s) \le \rho_\infty = \int_{\mathbb{T}^d} \tilde\mu(0)\tilde v(0) \quad \text{for all } s \le 0.$$

However $\int_{\mathbb{T}^d} \tilde\mu(s)\tilde v(s)$ is nonincreasing, so we also have the reverse inequality, and we deduce that this quantity must be constant in $(-\infty, 0]$. The duality relation (consisting as usual in multiplying the equation for $\tilde v$ by $\tilde\mu$ and the equation for $\tilde\mu$ by $\tilde v$, integrating in time and space and adding the resulting quantities) then implies $D\tilde v = 0$ for any $t \le 0$, which gives $\rho_\infty = 0$.

Next we claim that there exist $\gamma > 0$, $C > 0$ and $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, one has

$$\rho^\delta(t) \le C e^{-\gamma t} \quad \text{for all } t \ge 0. \tag{63}$$

Indeed, let $\varepsilon > 0$ small to be chosen later and let $T_0 > 0$, $\delta_0 > 0$ be such that

$$\rho^\delta(t) \le \varepsilon \quad \text{for all } t \ge T_0,\ \delta \in (0, \delta_0). \tag{64}$$

Fix $\delta \in (0, \delta_0)$ and let (v, μ) be a solution to (60). Inequalities (61) (combined with the fact that $\int_{\mathbb{T}^d} v\mu$ is nonnegative) and (64) imply

$$\int_{t_1}^{t_2} e^{-\delta s}\|Dv(s)\|_{L^2}^2\, ds \le C\varepsilon\|\mu_0\|_{L^2}^2 \quad \text{for all } t_1, t_2 \ge T_0,\ \delta \in (0, \delta_0).$$

Revisiting the estimate for μ, we have, for any $t_1 \ge 0$,

$$\|\mu(T_0+t_1)\|_{L^2} \le Ce^{-\omega t_1}\|\mu(T_0)\|_{L^2} + C\left[\int_{T_0}^{T_0+t_1} \|Dv(s)\|_{L^2}^2\, ds\right]^{1/2},$$

so, using Lemma 3.3 and the above estimate on Dv,

$$\begin{aligned}\|\mu(T_0+t_1)\|_{L^2} &\le Ce^{-\omega t_1+\delta T_0/2}\|\mu_0\|_{L^2} + Ce^{\delta(T_0+t_1)/2}\left[\int_{T_0}^{T_0+t_1} e^{-\delta s}\|Dv(s)\|_{L^2}^2\, ds\right]^{1/2} \\ &\le C\|\mu_0\|_{L^2}e^{\delta(T_0+t_1)/2}(e^{-(\omega+\delta/2)t_1} + \varepsilon^{1/2}).\end{aligned}$$

We choose t_1 large enough (independently of ε and $\delta \in (0, \omega)$) so that $Ce^{-\omega t_1} \le \frac{1}{4}$ and ε so small that $C\varepsilon^{1/2} \le \frac{1}{4}$. Setting $\tau := T_0 + t_1$, this yields

$$\|\mu(\tau)\|_{L^2} \le \tfrac{1}{2}\|\mu_0\|_{L^2}e^{\delta\tau/2}. \tag{65}$$

Fix (v, μ) a solution to (60). The pair $(\tilde v, \tilde\mu) := (v(\tau + \cdot), \mu(\tau + \cdot))$ is also a solution of (60) with initial condition $\tilde\mu(0) = \mu(\tau)$. Since the equation is linear in μ_0 and the quantity $\int_{\mathbb{T}^d} \mu(t)v(t)$is homogeneous of degree 2, we have therefore

$$e^{-\delta t}\int_{\mathbb{T}^d} \tilde\mu(t)\tilde v(t) \le \|\mu(\tau)\|_{L^2}^2\rho^\delta(t) \quad \text{for all } t \ge 0,$$

where

$$e^{-\delta t}\int_{\mathbb{T}^d} \tilde\mu(t)\tilde v(t) = e^{\delta\tau}e^{-\delta(t+\tau)}\int_{\mathbb{T}^d} \mu(t+\tau)v(t+\tau).$$

This implies

$$e^{-\delta(t+\tau)}\int_{\mathbb{T}^d} \mu(t+\tau)v(t+\tau) \le e^{-\delta\tau}\|\mu(\tau)\|_{L^2}^2 \rho^\delta(t).$$

Recalling estimate (65) and taking the supremum over $\|\mu_0\|_{L^2} \le 1$, we find

$$\rho^\delta(t+\tau) \le \tfrac{1}{2}\rho^\delta(t) \quad \text{for all } t \ge 0.$$

This easily implies (63).

We can now come back to the estimates of μ and v for a given solution (v, μ) of (60) with $\delta \in (0, \delta_0)$. For $t > 0$, we have, using Lemma 3.3, (61) and (63) successively,

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le Ce^{-\omega t/2}\|\mu(t/2)\|_{L^2} + C\left[\int_{t/2}^t \|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{-\omega t/2+\delta t/2}\|\mu_0\|_{L^2} + Ce^{\delta t/2}\left[\int_{t/2}^t e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le C\|\mu_0\|_{L^2}(e^{-\omega t/2+\delta t/2} + e^{\delta t/2-\gamma t/4}).\end{aligned}$$

For δ small enough, this implies

$$\|\mu(t)\|_{L^2} \le C\|\mu_0\|_{L^2}e^{-\lambda t} \quad \text{for all } t \ge 0,$$

for some $\lambda \in (0, \omega)$. Thus, by Lemma 3.3 applied on the time-interval $[t/2, +\infty)$,

$$\|Dv(t)\|_{L^2} \le C\|\mu(t/2)\|_{L^2}e^{\delta t/4} \le C\|\mu_0\|_{L^2}e^{-\lambda t}$$

for some possibly different $\lambda > 0$. The bound on $\|v\|_\infty$ follows directly from the equation for v and our regularity assumption on $\delta F/\delta m$, which implies

$$\left\|\frac{\delta F}{\delta m}(x, m^\delta)(\mu(t))\right\|_\infty \le C\|\mu(t)\|_{L^2} \le C\|\mu_0\|_{L^2}e^{-\lambda t} \quad \text{for all } t \ge 0. \qquad \square$$

In the next step we study a perturbed discounted linearized problem.

Proposition 3.5. *Let* (v, μ) *solve*

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, D\bar{u}^\delta)\cdot Dv = \dfrac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu(t)) + A(t,x) & \text{in } (0,+\infty)\times\mathbb{T}^d,\\ \partial_t\mu - \Delta\mu - \operatorname{div}(\mu H_p(x, D\bar{u}^\delta)) - \operatorname{div}(\bar{m}^\delta H_{pp}(x, D\bar{u}^\delta)Dv) = \operatorname{div}(B(t,x)) & \text{in } (0,+\infty)\times\mathbb{T}^d,\\ \mu(0,\cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded}, \end{cases} \tag{66}$$

with $\int_{\mathbb{T}^d}\mu_0 = 0$, $\|\mu_0\|_{L^2} \le 1$ *and assume that, for some* $\gamma > 0$,

$$\|A(t)\|_{L^2} + \|B(t)\|_{L^2} \le e^{-\gamma t} \quad \textit{for all } t \ge 0. \tag{67}$$

If $\delta \in (0, \delta_0)$, *then*

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C(1+t)e^{-\theta t}, \tag{68}$$

where $\theta := \gamma \wedge \lambda$ *and* $\delta_0, \lambda > 0$ *are defined in Proposition 3.4.*

Proof. Using Proposition 3.4 and the linearity of the equation, we can assume, without loss of generality, that $\mu_0 = 0$. We first assume that $A \equiv 0$. Throughout the proof, the constant C can depend on γ.

Let us start with preliminary estimates. The duality identity (i.e., the equality obtained by multiplying the equation for $e^{-\delta t}v$ by μ and the equation for μ by $e^{-\delta t}v$, integrating in time and space and adding the resulting quantities) here implies

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds \le -\left[e^{-\delta s}\int_{\mathbb{T}^d} v(s)\mu(s)\right]_{t_1}^{t_2} + C\int_{t_1}^{t_2} e^{-\delta s}\|B(s)\|_{L^2}^2\,ds. \tag{69}$$

One can check, exactly as for the proof of Lemma 3.3, that

$$\lim_{t\to+\infty} e^{-\delta t}\int_{\mathbb{T}^d}\mu(t)v(t) = 0.$$

Then the inequality (69) and our assumption (67) on B imply

$$\int_0^{+\infty} e^{-\delta s}\|Dv(s)\|_{L^2}^2\,ds \le C.$$

Arguing as before, we derive for μ that

$$\begin{aligned}\|\mu(t)\|_{L^2} &\le C\left[\int_0^t \|Dv(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2\,ds\right]^{1/2}\\ &\le Ce^{\delta t/2}\left[\int_0^t e^{-\delta s}(\|Dv(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2)\,ds\right]^{1/2} \le Ce^{\delta t/2}.\end{aligned}$$

Thus, applying Lemma 1.2 (with $T\to\infty$) to $e^{-\delta t}v$, we deduce

$$e^{-\delta t}\|v(t) - \langle v(t)\rangle\|_{L^2} \le C\int_t^{+\infty} e^{-\omega(s-t)}\|\mu(s)\|_{L^2}e^{-\delta s}\,ds \le Ce^{-\delta t/2},$$

which gives

$$\|v(t) - \langle v(t)\rangle\|_{L^2} \le Ce^{\delta t/2}.$$

We set

$$\rho^\delta(t) = \sup_B[e^{-\delta t}(\|\mu(t)\|_{L^2} + \|v(t) - \langle v(t)\rangle\|_{L^2})],$$

where the supremum is taken over the B that satisfy (67) and where (v,μ) solves (66) (with $A\equiv 0$ and $\mu_0 = 0$). Fix a solution (v,μ) to (66) with $A\equiv 0$ and $\mu_0 = 0$ and let us consider its restriction to a time interval $[\tau,+\infty)$. We can write

$$(v,\mu) = (v_1,\mu_1) + (v_2,\mu_2),$$

where (v_1,μ_1) solves on $[\tau,+\infty)$ the homogeneous equation (60) with initial condition $\mu_1(\tau) = \mu(\tau)$ and (v_2,μ_2) solves on $[\tau,+\infty)$ the inhomogeneous equation (66) with $\mu_2(\tau) = 0$ and $A\equiv 0$. By Proposition 3.4 we have, for $\delta\in(0,\delta_0)$,

$$\|\mu_1(\tau+t)\|_{L^2} + \|Dv_1(\tau+t)\|_{L^2} \le C_0e^{-\lambda t}\|\mu(\tau)\|_{L^2} \le C_0e^{-\lambda t}e^{\delta\tau/2} \quad \text{for all } t\ge 0,$$

while, as the restriction of B to $[\tau, +\infty)$ satisfies

$$\|B(\tau+t)\|_{L^2} \le e^{-\gamma\tau}e^{-\gamma t} \quad \text{for all } t \ge 0,$$

we have

$$\|\mu_2(\tau+t)\|_{L^2} + \|v_2(\tau+t) - \langle v_2(\tau+t)\rangle\|_{L^2} \le e^{-\gamma\tau}\rho^\delta(t)e^{\delta t} \quad \text{for all } t \ge 0.$$

So

$$\|\mu(\tau+t)\|_{L^2} + \|v(\tau+t) - \langle v(\tau+t)\rangle\|_{L^2} \le Ce^{-\lambda t}e^{\delta\tau/2} + e^{-\gamma\tau}\rho^\delta(t)e^{\delta t}.$$

Multiplying by $e^{-\delta(t+\tau)}$ and taking the supremum over B leads to

$$\rho^\delta(\tau+t) \le Ce^{-(\lambda+\delta)t} + e^{-(\gamma+\delta)\tau}\rho^\delta(t).$$

Setting $\theta := \gamma \wedge \lambda$ and considering the inequality satisfied by $e^{(\theta+\delta)t}\rho^\delta(t)$, we then obtain the exponential decay of ρ^δ

$$\rho^\delta(t) \le C(1+t)e^{-(\theta+\delta)t},$$

which implies, by the definition of $\rho^\delta(t)$, that

$$\sup_B(\|\mu(t)\|_{L^2} + \|v(t) - \langle v(t)\rangle\|_{L^2}) \le C(1+t)e^{-\theta t}.$$

Once more we observe that, by Lemma 1.2, we can estimate $\|Dv(t)\|_{L^2}$ in terms of $\|\mu(t)\|_{L^2}$ and $\|v(t) - \langle v(t)\rangle\|_{L^2}$. Hence (68) is proved when $A = 0$.

It remains to consider the case where $A \not\equiv 0$. Let v_1 be the unique bounded solution to

$$-\partial_t v_1 + \delta v_1 - \Delta v_1 + H_p(x, D\bar{u}^\delta)\cdot Dv_1 = A(t,x) \quad \text{in } (0,+\infty)\times\mathbb{T}^d.$$

Using as before Lemma 1.2 for $e^{-\delta t}v_1$ and with $T \to \infty$, we estimate

$$\|v_1(t) - \langle v_1(t)\rangle\|_{L^2} \le C\int_t^\infty e^{-(\omega+\delta)(s-t)}\|A(s)\|_{L^2}\,ds \le Ce^{-\gamma t}.$$

Finally, using again Lemma 1.2 gives

$$\|Dv_1(t)\|_{L^2} \le Ce^{-\gamma t}.$$

Note that, if (v, μ) is the solution to (66), then $(v - v_1, \mu)$ solves (66) with $A \equiv 0$ and $B' = B + \bar{m}^\delta H_{pp} Dv_1$, so, applying the above estimate gives

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \le C(1+t)e^{-\theta t},$$

where $\theta := \gamma \wedge \lambda$. □

3C. ***Exponential rate for the nonlinear system.*** We now consider the infinite-horizon discounted nonlinear MFG system (5). Let us recall that this system is well-posed and that we have Lipschitz estimates:

Lemma 3.6. *Under our standing assumptions, for any $\delta \in (0,1)$ there exists a unique solution (u^δ, m^δ) to (5). Moreover, for any $\alpha \in (0,1)$, there exists a constant $C > 0$, independent of δ, such that*

$$\|Du^\delta\|_{C^{(1+\alpha)/2,1+\alpha}} + \sup_{t\in[1,\infty)}\|m^\delta(t)\|_\infty \le C.$$

Proof. Existence and uniqueness of the solution rely on standard arguments, discussed for instance in [Lions 2010]. In particular, the unique solution can be obtained as limit of solutions in horizons $T_n \to \infty$ with the terminal condition $u(T_n) = 0$; this way one can prove, exactly as in Lemma 1.5, that Du^δ is uniformly bounded, and one also has a uniform bound for $\|\delta u^\delta\|_\infty$. As a consequence, m^δ is uniformly bounded in $[1, +\infty)$ thanks to Lemma 1.1 and is (uniformly) Hölder continuous in time with values in $\mathcal{P}(\mathbb{T}^d)$; see estimate (17). Finally, by considering the equation of $(u^\delta)_{x_i}$, namely

$$-\partial_t (u^\delta)_{x_i} + \delta (u^\delta)_{x_i} - \Delta (u^\delta)_{x_i} + H_{x_i} + H_p \cdot D(u^\delta)_{x_i} = F_{x_i},$$

the parabolic regularity applied in any interval $(t, t+1)$, jointly with the uniform bound already found for $\|(u^\delta)_{x_i}\|_\infty$, implies the desired estimate upon Du^δ. More precisely, by only using that $F_x(x, m)$ is uniformly bounded, and the bound on H_x and H_p, we deduce a bound for $(u^\delta)_{x_i}$ in $C^{(1+\alpha)/2, 1+\alpha}$ for any $\alpha \in (0, 1)$. □

The main result of this part is the following exponential convergence of the discounted problem.

Theorem 3.7. *Let (u^δ, m^δ) be the solution to the discounted MFG system* (5). *There exist $\gamma, \delta_0 > 0$ and $C > 0$ such that, if $\delta \in (0, \delta_0)$, then*

$$\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^\infty} \le Ce^{-\gamma t} \quad \textit{for all } t \ge 0, \tag{70}$$

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^\infty} \le Ce^{-\gamma t} \quad \textit{for all } t \ge 1. \tag{71}$$

Proof. The proof is very close to the proof of Theorem 2.6. Let

$$E := \{(v, \mu),\ \|Dv(t)\|_{L^\infty} + \|\mu(t)\|_{L^\infty} \le \widehat{K} e^{-\gamma t}\},$$

where $\widehat{K} > 0$ and $\gamma > 0$ are to be chosen below. We assume that $\widehat{K}$ is small enough so that

$$\bar{m}^\delta > \widehat{K} \quad \text{in } \mathbb{T}^d.$$

We also assume that the initial condition is close to $\bar{m}^\delta$, namely $\mu_0 := m_0 - \bar{m}^\delta$ satisfies

$$\|\mu_0\|_{L^\infty} \le \widehat{K}^2.$$

We consider the solution $(\tilde{v}, \tilde{\mu})$ to (66) with initial condition $\tilde{\mu}(0) = \mu_0$,

$$A(t,x) = -H(x, D(\bar{u}^\delta + v)) + H(x, D\bar{u}^\delta) + H_p(x, D\bar{u}^\delta)\cdot Dv + F(x, \bar{m}^\delta + \mu) - F(x, \bar{m}^\delta) - \frac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu),$$

$$B(t,x) = (\bar{m}^\delta + \mu) H_p(x, D(\bar{u}^\delta + v)) - \bar{m}^\delta H_p(x, D\bar{u}^\delta) - \mu H_p(x, D\bar{u}^\delta) - \bar{m}^\delta H_{pp}(x, D\bar{u}^\delta) Dv.$$

We note that

$$\|A(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} \le C\widehat{K}^2 e^{-2\gamma t}.$$

From Proposition 3.5 we have

$$\|\tilde{\mu}(t)\|_{L^2} + \|D\tilde{v}(t)\|_{L^2} \le C\widehat{K}^2 (1+t) e^{-\theta t},$$

where $\theta := 2\gamma \wedge \lambda$. Using the smoothing properties of $\delta F/\delta m$ and the parabolic regularity of the equation satisfied by $\tilde{v} - \langle \tilde{v} \rangle$, exactly as in Theorem 2.6 we can upgrade the above estimate to

$$\|\tilde{\mu}(t)\|_\infty + \|D\tilde{v}(t)\|_\infty \le C\widehat{K}^2(1+t)e^{-\theta t}.$$

So if one chooses $\gamma \in (0, \lambda)$, we infer that

$$\|\tilde{\mu}(t)\|_{L^\infty} + \|D\tilde{v}(t)\|_{L^\infty} \le C\widehat{K}^2 e^{-\gamma t}.$$

For $\widehat{K}$ small enough, this implies that $(\tilde{v}, \tilde{\mu})$ belongs to E. Note that $\tilde{v}$, $D\tilde{v}$ and $\tilde{\mu}$ are bounded in $C^{\alpha/2,\alpha}$ because they solve parabolic equations with bounded coefficients. So the map $(v, \mu) \to (\tilde{v}, \tilde{\mu})$ is compact (say in $W^{1,\infty} \times L^\infty$) and thus has a fixed point (v^δ, μ^δ). Then $(u^\delta, m^\delta) := (\bar{u}^\delta, \bar{m}^\delta) + (v^\delta, \mu^\delta)$ is a solution to (5) which satisfies the decay

$$\|m^\delta(t) - \bar{m}^\delta\|_\infty + \|D(u^\delta(t) - \bar{u}^\delta)\|_\infty \le Ce^{-\gamma t} \quad \text{for all } t \ge 0.$$

It remains to remove the assumption on the initial condition m_0. For this we only need to show that there exists a time $T > 0$ such that, for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, the solution (u^δ, m^δ) of (5) satisfies $\|m^\delta(T) - \bar{m}^\delta\|_\infty \le \widehat{K}^2$. Indeed, we can then apply the previous result to the restriction of (u^δ, m^δ) to the time interval $[T, +\infty)$.

By the duality relation (consisting here in multiplying the equation for $u^\delta - \bar{u}^\delta$ by $m^\delta - \bar{m}^\delta$ and the equation for $m^\delta - \bar{m}^\delta$ by $u^\delta - \bar{u}^\delta$, integrating in time and space and adding the resulting quantities), we have

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta t}\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^2}^2\, dt \le -\left[e^{-\delta t}\int_{\mathbb{T}^d}(u^\delta(t) - \bar{u}^\delta)(m^\delta(t) - \bar{m}^\delta)\right]_{t_1}^{t_2}. \tag{72}$$

Thus

$$C^{-1}\int_0^{+\infty} e^{-\delta t}\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^2}^2\, dt \le \int_{\mathbb{T}^d}(u^\delta(0) - \bar{u}^\delta)(m_0 - \bar{m}^\delta) \le C \tag{73}$$

because u^δ is uniformly Lipschitz continuous in space (see Lemma 3.6). As $\mu^\delta := m^\delta - \bar{m}^\delta$ satisfies

$$\partial_t \mu^\delta - \Delta\mu^\delta - \operatorname{div}(\mu^\delta H_p(x, Du^\delta)) = \operatorname{div}(\bar{m}^\delta(H_p(x, D\bar{u}^\delta) - H_p(x, Du^\delta))),$$

and still using the fact that Du^δ is bounded, Lemma 1.1 implies that, for any $t \ge 1$,

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^2} \le Ce^{-\omega(t-1)}\|m^\delta(1) - \bar{m}^\delta\|_{L^2} + Ce^{\delta t/2}\left[\int_1^t e^{-\delta s}\|D(u^\delta(s) - \bar{u}^\delta)\|_{L^2}^2\, ds\right]^{1/2}.$$

Recalling that m^δ is bounded in L^∞ (Lemma 1.1), we find

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^2} \le Ce^{\delta t/2} \quad \text{for all } t \ge 1.$$

Let $T \ge 2$ to be chosen below. Coming back to (73), there exist $t_1 \in [1, T]$ and $t_2 \in [3T+1, 4T]$ such that

$$e^{-\delta t_i}\|D(u^\delta(t_i) - \bar{u}^\delta)\|_{L^2}^2 \le \frac{C}{T}.$$

Then from (72) we deduce

$$C^{-1}\int_{t_1}^{t_2} e^{-\delta t}\|D(u^\delta(t)-\bar u^\delta)\|_{L^2}^2\,dt$$
$$\le e^{-\delta t_1}\|D(u^\delta(t_1)-\bar u^\delta)\|_{L^2}\|m^\delta(t_1)-\bar m^\delta\|_{L^2}+e^{-\delta t_2}\|D(u^\delta(t_2)-\bar u^\delta)\|_{L^2}\|m^\delta(t_2)-\bar m^\delta\|_{L^2}\le CT^{-1/2}.$$

Then, as $t_1\le T\le 3T+1\le t_2\le 4T$, we have, for any $t\in[2T,t_2]$,

$$\begin{aligned}\|m^\delta(t)-\bar m^\delta\|_{L^2}&\le Ce^{-\omega(2T-t_1)}\|m^\delta(t_1)-\bar m^\delta\|_{L^2}+Ce^{\delta t_2/2}\left[\int_{t_1}^{t_2} e^{-\delta t}\|D(u^\delta(t)-\bar u^\delta)\|_{L^2}^2\,dt\right]^{1/2}\\ &\le Ce^{-\omega T}e^{\delta T/2}+Ce^{2\delta T}T^{-1/4}.\end{aligned}\tag{74}$$

Notice that, by choosing T large, and then δ small, the above inequality implies that $m^\delta(t)-\bar m^\delta$ is sufficiently small for any $t\in[2T,3T]$. In order to conclude, we only need to upgrade this estimate to the L^∞ norm.

To this end, recall that $w^\delta:=u^\delta-\bar u^\delta$ solves the equation

$$-\partial_t w^\delta+\delta w^\delta-\Delta w^\delta+V^\delta\cdot Dw^\delta=F(x,m^\delta(t))-F(x,\bar m^\delta),$$

where $V^\delta=\int_0^1 H_p(x,D\bar u^\delta+sD(u^\delta-\bar u^\delta))\,ds$ is uniformly bounded. Since we have, by Poincaré's inequality,

$$e^{-\delta t_2}\|w^\delta(t_2)-\langle w^\delta(t_2)\rangle\|_{L^2}^2\le Ce^{-\delta t_2}\|Dw^\delta(t_2)\|_{L^2}^2\le\frac{C}{T},$$

applying Lemma 1.2 to $e^{-\delta t}w^\delta$ we deduce that, for $t\in[2T,2T+2]$,

$$\begin{aligned}\|w^\delta(t)-\langle w^\delta(t)\rangle\|_{L^2}&\le Ce^{-\omega(t_2-t)}\|w^\delta(t_2)-\langle w^\delta(t_2)\rangle\|_{L^2}e^{\delta(t-t_2)}+C\int_t^{t_2} e^{-\omega(s-t)}\|m^\delta(s)-\bar m^\delta\|_{L^2}e^{\delta(t-s)}\,ds\\ &\le Ce^{-\omega(t_2-t)}\frac{e^{\delta(t-t_2/2)}}{T^{1/2}}+C(e^{-\omega T}e^{\delta T/2}+e^{2\delta T}T^{-1/4})\int_t^{t_2} e^{-\omega(s-t)}e^{\delta(t-s)}\,ds,\end{aligned}$$

where we also used (74). Recalling that $t\in[2T,2T+2]$ and $t_2\in[3T+1,4T]$, we have $t-t_2/2\ge 0$, so if δ is small enough compared to ω we conclude that

$$\|w^\delta(t)-\langle w^\delta(t)\rangle\|_{L^2}\le C(e^{-\omega T/2}+e^{2\delta T}T^{-1/4}).$$

We apply once more Lemma 1.2 to estimate $Dw^\delta(t)$ in $(2T,2T+1)$: we deduce that

$$\|D(u^\delta(t)-\bar u^\delta)\|_{L^2}\le C(e^{-\omega T/2}+e^{2\delta T}T^{-1/4})$$

for every $t\in(2T,2T+1)$. In fact, since $D(u^\delta(t)-\bar u^\delta)$ is bounded, a similar estimate actually holds in L^p for all $p<\infty$:

$$\|D(u^\delta(t)-\bar u^\delta)\|_{L^p}\le C(e^{-\omega T/p}+e^{4\delta T/p}T^{-1/(2p)}).$$

Recalling the estimate (74), by parabolic regularity used for the equation of μ^δ in the interval $(2T,2T+1)$, we conclude that the L^∞ norm of μ^δ satisfies a similar estimate for, say, $t\in\left(2T+\frac12,2T+1\right)$. In particular, we can fix T large and $\delta_0>0$ small such that in this interval we have $\|m^\delta(t)-\bar m^\delta\|_{L^\infty}\le\widehat K^2$

for any $\delta \in (0, \delta_0)$. We notice that the choice of T (and so δ_0) only depends on $\widehat{K}$, which is only dependent on the data. This means that the estimates (70) and (71) have been proved to hold for $t \geq T_{\widehat{K}}$, for some $T_{\widehat{K}}$ only depending on the data. On the other hand, the global gradient bound implies

$$\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^\infty} \leq \widehat{C} e^{-\gamma T_{\widehat{K}}}$$

for some constant $\widehat{C} > 0$ and for every $t \in [0, T_{\widehat{K}}]$ and a similar estimate holds for $\|m^\delta(t) - \bar{m}^\delta\|_{L^\infty}$ for $t \in [1, T_{\widehat{K}}]$. Hence (70) and (71) are proved in the whole time range. □

Let us underline the following consequence of our estimates on the solution to the linearized system

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, Du^\delta) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m^\delta(t))(\mu(t)) \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, Du^\delta)) - \operatorname{div}(m^\delta H_{pp}(x, Du^\delta) Dv) = 0 \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded.} \end{cases} \tag{75}$$

Notice that the system has been now linearized around the pair (u^δ, m^δ) which solves the discounted MFG system (5).

Corollary 3.8. *There exist $\theta, \delta_0 > 0$ and a constant $C > 0$ such that, if $\delta \in (0, \delta_0)$, then the solution (v, μ) to* (75) *with $\int_{\mathbb{T}^d} \mu_0 = 0$ satisfies*

$$\|Dv(t)\|_{L^2} \leq C e^{-\theta t} \|\mu_0\|_{L^2} \quad \textit{for all } t \geq 0,$$
$$\|\mu(t)\|_{L^2} \leq C e^{-\theta t} \|\mu_0\|_{L^2} \quad \textit{for all } t \geq 1.$$

In addition, for any $\alpha \in (0, 1)$, there is a constant C (independent of $\delta \in (0, \delta_0)$) such that

$$\sup_{t \geq 0} \|v(t)\|_{C^{2+\alpha}} \leq C \|\mu_0\|_{(C^{2+\alpha})'}.$$

Proof. As in the proof of Lemma 3.3, we have a preliminary estimate:

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C_0 \|\mu_0\|_{L^2} e^{\delta t/2}.$$

We rewrite system (75) in the form (66) with

$$A(t, x) := -(H_p(x, Du^\delta) - H_p(x, D\bar{u}^\delta)) \cdot Dv + \frac{\delta F}{\delta m}(x, m^\delta(t))(\mu(t)) - \frac{\delta F}{\delta m}(x, \bar{m}^\delta)(\mu(t)),$$
$$B(t, x) := -\mu(H_p(x, Du^\delta) - H_p(x, D\bar{u}^\delta)) - (m^\delta H_{pp}(x, Du^\delta) - \bar{m}^\delta H_{pp}(x, D\bar{u}^\delta)) Dv.$$

From Theorem 3.7, we have, for δ small enough,

$$\|A(t)\|_{L^2} \leq C e^{-\gamma t}(\|Dv\|_{L^2} + \|\mu(t)\|_{L^2}) \leq C e^{-(\gamma - \delta)t} \|\mu_0\|_{L^2} \leq C e^{-\gamma t/2} \|\mu_0\|_{L^2}.$$

In the same way,

$$\|B(t)\|_{L^2} \leq C e^{-\gamma t/2} \|\mu_0\|_{L^2}.$$

Then Proposition 3.5 implies

$$\|\mu(t)\|_{L^2} + \|Dv(t)\|_{L^2} \leq C(1 + t) e^{-\gamma t/2} \|\mu_0\|_{L^2}.$$

The above estimates combined with the maximum principle imply that v is bounded in L^∞ by

$$\sup_{t\in[0,T]} \|v(t)\|_\infty \le C\|\mu_0\|_{L^2}.$$

In order to change the left-hand side $\|v(t)\|_\infty$ into $\|v(t)\|_{C^{2+\alpha}}$ and the right-hand side $\|\mu_0\|_{L^2}$ into $\|\mu_0\|_{(C^{2+\alpha})'}$, one can proceed as in Corollary 2.8. □

4. The master cell problem

In this section we study the master cell problem:

$$\lambda - \Delta_x \chi(x,m) + H(x, D_x\chi(x,m)) - \int_{\mathbb{T}^d} \mathrm{div}_y(D_m\chi(x,m,y))\, dm(y) \\ + \int_{\mathbb{T}^d} D_m\chi(x,m,y)\cdot H_p(y, D_x\chi(y,m))\, dm(y) = F(x,m) \quad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \quad (76)$$

We prove that this equation is well-defined in a suitable sense: there is a unique constant $\bar\lambda$ for which the master cell problem has a "weak" solution in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$. Moreover we prove that $\bar\lambda$ is also the unique constant for which the ergodic mean field game system (2) has a solution $(\bar\lambda, \bar u, \bar m)$.

Let us stress that a weak solution of (76), according to our next definition, is not necessarily C^1 with respect to m, so (76) is not formulated classically. Instead, the equation is interpreted as is often done with transport equations, by requiring somehow that the value of the solution is obtained through the characteristic curves. By considering weak solutions, we avoid some lengthy and involved estimates which are needed to achieve the C^1 character with respect to m. The reader is referred to [Cardaliaguet et al. 2019] for this issue. For our purposes, the context of weak solutions is enough to characterize the ergodic limit.

Definition 4.1. We say that the pair (λ, χ), with $\lambda \in \mathbb{R}$ and $\chi : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ a map, is a weak solution to the master cell problem (76) if χ and $D_x\chi$ are globally Lipschitz continuous in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$ and if χ satisfies the two conditions

(i) χ is monotone, i.e.,

$$\int_{\mathbb{T}^d} (\chi(x,m) - \chi(x,m'))\, d(m-m')(x) \ge 0 \quad \text{for all } m, m' \in \mathcal{P}(\mathbb{T}^d),$$

(ii) for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, and any $T > 0$, whenever we consider the unique solution (u, m) to

$$\begin{cases} -\partial_t u + \lambda - \Delta u + H(x, Du) = F(x,m) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \mathrm{div}(mH_p(x,Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = m_0, \quad u(T,\cdot) = \chi(x, m(T)) & \text{in } \mathbb{T}^d, \end{cases} \quad (77)$$

then we have $\chi(x, m_0) = u(0,x)$ for any $x \in \mathbb{T}^d$.

Let us make some comments about the above definition. Firstly, the monotonicity condition on χ ensures the uniqueness of the solution (u, m) to (77). Secondly, if $\chi = \chi(x,m)$ is a weak solution, then χ is actually C^2 in the space variable x because so is the solution u of (77) at time $t = 0$. Thirdly, condition (ii) implies that in (77) one actually has $\chi(x, m(t)) = u(t, x)$ for any $(t, x) \in [0, T] \times \mathbb{T}^d$, so m solves the McKean–Vlasov equation

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D\chi(x, m(t)))) = 0, \quad m(0, \cdot) = m_0. \tag{78}$$

The Lipschitz continuity of $D_x\chi$ ensures that this equation has a unique solution.

Theorem 4.2. *There is a unique constant $\bar\lambda \in \mathbb{R}$ for which the master cell problem* (76) *has a weak solution. The constant $\bar\lambda$ is also the unique constant for which the ergodic MFG problem* (2) *has a solution. Besides, if χ is a solution to* (76), *then $\chi(\cdot, m)$ is of class C^2 for any $m \in \mathcal{P}(\mathbb{T}^d)$ and*

$$D_x\chi(x, \bar m) = D\bar u(x) \quad \textit{for all } x \in \mathbb{T}^d,$$

where $(\bar u, \bar m)$ is a solution to (2).

The proof requires several steps. As usual, we build the solution through the discounted problem, for which we have to show uniform regularity estimates (independent of the discount factor).

4A. ***Estimates for the discounted master equation.*** In order to build a solution to the cell problem, we consider, for $\delta > 0$, the discounted master equation (6). Let us recall, see [Cardaliaguet et al. 2019], that U^δ can be built as follows: for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let (u^δ, m^δ) be the solution to (5). Then

$$U^\delta(x, m_0) = u^\delta(0, x). \tag{79}$$

The next lemma collects standard estimates on U^δ.

Lemma 4.3. *Let U^δ be the solution to* (6). *Then, for any $\alpha \in (0, 1)$, there is a constant C, independent of m_0 and δ, such that*

$$\|\delta U^\delta(\cdot, m)\|_\infty + \|D_x U^\delta(\cdot, m)\|_{C^{1+\alpha}} \le C \quad \textit{for all } m \in \mathcal{P}(\mathbb{T}^d).$$

Proof. Let (u^δ, m^δ) be a solution to (5). As u^δ is a bounded solution to the first equation in (5), it is well known that

$$\sup_{(t,x)\in[0,+\infty)\times\mathbb{T}^d} |\delta u^\delta(t, x)| \le \sup_{x\in\mathbb{T}^d} |H(x, 0)| + \sup_{(x,m)\in\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)} |F(x, m)|.$$

This yields the uniform estimate on $\|\delta U^\delta\|_\infty$. From Lemma 3.6, we know that Du^δ is bounded in $C^{(1+\alpha)/2,1+\alpha}$ for any $\alpha \in (0, 1)$; this implies the same bound on $D_x U^\delta$. □

The next result states that U^δ is uniformly Lipschitz continuous with respect to m.

Proposition 4.4. *Let U^δ be the solution to* (6). *Then, for any $\alpha \in (0, 1)$, there exists a constant C, depending on α and on the data only, such that*

$$\|D_m U^\delta(\cdot, m, \cdot)\|_{2+\alpha,1+\alpha} \le C. \tag{80}$$

In particular, $U^\delta(\cdot, \cdot)$ and $D_x U^\delta(\cdot, \cdot)$ are uniformly Lipschitz continuous.

Proof. Let us fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$, and let (u^δ, m^δ) be the solution to (5). We use the following representation formula, see [Cardaliaguet et al. 2019]: for any smooth map μ_0, we have

$$\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(x, m_0, y)\mu_0(y)\, dy = v(0, x), \tag{81}$$

where (v, μ) is the unique solution to the linearized system

$$\begin{cases} -\partial_t v + \delta v - \Delta v + H_p(x, Du^\delta) \cdot Dv = \dfrac{\delta F}{\delta m}(x, m^\delta(t))(\mu(t)) & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \partial_t \mu - \Delta\mu - \operatorname{div}(\mu H_p(x, Du^\delta)) - \operatorname{div}(m^\delta H_{pp}(x, Du^\delta) Dv) = 0 & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu(0, \cdot) = \mu_0 \quad \text{in } \mathbb{T}^d, \qquad v \text{ bounded.} \end{cases} \tag{82}$$

If we suppose that $\int_{\mathbb{T}^d} \mu_0 = 0$, Corollary 3.8 states that

$$\sup_{t \geq 0} \|v(t)\|_{C^{2+\alpha}} \leq C\|\mu_0\|_{(C^{2+\alpha})'}$$

for any $\alpha > 0$. By (81) and

$$D_y \frac{\delta U^\delta}{\delta m}(x, m_0, y) = D_m U^\delta(x, m_0, y),$$

we infer exactly as in [Cardaliaguet et al. 2019] that

$$\|D_m U^\delta(\cdot, m_0, \cdot)\|_{2+\alpha, 1+\alpha} \leq C. \qquad \square$$

Remark 4.5. We stress that the uniform Lipschitz continuity of $U^\delta(\cdot, \cdot)$ and $D_x U^\delta(\cdot, \cdot)$ would require milder assumptions than those needed to prove (80). Indeed, by only using condition (FGb) on the couplings, we can replace the conclusion of Corollary 3.8 with the estimate

$$\sup_{t \geq 0} \|v(t)\|_{C^1} \leq C\|\mu_0\|_{(C^1)'},$$

which would follow as explained in Remark 2.10. With this latter estimate in hand, using (81) with $\mu_0 = D_y\psi(y)$ (for ψ smooth), it follows that

$$\int_{\mathbb{T}^d} D_y D_x \frac{\delta U^\delta}{\delta m}(x, m_0, y)\psi(y)\, dy \leq C\|\mu_0\|_{(C^1)'} \leq C\|\psi\|_{L^1},$$

which yields

$$\|D_m D_x U^\delta(x, m_0)\|_\infty \leq C.$$

Since $D^2_{xx} U^\delta(x, m)$ is estimated from Lemma 4.3, this would imply the Lipschitz uniform bound for $D_x U^\delta(\cdot, \cdot)$.

In the following, we actually only use this information in order to prove the existence of a weak solution to the master equation and the convergence of the ergodic limit.

We finally establish that U^δ is monotone:

Lemma 4.6. *For any $\delta > 0$ the map U^δ is monotone.*

Proof. Fix $m_0, m_0' \in \mathcal{P}(\mathbb{T}^d)$. Let us recall that $U^\delta(x, m_0) = u^\delta(0, x)$, where the pair (u^δ, m^δ) solves (5) with initial condition m_0. We denote by (u', m') the solution of (5) with initial condition m_0'. Then by duality (consisting here in multiplying the equation for $u^\delta - u'$ by $m^\delta - m'$ and the equation by $m^\delta - m'$ by $u^\delta - u'$, integrating in time and space and adding the resulting quantities), we have

$$\frac{d}{dt} e^{-\delta t} \int_{\mathbb{T}^d} (u^\delta(t, x) - u'(t, x))(m^\delta(t, x) - m'(t, x))\, dx \le 0,$$

where, as u^δ and u' are bounded and m^δ and m' are probability measures,

$$\lim_{t\to+\infty} e^{-\delta t} \int_{\mathbb{T}^d} (u^\delta(t, x) - u'(t, x))(m^\delta(t, x) - m'(t, x))\, dx = 0.$$

This proves that

$$\int_{\mathbb{T}^d} (U^\delta(x, m_0) - U^\delta(x, m_0'))\, d(m_0 - m_0')(x) = \int_{\mathbb{T}^d} (u^\delta(0, x) - u'(0, x))\, d(m_0 - m_0')(x) \ge 0. \qquad \square$$

4B. ***Existence of a solution for the master cell problem.***

Proof of Theorem 4.2. Let us start with the proof of the existence of the solution to the master cell problem. The proof of the uniqueness of the ergodic constant is given in Proposition 4.7 below.

For $\delta > 0$, let U^δ be the solution to the discounted master equation (6). We have seen in Lemma 4.3 and Proposition 4.4 that U^δ and $D_x U^\delta$ are uniformly Lipschitz continuous and that δU^δ is bounded. We set $W^\delta(x, m) = U^\delta(x, m) - U^\delta(0, \bar{m})$. Then W^δ is bounded and uniformly Lipschitz continuous on the compact space $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$, so it converges, up to a subsequence, to a continuous map $\chi : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. Since $D_x W^\delta$ is also bounded in Lipschitz norm, we deduce that $D_x \chi$ is Lipschitz continuous (in $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$). Moreover $(\delta U^\delta(0, \bar{m}))$ converges (along the same subsequence, without loss of generality) to some constant λ.

Next we prove that χ is a weak solution to (76). We already know that χ and $D_x \chi$ are Lipschitz continuous with respect to both variables. In addition, χ is monotone thanks to Lemma 4.6. Let $T > 0$, $m_0 \in \mathcal{P}(\mathbb{T}^d)$ with a smooth density and (w^δ, m^δ) be the solution to

$$\begin{cases} -\partial_t w^\delta + \delta w^\delta + \delta U^\delta(0, \bar{m}) - \Delta w^\delta + H(x, Dw^\delta) = F(x, m^\delta) & \text{on } (0, T) \times \mathbb{T}^d, \\ \partial_t m^\delta - \Delta m^\delta - \operatorname{div}(m^\delta H_p(x, Dw^\delta)) = 0 & \text{on } (0, T) \times \mathbb{T}^d, \\ m^\delta(0, \cdot) = m_0, \quad w^\delta(T, \cdot) = W^\delta(x, m^\delta(T)) & \text{on } \mathbb{T}^d. \end{cases}$$

By definition we have $W^\delta(x, m^\delta(T)) = U^\delta(x, m^\delta(T)) - U^\delta(0, \bar{m})$ and we know that $U^\delta(x, m^\delta(t)) = u^\delta(t, x)$ for all t, where u^δ is the solution to (5). Hence we deduce that

$$w^\delta(t, x) = u^\delta(t, x) - U^\delta(0, \bar{m}) = W^\delta(x, m^\delta(t))$$

for all $(t, x) \in (0, T) \times \mathbb{T}^d$. In particular, by Lemma 3.6, w^δ is uniformly bounded in $C^{1+\alpha/2, 2+\alpha}$ for some $\alpha \in (0, 1)$, while m^δ is uniformly bounded and uniformly continuous on $[0, T]$ with values in $\mathcal{P}(\mathbb{T}^d)$. So there exists a subsequence, still denoted for simplicity by (w^δ, m^δ), such that w^δ converges in $C^{1,2}$ to a map w and m^δ converges in $C^0([0,T],\mathcal{P}(\mathbb{T}^d))$ to a map m. The pair (w,m) is a solution to

$$\begin{cases} -\partial_t w + \lambda - \Delta w + H(x, Dw) = F(x,m) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(mH_p(x,Dw)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = m_0, \quad w(T,\cdot) = \chi(x, m(T)) & \text{in } \mathbb{T}^d. \end{cases}$$

As the solution to this equation is unique (because χ is monotone), we derive that (w,m) is the unique solution to (77). Moreover, as $w^\delta(0,x) = W^\delta(x,m_0)$, we also have at the limit $w(0,x) = \chi(x,m_0)$. This proves that χ is a weak solution to (76). □

Let us now come back to the ergodic MFG problem (2). We denote by $(\bar\lambda, \bar u, \bar m)$ the solution to this equation.

Proposition 4.7. *Let (λ,χ) be a solution of the ergodic master equation. Then we have $\lambda = \bar\lambda$ and $D_x\chi(x,\bar m) = D\bar u(x)$.*

Proof. Let us fix $T > 0$ and let (u,m) be the solution to

$$\begin{cases} -\partial_t u + \lambda - \Delta u + H(x, Du) = F(x,m) & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(mH_p(x,Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d, \\ m(0,\cdot) = \bar m, \quad u(T,\cdot) = \chi(x, m(T)) & \text{in } \mathbb{T}^d. \end{cases} \tag{83}$$

We have already noticed that m is the solution to the McKean–Vlasov equation

$$\partial_t m - \Delta m - \operatorname{div}(mH_p(x, D_x\chi(x,m(t)))) = 0, \quad m(0,\cdot) = \bar m,$$

which has a unique solution because $D_x\chi$ is Lipschitz continuous. This means that m is defined independently of the horizon T. As we know that $u(t,x) = \chi(x,m(t))$, the same holds for u. Then, from the usual energy inequality applied to $(u-\bar u, m-\bar m)$, we have, for any $0 \le t_1 \le t_2 \le T$,

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^d} \frac{m+\bar m}{2}|Du - D\bar u|^2 \le -C\left[\int_{\mathbb{T}^d}(u-\bar u)(m-\bar m)\right]_{t_1}^{t_2}. \tag{84}$$

The right-hand side is bounded because $u(t,\cdot) = \chi(\cdot, m(t))$ and $\bar u$ are bounded, so

$$\int_0^T\int_{\mathbb{T}^d} \bar m|Du - D\bar u|^2 \le C. \tag{85}$$

By Lemma 1.4 we have

$$\sup_{t\in[0,T]} \|m(t) - \bar m\|_{L^2} \le C. \tag{86}$$

As $\bar m$ is bounded below, (85) implies that there exists $t_T \in [T/2, T]$ such that $\int_{\mathbb{T}^d} |Du(t_T) - D\bar u|^2 \le 2C/T$. In particular, for T large enough, we have, by (84) applied with $t_1 = 0$ and $t_2 = t_T$,

$$\begin{aligned} \int_0^1\int_{\mathbb{T}^d} |Du - D\bar u|^2 &\le \int_0^{t_T}\int_{\mathbb{T}^d} |Du - D\bar u|^2 \le -C\int_{\mathbb{T}^d}(u(t_T)-\bar u)(m(t_T)-\bar m) \\ &\le -C\int_{\mathbb{T}^d}(u(t_T) - \bar u - \langle u(t_T) - \bar u\rangle)(m(t_T) - \bar m) \\ &\le C\|Du(t_T) - D\bar u\|_{L^2} \le CT^{-1/2}, \end{aligned}$$

by Poincaré's inequality, (86) and our choice of t_T. Letting $T \to \infty$ we can conclude that $Du = D\bar{u}$ on $[0, 1] \times \mathbb{T}^d$. Therefore, m satisfies

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D\bar{u}(x))) = 0 \quad \text{on } (0, 1) \times \mathbb{T}^d, \quad m(0, \cdot) = \bar{m}.$$

But this equation has $\bar{m}$ as a unique solution, which shows that $m(t, x) = \bar{m}(x)$ on $[0, 1] \times \mathbb{T}^d$. Since the McKean–Vlasov equation (78) is autonomous, we finally have $m(t) = \bar{m}$ and $Du(t, x) = D_x\chi(x, \bar{m}) = D\bar{u}(x)$ for any $(t, x) \in [0, T] \times \mathbb{T}^d$ and, as a consequence, $\lambda = \bar{\lambda}$. □

5. The long time behavior

We now fix a solution χ to the master cell problem and, given a terminal condition $G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ satisfying our standing assumptions (see Section 1A), we consider the solution to the backward equation

$$\begin{cases} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + H(x, D_x U(t,x,m)) \\ \quad - \displaystyle\int_{\mathbb{T}^d} \operatorname{div}(D_m U(t,x,m,y))\, dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot H_p(y, D_x U(t,y,m))\, dm(y) = F(x,m) & \text{in } (-\infty, 0) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(0,x,m) = G(x,m) & \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{cases} \tag{87}$$

We recall that the existence of a unique classical solution to (87) was proved in [Cardaliaguet et al. 2019]. Here is our main convergence result.

Theorem 5.1. *Let χ be a weak solution to the master cell problem* (76). *Then, there exists a constant $c \in \mathbb{R}$ such that*

$$\lim_{t \to -\infty} U(t, x, m) + \bar{\lambda} t = \chi(x, m) + c,$$

uniformly with respect to $(x, m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.

Moreover, we also have that $D_x U(t, x, m) \to D_x \chi(x, m)$ as $T \to \infty$, uniformly with respect to (x, m).

Theorem 5.1 implies the convergence of the solution of the MFG system as $T \to +\infty$.

Corollary 5.2. *Let c be the constant given in Theorem 5.1. For $T > 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let (u^T, m^T) be the solution to* (1). *Then, for any $t \geq 0$,*

$$\lim_{T \to +\infty} u^T(t, x) - \bar{\lambda}(T - t) = \chi(x, m(t)) + c,$$

where the convergence is uniform in x and m solves

$$\partial_t m - \Delta m - \operatorname{div}(m H_p(x, D_x \chi(x, m))) = 0, \quad m(0) = m_0. \tag{88}$$

Moreover, for any $\delta \in (0, 1)$,

$$\lim_{T \to +\infty} u^T(\delta T, x) - (1 - \delta)\bar{\lambda} T = \chi(x, \bar{m}) + c,$$

where $(\bar{u}, \bar{m})$ solves (2) *and where the convergence is uniform in x.*

In particular, when $t = 0$, we get

$$\lim_{T\to+\infty} u^T(0,x) - \bar\lambda T = \chi(x, m_0) + c.$$

Proof of Corollary 5.2. We know that $u^T(t,x) = U(t-T, x, m^T(t))$ and that m^T solves the McKean–Vlasov equation

$$\partial_t m^T - \Delta m^T - \operatorname{div}(m^T H_p(x, D_x U(t-T, x, m))) = 0, \quad m^T(0) = m_0.$$

As $x \to D_x U(t,x,m)$ is bounded in C^1 (see Proposition 5.3 below), we know from Theorem 5.1 that, as $T \to +\infty$, $(D_x U(t-T,\cdot,\cdot))$ converges uniformly to $D_x\chi$. So, for any $t \ge 0$, m^T converges in $C^0([0,t], \mathcal{P}(\mathbb{T}^d))$ towards a solution m of (88). Then again by Theorem 5.1, we have

$$\lim_{T\to+\infty} u^T(t,x) + \bar\lambda(t-T) = \lim_{T\to+\infty} U(t-T, x, m^T(t)) + \bar\lambda(t-T) = \chi(x, m(t)) + c.$$

Let us now fix $\delta > 0$. From Theorem 2.6, we have that $m^T(\delta T)$ converges (exponentially fast) to $\bar m$. Hence, by Theorem 5.1 again, we have

$$\lim_{T\to+\infty} u^T(\delta T, x) - (1-\delta)\bar\lambda T = \lim_{T\to+\infty} U(-(1-\delta)T, x, m^T(\delta T)) - (1-\delta)\bar\lambda T = \chi(x,\bar m) + c. \quad \square$$

The proof of Theorem 5.1 relies on estimates on $U(t,\cdot,\cdot)$ (independent of t) developed in the next section.

5A. ***Lipschitz estimates of the solution U.*** We collect here the main estimates satisfied by the solution of (87). They actually follow from the estimates developed in Section 2B for the solution (u,m) of the MFG system.

Proposition 5.3. *Let U be the solution to the master equation* (87). *Then there exists a constant C such that*

$$\sup_{t\le 0,\ m\in\mathcal{P}(\mathbb{T}^d)} \|U(t,\cdot,m) + \bar\lambda t\|_{C^{2+\alpha}} + \|D_m U(t,\cdot,m,\cdot)\|_{2+\alpha,1+\alpha} \le C, \tag{89}$$

while

$$\sup_{(x,m)\in\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)} |U(t,x,m) - U(s,x,m)| \le C|t-s|^{1/2} \quad \textit{for all } s,t \le 0,\ |s-t| \le 1.$$

Proof. Let us recall that, for any $t_0 \le 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, one has $U(t_0, x, m_0) = u(t_0, x)$, where (u,m) is the solution to the MFG system

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x,m) & \text{in } (t_0,0)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (t_0,0)\times\mathbb{T}^d, \\ m(t_0,\cdot) = m_0, \quad u(0,\cdot) = G(x, m(0)) & \text{in } \mathbb{T}^d. \end{cases}$$

By Lemma 1.5, we have the Lipschitz bound $\|Du\|_\infty \le C$, uniform with respect to the horizon t_0. This proves that $\|D_x U\|_\infty \le C$ and, in turn, that m is uniformly Hölder continuous in time with values in

$\mathcal{P}(\mathbb{T}^d)$; see (17). Furthermore, from Theorem 2.6 we get an estimate for $U(t-T,x,m)$ at time $t=0$; namely, that there exists a constant C, independent of T, such that

$$\|D_x U(-T,\cdot,m_0)\|_{C^{1+\alpha}} \le C,$$
$$\|U(-T,x,m_0) - \bar\lambda T\|_\infty \le C.$$

Therefore, we deduce that

$$\sup_{t\le 0,\ m\in\mathcal{P}(\mathbb{T}^d)} \|U(t,\cdot,m) + \bar\lambda t\|_{C^{2+\alpha}} \le C.$$

Following [Cardaliaguet et al. 2019], the derivative of U with respect to m can be represented as

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(t_0,x,m_0,y)\mu_0(y)\,dy = v(t_0,x), \tag{90}$$

where, for any smooth map $\mu_0 : \mathbb{T}^d \to \mathbb{R}$, (v,μ) solves the linearized problem

$$\begin{cases} -\partial_t v - \Delta v + H_p(x,Du)\cdot Dv = \dfrac{\delta F}{\delta m}(x,m)(\mu) & \text{in } (t_0,0)\times\mathbb{T}^d,\\ \partial_t\mu - \Delta\mu - \operatorname{div}(\mu H_p(x,Du)) - \operatorname{div}(mH_{pp}(x,Du)Dv) = 0 & \text{in } (t_0,0)\times\mathbb{T}^d,\\ \mu(t_0,\cdot) = \mu_0, \quad v(0,\cdot) = \dfrac{\delta G}{\delta m}(x,m(0))(\mu(0)) & \text{in } \mathbb{T}^d. \end{cases}$$

Our aim is to provide estimates on v in order to show the uniform Lipschitz regularity of U with respect to m. We assume that $\int_{\mathbb{T}^d}\mu_0 = 0$ since we are only interested in $D_m U = D_y(\delta U/\delta m)$. Then Corollary 2.8 states that

$$\sup_{t\in[0,T]} \|v(t)\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

This proves that

$$\left\| \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(t_0,\cdot,m_0,y)\mu_0(y)\,dy \right\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}$$

for any smooth map μ_0 with $\int_{\mathbb{T}^d}\mu_0 = 0$. Therefore, as in [Cardaliaguet et al. 2019], we obtain

$$\|D_m U(t_0,\cdot,m_0,\cdot)\|_{2+\alpha,1+\alpha} \le C. \tag{91}$$

It remains to check the time regularity of U. For this, let us first check that u is globally $\frac{1}{2}$-Holder in time. Let us recall that u is globally Lipschitz continuous in space. So, integrating in space the equation for u, the map $t \to \langle u(t)\rangle$ is globally Lipschitz continuous. Then the map $(t,x) \to u(t,x) - \langle u(t)\rangle$ is globally bounded in L^∞, is globally Lipschitz continuous in space and solves a heat equation with bounded right-hand side; therefore it is $\frac{1}{2}$-Holder continuous in time. This implies the global Holder continuity in time for u. As $U(t,x,m(t)) = u(t,x)$ and U is uniformly Lipschitz continuous in m, we have, for $t_0 \le s \le t_0+1$,

$$\begin{aligned} |U(s,x,m_0) - U(t_0,x,m_0)| &\le |U(s,x,m_0) - U(s,x,m(s))| + |U(s,x,m(s)) - U(t_0,x,m_0)| \\ &\le C\boldsymbol{d}_1(m_0,m(s)) + |u(s,x) - u(t_0,x)| \\ &\le C|s-t_0|^{1/2} + |u(s,x) - u(t_0,x)| \le C|s-t_0|^{1/2}, \end{aligned}$$

where we used the uniform regularity of m in time (since $H_p(\cdot, Du)$ is bounded, see Remark 1.6) for the second inequality, and the uniform Holder regularity in time of u in the last one. □

Remark 5.4. We stress that if we only use the regularity condition (FGb) on the couplings, then we can replace the conclusion of Corollary 2.8 with the first-order estimate (57) and obtain, rather than (91), the milder estimate $\|D_m D_x U(t,x,m)\|_\infty \le C$. This is actually enough to conclude with the uniform Lipschitz bound for U and $D_x U$, which is what is only needed in the proof of Theorem 5.1.

5B. ***Proof of Theorem 5.1.*** We are now ready to prove our main result.

Proof of Theorem 5.1. Let χ be a weak solution to the master cell problem (76). For $T>0$, let us consider

$$U^T(t,x,m) = U(t-T,x,m) \quad \text{for } (t,x,m)\in(-\infty,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d).$$

Then U^T solves

$$\begin{cases} -\partial_t U^T - \Delta_x U^T + H(x, D_x U) \\ \quad -\displaystyle\int_{\mathbb{T}^d} \operatorname{div}(D_m U^T(t,x,m,y))\,dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m U^T(t,x,m,y)\cdot H_p(D_x U(t,y,m,y))\,dm(y) = F(x,m) & \text{in } (-\infty,T)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d), \\ U^T(T,x,m) = G(x,m) & \text{in } \mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d). \end{cases}$$

By the Lipschitz regularity of U and $D_x U$ and the bound in (89) (Proposition 5.3), the family $\{U^T(\cdot,\cdot,\cdot)+\bar\lambda(\cdot - T)\}_T$ is relatively compact in $C^0(\mathbb{R}\times\mathbb{T}\times\mathcal{P}(\mathbb{T}^d))$. Let $T_n\to+\infty$ be any sequence such that $(t,x,m)\to U^{T_n}(t,x,m)+\bar\lambda(t-T_n)$ locally uniformly converges to some $V(t,x,m)$. Then V is a weak solution to

$$-\partial_t V + \bar\lambda - \Delta_x V + H(x, D_x V) - \int_{\mathbb{T}^d} \operatorname{div}(D_m V(t,x,m,y))\,dm(y) \\ + \int_{\mathbb{T}^d} D_m V(t,x,m,y)\cdot H_p(y, D_x U(t,y,m,y))\,dm(y) = F(x,m) \quad \text{in } \mathbb{R}\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d) \tag{92}$$

in the sense that V satisfies similar requirements to those in Definition 4.1. Namely, V and $D_x V$ are uniformly Lipschitz continuous in x and m and $\frac{1}{2}$-Hölder continuous in the time variable, V is monotone in m and satisfies that, for any $t_1\le t_2$ and if (u,m) solves the MFG system

$$\begin{cases} -\partial_t u + \bar\lambda - \Delta u + H(x,Du) = F(x,m) & \text{in } (t_1,t_2)\times\mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m H_p(x,Du)) = 0 & \text{in } (t_1,t_2)\times\mathbb{T}^d, \\ m(t_1,\cdot) = m_0, \quad u(t_2,\cdot) = V(t_2,x,m(t_2)) & \text{in } \mathbb{T}^d, \end{cases} \tag{93}$$

we have $V(t_1,x,m_0)=u(t_1,x)$ (and so $V(t,x,m(t))=u(t,x)$ for any $t\in[t_1,t_2]$).

Our goal is to show that $V(t,x,m)-\chi(x,m)$ is constant. Let us recall that Proposition 2.7 implies that $U^T(0,x,\bar m)-\bar\lambda T-\bar u$ converges to a constant $\bar c$ as $T\to+\infty$. Hence $V(0,x,\bar m)=\bar u(x)+\bar c$. Since $\chi(x,\bar m)=\bar u$, this shows that, if $V(t,x,m)-\chi(x,m)$ is proved to be constant, then this constant will be equal to $\bar c$, and independent of the subsequence (T_n).

Let us fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$. Let $T > 0$ be large and (u, m) be the solution to the MFG system (93) with $t_1 = 0$ and $t_2 = T$. We note that m is the unique solution to the McKean–Vlasov equation

$$\begin{cases} \partial_t m - \Delta m - \operatorname{div}(m H_p(x, D_x V(t, x, m))) = 0 & \text{on } [0, T] \times \mathbb{T}^d, \\ m(0) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{94}$$

In particular, since V and $D_x V$ are globally Lipschitz in m, this implies that m and u are defined independently of the horizon T (meaning that, for $t \in [0, T]$, $u(t, \cdot) := V(t, \cdot, m(t))$ and $m(t, \cdot)$ do not depend on T).

In the same way we define $(\tilde{u}, \tilde{m})$ to be the solution to the MFG system

$$\begin{cases} -\partial_t \tilde{u} + \bar{\lambda} - \Delta \tilde{u} + H(x, D\tilde{u}) = F(x, \tilde{m}) & \text{in } (0, T) \times \mathbb{T}^d, \\ \partial_t \tilde{m} - \Delta \tilde{m} - \operatorname{div}(\tilde{m} H_p(x, D\tilde{u})) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\ \tilde{m}(0, \cdot) = m_0, \quad \tilde{u}(T, \cdot) = \chi(x, \tilde{m}(T)) & \text{in } \mathbb{T}^d. \end{cases}$$

As before we note that $(\tilde{u}, \tilde{m})$ does not depend on the horizon T, that $\tilde{u}(t, x) = \chi(x, \tilde{m}(t))$ for any $t \in [0, T]$ and that $\tilde{m}$ is the unique solution to the McKean–Vlasov equation

$$\partial_t \tilde{m} - \Delta \tilde{m} - \operatorname{div}(m H_p(x, D_x \chi(x, \tilde{m})) = 0 \quad \text{on } [0, T],\ \tilde{m}(0) = m_0. \tag{95}$$

Using the result of Theorem 2.6 with both $G(x, \cdot) = V(T, x, \cdot)$ and $G = \chi(x, \cdot)$, we have (changing u into $u + \bar{\lambda}(T - t)$ and $\tilde{u}$ into $\tilde{u} + \bar{\lambda}(T - t)$),

$$\|m(t) - \bar{m}\|_\infty + \|\tilde{m}(t) - \bar{m}\|_\infty \le C(e^{-\gamma t} + e^{-\gamma(T-t)}), \quad t \in [1, T],$$

where $(\bar{u}, \bar{m})$ is the solution to the ergodic MFG system (2). But since m and $\tilde{m}$ do not depend on the horizon T, here we can let first $T \to \infty$, and then $t \to \infty$, so we conclude that both $m(t)$ and $\tilde{m}(t)$ converge to $\bar{m}$ as $t \to +\infty$.

Applying once more the standard estimates on the MFG systems, we have

$$\int_0^T \int_{\mathbb{T}^d} (m + \tilde{m}) |Du - D\tilde{u}|^2 \le -C \left[\int_{\mathbb{T}^d} (u - \tilde{u})(m - \tilde{m}) \right]_0^T = -C \int_{\mathbb{T}^d} (u(T) - \tilde{u}(T))(m(T) - \tilde{m}(T))$$

since $m(0) = \tilde{m}(0) = m_0$. As u and $\tilde{u}$ are uniformly Lipschitz continuous in space and $m(T)$ and $\tilde{m}(T)$ have the same limit $\bar{m}$ as $T \to +\infty$, we deduce that

$$\lim_{T \to +\infty} \int_0^T \int_{\mathbb{T}^d} (m + \tilde{m}) |Du - D\tilde{u}|^2 = 0.$$

In particular, as m (and $\tilde{m}$) are regular and bounded below by a positive constant on intervals of the form $[\varepsilon, T]$ with $\varepsilon > 0$, we deduce that $Du = D\tilde{u}$ on $[\varepsilon, T]$ and thus on $[0, T]$. Therefore m and $\tilde{m}$ solve the same equation, which implies $m(t) = \tilde{m}(t)$ for any $t \ge 0$. Coming back to the equations satisfied by u and $\tilde{u}$ gives $\partial_t u = \partial_t \tilde{u}$, so there is a constant c such that $u(t, x) = \tilde{u}(t, x) + c$. In other words

$$V(t, x, m(t)) = \chi(x, m(t)) + c \quad \text{for all } t \ge 0.$$

Notice that the above conclusion holds for any given $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and the constant c could depend on m_0 at this stage. But we are going to show that this is actually not the case.

Indeed, let us choose $m_0 = \bar{m}$. Then Proposition 4.7 says that $m(t) = \tilde{m}(t) = \bar{m}$. We denote by $\bar{c}$ the constant found above, i.e., $u(t,x) = \tilde{u}(t,x) + \bar{c}$. By definition, this implies $V(t,x,\bar{m}) = \chi(x,\bar{m}) + \bar{c}$. Now, for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, we recall that the solution $m(t) = \tilde{m}(t)$ converges to $\bar{m}$ as $t \to +\infty$. By the uniform Lipschitz continuity of χ and V with respect to m (uniform in (t,x)), this implies

$$|V(t,x,m(t)) - V(t,x,\bar{m})| + |\chi(x,m(t)) - \chi(x,\bar{m})| \to 0 \quad \text{as } t \to \infty.$$

Since

$$|c - \bar{c}| = |V(t,x,m(t)) - \chi(x,m(t)) - (V(t,x,\bar{m}) - \chi(x,\bar{m}))|,$$

by letting $t \to \infty$ we deduce that $c = \bar{c}$. In particular, we have proved that

$$V(0,x,m_0) = \chi(x,m_0) + \bar{c} \quad \text{for all } m_0 \in \mathcal{P}(\mathbb{T}^d).$$

Finally, we can apply the above reasoning to the translation $V(\cdot + t_0, x, m)$ for any $t_0 \in \mathbb{R}$. It turns out that $\bar{c} = \lim_{t\to\infty} V(t+t_0,x,m(t)) - \chi(x,m(t))$, which is clearly independent of t_0. Therefore we conclude that

$$V(t_0,x,m_0) = \chi(x,m_0) + \bar{c} \quad \text{for all } (t_0,x,m_0) \in \mathbb{R} \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \qquad \square$$

Let us point out that any weak solution of the ergodic master equation solves (92). So the above proof actually shows that two solutions of the ergodic master equation differ only by a constant:

Corollary 5.5. *If χ_1 and χ_2 are weak solutions of the ergodic master equation* (76), *then there exists a constant $\bar{c}$ such that*

$$\chi_2(x,m) = \chi_1(x,m) + \bar{c} \quad \textit{for all } (x,m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

6. The discounted problem

We now investigate the behavior, as $\delta \to 0^+$, of the solution U^δ of the discounted master equation (6). Our main result is:

Theorem 6.1. *Let U^δ be the solution to the discounted master equation* (6) *and $(\bar{\lambda}, \bar{u}, \bar{m})$ the solution of the ergodic problem* (2). *Then, as $\delta \to 0^+$, $U^\delta - \bar{\lambda}/\delta$ converges uniformly to the solution χ to the master cell problem* (76) *such that $\chi(x,\bar{m}) = \bar{u}(x) + \bar{\theta}$, where $\bar{\theta}$ is the unique constant for which the following linearized ergodic problem has a solution $(\bar{v}, \bar{\mu})$:*

$$\begin{cases} \bar{u} + \bar{\theta} - \Delta\bar{v} + H_p(x,D\bar{u})\cdot D\bar{v} = \dfrac{\delta F}{\delta m}(x,\bar{m})(\bar{\mu}) & \textit{in } \mathbb{T}^d, \\ -\Delta\bar{\mu} - \operatorname{div}(\bar{\mu}H_p(x,D\bar{u})) - \operatorname{div}(\bar{m}H_{pp}(x,D\bar{u})D\bar{v}) = 0 & \textit{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d}\bar{\mu} = \int_{\mathbb{T}^d}\bar{v} = 0. \end{cases} \tag{96}$$

Let us comment a bit more on the normalization condition $\chi(x,\bar{m}) = \bar{u}(x) + \bar{\theta}$ which selects the unique limit of the discounted master equation (6), according to the above result. As we shall see in the next section, given any (not necessarily normalized with zero average) solution $\bar{u}$ to

$$\bar{\lambda} - \Delta\bar{u} + H(x,D\bar{u}) = F(x,\bar{m}) \text{in } \mathbb{T}^d, \tag{97}$$

there is a unique constant $\bar{\theta}$ for which (96) admits a solution. However, since $\bar{u}$ is unique up to addition of a constant, the sum $\bar{u}+\bar{\theta}$ will be uniquely determined. Indeed, by changing $\bar{u}$ through the addition of a constant, the value $\bar{\theta}$ will be translated accordingly. In other words, one can say that the limit of $U^\delta - \bar{\lambda}/\delta$ is the solution χ of the master cell problem (76) such that $\chi(x,\bar{m})$ coincides with the unique solution of (97) for which the constant $\bar{\theta}$ vanishes.

Exactly as for the time-dependent problem, we can infer from Theorem 6.1 the limit behavior of the solution of the discounted MFG system:

Corollary 6.2. *Let $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and, for $\delta > 0$, let (u^δ, m^δ) be the solution to the discounted MFG system* (5). *Then*

$$\lim_{\delta\to 0} u^\delta(0,x) - \bar{\lambda}/\delta = \chi(x,m_0),$$

uniformly with respect to x, where χ is the solution of the ergodic cell problem (76) *given in Theorem 6.1.*

6A. ***An additional ergodic system.*** Given a solution $\bar{u}$ of the MFG ergodic problem (2), we investigate the ergodic problem (96). The heuristic justification of (96) is that we expect the solution $(\bar{u}^\delta, \bar{m}^\delta)$ of (7) to be of the form

$$\bar{u}^\delta \sim \frac{\bar{\lambda}}{\delta} + \bar{u} + \bar{\theta} + \delta\bar{v}, \quad \bar{m}^\delta \sim \bar{m} + \delta\bar{\mu}, \tag{98}$$

and, in view of (7), the equation satisfied by $(\bar{\theta}, \bar{v}, \bar{\mu})$ should be (96).

We start the proof of the existence for (96) as usual, by a discounted problem:

Lemma 6.3. *Let $A, B \in L^\infty(\mathbb{T}^d)$. For $\delta > 0$ small, there is a unique solution $(v^\delta, \mu^\delta) \in W^{1,\infty}(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d)$ to the discounted system*

$$\begin{cases} \bar{u} + \delta v^\delta - \Delta v^\delta + H_p(x, D\bar{u})\cdot Dv^\delta = \dfrac{\delta F}{\delta m}(x,\bar{m})(\mu^\delta) + A & \text{in } \mathbb{T}^d, \\ -\Delta\mu^\delta - \operatorname{div}(\mu^\delta H_p(x,D\bar{u})) - \operatorname{div}(\bar{m}H_{pp}(x,D\bar{u})Dv^\delta) = \operatorname{div}(B) & \text{in } \mathbb{T}^d, \end{cases} \tag{99}$$

with $\int_{\mathbb{T}^d} \mu^\delta = 0$. Moreover, there is a constant $C > 0$ (independent of δ, A and B) such that

$$\|\delta v^\delta\|_\infty + \|Dv^\delta\|_\infty + \|\mu^\delta\|_\infty \le C(1 + \|A\|_\infty + \|B\|_\infty).$$

Proof. Existence of a solution runs with a standard fixed point, so we omit it. The duality relation (here between v^δ and μ^δ) gives (using Poincaré's inequality)

$$\begin{aligned} C^{-1}\|Dv^\delta\|_{L^2}^2 &\le \int_{\mathbb{T}^d} (\bar{u} + \delta v^\delta - A)\mu^\delta + B\cdot Dv^\delta \\ &\le (\|D\bar{u}\|_{L^2} + \delta\|Dv^\delta\|_{L^2} + \|A\|_{L^2})\|\mu^\delta\|_{L^2} + \|B\|_{L^2}\|Dv^\delta\|_{L^2}, \end{aligned}$$

so

$$\|Dv^\delta\|_{L^2} \le C\big((\|D\bar{u}\|_{L^2}^{1/2} + \|A\|_{L^2}^{1/2})\|\mu^\delta\|_{L^2}^{1/2} + \delta\|\mu^\delta\|_{L^2} + \|B\|_{L^2}\big).$$

By Corollary 1.3, we have

$$\|\mu^\delta\|_{L^2} \le C(\|Dv^\delta\|_{L^2} + \|B\|_{L^2}) \le C\big((\|D\bar{u}\|_{L^2}^{1/2} + \|A\|_{L^2}^{1/2})\|\mu^\delta\|_{L^2}^{1/2} + \delta\|\mu^\delta\|_{L^2} + \|B\|_{L^2}\big).$$

So, for $\delta > 0$ small enough, we obtain

$$\|\mu^\delta\|_{L^2} \le C(\|D\bar{u}\|_{L^2} + \|A\|_{L^2} + \|B\|_{L^2}).$$

This implies the same bound for Dv^δ and, by the maximum principle, the estimate

$$\|\delta v^\delta\|_\infty \le C(\|\bar{u}\|_{L^\infty} + \|D\bar{u}\|_{L^2} + \|B\|_{L^2} + \|A\|_{L^\infty}).$$

Moreover, considering the equation satisfied by $w := v^\delta - \langle v^\delta\rangle$, we have by local regularity for weak solutions [Gilbarg and Trudinger 1977, Theorem 8.17] and Poincaré's inequality

$$\|v^\delta - \langle v^\delta\rangle\|_\infty \le C(1 + \|v^\delta - \langle v^\delta\rangle\|_{L^2}) \le C(1 + \|Dv^\delta\|_{L^2}) \le C(1 + \|\bar{u}\|_{W^{1,\infty}} + \|A\|_{L^\infty} + \|B\|_{L^2}).$$

Then by classical elliptic regularity [Gilbarg and Trudinger 1977, Theorem 8.32], we have, for any $\alpha \in (0, 1)$,

$$\|v^\delta - \langle v^\delta\rangle\|_{C^{1+\alpha}} \le C(1 + \|\bar{u}\|_{W^{1,\infty}} + \|A\|_{L^\infty} + \|B\|_{L^2}).$$

We can now apply the local regularity for weak solutions to μ^δ [Gilbarg and Trudinger 1977, Theorem 8.17]) and infer that

$$\|\mu^\delta\|_{C^\alpha} \le C(\|Dv^\delta\|_\infty + \|B\|_\infty) \le C(\|\bar{u}\|_{W^{1,\infty}} + \|A\|_{L^\infty} + \|B\|_{L^\infty}). \qquad \square$$

Proposition 6.4. *Let* $(\bar{\lambda}, \bar{u}, \bar{m})$ *be a solution of the ergodic system* (2) *and* (v^δ, μ^δ) *be the solution to* (99) *for A and B satisfying*

$$\|A\|_\infty + \|B\|_\infty \le C\delta$$

for some constant C. Then, as $\delta \to 0^+$,

$$\delta\langle v^\delta\rangle \longrightarrow \bar{\theta}, \quad (v^\delta - \langle v^\delta\rangle) \xrightarrow{L^\infty} \bar{v}, \quad \mu^\delta \xrightarrow{L^\infty} \bar{\mu},$$

where $(\bar{\theta}, \bar{v}, \bar{\mu})$ *is the unique solution to* (96).

Proof of Proposition 6.4. Passing to the limit in (99) (up to a subsequence) provides a constant $\bar{\theta}$ (limit of $\delta\langle v^\delta\rangle$), a map $\bar{v} \in W^{1,\infty}$ (limit of $v^\delta - \langle v^\delta\rangle$) and a map $\bar{\mu} \in L^\infty$ (limit of μ^δ) which solve (96). The uniqueness of $D\bar{v}$ (and hence of $\bar{v}$) and of $\bar{\mu}$ can be established by the standard duality argument of [Lasry and Lions 2007]. Then $\bar{\theta}$ is unique by the equation. The full convergence of $(\delta\langle v^\delta\rangle, v^\delta - \langle v^\delta\rangle, \mu^\delta)$ holds by uniqueness of the limit. $\square$

6B. ***Proof of Theorem 6.1.*** The proof of Theorem 6.1 consists mostly in showing that the heuristic relation (98) holds.

Proposition 6.5. *Let* $(\bar{\lambda}, \bar{u}, \bar{m})$, $(\bar{u}^\delta, \bar{m}^\delta)$ *and* $(\bar{\theta}, \bar{v}, \bar{\mu})$ *be respectively solutions to* (2), (7) *and* (96). *Then*

$$\lim_{\delta\to 0^+} \left\| \bar{u}^\delta - \frac{\bar{\lambda}}{\delta} - \bar{u} - \bar{\theta} \right\|_\infty + \|\bar{m}^\delta - \bar{m}\|_\infty = 0.$$

Proof. The argument is very close to the proof of the exponential rate (see Theorem 2.6). Let

$$E = \{(v, \mu) \in W^{1,\infty}(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d) : \|\delta v\|_\infty + \|Dv\|_\infty + \|\mu\|_\infty \le \widehat{C}\},$$

where $\widehat{C}$ is to be chosen below. For $(v, \mu) \in E$, we consider the solution $(\hat{v}, \hat{\mu})$ to (99) with

$$A(x) := \delta^{-1}\Big(-(H(x, D(\bar{u}+\delta v)) - H(x, D\bar{u}) - \delta H_p(x, D\bar{u}) \cdot Dv) + F(x, \bar{m}+\delta\mu) - F(x, \bar{m}) - \delta \frac{\delta F}{\delta m}(x, \bar{m})(\mu)\Big),$$

$$B(x) := \delta^{-1}\big((\bar{m}+\delta\mu)H_p(x, D(\bar{u}+\delta v)) - \bar{m}H_p(x, D\bar{u}) - \delta\mu H_p(x, D\bar{u}) - \delta\bar{m}H_{pp}(x, \bar{m})Dv\big).$$

As

$$\|A\|_\infty + \|B\|_\infty \le C\widehat{C}^2\delta,$$

we have, by Lemma 6.3 (and for δ small enough),

$$\|\delta\hat{v}\|_\infty + \|D\hat{v}\|_\infty + \|\hat{\mu}\|_\infty \le C(1 + \|A\|_\infty + \|B\|_\infty) \le C(1 + \widehat{C}^2\delta).$$

We can choose $\widehat{C}$ such that, for δ small enough, the right-hand side is less than $\widehat{C}$. Then we can easily conclude that the map $(v, \mu) \to (\hat{v}, \hat{\mu})$ has a fixed point (v^δ, μ^δ). Note that $(\bar{\lambda}/\delta + \bar{u} + \delta v^\delta, \bar{m} + \delta\mu^\delta)$ solves (7) and therefore is equal to $(\bar{u}^\delta, \bar{m}^\delta)$. Hence, by Proposition 6.4, we deduce

$$\left\|\bar{u}^\delta - \frac{\bar{\lambda}}{\delta} - \bar{u} - \bar{\theta}\right\|_\infty = \|\delta v^\delta - \bar{\theta}\|_\infty \le \|\delta(v^\delta - \langle v^\delta\rangle)\|_\infty + |\delta\langle v^\delta\rangle - \bar{\theta}| \to 0 \quad \text{as } \delta \to 0,$$

which completes the proof. □

Proof of Theorem 6.1. Recall that we have uniform Lipschitz estimates on U^δ and on D_xU^δ (Lemma 4.3 and Proposition 4.4) and that any converging subsequence is a weak solution of the ergodic master equation (proof of Theorem 4.2). Therefore, we only need to show that $U^\delta - \delta^{-1}\bar{\lambda}$ has a limit when evaluated at some value. For this, let $(\bar{u}^\delta, \bar{m}^\delta)$ be the solution to (7). As $(\bar{u}^\delta, \bar{m}^\delta)$ is also a stationary solution to (5), we have

$$U^\delta(x, \bar{m}^\delta) = \bar{u}^\delta(x) \quad \text{for all } x \in \mathbb{T}^d.$$

We have seen in Proposition 6.5 that, as $\delta \to 0$, $\bar{m}^\delta$ converges to $\bar{m}$, while $\bar{u}^\delta - \delta^{-1}\bar{\lambda}$ converges to $\bar{u} + \bar{\theta}$. □

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